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Blow-up of solutions to parabolic equations with nonstandard growth conditions

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ABSTRACT

We study the phenomenon of finite time blow-up in solutions of the homogeneous Dirichlet problem for the parabolic equation

$$u_t = \operatorname{div}\left(a(x,t)|\nabla u|^{p(x)-2}\nabla u\right) + b(x,t)|u|^{\sigma(x,t)-2}u$$

with variable exponents of nonlinearity $p(x), \sigma(x,t) \in (1,\infty)$. Two different cases are studied. In the case of semilinear equation with $p(x) \equiv 2$, $a(x,t) \equiv 1$, $b(x,t) \geq b^- > 0$ we show that the finite time blow-up happens if the initial function is sufficiently large and either $\min_{\Omega} \sigma(x,t) = \sigma^-(t) > 2$ for all $t > 0$, or $\sigma^-(t) \geq 2$, $\sigma^-(t) \searrow 2$ as $t \to \infty$ and $\int_1^\infty e^{s(2-\sigma^-(s))}\,ds < \infty$. In the case of the evolution $p(x)$-Laplace equation with the exponents $p(x), \sigma(x)$ independent of t, we prove that every solution corresponding to a sufficiently large initial function exhibits a finite time blow-up if $b(x,t) \geq b^- > 0$, $a_t(x,t) \leq 0$, $b_t(x,t) \geq 0$, $\min \sigma(x) > 2$ and $\max p(x) \leq \min \sigma(x)$.

1. Introduction

This work addresses the blow-up phenomenon in solutions of nonlinear parabolic equations with variable nonlinearity. We consider the Dirichlet problem

$$\begin{cases} u_t = \operatorname{div}\left(a(x,t)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + b(x,t)\,|u|^{\sigma(x,t)-2}\,u \quad \text{for } (x,t) \in Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_T, \end{cases} \tag{1.1}$$

where $Q_T = \Omega \times (0,T]$ is the cylinder with the base $\Omega \subset \mathbb{R}^n$, $\Gamma = \partial\Omega$ and $\Gamma_T = \Gamma \times (0,T]$. The coefficients a, b and the exponents p, σ are given measurable functions of their arguments. It is assumed that these functions satisfy the following conditions:

$$\begin{array}{ll} 0 < a^- \leq a(x,t) \leq a^+ \leq \infty, & 0 \leq b^- \leq b(x,t) \leq b^+ \leq \infty, \\ 1 < p^- \leq p(x) \leq p^+ < \infty, & 1 < \sigma^- \leq \sigma(x,t) \leq \sigma^+ < \infty. \end{array} \tag{1.2}$$

Equations of the type (1.1) appear in the mathematical modelling of various physical phenomena such as flows of electro-rheological or thermo-rheological fluids [1–3], processes of filtration through a porous medium. They are frequently used in the processing of digital images [4–6]. For a more detailed information on the possible applications of these models to the study of the real world processes we refer the reader to the papers [7,3,8] and the further references therein.

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Equations of the type (1.1) are usually referred to as equations with nonstandard growth conditions. In the recent years, PDEs of this type have been intensively studied. The questions of existence, uniqueness and qualitative properties of solutions for elliptic and parabolic equations with variable nonlinearity were discussed by many authors and under different conditions on the data; see, for example, [9–13,7,14–17].

It is known that parabolic equations with variable nonlinearity may possess, for certain ranges of the exponents, the localization (alias vanishing) properties which are intrinsic for the solutions of nonlinear equations with constant nonlinearity such as vanishing in a finite time, finite speed of propagation on disturbances from the data or waiting time phenomena (see [14–16,18]), but thus far only one work [19] has addressed the question of possible blow-up of solutions of the parabolic PDEs with nonstandard growth conditions. An excellent insight into the theory of blow-up behavior of solutions to parabolic equations with constant nonlinearity can be found in the monographs [20–22] (see also [23–30]). Paper [19] deals with the solutions of the homogeneous Dirichlet problem for the semilinear parabolic equation

$$\begin{cases} u_t = \Delta u + f(x,u) & \text{in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, & u = 0 \text{ on } \Gamma_T, \end{cases}$$

where the source term is either a power,

$$f(x,u) = a(x)u^{p(x)} \quad \text{or is nonlocal: } f(x,u) = a(x)\int_{\Omega} u^{q(y)}(y,t)\mathrm{d}y.$$

In the present paper, we consider a more general class of parabolic equations with nonstandard growth conditions. In Section 2 we give the definition of weak solution to problem (1.1) and remind the existence theorem. In Section 3 we study problem (1.1) for the semilinear equations with $a(x,t) = 1$, $p(x,t) = 2$, and variable $\sigma(x,t)$, $b(x,t)$. The first result of this section extends the assertion of [19] to the case when the exponent of nonlinearity in the source term may depend on t. The second result is specific to the case when the exponent σ depends on t. We show that the solutions of the semilinear problem (1.1) may blow-up even in the case when $\sigma(x,t) \searrow 2$ as $t \to \infty$ and the equation eventually becomes linear. In Section 4 we present some examples and generalize the conclusions of Section 3 to the case when the Laplace operator is substituted by a linear elliptic operator of general form but with the coefficients independent of t. Another generalization concerns the form of the source term which can be nonlocal. In Section 5 we establish sufficient conditions of the blow-up for solutions of problem (1.1) assuming that the exponents of nonlinearity $p(x)$ and $\sigma(x)$ are independent of t, and satisfy the conditions $p^+ \le \sigma^-, \sigma^- > 2$. The coefficients $a(x,t), b(x,t)$ are assumed differentiable in t and monotone: $a_t(x,t) \le 0$, $b_t(x,t) \ge 0$. Results of this work were announced in the preprint [31].

The proofs of the main results are based on the reduction of the problem to the study of a nonlinear ordinary differential inequality for a suitably chosen function associated with the solution. It happens so that every function satisfying such an inequality becomes unbounded in a finite time which yields the finite time blow-up of the corresponding weak solution. The technical implementation of this idea is different in the cases of the semilinear and quasilinear equations. In the case of the semilinear equation we follow the eigenfunction method of S. Kaplan [32] and choose the first eigenfunction of the Dirichlet problem for the Laplace operator for the test-function in the definition of weak solution. To deal with the evolution $p(x)$-Laplace equation we choose the solution for the test-function and claim that the initial function possesses some specific properties. In the result we obtain the ordinary differential inequality for the function $\|u(\cdot,t)\|^2_{2,\Omega}$ and then show that this function becomes infinite in a finite time.

