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Departamento de Matemática Aplicada y Ciencias de la Computación

Optimal Control Problems Governed by Semilinear Equations with Integral Constraints on the Gradient of the State

PhD Thesis

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... to Cristina

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Chapter 1

Introduction

This report is devoted to the study of optimal control problems governed by partial differential equations. In an optimal control problem we have to minimize a functional which depends on two variables. The control variable, which will be denoted by u and the state variable, which will be denoted by y. The state and the control are related by some functional equation, where the control stands for some data of the equation and the state, which will be called *associate state* is the solution of the equation. In the problems here treated, for each control u there is a unique associate state, which will be denoted by y_u. Normally we will choose the control in a family of *admissible* controls $\mathbb{K}$, and we will have certain constraints on the state $y \in C$.

One of the first examples that come up is that of a control problem governed by an ordinary differential equation. Let f and g be functions, $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\mathbb{K} \subset \mathbb{R}^m$ non empty and a a given initial state. We can formulate the control problem as:

$$\left\{ \begin{array}{l} \text{Find } y \in W^{1,\infty}(0,T;\mathbb{R}^n), \quad u \in L^\infty(0,T;\mathbb{R}^m) \\ \text{which minimize } J(y,u) = \displaystyle\int_0^T g(t,y(t),u(t))\,dt, \\ \text{where } u(t) \in \mathbb{K} \text{ for a.e. } t \in [0,T], \\ y(0) = a, \\ \dot{y}(t) = f(t,y(t),u(t)) \text{ for a.e. } t \in [0,T], \end{array} \right.$$

The optimal control theory started with the study of problems governed by ordinary differential equations, and still today this kind of problems is object of study. Basic references about this topic are the books by Fleming [57], Pontryagin [73] or Cesari [40]. The range of applications of control problems is very wide. See for instance [58].

We will dedicate to control problems governed by partial differential equations. The reference point for the study of this kind of problems is the book by J. L. Lions [66]. May be one of the most simple examples of control problems governed by partial differential equations is the so called linear-quadratic problem with pointwise constraints on the control and without constraints on the state

$$\begin{cases} \text{Find } y \in L^2(\Omega), u \in L^\infty(\Omega) \\ \text{which minimize } J(y,u) = \int_\Omega |y(x) - y_d(x)|^2\,dx + \frac{k}{2}\int_\Omega u(x)^2\,dx \\ \text{where } a \le u(x) \le b \text{ for a.e. } x \in \Omega, \\ -\Delta y = u \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma. \end{cases}$$

The problem becomes more complicated when we add constraints on the state. Control problems governed by partial differential equations for different kinds of constraints of the state have been studied. For instance, integral constraints, both inequality and equality constraints

$$\int_\Omega |y(x)|^p\,dx \le \delta, \quad \int_\Omega |y(x)|^p = \delta;$$

pointwise constraints on a finite number of points

$$y(x_j) = \delta_j \ \text{ for } j = 1, \ldots, n;$$

pointwise constraints on an infinite number of points

$$y(x) \le \delta \text{ for all } x \in \bar{\Omega}.$$

Chapter 9 is devoted to the study for the numerical analysis of a problem with this kind of constraints.

Another kind of constraints are the integral constraints on the gradient of the state

$$\int_\Omega |\nabla y(x)|^p\,dx \le \delta.$$

This thesis is mainly devoted to problems with this kind of constraints. There are few results avalaible for problems with constraints on the gradient of the state. Casas and

Fernández [29] treat a problem with constraints on the gradient of the state in which, due to the assumptions made, you can assure that the the solution is C^1, simplifying in an important way the difficulties that appear. Fattorini [53, 54] deals with control problems formulated in an abstract frame. The adjoint state equation is not a partial differential equation and must be understood in a formal way.

Other of the difficulties that can be added to this kind of problem is considering that the equation that relates the control and the state is nonlinear. Control problems governed by quasilinear equations have been studied by Fernández [56], Casas and Fernández [24, 23, 25, 28, 26, 27, 30], Casas, Fernández and Yong [32], Hu and Yong [60] or Casas and Yong [38]. In this thesis we study control problems governed by semilinear equations, both elliptic and parabolic. There is also bibliography about this topic. Let us cite here Lions [67], Bonnans [7], Bonnans and Casas [8, 9, 11], Casas [19, 20, 21, 22], Casas and Fernández [29], Fattorini [55, 52], Yong [92], Casas and Mateos [33], Hu and Yong [60], Raymond [75], Raymond and Zidani [78, 79], Unger [88] or Casas and Tröltzsch [37].

Finally, we will say that the functional J can be more complicates than the above exposed. Usually J is a functional that depends both on the control and on the associate state.

1.1 Notation

We will introduce now the spaces we are going to use in this thesis. There exist many references where properties of these spaces can be found. See for instance [2, 70, 43, 68, 13, 86] among others. Let Ω be an open set of $\mathbb{R}^N$. We will denote $\bar{\Omega}$ its closure and Γ its boundary. On this set we can define the function spaces

$$C(\bar{\Omega}) = \{y : \bar{\Omega} \longrightarrow \mathbb{R}, \text{ continuous}\},$$

and for $m \in \mathbb{N} = \{1, 2, \ldots\}$,

$$C^m(\bar{\Omega}) = \{y : \bar{\Omega} \longrightarrow \mathbb{R}, \text{ such that } \partial^\alpha y \in C(\bar{\Omega}) \text{ for every multiindex } |\alpha| \leq m\}.$$

For $1 \leq p \leq \infty$

$$L^p(\Omega) = \{y : \Omega \longrightarrow \mathbb{R}, \text{ Lebesgue measurable, such that } \|y\|_{L^p(\Omega)} < \infty\},$$

where

$$\|y\|_{L^p(\Omega)} = \left(\int_\Omega |y(x)|^p dx\right)^{\frac{1}{p}},$$

if $1 \le p < \infty$ and

$$\|y\|_{L^\infty(\Omega)} = \sup \text{ess}\{|y(x)| : x \in \Omega\}.$$

Remember that an element in a Lebesgue space is a class of functions that are equal in almost every point, i.e., but on a set of zero Lebesgue measure. Normally we will write a.e. to shorten *almost every point.* The Lebesgue measure of a set A will be denoted by $|A|$.

