# Central tendency for symmetric random fuzzy numbers [1]

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**Abstract**

Random fuzzy numbers are becoming a valuable tool to model and handle fuzzy-valued data generated through a random process. Recent studies have been devoted to introduce measures of the central tendency of random fuzzy numbers showing a more robust behaviour than the so-called Aumann-type mean value. This paper aims to deepen in the (rather comparative) analysis of these centrality measures and the Aumann-type mean by examining the situation of symmetric random fuzzy numbers. Similarities and differences with the real-valued case are pointed out and theoretical conclusions are accompanied with some illustrative examples.

*Key words:* Aumann-type mean of a random fuzzy number, fuzzy number, $L^1$ medians of a random fuzzy number, random fuzzy number, symmetric fuzzy number, symmetric random fuzzy number

## 1 Introduction

Symmetric random variables sometimes appear exactly in real-life situations, but they mainly correspond to either an idealized or an approximate model for many of them. Symmetric random variables show several interesting properties, especially in connection with their central tendencies. More specifically, the behaviour of the two most popular central tendency measures, the mean and the median, in dealing with symmetric distributions of random variables becomes one of the soundest arguments supporting their adequacy to summarize the central tendency of these variables.

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On the other hand, in the last decades fuzzy data have been shown to be a suitable tool in modeling imprecise data coming from judgements/opinions/ ratings/valuations/etc. The flexibility of fuzzy numbers allows us to capture the intrinsic imprecision of such data by means of the use of $[0,1]$-valued functions, leading to a powerful and expressive way to 'wording' such ratings/valuations... and to an ease-to-develop computation setting.

Random fuzzy numbers, as a special case of the so-coined fuzzy random variables by Puri and Ralescu [19] model a random mechanism generating fuzzy data and extend real-valued random variables (and also random intervals) by allowing data to be fuzzy-valued.

In Section 2 some preliminaries concerning random fuzzy numbers will be recalled along with convenient extensions of the mean and median values for them. Section 3 introduces and examines the notion of symmetric random fuzzy number about a real value. In Section 4 a discussion is presented on the values these centrality measures take on for symmetric random fuzzy numbers, and the obtained conclusions are compared with those for the real-valued case. In Section 5 a comparative study is developed to examine the proximity of the central tendency measures to 'central position' values of some symmetric random fuzzy numbers. The paper ends with some concluding remarks.

## 2 Preliminaries

Let $\mathcal{F}_c(\mathbb{R})$ denote the space of fuzzy numbers, where a *fuzzy number* (also called bounded fuzzy number) is a mapping $\widetilde{U}:\mathbb{R}\to[0,1]$ so that for each $\alpha\in[0,1]$ the *$\alpha$-level set*

$$\widetilde{U}_\alpha = \begin{cases} \{x\in\mathbb{R} : \widetilde{U}(x)\geq\alpha\} & \text{if } \alpha>0 \\ \\ \mathrm{cl}\{x\in\mathbb{R} : \widetilde{U}(x)>0\} & \text{if } \alpha=0 \end{cases}$$

is a nonempty compact interval.

Equivalently, Goetschel and Voxman [15] proved that a fuzzy number is a mapping $\widetilde{U}:\mathbb{R}\to[0,1]$ such that

- $\inf\widetilde{U}_{(\cdot)}:[0,1]\to\mathbb{R}$ is a bounded non-decreasing function,
- $\sup\widetilde{U}_{(\cdot)}:[0,1]\to\mathbb{R}$ is a bounded non-increasing function,
- $\inf\widetilde{U}_1\leq\sup\widetilde{U}_1$,
- $\inf\widetilde{U}_{(\cdot)}$ and $\sup\widetilde{U}_{(\cdot)}$ are left-continuous on $(0,1]$ and right-continuous at 0.

When fuzzy data are described by means of elements in $\mathcal{F}_c(\mathbb{R})$, the statistical data analysis involves the usual fuzzy arithmetic based on Zadeh's extension principle [29]. The two key operations, the sum and the product by a scalar, can be equivalently formalized as the level-wise extensions of the usual interval-valued operations, i.e., for $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$ and $\lambda \in \mathbb{R}$, for each $\alpha \in [0,1]$

$$
\begin{aligned}
(\widetilde{U} + \widetilde{V})_\alpha &= (\text{Minkowski sum of } \widetilde{U}_\alpha \text{ and } \widetilde{V}_\alpha) = \{y + z \,:\, y \in \widetilde{U}_\alpha, z \in \widetilde{V}_\alpha\} \\
&= [\,\inf \widetilde{U}_\alpha + \inf \widetilde{V}_\alpha, \sup \widetilde{U}_\alpha + \sup \widetilde{V}_\alpha],
\end{aligned}
$$

$$
(\lambda \cdot \widetilde{U})_\alpha = \lambda \cdot \widetilde{U}_\alpha = \{\lambda \cdot y \,:\, y \in \widetilde{U}_\alpha\} = \begin{cases} [\lambda \inf \widetilde{U}_\alpha, \lambda \sup \widetilde{U}_\alpha] & \text{if } \lambda \geq 0 \\ \\ [\lambda \sup \widetilde{U}_\alpha, \lambda \inf \widetilde{U}_\alpha] & \text{otherwise.} \end{cases}
$$

As a consequence of this arithmetic, if $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$, then the difference $\widetilde{U} - \widetilde{V}$ can be immediately established by considering $\widetilde{U} - \widetilde{V} = U + (-1) \cdot \widetilde{V}$. At this point, it should be pointed out that $\widetilde{U} - \widetilde{V} + \widetilde{V} \neq \widetilde{U}$. More precisely, $\widetilde{V} - \widetilde{V} \neq \mathbb{1}_{\{0\}}$, but $\widetilde{V} - \widetilde{V} = \mathcal{O}_{\widetilde{V}}$, where for any $\alpha \in [0,1]$ corresponds to the centrally symmetric about 0 interval given by

$$
(\mathcal{O}_{\widetilde{V}})_\alpha = [\,\inf \widetilde{V}_\alpha - \sup \widetilde{V}_\alpha, \sup \widetilde{V}_\alpha - \inf \widetilde{V}_\alpha],
$$

whence $\mathcal{O}_{\widetilde{V}}$ is a symmetric fuzzy number about 0 which only reduces to $\mathbb{1}_{\{0\}}$ if, and only if, $\widetilde{V}$ reduces to the indicator function of a singleton $\mathbb{1}_{\{v\}}$ $(v \in \mathbb{R})$.

Random elements taking on intrinsic fuzzy number values can be suitably formalized in terms of random fuzzy numbers, a notion which was coined as fuzzy random variable and which was stated in a more general space of fuzzy sets by Puri and Ralescu [19]. The particularization to the case of fuzzy number-valued random elements lead to the following concept:

**Definition 2.1** *Given a probability space* $(\Omega, \mathcal{A}, P)$ *modeling a random experiment, an associated* ***random fuzzy number*** *is a mapping* $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ *such that for all* $\alpha \in [0,1]$ *the interval-valued* $\alpha$*-level mapping* $\mathcal{X}_\alpha = [\inf \mathcal{X}_\alpha, \sup \mathcal{X}_\alpha]$ *is a compact random interval (that is,* $\inf \mathcal{X}_\alpha$ *and* $\sup \mathcal{X}_\alpha$ *are two random variables satisfying that* $\inf \mathcal{X}_\alpha \leq \sup \mathcal{X}_\alpha$*).*

If $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ is a random fuzzy number, then one can prove (see Colubi *et al.* [5]) that it is a Borel-measurable mapping with respect to the Borel $\sigma$-field generated on $\mathcal{F}_c(\mathbb{R})$ by the topology associated with several different metrics. The Borel-measurability of random fuzzy numbers allows us to properly establish the *induced distribution of a random fuzzy number*, the *independence of random fuzzy numbers*, and others.

**Remark 2.1** It should be emphasized that, although the induced distribution of a random fuzzy number is well-defined, one cannot universally characterize it by means of a distribution function like in the real-valued case. This is due to the fact that there is no ranking for fuzzy numbers which is universally accepted. Indeed, one can define different complete orderings between fuzzy numbers, which show reasonable properties in many cases and being valuable for problems inexcusably requiring a ranking, but none of them can be considered as generally acceptable. For this reason, there is no formal definition of continuous random fuzzy numbers. However, one can consider a rather formal approach to discrete random fuzzy numbers and one can also compute the probability function by using the induced probabilities based on the Borel-measurability.

In summarizing the central tendency of the distribution of a random fuzzy number we can consider extensions of the two main measures in the real-valued case, namely, the mean value and the median.