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2. Preliminaries

2.1. The function spaces

Throughout this paper, we assume that

$$\Omega \subset \mathbb{R}^n \text{ is a bounded domain with the Lipschitz-continuous boundary } \Gamma = \partial\Omega. \tag{2.3}$$

Let $p(x,t) \ge 1$ be a measurable bounded function defined in the cylinder $Q_T = \Omega \times (0,T]$. We introduce the set of functions

$$L^{p(\cdot)}(Q_T) = \left\{ u(x,t) : \ u \text{ is measurable in } Q_T, \ A_{p(\cdot)}(u) \equiv \int_{Q_T} |u|^{p(x,t)}\,\mathrm{d}x\mathrm{d}t < \infty \right\}.$$

The set $L^{p(\cdot)}(Q_T)$ equipped with the norm (Luxemburg's norm)

$$\|u\|_{p(\cdot),Q_T} = \inf\left\{ \lambda > 0 : \int_{Q_T} \left|\frac{u}{\lambda}\right|^{p(x,t)} \mathrm{d}x\mathrm{d}t < 1 \right\}$$

becomes a Banach space. The set $C^{\infty}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$, provided that the exponent $p(x,t) \in C^0(\overline{Q}_T)$. For the elements of these spaces Hölder's inequality holds in the following form: for $f \in L^{p(\cdot)}(Q_T)$, $g \in L^{q(\cdot)}(Q_T)$ with $p(x,t) \in [p^-, p^+] \subset$

$(1, \infty)$, $q(x,t) = \frac{p(x,t)}{p(x,t)-1} \in [q^-, q^+] \subset (1, \infty)$

$$\left| \int_{Q_T} f\, g \,\mathrm{d}x\mathrm{d}t \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|f\|_{p(\cdot),Q_T} \|g\|_{q(\cdot),Q_T}. \tag{2.4}$$

The norms $\|\cdot\|_{p(\cdot),Q_T}$ can be estimated in terms of the integrals $A_{p(\cdot)}(u)$: for every $u \in L^{p(\cdot)}(Q_T)$

$$\min\left\{ \|u\|^{p^+}_{p(\cdot),Q_T},\ \|u\|^{p^-}_{p(\cdot),Q_T} \right\} \le A_{p(\cdot)}(u) \le \max\left\{ \|u\|^{p^+}_{p(\cdot),Q_T},\ \|u\|^{p^-}_{p(\cdot),Q_T} \right\}. \tag{2.5}$$

By $\mathbf{W}(Q_T)$ we denote the Banach space

$$\begin{cases} \mathbf{W}(Q_T) = \left\{ u(x,t) :\ u \in L^2(Q_T),\ |\nabla u| \in L^{p(x,t)}(Q_T),\ u = 0 \text{ on } \Gamma_T \right\}, \\ \|u\|_{\mathbf{W}(Q_T)} = \|u\|_{2,Q_T} + \|\nabla u\|_{p(\cdot),Q_T}, \end{cases}$$

and by $\mathbf{W}'(Q_T)$ we denote the dual of $\mathbf{W}(Q_T)$ with respect to the scalar product in $L^2(Q_T)$. The set $C_0^\infty(Q_T)$ is dense in $\mathbf{W}(Q_T)$ if $p(x,t)$ is Log-continuous in $\overline{Q}_T$ (see condition (2.8) below, [15, Sec. 2]).

2.2. Existence of weak solutions

Let us consider the following problem:

$$\begin{cases} u_t = \operatorname{div}\left(a(x,t,u)|\nabla u|^{p(x,t)-2}\nabla u \right) + d(x,t,u) \quad \text{in } Q_T, \\ u = 0 \text{ on } \Gamma_T, \quad u(x,0) = u_0(x) \text{ in } \Omega. \end{cases} \tag{2.6}$$

Definition 1. A function $u(x,t) \in \mathbf{W}(Q_T) \cap L^\infty(0,T;\ L^2(\Omega))$ is called weak solution of problem (2.6) if for every test-function $\zeta \in \{\eta(z) :\ \eta \in \mathbf{W}(Q_T) \cap L^\infty\left(0,T; L^2(\Omega)\right),\ \eta_t \in \mathbf{W}'(Q_T)\}$, and every $t_1, t_2 \in [0,T]$, the following identity holds:

$$\int_{t_1}^{t_2} \int_\Omega \left(u\zeta_t - a(x,t,u)|\nabla u|^{p(x,t)-2}\nabla u \cdot \nabla\zeta + d(x,t,u)\zeta \right) \mathrm{d}z = \int_\Omega u\zeta \,\mathrm{d}x \Big|_{t_1}^{t_2}. \tag{2.7}$$

Let us assume that the exponent $p(x,t)$ and the coefficient $a(x,t,u)$ are subject to the conditions

$$\begin{cases} a(x,t,u) \text{ and } d(x,t,u) \text{ are Carathéodory functions} \\ \quad \text{(measurable in } (x,t) \text{ for every } s \in \mathbb{R}, \text{ continuous in } u \text{ for a.a. } (x,t) \in Q_T), \\ p(\cdot) \text{ is continuous in } \overline{Q}_T \text{ with the logarithmic module of continuity:} \\ \forall\, z,\ \zeta \in \overline{Q}_T,\ |z-\zeta| < 1, \quad |p(z) - p(\zeta)| \le \omega(|z-\zeta|) \quad \text{where } \overline{\lim_{\tau\to 0^+}}\, \omega(\tau)\ln\frac{1}{\tau} < +\infty. \end{cases} \tag{2.8}$$

Theorem 1 (*[15, Theorem 4.3]*)**.** *Let assumptions (1.2), (2.3) and (2.8) be fulfilled and let the function $d(x,t,u)$ satisfy the growth condition*

$$|d(x,t,s)| \le d_0|s|^{\delta-1} + h(x,t) \quad \textit{with some constants } d_0 > 0 \textit{ and } \delta > 2.$$

Then for every $u_0 \in L^\infty(\Omega)$ there exists $\theta \in (0,T]$, depending on δ, d_0, $\|u_0\|_{L^\infty(\Omega)}$, $\|h\|_{L^1(0,\theta;L^\infty(\Omega))}$, such that problem (2.6) has at least one weak solution $u \in \mathbf{W}(Q_\theta)$ with $u_t \in \mathbf{W}'(Q_\theta)$ and $\|u\|_{\infty,Q_\theta} < \infty$. The solution can be continued to the interval $[0,T^*]$ where

$$T^* = \sup\{\theta > 0 :\ \|u\|_{\infty,Q_\theta} < \infty\}.$$

Remark 1. Let $p(x,t)$ satisfy the log-continuity condition (2.8). Then for every $u,\ \phi \in \mathbf{W}(Q_T)$ with $\partial_t\phi,\ \partial_t u \in \mathbf{W}'(Q_T)$ the formula of integration by parts hold (see [15, Proposition 2.5])

$$\int_{Q_T} u\,\partial_t\phi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \phi\,\partial_t u \,\mathrm{d}x\mathrm{d}t = \int_\Omega u\,\phi \,\mathrm{d}x \Big|_{t=t_1}^{t=t_2},$$

and identity (2.7) can be written in the form

$$\int_{t_1}^{t_2} \int_\Omega \left(\zeta\,\partial_t u + a(x,t,u)|\nabla u|^{p(x,t)-2}\nabla u \cdot \nabla\zeta - d(x,t,u)\zeta \right) \mathrm{d}z = 0. \tag{2.9}$$

It follows from identity (2.7) that $\int_\Omega \phi(x,t)\,u(x,t)\,\mathrm{d}x \in C^0(0,T)$ for every $\phi \in \mathbf{W}(Q_T)$.

Remark 2. In the rest of this paper we assume that the data of problem (2.6) satisfy the conditions of Theorem 1. Although these conditions are not explicitly used anymore, they provide the analytical frame for the further study and ensure the existence of weak solutions in the sense of Definition 2, guarantee the possibility of integration by parts and make possible the derivation of a priori estimates needed in the proof of Theorem 4.