We define the Sobolev norms on $C^m(\bar{\Omega})$ as

$$\|y\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \int_\Omega |\partial^\alpha y|^p \, dx \right)^{\frac{1}{p}}$$

if $1 \le p < \infty$ and

$$\|y\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha| \le m} \{\sup \text{ess}\{|\partial^\alpha y(x)| : x \in \Omega\}\}.$$

With this norms, the spaces $C^m(\bar{\Omega})$ are not complete. We will denote

$$W^{m,p}(\Omega) = \overline{C^m(\bar{\Omega})},$$

where the bar indicates the closure in the sense of the Sobolev norm above defined. For $p = 2$, we will usually write

$$W^{m,2}(\Omega) = H^m(\Omega).$$

Given $\sigma \in (0, 1]$, we will say that the boundary of Ω is of class $C^{m,\sigma}$ [resp. C^m] if there exist numbers $\alpha > 0$, $\beta > 0$, coordinate systems $(x_{k1}, x_{k2}, \dots, x_{kN})$, short (x'_k, x_{kN}), $k = 1, 2, \dots, \Lambda$, and functions b_k of class $C^{m,\sigma}$ [resp. C^m] in the closed $N-1$ dimensional cubes $|x_{ki}| \le \alpha$, $i = 1, 2, \dots, N-1$, in such a way that every point x of Γ can be represented at least in one of these systems as $x = (x'_k, b_k(x'_k))$. It is also supposed that the points (x'_k, x_{kN}) such that $x'_k \in [-\alpha, \alpha]^{N-1}$, $b_k(x'_k) < x_{kN} < b_k(x'_k) + \beta$ are in Ω, meanwhile the points (x'_k, x_{kN}) such that $x'_k \in [-\alpha, \alpha]^{N-1}$, $b_k(x'_k) - \beta < x_{kN} < b_k(x'_k)$ are out of $\bar{\Omega}$ (cf: Nečas [72]). If the boundary is of class $C^{0,1}$ we will say it is Lipschitz.

A rigorous definition of the Lebesgue spaces on the boundary using partitions of the unity and coordinate systems associated to a covering can be found in [72, pp. 82,83].

If Ω is of class C^m, we can define the trace mapping for $l < m$

$$\gamma_l : C^m(\bar{\Omega}) \longrightarrow \prod_{j=0}^{l} L^p(\Gamma)$$

as

$$\gamma_l y = \left(y, \frac{\partial y}{\partial n}, \ldots, \frac{\partial^l y}{\partial n^l} \right),$$

where n is the outer unitary vector normal to Γ. This mapping is extended in a continuous way to $W^{m,p}(\Omega)$. The image of $W^{m,p}(\Omega)$ by γ_l is

$$\gamma_l(W^{m,p}(\Omega)) = \prod_{j=0}^{l} W^{m-j-\frac{1}{p},p}(\Gamma).$$

Normally we will write γ with no subindex for γ_0. To define γ it is enough that Γ is Lipschitz.

We define now

$$W_0^{m,p}(\Omega) = \{ y \in W^{m,p}(\Omega) : \ \gamma_{m-1} y = 0, \}$$

with the same norm than $W^{m,p}(\Omega)$. It is known that if Γ is Lipschitz,

$$C_0^m(\Omega) = \{ y \in C^m(\bar{\Omega}) : \ \text{supp}\, y \subset \Omega \text{ is compact } \}$$

and if we denote

$$\mathcal{D}(\Omega) = \bigcap_{m \geq 1} C_0^m(\Omega),$$

then

$$W_0^{m,p}(\Omega) = \overline{\mathcal{D}(\Omega)},$$

see Nečas [72].

The space of continuous and bounded functions on Ω is named $C_b(\Omega)$.

Given a normed space X we will denote by X' its dual, i.e., the space of continuous and linear functionals on X. We define

$$W^{-m,p}(\Omega) = (W_0^{m,p}(\Omega))'.$$

Given $\sigma \in (0,1)$ we define the Hölder functions spaces as

$$C^{0,\sigma}(\bar{\Omega}) = \{ y \in C(\bar{\Omega}) : \ \sup_{x,x' \in \bar{\Omega}} \frac{|y(x) - y(x')|}{|x - x'|^\sigma} < \infty \}.$$

The norm in this space is

$$\|y\|_{C^{0,\sigma}(\bar{\Omega})} = \sup_{x,x' \in \bar{\Omega}} \frac{|y(x) - y(x')|}{|x - x'|^\sigma}$$

For $\sigma = 1$, $C^{0,1}(\bar{\Omega})$ is named space of Lipschitz functions, and coincides with $W^{1,\infty}(\Omega)$. Also

$$C^{m,\sigma}(\bar{\Omega}) = \{y \in C^m(\bar{\Omega}) : \ \partial^\alpha y \in C^{0,\sigma}(\bar{\Omega}) \text{ for } |\alpha| = m\}.$$

