S E
I O

Estadística

Random fuzzy sets: why, when, how [1]

María Ángeles Gil, Ana Colubi and Pedro Terán

Departamento de Estadística e I.O. y D.M.

Universidad de Oviedo

✉ magil@uniovi.es, colubi@uniovi.es,   teranpedro@uniovi.es

Abstract

Random elements of non-Euclidean spaces have reached the forefront of statistical research with the extension of continuous process monitoring, leading to a lively interest in functional data. A fuzzy set is a generalized set for which membership degrees are identified by a $[0, 1]$-valued function. The aim of this review is to present random fuzzy sets (also called fuzzy random variables) as a mathematical formalization of data-generating processes yielding fuzzy data. They will be contextualized as Borel measurable random elements of metric spaces endowed with a special convex cone structure. That allows one to construct notions of distribution, independence, expectation, variance, and so on, which mirror and generalize the literature of random variables and random vectors. The connections and differences between random fuzzy sets and random elements of classical function spaces (functional data) will be underlined. The paper also includes some bibliometric remarks, comments on the statistical analysis of fuzzy data, and pointers to the literature for the interested reader.

Keywords: Functional data, Fuzzy statistics, Metric space, Random elements

AMS Subject classifications: 60B05, 60B99, 60F17, 62A86, 52A22

1. Why random fuzzy sets?

Random elements taking on values in metric spaces were introduced by Fréchet (1948, 1950). He pointed out the valuable implications derived from the introduction of a distance between elements in the considered space. In accordance with their current usage, a random element is defined to be a measurable function between a sample space and a metric space equipped with its Borel σ-algebra.

---

[1] This paper is dedicated in his 65th birthday to our former Head of the Department and scientific father/grandfather/great-grandfather, Professor Pedro Gil, who introduced us in the knowledge of Fuzzy Sets.

Fréchet anticipated that future mathematics would have to incorporate new and unexpected sorts of objects quite beyond numbers, vectors, curves and functions. Therefore it was worth developing a theory of random variables in a space of elements 'of an arbitrary nature'. Maybe half-jokingly, he envisaged an eventual mathematical formalization of such magnitudes as those related to moral opinions, political spirit and aesthetic judgements. The only requirement is the definition of a distance between the elements of such spaces.

Despite Fréchet's unifying drive, it is clear that a theory of random elements of a given kind must also include notions and methods particular to the specific context being considered. Zadeh (1965) introduced *fuzzy sets* as a way to model vague or poorly defined properties for situations in which it is not possible to fully discriminate between having and not having said properties (incidentally, one can find in Kosko (1999), Chapter 3, a fuzzy formalization of Fréchet's 'political spirit').

A fuzzy set, a set with unclear boundaries, is formalized as a $[0,1]$-valued mapping on the reference set or universe $\mathbb{X}$. More specifically, a fuzzy subset $\widetilde{U} \subset \mathbb{X}$ is a mapping

$$\widetilde{U} : \mathbb{X} \to [0,1]$$

so that for each $x \in \mathbb{X}$ the value $\widetilde{U}(x)$ means the degree of membership of x to $\widetilde{U}$ (or, more intuitively, the degree of compatibility of x with the property $\widetilde{U}$ stands for, or degree of truth of the assertion "x has the defining property of $\widetilde{U}$"). This definition corresponds to what is called the 'vertical view' of fuzzy sets.

Alternatively, one can define fuzzy sets by means of the 'horizontal view' which is determined by their level sets. The α-level set of $\widetilde{U}$ is given by the set

$$\widetilde{U}_\alpha = \{x \in \mathbb{X} : \widetilde{U}(x) \geq \alpha\}$$

for any $\alpha \in [0,1]$, and the nondecreasing set-valued mapping $L_{\widetilde{U}} : [0,1] \to \mathcal{P}(\mathbb{X})$ such that $L_{\widetilde{U}}(\alpha) = \widetilde{U}_\alpha$ characterizes the fuzzy set $\widetilde{U}$. Often, $\widetilde{U}_0$ denotes, instead of $\mathbb{X}$, the closure of the support set $\mathrm{cl}(\mathrm{supp}\,\widetilde{U}) = \mathrm{cl}\{x \in \mathbb{R}^p : \widetilde{U}(x) > 0\}$.

Fuzzy sets have been used in many different fields (Social and Health Sciences, Engineering, etc.). Many real-life situations and problems involve ratings, judgements, perceptions, etc. (mostly related to human valuations) which are hard to quantify in terms of precise numbers or vectors but can be suitably described as fuzzy subsets of $\mathbb{R}$ or $\mathbb{R}^p$ ($p \in \mathbb{N}$). The mathematical advantage of using such a description is mainly due to the fact that the mathematical 'language' of fuzzy sets is more expressive, flexible and richer than natural languages. Furthermore, statistical techniques can be generally better and more widely adapted to deal with real-valued functions than to deal with categories or linguistic labels.

When random experiments involve elements which can be properly described by means of fuzzy values, the mechanisms generating such elements can be treated as fuzzy-valued random elements in accordance with Fréchet's approach.

With the purpose of formalizing random elements taking on fuzzy values and following some former ideas by Féron (1976a), Puri and Ralescu (1986) introduced the concept of *random fuzzy set* that they coined as fuzzy random variables.

2. When random fuzzy sets?

In the literature one can distinguish two main approaches to establish fuzzy set-valued random attributes. The two approaches were rather contemporary and they are rigourously stated within the probabilistic setting. We now briefly explain the essential differences and analogies between both approaches.

On one hand, Kwakernaak (1978, 1979) introduced the so-called *fuzzy random variable* (for short FRV) to model the fuzzy perception or description of a real-valued random variable. In this respect, for any FRV in Kwakernaak's sense there exists an underlying real-valued random variable (that is referred to as the original). Since the actual values of the variable are not observed, this has a connection to coarse data (made explicit e.g. by Nguyen and Wu, 2006) and censored data. Kwakernaak's ideas were formalized in a clearer mathematical way by Kruse and Meyer (1987).

This approach is used when precise values exist but the corresponding observed/reported information (often in natural language) is written in the form of fuzzy intervals. In other words, FRV is an appropriate model for fuzzy-valued random elements when the 'epistemic' perspective is considered.

In Kwakernaak's approach, the distribution of the FRVs and the related summary measures are based on Zadeh's extension principle, and the statistical analysis is generally focused on drawing conclusions about the original on the basis of the available fuzzy information (Kruse and Meyer, 1987). Note that this notion of distribution conveys the fuzzy information about the original distribution and is not a probability distribution in the usual mathematical sense.

On the other hand, Féron (1976ab, 1979) introduced the notion of a *random fuzzy set* (RFS) to model a random mechanism generating fuzzy set values (a rather general type of fuzzy sets of $\mathbb{R}^p$, $p \in \mathbb{N}$, or even of a more general metric space). Two different definitions were proposed, namely,

- the definition formalizing RFSs as random elements taking on values on spaces of fuzzy sets endowed with certain Borel σ-fields (i.e., by following Fréchet's theory),
- the definition formalizing RFSs as extending levelwise the notion of random sets.

Féron's ideas were reprised by Puri and Ralescu (1985, 1986), who considered the specific metrics missing in Féron's papers and introduced key notions like expectation, conditional expectation, and so on.

Although concurrent with Féron's notion of a random fuzzy set, Puri and Ralescu kept using the terminology 'fuzzy random variables' from Kwakernaak.

This may have introduced some confusion between both approaches at the conceptual level. Formally, when both definitions apply they are equivalent, as we will show.