3. Semilinear equation with nonlinear source

3.1. Statement of problem and results

Let us consider the semilinear problem

$$\begin{cases} u_t = \Delta u + b(x,t)u^{\sigma(x,t)-1} & \text{in } Q_T, \\ u(x,0) = u_0(x) \geq 0 \text{ in } \Omega, & u = 0 \text{ on } \Gamma_T \end{cases} \tag{3.10}$$

with the coefficients $b(x,t)$, $\sigma(x,t)$ satisfying conditions (1.2) and (2.8). Under these conditions problem (3.10) has a local in time solution for every $u_0 \in L^\infty(\Omega)$. Moreover, $u \geq 0$ a.e. in Q_T, provided that $u_0 \geq 0$ in Ω [15, Theorem 4.1].

To study the possibility of the blow-up we will apply the eigenvalue method of S. Kaplan [32]. Let $\lambda > 0$, $\phi(x) \geq 0$ be the first eigenvalue and the corresponding eigenfunction of the Dirichlet problem for the Laplace operator in Ω:

$$-\Delta\phi = \lambda\phi \text{ in } \Omega, \quad \phi = 0 \text{ on } \Gamma. \tag{3.11}$$

We normalize ϕ by the condition $\int_\Omega \phi(x)\,dx = 1$. Introduce the functions

$$\begin{aligned} &\mu(t) = \int_\Omega u(x,t)\phi(x)\,dx, \qquad \alpha(t) = \left(\int_\Omega b^{\frac{1}{2-\sigma^-(t)}}(x,t)\phi(x)dx\right)^{2-\sigma^-(t)}, \\ &\sigma^-(t) = \min_{x\in\Omega}\sigma(x,t), \qquad \beta(t) = \int_\Omega b(x,t)\phi(x)dx \end{aligned} \tag{3.12}$$

and

$$A(t) = \left(\alpha(t) - \frac{\lambda^{\sigma^-(t)-1}}{\sigma^-(t)-1}\right), \qquad B(t) = \beta(t) + \frac{\sigma^-(t)-2}{\sigma^-(t)-1}. \tag{3.13}$$

We will assume that

$$A^- = \min_{t\geq 0} A(t) > 0, \qquad B^+ = \max_{t\geq 0} B(t) < \infty. \tag{3.14}$$

Definition 2. We say that the solution $u(x,t)$ blows up in a finite time if there exists an instant $t^* < \infty$ such that

$$\|u(\cdot,t)\|_{\infty,\Omega} \to \infty \quad \text{as } t \to t^*.$$

It is easy to see that the finite time blow-up happens if, say, there exists a moment $t^* < \infty$ such that $\mu(t^*) = \infty$. Indeed:

$$\mu(t) = \int_\Omega u(x,t)\,\phi(x)\,dx \leq \|u(\cdot,t)\|_{\infty,\Omega}\int_\Omega \phi\,dx = \|u(\cdot,t)\|_{\infty,\Omega} \to \infty \quad \text{as } t \to t^*.$$

This observation allows us to characterize blow-up of the solution $u(x,t)$ in terms of the function $\mu(t)$.

Theorem 2. *Let the data of problem (3.10) satisfy the conditions*

$$\begin{cases} \forall\, t \geq 0 \quad 2 < \sigma^- = \min\limits_{x\in\Omega}\sigma(x,t) = const, \\ 0 < b^- \leq b(x,t) \leq b^+ < \infty, \end{cases} \tag{3.15}$$

$$\begin{cases} -\lambda\mu(0) + b^-\mu^{\sigma^--1}(0) - b^+ > 0, \\ -\lambda + b^-(\sigma^- - 1)\mu^{\sigma^--1}(0) > 0. \end{cases} \tag{3.16}$$

Then every weak solution blows up at a moment $t^* \equiv t^*(\mu(0), \sigma^-, b^\pm) < \infty$.

Theorem 3. *Let condition (3.14) be fulfilled and, in addition,*

$$g(t,\mu(0)) = A^-\mu^{\sigma^-(t)-1}(0) - B^+ > 0 \quad \textit{for every } t \geq 0. \tag{3.17}$$

If either

$$\sigma^-(t) = \min_{x\in\Omega}\sigma(x,t) \geq \sigma^- = const > 2, \tag{3.18}$$

or

$$\begin{cases} \mu(0) > 1, \quad \sigma^-(t) \geq 2, \\ \sigma^-(t) \textit{ is monotone decreasing and } \sigma^-(t) \to 2 \textit{ as } t \to \infty, \\ \displaystyle\int_{\ln\mu(0)}^\infty e^{s(2-\sigma^-(s))}ds < \infty, \end{cases} \tag{3.19}$$

then every weak solution of problem (3.10) blows up at a finite moment $t^* < \infty$.

3.2. Proof of Theorems 2 and 3

3.2.1. The differential inequality for $\mu(t)$.

Let $u(x,t)$ be a weak solution of the semilinear problem (3.10) with $p(x,t)\equiv 2$ and $a(x,t)\equiv 1$. According to (2.9), for every test-function $\phi(x)\in W_0^{1,2}(\Omega)$ and every $t, t+h<T^*$

$$\int_t^{t+h}\int_\Omega \left(u_t\phi+\nabla u\cdot\nabla\phi-b(x,t)\,u^{\sigma(x,t)-1}\,\phi\right)\mathrm{d}x\mathrm{d}t=0. \tag{3.20}$$

Let us choose the eigenfunction ϕ for the test-function in (3.20), divide the resulting equality by h, and let $h\to 0$. Applying the Lebesgue differentiation theorem we find that for a.e. $t<T^*$

$$\mu'(t)=\int_\Omega u_t\phi\mathrm{d}x=-\int_\Omega\nabla u\cdot\nabla\phi\mathrm{d}x+\int_\Omega b(x,t)u^{\sigma(x,t)-1}\phi(x)\mathrm{d}x=-\lambda\mu+\int_\Omega bu^{\sigma(x,t)-1}\phi\mathrm{d}x. \tag{3.21}$$

Using the representation

$$I=\int_\Omega bu^{\sigma(x,t)-1}\phi\mathrm{d}x=\int_{\Omega\cap(u\geq 1)}bu^{\sigma(x,t)-1}\phi\mathrm{d}x+\int_{\Omega\cap(u<1)}bu^{\sigma(x,t)-1}\phi\mathrm{d}x, \tag{3.22}$$

we evaluate I in the following way: since $\sigma^->2$

$$I\geq\int_{\Omega\cap(u\geq 1)}bu^{\sigma^--1}\phi\mathrm{d}x=\int_\Omega bu^{\sigma^--1}\phi\mathrm{d}x-\int_{\Omega\cap(u<1)}bu^{\sigma^--1}\phi\mathrm{d}x\geq\int_\Omega bu^{\sigma^--1}\phi\mathrm{d}x-\int_\Omega b\phi\mathrm{d}x.$$

