

The Freudenthal gauge symmetry of the black holes of $\mathcal{N} = 2$, $d = 4$ supergravity

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ABSTRACT: We show that the representation of black-hole solutions in terms of the variables H^M which are harmonic functions in the supersymmetric case is non-unique due to the existence of a local symmetry in the effective action. This symmetry is a continuous (and local) generalization of the discrete Freudenthal transformations initially introduced for the black-hole charges and can be used to rewrite the physical fields of a solution in terms of entirely different-looking functions.

KEYWORDS: Black Holes, String Duality, Gauge Symmetry, Supergravity Models

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1 Introduction

The FGK formalism developed in ref. [1] reduces the problem of finding single, static, charged, spherically-symmetric black-hole solutions of a generic 4-dimensional theory of gravity coupled to a number of Abelian vectors A^Λ_μ and scalars ϕ^i (without scalar potential) to the simpler problem of finding solutions to a dynamical system whose dynamical variables are just the metric function $U(\tau)$ and the scalar fields $\phi^i(\tau)$; the evolution parameter τ corresponds to a radial coordinate in the black hole spacetime metric. This dramatic simplification allowed the authors of ref. [1] to derive the very important result, valid for the extremal black-hole solutions of any of these theories including all the 4-dimensional ungauged supergravity theories, relating the attractor values of the scalars on the event horizon with the entropy through the so-called black-hole potential. We will refer to this famous result as the *FGK theorem*.

Following these results, most of the work in this field has focused on extremal black holes (supersymmetric and non-supersymmetric) since they can be characterized, to a large extent, by the possible attractors and the entropy which, in many supersymmetric theories with large enough duality groups, can be determined by purely algebraic methods.

The FGK formalism was not used for the explicit construction of the extremal solutions, though. The dynamical system is simpler than the original equations but still very non-linear and complicated. The supersymmetric extremal solutions were constructed by methods based on the study of the consistency conditions of the Killing spinor equations. Even though the form of these solutions is known, showing that they solve the equations of motion of the FGK formalism is not a simple task.

Non-supersymmetric extremal solutions have received a lot of attention in the last few years: there are more of these than supersymmetric ones and, furthermore, they have a richer structure. A first-order formalism has been constructed for them starting from the FGK dynamical system and a lot has been learned about the possible attractors, entropies

etc., see e.g. refs. [2–7]. However, not many explicit solutions have been constructed since the first-order equations are not easy to integrate.

Non-extremal black-hole solutions have been left untouched by these developments since the FGK theorem does not apply to them: one needs to construct the explicit solution in order to compute the entropy, the temperature and the dependence of the very important *non-extremality* parameter r_0 on the physical constants, i.e. mass, electric and magnetic charges and the values of the scalars at infinity. In ref. [8] a general ansatz for non-extremal black holes of ungauged $N = 2, d = 4$ supergravity was proposed and it was shown that using this ansatz the equations of motion of the FGK formalism can be solved at least for some simple theories.¹ Non-extremal solutions interpolate between different extremal solutions, supersymmetric and non-supersymmetric alike, that can be recovered by taking the extremal limit. This provides a new method for constructing the extremal non-supersymmetric solutions.

The *hyperbolic ansatz* proposed in ref. [8] was based on the assumption that all the black-hole solutions of a given theory have exactly the same expression in terms of some functions $H^M(\tau)$, called *seed functions*. Different solutions correspond to different profiles for the seed functions, since they will satisfy different equations. For supersymmetric solutions, the functions $H^M(\tau)$ will just be harmonic functions (linear in the coordinate τ). For non-extremal solutions, ref. [8] proposed that the seed functions $H^M(\tau)$ should be linear combinations of hyperbolic functions. The hyperbolic ansatz was known to be valid in the few non-extremal solutions known to the literature [10, 11]. Furthermore, the expression of the physical fields in terms of the $H^M(\tau)$ was known to remain the same after the gauging of global symmetries [12–14].

The assumption that the black hole solutions have the same form in terms of the seed functions was proven in the formulation of the H-FGK formalism for $N = 2, d = 4$ supergravity theories, developed in refs. [15, 16]: this formalism is obtained from the standard FGK one by a change of variables, the new variables being, precisely, the H^M s mentioned above.² The very existence of the change of variables in all $N = 2, d = 4$ theories proves the assumption. However, the new formulation has additional advantages: since the new variables are, somehow, the “right” variables, finding new solutions and general results (attractor theorems, first-order flow equations etc.) becomes much simpler [18].³ In particular, it is extremely easy to prove that the supersymmetric extremal black-hole solutions with harmonic H^M s are solutions of the equations of motion; the situation w.r.t. extremal non-supersymmetric black hole solutions is more complicated.