RFSs are used when the imprecise data are assumed to have been generated without regard to an underlying precise random variable or vector. This is usually referred to as the 'ontic' view. Fuzzy sets are then used as a means to model real entities. For a recent and clarifying discussion on the epistemic and ontic fuzzy sets, readers are referred to Dubois and Prade (2012). Note the shift in the intent of the analysis, which is no longer to identify features of an underlying classical distribution (e.g. parameter estimation) but to identify features of a probability distribution in a space of fuzzy sets. This distribution arises naturally from the formalization of random fuzzy sets as Borel measurable mappings.

The theory of random fuzzy sets extends the existing theories of random variables/vectors and random sets and is connected with that of random functions, as will be shown in the next section. RFSs allow us to develop statistical methods for fuzzy data within an appropriate probabilistic setting and to preserve the key ideas and notions from the real/vectorial-valued case.

To illustrate the concept of RFS, which will be formally presented later, most of the real-life examples one can think about involve human valuations which frequently arise in Social Sciences, Medicine, Decision Making, Control Engineering, and so on.

The following situation exemplifies an application of RFSs. It corresponds to a psychometrical study, and although RFSs were not explicitly mentioned in it, they can be definitely applied for further statistical analysis. Most of the content have been directly extracted from the referenced study.

Example 2.1. A *fuzzy rating scale*, as referred to by Hesketh *et al.* (1988, 1992, 1994, 1995, 2011), is a method of eliciting a 'preference' that allows for a degree of vagueness or uncertainty, but with the possibilities that the uncertainty may be gradual and symmetric/asymmetric around an assumed 'preferred' point or interval (actually, the 1-level, i.e., the singleton or interval of real values which are considered to be fully compatible with the preference or answer).

The fuzzy rating may be used as a way of capturing flexibility of requirements/perceptions/responses/.... The concept of fuzzy rating allows for statements such as 'someone who is high on X, and moderate on Y, while not being low on Z is likely to have a moderately high level of coping performance'. The fuzzy sets modelling the words HIGH, MODERATE, and LOW can be somehow elicited graphically, providing a measurement approach which is more manageable from a mathematical perspective (and, ultimately, for statistical purposes).

Verbal anchors used on the five prestige and three sex-type scales

1.
Not very well paid |.................|.................| Very well paid

2.
This occupation   This occupation
   requires no education   requires a high level
   for entry |.................|.................| of education

3.
Most people think this   Most people think this
   occupation has low   occupation has high
   status |.................|.................| status

4.
Most people think this   Most people think this
   occupation does not   occupation has much
   have much power |.................|.................| power

5.
Most people do not   Most people think
   think highly of people   highly of people in
   in this occupation |.................|.................| this occupation

6.
Most people think this   Most people think this
   occupation suits   occupation suits
   men |.................|.................| women

7.
Men usually choose this   Women usually choose
   occupation |.................|.................| this occupation

8.
Generally considered   Generally considered
   men's work |.................|.................| women's work
                         0...................................100
                         Left   Right

Figure 1: Example of a fuzzy rating scale-based psychometric study by Hesketh *et al.* (1988)

In psychometric studies the semantic differential (Osgood *et al.*, 1975) has been widely used to rate a variety of stimuli and attitudes. That can be adapted to provide a fuzzy rating scale which may be graphically represented on Cartesian axes. For instance, a study carried out by Hesketh and collaborators (Hesketh *et al.*, 1988) was as follows.

Several occupations were chosen to represent levels of prestige and levels of sex-type. For each of them, five anchors were developed to measure prestige and three to measure sex-type (Figure 1 reproduces these anchors).

The combination of the eight scales and the occupations was randomized, so that respondents were asked how they thought people generally viewed each occupation in relation to the scales. An interactive computerized fuzzy graphic rating scale was designed to get the responses.

In this particular study respondents were asked to consider triangular fuzzy numbered responses by indicating where the '∨' pointer (i.e., the upper vertex) might fall between the two anchors, and then which the spread to the left and the right were by using left and right arrow keys. The responses were referred to the interval $[0, 100]$ (see Figure 2).

The computerized graphic rating procedure was explained to respondents using a trial occupation not included in the stimuli. The opportunity was used to ensure that respondents understood that the fuzzy rating represented their estimate of how people generally view occupations.

In summary, the fuzzy rating scale provides a mathematical language which, in a rather friendly way, allows capturing and managing the imprecision associated with many experimental data (say, those related to perceptions, opinion, ratings, etc.). The quantitative, though not numerical, management using the fuzzy rating scale is richer and more expressive than natural language, since it allows for a continuum of modifiers and nuances. Further, the fuzzy rating lends itself better than the linguistic scale to statistical handling.
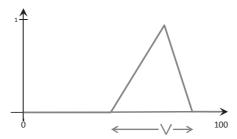

Figure 2: Representation of a triangular response by Hesketh *et al.* (1995)

3. How random fuzzy sets?

As we have indicated, random fuzzy sets were introduced by Féron (1976ab, 1979) in a double way: as a Borel-measurable function (i.e., following Fréchet's approach), and as a levelwise extension of random sets. Féron sketched the guiding idea in the notion, but without specifying some key terms like the involved metrics. This specification was made by Puri and Ralescu (1986).

Let $\mathcal{F}(\mathbb{R}^p)$ be the class of fuzzy subsets $\widetilde{U} : \mathbb{R}^p \to [0, 1]$ such that $\widetilde{U}_\alpha$ is compact for each $\alpha \in [0, 1]$ and $\widetilde{U}_1 \neq \emptyset$. In other words, $\mathcal{F}(\mathbb{R}^p)$ is the class of normalized upper semicontinuous elements of $[0, 1]^{\mathbb{R}^p}$ with bounded 0-level (i.e., with bounded 'support set').

Definition 3.1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. A random fuzzy set (or fuzzy random variable in Puri and Ralescu's sense) associated with $(\Omega, \mathcal{A}, P)$ is a mapping $\mathcal{X} : \Omega \to \mathcal{F}(\mathbb{R}^p)$ such that for each $\alpha \in [0, 1]$ the set-valued mapping $\mathcal{X}_\alpha : \Omega \to \mathcal{P}(\mathbb{R}^p)$ (with $\mathcal{X}_\alpha(\omega) = \big(\mathcal{X}(\omega)\big)_\alpha$) is a random compact set.*

Random sets were rigorously formalized as random elements of a space of sets by Matheron (1975), although informal instances of the notion go back several decades to Kolmogorov and Robbins. In this way, Matheron stated the fundamentals of the theory of random closed sets, as well as the appropriate model and basic tools within the probabilistic setting. In Molchanov (2005) one can find a wide and quite updated monograph on random sets.

If $\mathcal{K}(\mathbb{R}^p)$ is the class of the nonempty compact subsets of $\mathbb{R}^p$ and $(\Omega, \mathcal{A}, P)$ is a probability space, then a mapping $X : \Omega \to \mathcal{K}(\mathbb{R}^p)$ is said to be a *random compact set* associated with the probability space if the graph $G(X) = \{(\omega, x) : x \in X(\omega)\}$ is in the product σ-algebra $\mathcal{A} \otimes \mathcal{B}_{\mathbb{R}^p}$. Equivalently (see, for instance, Hiai and Umegaki (1977)), X is said to be a random compact set if X is measurable with respect to the Borel σ-algebra of the Hausdorff metric on $\mathcal{K}(\mathbb{R}^p)$ given by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\},$$

where $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^p$.

Remark 3.1. Particular cases of random sets/fuzzy sets frequently considered in practice are those taking on convex values. More concretely, when $\mathcal{K}(\mathbb{R}^p)$ is replaced by $\mathcal{K}_c(\mathbb{R}^p) = \{A \in \mathcal{K}(\mathbb{R}^p) : A \text{ convex set}\}$ we refer to random convex compact sets. The convexity of fuzzy sets is defined in terms of the convexity of their α-levels (and thus coincides with quasiconcavity, not with the usual convexity of functions), so that the class of the convex compact normal fuzzy values is

$$\mathcal{F}_c(\mathbb{R}^p) = \{\widetilde{U} \in \mathcal{F}(\mathbb{R}^p) : \widetilde{U}_\alpha \in \mathcal{K}_c(\mathbb{R}^p) \text{ for all } \alpha \in [0, 1]\}.$$

Many of the developments on random fuzzy sets, especially those related to Statistics, assume their convexity.