We define the fractionary Sobolev spaces as follows. Let $\sigma \in (0,1)$. Let us take

$$I_{\sigma,p}(y) = \int_{\Omega\times\Omega} \frac{|y(x) - y(x')|^p}{|x - x'|^{N+\sigma p}} dx\, dx',$$

and for $s > 0$

$$W^{s,p}(\Omega) = \left\{y \in W^{[s],p}(\Omega) : \ I_{s-[s],p}(\partial^\alpha y) < \infty \text{ for } |\alpha| = [s]\right\},$$

where $[s]$ is the integer part of s. The norm in this space is given by

$$\|y\|_{W^{s,p}(\Omega)} = \left(\|y\|^p_{W^{[s],p}(\Omega)} + \sum_{|\alpha|=[s]} I_{s-[s],p}(\partial^\alpha y)^p \right)^{\frac{1}{p}}.$$

We have the following result of continuous inclusion

$$W^{s,p}(\Omega) \subset L^q(\Omega) \text{ for } q \le \frac{Np}{N - sp} \text{ if } N - sp > 0,$$

$$W^{s,p}(\Omega) \subset C^{0,\lambda}(\bar{\Omega}) \text{ for } 0 < \lambda < s - \frac{N}{p} \text{ if } sp - N > 0.$$

If Γ is Lipschitz, the following inclusion is compact

$$W^{1+\sigma,p}(\Omega) \subset W^{1,p}(\Omega) \text{ for } \sigma > 0.$$

Given $T > 0$, we define the Lebesgue vector spaces, for $1 \le \tau \le \infty$ as

$$L^\tau(0,T; W^{s,p}(\Omega)) = \left\{y : (0,T) \times \Omega \longrightarrow \mathbb{R} : \ \|y\|_{L^\tau(0,T;W^{s,p}(\Omega))} < \infty\right\},$$

where

$$\|y\|_{L^\tau(0,T;W^{s,p}(\Omega))} = \left(\int_0^T \|y(t,\cdot)\|^\tau_{W^{s,p}(\Omega)} d\tau \right)^{\frac{1}{\tau}}$$

if $1 \le \tau < \infty$ and

$$\|y\|_{L^\infty(0,T;W^{s,p}(\Omega))} = \sup\text{ess}\left\{\|y(t,\cdot)\|_{W^{s,p}(\Omega)} : \ t \in (0,T)\right\}.$$

We can also define Sobolev vector spaces:

$$W^{1,\tau}(0,T;W^{s,p}(\Omega)) = \left\{ y \in L^{\tau}(0,T;W^{s,p}(\Omega)) \text{ such that } \frac{dy}{dt} \in L^{\tau}(0,T;W^{s,p}(\Omega)) \right\},$$

where the derivative is taken in the distributions sense.

We also define

$$C([0,T],C^{0,\sigma}(\bar{\Omega})) = \{y : [0,T] \times \bar{\Omega} \longrightarrow \mathbb{R} : \ \|y\|_{C([0,T],C^{0,\sigma}(\bar{\Omega}))} < \infty\},$$

where

$$\|y\|_{C([0,T],C^{0,\sigma}(\bar{\Omega}))} = \sup_{t\in[0,T]} \|y(t,\cdot)\|_{C^{0,\sigma}(\bar{\Omega})}.$$

In this thesis, and if this does not lead to confusion, we will use the following shortening: $L^{\tau}(W^{s,p})$, $L^2(H^1)$, $W^{1,\tau}((W^{1,p})')$, $L^{\bar{k}}(L^k(\Omega))$, $L^{\bar{\sigma}}(L^{\sigma}(\Gamma))$, and $C(C^{0,\varepsilon}(\bar{\Omega}))$ respectively for $L^{\tau}(0,T;W^{s,p}(\Omega))$, $L^2(0,T;H^1(\Omega))$, $W^{1,\tau}(0,T;(W^{1,p}(\Omega))')$, $L^{\bar{k}}(0,T;L^k(\Omega))$, $L^{\bar{\sigma}}(0,T;L^{\sigma}(\Gamma))$ y $C([0,T];C^{0,\varepsilon}(\bar{\Omega}))$, for τ, s, p, $\bar{k}$, k, $\bar{\sigma}$, σ y ε real numbers. We will also denote, as it is usual

$$W(0,T) = \{y \in L^2(0,T;H^1(\Omega)) : \frac{dy}{dt} \in L^2(0,T;H^1(\Omega)')\}.$$

Given a metric space X, we will denote the ball of center x and radius r by $B_X(x,r)$.

As it is usual, we will write $\mathbb{R}^N_+ = \{x = (x_1, \cdots, x_N) \in \mathbb{R}^N$ such that $x_N > 0\}$.

1.2 Plan of exposition

The aim of this thesis is to study is to study the following control problems:

Elliptic problem Let Ω be an open set in $\mathbb{R}^N$, Γ its boundary, A an elliptic operator and f, g and L functions $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, $g : \Gamma \to \mathbb{R}$, $L : \Omega \times \mathbb{R}^2 \to \mathbb{R}$. Let n_i[1], n_d[2] be nonnegative integers and let $g_j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be functions for $1 \leq j \leq n_i + n_d$. Our

[1] número de igualdades=number of equalities
[2] número de desigualdes=number of inequalities

first control problem is formulated as

$$
(\mathbf{P_e}) \begin{cases} \text{Minimize } J(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))dx, \\ u \in U_{ad} = \{u : \Omega \to \mathbb{R} : \ u(x) \in K_\Omega(x) \text{ a.e. } x \in \Omega\}, \\ \displaystyle\int_\Omega g_j(x, \nabla y_u(x))\, dx = 0, \ 1 \le j \le n_i, \\ \displaystyle\int_\Omega g_j(x, \nabla y_u(x))\, dx \le 0, \ n_i + 1 \le j \le n_i + n_d, \end{cases}
$$

where

$$
\begin{cases} Ay_u = f(x, y_u, u) & \text{in } \Omega \\ \partial_{n_A} y_u = g & \text{on } \Gamma, \end{cases}
$$

and K_Ω is a measurable multimapping with nonempty closed image in $\mathcal{P}(\mathbb{R})$.