There are, however, some loose ends in these developments: in ref. [22, 23] an extremal non-supersymmetric solution for cubic models was constructed in which one of the $H^M(\tau)$ s, rather than being harmonic, has been shown in ref. [7] to be the inverse of a harmonic function. Ratios of harmonic functions have been later on discussed and con-

¹A generalization of the FGK formalism for higher-dimensional theories as made in ref. [9], where a similar ansatz was shown to work in a simple $N = 2, d = 5$ supergravity theory.

²This formulation is clearly related to the real formulation of local special geometry of ref. [17].

³There is also an H-FGK formulation for black holes and black strings of $N = 2, d = 5$ supergravity [16, 19, 20]. The derivation of the attractor theorem, first-order flow equations etc. has been done in ref. [21].

firmed in ref. [24, 25]. On the other hand, the general study performed in [18] suggests that in extremal black holes, supersymmetric or not, all the H^M s should be harmonic.⁴ Furthermore, the hyperbolic ansatz is used together with a simplifying constraint on the variables H^M which arises quite naturally in the supersymmetric case [26], but which has no justification in the non-supersymmetric cases, both extremal and non-extremal. The non-harmonic solutions of refs. [7, 22–25] do not satisfy said constraint.

In this paper we take a first step towards the clarification of the situation by showing how the description of a solution in terms of the variables H^M is not unique. We are going to show the existence of a gauge symmetry in the 4-dimensional H-FGK formalism that acts on the variables H^M in a highly non-trivial and non-linear way but preserves the physical fields of the black-hole solution: the metric function $U(\tau)$ and the complex scalar fields $Z^i(\tau)$. This symmetry does not preserve the above-mentioned constraint and, as we are going to see, it can relate a configuration of the H^M s that does not satisfy it to another configuration that does: both configurations, however, describe the same physical black-hole solution. Whether the transformed H^M that do satisfy the constraint are harmonic is more difficult to prove in general and we will study this problem in another publication [27].

An interesting aspect of the gauge symmetry that we have discovered is that it is based on a generalization of the Freudenthal duality transformation discovered in ref. [28] and generalized in the context of $\mathcal{N} = 8, d = 4$ supergravity and generalized to $\mathcal{N} \geq 2, d = 4$ supergravities in ref. [29]. The original Freudenthal transformation is a discrete transformation that acts on the symplectic vector of magnetic and electric charges of a given theory⁵ but one can define the same action on any other symplectic vector of the same theory and, in particular on the variables H^M . As we will show, the discrete transformations are a particular case of a continuous local symmetry of the H-FGK.

We start by reviewing in depth the H-FGK formalism for $N = 2, d = 4$ theories in section (2). In section (3) we discuss the discrete Freudenthal transformations and in section (4) we show that the HFGK action has a Freudenthal gauge symmetry. In section (5) we discuss the interplay of the Freudenthal gauge symmetry with the constraint, identifying the latter as a gauge fixing condition. Finally, in section (6) we present our conclusions and discuss, briefly, the implications of the local Freudenthal symmetry for the extremal solutions.

2 The H-FGK formalism for $\mathcal{N} = 2, d = 4$ supergravity revisited

The action of all ungauged $N = 2, d = 4$ supergravity theories coupled to n vector multiplets takes the form⁶

$$I[g_{\mu\nu}, A^\Lambda_\mu, Z^i] = \int d^4x \sqrt{|g|} \left\{ R + 2\mathcal{G}_{ij^*} \partial_\mu Z^i \partial^\mu Z^{*j^*} + 2\Im \mathcal{N}_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu} - 2\Re \mathcal{N}_{\Lambda\Sigma} F^\Lambda_{\mu\nu} \star F^{\Sigma\mu\nu} \right\}, \quad (2.1)$$

⁴Observe that the hyperbolic ansatz always gives harmonic functions in the extremal limit.

⁵The transformation depends on the particular theory under consideration.