Remark 3.2. Although Euclidean spaces are most commonly used, the concept of a random fuzzy set can be established in more general spaces, like separable Banach spaces (see, for instance, Puri and Ralescu (1991), Colubi *et al.* (2001), Li *et al.* (2002)) or metric spaces (Terán, 2013). The compactness assumption has been removed in some studies too (e.g. Li and Ogura (1999), Ogura and Li (2001)).

In addition to the levelwise measurability in Definition 3.1, Puri and Ralescu considered also to define random fuzzy sets as Borel measurable functions, as suggested also by Féron and in agreement with Fréchet's approach. Thereby, notions like the induced distribution, independence and others are inherited from those in the sample space. In this respect, Puri and Ralescu (1985) define a random fuzzy set associated with the probability space $(\Omega, \mathcal{A}, P)$ as a mapping $\mathcal{X} : \Omega \to \mathcal{F}(\mathbb{R}^p)$ which is measurable with respect to the Borel σ-algebra of the metric

$$d_\infty(\widetilde{U}, \widetilde{V}) = \sup_{\alpha \in [0,1]} d_H(\widetilde{U}_\alpha, \widetilde{V}_\alpha)$$

in $\mathcal{F}(\mathbb{R}^p)$ (see, also, Klement *et al.* (1986)).

$(\mathcal{F}(\mathbb{R}^p), d_\infty)$ is a complete non-separable metric space (Klement *et al.*, 1986). Colubi *et al.* (2001, 2002), Kim (2002) and Terán (2006) established that d_∞ measurability implies levelwise measurability, with equivalence if and only if the range of the RFS is essentially d_∞-separable. Based upon results by Colubi *et al.* (2001, 2002) and Kim (2002), one can characterize RFSs using a complete separable metric space as those measurable with respect to the Borel σ-field of the Skorohod metric

$$d_S(\widetilde{U}, \widetilde{V}) = \inf_{\lambda \in \Lambda} \max \left\{ \sup_{\alpha \in [0,1]} |\lambda(\alpha) - \alpha|, \ \sup_{\alpha \in [0,1]} d_H(\widetilde{U}_\alpha, \widetilde{V}_{\lambda(\alpha)}) \right\},$$

where Λ is the class of increasing bijections from $[0,1]$ to $[0,1]$.

When we deal with convex fuzzy values (i.e. we work on $\mathcal{F}_c(\mathbb{R}^p)$), other equivalences to the levelwise measurability can be stated in terms of metrics like e.g. Krätschmer (2001) and Trutschnig *et al.* (2009) (see González-Rodríguez *et al.* (2012) for a detailed study on this equivalence and some interesting implications).

Let $\mathbb{S}^{p-1}$ denote the unit sphere of $\mathbb{R}^p$. The *support function* of $\widetilde{U} \in \mathcal{F}_c(\mathbb{R}^p)$ (see Puri and Ralescu, 1985) extends levelwise the notion of the support function of a set (see, for instance, Castaing and Valadier (1977)) and is given by the mapping $s_{\widetilde{U}} : \mathbb{S}^{p-1} \times (0,1] \to \mathbb{R}$ defined by

$$s_{\widetilde{U}}(u, \alpha) = \sup_{v \in \widetilde{U}_\alpha} \langle u, v \rangle$$

for all $u \in \mathbb{S}^{p-1}, \alpha \in (0,1]$, $\langle \cdot, \cdot \rangle$ denoting the inner product on $\mathbb{R}^p$. In general, $s_{\widetilde{U}}(u, \alpha)$ represents the signed (i.e., oriented) distance from $\mathbf{0} \in \mathbb{R}^p$ to the supporting hyperplane of $\widetilde{U}_\alpha$ which is orthogonal to u. Figures 3 and 4 show the graphical interpretation of the support function of a level set in dimension $p = 1$ and $p = 2$, respectively.
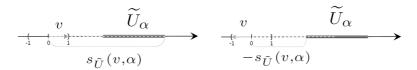
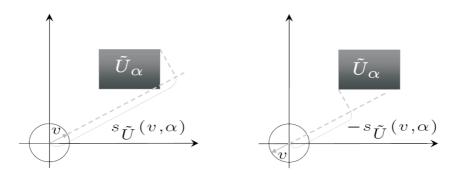Figure 3: Support function of a level set. Case $p = 1$, $\mathbb{S}^0 = \{-1, 1\}$

Figure 4: Support function of a level set. Case $p = 2$, $\mathbb{S}^1 =$ circumference (center $= (0, 0)$, radius $= 1$)

The support function of any $\widetilde{U} \in \mathcal{F}_c(\mathbb{R}^p)$ can be expressed in accordance with its mid/spr decomposition, given by

$$s_{\widetilde{U}} = \text{mid } s_{\widetilde{U}} + \text{spr } s_{\widetilde{U}}$$

where, for all $u \in \mathbb{S}^{p-1}$ and $\alpha \in (0, 1]$, $\Pi_u \widetilde{U}_\alpha$ is the projection of $\widetilde{U}_\alpha$ over the direction $u \in \mathbb{S}^{p-1}$ and

$$\text{mid } s_{\widetilde{U}}(u, \alpha) = \frac{s_{\widetilde{U}}(u, \alpha) - s_{\widetilde{U}}(-u, \alpha)}{2} = \text{ mid-point/center of } \Pi_u \widetilde{U}_\alpha,$$

$$\text{spr } s_{\widetilde{U}}(u, \alpha) = \frac{s_{\widetilde{U}}(u, \alpha) + s_{\widetilde{U}}(-u, \alpha)}{2} = \text{ spread/radius of } \Pi_u \widetilde{U}_\alpha.$$

On the basis of this representation, Trutschnig *et al.* (2009) introduced in a more general space (allowing unbounded 0-level for the fuzzy values) the following metric:

Definition 3.2. *Let $\theta \in (0, +\infty)$ and let φ be an absolutely continuous probability measure on $([0, 1], \mathcal{B}_{[0,1]})$ with the density function being positive in $(0, 1)$. Then, the (θ, φ)-metric on $\mathcal{F}_c(\mathbb{R}^p)$ is the mapping $D_\theta^\varphi : \mathcal{F}_c(\mathbb{R}^p) \times \mathcal{F}_c(\mathbb{R}^p) \to [0, +\infty)$ given by*

$$D_\theta^\varphi(\widetilde{U}, \widetilde{V}) = \sqrt{\left(\|\text{mid } s_{\widetilde{U}} - \text{mid } s_{\widetilde{V}}\|^\varphi \right)^2 + \theta \left(\|\text{spr } s_{\widetilde{U}} - \text{spr } s_{\widetilde{V}}\|^\varphi \right)^2},$$

where

$$\|f - g\|^{\varphi} = \sqrt{\int_{(0,1]} \int_{\mathbb{S}^{p-1}} [f(u,\alpha) - g(u,\alpha)]^2 \, d\lambda_p(u) \, d\varphi(\alpha)},$$

where λ_p denotes the uniform distribution on $\mathbb{S}^{p-1}$.

Remark 3.3. The D_{θ}^{φ} metric on $\mathcal{F}_c(\mathbb{R}^p)$ is defined so that, for each level, the choice of θ allows us to weight the effect of the deviation between spreads (which could be intuitively translated into the difference in 'shape' or 'imprecision') in contrast to the effect of the deviation between mid's (which can be intuitively translated into the difference in 'location').