Parabolic problem Let Ω be an open set in $\mathbb{R}^N$, Γ its boundary and $T > 0$. Let us state $Q = \Omega \times]0, T[$ and $\Sigma = \Gamma \times]0, T[$. Let A be an elliptic operator. Let us consider functions $F : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $G : \Sigma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, $f : Q \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \Sigma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $y_0 : \Omega \longrightarrow \mathbb{R}$. The control problem is the following:

$$
(\mathbf{P_p}) \begin{cases} \min J(v) = \displaystyle\int_0^T \int_\Omega F(x, t, y_v)\, dx\, dt + \int_0^T \int_\Gamma G(s, t, y_v, v)\, ds\, dt \\ \qquad\qquad + \displaystyle\int_\Omega L(x, y_v(x, T))\, dx \\ v \in V_{ad} = \{v \in L^\infty(\Sigma) : \ v(s,t) \in K_\Sigma(s,t) \text{ for a.e. } (s,t) \in \Sigma\}, \\ \nabla_x y_v \in C \subset (L^r(0, T; L^p(\Omega)))^N, \end{cases}
$$

where

$$
\begin{cases} \dfrac{\partial y_v}{\partial t} + Ay_v = f(x, t, y_v) & \text{in } Q, \\ \dfrac{\partial y_v}{\partial n_A} = g(s, t, y_v, v) & \text{on } \Sigma, \\ y_v(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}
$$

K_Σ is a measurable multimapping with nonempty compact image in $\mathcal{P}(\mathbb{R})$ and C is closed convex and with nonempty interior subset of $(L^r(0, T; L^p(\Omega)))^N$.

We have decided to introduce a distributed control for the elliptic problem and a boundary control for the parabolic case just to illustrate these two cases, since writing all the possible cases would have increased the length of the thesis. Nevertheless, after the detailed study of these problems we will state results for other problems that can be treated following the same techniques.

The plan of the work is the following:

In the first part we study the equations that appear in the studied control problems. In Chapter 2 we make an study on regularity for linear equations. These results will be applied later to state the regularity both for the state and for the adjoint state. In Chapter 3 we study the state equations that govern the control problems. We show the continuity and differentiability relations between that state and the control. We also perform a sensitivity analysis of the state with respect to diffuse perturbations of the control.

The second part constitutes the central kernel of the thesis. Here we study optimality conditions, both necessary and sufficient, for the control problems. In Chapter 4 we expose the properties of the functionals that appear in the control problems: The objective functional and the constraints. We study under what conditions they are differentiable and, since we expect to prove Pontryagin's Principle, we ,make a sensitivity analysis with respect to diffuse perturbations of the control. In Chapter 5 we expose Pontryagin's Principle. In Chapter 6 we introduce first and second order optimality conditions. Finally, in Chapter 7 we introduce a new type of second order conditions in which the Hamiltonian is involved.

In every chapter we intercalate the elliptic and the parabolic case.

In the third part we make a study of the numerical approximations of the following control problem: Let Ω be an open set in $\mathbb{R}^N$, Γ its boundary, A an elliptic operator, U_{ad} a subset of $L^\infty(\Omega)$ and $L : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ a function. Let $g : \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function. We formulate the optimal control problem

$$(P_\delta) \begin{cases} \min J(u) = \displaystyle\int_\Omega L\left(x, y_u(x), u(x)\right) dx \\ u \in K \quad g\left(x, y_u(x)\right) \leq \delta \quad \forall x \in \bar{\Omega}, \end{cases} \tag{1.2.1}$$

where

$$\begin{cases} Ay &= f(x,y) + u \quad \text{in } \Omega \\ y &= 0 \quad\quad\quad\quad \text{on } \Gamma \end{cases}$$

The topics about existence of solution and optimality conditions for this problem have already been treated by Casas in [18].

Part I

Study of the equations

In the first part of the thesis we study the equations that appear in the control problems we are going to deal with. This study is divided into two main parts. First, we make the study of linear equations, which will allow us to treat later the linearized state equation and the adjoint state equation. Finally, we will establish the properties of the mapping that relates the control and the state.

In our case, since we are studying control problems with integral constraints on the gradient of the state, the study of equations (linearized and state equation) is very similar, since, *grosso modo*, we have to prove $W^{1,p}(\Omega)$ regularity of the solution of a linear equation, for $p \in (1, \infty)$.

The second part is the study of the relation between the control and the state. In our case, for every control there exists a unique state. There exist studies for control problems where this is not verified. For instance, Casas and Fernández [24] or Bonnans and Casas [8] study a multistate control problem. Abergel and Casas [1] study multistate control problems which appear in fluid mechanics.

In our case, since we deal problems governed by semilinear equations, the functional, let us name it G, that relates the state y with the control u is nonlinear. We must prove that there exist a unique solution, that it is in the correct space and that it depends continuously on the control. In the second part of the thesis we obtain first and second order conditions. To do that we also study under what conditions G is C^1 or C^2. If we write the functional that we want to to minimize as

$$J(u) = F(y_u, u) = F(G(u), u),$$

using the chain rule, we can prove that some of the properties of G are inherited by J. This is seen in detail in the second part of the thesis.