⁶We will follow the notation and conventions of ref. [16].

where $i, j = 1, \dots, n$ and $\Lambda, \Sigma = 0, 1, \dots, n$. The scalar-dependent Kähler metric \mathcal{G}_{ij^*} and period matrix $\mathcal{N}_{\Lambda\Sigma}$ are related by supersymmetry and can be derived, in general, from a holomorphic prepotential function $\mathcal{F}(\mathcal{X})$ homogeneous of degree 2 in the coordinates \mathcal{X}^Λ or, equivalently, from a canonically normalized, covariantly holomorphic symplectic section $(\mathcal{V}^M) = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix}$. Here M, N, \dots are $(2n + 2)$ -dimensional symplectic indices and we use the symplectic metric $(\Omega_{MN}) \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$ and $\Omega^{MP}\Omega_{NP} = \delta^M_N$ to lower and raise the symplectic indices according to the convention

$$\mathcal{V}_M = \Omega_{MN}\mathcal{V}^N, \quad \mathcal{V}^M = \mathcal{V}_N\Omega^{NM}. \quad (2.2)$$

The metrics of any single, static, 4-dimensional black-hole solutions to these theories can be put in the form

$$ds^2 = e^{2U}dt^2 - e^{-2U}\gamma_{mn}dx^m dx^n, \quad (2.3)$$

$$\gamma_{mn}dx^m dx^n = \frac{r_0^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau}d\Omega_{(2)}^2,$$

where r_0 is the so-called *non-extremality parameter* and $U(\tau)$ the metric function that characterizes a particular solution.⁷ Assuming that all the fields are static and spherically symmetric, so that they only depend on the radial coordinate τ , the action (2.1) reduces to the FGK effective action [1]

$$I_{\text{FGK}}[U, Z^i] = \int d\tau \left\{ (\dot{U})^2 + \mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*j^*} - e^{2U}V_{\text{bh}}(Z, Z^*, \mathcal{Q}) \right\}, \quad (2.4)$$

which has to be supplemented by the Hamiltonian constraint

$$(\dot{U})^2 + \mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*j^*} + e^{2U}V_{\text{bh}}(Z, Z^*, \mathcal{Q}) = r_0^2. \quad (2.5)$$

In the above formulae $V_{\text{bh}}(Z, Z^*, \mathcal{Q})$ is the so-called *black-hole potential* and is given by

$$-V_{\text{bh}}(Z, Z^*, \mathcal{Q}) = -\frac{1}{2}\mathcal{M}_{MN}(\mathcal{N})\mathcal{Q}^M\mathcal{Q}^N; \quad (2.6)$$

\mathcal{Q}^M is the $(2n + 2)$ -dimensional symplectic vector of electric q and magnetic p charges $(\mathcal{Q}^M) = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}$ and $\mathcal{M}_{MN}(\mathcal{N})$ is the symmetric, symplectic matrix defined by

$$(\mathcal{M}_{MN}(\mathcal{N})) \equiv \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}, \quad R \equiv \Re\mathcal{N}, \quad I \equiv \Im\mathcal{N}. \quad (2.7)$$

Observe that since there is no explicit τ dependence in the effective action (2.4), the corresponding Hamiltonian must take a constant value: the Hamiltonian constraint (2.5) fixes this *a priori* unconstrained value to be r_0^2 .

The change of variables that brings us to the H-FGK formalism is inspired in the general form of the timelike supersymmetric solutions of these theories obtained by analyzing the consistency of the Killing spinor equations (see e.g. ref. [30]): given an $\mathcal{N} = 2$, $d = 4$

⁷More information about this metric can be found in ref. [8].

theory with canonical symplectic section \mathcal{V}^M , introducing a complex variable X with the same Kähler weight as \mathcal{V}^M , we can define the real Kähler-neutral symplectic vectors

$$\mathcal{R}^M \equiv \Re(\mathcal{V}^M/X), \quad \mathcal{I}^M \equiv \Im(\mathcal{V}^M/X). \quad (2.8)$$

The components \mathcal{R}^M can be expressed in terms of the \mathcal{I}^M by solving a set of algebraic equations commonly called the stabilization equations [31–33] (although this name is used with a different meaning in part of the literature), but to which we shall refer henceforth, for reasons that will become clear in the following and to avoid confusion, as the *Freudenthal duality equations*. The functions $\mathcal{R}^M(\mathcal{I})$ are characteristic of each theory, but they are always homogeneous of first degree in the \mathcal{I}^M .