Trutschnig *et al.* (2009) and González-Rodríguez *et al.* (2012) proved

Theorem 3.1. *The metric D_{θ}^{φ} satisfies*

i) *$(\mathcal{F}_c(\mathbb{R}^p), D_{\theta}^{\varphi})$ is a separable metric space.*

ii) *An $\mathcal{F}(\mathbb{R}^p)$-valued mapping is an RFS if, and only if, it is measurable with respect to the Borel σ-field of the metric D_{θ}^{φ} on $\mathcal{F}_c(\mathbb{R}^p)$.*

Observe that D_{θ}^{φ} identifies each element of $\mathcal{F}_c(\mathbb{R}^p)$ with an element of the Hilbert space

$$\mathcal{H} = \mathcal{L}^2(\mathbb{S}^{p-1} \times (0,1], \lambda_p \otimes \nu) \oplus_2 \mathcal{L}^2(\mathbb{S}^{p-1} \times (0,1], \theta \cdot (\lambda_p \otimes \nu)),$$

where ν denotes the uniform distribution on $(0,1]$. Krätschmer (2001) also presented a similar approach, although not through the mid/spread decomposition.

Remark 3.4. In case the 0-level of the fuzzy values is allowed to be unbounded, as in Trutschnig *et al.* (2009) or González-Rodríguez *et al.* (2012), the metric space in Theorem 3.1 *i)* becomes complete.

On the other hand, one can consider on $\mathcal{F}(\mathbb{R}^p)$ the usual fuzzy arithmetic based on Zadeh's extension principle (Zadeh, 1975).

Definition 3.3. *Given $\widetilde{U}, \widetilde{V} \in \mathcal{F}(\mathbb{R}^p)$ and $\gamma \in \mathbb{R}$, the sum of $\widetilde{U}$ and $\widetilde{V}$ is defined as the fuzzy set $\widetilde{U} + \widetilde{V} \in \mathcal{F}(\mathbb{R}^p)$ such that*

$$(\widetilde{U} + \widetilde{V})(t) = \sup_{y+z=t} \min \left\{ \widetilde{U}(y), \widetilde{V}(z) \right\}$$

or, equivalently and based on Nguyen (1978), for each $\alpha \in [0,1]$

$$(\widetilde{U} + \widetilde{V})_{\alpha} = \text{Minkowski sum of } \widetilde{U}_{\alpha} \text{ and } \widetilde{V}_{\alpha} = \left\{ y + z \, : \, y \in \widetilde{U}_{\alpha}, z \in \widetilde{V}_{\alpha} \right\}.$$

The product of $\widetilde{U}$ by the scalar γ is defined as the fuzzy value $\gamma \cdot \widetilde{U} \in \mathcal{F}(\mathbb{R}^p)$ such that

$$(\gamma \cdot \widetilde{U})(t) = \sup_{y \in \mathbb{R}^p \, : \, y = \gamma t} \widetilde{U}(y) = \begin{cases} \widetilde{U}\left(\dfrac{t}{\gamma}\right) & \text{if } \gamma \neq 0 \\[2ex] \mathbf{1}_{\{0\}}(t) & \text{if } \gamma = 0 \end{cases}$$

or, equivalently and based on Nguyen (1978), for each $\alpha \in [0,1]$

$$(\gamma \cdot \widetilde{U})_\alpha = \gamma \cdot \widetilde{U}_\alpha = \{\gamma \cdot y \,:\, y \in \widetilde{U}_\alpha\}.$$

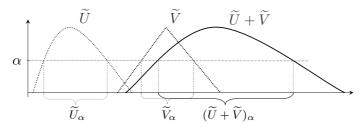Figures 5 and 6 display graphically the sum and the product by a scalar when $p = 1$.

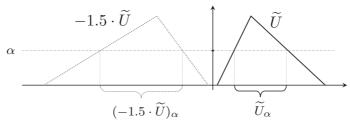Figure 5: Sum of two elements in $\mathcal{F}_c(\mathbb{R})$

Figure 6: Product by a real number of an element in $\mathcal{F}_c(\mathbb{R})$

Remark 3.5. As is clear from the figures, the arithmetic above differs from the usual arithmetic with functions. In general, the application of the function arithmetic in $\mathcal{F}_c(\mathbb{R}^p)$ would lead to elements out of this space and the fuzzy set semantics would be lost. This is why developments in Functional Data Analysis have not been applied directly to fuzzy data.

Remark 3.6. The space $\mathcal{F}(\mathbb{R}^p)$ of fuzzy values, endowed with the operations above, $(\mathcal{F}(\mathbb{R}^p), +, \cdot)$, has not a linear space (only a convex cone) structure, since in general fuzzy sets cannot be subtracted. This is due to the fact that the sum extends level-wise the Minkowski sum of sets which is not linear: $\{0,1\} + (-1) \cdot \{0,1\} \neq \{0\}$.

When we constrain again to the convex case, the following results can be obtained (González-Rodríguez *et al.*, 2012).

Theorem 3.2. *Let $\theta \in (0, +\infty)$ and let φ be an absolutely continuous probability measure on $([0,1], \mathcal{B}_{[0,1]})$ with the density function being positive on $(0,1)$.*

i) *The mapping $s : \widetilde{U} \in \mathcal{F}_c(\mathbb{R}^p) \mapsto (\mathrm{mid}, \mathrm{spread}) \in \mathcal{H}$ states an isometrical embedding of $\mathcal{F}_c(\mathbb{R}^p)$ (with the fuzzy arithmetic and D_θ^φ) onto a convex cone of the Hilbert space $\mathcal{H}$.*

ii) *$\mathcal{X}$ is an RFS if, and only if, $s_\mathcal{X} = s \circ \mathcal{X} : \Omega \to \mathcal{H}$ is a random element of $\mathcal{H}$.*

iii) *For each $\alpha \in (0,1]$ and $u \in \mathbb{S}^{p-1}$, the real functions $\mathrm{mid}\, s_\mathcal{X}(u, \alpha)$ and $\mathrm{spr}\, s_\mathcal{X}(u, \alpha)$ are real random variables.*

Remark 3.7. In case the 0-level of the fuzzy values is allowed to be unbounded, the closeness of the convex cone in Theorem 3.2 holds (see Trutschnig *et al.* (2009), and González-Rodríguez *et al.* (2012)).

Remark 3.8. An immediate and crucial implication is that data in the setting of fuzzy convex values with the fuzzy arithmetic and the metric D_θ^φ can be systematically translated into the setting of functional data with the usual arithmetic and a Hilbert norm (see also Krätschmer, 2004, for related work).

When analyzing fuzzy data, or the corresponding RFSs, the two most usual *summary measures* are the mean and variance. The notions have often been based on Fréchet's abstract approach. The mean value of an RFS can be presented in three equivalent ways, either (Puri and Ralescu, 1986) as a levelwise extension of the set-valued expectation of Aumann (1965), or (in the convex case) as the Fréchet mean (Körner, 1997) or as induced from the Bochner expectation in a Banach space via an appropriate embedding. Thus,

Definition 3.4. *Given a probability space $(\Omega, \mathcal{A}, P)$ and an associated RFS $\mathcal{X} : \Omega \to \mathcal{F}(\mathbb{R}^p)$, the (Aumann type) mean value or expected value of $\mathcal{X}$ is the fuzzy value $\widetilde{E}(\mathcal{X}) \in \mathcal{F}(\mathbb{R}^p)$, if it exists, such that for all $\alpha \in (0,1]$*

$$\left(\widetilde{E}(\mathcal{X}) \right)_\alpha = \left\{ E(X) \mid X : \Omega \to \mathbb{R}^p, X \in L_{\mathbb{R}^p}^1(\Omega, \mathcal{A}, P), X \in \mathcal{X}_\alpha \text{ a.s. } [P] \right\}.$$

If $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R}^p)$, then the mean value can be equivalently defined as the fuzzy value $\widetilde{E}(\mathcal{X}) \in \mathcal{F}_c(\mathbb{R}^p)$ such that

$$s_{\widetilde{E}(\mathcal{X})} = E(s_\mathcal{X}),$$

where the right-hand side is a Bochner integral. And, also, as the Fréchet mean

$$\widetilde{E}(\mathcal{X}) = \arg \min_{\widetilde{U} \in \mathcal{F}_c(\mathbb{R}^p)} E \left(\left[D_\theta^\varphi(\mathcal{X}, \widetilde{U}) \right]^2 \right)$$

(provided the expectation in the right-hand side is finite).