Finally, to deal with the non convex case, we introduce a Taylor expansion based in diffuse perturbations of the control. The aim is to deduce a Pontryagin Principle. To do this, we use the Taylor expansions (Theorems 3.3.2 and 3.3.4) for the solution of the state equation with a remainder converging to zero in the norm of $L^\tau(0,T;W^{1,p}(\Omega))$ in the parabolic case and in the norm of $W^{1,p}(\Omega)$ in the elliptic case (the norm corresponding to the state constraint).

In order to state this result, in the parabolic case, we use the compact injection of $L^\tau(0,T;W^{1+\varepsilon,p}(\Omega)) \cap W^{1,\tau}(0,T,(W^{1,p'}(\Omega))')$ in $L^\tau(0,T;W^{1,p}(\Omega))$ (see the proof of Theorem 3.3.4). To do that we have to establish regularity results in $L^\tau(0,T;W^{1+\varepsilon,p}(\Omega))$ for the linearized state equation in section 2.2.

Chapter 2

Regularity results for linear equations

2.1 Elliptic equations

In this section, we are concerned with the $W^{1,p}(\Omega)$ regularity of the solutions of Dirichlet and Neumann problems. This section comes to fill up the gap between some known results and counterexamples to this regularity. The aim is to deduce the existence, uniqueness and estimates in $W^{1,p}(\Omega)$ of the solution under minimal regularity assumptions on the coefficients of the main part of the elliptic operator and on the boundary of the domain. Continuous coefficients and C^1 boundary is enough for this regularity. The case of a Lipschitz boundary is investigated too.

Although the results exposed here are more or less known by the specialists in PDE, we have not found a clear reference for them. We introduce them here for completeness and clearness in the exposition.

Introduction and main results

Let Ω be a bounded open set in $\mathbb{R}^N$ with boundary Γ and let us set

$$Ay = -\sum_{i,j=1}^{N} \partial_{x_j} \left[a_{ij} \partial_{x_i} y \right], \tag{2.1.1}$$

where the coefficients a_{ij} belong to $L^\infty(\Omega)$ and satisfy

$$m\|\xi\|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq M\|\xi\|^2 \;\forall\xi \in \mathbb{R}^N \text{ and } \forall x \in \Omega. \tag{2.1.2}$$

for some $m, M > 0$. We also introduce $a_0 \in L^r(\Omega)$, $a_0(x) \geq 0$ in Ω, where we choose $r \geq Np/(N+p)$ if $p > N$, $r \geq N/2$ if $N/(N-1) \leq p \leq N$ and $r \geq Np'/(N+p')$ if $p < N/(N-1)$, with $p' = p/(p-1)$. For instance, if $p > N$, we can choose $r = p/2$.

Let $f_D \in W^{-1,p}(\Omega)$, $f \in \left(W^{1,p'}(\Omega)\right)'$ with $1/p + 1/p' = 1$ and $g \in W^{-\frac{1}{p},p}(\Gamma)$, with $p \in (1,\infty)$.

The purpose of this section is to study $W^{1,p}(\Omega)$ regularity for the solution of Dirichlet's problem

$$\left\{\begin{array}{rcll} Ay + a_0y &=& f_D & \text{in } \Omega \\ y &=& 0 & \text{on } \Gamma. \end{array}\right. \tag{2.1.3}$$

and, assuming $a_0 \not\equiv 0$, of Neumann's problem

$$\left\{\begin{array}{rcll} Ay + a_0y &=& f & \text{in } \Omega \\ \partial_{n_A}y &=& g & \text{on } \Gamma. \end{array}\right. \tag{2.1.4}$$

The existence, uniqueness and regularity of u in $W^{1,p}(\Omega)$ depends on the regularity of Γ and the coefficients a_{ij} and a_0.

If $p \geq 2$, we can reduce Dirichlet's problem to the case $a_0 = 0$ and Neumann's problem to the case $a_0 = 1$: if $p \geq 2$, then, due to Lemmas 2.1.4 and 2.1.12, there exists a unique solution $y \in H^1(\Omega) \cap L^{p^*}(\Omega)$, where $p^* = \infty$ if $p > N$, p^* is any number in $[1,\infty)$ if $p = N$ y $p^* = Np/(N-p)$ if $2 \leq p < N$. Therefore $a_0y \in L^{\frac{Np}{N+p}}(\Omega)$. So, due to Sobolev inequalities, for Dirichlet's problem $a_0y \in W^{-1,p}(\Omega)$ and we can add $-a_0y$ to equation (2.1.2) and if we rename f_D as $f_D - a_0y$, we will have to solve the problem

$$\left\{\begin{array}{rcll} Ay &=& f_D & \text{in } \Omega \\ y &=& 0 & \text{on } \Gamma. \end{array}\right. \tag{2.1.5}$$

And for Neumann's problem, we can replace f for $f - a_0y + y \in \left(W^{1,p'}(\Omega)\right)'$ and so we will have

$$\left\{\begin{array}{rcll} Ay + y &=& f & \text{in } \Omega \\ \partial_{n_A}y &=& g & \text{on } \Gamma. \end{array}\right. \tag{2.1.6}$$

For $p < 2$ the result is achieved by duality and transposition.