Given the fact that, in supersymmetric solutions, the \mathcal{I}^M are harmonic functions, it is customary to relabel these variables as

$$H^M \equiv \mathcal{I}^M, \quad \tilde{H}^M \equiv \mathcal{R}^M. \quad (2.9)$$

Given those functions we can define the *Hesse potential* $W(H)$ [15, 16, 34]

$$W(H) \equiv \langle \tilde{H} | H \rangle \equiv \tilde{H}_M H^M, \quad (2.10)$$

which is homogeneous of second degree in H^M . The relation between \tilde{H}^M and H^M can be inverted and the Hesse potential can also be written as $W(\tilde{H})$; from the homogeneity of W one can deduce that

$$\tilde{H}_M = \frac{1}{2} \frac{\partial W}{\partial H^M} \equiv \frac{1}{2} \partial_M W, \quad H^M = \frac{1}{2} \frac{\partial W}{\partial \tilde{H}_M}. \quad (2.11)$$

Of special importance to the H-FGK formalism is the symmetric symplectic matrix $\mathcal{M}_{MN}(\mathcal{F})$ which is obtained by replacing in the expression (2.7) the period matrix $\mathcal{N}_{\Lambda\Sigma}$ by

$$\mathcal{F}_{\Lambda\Sigma} \equiv \frac{\partial^2 \mathcal{F}(\mathcal{X})}{\partial \mathcal{X}^\Lambda \partial \mathcal{X}^\Sigma}, \quad (2.12)$$

where $\mathcal{F}(\mathcal{X})$ is the prepotential of the theory; the relation between them can be seen to be

$$\mathcal{M}_{MN}(\mathcal{F}) = -\mathcal{M}_{MN}(\mathcal{N}) - 2W^{-1} (H_M H_N + \tilde{H}_M \tilde{H}_N). \quad (2.13)$$

From the fundamental properties of the matrix $\mathcal{M}(\mathcal{F})$, namely

$$\begin{aligned} \tilde{H}_M &= -\mathcal{M}_{MN}(\mathcal{F}) H^N, & d\tilde{H}_M &= -\mathcal{M}_{MN}(\mathcal{F}) dH^N, \\ H_M &= \mathcal{M}_{MN}(\mathcal{F}) \tilde{H}^N, & dH_M &= \mathcal{M}_{MN}(\mathcal{F}) d\tilde{H}^N, \end{aligned} \quad (2.14)$$

one can infer that

$$\mathcal{M}_{MN}(\mathcal{F}) = -\frac{1}{2} \frac{\partial^2 W}{\partial H^M \partial H^N} = \frac{1}{2} \frac{\partial^2 W}{\partial \tilde{H}^M \partial \tilde{H}^N}, \quad (2.15)$$

this equation can be rewritten using eqs. (2.11) as

$$\frac{\partial \tilde{H}_N}{\partial H^M} = \Omega_{MP} \Omega_{NQ} \frac{\partial H^Q}{\partial \tilde{H}_P}, \quad (2.16)$$

which is equivalent to saying that \mathcal{M} is a symplectic matrix.

Eq. (2.15) tells us that the Hesse potential W is closely related to the prepotential and is to be considered a *real prepotential*.

Observe that the above discovered Hessicity implies that $\partial_P \mathcal{M}_{MN}(\mathcal{F}) = \partial_{(P} \mathcal{M}_{MN)}(\mathcal{F})$, whereas the homogeneity implies

$$0 = H^P \partial_P \mathcal{M}_{MN}(\mathcal{F}) = \tilde{H}^P \partial_P \mathcal{M}_{MN}(\mathcal{F}). \quad (2.17)$$

Now, using general properties of Special Geometry and the above properties one can rewrite the effective action (2.4) and Hamiltonian constraint (2.5) entirely in terms of the new variables H^M [16]:

$$-I_{\text{H-FGK}}[H] = \int d\tau \left\{ \frac{1}{2} g_{MN} \dot{H}^M \dot{H}^N - V \right\}, \quad (2.18)$$

$$-r_0^2 = \frac{1}{2} g_{MN} \dot{H}^M \dot{H}^N + V, \quad (2.19)$$

where we have defined the H -dependent metric

$$g_{MN} \equiv \partial_M \partial_N \log W - 2 \frac{H_M H_N}{W^2} = \frac{\partial_M \partial_N W}{W} - 2 \frac{H_M H_N}{W^2} - 4 \frac{\tilde{H}_M \tilde{H}_N}{W^2}, \quad (2.20)$$

and the potential

$$V(H) = \left\{ -\frac{1}{4} \partial_M \partial_N \log W + \frac{H_M H_N}{W^2} \right\} \mathcal{Q}^M \mathcal{Q}^N = \left\{ -\frac{1}{4} g_{MN} + \frac{1}{2} \frac{H_M H_N}{W^2} \right\} \mathcal{Q}^M \mathcal{Q}^N. \quad (2.21)$$