On one hand, the mean value is suitably extended by means of the Aumann-type mean value of a random fuzzy number (Puri and Ralescu [19]), which for *integrably bounded random fuzzy numbers* (i.e., for random fuzzy numbers $\mathcal{X}$ such that $\inf \mathcal{X}_0$ and $\sup \mathcal{X}_0$ are integrable w.r.t. the corresponding probability space) are defined as follows:

**Definition 2.2** *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ be an associated integrably bounded random fuzzy number. The* ***Aumann-type mean value*** *of $\mathcal{X}$ is the fuzzy number $\widetilde{E}(\mathcal{X}) \in \mathcal{F}_c(\mathbb{R})$, if it exists, such that for all $\alpha \in [0, 1]$*

$$\left(\widetilde{E}(\mathcal{X})\right)_\alpha = \textit{Aumann integral of } \mathcal{X}_\alpha \textit{ in } (\Omega, \mathcal{A}, P) = [E(\inf \mathcal{X}_\alpha), E(\sup \mathcal{X}_\alpha)].$$

The Aumann-type mean value of a random fuzzy number is the Fréchet's expectation with respect to many $L^2$ metrics on $\mathcal{F}_c(\mathbb{R})$. In particular, it satisfies that

$$\widetilde{E}(\mathcal{X}) = \arg \min_{\widetilde{U} \in \mathcal{F}_c(\mathbb{R})} E\left(\left[D_\theta(\mathcal{X}, \widetilde{U})\right]^2\right),$$

with

$$D_\theta(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{[0,1]} \left(\left[\operatorname{mid} \widetilde{U}_\alpha - \operatorname{mid} \widetilde{V}_\alpha\right]^2 + \theta \left[\operatorname{spr} \widetilde{U}_\alpha - \operatorname{spr} \widetilde{V}_\alpha\right]^2\right) d\alpha}$$

where $\operatorname{mid} \widetilde{U}_\alpha =$ mid-point of $\widetilde{U}_\alpha = (\inf \widetilde{U}_\alpha + \sup \widetilde{U}_\alpha)/2$, $\operatorname{spr} \widetilde{U}_\alpha =$ radius of $\widetilde{U}_\alpha$ $= (\sup \widetilde{U}_\alpha - \inf \widetilde{U}_\alpha)/2$ and $\theta \in (0, 1]$ is a weighting parameter (see Bertoluzza *et al.* [2], for the first reference, and Näther [17], Trutschnig *et al.* [27] and Gil *et al.* [11] for subsequent generalizations and studies).

The operator $\widetilde{E}$ is equivariant under 'affine' (in accordance with the usual fuzzy arithmetic) transformations, it is additive, and it is coherent with the usual arithmetic with fuzzy numbers. $\widetilde{E}(\mathcal{X})$ is also supported by Strong Laws of Large Numbers, so that the sample mean value is a strongly consistent estimator of the population one.

On the other hand, and because of the problems encountered in ranking fuzzy numbers, the median has been already extended by considering some $L^1$ metrics on $\mathcal{F}_c(\mathbb{R})$, based on some different representations of fuzzy numbers, as the fuzzy number(s) minimizing the expected distance to the random fuzzy number. With this purpose two approaches have been recently introduced. In accordance with the first one (Sinova *et al.* [23]):

**Definition 2.3** *Given a probability space* $(\Omega, \mathcal{A}, P)$ *and an associated random fuzzy number* $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$*, the **1-norm** (inf/sup-type) **median** of* $\mathcal{X}$ *is the fuzzy number* $\widetilde{\mathrm{Me}}(\mathcal{X}) \in \mathcal{F}_c(\mathbb{R})$ *such that for all* $\alpha \in (0,1]$

$$\left(\widetilde{\mathrm{Me}}(\mathcal{X})\right)_\alpha = [\mathrm{Me}(\inf \mathcal{X}_\alpha), \mathrm{Me}(\sup \mathcal{X}_\alpha)],$$

*where in case either* $\mathrm{Me}(\inf \mathcal{X}_\alpha)$ *or* $\mathrm{Me}(\sup \mathcal{X}_\alpha)$ *are not unique the usual criterion of selecting the mid-point of the interval of medians is applied.*

The 1-norm median satisfies that

$$\widetilde{\mathrm{Me}}(\mathcal{X}) = \arg \min_{\widetilde{U} \in \mathcal{F}_c(\mathbb{R})} E\left(\rho_1(\mathcal{X}, \widetilde{U})\right),$$

with

$$\rho_1(\widetilde{U}, \widetilde{V}) = \frac{1}{2} \int_{(0,1]} \left(\left|\inf \widetilde{U}_\alpha - \inf \widetilde{V}_\alpha\right| + \left|\sup \widetilde{U}_\alpha - \sup \widetilde{V}_\alpha\right|\right) d\alpha$$

(see Diamond and Kloeden [8]).

The operator $\widetilde{\mathrm{Me}}$ is equivariant under 'affine' transformations and the sample 1-norm median is a strongly consistent estimator of the population one in the $\rho_1$ sense.

In accordance with the second approach (Sinova *et al.* [24]):

**Definition 2.4** *Given a probability space* $(\Omega, \mathcal{A}, P)$ *and an associated random fuzzy number* $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$*, the* $0.5$***-median** of* $\mathcal{X}$ *is the fuzzy number* $\widetilde{\mathrm{Me}}^{0.5}(\mathcal{X}) \in \mathcal{F}_c(\mathbb{R})$ *such that for all* $\alpha \in (0,1]$

$$\left(\widetilde{\mathrm{Me}}^{0.5}(\mathcal{X})\right)_\alpha = \left[\mathrm{Me}(\mathrm{mid}\, \mathcal{X}_{0.5}) - \mathrm{Me}(\mathrm{ldev}^{0.5}\, \mathcal{X}_\alpha), \mathrm{Me}(\mathrm{mid}\, \mathcal{X}_{0.5}) + \mathrm{Me}(\mathrm{rdev}^{0.5}\, \mathcal{X}_\alpha)\right],$$

*where* $\mathrm{ldev}^{0.5}\, \mathcal{X}_\alpha = \mathrm{mid}\, \mathcal{X}_{0.5} - \inf \mathcal{X}_\alpha$*,* $\mathrm{rdev}^{0.5}\, \mathcal{X}_\alpha = \sup \mathcal{X}_\alpha - \mathrm{mid}\, \mathcal{X}_{0.5}$*, and where in case the involved median of a real-valued random variable is not unique the usual criterion of selecting the mid-point of the interval of medians is applied.*

The 0.5-median satisfies that

$$\widetilde{\mathrm{Me}}^{0.5}(\mathcal{X}) = \arg \min_{\widetilde{U} \in \mathcal{F}_c(\mathbb{R})} E\left(\mathfrak{D}_\theta^{0.5}(\mathcal{X}, \widetilde{U})\right),$$

with

$$\mathfrak{D}_\theta^{0.5}(\widetilde{U},\widetilde{V}) = |\text{mid}\,\widetilde{U}_{0.5} - \text{mid}\,\widetilde{V}_{0.5}|$$

$$+\frac{\theta}{2}\int_{(0,1]} |\text{ldev}^{0.5}\,\widetilde{U}_\alpha - \text{ldev}^{0.5}\,\widetilde{V}_\alpha|\,d\alpha + \frac{\theta}{2}\int_{(0,1]} |\text{rdev}^{0.5}\,\widetilde{U}_\alpha - \text{rdev}^{0.5}\,\widetilde{V}_\alpha|\,d\alpha$$

(see Sinova *et al.* [24]).

The operator $\widetilde{\text{Me}}^{0.5}$ is equivariant under 'affine' transformations and the sample 0.5-norm median is a strongly consistent estimator of the population one in the $\mathfrak{D}_\theta^{0.5}$ sense.

The two sample medians have shown a more robust behaviour than the Aumann-type sample mean (see Sinova *et al.* [23,24]).

**Remark 2.2** It should be pointed out that one of the main advantages of the 1-norm and the 0.5 median is that they can be computed on the basis of the medians for certain real-valued random variables, and we can prove that they both determine fuzzy numbers, as formally shown in Sinova *et al.* [23,24].

This makes computations rather easy-to-perform and, mainly, easy to implementing and programming in R or others. Actually, there is no need to solve a minimization problem to find the median of the random fuzzy number but simply applying the general solution, which is known.

At this point, we should indicate that when the involved $L^1$ metrics are replaced for some other ones, the minimization problem can become a very difficult task, and often infeasible at least to get the exact solution. In this respect, if we consider the $L^1$ metric given by

$$\mathbf{D}_\theta(\widetilde{U},\widetilde{V}) = \int_{[0,1]} \left(\left|\text{mid}\,\widetilde{U}_\alpha - \text{mid}\,\widetilde{V}_\alpha\right| + \theta\left|\text{spr}\,\widetilde{U}_\alpha - \text{spr}\,\widetilde{V}_\alpha\right|\right)\,d\alpha$$

one cannot reason as for $\rho_1$ and $\mathfrak{D}_\theta^{0.5}$ since there are not sufficient conditions for the mid/spr representation of fuzzy numbers so that these conditions characterize them. More concretely, if following the solutions for $\rho_1$ and $\mathfrak{D}_\theta^{0.5}$ one is tempted to use as a possible solution minimizing $E\left(\left[\mathbf{D}_\theta(\mathcal{X},\widetilde{U})\right]\right)$ over $\widetilde{U} \in \mathcal{F}_c(\mathbb{R})$, the level-wise solution $\text{M}_\alpha = [\text{Me}(\text{mid}\,\mathcal{X}_\alpha) - \text{Me}(\text{spr}\,\mathcal{X}_\alpha), \text{Me}(\text{mid}\,\mathcal{X}_\alpha) + \text{Me}(\text{spr}\,\mathcal{X}_\alpha)]$ for each $\alpha$, the class $\{\text{M}_\alpha\}_\alpha$ does not define in general a fuzzy number.