It is known (Troianiello [87, Th. 3.16(iv)]) that if the coefficients a_{ij} are Hölder continuous and the domain is of class $C^{1,\delta}$, with $0 < \delta < 1$, then $W^{1,p}(\Omega)$ regularity of the solution can be assured, both for Dirichlet's and for Neumann's problem. It is also known (Serrin [81]) that if the coefficients are not continuous, this can fail.

Example 2.1.1 *Let Ω be the unit ball in $\mathbb{R}^N$, $N > 1$ and $v(x) = x_1(|x|^\lambda - 1)$ with $\lambda = \frac{1}{2} - N$. We have that $v \in W_0^{1,r}(\Omega)$ for all $r \in \left[1, \frac{2N}{2N-1}\right)$ and $v \notin W_0^{1,p}(\Omega)$ for any $p \geq \frac{2N}{2N-1}$. Let us set $a = \frac{4(N-1)}{2N-3}$ and $a_{ij} = \delta_{ij} + (a-1)\frac{x_i x_j}{|x|^2}$. Then coefficients a_{ij} are bounded and* (2.1.2). *holds. Now it is easy to check that v solves the following Dirichlet problem*

$$\begin{cases} Ay = f_D & \text{in } \Omega \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{2.1.7}$$

where

$$f_D(x) = \frac{(a-1)(N-2)x_1}{|x|^2}.$$

Function f_D is in $L^q(\Omega)$ for every $q < N$, therefore $f_D \in W^{-1,p}(\Omega)$ for all $p < +\infty$. This proves that the regularity fails for non continuous coefficients.

On the other hand, we know that there exists a unique solution y in $H_0^1(\Omega) \subset W_0^{1,r}(\Omega)$ to the previous problem. Since $v \notin H_0^1(\Omega)$, then $y \neq v$ and both are solutions in $W_0^{1,r}(\Omega)$ to (2.1.7), so we deduce that uniqueness fails in this space.

Our results come to fill up this gap between Troianiello's result and previous counterexample. We will see below that continuity of the coefficients is enough to obtain uniqueness and regularity.

On the other hand, the $C^{1,\delta}$ regularity of the boundary Γ assumed by Troianiello [87, Th. 3.16(iv)] can be relaxed. Indeed Theorems 2.1.1 and 2.1.3 state the $W^{1,p}(\Omega)$ regularity of the solutions of problems (2.1.3) and (2.1.4) assuming C^1 regularity of Γ. Theorem 2.1.1 was established by Simader [82] and Jerison and Kenig [62] for Laplace operator, $A = -\Delta$ and by Morrey [71, page 156].

The question is whether the same result can be achieved just by supposing Γ to be Lipschitz. Jerison and Kenig [62, Th. 0.5, 1.1, 1.3] answered this question for problem (2.1.3) in the case of Laplace operator, $A = -\Delta$. They proved that if the boundary Γ is Lipschitz, then we can only assure $W^{1,p}(\Omega)$ regularity for $p_1' < p < p_1$, with $p_1 = 4+\varepsilon(\Omega)$ if $N = 2$ and $p_1 = 3 + \varepsilon(\Omega)$ if $N \geq 3$, with $0 < \varepsilon(\Omega) \leq 1/2$. Furthermore this result is sharp. Indeed, in [62], it is proved that for any $p > 4$ if $N = 2$, or $p > 3$ if $N \geq 3$, there exists a Lipschitz domain Ω and a function $f_D \in C^\infty(\bar{\Omega})$ such that the solution of (2.1.3)is not in $W^{1,p}(\Omega)$. Theorem 2.1.2 extends [62] to the case of an elliptic operator A with continuous coefficients.

It has also been proved (Dauge [47]) that if Ω is a convex polyhedrical domain ($N \leq 3$) and the coefficients of the operator are continuous, then $y \in W_0^{1,p}(\Omega)$, with $1 < p < \infty$ for Dirichlet problem, and with $6/(3+\sqrt{5}) < p < 6/(3-\sqrt{5})$ for Neumann problem.

The continuity of the coefficients a_{ij} is relaxed by Chiarenza [41] by assuming that a_{ij} are bounded mean oscillation functions whose integral oscillation over balls shrinking to a point converge uniformly to zero. This is made for Dirichlet problem under $C^{1,1}$ regularity of Γ

In all the above cited references, except in [87], the symmetry of the operator A was assumed, $a_{ij} = a_{ji}$. We remove this assumption, which does not change the proof for Dirichlet problem, but it introduces some extra difficulties when dealing with Neumann problem; see Remark 2.1.3. Let us mention that the proof of regularity for Neumann problem is not carried out in [87].

There exist estimates in $W^{1,p}(\Omega)$ for continuous coefficients which could lead to the results here introduced (cf. [3, Theorems 15.3',15.1"]), at least in the case of symmetric coefficients. Nevertheless, we have decided to include here the proofs, since we have not been able to find a detailed proof of the method, and we think that the case of non symmetric coefficients is interesting enough and it is not treated in the existent literature

Let us state the theorems to be proved in this section.

Theorem 2.1.1 *If Γ is of class C^1 and the coefficients $a_{ij} \in C(\bar{\Omega})$, then there exists a unique solution $y \in W_0^{1,p}(\Omega)$ to Dirichlet's problem* (2.1.5). *Moreover, the estimate*

$$\|y\|_{W_0^{1,p}(\Omega)} \leq C\|f_D\|_{W^{-1,p}(\Omega)} \tag{2.1.8}$$

holds, where C is a constant which only depends on p, the dimension N, the coefficients a_{ij} and Ω.