Applying the inverse Hölder's inequality

$$\int_\Omega|u||v|\mathrm{d}x\geq\left(\int_\Omega|u|^q\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_\Omega|v|^{\frac{q}{q-1}}\mathrm{d}x\right)^{\frac{q-1}{q}} \tag{3.23}$$

with the exponent $q=\frac{1}{\sigma^--1}\in(0,1)$, we may estimate

$$\begin{aligned}\int_\Omega bu^{\sigma^--1}\phi\,\mathrm{d}x&=\int_\Omega u^{\sigma^--1}\phi^{\sigma^--1}b\phi^{2-\sigma^-}\mathrm{d}x\,\geq\left(\int_\Omega u\,\phi\,\mathrm{d}x\right)^{\sigma^--1}\left(\int_\Omega b^{\frac{1}{2-\sigma^-}}(x,t)\,\phi\,\mathrm{d}x\right)^{2-\sigma^-}\\&=\alpha(t)\mu^{\sigma^--1}(t).\end{aligned} \tag{3.24}$$

Gathering these formulas we arrive at the ordinary differential inequality for the function $\mu(t)$:

$$\mu'(t)\geq-\lambda\mu(t)+\alpha(t)\mu^{\sigma^--1}(t)-\beta(t)\equiv f(\mu(t)), \tag{3.25}$$

with the functions $\alpha(t)$, $\beta(t)$ defined in (3.12). Notice that if condition $\min_{x\in\Omega}\sigma(x,t)=\sigma^-=const$ is substituted by the weaker condition $\min_{x\in\Omega}\sigma(x,t)=\sigma^-(t)>2$ the same arguments lead to the inequality

$$\mu'(t)\geq-\lambda\mu(t)+\alpha(t)\mu^{\sigma^-(t)-1}(t)-\beta(t)\equiv F(t,\mu(t)). \tag{3.26}$$

3.2.2. Analysis of the differential inequality — Theorem 2

Under conditions (3.15)

$$0<b^-\leq\alpha(t)=\left(\int_\Omega b^{\frac{1}{2-\sigma^-}}(x,t)\phi(x)\,\mathrm{d}x\right)^{2-\sigma^-},$$

$$0\leq\beta(t)=\int_\Omega b(x,t)\phi(x)\,\mathrm{d}x\leq b^+<\infty,$$

and (3.25) yields the inequality with constant coefficients and exponents:

$$\mu'(t)\geq-\lambda\mu(t)+b^-\mu^{\sigma^--1}(t)-b^+\equiv f(\mu(t)),\quad\sigma^-=const>2. \tag{3.27}$$

The function $f(s)$ is concave and attains its minimum at the point

$$y_*=\left(\frac{\lambda}{b^-(\sigma^--1)}\right)^{\frac{1}{\sigma^--2}}.$$

Conditions (3.16) mean that $f(\mu(0)) > 0$, $\partial_\mu f(\mu(0)) > 0$ and inequality (3.27) guarantees that $\mu(t)$ is a strictly positive and increasing function of t, and $f(\mu(t))$ is strictly positive for all $t \geq 0$. Dividing the both parts of (3.27) by $f(\mu(t))$ and integrating, we have:

$$J(\mu(t)) = \int_{\mu(0)}^{\mu(t)} \frac{\mathrm{d}s}{f(s)} \geq t.$$

Since the integral $J(s)$ is convergent at $s = \infty$, this inequality is possible only if there exists t^* such as $\mu(t) \to \infty$ as $t \to t^*$.

Remark 3. Conditions (3.16) are surely fulfilled for all sufficiently large $\mu(0)$.

3.2.3. Analysis of the differential inequality — Theorem 3

In this case $\mu(t)$ satisfies (3.26). Applying Young's inequality

$$a\,b \leq \frac{1}{p}a^p + \frac{p-1}{p}b^{\frac{p}{p-1}}$$

with $a = \lambda\,\mu(t)$, $b = 1$, $p = \sigma^-(t) - 1$, we have:

$$\mu(t)\lambda \leq \frac{1}{\sigma^-(t)-1}\,(\lambda\mu(t))^{\sigma^-(t)-1} + \frac{\sigma^-(t)-2}{\sigma^-(t)-1}.$$

Plugging this inequality into (3.26), we obtain

$$\mu'(t) \geq F(t,\mu(t)) \geq A(t)\mu^{\sigma^-(t)-1}(t) - B(t) \geq A^-\mu^{\sigma^-(t)-1}(t) - B^+ \equiv g(t,\mu(t)) \tag{3.28}$$

with the coefficients $A(t), B(t), A^-, B^+$ defined in (3.13) and (3.14). Since

$$\partial_\mu g(t,\mu) = A^-\left(\sigma^-(t)-1\right)\mu^{\sigma^-(t)-1} > 0,$$

the function $g(t,\mu)$ is increasing as a function of μ. Recall that $\mu(t) \in C^0(0,T)$ (see Remark 1), $g(t,\mu(0)) > 0$ for all $t \geq 0$ by the assumption and $g(t,\mu)$ is continuous with respect to μ. It follows that there exists $t_* > 0$ such that $g(t,\mu(t)) > 0$ for all $s \in [0,t_*]$:

$$g(t,\mu(t)) \geq g(t,\mu(0)) - |g(t,\mu(t)) - g(t,\mu(0))| \geq \frac{1}{2}g(t,\mu(0)) > 0 \quad \text{for all } t \in [0,t_*].$$

By virtue of (3.28) $\mu'(t) > 0$ for a.a. $t \in (0,t_*)$, which yields strict monotonicity of $\mu(t)$ on the interval $(0,t_*)$. Since $\mu(t_*) > \mu(0)$ and $g(t,\mu(t_*)) > g(t,\mu(0))$, we may now take t_* for the initial moment and repeat the above arguments to extend the conclusion to an interval $(t_*, t_*+\epsilon)$ with some $\epsilon >$. Continuing this process, we find that $\mu'(t) > 0$ for a.a. $t \in (0,t^+)$ where $t^+ = \sup\{t > 0 :\ \mu(t) < \infty\}$ is the right endpoint of the interval of existence of $\mu(t)$.

If condition (3.18) is fulfilled, the conclusion about the finite time blow-up of the solution u follows exactly as in the proof of Theorem 2. Let us assume that condition (3.19) is fulfilled. Since $g(t,\mu)$ in increasing as a function of μ, μ is an increasing function of t, and $g(t,\mu) \to \infty$ as $\mu \to \infty$, there exists t' such that $g(t,\mu) \geq \frac{1}{2}A^-\mu^{\sigma^-(t)-1}(t)$ for all $t \geq t'$. Inequality (3.28) gives

$$\mu'(t) \geq \frac{1}{2}A^-\mu^{\sigma^-(t)-1}(t) \quad \text{for } t \geq t', \qquad \mu(t') \geq \mu(0) > 1.$$