As a counterexample illustrating this assertion, we can consider the following:

**Example 2.1** Consider a random fuzzy number $\mathcal{X}$ taking with probability 0.2 each of five different values $\widetilde{x}_i$, $i \in \{1, \dots, 5\}$ which, in accordance with their horizontal view, are given by

$$(\text{mid}\,\widetilde{x}_1)_\alpha = 1 - \alpha/2, \quad (\text{spr}\,\widetilde{x}_1)_\alpha = 1.1 - \alpha,$$

$$(\text{mid}\,\widetilde{x}_2)_\alpha = \begin{cases} 0.75 & \text{if } \alpha \leq 0.5 \\ 1.25 - \alpha & \text{otherwise} \end{cases}, \quad (\text{spr}\,\widetilde{x}_2)_\alpha = 1.1 - \alpha,$$

$$(\text{mid}\,\widetilde{x}_3)_\alpha = 0.6 + 0.3\,\alpha, (\text{mid}\,\widetilde{x}_4)_\alpha = 0, \; (\text{mid}\,\widetilde{x}_5)_\alpha = 2,$$

$$(\text{spr}\,\widetilde{x}_3)_\alpha = (\text{spr}\,\widetilde{x}_4)_\alpha = (\text{spr}\,\widetilde{x}_5)_\alpha = \begin{cases} 0.75 - \alpha & \text{if } \alpha \leq 0.5 \\ 0.4 - 0.3\,\alpha & \text{otherwise,} \end{cases}$$

which are graphically displayed in Figure 1

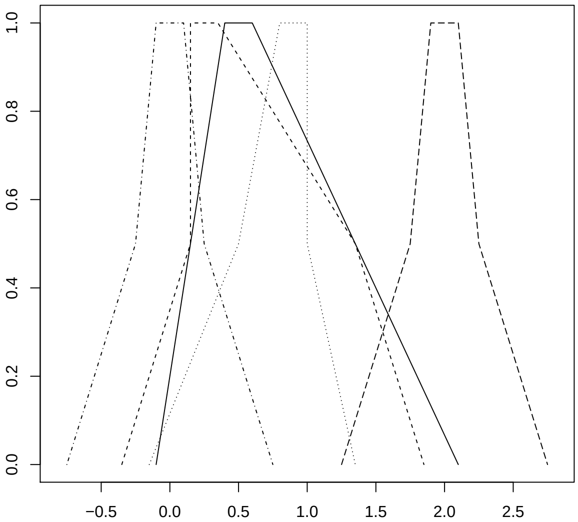


Fig. 1. Five different values of a random set (that takes them with the same probability)

The value for the mean, 1-norm median and $\beta = 0.5$-median can be found graphically displayed in Figure 2

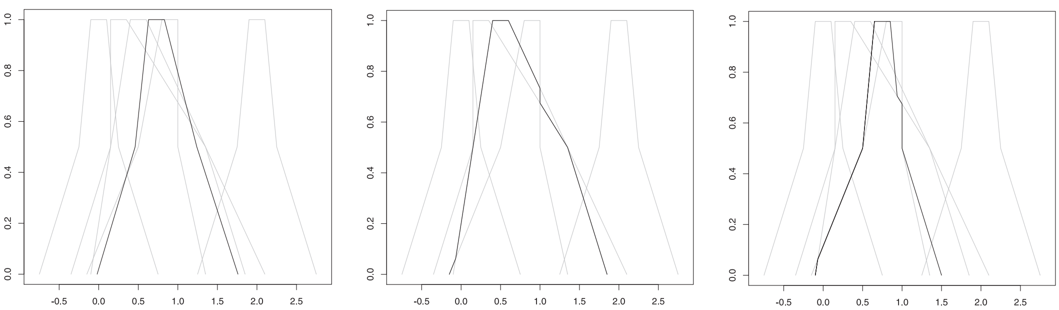

Fig. 2. Mean, 1-norm median and 0.5 median of the random fuzzy number being uniformly distributed on the set of fuzzy number values in Figure 1

In this case we have that

$$\text{Me}(\text{mid}\,\mathcal{X}_\alpha) = \begin{cases} 0.75 & \text{if } \alpha \leq 0.5 \\ 1 - \alpha/2 & \text{otherwise,} \end{cases} \quad \text{Me}(\text{spr}\,\mathcal{X}_\alpha) = \begin{cases} 0.75 - \alpha & \text{if } \alpha \leq 0.5 \\ 0.4 - 0.3\,\alpha & \text{otherwise,} \end{cases}$$

whence the intervals $[\text{Me}(\text{mid}\,\mathcal{X}_\alpha) - \text{Me}(\text{spr}\,\mathcal{X}_\alpha), \text{Me}(\text{mid},\mathcal{X}_\alpha) + \text{Me}(\text{spr}\,\mathcal{X}_\alpha)]$ do not lead to a fuzzy number, but to the function in Figure 3.

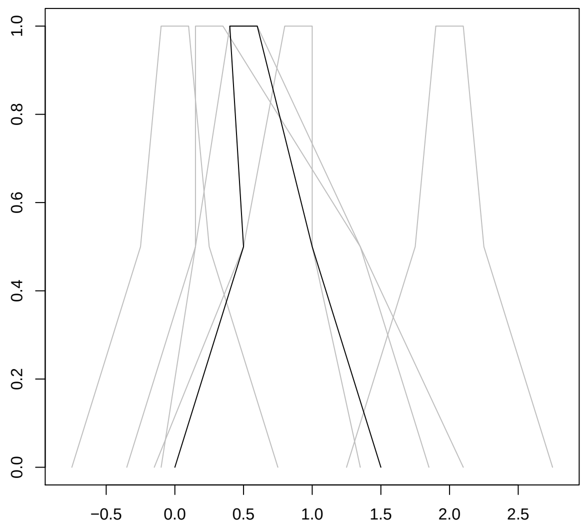


Fig. 3. Result of representing the function (fuzzy number?) with $\alpha$-levels $[\text{Me}(\text{mid}\,\mathcal{X}_\alpha) - \text{Me}(\text{spr}\,\mathcal{X}_\alpha), \text{Me}(\text{mid},\mathcal{X}_\alpha) + \text{Me}(\text{spr}\,\mathcal{X}_\alpha)]$, which is not a fuzzy number

## 3 Symmetric random fuzzy numbers

The notion of symmetry of a probability distribution can be naturally extended from the real- to the fuzzy-valued case. To formalize this extension in the most general way one can state the following:

**Definition 3.1** *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space, and let* $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ *be a* ***random fuzzy number*** *associated with* $(\Omega, \mathcal{A}, P)$. $\mathcal{X}$ *is said to be* ***symmetric about*** $c \in \mathbb{R}$ *if, and only if,* $\mathcal{X} - c \overset{d}{=} c - \mathcal{X}$ *or, equivalently,* $\mathcal{X} \overset{d}{=} 2c - \mathcal{X}$, $\overset{d}{=}$ *denoting identity in distribution.*

Obviously, $\mathcal{X}$ is a symmetric random fuzzy number about $c$ if, and only if, $\mathcal{X} - c$ is a symmetric random fuzzy number about 0. Consequently, if $\mathcal{X}$ is a symmetric random fuzzy number about $c$, then it can be rewritten as $\mathcal{X} = \mathcal{E} + c$, where $\mathcal{E}$ is a symmetric random fuzzy number about 0.

To easily interpret this notion we can consider three examples. The first example is a simple one which indicates a key divergence with respect to the real-valued case.

**Example 3.1** Let $\mathcal{X}$ be a random fuzzy number associated with a probability space $(\Omega, \mathcal{A}, P)$ and let $\mathcal{O}_{\mathcal{X}} = \mathcal{X} - \mathcal{X}$. For any $\omega \in \Omega$, we have that $\mathcal{O}_{\mathcal{X}}(\omega)$ is a symmetric random fuzzy number about 0.

The second example is based on the one supplied by Chou [7].

**Example 3.2** In many social surveys, respondents are customarily asked by means of a questionnaire to indicate their choices from a set of prefixed Likert-type items. Many researchers consider Likert-type labels as fuzzy number-valued ones, by identifying the generic response to a question with a fuzzy linguistic variable (see, for instance, Serrano-Guerrero *et al.* [21], Porcel *et al.* [18], for recent studies about). As indicated by Chou [7], "often the wording of response levels clearly implies a symmetry of response levels about a middle category; at the very least, such an item would fall between ordinal-level and interval-level measurement... The use of fuzzy sets is central in computing with words or labels as they provide a means of modeling the vagueness underlying most natural linguistic terms (see, for instance, Zadeh [30]). The semantic elements of the term set are given by fuzzy numbers defined on a bounded interval (say $[0, 1]$). In practice, triangular fuzzy numbers are a uniformly distributed ordered set of linguistic terms, so they provide a relatively simple way to capture the vagueness of linguistic assessments,..." like the ones which are graphically displayed in Figure 4, where VD = STRONGLY DISAGREE, D = DISAGREE, SD = SOMEWHAT DISAGREE, N = NEITHER AGREE NOR DISAGREE, SA = SOMEWHAT AGREE, A = AGREE and VA = STRONGLY AGREE.