Theorem 2.1.2 *If Γ is Lipschitz and the coefficients $a_{ij} \in C(\bar{\Omega})$ then there exist $\varepsilon(\Omega) >$ 0 and a unique solution $y \in W_0^{1,p}(\Omega)$ to Dirichlet's problem* (2.1.5) *for all $p_1' < p < p_1$, where $p_1 = 4 + \varepsilon(\Omega)$ if $N = 2$ y $p_1 = 3 + \varepsilon(\Omega)$ if $N \geq 3$. Moreover, the estimate*

$$\|y\|_{W_0^{1,p}(\Omega)} \leq C\|f_D\|_{W^{-1,p}(\Omega)}$$

holds, where C is a constant which only depends on p, the dimension N, the coefficients a_{ij} and Ω.

Theorem 2.1.3 *If Γ is of class C^1 and the coefficients $a_{ij} \in C(\bar{\Omega})$, then there exist a unique variational solution $y \in W^{1,p}(\Omega)$ of Neumann's problem* (2.1.6). *Moreover, the estimate*

$$\|y\|_{W^{1,p}(\Omega)} \leq C(\|f\|_{(W^{1,p'}(\Omega))'} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)})$$

holds, where C is a constant which only depends on p, the dimension N, the coefficients a_{ij} and Ω.

In this level of regularity the normal derivative has no sense (cf. Lions y Magenes [68]). Let us precise what we mean with variational solution to the problem (2.1.6).

Definition 2.1.1 *We shall call variational solution of* (2.1.6) *to the solution of the variational problem*

$$a(y,z) = \langle f, z\rangle_{(W^{1,p'}(\Omega))' \times W^{1,p'}(\Omega)} + \langle g, \gamma v\rangle_{W^{-\frac{1}{p},p}(\Gamma)\times W^{\frac{1}{p},p'}(\Gamma)} \ \forall z \in W^{1,p'}(\Omega), \tag{2.1.9}$$

where

$$a(y,z) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}\partial_{x_i} y \partial_{x_j} z + \int_\Omega a_0 y z \tag{2.1.10}$$

is the bilinear form associated to the operator A and $\gamma : W^{1,p'}(\Omega) \to W^{\frac{1}{p},p'}(\Gamma)$ is the trace operator.

In the previous theorems the dependence of the estimates with respect to the coefficients a_{ij} is through m, M and their continuity modulus.

Remark 2.1.1 *Some authors have studied the case corresponding to data f and g, in the above problems, which are measures in Ω and Γ respectively; see, for instance, Casas [16] or Boccardo [6]. Since a measure in Ω is an element of $(W^{1,p'}(\Omega))'$ and a measure on Γ belongs to $W^{-1/p,p}(\Gamma)$ for every $p < N/(N-1)$, then Theorems 2.1.1 and 2.1.3 state the existence and uniqueness of solutions in $W^{1,p}(\Omega)$ for every $p < N/(N-1)$, which is the classical result.*

Dirichlet problem. Proof of Theorems 2.1.1 and 2.1.2

For the proof of Theorems 2.1.1 and 2.1.2 we shall use the following result, due to Stampacchia [84].

Lemma 2.1.4 *Let us suppose $p \geq 2$. Then there exists a unique function $y \in H_0^1(\Omega) \cap L^{p^*}(\Omega)$, where $p^* = \infty$ if $p > N$, p^* is any number in $[1, \infty)$ if $p = N$ y $p^* = Np/(N-p)$ if $2 \leq p < N$, satisfying the equation* (2.1.3). *Moreover, the estimate*

$$\|y\|_{L^{p^*}(\Omega)} \leq C\|f_D\|_{W^{-1,p}(\Omega)}$$

holds, where C is a constant which only depends on p, the dimension N, m, M and the measure of Ω. Notice that obviously also $y \in L^p(\Omega)$ and

$$\|y\|_{L^p(\Omega)} \leq C\|f_D\|_{W^{-1,p}(\Omega)}.$$

We shall also use the following lemma about operators with constant coefficients.

Lemma 2.1.5 *Let us suppose that the coefficients a_{ij} of the operator A are constant for $1 \leq i, j \leq N$. If*

1. *Γ is of class C^1 and $1 < p < \infty$ or*
2. *Γ is Lipschitz and $p_1' < p < p_1$, where p_1 depends on Ω, $p_1 > 3$ if $N = 3$ and $p_1 > 4$ if $N = 2$,*

then there exists a unique function $y \in W_0^{1,p}(\Omega)$ satisfying the partial differential equation

$$\left\{ \begin{array}{rcll} Ay & = & f_D & \text{in } \Omega \\ y & = & 0 & \text{on } \Gamma. \end{array} \right. \tag{2.1.11}$$

Moreover, the estimate

$$\|y\|_{W_0^{1,p}(\Omega)} \leq C_0\|f_D\|_{W^{-1,p}(\Omega)}$$

holds, where C_0 depends on a_{ij}, Ω, N and p.

Proof. There is no loss of generality in assuming that $a_{ij} = a_{ji}$. Then hypothesis (2.1.2) implies that $\hat{A} = (a_{ij})$ is symmetric and positive definite, therefore there exists a real and regular matrix P such that $\hat{A} = PP^T$. Let $T = P^{-1}$. Through a linear change of variable

$$\hat{x} = Tx$$

we can transform problem (2.1.11) into

$$\begin{cases} -\Delta\hat{y} = \hat{f}_D & \text{in } \hat{\Omega} \\ \hat{y} = 0 & \text{on } \partial\hat{\Omega}, \end{cases} \tag{2.1.12}$$

where $\hat{y} = y \circ T^{-1}$, $\hat{f}_D = f_D \circ T^{-1}$ y $\hat{\Omega} = T(\Omega)$.