Let us introduce the new independent variable $\theta = \frac{A^-}{2}(t-t')$ and denote $\gamma(\theta) = \sigma^-(t)$, $\nu(\theta) \equiv \mu(t)$. For the function $\nu(\theta)$ we have the conditions

$$\nu'(\theta) \geq \nu^{\gamma(\theta)-1}(\theta), \qquad \nu(\theta) \geq 1 \quad \text{for } \theta > 0, \tag{3.29}$$

which yield the inequality $\nu'(\theta) \geq \nu(\theta)$. Integration of this inequality gives

$$\ln\,\nu(\theta) \geq \ln\left(\frac{\nu(\theta)}{\nu(0)}\right) \geq \theta,$$

and for the monotone decreasing function $\gamma(\theta)$ we have: $\gamma(\theta) \geq \gamma\,(\ln\,\nu(\theta))$. In the result we have the autonomous inequality for the $\nu(\theta)$:

$$\nu'(\theta) \geq \nu^{\gamma(\ln\,\nu(\theta))-1}(\theta), \qquad \nu(\theta) \geq 1 \quad \text{for } \theta \geq 0.$$

Integrating and changing the variable of integration we finally obtain the inequality

$$I(\ln\,\nu(\theta)) \equiv \int_{\ln\,\nu(0)}^{\ln\,\nu(\theta)} \frac{\mathrm{d}\tau}{\mathrm{e}^{\tau(\gamma(\tau)-2)}} \geq \int_{\nu(0)}^{\nu(\theta)} \frac{\mathrm{d}s}{s^{\gamma(\ln\,s)-1}} \geq \theta. \tag{3.30}$$

Using assumption (3.19), from the last inequality we infer (recall that $\nu(0) = \mu(t') \geq \mu(0) > 1$)

$$\theta \leq I(\ln\nu(\theta)) < I(\infty) = \int_{\ln\nu(0)}^{\infty} \frac{\mathrm{d}\tau}{\mathrm{e}^{\tau(\gamma(\tau)-2)}} \leq \int_{\ln\mu(0)}^{\infty} \frac{\mathrm{d}\tau}{\mathrm{e}^{\tau(\gamma(\tau)-2)}} < \infty,$$

which is impossible unless there exists a finite θ^* such that $\nu(\theta) \to \infty$ as $\theta \to \theta^*$. The proof of Theorem 3 is completed.

4. Generalizations

4.1. An example

Let us illustrate the assertion of Theorem 3 by the following example: let u be a weak solution of problem (3.10) for the equation

$$u_t = \Delta u + u^{1+\epsilon(t)}, \tag{4.31}$$

where $\epsilon(t)$ is a monotone decreasing positive function such that $\epsilon(t) \to 0$ as $t \to \infty$. We assume that the initial function u_0 is as large as is required in Theorem 3. Moreover, increasing $\mu(0)$ we may guarantee that in (3.28) $g(t, \mu(t)) \geq \frac{1}{2}A^{-}\mu^{1+\epsilon(t)}(t)$ for all $t \geq 0$, so that the sufficient condition of the finite time blow-up of the solution u reduces to the following claim:

$$\int_{\ln \mu(0)}^{\infty} \frac{\mathrm{d}\tau}{\mathrm{e}^{\epsilon(\tau)\tau}} < \infty.$$

Since $\mu(0) > 1$, the simplest convergence test shows that this condition is fulfilled if, say, $\epsilon(\tau) = \alpha \frac{\ln \tau}{\tau}$ with any $\alpha > 1$.

4.2. The nonlinear ordinary differential inequality

The proof of Theorems 2 and 3 is based on the study of the properties of the functions satisfying the ordinary differential inequality (3.26). The study of such inequalities has an independent interest. Under the assumptions of Theorems 2 and 3 inequality (3.26) reduces to inequalities (3.27) or (3.28) with constant coefficients, and integration of these inequalities shows that the functions satisfying become infinite in a finite time. The same effect takes place if the coefficient in the differential inequality is nonnegative but not necessarily separated away from zero. Let us consider the simplified inequality

$$\mu'(t) \geq \alpha(t)\mu^{\sigma^{-}(t)-1}, \qquad \mu(0) > 1,$$

with a nonnegative coefficient $\alpha(t)$ such that $\alpha(t) \to 0$ as $t \to \infty$. Introducing the new independent variable

$$\tau = \int_0^t \alpha(s)\,\mathrm{d}s$$

and the functions $\nu(\tau) = \mu(t)$, $\gamma(\tau) = \sigma^{-}(t)$, we arrive at the inequality

$$\nu'(\tau) \geq \nu^{\gamma(\tau)-1}, \qquad \nu(\tau) \geq 1.$$

Arguing as in the proof of Theorem 3, we then conclude that the function $\mu(t)$ blows up at a finite instant t^* if, for example,

$$\infty > \int_{\ln \mu(0)}^{\infty} \frac{\mathrm{d}\tau}{\mathrm{e}^{\tau(\gamma(\tau)-2)}} \geq \tau = \int_0^t \alpha(s)\,\mathrm{d}s \to \infty \quad \text{as } t \to \infty.$$

This condition indicates the admissible rate of vanishing of the coefficient $\alpha(t)$ as $t \to \infty$ for the finite time blow-up of $\mu(t)$.

4.3. Regional blow-up

The conclusions about the blow-up of solutions of problem (3.10) remain true if instead of the whole domain Ω we restrict the study to a subdomain. Let us assume that there exists a subdomain $D \subset \Omega$, meas $D > 0$, $\partial D \in C^1$, and let $\phi > 0$ in D and λ be the first eigenfunction and the corresponding eigenvalue of the problem

$$-\Delta\phi = \lambda\phi \quad \text{in } D, \qquad \phi = 0 \quad \text{on } \partial D. \tag{4.32}$$

Let us introduce the function

$$\mu(t) = \int_D u(x,t)\,\phi(x)\,\mathrm{d}x.$$

Given a solution $u \in \mathbf{W}(Q_T)$, we may formally consider the semilinear equation (3.10) (at least for small times) as the heat equation with the bounded free term $f(x,t) \equiv b(x,t)\,u^{\sigma(x,t)-1}$. It follows then from the classical parabolic theory that $u \in W_2^{1,2}(\omega \times \theta)$ for every subdomain $\omega \subset \Omega$ with the sufficiently smooth boundary $\partial\omega$ and every $\theta < t^*$. This observation justifies the forthcoming arguments. Let us multiply Eq. (3.10) by the function ϕ and integrate over D:

$$\begin{aligned}\mu'(t) &= \int_D u_t\phi\,\mathrm{d}x = \int_D u\Delta\phi\,\mathrm{d}x - \int_{\partial D} u\,(\nabla\phi,\,\mathbf{n})\,\mathrm{d}S + \int_D b\,u^{\sigma(x,t)-1}\,\phi\,\mathrm{d}x \\ &= -\lambda\mu - \int_{\partial D} u\,(\nabla\phi,\,\mathbf{n})\,\mathrm{d}S + \int_D b\,u^{\sigma(x,t)-1}\phi\,\mathrm{d}x,\end{aligned}$$

where $\mathbf{n}$ denotes the outward normal to ∂D. Since $\phi \geq 0$ in D, then $(\nabla\phi, \mathbf{n}) \leq 0$ on ∂D, and for the nonnegative solution u

$$-\int_{\partial D} u\,(\nabla\phi, \mathbf{n})\,\mathrm{d}S \geq 0.$$

The differential inequality for $\mu(t)$ takes on the form (cf. with (3.21) and (3.25))

$$\mu'(t) \geq -\lambda\mu + \int_D b\,u^{\sigma(x,t)-1}\,\phi\,\mathrm{d}x.$$

The analysis of this inequality is performed in the proofs of Theorems 2 and 3.