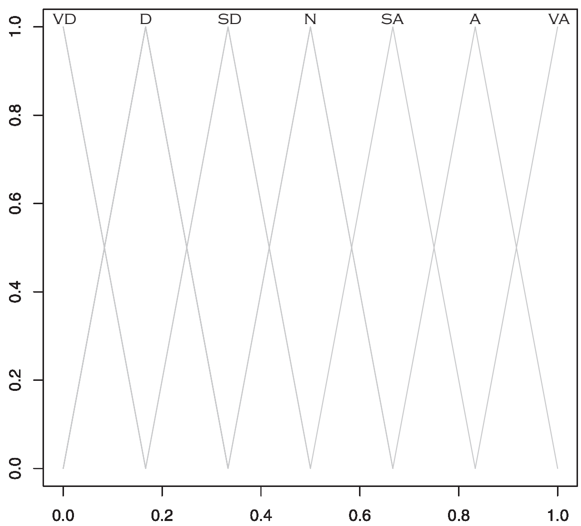


Fig. 4. Semantic elements of a term set given by 7 fuzzy triangular numbers

Assume that the survey has been performed in a population/sample $\{\omega_1, \ldots, \omega_{953}\}$ of 953 respondents and that the distribution of the random response $\mathcal{X}$ is the following:

| label | VD | D | SD | N | SA | A | VA |
|---|---|---|---|---|---|---|---|
| absol. freq. | 38 | 143 | 207 | 177 | 207 | 143 | 38 |

Since

$$\text{VD} = 1 - \text{VA}, \ \ \text{D} = 1 - \text{A}, \ \ \text{SD} = 1 - \text{SA},$$

and

$$\#\{\omega_j : \mathcal{X}(\omega_j) = \text{VD}\} = 38 = \#\{\omega_j : \mathcal{X}(\omega_j) = \text{VA}\},$$
$$\#\{\omega_j : \mathcal{X}(\omega_j) = \text{D}\} = 143 = \#\{\omega_j : \mathcal{X}(\omega_j) = \text{A}\},$$
$$\#\{\omega_j : \mathcal{X}(\omega_j) = \text{SD}\} = 207 = \#\{\omega_j : \mathcal{X}(\omega_j) = \text{SA}\},$$

then the random fuzzy set $\mathcal{X}$ is symmetric about $c = 0.5$.

The third example has been drawn from the so-called ‘characterizing fuzzy representation’ of real-valued random variables (see González-Rodríguez *et al.* [13]).

**Example 3.3** Let $X$ be a random variable associated with a probability space and assume that $X$ has a Binomial distribution $\text{Bin}(4, 0.5)$. González-Rodríguez *et al.* [13] have introduced a generalized fuzzy representation which characterizes the distribution of a real-valued random variable by means of the Aumann-type expected value of random fuzzy set corresponding to the composition of the fuzzy representation and the random variable.

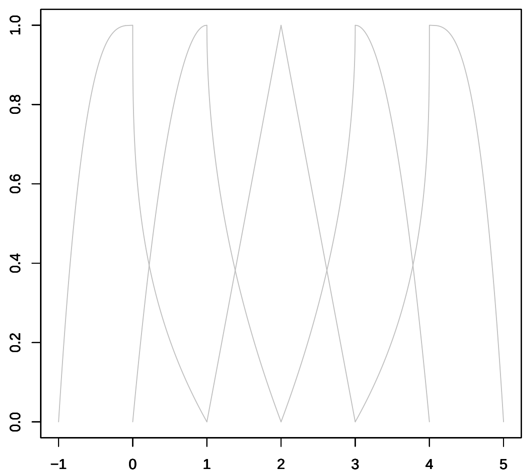


Fig. 5. Values of a characterizing fuzzy representation of an RV taking on values 0, 1, 2, 3 and 4

In this way, if $X \sim \text{Bin}(4, 0.5)$, then the random fuzzy number $\mathcal{X} = \gamma_{(2)} \circ X$ such that for all $\alpha \in [0, 1]$

$$(\gamma_{(2)}(x))_\alpha = \begin{cases} \left[x - (1-\alpha)^{1/(3-x)}, x + (1-\alpha)^{3-x}\right] & \text{if } x = 0, 1 \\ \\ \left[x - (1-\alpha)^{x-1}, x + (1-\alpha)^{1/(x-1)}\right] & \text{if } x = 2, 3, 4 \end{cases}$$

is a symmetric random fuzzy number about $c = 2$. Its five different values have been graphically displayed on Figure 5 and their corresponding probabilities are those associated with the five values of the Binomial.

The symmetry of a random fuzzy number entails that of all their $\alpha$-level random intervals. Thus,

**Proposition 3.1** *Let $\mathcal{X}$ be a random fuzzy number associated with a probability space $(\Omega, \mathcal{A}, P)$. If $\mathcal{X}$ is symmetric about $c \in \mathbb{R}$, then for all $\alpha \in [0, 1]$, the $\alpha$-level random interval $\mathcal{X}_\alpha$ is symmetric about $c$.*

*Proof.* Indeed, whatever $\alpha \in [0, 1]$ and the compact interval $K$ may be, we have that

$$\mathcal{X}_\alpha^{-1}(K) = \{\omega \in \Omega : (\mathcal{X}(\omega))_\alpha = K\}$$

$$= \bigcup_{\widetilde{x} \in \mathcal{X}(\Omega) : \widetilde{x}_\alpha = K} \{\omega \in \Omega : \mathcal{X}(\omega) = \widetilde{x}\} = \bigcup_{\widetilde{x} \in \mathcal{X}(\Omega) : \widetilde{x}_\alpha = K} \mathcal{X}_\alpha^{-1}(\{\widetilde{x}\}),$$

$$(2c - \mathcal{X})_\alpha^{-1}(K) = \{\omega \in \Omega : (2c - \mathcal{X}(\omega))_\alpha = K\}$$

$$= \bigcup_{\widetilde{x} \in (2c-\mathcal{X})(\Omega) : \widetilde{x}_\alpha = K} \{\omega \in \Omega : (2c - \mathcal{X})(\omega) = \widetilde{x}\}$$

$$= \bigcup_{\widetilde{x} \in (2c-\mathcal{X})(\Omega) : \widetilde{x}_\alpha = K} \{\omega \in \Omega : \mathcal{X}(\omega) = 2c - \widetilde{x}\}$$

$$= \bigcup_{\widetilde{x} \in (2c-\mathcal{X})(\Omega) : \widetilde{x}_\alpha = K} \mathcal{X}_\alpha^{-1}(\{2c - \widetilde{x}\})$$

whence, because of the symmetry of $\mathcal{X}$ about $c$, we can conclude that $\mathcal{X}_\alpha^{-1}(K) \overset{a.s.[P]}{=} (2c - \mathcal{X})_\alpha^{-1}(K)$ and, consequently, $\mathcal{X}_\alpha$ and $(2c - \mathcal{X})_\alpha$ are identically distributed. □

The converse assertion is not true. Thus, if for each $\alpha \in [0, 1]$, the random interval $\mathcal{X}_\alpha$ is symmetric about $c$, the random fuzzy number $\mathcal{X}$ is not necessarily symmetric about $c$. As a counterexample we can consider the following:

**Example 3.4** Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\mathcal{A} = \mathcal{P}(\Omega)$, and $P$ being associated with a uniform distribution on $\Omega$. Let $\mathcal{X}$ be the random fuzzy number such that

$$\mathcal{X}(\omega_1)(x) = \begin{cases} x^2/2 & \text{if } x \in [0,1] \\ -(x^2 - 4x + 2)/2 & \text{if } x \in (1,3] \\ (x-4)^2/2 & \text{si } x \in (3,4] \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{X}(\omega_2)(x) = \text{Tri}(0,2,4),$$

where $\text{Tri}(a,b,c)$ denotes the triangular fuzzy number such that $\text{Tri}(a,b,c)_0 = [a,c]$, $\text{Tri}(a,b,c)_1 = \{b\}$,

$$\mathcal{X}(\omega_3)(x) = \begin{cases} -x/2 & \text{if } x \in (-1,0] \\ -(x^2 + 4x + 2)/2 & \text{if } x \in (-3,-1] \\ (x+4)/2 & \text{si } x \in [-4,-3] \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{X}(\omega_4)(x) = \begin{cases} x^2/2 & \text{if } x \in (-1,0] \\ -x/2 & \text{if } x \in (-2,-1] \\ (x+4)/2 & \text{if } x \in (-3,-2] \\ (x+4)^2/2 & \text{si } x \in [-4,-3] \\ 0 & \text{otherwise} \end{cases}$$

One can easily prove that, for all $\alpha \in [0,1]$ the random interval $\mathcal{X}_\alpha$ is symmetric about 0. Thus,

- on one hand, for all $\alpha \in [0, 0.5]$

$$\mathcal{X}_\alpha(\omega_1) = -\mathcal{X}_\alpha(\omega_4),\ \mathcal{X}_\alpha(\omega_2) = -\mathcal{X}_\alpha(\omega_3)$$

  whence, because of $P$ being associated with the uniform distribution on $\Omega$, $\mathcal{X}_\alpha$ and $-\mathcal{X}_\alpha$ are identically distributed, i.e., $\mathcal{X}_\alpha$ is symmetric about 0;
- on the other hand, for all $\alpha \in (0.5, 1]$

$$\mathcal{X}_\alpha(\omega_1) = -\mathcal{X}_\alpha(\omega_3),\ \mathcal{X}_\alpha(\omega_2) = -\mathcal{X}_\alpha(\omega_4)$$

  whence, because of $P$ being associated with the uniform distribution on $\Omega$, $\mathcal{X}_\alpha$ and $-\mathcal{X}_\alpha$ are identically distributed, i.e., $\mathcal{X}_\alpha$ is symmetric about 0.