The relation of this potential to the black-hole potential (2.6) is given by

$$V_{\text{bh}} = -W V. \quad (2.22)$$

3 Discrete Freudenthal transformations

The relation between the tilded and untilded variables can be understood as a duality transformation $H^M \rightarrow \tilde{H}^M$ which can be iterated if we define $\tilde{\tilde{H}}^M \equiv \tilde{H}^M(\tilde{H})$. Using the properties in eqs. (2.11)–(2.17), we find that this duality is an anti-involution, e.g.

$$\tilde{\tilde{H}}^M = -H^M. \quad (3.1)$$

It is not difficult to see that the duality transformation is just the generalization to $\mathcal{N} = 2$, $d = 4$ supergravity theories made in ref. [29] of the *Freudenthal duality* introduced in ref. [28] in the context of $\mathcal{N} = 8$, $d = 4$ supergravity. The same operation can be performed on any symplectic vector of a given theory and, in particular, on the charge vector \mathcal{Q} .

In ref. [29] it was shown that the entropy and the critical points of the black-hole potential are invariant under Freudenthal duality. We will recover this result later as a particular case of the invariance of the H-FGK system under *local Freudenthal rotations*.

The variables we have just defined are related to the physical variables of the FGK formalism U, Z^i by [16]⁸

$$e^{-2U} \equiv W(H) = \tilde{H}_M H^M, \quad Z^i \equiv \frac{\tilde{H}^i + iH^i}{\tilde{H}^0 + iH^0}. \quad (3.2)$$

We can immediately see that the physical variables are invariant under the above Freudenthal duality transformations, i.e.

$$e^{-2U}(\tilde{H}) = e^{-2U}(H), \quad Z^i(\tilde{H}) = Z^i(H), \quad (3.3)$$

It is interesting to study how the central charge changes under Freudenthal duality: first, we rewrite the central charge, whose definition is $\mathcal{Z}(\phi, \mathcal{Q}) \equiv \langle \mathcal{V} | \mathcal{Q} \rangle$ in the form

$$\mathcal{Z}(\phi, \mathcal{Q}) = \frac{e^{i\alpha}}{\sqrt{2W(H)}} (\tilde{H}_M + iH_M) \mathcal{Q}^M, \quad (3.4)$$

where $e^{i\alpha}$ is the phase of X and satisfies the equation [30]

$$\dot{\alpha} = W^{-1} \dot{H}^M H_M - \mathcal{Q}_*, \quad (3.5)$$

where \mathcal{Q}_* is the pullback of the Kähler connection 1-form

$$\mathcal{Q}_* = \frac{1}{2i} \dot{Z}^i \partial_i \mathcal{K} + \text{c.c.} \quad (3.6)$$

Under discrete Freudenthal duality transformations, $W(H)$, the scalars and the Kähler potential are invariant. α is also invariant and

$$(\tilde{H}_M + iH_M)' = -i(\tilde{H}_M + iH_M), \quad (3.7)$$

which implies that

$$\mathcal{Z}'(\phi, \mathcal{Q}) = -i\mathcal{Z}(\phi, \mathcal{Q}), \quad (3.8)$$

but its absolute value will remain invariant.

Observe that when these Freudenthal transformations are non-linear (which is the general case), if we transform a supersymmetric solution, which must have harmonic H^M s of the form

$$H^M = A^M - \frac{1}{\sqrt{2}} \mathcal{Q}^M \tau, \quad (3.9)$$

we will obtain non-harmonic H^M and the transformed solution couldn't possibly be supersymmetric. We must remember, however, that all the physical fields are invariant, whence their supersymmetry properties must also remain invariant. This implies that the variables H^M cannot immediately be identified with those appearing in the analysis of the Killing spinor equations: this is possible only up to discrete Freudenthal transformations.

The near-horizon limit of the transformed H^M s is dominated by the Freudenthal dual of the charges \mathcal{Q}^M , defined in refs. [28, 29], namely

$$\tilde{\mathcal{Q}}^M \equiv -\frac{1}{2} \Omega^{MN} \frac{\partial W(\mathcal{Q})}{\partial \mathcal{Q}^N}. \quad (3.10)$$

⁸The expression for the scalars is not unique (only up to reparametrizations). The expression we give is, however, convenient and simple.