The mean value of an RFS is well-defined when the RFS is *integrably bounded* (see Puri and Ralescu, 1986), i.e. there exists $h \in L^1(\Omega, \mathcal{A}, P)$ such that

$\sup_{x \in \mathcal{X}_0} \|x\| \leq h$ almost surely. The mean value of an RFS satisfies several valuable properties similar to those in the classical case.

Proposition 3.1. $\widetilde{E}$ *is equivariant under affine transformations: if* $\gamma \in \mathbb{R}$, $\widetilde{U} \in \mathcal{F}(\mathbb{R}^p)$ *and* $\mathcal{X}$ *is an integrably bounded RFS, then*

$$\widetilde{E}(\gamma \cdot \mathcal{X} + \widetilde{U}) = \gamma \cdot \widetilde{E}(\mathcal{X}) + \widetilde{U}.$$

Proposition 3.2. $\widetilde{E}$ *is additive: for RFSs* $\mathcal{X}$ *and* $\mathcal{Y}$ *on the same probability space and such that* $\widetilde{E}(\mathcal{X})$ *and* $\widetilde{E}(\mathcal{X})$ *exist,*

$$\widetilde{E}(\mathcal{X} + \mathcal{Y}) = \widetilde{E}(\mathcal{X}) + \widetilde{E}(\mathcal{Y}).$$

Proposition 3.3. $\widetilde{E}$ *is coherent with the usual fuzzy arithmetic, so that if* $\mathcal{X}$ *is a convex-valued RFS and* $\mathcal{X}(\Omega) = \{\widetilde{x}_1, \dots, \widetilde{x}_m\} \subset \mathcal{F}_c(\mathbb{R}^p)$, *then if* $p_i = P(\{\omega \in \Omega : \mathcal{X}(\omega) = \widetilde{x}_i\})$ *we have*

$$\widetilde{E}(\mathcal{X}) = p_1 \cdot \widetilde{x}_1 + \dots + p_m \cdot \widetilde{x}_m.$$

The above mentioned definition for the mean value is supported by *Strong Laws of Large Numbers* for FRS's (cf., Colubi *et al.*, 1999, Molchanov, 1999, Proske and Puri, 2003, Li and Ogura, 2006, Terán, 2010, etc.). The mean value is the almost sure limit of the 'sample fuzzy mean'. Further limit theorems include the Central Limit Theorem (Li *et al.*, 2003, Terán, 2007), the Law of the Iterated Logarithm (Colubi, 2002) and the Large Deviation Principle (Terán, 2006, Ogura and Setokuchi, 2009).

Proposition 3.4. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space,* $\mathcal{X} : \Omega \to \mathcal{F}(\mathbb{R}^p)$ *an integrably bounded RFS and* $\{\mathcal{X}_n\}_n$ *a sequence of independent RFSs identically distributed as* $\mathcal{X}$. *If* $\overline{\mathcal{X}_n}$ *denotes the 'sample fuzzy mean'* $\overline{\mathcal{X}_n} = \frac{1}{n} \cdot (\mathcal{X}_1 + \dots + \mathcal{X}_n)$, *then*

$$\lim_{n \to \infty} d_\infty\left(\overline{\mathcal{X}_n}, \widetilde{E}(\mathcal{X})\right) = 0 \quad a.s.\,[P].$$

Conversely, if $\{\mathcal{X}_n\}_n$, *with* $\mathcal{X}_n : \Omega \to \mathcal{F}_c(\mathbb{R}^p)$, *is a sequence of pairwise independent RFSs which are identically distributed as an RFS* $\mathcal{X}$, *and there exists* $\widetilde{U} \in \mathcal{F}(\mathbb{R}^p)$ *so that* $\lim_{n \to \infty} d_\infty\left(\overline{\mathcal{X}_n}, \widetilde{U}\right) = 0$ *a.s.*\,$[P]$, *then,* $\mathcal{X}$ *is integrably bounded and* $\widetilde{U} = \widetilde{E}(\mathcal{X})$.

In formalizing the variance of an RFS in the convex case, Fréchet's approach was considered (see Körner (1997), Lubiano *et al.* (2000), Körner and Näther (2002), González-Rodríguez *et al.* (2012))). With this approach the

variance can be interpreted as a measure of the 'least squares error' in approximating/estimating the values of the RFS by a non-random fuzzy set. When considering the metric space $(\mathcal{F}_c(\mathbb{R}^p), D_\theta^\varphi)$ in the Fréchet approach, one defines

Definition 3.5. *Given a probability space $(\Omega, \mathcal{A}, P)$ and an associated integrably bounded convex-valued RFS $\mathcal{X}$, the (θ, φ)-Fréchet variance of $\mathcal{X}$ is the real number, if it exists, given by*

$$\sigma_\mathcal{X}^2 = E\left(\left[D_\theta^\varphi\left(\mathcal{X}, \widetilde{E}(\mathcal{X})\right)\right]^2\right)$$

or, equivalently,

$$\sigma_\mathcal{X}^2 = \mathrm{Var}(s_\mathcal{X}) = \mathrm{Var}(\mathrm{mid}\, s_\mathcal{X}) + \theta\, \mathrm{Var}(\mathrm{spr}\, s_\mathcal{X}).$$

As suggested by the last expression in terms of variances of real random variables, the (θ, φ)-Fréchet variance satisfies the usual properties. In this way,

Proposition 3.5. *$\sigma_\mathcal{X}^2 \geq 0$ with $\sigma_\mathcal{X}^2 = 0$ if, and only if, there exists $\widetilde{U} \in \mathcal{F}_c(\mathbb{R}^p)$ such that $\mathcal{X} = \widetilde{U}$ a.s. $[P]$.*

Proposition 3.6. *If $\gamma \in \mathbb{R}$, $\widetilde{U} \in \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{X}$ is an RFS associated with the probability space $(\Omega, \mathcal{A}, P)$ and such that $\sigma_\mathcal{X}^2$ exists, then*

$$\sigma_{\gamma \cdot \mathcal{X} + \widetilde{U}}^2 = \gamma^2 \cdot \sigma_\mathcal{X}^2.$$

Proposition 3.7. *For independent RFSs $\mathcal{X}$ and $\mathcal{Y}$ associated with the same probability space $(\Omega, \mathcal{A}, P)$ and such that $\sigma_\mathcal{X}^2$ and $\sigma_\mathcal{Y}^2$ exist, we have that*

$$\sigma_{\mathcal{X}+\mathcal{Y}}^2 = \sigma_\mathcal{X}^2 + \sigma_\mathcal{Y}^2.$$

To illustrate some of the key ideas in this section we will consider a real-life example.

Example 3.1. A wide study about the progress of a reforestation performed around two decades ago in Huerna Valley (between the provinces of Asturias and León, in the North of Spain) is being carried out by the Research Institute INDUROT (University of Oviedo). In this study, experts were interested in rating, among others, the quality of trees. The quality varies from tree to tree, rating can even vary from expert to expert, and quality assessments are naturally imprecise.

Traditionally, the way to proceed was to consider a Likert 1-5 (or 1-7) scale and the associated integer codings. Recently, environmental experts were informed of the possibility of rating using fuzzy numbers with (for instance) 0-level [0,100] (0 meaning the lowest quality, 100 the highest).