However, each of the four distinct values of $\mathcal{X}$ differ from the four distinct values of $-\mathcal{X}$, and hence $\mathcal{X}$ is not symmetric about 0.

## 4 Special features of the central tendency of symmetric random fuzzy numbers

In the real-valued case a well-known and valuable result is that the two main central tendency measures of a symmetric random variable, namely, the mean and the median, coincide with the point the variable is symmetric about whenever the latter is unique.

This section aims to show that in case of considering random fuzzy numbers this assertion should be slightly modified, due to the involved fuzziness. Thus, the extended central tendency measures show a suitable central tendency behaviour since they lead to fuzzy numbers which are symmetric about the symmetry point. Moreover, we will see that the fuzzy values of these measures neither necessarily coincide nor correspond to any of the values the random fuzzy number takes on.

The discussion about this failure starts with the analysis of the Aumann-type mean value of a symmetric random fuzzy number.

**Proposition 4.1** *Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\mathcal{X}$ be an integrably bounded symmetric random fuzzy number about $c \in \mathbb{R}$. Then, the Aumann-type mean value of $\mathcal{X}$ is a symmetric fuzzy number about $c$.*

*Proof.* Since $\mathcal{X} \stackrel{d}{=} 2c - \mathcal{X}$, then $\widetilde{E}(\mathcal{X}) = \widetilde{E}(2c - \mathcal{X})$ whence, because of the equivariance of the Aumann-type mean value under affine transformations, we have that

$$\widetilde{E}(\mathcal{X}) = 2c - \widetilde{E}(\mathcal{X}).$$

By adding $\widetilde{E}(\mathcal{X})$ to the two members in the last equality $2\,\widetilde{E}(\mathcal{X}) = 2c+ \widetilde{E}(\mathcal{X})$ $-\ \widetilde{E}(\mathcal{X})$ and, hence,

$$\widetilde{E}(\mathcal{X}) = c + \frac{1}{2} \cdot \mathcal{O}_{\widetilde{E}(\mathcal{X})},$$

that is, for all $\alpha \in [0, 1]$

$$(\widetilde{E}(\mathcal{X}))_\alpha = [c - \operatorname{spr}(\widetilde{E}(\mathcal{X}))_\alpha, c + \operatorname{spr}(\widetilde{E}(\mathcal{X}))_\alpha].$$

Consequently, $\widetilde{E}(\mathcal{X})$ is a symmetric fuzzy number about $c$. □

The result in Proposition 4.1 is now illustrated by computing the mean values of the symmetric random fuzzy numbers in Examples 3.2 and 3.3.

**Example 4.1** The Aumann-type mean value of the symmetric random fuzzy number about 0.5 in Example 3.2 is graphically displayed in Figure 6:

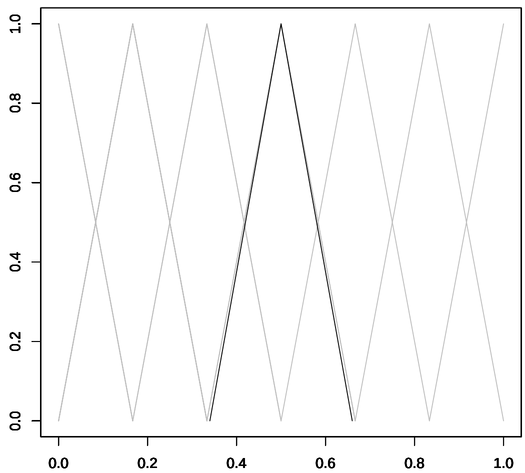


Fig. 6. Aumann-type mean (in black) of the 953 responses in Example 3.2

The Aumann-type mean expected value of the symmetric random fuzzy number about 2 in Example 3.3 is graphically displayed in Figure 7:

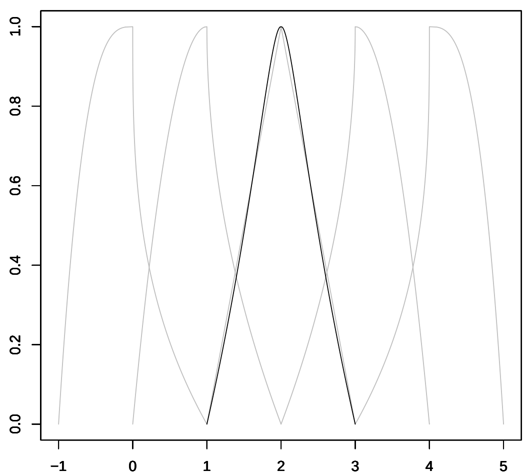


Fig. 7. Aumann-type mean (in black) of the characterizing fuzzy representation of the Bin$(4, 0.5)$ in Example 3.3

The discussion about the median of a symmetric random fuzzy number depends on the approach for the median we consider, but conclusions are the same.

**Proposition 4.2** *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space, and let* $\mathcal{X}$ *be a symmetric random fuzzy number about* $c \in \mathbb{R}$*. Then, both the 1-norm median and the* $\beta = 0.5$*-median of* $\mathcal{X}$ *are symmetric fuzzy numbers about* $c$*.*

*Proof.* Since $\mathcal{X} \stackrel{d}{=} 2c-\mathcal{X}$, then $\widetilde{\mathrm{Me}}(\mathcal{X}) = \widetilde{\mathrm{Me}}(2c-\mathcal{X})$ and $\widetilde{\mathrm{Me}}^{0.5}(\mathcal{X}) = \widetilde{\mathrm{Me}}^{0.5}(2c - \mathcal{X})$, whence because of the equivariance properties of $\widetilde{\mathrm{Me}}$ and $\widetilde{\mathrm{Me}}^{0.5}$ under affine transformations, we have that

$$\widetilde{\mathrm{Me}}(\mathcal{X}) = 2c - \widetilde{\mathrm{Me}}(\mathcal{X}) \text{ and } \widetilde{\mathrm{Me}}^{0.5}(\mathcal{X}) = 2c - \widetilde{\mathrm{Me}}^{0.5}(\mathcal{X}).$$

By adding $\widetilde{\mathrm{Me}}(\mathcal{X})$ and $\widetilde{\mathrm{Me}}^{0.5}(\mathcal{X})$ to the two members in the last two equalities, respectively, $2\,\widetilde{\mathrm{Me}}(\mathcal{X}) = 2c + \widetilde{\mathrm{Me}}(\mathcal{X}) - \widetilde{\mathrm{Me}}(\mathcal{X})$ and $2\,\widetilde{\mathrm{Me}}^{0.5}(\mathcal{X}) = 2c + \widetilde{\mathrm{Me}}^{0.5}(\mathcal{X}) - \widetilde{\mathrm{Me}}^{0.5}(\mathcal{X})$ and, hence,

$$\widetilde{\mathrm{Me}}(\mathcal{X}) = c + \frac{1}{2} \cdot \mathcal{O}_{\widetilde{\mathrm{Me}}(\mathcal{X})} \text{ and } \widetilde{\mathrm{Me}}^{0.5}(\mathcal{X}) = c + \frac{1}{2} \cdot \mathcal{O}_{\widetilde{\mathrm{Me}}^{0.5}(\mathcal{X})}.$$

By arguing like in the proof of Proposition 4.1 we can immediately conclude that both $\widetilde{\mathrm{Me}}(\mathcal{X})$ and $\widetilde{\mathrm{Me}}^{0.5}(\mathcal{X})$ are symmetric fuzzy numbers about $c$. □

The result in Proposition 4.2 is now illustrated by computing the two medians of the symmetric random fuzzy numbers in Examples 3.2 and 3.3.