4.4. Equations with nonlocal reaction terms

Let us consider the problem

$$\begin{cases} u_t = \triangle u + f(x,t,u) & \text{in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, & u = 0 \text{ on } \Gamma_T, \end{cases} \tag{4.33}$$

where

$$f(x,t,u) = \sum_{k=1}^{N} b_k(x,t)u^{\sigma_k(x,t)-1} + \sum_{i=N+1}^{Q} c_i(x,t)\int_\Omega d_i(s,t)u^{\sigma_i(s,t)-1}\mathrm{d}s,$$

with $b_k \geq 0$, $c_i \geq 0$, $d_i \geq 0$, $Q \leq n$. Multiplying (4.33) by the first eigenfunction ϕ of problem (3.11) and integrating over Ω we arrive at the relation (cf. with (3.21))

$$\mu'(t) = -\lambda\mu + I_1 + I_2, \tag{4.34}$$

where

$$I_1 = \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t)u^{\sigma_k(x,t)-1}\right)\phi\,\mathrm{d}x, \qquad I_2 = \int_\Omega \left(\sum_{i=N+1}^{Q} c_i(x,t)\int_\Omega d_i(s,t)u^{\sigma_i(s,t)-1}\mathrm{d}s\right)\phi\,\mathrm{d}x.$$

I_1, I_2 are estimated from below in the following way (cf. with (3.22)–(3.24)):

$$\begin{aligned}
I_1 &\geq \int_{\Omega\cap(u\geq 1)} \left(\sum_{k=1}^{N} b_k(x,t)u^{\sigma_k(x,t)-1}\right)\phi(x)\,\mathrm{d}x \geq \int_{\Omega\cap(u\geq 1)} \left(\sum_{k=1}^{N} b_k(x,t)u^{\sigma_k^-(t)-1}\right)\phi(x)\,\mathrm{d}x \\
&= \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t)u^{\sigma_k^-(t)-1}\right)\phi(x)\,\mathrm{d}x - \int_{\Omega\cap(u<1)} \left(\sum_{k=1}^{N} b_k(x,t)u^{\sigma_k^-(t)-1}\right)\phi(x)\,\mathrm{d}x \\
&\geq \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t)u^{\sigma_k^-(t)-1}\right)\phi(x)\mathrm{d}x - \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t)\right)\phi(x)\,\mathrm{d}x \\
&\geq \sum_{k=1}^{N}\left(\int_\Omega b_k^{\frac{1}{2-\sigma_k^-(t)}}(x,t)\phi(x)\mathrm{d}x\right)^{2-\sigma_k^-(t)}\left(\int_\Omega u(x,t)\phi(x)\,\mathrm{d}x\right)^{\sigma_k^-(t)-1} \\
&\quad - \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t)\right)\phi(x)\,\mathrm{d}x = \sum_{k=1}^{N}\alpha_k(t)\mu^{\sigma_k^-(t)-1} - \beta(t),
\end{aligned}$$

$$\begin{aligned}
I_2 &\geq \sum_{i=N+1}^{Q}\int_\Omega c_i(x,t)\phi(x,t)\left(\int_\Omega u(s,t)\phi(s)\,\mathrm{d}s\right)^{\sigma_i^-(t)-1}\left(\int_\Omega d_i^{\frac{1}{2-\sigma_i^-(t)}}(s,t)\phi^{\frac{\sigma_i(t)-1}{\sigma_i(t)-2}}(s,t)\mathrm{d}s\right)^{2-\sigma_i^-(t)}\mathrm{d}x \\
&\quad - \sum_{i=N+1}^{Q}\int_\Omega \left(c_i(x,t)\int_\Omega d_i(s,t)\,\mathrm{d}s\right)\phi(x)\,\mathrm{d}x = \sum_{i=N+1}^{Q}\mu^{\sigma_i^-(t)-1}(t)\theta_i(t)\int_\Omega c_i(x,t)\phi(x)\,\mathrm{d}x - \beta(t) \\
&= \sum_{i=N+1}^{Q}\alpha_i(t)\mu^{\sigma_i^-(t)-1} - \beta(t),
\end{aligned}$$

where for $k = 1, \ldots, N$

$$\alpha_k(t) = \left(\int_\Omega b_k^{\frac{1}{2-\sigma_k^-(t)}}(x,t)\phi(x)\mathrm{d}x\right)^{2-\sigma_k^-(t)}, \qquad \beta(t) = \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t)\right)\phi(x)\,\mathrm{d}x,$$

and for $k = N + 1, \dots, Q$

$$\alpha_k(t) = \int_\Omega c_k \phi(x)\,\mathrm{d}x \left(\int_\Omega d_k^{\frac{1}{2-\sigma_k^-(t)}} \phi^{\frac{\sigma_k(t)-1}{\sigma_k(t)-2}}(x)\,\mathrm{d}x \right)^{2-\sigma_k^-(t)}, \qquad \beta(t) = \sum_{i=N+1}^{Q} \int_\Omega \left(c_i \int_\Omega d_i(s,t)\,\mathrm{d}s \right) \phi(x)\,\mathrm{d}x.$$

Gathering these formulas we arrive at the nonlinear ordinary differential inequality for the function $\mu(t)$:

$$\mu'(t) \geq -\lambda\mu + \sum_{i=1}^{Q} \alpha_i(t) \mu^{\sigma_i^-(t)-1}(t)(t) - \beta(t),$$

which can be studied as (3.26).

5. Evolution equations of $p(x)$-Laplace type

5.1. Assumptions and result

Let us consider the problem

$$\begin{cases} u_t = \operatorname{div}\left(a(x,t)|\nabla u|^{p(x)-2}\nabla u\right) + b(x,t)|u|^{\sigma(x)-2}u \quad \text{in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_T \end{cases} \tag{5.35}$$

with the coefficients and exponents satisfying conditions (1.2). Let us introduce the functions

$$f(t) = \frac{1}{2}\int_0^t \int_\Omega u^2(x,\tau)\,\mathrm{d}x\mathrm{d}\tau, \qquad E(t) = \int_\Omega \left(\frac{a}{p}|\nabla u|^p - \frac{b}{\sigma}|u|^\sigma \right) \mathrm{d}x. \tag{5.36}$$

Theorem 4. *Let conditions (1.2), (2.3) and (2.8) be fulfilled, and let the exponents $p(x)$, $\sigma(x)$ satisfy the conditions*

$$\sigma^- > 2 \quad \textit{and} \quad p^+ = \max_\Omega p(x) \leq \sigma^- = \min_\Omega \sigma(x). \tag{5.37}$$

Let us assume, in addition, that the coefficients a, b are differentiable in t and monotone:

$$a_t(x,t) \leq 0, \quad b_t(x,t) \geq 0, \quad \int_0^T \left(\max_{x\in\Omega} |a_t(x,t)| + |b_t(x,t)| \right) \mathrm{d}t < \infty. \tag{5.38}$$