4 Local Freudenthal rotations

In the change of variables taking us to the H-FGK formalism, we have gone from a formulation based on $2n + 1$ real variables, namely U and the Z^i , to one which is based on $2n + 2$ variables, whence we obtained an over-complete formulation. This suggests that there should be a local symmetry in the H-FGK formalism allowing the elimination of one of its degrees of freedom. The variables H^M , on the other hand, transform linearly under the duality group (embedded in $\text{Sp}(n + 1; \mathbb{R})$), as follows from its definition.

The looked-for gauge symmetry can be found by observing that the metric g_{MN} is singular: using the properties (2.14)–(2.17) it is easy to show that it always admits an eigenvector with zero eigenvalue, namely:⁹

$$\tilde{H}^M g_{MN} = 0. \tag{4.2}$$

The equations of motion in the H-FGK formalism are

$$\frac{\delta I_{\text{H-FGK}}}{\delta H^M} = g_{MN} \ddot{H}^N + [PQ, M] \dot{H}^P \dot{H}^Q + \partial_M V = 0, \tag{4.3}$$

where, as g_{MN} is not invertible, we have used the Christoffel symbol of the first kind, i.e.

$$[PQ, M] \equiv \partial_{(P} g_{Q)M} - \frac{1}{2} \partial_M g_{PQ}. \tag{4.4}$$

Using the properties (2.14)–(2.17) it is not difficult to show that

$$\left. \begin{aligned} [PQ, M] \tilde{H}^M &= 0 \\ \tilde{H}^M \partial_M V &= 0 \end{aligned} \right\} \xrightarrow{\text{so that}} \tilde{H}^M \frac{\delta I_{\text{H-FGK}}}{\delta H^M} = 0. \tag{4.5}$$

This is a constraint that relates the equations of motion of the H-FGK formalism. This kind of constraints arises in systems with gauge symmetries, as a consequence of Noether’s second theorem and is a *gauge identity*. Indeed, multiplying the constraint by an arbitrary infinitesimal function $f(\tau)$ and integrating over τ we find that eq. (4.5) implies

$$\delta_f I_{\text{H-FGK}} = \int d\tau \delta_f H^M \frac{\delta I_{\text{H-FGK}}}{\delta H^M} = 0, \tag{4.6}$$

where we have defined the local infinitesimal transformations

$$\delta_f H^M \equiv f(\tau) \tilde{H}^M. \tag{4.7}$$

As one can expect from a gauge invariance, this transformation leaves invariant the physical variables of the FGK formalism U, Z^i . To check it, it is enough to use

$$\delta_f \tilde{H}^M \equiv -f(\tau) H^M, \tag{4.8}$$

⁹For the sake of completeness we also quote the relation

$$g_{MN} H^N = -2\tilde{H}_M/W \Rightarrow g_{MN} H^M H^N = -2. \tag{4.1}$$

which follows from eq. (3.1) and eqs. (3.2).

The finite gauge transformations can be obtained by exponentiating the infinitesimal ones:

$$\delta_f H^M \equiv f(\tau) \mathcal{L}_K H^M \quad \longrightarrow \quad H'^M = e^{f(\tau) \mathcal{L}_K} H^M \quad \text{where} \quad K^M(H) = \tilde{H}^M. \quad (4.9)$$

It is not difficult to see that the finite transformations are

$$\begin{cases} H'^M = \cos f H^M - \sin f \Omega^{MN} \tilde{H}_N, \\ \tilde{H}'_M = -\sin f \Omega_{MN} H^N + \cos f \tilde{H}_M. \end{cases} \quad (4.10)$$

By defining the complex variables $\mathcal{H}^M \equiv \tilde{H}^M + iH^M$ we can write the transformation as

$$\mathcal{H}'^M = e^{if(\tau)} \mathcal{H}^M. \quad (4.11)$$

Using this form of the transformation and expressing the scalars and the metric function in the forms

$$e^{-2U} = W(H) = \frac{i}{2} \mathcal{H}_M \mathcal{H}^{*M}, \quad Z^i \equiv \mathcal{H}^i / \mathcal{H}^0, \quad (4.12)$$

the invariance of the physical fields under this gauge symmetry is paramount.