**Example 4.2** To compute the median of the symmetric random fuzzy number about 0.5 in Example 3.2 we should take into account that

| label | VD | D | SD | N | SA | A | VA |
|---|---|---|---|---|---|---|---|
| absol. freq. | 38 | 143 | 207 | 177 | 207 | 143 | 38 |
| $\inf_\alpha$ | 0 | $\frac{\alpha}{6}$ | $\frac{\alpha+1}{6}$ | $\frac{\alpha+2}{6}$ | $\frac{\alpha+3}{6}$ | $\frac{\alpha+4}{6}$ | $\frac{\alpha+5}{6}$ |
| $\sup_\alpha$ | $\frac{1-\alpha}{6}$ | $\frac{2-\alpha}{6}$ | $\frac{3-\alpha}{6}$ | $\frac{4-\alpha}{6}$ | $\frac{5-\alpha}{6}$ | $\frac{6-\alpha}{6}$ | 1 |
| $\mathrm{mid}_{0.5}$ | $\frac{1}{24}$ | $\frac{4}{24}$ | $\frac{8}{24}$ | $\frac{12}{24}$ | $\frac{16}{24}$ | $\frac{20}{24}$ | $\frac{23}{24}$ |
| $\mathrm{ldev}_\alpha^{0.5}$ | $\frac{1}{24}$ | $\frac{1-\alpha}{6}$ | $\frac{1-\alpha}{6}$ | $\frac{1-\alpha}{6}$ | $\frac{1-\alpha}{6}$ | $\frac{1-\alpha}{6}$ | $\frac{3-4\alpha}{24}$ |
| $\mathrm{rdev}_\alpha^{0.5}$ | $\frac{3-4\alpha}{6}$ | $\frac{1-\alpha}{6}$ | $\frac{1-\alpha}{6}$ | $\frac{1-\alpha}{6}$ | $\frac{1-\alpha}{6}$ | $\frac{1-\alpha}{6}$ | $\frac{1}{24}$ |

whence, by developing a comparison of the values in each row as a function of $\alpha$, one can easily conclude that

$$\widetilde{\mathrm{Me}}(\mathcal{X}) = \widetilde{\mathrm{Me}}^{0.5}(\mathcal{X}) = \mathrm{N}.$$

To compute the median of the symmetric random fuzzy number about 2 in Example 3.3 we should take into account that

| label | $\gamma_{(2)}(0)$ | $\gamma_{(2)}(1)$ | $\gamma_{(2)}(2)$ | $\gamma_{(2)}(3)$ | $\gamma_{(2)}(4)$ |
|---|---|---|---|---|---|
| probab. | .0625 | .25 | .375 | .25 | .0625 |
| $\inf_\alpha$ | $-\sqrt[3]{1-\alpha}$ | $1-\sqrt{1-\alpha}$ | $1+\alpha$ | $3-(1-\alpha)^2$ | $4-(1-\alpha)^3$ |
| $\sup_\alpha$ | $(1-\alpha)^3$ | $1+(1-\alpha)^2$ | $3-\alpha$ | $3+\sqrt{1-\alpha}$ | $4+\sqrt[3]{1-\alpha}$ |
| $\mathrm{mid}_{0.5}$ | $\frac{\sqrt[3]{2}-8}{16\sqrt[3]{2}}$ | $\frac{\sqrt{2}-4}{8\sqrt{2}}$ | 2 | $\frac{23\sqrt{2}+4}{8\sqrt{2}}$ | $\frac{63\sqrt[3]{2}+8}{16\sqrt[3]{2}}$ |
| $\mathrm{ldev}_\alpha^{0.5}$ | $\frac{\sqrt[3]{2}+16\sqrt[3]{2(1-\alpha)}-8}{16\sqrt[3]{2}}$ | $\frac{8\sqrt{2(1-\alpha)}-7\sqrt{2}-4}{8\sqrt{2}}$ | $1-\alpha$ | $\frac{4-\sqrt{2}+8\sqrt{2}(1-\alpha)^2}{8\sqrt{2}}$ | $\frac{8-\sqrt[3]{2}-16\sqrt[3]{2}(1-\alpha)^3}{16\sqrt[3]{2}}$ |
| $\mathrm{rdev}_\alpha^{0.5}$ | $\frac{8-\sqrt[3]{2}-16\sqrt[3]{2}(1-\alpha)^3}{16\sqrt[3]{2}}$ | $\frac{4-\sqrt{2}+8\sqrt{2}(1-\alpha)^2}{8\sqrt{2}}$ | $1-\alpha$ | $\frac{8\sqrt{2(1-\alpha)}-7\sqrt{2}-4}{8\sqrt{2}}$ | $\frac{\sqrt[3]{2}+16\sqrt[3]{2(1-\alpha)}-8}{16\sqrt[3]{2}}$ |

whence, by developing a comparison of the values in each row as a function of $\alpha$ for the 1-norm median, one can easily conclude that

$$\widetilde{\mathrm{Me}}(\mathcal{X}) = \gamma_{(2)}(2),$$

whereas by using an R function in line with the one in the SAFD package (see Sinova *et al.* [25], Trutschnig and Lubiano [28]) the $\beta = 0.5$-median of $\gamma_{(2)} \circ \mathrm{Bin}(4, 0.5)$ has been graphically displayed in Figure 8, and it is very close but not coincident with any random fuzzy number value.

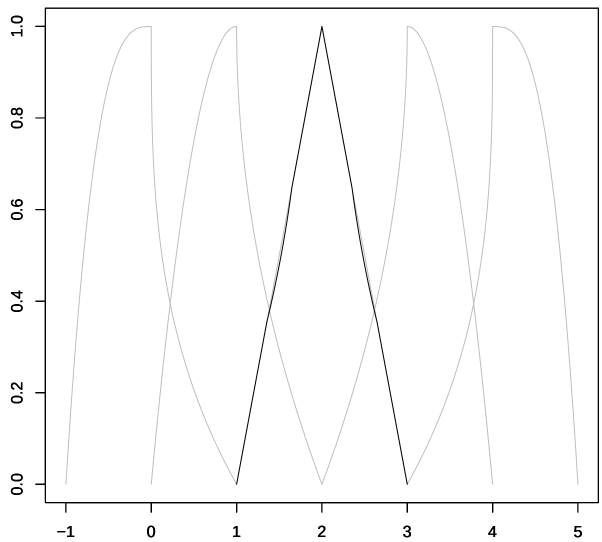


Fig. 8. The $\beta = 0.5$-median (in black) of the characterizing fuzzy representation of the $\mathrm{Bin}(4, 0.5)$ in Example 3.3

Consequently, one can assert that for symmetric random fuzzy numbers about $c$ the two main central tendencies (i.e., the Aumann-type mean and the median, defined in accordance with the two approaches based on $L^1$ metrics between fuzzy numbers) are symmetric fuzzy numbers about $c$, but they do not necessarily coincide.

The above described examples show that the Aumann-type mean is not necessarily a value of the symmetric random fuzzy number, even in case the number of different values is an odd one, and the same happens (although not that frequently) with the median. Anyway, the examined examples make us think that the behaviour of the median seems to be closer to that of the real-valued case than the behaviour of the mean value. The next section will present an empirical discussion on this point.

## 5 Illustrative comparative study between the central tendency measures

This section aims to show, through some examples, that in measuring the central tendency for symmetric random fuzzy numbers the medians (especially the 1-norm one) behave in a more suitable and advisable way than the Aumann-type mean. Thus, in addition to provide us with more robust estimates (see Sinova *et al.* [23] [24]) than the mean, the medians also lead to a fuzzy value which is closer to the one which occupies the 'central position'.

To illustrate this assertion we have considered three different random fuzzy numbers that are symmetric about 0, assumption made for the sake of simplicity and unification although not being relevant. These symmetric random fuzzy numbers have been obtained by composing the already-mentioned characterizing fuzzy representation (González-Rodríguez *et al.* [13]) given by

$$(\gamma_{(0)}(x))_\alpha = \begin{cases} \left[x-(1-\alpha)^{1+x}, x+(1-\alpha)^{1/(1+x)}\right] & \text{if } x \geq 0 \\ \\ \left[x-(1-\alpha)^{1/(1-x)}, x+(1-\alpha)^{1-x}\right] & \text{if } x < 0 \end{cases}$$

with three symmetric real-valued random variables: a standard normal $X \sim \mathcal{N}(0,1)$;, a uniform $X \sim \text{Uniform}(-0.5, 0.5)$; and a translated binomial $X \sim \text{Bin}(5, 0.5) - 2.5$.

After representing the (population) 1-norm median, 0.5-median and Aumann-type mean of each of the random fuzzy numbers $\gamma_{(0)} \circ X$ graphically, distances between each of the summary measures and the correspondent central position value $\gamma_{(0)}(0)$ have been computed and graphically displayed (as functions of $\theta$ when the distance is parameterized). Conclusions are now presented.

Figure 9 shows that when the considered random fuzzy number is $\gamma_{(0)} \circ \mathcal{N}(0,1)$, then the 1-norm median coincides with the central position value, and the 0.5-median is quite close to it, whereas the Aumann-type mean is not that close.

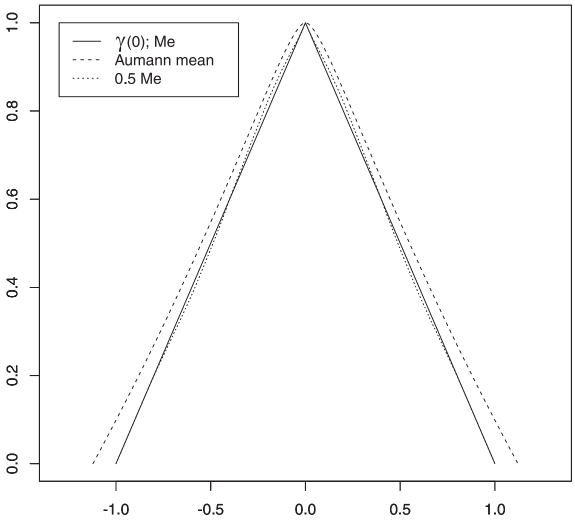


Fig. 9. Aumann-type mean (- - - -), 0.5-median ($\cdots\cdots$) and 1-norm median= $\gamma_{(0)}(0)$ (——) of the random fuzzy number $\gamma_{(0)} \circ \mathcal{N}(0,1)$

This is also corroborated by computing the $D_\theta$-, $\mathfrak{D}_\theta^{0.5}$- and $\rho_1$-distances between each summary measure and $\gamma_{(0)}(0)$, the two first distances as functions of the weighting parameter $\theta$. Thus,

$$\rho_1\left(\widetilde{\mathrm{Me}}(\gamma_{(0)} \circ \mathcal{N}(0,1)), \gamma_{(0)}(0)\right) = 0,$$

$$\rho_1\left(\widetilde{\mathrm{Me}}^{0.5}(\gamma_{(0)} \circ \mathcal{N}(0,1)), \gamma_{(0)}(0)\right) = 0.0078,$$

$$\rho_1\left(\widetilde{E}(\gamma_{(0)} \circ \mathcal{N}(0,1)), \gamma_{(0)}(0)\right) = 0.0751,$$

and the $D_\theta$- and the $\mathfrak{D}_\theta^{0.5}$-distances have been displayed in Figure 10.