Finally, let $|u_0|^{\sigma(x)} \in L^1(\Omega)$, $|\nabla u_0|^{p(x)} \in L^1(\Omega)$. If

$$E(0) = \int_\Omega \left(\frac{a(x,0)}{p(x)}|\nabla u_0|^{p(x)} - \frac{b(x,0)}{\sigma(x)}|u_0|^{\sigma(x)} \right) \mathrm{d}x \leq 0, \tag{5.39}$$

then every nonstationary weak solution $u \in \mathbf{W}(Q_T)$ blows up in a finite time:

$$\exists\, t^* \equiv t^*(\Omega, \|u_0\|_\infty) < \infty : \ \|u(\cdot,t)\|_{\infty,\Omega} \to \infty \quad \textit{as } t \to t^*.$$

5.2. The energy relations

According to Theorem 1 the solution $u \in \mathbf{W}(Q_T)$ can be taken for the test-function in the integral identity (2.7), which gives the first energy relation:

$$\frac{1}{2}\int_\Omega u^2\,\mathrm{d}x + \int_0^t \int_\Omega \left(a|\nabla u|^p - b|u|^\sigma \right) \mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_\Omega u_0^2\,\mathrm{d}x. \tag{5.40}$$

To derive the second energy estimate we rely on the following result:

Lemma 1. *Let the exponents and coefficients of problem (5.35) satisfy the conditions of Theorem 4. Then the weak solution of problem (5.35) satisfies the estimate*

$$\forall\,\text{a.e. } t > 0 \qquad E(t) + \int_0^t \int_\Omega u_t^2\,\mathrm{d}x\mathrm{d}t \leq E(0). \tag{5.41}$$

Proof. The assertion is a simplified version of the estimate proved in [15, Theorem 6.1], that is why we limit ourselves by giving a sketch of the arguments and skip the details. The proof of existence of a weak solution to problem (5.35) (in a more general setting) is performed with the Galerkin–Faedo method. The solution is obtained as the limit of the sequence of functions $u^{(k)} = \sum_1^k c_{i,k}(t)\psi_i(x)$, $\{\psi_i\}$ is the orthogonal basis of the function space $L^{p^+}(\Omega)$, which is dense in $L^{p(x)}(\Omega)$. In this approach estimates on the limit function result from the uniform in k estimates for the approximate solutions $u^{(k)}$. Let u be a sufficiently regular solution of problem (5.35) (or the approximate solution $u^{(k)}$). Multiplying the equation by u_t, integrating by parts, and using the obvious relations

$$\partial_t \left(a \frac{|\nabla u\,|^p}{p} \right) = a_t \frac{|\nabla u\,|^p}{p} + a\left(|\nabla u|^{p-2}\nabla u\, \nabla u_t\right), \qquad \partial_t \left(\frac{b}{\sigma}\,|u|^\sigma \right) = b_t \frac{|u|^\sigma}{\sigma} + b\left(|u|^{\sigma-2} u\, u_t\right),$$

we have:

$$E'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{a}{p}\,|\nabla u|^p - \frac{b}{\sigma}\,|u|^\sigma \right) = -\int_\Omega u_t^2\, \mathrm{d}x + \Lambda_1(t) + \Lambda_2(t) \le -\int_\Omega u_t^2\, \mathrm{d}x,$$

because

$$\Lambda_1(t) = \int_\Omega a_t \frac{|\nabla u\,|^p}{p}\, \mathrm{d}x \le 0, \qquad \Lambda_2(t) = -\int_\Omega b_t \frac{|u|^\sigma}{\sigma}\, \mathrm{d}x \le 0$$

by assumption. Inequality (5.41) follows after integration in t. □

Remark 4. In the case that a, p, b, σ are independent of t, the energy relation takes on the form

$$E(t) + \int_0^t \int_\Omega u_t^2\, \mathrm{d}x\mathrm{d}t = E(0).$$

5.3. *Ordinary differential inequality for $f(t)$.*

Let us consider the function $f(t)$ defined in (5.36). Under the conditions of Lemma 1, for every solution of problem (5.35) and for a.e. $t > 0$

$$f'(t) = \frac{1}{2} \int_\Omega u^2(\cdot, t)\, \mathrm{d}x \ge 0, \qquad f''(t) = \int_\Omega u\, u_t\, \mathrm{d}x = \int_\Omega \left(-a\,|\nabla u|^p + b\,|u|^\sigma\right)\, \mathrm{d}x. \tag{5.42}$$

The former equality follows from the definition of $f(t)$ and the inclusion $u \in L^\infty(0, T; L^2(\Omega))$. The latter one is a byproduct of the definitions of $f(t)$ and weak solution of problem (5.35). Let us choose the solution $u \in \mathbf{W}(Q_T)$ for the test-function in (2.9), integrate over the cylinder $Q_T \cap \{\tau < t < \tau + h\}$ with some $h > 0$ (small), and then divide the resulting equality by h:

$$\frac{1}{h} \int_\tau^{\tau+h} \int_\Omega u\, u_t\, \mathrm{d}x\mathrm{d}t = -\frac{1}{h} \int_\tau^{\tau+h} \int_\Omega a\,|\nabla u|^p\, \mathrm{d}x\mathrm{d}t + \frac{1}{h} \int_\tau^{\tau+h} \int_\Omega b\,|u|^\sigma\, \mathrm{d}x\mathrm{d}t.$$

Since

$$\int_\Omega a\,|\nabla u|^p\, \mathrm{d}x, \ \int_\Omega b\,|u|^\sigma\, \mathrm{d}x \in L^1(0, T),$$

by the Lebesgue differentiation theorem each of the two terms on the right-hand side has a limit as $h \to 0$ for a.e. $\tau > 0$. It follows that so does the term on left-hand side, whence the second equality of (5.42).

Gathering (5.42) with (5.40), we find that

$$0 \le f'(t) = \frac{1}{2} \int_\Omega u_0^2\, \mathrm{d}x + \int_0^t \int_\Omega \left(-a\,|\nabla u|^p + b\,|u|^\sigma\right)\, \mathrm{d}x\mathrm{d}t = \frac{1}{2} \int_\Omega u_0^2\, \mathrm{d}x + \int_0^t f''(t)\, \mathrm{d}t.$$

Let us take a constant $\lambda > 0$ such that

$$\frac{1}{\sigma^-} \le \lambda \le \frac{1}{p^+}.$$

Multiplying the second equality of (5.42) by λ and adding the result to (5.41), we obtain the inequality

$$E(t) + \lambda \int_\Omega \left(-a\,|\nabla u|^p + b\,|u|^\sigma\right)\, \mathrm{d}x + \int_0^t \int_\Omega u_t^2\, \mathrm{d}x\mathrm{d}t \le \lambda f''(t) + E(0), \tag{5.43}$$

which leads to the following one: for $E(0) \le 0$

$$\left(\frac{1}{p^+} - \lambda \right) a^- \int_\Omega |\nabla u|^p\, \mathrm{d}x + \left(\lambda - \frac{1}{\sigma^-} \right) b^- \int_\Omega |u|^\sigma\, \mathrm{d}x + \int_0^t \int_\Omega u_t^2\, \mathrm{d}x\mathrm{d}t \le \lambda f''(t). \tag{5.44}$$

5.4. *Lower estimates on the growth of $f(t)$. Proof of Theorem 4*

The proof of Theorem 4 is based on the analysis of behavior of the function $f(t)$. There are two possibilities which are considered separately: either $p^+ > 2$, or $p^+ \le 2$.