A direct proof of the invariance of the H-FGK effective action is also desirable: the invariance of the kinetic term, i.e. $\frac{1}{2} g_{MN} \dot{H}^M \dot{H}^N$, follows from the identities

$$(\tilde{H}_M \dot{H}^M)' = \tilde{H}_M \dot{H}^M, \quad \dot{\tilde{H}}^M \mathcal{M}_{MN}(\mathcal{F}) = \dot{H}_N, \quad \dot{H}^M \mathcal{M}_{MN}(\mathcal{F}) = -\dot{\tilde{H}}_N, \quad (4.13)$$

which can be derived from eqs. (2.14). The invariance of the potential $V(H)$ follows from eq. (2.17).

The existence of this symmetry does not help in solving the equations of motion as the Noether charge associated to the invariance under the global Freudenthal rotations vanishes identically:

$$Q = \delta_f H^M \frac{\partial L}{\partial H^M} \sim f \tilde{H}^M g_{MN} \dot{H}^N = 0. \quad (4.14)$$

We have already said that the origin of this gauge symmetry is the introduction of one additional degree of freedom in the passage from the FGK to the H-FGK formalism. Had the original FGK formulation contained the full complex variable $X = e^{U+i\alpha}$ instead of just U , the change of variables would, actually, have been much simpler; alas, the phase α is completely absent from the FGK effective action. The local Freudenthal symmetry is associated to this absence, which allows to change α arbitrarily leaving everything else invariant. Indeed, from eq. (3.5) that defines α , we can easily see that

$$\delta_f \dot{\alpha} = -\dot{f}. \quad (4.15)$$

On the other hand, the Freudenthal gauge symmetry can be made manifest as follows: first, observe that the metric

$$G_{MN}(H) \equiv \partial_M \partial_N \log W - 2(1 + \varepsilon) \frac{H_M H_N}{W}, \quad \varepsilon = \pm 1 \quad (4.16)$$

always admits $K^M(H) = \tilde{H}^M$ as a Killing vector. Then, consider the action

$$-I_{\text{ungauged}}[H] = \int d\tau \left\{ \frac{1}{2} G_{MN} \dot{H}^M \dot{H}^N - V \right\}, \quad (4.17)$$

which has a global Freudenthal symmetry generated by $\delta H^M = f \tilde{H}^M$ with $\dot{f} = 0$. To gauge the Freudenthal symmetry, we just have to replace in this action the derivatives with respect to τ by the covariant derivatives

$$\begin{aligned} \dot{H}^M &\rightarrow \mathcal{D}H^M \equiv \dot{H}^M + A \tilde{H}^M, \\ \dot{\tilde{H}}^M &\rightarrow \mathcal{D}\tilde{H}^M \equiv \dot{\tilde{H}}^M - A H^M, \end{aligned} \quad (4.18)$$

which transform covariantly under the infinitesimal transformations eq. (4.8)

$$\begin{aligned} \delta_f \mathcal{D}H^M &= f \mathcal{D}\tilde{H}^M, \\ \delta_f \mathcal{D}\tilde{H}^M &= -f \mathcal{D}H^M, \end{aligned} \quad (4.19)$$

if the 1-form A transforms as

$$\delta_f A = -\dot{f}(\tau). \quad (4.20)$$

The action

$$-I_{\text{gauged}}[H, A] = \int d\tau \left\{ \frac{1}{2} G_{MN} \mathcal{D}H^M \mathcal{D}H^N - V \right\}, \quad (4.21)$$

is manifestly invariant under local Freudenthal rotations and equivalent to the effective H-FGK action eq. (2.18) as one can see by integrating out the auxiliary field A : its equation of motion is solved by

$$A = \frac{H_N \dot{H}^N}{W}, \quad (4.22)$$

and, upon this substitution

$$G_{MN} \mathcal{D}H^M \mathcal{D}H^N = \left(G_{MN} + 2\varepsilon \frac{H_M H_N}{W} \right) \dot{H}^M \dot{H}^N = g_{MN} \dot{H}^M \dot{H}^N. \quad (4.23)$$

The choice $\varepsilon = +1$, which leads to $G_{MN} = 2W^{-1} \mathcal{M}_{MN}(\mathcal{N})$ is, perhaps, the most natural since the same metric would then occur in the kinetic term and in the potential. It follows that we can rewrite the effective action eq. (2.18) and the Hamiltonian constraint eq. (2.19) in the suggestive form

$$I_{\text{H-FGK}}[H] = \int d\tau \left\{ V(H, \sqrt{2} \mathcal{D}H) + V(H, \mathcal{Q}) \right\}, \quad (4.24)$$

$$r_0^2 = V(H, \sqrt{2} \mathcal{D}H) - V(H, \mathcal{Q}), \quad (4.25)$$

with

$$\mathcal{D}H^M = \dot{H}^M + \frac{H_N \dot{H}^N}{W} \tilde{H}^M. \quad (4.26)$$

Finally, it is worth noting that this Freudenthal gauge theory is unrelated to the one constructed in ref. [35].