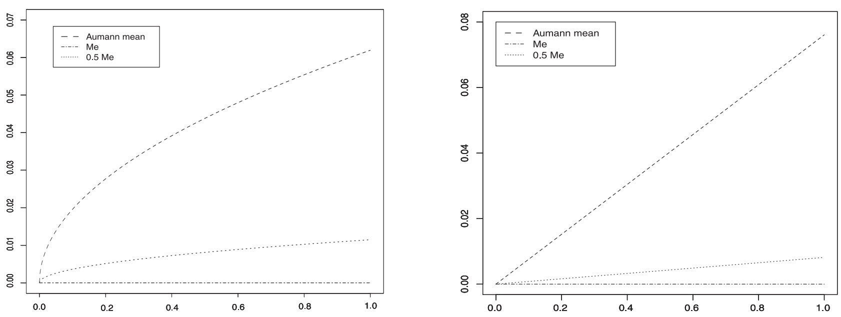


Fig. 10. $D_\theta$-distance (on the left) and the $\mathfrak{D}_\theta^{0.5}$-distance (on the right) between $\gamma_{(0)}(0)$ and the Aumann-type mean (- - - -), 0.5-median ($\cdots\cdots$) and 1-norm median (—·—·) of the random fuzzy number $\gamma_{(0)} \circ \mathcal{N}(0,1)$ as functions of $\theta$

Analogously, Figure 11 shows that when the considered random fuzzy number is $\gamma_{(0)} \circ \text{Uniform}(-0.5, 0.5)$, then the 1-norm median coincides with the central position value, and the 0.5-median is quite close to it, whereas the Aumann-type mean is not that close.

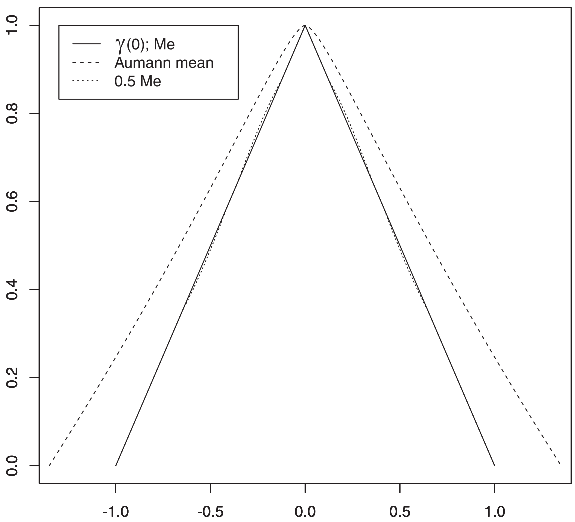


Fig. 11. Aumann-type mean (- - - -), 0.5-median $(\cdots\cdots)$ and 1-norm median= $\gamma_{(0)}(0)$ (——) of the random fuzzy number $\gamma_{(0)} \circ \text{Uniform}(-0.5, 0.5)$

This is also corroborated by computing the $D_\theta$-, $\mathfrak{D}_\theta^{0.5}$- and $\rho_1$-distances between each summary measure and $\gamma_{(0)}(0)$, the two first distances as functions of the weighting parameter $\theta$. Thus,

$$\rho_1\left(\widetilde{\text{Me}}(\gamma_{(0)} \circ \text{Uniform}(-0.5, 0.5)), \gamma_{(0)}(0)\right) = 0,$$

$$\rho_1\left(\widetilde{\text{Me}}^{0.5}(\gamma_{(0)} \circ \text{Uniform}(-0.5, 0.5)), \gamma_{(0)}(0)\right) = 0.0034,$$

$$\rho_1\left(\widetilde{E}(\gamma_{(0)} \circ \text{Uniform}(-0.5, 0.5)), \gamma_{(0)}(0)\right) = 0.1771,$$

and the $D_\theta$- and the $\mathfrak{D}_\theta^{0.5}$-distances have been displayed in Figure 12.

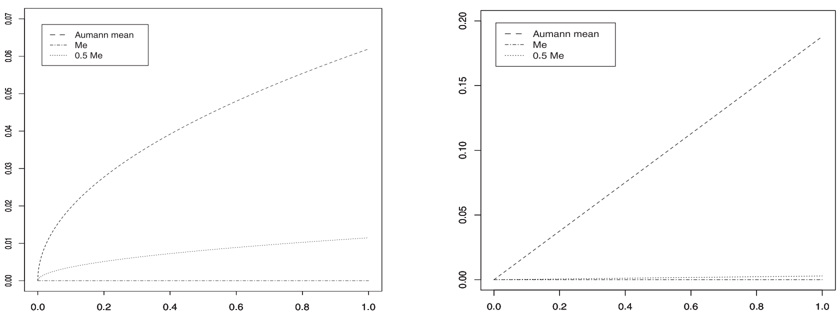


Fig. 12. $D_\theta$-distance (on the left) and the $\mathfrak{D}_\theta^{0.5}$-distance (on the right) between $\gamma_{(0)}(0)$ and the Aumann-type mean (- - - -), 0.5-median $(\cdots\cdots)$ and 1-norm median (—·—·) of the random fuzzy number $\gamma_{(0)} \circ \text{Uniform}(-0.5, 0.5)$ as functions of $\theta$

**Remark 5.1** It should be emphasized that the coincidence $\widetilde{\mathrm{Me}}(\gamma_{(0)} \circ X) = \gamma_{(0)}(0)$ is not at all casual. Whenever $X$ has a either a symmetric continuous distribution or a discrete one with an odd number of distinct values, then the equality holds. This is due to the fact that both $\inf(\gamma_{(0)} \circ X)_\alpha$ and $\sup(\gamma_{(0)} \circ X)_\alpha$ are strictly increasing functions of $X$, whence for each $\alpha \in [0, 1]$ we have that $\mathrm{Me}(\,\inf(\gamma_{(0)} \circ X)_\alpha) = \inf\left(\gamma_{(0)}(0)\right)_\alpha$ and $\mathrm{Me}(\,\sup(\gamma_{(0)} \circ X)_\alpha) = \sup\left(\gamma_{(0)}(0)\right)_\alpha$.

The above-mentioned coincidence does not hold in general when the number of distinct values of $X$ is even. In such a case, we cannot properly talk about 'central position' and conventions should be made, so the use of $\gamma_{(0)}(0)$ as the central position value is not completely fair. Anyway, ir serves us to illustrate that the behavior of the medians in contrast to that of the mean is preserved although advantages for the 0.5-median are not so obvious.

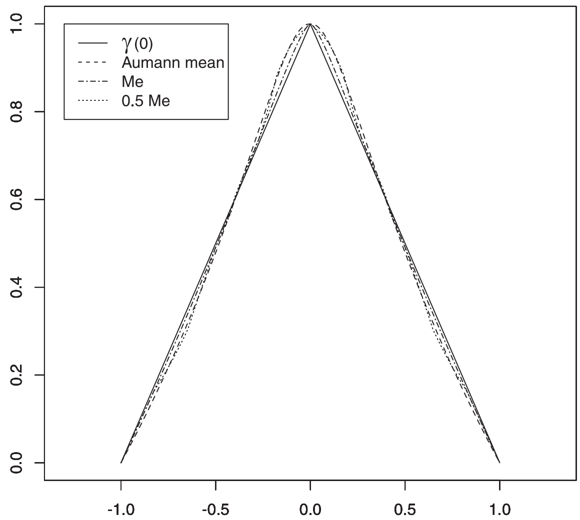


Fig. 13. Aumann-type mean (- - - -), 0.5-median $(\cdots\cdots)$ and 1-norm median$= \gamma_{(0)}(0)$ (—–) of the random fuzzy number $\gamma_{(0)} \circ [\mathrm{Bin}(5, 0.5) - 2.5]$

In this way, Figure 13 shows the scenario when the considered random fuzzy number is $\gamma_{(0)} \circ [\mathrm{Bin}(5, 0.5) - 2.5]$, a random fuzzy number symmetric about 0 and taking on 6 different values. In this case, the 1-norm median is very close to $\gamma_{(0)}(0)$ (which does not exactly correspond to a central position) but the 0.5-median is not very close to it, and the Aumann-type mean is not close either.