5.4.1. Case 1: $p^+ > 2$.

Notice that the first two terms on the left-hand side of (5.44) are always nonnegative because of (5.37), but they need not be strictly positive if $p^+ > 2$. Dropping these terms, we obtain the inequality

$$\int_0^t \int_\Omega u_t^2 \,dx \le \lambda f''(t).$$

If u is a nonstationary weak solution, there exist a number $\epsilon > 0$ and a moment $t' > 0$ such that $f''(t) \ge \epsilon$ and $f(t) \ge \epsilon$ for all $t \ge t'$. Let us denote by t^* the time of existence of the solution u,

$$t^* = \sup\{t > 0 : \|u(\cdot, t)\|_{\infty,\Omega} < \infty \text{ for all } t < t^*\},$$

and assume, for contradiction, that $t^* = \infty$, i.e., there is no finite time blow-up (see Definition 2). Using Hölder's inequality, we obtain the chain of relations

$$\begin{aligned}(f'(t) - f'(t'))^2 &= \left(\int_{t'}^t \frac{d}{dt}\left(\frac{1}{2}\int_\Omega u^2 \,dx\right) dt\right)^2 = \left(\int_{t'}^t \int_\Omega u u_t \,dx\right)^2 \le \left(\int_{t'}^t \|u_t\|_{2,\Omega}\|u\|_{2,\Omega} \,dt\right)^2 \\ &\le \|u_t\|^2_{2,\Omega\times(t',t)}\|u\|^2_{2,\Omega\times(t',t)} \\ &\le \lambda f''(t)\int_{t'}^t \int_\Omega u^2 \,dxdt = 2\lambda f''(t)f(t) \le \frac{2}{p^+} f''(t)f(t) \quad \text{for all } t > t'. \end{aligned} \tag{5.45}$$

Since $f(t) \ge \epsilon, f'(t) > 0, f''(t) \ge \epsilon$ for all $t > t'$, it is necessary that $f'(t) \nearrow \infty$ as $t \to \infty$. Notice that for every $1 < \nu < p^+/2$

$$1 - \sqrt{\frac{2\nu}{p^+}} \ge \frac{f'(t')}{f'(t)} \searrow 0 \quad \text{as } t \to \infty.$$

It follows that for every fixed $\nu \in (1, p^+/2)$ there exists a moment $t_0 > t'$ such that

$$(f'(t) - f'(t'))^2 \ge \frac{2\nu}{p^+} (f'(t))^2 \quad \text{for } t \ge t_0,\ f(t_0) > 0.$$

Using this inequality, we continue (5.45) as follows:

$$\nu(f'(t))^2 \le \frac{p^+}{2}\left(f'(t) - f'(t')\right)^2 \le f''(t)f(t) \qquad \text{for all } t \ge t_0,$$

that is,

$$(\ln f^\nu(t))' = \nu\frac{f'(t)}{f(t)} \le \frac{f''(t)}{f'(t)} = \left(\ln f'(t)\right)' \quad \Longrightarrow \quad \left(\frac{f'(t_0)}{f^\nu(t_0)}\right) f^\nu(t) \le f'(t) \quad \text{for all } t > t_0.$$

The straightforward integration leads to the inequality

$$f^{\nu-1}(t) \ge \frac{f^{\nu-1}(t_0)}{1 - (t - t_0)\,(\nu - 1)\frac{f'(t_0)}{f(t_0)}} \to \infty \quad \text{as } t \nearrow T = t_0 + \frac{f(t_0)}{(\nu - 1)f'(t_0)},$$

which contradicts the assumption $t^* = \infty$ because

$$\infty > \frac{1}{2} T\,|\Omega| \sup_{(0,T)} \|u\|^2_{\infty,\Omega} \ge \frac{1}{2}\int_0^t \int_\Omega u^2 \,dxdt \equiv f(t) \to \infty \quad \text{as } t \nearrow T.$$

This completes the proof of Theorem 4 in the case $p^+ > 2$.

5.4.2. Case 2: $1 < p^+ \le 2$.

In this case $\sigma^- > p^+$ and there exists λ such that $\frac{1}{\sigma^-} < \lambda < \frac{1}{p^+}$. Under this choice of λ (5.44) yields

$$\left(\lambda - \frac{1}{\sigma^-}\right) b^- \int_\Omega |u|^\sigma \le \lambda f''(t).$$

Since $\sigma^- > 2$, applying Hölder's inequality (2.4) and using (2.5) we find that

$$f'(t) \equiv \frac{1}{2}\|u(\cdot, t)\|^2_{2,\Omega} \le C\,\|u\|^2_{\sigma(\cdot),\Omega} \le C\,\max\left\{\left(\int_\Omega |u|^{\sigma(x)} \,dx\right)^{\frac{2}{\sigma^+}}, \left(\int_\Omega |u|^{\sigma(x)} \,dx\right)^{\frac{2}{\sigma^-}}\right\}$$

with the constant $C = 2\,\|1\|^2_{\sigma'(\cdot),\Omega}$. Gathering this inequality with (5.42) we have:

$$\left(\lambda - \frac{1}{\sigma^-}\right) b^- \min\left\{\left(\frac{f'(t)}{C}\right)^{\frac{\sigma^+}{2}}, \left(\frac{f'(t)}{C}\right)^{\frac{\sigma^-}{2}}\right\} \le \left(\lambda - \frac{1}{\sigma^-}\right) b^- \int_\Omega |u|^{\sigma(x)}\,dx \le \lambda f''(t).$$

Let us assume that the blow-up does not occur and $t^* = \infty$. As in the case $p^+ > 2$ we first notice that, by virtue of (5.44) and due to the choice of λ, for every nonstationary solution u one may indicate $t' > 0$ such that the functions $f(t), f'(t)$, $f''(t)$ are strictly positive for all $t \ge t'$. It follows that $f'(t) \nearrow \infty$ as $t \to \infty$ and there exist a moment $t_0 > t'$ and a constant $C > 0$ such that $f'(t) \ge C$ for all $t \ge t_0$, which leads to the inequality

$$K\,(f'(t))^{\frac{\sigma^-}{2}} \le f''(t) \quad \text{for } t > t_0, \qquad K = \left(\lambda - \frac{1}{\sigma}\right)\frac{b^-}{\lambda}.$$

The direct integration leads to the inequality

$$(f'(t))^{\frac{\sigma^-}{2}-1} \ge \frac{(f'(t_0))^{\frac{\sigma^-}{2}-1}}{1-(t-t_0)K\left(\frac{\sigma^-}{2}-1\right)(f'(t_0))^{\frac{\sigma^-}{2}-1}} \to \infty \quad \text{as } t \to T = t_0 + \frac{2\,(f'(t_0))^{1-\frac{\sigma^-}{2}}}{K(\sigma^- - 2)},$$

which contradicts the assumption $t^* = \infty$:

$$\infty > \frac{1}{2}\,|\Omega|\,\|u(\cdot,t)\|_{\infty,\Omega} \ge \frac{1}{2}\|u(\cdot,t)\|^2_{2,\Omega} \equiv f'(t) \to \infty \quad \text{as } t \to T.$$

The proof of Theorem 4 is completed.

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