5 Unconventional solutions and Freudenthal gauge freedom

If we contract the equations of motion (4.3) with H^P and use the homogeneity properties of the different terms and the Hamiltonian constraint eq. (2.19), we find a useful equation

$$\tilde{H}_M \left(\ddot{H}^M - r_0^2 H^M \right) + \frac{(\dot{H}^M H_M)^2}{W} = 0, \quad (5.1)$$

which corresponds to that of the variable U in the FGK formulation.

In the supersymmetric (hence, extremal) case, the constraint

$$\dot{H}^M H_M = 0, \quad (5.2)$$

enforcing the absence of NUT charge must be satisfied, in agreement with the assumption of staticity of the metric [26]. Using this constraint the above equation takes the form

$$\tilde{H}_M \left(\ddot{H}^M - r_0^2 H^M \right) = 0, \quad (5.3)$$

and can be solved in the extremal case by assuming that the H^M are linear in τ , whence they are harmonic, and in the non-extremal case by assuming that the H^M are linear combinations of hyperbolic functions of $r_0\tau$ (the hyperbolic ansatz). The solutions that one can get with these assumptions have been intensively studied in ref. [18].

The constraint eq. (5.2) is not preserved by the local Freudenthal symmetry: a small calculation gives

$$\delta_f(\dot{H}^M H_M) = -\dot{f}W, \quad (5.4)$$

which can be integrated straightforwardly to a finite rotation, namely

$$(\dot{H}^M H_M)' = -\dot{f}W + \dot{H}^M H_M. \quad (5.5)$$

This equation implies that given a configuration H^M with $\dot{H}^M H_M \neq 0$, we can find another configuration H'^M with $\dot{H}'^M H'_M = 0$ describing exactly the same configuration of physical fields by performing a finite local Freudenthal transformation with a parameter $f(\tau)$ satisfying

$$\dot{f} = \frac{\dot{H}^M H_M}{W}. \quad (5.6)$$

This shows that it is always possible to impose the constraint eq. (5.2) without loss of generality because it can be understood as just a good gauge-fixing condition.

6 Conclusions

The extremal static black-hole solutions of $\mathcal{N} = 2, d = 4$ supergravity constructed so far in the literature and written in terms of the variables H^M can be classified using two criteria: the harmonicity of the H^M s and whether they satisfy the constraint $H_M \dot{H}^M = 0$ or not. Out of the four possible cases, represented in table (1), the equation of motion eq. (5.1) excludes the one corresponding to the upper right corner. The upper left corner

	$H_M \dot{H}^M = 0$	$H_M \dot{H}^M \neq 0$
$\ddot{H}^M = 0$	BPS and some non-BPS	no solutions
$\ddot{H}^M \neq 0$		some non-BPS

Table 1. Classification of the extremal static black-hole solutions of $\mathcal{N} = 2, d = 4$ supergravity according to their representation in terms of the variables H^M . It must be taken into account that they satisfy eq. (5.1) with $r_0 = 0$.

corresponds to the supersymmetric black-hole solutions and, as shown in ref. [8], also to some non-BPS solutions as well. The lower-right corner corresponds to the extremal non-BPS solutions discovered in refs. [7, 22–25] and the lower-left corner does not correspond to any known solution.

In this paper we have shown that the representation of the solutions in terms of the H -variables is non-unique due to the presence of the local Freudenthal invariance. Furthermore, we have shown that this symmetry can be used to transform all the solutions in the lower-right corner to solutions in the left column. It is not yet clear whether they will be transformed into solutions in the upper or lower row although preliminary results in simple examples suggest that, typically, they will be transformed into solutions in the lower-left corner. The form of the H^M s in this class is probably quite complicated as they must satisfy the equation

$$\tilde{H}_M \ddot{H}^M = 0, \tag{6.1}$$

and, at the same time, $\ddot{H}^M \neq 0$. Furthermore, solutions of this kind must be possible only in very special cases and only in some theories, as it happens for the solutions in the lower-right corner. Clearly, more work is needed to arrive at a complete understanding of the situation and to chart the space of extremal black-hole solutions of these theories. The non-extremal case is even more challenging. Work in these directions is in progress [27].

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