To facilitate the graphical representation, they were recommended to draw trapezoidal fuzzy numbers, by stating for each tree the 1-level (or closed interval of values which are viewed as being 'fully compatible' with their rating of the quality of the tree), the 0-level (or closed interval of values such that all those in the corresponding open interval are viewed as being 'compatible to some extent' with their rating of the quality of the tree), and finally the two closed intervals are linearly 'interpolated' to build a trapezoidal fuzzy set. Figure 7 displays a dataset of quality ratings of 10 birches (*Betula celtiberica*) from an expert.
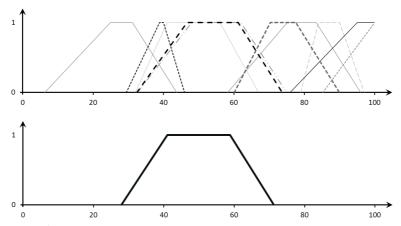

Figure 7: Above, graphical dataset rating the quality of 10 birches in a reforestation in Valle del Huerna (Asturias, Spain). Below, their sample mean.

If we assume Ω to be the sample of 10 birches, then the rating of quality of the trees can be formalized as an RFS taking on 10 different values with probabilities (actually, sample frequencies) equal to 1/10. The mean value of this RFS (actually, the sample mean quality rating) is graphically displayed on the bottom of Figure 7, and the Fréchet variance for φ the uniform distribution and $\theta = 1/3$ equals

$$\sigma_{\mathcal{X}}^2 = \frac{1}{10} \sum_{i=1}^{10} \left[D_{1/3}^{\nu}(\mathcal{X}(\omega_i), \overline{\mathcal{X}_{10}}) \right]^2 = 525.5576.$$

Notice that for $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$

$$D_{1/3}^{\nu}(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{(0,1]} \int_{[0,1]} \left[\widetilde{U}_{\alpha}^{[\tau]} - \widetilde{V}_{\alpha}^{[\tau]} \right]^2 d\nu(\tau) \, d\nu(\alpha)},$$

where $\widetilde{U}_\alpha^{[\tau]} = \tau \cdot \sup \widetilde{U}_\alpha + (1 - \tau) \cdot \inf \widetilde{U}_\alpha$.

Other summary measures/parameters associated with the distribution of an RFS or several RFSs have been introduced (see, among others, the covariance of two RFS defined by Körner and Näther (2002), González-Rodríguez *et al.* (2009), Blanco-Fernández *et al.* (2013, 2014), or the inequality of an RFS Gil *et al.* (1998), Alonso *et al.* (2001)).

4. Literature on random fuzzy sets

This section undertakes a brief bibliometric description of the literature on RFSs. To this purpose, a thorough literature search of papers published before 2013 (actually, before the 15th of December 2013) on the topic has been conducted by using Thomson–Reuters's Web of Science (WoS) and Elsevier's SCOPUS.

For the first source, the search is constrained to articles in JCR-SCI journals for which the topic appears as "random fuzzy set(s)" or "fuzzy random variable(s)" (minority variants like "fuzzy valued random variable(s)", "fuzzy set valued random variable(s)", "random upper semicontinuous function(s)", "fuzzy random set(s)", "fuzzy random element(s)" or "random fuzzy variable(s)" have been added to the search). For the second one, the search used the same inputs within the title, abstract or keywords but referred to different types of documents (including conference proceedings and others).

Regarding the global *chronological evolution* of the topic, by looking at Figures 8 and 9 one can conclude that the research on RFSs, although still dealing with a rather specialized subject, is becoming more and more active.
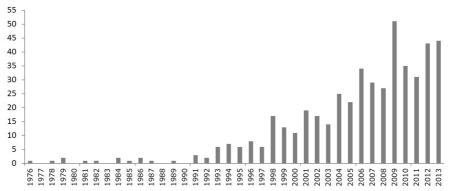

Figure 8: Chronological evolution of articles published in JCR-SCI journals on the topic of RFSs. Information Source: Web of Science

For a more detailed analysis on the scientific evolution, we have considered a rather personal nonfuzzy classification of the articles from the WoS search, the classes being the following: probabilistic aspects of RFSs (mostly referring to measurability, limit theorems, embedding results, etc.); statistics with RFSs (mostly referring to statistical inferences on the means, and regression analysis); applications of RFSs to other fields (like optimization, reliability, etc.); others (concerning related concepts, tools and developments but referring to non-probabilistic approaches). The evolution of the first three categories is shown in Figure 10.
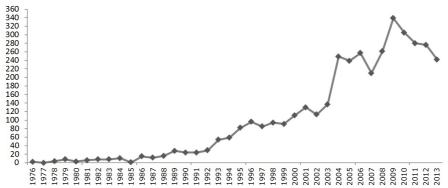

Figure 9: Chronological evolution of documents on the topic of RFSs. Information Source: SCOPUS

Probabilistic aspects (around 37% of the articles) have received an earlier attention starting by 1976 with Féron's contributions and continuing along the late seventies and eighties with Kwakernaak's (1978, 1979), Puri and Ralescu (1985, 1986), Kruse and Meyer (1987), etc. Statistics and Applications (around 25% and 38% of the articles, respectively) start receiving attention by mid nineties and are gradually rising since 2000. It should be clarified that, in accordance with our subjective classification, applications include a very wide range of topics, such as Mathematical Programming, Renewal Processes, Portfolio Selection and so on.
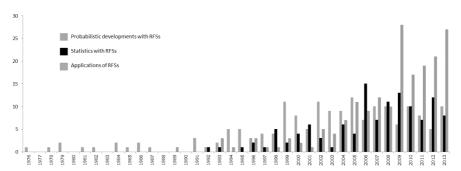
Figure 10: Separate chronological evolution of articles published in JCR-SCI journals on the main subjects related to RFSs. Information Source: Web of Science

In the WoS data, the top five *countries* by number of papers have been China ($\simeq 29.8\%$), Spain ($\simeq 14.8\%$), Japan ($\simeq 10.8\%$), USA ($\simeq 8.5\%$) and Germany ($\simeq 6.5\%$), a Spanish institution leading the list of *research organizations* on the topic.

Regarding *Web of Science categories*, Statistics & Probability is first (32.1%), Mathematics-Applied second (29.6%), the third and fourth positions correspond to different sections of Computer Science (Theory and Methods, and Artificial Intelligence with 21.7% and 20.8%, respectively) and the fifth position is associated with Operation Research & Management Science (11.9%).

Finally, concerning *source titles* the top five journals are Fuzzy Sets and Systems, Information Sciences, the European Journal of Operational Research, the International Journal of Approximate Reasoning, and IEEE Transactions on Instrumentation and Measurement, all of them on the top 25 % of their JCR-SCI categories.

5. Remarks on the statistical analysis of fuzzy data based on random fuzzy sets

As we have commented in Section 1, results, methods and conclusions related to random elements must often be established *ad hoc* depending on the type of values random elements take on. In this respect, in managing fuzzy data and random fuzzy sets several key distinctive features should be taken into account:

- because of the nonlinearity of the space $\mathcal{F}(\mathbb{R}^p)$, it is not possible to state an always well-defined 'difference' between fuzzy values preserving the main properties of the difference between real numbers (more concretely, with $\widetilde{V} + (\widetilde{U} - \widetilde{V}) = \widetilde{U}$);

- fuzzy values cannot be totally ordered in a way meaningful for all applications;

- there are not realistic general parametric models for RFSs (although some attempts have been made in this matter, like the normal definition stated by Puri and Ralescu, 1985, they become unrealistic or too restrictive in practice);

- the lack of Central Limit Theorems for RFSs which are directly applicable for inferential purposes (some CLTs for RFSs have been established, in which the normalized distance between the sample and the population means converges in law to the norm of a Gaussian random element of a larger Banach space including the support functions).