This is better confirmed by computing the $D_\theta$-, $\mathfrak{D}_\theta^{0.5}$- and $\rho_1$-distances between each summary measure and $\gamma_{(0)}(0)$, the two first distances as functions of the weighting parameter $\theta$. Thus,

$$\rho_1\left(\widetilde{\mathrm{Me}}(\gamma_{(0)} \circ [\mathrm{Bin}(5, 0.5) - 2.5]), \gamma_{(0)}(0)\right) = 0.0108,$$
$$\rho_1\left(\widetilde{\mathrm{Me}}^{0.5}(\gamma_{(0)}[\mathrm{Bin}(5, 0.5) - 2.5]), \gamma_{(0)}(0)\right) = 0.0234,$$
$$\rho_1\left(\widetilde{E}(\gamma_{(0)} \circ [\mathrm{Bin}(5, 0.5) - 2.5]), \gamma_{(0)}(0)\right) = 0.0274,$$

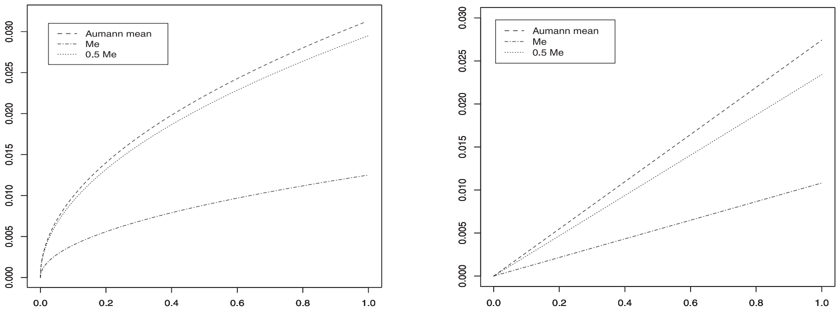


Fig. 14. $D_\theta$-distance (on the left) and the $\mathfrak{D}_\theta^{0.5}$-distance (on the right) between $\gamma_{(0)}(0)$ and the Aumann-type mean (- - - -), 0.5-median $(\cdots\cdots)$ and 1-norm median $(-\cdot-\cdot)$ of the random fuzzy number $\gamma_{(0)} \circ [\text{Bin}(5, 0.5) - 2.5]$ as functions of $\theta$

and the $D_\theta$- and the $\mathfrak{D}_\theta^{0.5}$-distances have been displayed in Figure 14.

## 6 Concluding remarks

In this paper the adequacy of central tendency measures for random fuzzy numbers has been discussed by considering their behaviour in case of symmetric distributions. Random fuzzy numbers are becoming an appealing tool to statistically analyze imprecise data which can be suitably formalized by means of fuzzy numbers. Several problems and techniques are being studied and developed along this century. For instance, testing about means (see Colubi *et al.* [4], González-Rodríguez *et al.* [14] or the recent review by Blanco-Fernández *et al.* [1]), regression analysis (see, for instance, Ferraro *et al.* [9], Ferraro and Giordani [10]), clustering (see, for instance, González-Rodríguez *et al.* [12]), Bayesian analysis (see Stein *et al.* [26]), actuarial developments, portfolio selection and mathematical programming (see, for instance, Shapiro [22], Li and Xu [16], Sakawa and Matsui [20]), and so on.

An open direction from the study in this paper would be extending the notion of symmetry about a real number to that about a fuzzy number. Special care should be put for this extension with the nonlinearity of the space of fuzzy numbers when it is endowed with the usual fuzzy arithmetic, so the extension is not a trivial task.

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## References


[1] Blanco-Fernández, A., Casals, M.R., Colubi, A., Corral, N., García-Bárzana, M., Gil, M.A., González-Rodríguez, G., López, M.T., Lubiano, M.A., Montenegro, M., Ramos-Guajardo, A.B., de la Rosa de Sáa, S., Sinova, B., 2014. A distance-based statistical analysis of fuzzy number-valued data. Int. J. Approx. Reas. (doi:10.1016/j.ijar.2013.09.020).

[2] Bertoluzza, C., Corral, N., Salas, A., 1995. On a new class of distances between fuzzy numbers. Mathw. & Soft Comp. 2, 71–84.

[3] Casals, M.R., Corral, N., Gil, M.A., López, M.T., Lubiano, M,A,, Montenegro, M., Naval, G., Salas, A., 2013. Bertoluzza *et al.*'s metric as a basis for analyzing fuzzy data. Metron 71 (3), 307–322.

[4] Colubi, A., Coppi, R., D'Urso, P., Gil, M.A., 2007. Statistics with fuzzy random variables. Metron. LXV (3), 277–303.

[5] Colubi, A., Domínguez-Menchero, J.S., López-Díaz, M., Ralescu, D.A., 2001. On the formalization of fuzzy random variables. Inform. Sci. 133, 3–6.

[6] Chaherian, G.D., 1967. Inequalities for the difference body of a convex body. Proc. Am. Math. Soc. 18 (5), 879–884.

[7] Chou, J.-R., 2012. A linguistic evaluation approach for universal design. Inform. Sci. 190, 76–94.

[8] Diamond, P., Kloeden, P., 1999. Metric spaces of fuzzy sets. Fuzzy Set Syst. 100, 63–71.

[9] Ferraro, M.B., Coppi, R., González Rodríguez, G., Colubi, A., 2010. A linear regression model for imprecise response. Int. J. Approx. Reas. 51(7), 759–770.

[10] Ferraro, M.B., Giordani, P., 2012. A multiple linear regression model for imprecise information. Metrika 75(8), 1049–1068.

[11] Gil, M.A., Colubi, A., Terán, P., 2014. Random fuzzy sets: why, when, how. BEIO 30 (1), 5–29.

[12] González-Rodríguez, G., Colubi, A., D'Urso, P., Montenegro, M., 2009. Multi-sample test-based clustering for fuzzy random variables. Int. J. Approx. Reas. 50(5), 721–731.

[13] González-Rodríguez, G., Colubi, A., Gil, M.A., 2006. A fuzzy representation of random variables: an operational tool in exploratory analysis and hypothesis testing. Comput. Stat. Data Anal. 51(1), 163–176.

[14] González-Rodríguez, G., Colubi, A., Gil, M.A., 2012. Fuzzy data treated as functional data. A one-way ANOVA test approach. Comp. Stat. Data Anal. 56(4), 943-955.

[15] Goetschel, R. Jr, Voxman, W., 1986. Elementary fuzzy calculus. Fuzzy Sets Syst. 18, 31–43.

[16] Li, J., Xu, J., 2013. Multi-objective portfolio selection model with fuzzy random returns and a compromise approach-based genetic algorithm. Inform. Sci. 220, 507–521.

[17] Näther, W., 2000. On random fuzzy variables of second order and their application to linear statistical inference with fuzzy data. Metrika 51, 201–221.

[18] Porcel, C., Tejeda-Lorente, A., Martínez, M.A., Herrera-Viedma, E. 2012. A hybrid recommender system for the selective dissemination of research resources in a Technology Transfer Office. Inform. Sci. 184, 1–19.

[19] Puri, M.L., Ralescu, D.A., 1986. Fuzzy random variables. J. Math. Anal. Appl. 114, 409–422.

[20] Sakawa, M., Matsui, T., 2013. Interactive fuzzy random two-level linear programming based on level sets and fractile criterion optimization. Inform. Sci. 238, 163–175.

[21] Serrano-Guerrero, J., Herrera-Viedma, E., Olivas, J.A., Cerezo, A., Romero, F.P., 2011. A google wave-based fuzzy recommender system to disseminate information in University Digital Libraries 2.0. Inform. Sci. 181, 1503–1516.

[22] Shapiro, A.F., 2009. Fuzzy random variables. Insur.: Math. Econ. 44(2), 307–314.

[23] Sinova, B., Gil, M.A., Colubi, A., Van Aelst, S., 2012. The median of a random fuzzy number. The 1-norm distance approach. Fuzzy Sets and Systems 200, 99-115.

[24] Sinova, B., de la Rosa de Sáa, S., Gil, M.A., 2013. A generalized $L^1$-type metric between fuzzy numbers for an approach to central tendency of fuzzy data. Inform. Sci. 242, 22–34.

[25] Sinova, B., Lubiano, M.A., Trutschnig, W., 2011. A function to calculate the median of a sample of fuzzy numbers in R. In: Abst. 58th Session Int. Statist. Inst. (ISI 2011, Dublin, Ireland), CPS065–06.

[26] Stein, M., Beer, M., Kreinovich, V., 2013. Bayesian approach for inconsistent information. Inform. Sci. 245, 96–111.

[27] Trutschnig, W., González-Rodríguez, G., Colubi, A., Gil, M.A., 2009. A new family of metrics for compact, convex (fuzzy) sets based on a generalized concept of mid and spread. Inform. Sci. 179, 3964–3972.

[28] Trutschnig, W., Lubiano, M.A., 2014 (last update). SAFD: Statistical Analysis of Fuzzy Data (http://cran.r-project.org/web/packages/SAFD/index.html)

[29] Zadeh, L.A., 1975. The concept of a linguistic variable and its application to approximate reasoning, Part 1. Inform. Sci. 8, 199–249; Part 2. Inform. Sci. 8, 301–353; Part 3. Inform. Sci. 9, 43–80.

[30] Zadeh, L.A., 1996. Fuzzy logic = computing with words. IEEE Trans. Fuzzy Syst. 4(2), 103-111.