To overcome the difficulties arising from these distinctive features, a crucial role is played by appropriate metrics between fuzzy data, like the (θ, φ)-distance and, for the inferential developments, by the existence of CLTs for Hilbert space-valued random elements, particularly the bootstrapped CLTs ones (see, for instance, Giné and Zinn (1990)).

Anyway, sometimes (see, for instance, González-Rodríguez *et al.* (2012)), techniques developed for Functional Data Analysis can be particularized through the support function connection to handling fuzzy data.

Blanco-Fernández *et al.* (2013, 2014) survey many of the discussed problems and the methods for statistical analysis of fuzzy data based on random fuzzy sets. Most of the concepts and methods described in these reviews can be applied by using a recently developed R-package called SAFD (Statistical Analysis of Fuzzy Data), which has been designed by Trutschnig and Lubiano (2012) to perform statistical computations with RFSs. It is being updated periodically.

Former review papers, with different emphases, include those of Gil *et al.* (2006), López-Díaz and Ralescu (2006), Ogura and Li (2004), and Colubi *et al.* (2007). Four books have been written which are primarily about random fuzzy sets: Kruse and Meyer (1987), Bandemer and Näther (1992), Li *et al.* (2002), and Möller and Beer (2004).

Acknowledgements

References

[1] Alonso, M.C., Brezmes, T., Lubiano, M.A. and Bertoluzza, C. (2001). A generalized real-valued measure of the inequality associated with a fuzzy random variable. *Int. J. Approx. Reasoning*, **26**, 47–66.

[2] Aumann, R.J. (1965). Integrals of set-valued functions. *J. Math. Anal. Appl.*, **12**, 1–12.

[3] Bandemer, H. and Näther, W. (1992). *Fuzzy Data Analysis*. Kluwer, Dordrecht.

[4] Blanco-Fernández, A., Casals, M.R., Colubi, A., Corral, N., García-Bárzana, M., Gil, M.A., González-Rodríguez, G., López, M.T., Lubiano, M.A., Montenegro, M., Ramos-Guajardo, A.B., de la Rosa de Sáa, S. and Sinova, B. (2013). Random fuzzy sets: a mathematical tool to develop statistical fuzzy data analysis. *Iranian J. Fuzzy Syst.*, **10**, 1–28.

[5] Blanco-Fernández, A., Casals, M.R., Colubi, A., Corral, N., García-Bárzana, M., Gil, M.A., González-Rodríguez, G., López, M.T., Lubiano, M.A., Montenegro, M., Ramos-Guajardo, A.B., de la Rosa de Sáa, S. and Sinova, B. (2014). A distance-based statistical analysis of fuzzy number-valued data. *Int. J. Approx. Reas.* (doi:10.1016/j.ijar.2013.09.020)

[6] Castaing, C. and Valadier, M. (1977). *Convex Analysis and Measurable Multifunctions.* Lect. Notes Math., 580, Springer, Berlin.

[7] Colubi, A. (2002). Traditional techniques to prove some limit theorems for fuzzy random variables. In: Bertoluzza, C., Gil, M. A. and Ralescu, D. A. (eds.), *Statistical Modeling, Analysis and Management for Fuzzy Data*, 64–71. Physica, Heidelberg.

[8] Colubi, A., Coppi, R., D'Urso, P. and Gil, M.A. (2007). Statistics with fuzzy random variables. *Metron*, **65**, 277–303.

[9] Colubi, A., Domínguez-Menchero, J.S., López-Díaz, M. and Ralescu, D.A. (2001). On the formalization of fuzzy random variables. *Inform. Sci.*, **133**, 3–6.

[10] Colubi, A., Domínguez-Menchero, J.S., López-Díaz, M. and Ralescu, D.A. (2002). A $D_E[0,1]$ representation of random upper semicontinuous functions. *Proc. Amer. Math. Soc.*, **130**, 3237–3242.

[11] Colubi, A., López-Díaz, M., Domínguez-Menchero, J.S. and Gil, M.A. (1999). A generalized Strong Law of Large Numbers. *Probab. Theory Relat. Fields*, **114**, 401–417.

[12] Dubois, D. and Prade, H. (2012). Gradualness, uncertainty and bipolarity: making sense of fuzzy sets. *Fuzzy Sets and Systems*, **192**, 3–24.

[13] Féron, R. (1976a). Ensembles aléatoires flous. *C. R. Acad. Sc. Paris A*, **282**, 903–906.

[14] Féron, R. (1976b). Ensembles flous attachés à un ensemble aléatoire flou. *Publ. Econometr.*, **9**, 51–66.

[15] Féron, R. (1979). Sur les notions de distance et d'ecart dans une structure floue et leurs applications aux ensembles aléatoires flous. *C. R. Acad. Sc. Paris A*, **289**, 35–38.

[16] Fréchet, M. (1948). Les éléments aléatoires de nature quelconque dans un espace distancié. *Ann. L'Inst. H. Poincaré*, **10**, 215–310.

[17] Fréchet, M. (1950). Conferencias sobre los elementos aleatorios de naturaleza cualquiera. *Trabajos de Estadística*, **1**, 157–181.

[18] Gil, M.A., López-Díaz, M. and López-García, H. (1998). The fuzzy hyperbolic inequality index associated with fuzzy random variables. Eur. J. Oper. Res., **110**, 377–391.

[19] Gil, M.A., López-Díaz, M. and Ralescu, D.A. (2006). Overview on the development of fuzzy random variables. *Fuzzy Sets and Systems*, **157**, 2546–2557.

[20] Giné, E. and Zinn, J. (1990). Bootstrapping general empirical measures. *Ann. Probab.*, **18**, 851–869.

[21] González-Rodríguez, G., Blanco, A., Colubi, A. and Lubiano, M.A. (2009). Estimation of a simple linear regression model for fuzzy random variables. *Fuzzy Sets and Systems*, **160**, 357–370.

[22] González-Rodríguez, G., Colubi, A. and Gil, M.A. (2012). Fuzzy data treated as functional data. A one-way ANOVA test approach. *Comp. Stat. Data Anal.*, **56**, 943–955.

[23] Hesketh, B., Griffin, B. and Loh, V. (2011). A future-oriented retirement transition adjustment framework. *J. Vocat. Behav.*, **79**, 303–314.

[24] Hesketh, B., McLachlan, K. and Gardner, D. (1992). Work adjustment theory: An empirical test using a fuzzy rating scale. *J. Vocat. Behav.*, **40**, 318–337.

[25] Hesketh, B., Hesketh, T., Hansen, J.-I. and Goranson, D. (1995). Use of fuzzy variables in developing new scales from the strong interest inventory. *J. Couns. Psychol.*, **42**, 85–99.

[26] Hesketh, T. and Hesketh, B. (1994). Computerized fuzzy ratings: the concept of a fuzzy class. *Behav. Res. Meth., Instr., & Comp.*, **26**, 272–277.

[27] Hesketh, T., Pryor, R. and Hesketh, B. (1988). An application of a computerized fuzzy graphic rating scale to the psychological measurement of individual differences. *Int. J. Man–Machine Studies*, **29**, 21–35.

[28] Hiai, F. and Umegaki, H. (1977). Integrals, conditional expectations, and martingales of multivalued junctions. *J. Multiv. Anal.*, **7**, 149–182.

[29] Kim, Y.-K. (2002). Measurability for fuzzy valued functions. *Fuzzy Sets and Systems*, **129**, 105–109.

[30] Klement, E.P., Puri, M.L. and Ralescu, D.A. (1986). Limit theorems for fuzzy random variables. *Proc. Royal Soc. London A*, **407**, 171–182.

[31] Körner, R. (1997). On the variance of fuzzy random variables. *Fuzzy Sets and Systems*, **92**, 83–93.

[32] Körner, R. and Näther, W. (2002). On the variance of random fuzzy variables. In: *Statistical Modeling, Analysis and Management of Fuzzy Data* (Bertoluzza, C., Gil, M.A. and Ralescu, D.A., eds.), 22–39. Physica-Verlag, Heidelberg.

[33] Kosko, B. (1999). *The fuzzy future: from society and science to heaven in a chip.* Harmony, New York.

[34] Krätschmer, V. (2001). A unified approach to fuzzy random variables. *Fuzzy Sets and Systems*, **123**, 1–9.

[35] Krätschmer, V. (2004). Probability theory in fuzzy sample spaces. *Metrika*, **60**, 167–189.

[36] Kruse, R. and Meyer, K.D. (1987). *Statistics with Vague Data.* Reidel, Dordrecht.

[37] Kwakernaak, H. (1978). Fuzzy random variables - I. Definitions and Theorems. *Inform. Sci.*, **15**, 1–29.

[38] Kwakernaak, H. (1979). Fuzzy random variables - II. Algorithms and examples for the discrete case. *Inform. Sci.*, **17**, 253–278.

[39] Li, S. and Ogura, Y. (1999). Convergence of set-valued and fuzzy-valued martingales. *Fuzzy Sets and Systems*, **101**, 453–461.

[40] Li, S. and Ogura, Y. (2006). Strong laws of large numbers for independent fuzzy set-valued random variables. *Fuzzy Sets and Systems*, **157**, 2569–2578.

[41] Li, S., Ogura, Y. and Kreinovich, V. (2002). Limit theorems and applications of set-valued and fuzzy set-valued random variables. Kluwer, Dordrecht.

[42] Li, S., Ogura, Y., Proske, F. and Puri, M. L. (2003). Central limit theorems for generalized set-valued random variables. *J. Math. Anal. Appl.*, **285**, 250–263.

[43] López-Díaz, M. and Ralescu, D.A. (2006). Tools for fuzzy random variables: Embeddings and measurabilities. *Comp. Stat. Data Anal.*, **51**, 109–114.

[44] Lubiano, M.A., Gil, M.A., López-Díaz, M. and López, M.T. (2000). The $\overrightarrow{\lambda}$-mean squared dispersion associated with a fuzzy random variable. *Fuzzy Sets and Systems*, **111**, 307–317.

[45] Matheron, G. (1975). *Random Sets and Integral Geometry.* Wiley, New York.

[46] Molchanov, I. (2005). *Theory of Random Sets.* Springer, London.

[47] Molchanov, I.S. (1999). On Strong Laws of Large Numbers for random upper semicontinuous functions. *J. Math. Anal. Appl.*, **235**, 349–355.

[48] Möller, B. and Beer, M. (2004). *Fuzzy Randomness: Uncertainty in Civil Engineering and Computational Mechanics.* Springer, Berlin.

[49] Nguyen, H.T. (1978). A note on the extension principle for fuzzy sets. *J. Math. Anal. Appl.*, **64**, 369–380.

[50] Nguyen, H.T. and Wu, B. (2006). *Fundamentals of Statistics with Fuzzy Data.* Springer, Berlin.

[51] Ogura, Y. and Li, S. (2001). Separability for graph convergence of sequences of fuzzy-valued random variables. *Fuzzy Sets and Systems*, **123**, 19–27.

[52] Ogura, Y. and Li, S. (2004). On limit theorems for random fuzzy sets including large deviation principles. In: López-Díaz, M., Gil, M.A., Grzegorzewski, P., Hyrniewicz, O. and Lawry, J. (eds.), *Soft Methodology and Random Information Systems.* Springer, Berlin.

[53] Ogura, Y. and Setokuchi, T. (2009). Large deviations for random upper semicontinuous functions. *Tohoku Math. J.*, **61**, 213–223.

[54] Osgood, C. E., May, W. H., and Miron, M. S. (1975) *Cross-Cultural Universals of Affective Meaning.* University of Illinois Press, Urbana.

[55] Proske, F.N. and Puri, M.L. (2003). A strong law of large numbers for generalized random sets from the viewpoint of empirical processes. *Proc. Amer. Math. Soc.*, **131**, 2937–2944.

[56] Puri, M.L. and Ralescu, D.A. (1985). The concept of normality for fuzzy random variables. *Ann. Probab.*, **11**, 1373–1379.

[57] Puri, M.L. and Ralescu, D.A. (1986). Fuzzy random variables. *J. Math. Anal. Appl.*, **114**, 409–422.

[58] Puri, M.L. and Ralescu, D.A. (1991). Convergence theorem for fuzzy martingales *J. Math. Anal. Appl.*, **160**, 107–122.

[59] Terán, P. (2007). Probabilistic foundations for measurement modelling with fuzzy random variables. *Fuzzy Sets and Systems*, **158**, 973–986.

[60] Terán, P. (2013). Algebraic, metric and probabilistic properties of convex combinations based on the t-normed extension principle. *Fuzzy Sets and Systems*, **223**, Pages 1–25.

[61] Terán Agraz, P. (2006). On Borel measurability and large deviations for fuzzy random variables. *Fuzzy Sets and Systems*, **157**, 2558–2568.

[62] Trutschnig, W. and Lubiano, M.A. (2012). *SAFD: Statistical Analysis of Fuzzy Data*, http://cran.r-project.org/web/packages/SAFD/index.html

[63] Trutschnig, W., González-Rodríguez, G., Colubi, A. and Gil, M.A. (2009). A new family of metrics for compact, convex (fuzzy) sets based on a generalized concept of mid and spread. *Inform. Sci.* **179**, 3964–3972.

[64] Zadeh, L.A. (1965). Fuzzy sets. *Inform. Control*, **8**, 338–353.

[65] Zadeh, L.A. (1975). The concept of a linguistic variable and its application to approximate reasoning. Part 1: *Inform. Sci.*, **8**, 199–249; Part 2: *Inform. Sci.*, **8**, 301–353; Part 3: *Inform. Sci.*, **9**, 43–80.

About the authors

María Ángeles Gil is a full professor of Statistics and O.R. at the University of Oviedo (Spain). Her research was initially focused on the measurement and statistical developments concerning the amount of information and inequality of random variables. Since 1983, her research interests have been mostly related to the statistical analysis of imprecise data (mainly set-valued and fuzzy set-valued). On this topic she heads the research group SMIRE (http://bellman.ciencias.uniovi.es) and she is a member of the Core SMPS Group steering the biennial conferences on Soft Methods in Probability and Statistics. She was co-editor-in-chief of the journal *TEST* (2005-2008), and is currently associate or area editor for *TEST*, *Computational Statistics and Data Analysis*, *Fuzzy Sets and Systems*, *Information Sciences* and *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*.

Ana Colubi is a full professor at the Department of Statistics and O.R. of the University of Oviedo (Spain). She is also member of the management committee and responsible of the area of statistics of the Institute of Zoning and Natural Resources (INDUROT). Her main research is in the areas of computational statistics with imperfect/incomplete data, nonparametric statistics and data analysis for environmental and soil sciences. Currently she is an associate editor of *Computational Statistics and Data Analysis* and the *International Journal of Computers Mathematics*, and area editor of Statistics in *Fuzzy Sets and Systems*. She has co-chaired and co-organized several international conferences and she is the IASC-ERS BoD Chairman Elect (2014-2016).

Pedro Terán has worked on topics ranging from group decision making to stochastic optimization, and from image processing to Banach space geometry, with a special interest on the interface between probability theory and fuzzy sets. He is a former recipient of the Melendreras Award for Young Researchers, co-chair of the European Young Statisticians Meeting, and plenary lecturer at the Young Researchers Congress celebrating the centennial of the Spanish Royal Mathematical Society.