Applying Jerison and Kenig's result [62] we have that (2.1.12) has a unique solution $\hat{y} \in W_0^{1,p}(\hat{\Omega})$ and that

$$\|\hat{y}\|_{W_0^{1,p}(\hat{\Omega})} \leq C\|\hat{f}_D\|_{W^{-1,p}(\hat{\Omega})},$$

where C depends on p, N and on the Lipschitz constant of the boundary $\hat{\Omega}$.

Undoing the change of variable we obtain that $y \in W_0^{1,p}(\Omega)$ and the estimate

$$\|y\|_{W_0^{1,p}(\Omega)} \leq (\det T)^{\frac{1}{p'}} C\|f_D\|_W^{-1,p}(\Omega)$$

holds □

We are now ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. Thanks to the continuity of the coefficients, we know that for all $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\sum_{i,j=1}^{N} |a_{ij}(x_1) - a_{ij}(x_2)| < \varepsilon \; \forall x_1, x_2 \in \bar{\Omega}, \text{ con } |x_1 - x_2| < \rho. \tag{2.1.13}$$

Let $\{C_\rho^s\}_{s=1}^{\mu}$ be a collection of open sets covering Ω, every set C_ρ^s having a boundary of class C^1 which leaves the interior of the set at one side of the boundary and its diameter is less or equal than ρ. Let us choose $x_s \in C_\rho^s$ a fixed point, and let $\{\varphi_s\}_{s=1}^{\mu}$ be a partition of the unity relative to the covering.

First let us consider the case $p \geq 2$. Let us take $y \in H_0^1(\Omega) \cap L^p(\Omega)$as in Lemma 2.1.4 and let us set

$$y_s = \varphi_s y, \text{ for } 1 \leq s \leq \mu. \tag{2.1.14}$$

We have that y_s verifies the equation

$$\begin{cases} A_s y_s = \varphi_s f_D - \sum_{i,j=1}^{N} a_{ij}(x)\partial_{x_j}\varphi_s\partial_{x_i}y - \sum_{i,j=1}^{N}\partial_{x_j}\left(a_{ij}(x)y\partial_{x_i}\varphi_s\right) - \\ \quad \sum_{i,j=1}^{N}\partial_{x_j}\left[(a_{ij}(x_s)-a_{ij}(x))\partial_{x_i}y_s\right] & \text{in } C^s_\rho \\ y_s = 0 & \text{on } \partial C^s_\rho, \end{cases} \tag{2.1.15}$$

where A_s is the operator associated to the constant coefficients matrix $(a_{ij}(x_s))$. In the case $N \geq 3$, in a first stage we shall assume that

$$p \leq \frac{2N}{N-2}.$$

Lemma 2.1.4, the conditions imposed to p and the conditions on the support of φ_s allow us deduce that

$$\varphi_s f_D - \sum_{i,j=1}^{N} a_{ij}(x)\partial_{x_j}\varphi_s\partial_{x_i}y - \sum_{i,j=1}^{N}\partial_{x_j}\left(a_{ij}(x)y\partial_{x_i}\varphi_s\right) \in W^{-1,p}(\Omega).$$

Firstly, we have the inequality

$$\|\varphi_s f_D\|_{W^{-1,p}(\Omega)} \leq C\left(\|\varphi_s\|_{W^{1,\infty}(\Omega)}\right)\|f_D\|_{W^{-1,p}(\Omega)}. \tag{2.1.16}$$

Also, thanks to Lemma 2.1.4, we have

$$\begin{aligned} \|\sum_{i,j=1}^{N}\partial_{x_j}\left(a_{ij}y\partial_{x_i}\varphi_s\right)\|_{W^{-1,p}(\Omega)} &\leq \sum_{i,j=1}^{N}\|a_{ij}y\partial_{x_i}\varphi_s\|_{L^p(\Omega)} \leq \\ &\leq C\left(\|a_{ij}\|_{L^\infty(\Omega)}, \|\varphi_s\|_{W^{1,\infty}(\Omega)}\right)\|f_D\|_{W^{-1,p}(\Omega)}. \end{aligned} \tag{2.1.17}$$

On the other hand, the conditions imposed to p imply that $L^2(\Omega) \subset W^{-1,p}(\Omega) \subset H^{-1}(\Omega)$, the inclusions being continuous. Using the usual estimates in $H^1_0(\Omega)$ we have

$$\begin{aligned} \|\sum_{i,j=1}^{N} a_{ij}\partial_{x_i}\varphi_s\partial_{x_j}y\|_{W^{-1,p}(\Omega)} &\leq \|\sum_{i,j=1}^{N} a_{ij}\partial_{x_i}\varphi_s\partial_{x_j}y\|_{L^2(\Omega)} \leq \\ \sum_{i,j=1}^{N}\|a_{ij}\|_{L^\infty(\Omega)}\|\varphi_s\|_{W^{1,\infty}(\Omega)}\|\partial_{x_j}y\|_{L^2(\Omega)} &\leq \\ \leq C\left(\|a_{ij}\|_{L^\infty(\Omega)}, \|\varphi_s\|_{W^{1,\infty}(\Omega)}\right)\|y\|_{H^1_0(\Omega)} &\leq \\ C\left(\|a_{ij}\|_{L^\infty(\Omega)}, \|\varphi_s\|_{W^{1,\infty}(\Omega)}\right)\|f_D\|_{H^{-1}(\Omega)} &\leq C\|f_D\|_{W^{-1,p}(\Omega)} \end{aligned} \tag{2.1.18}$$

Let us see that $y_s \in W^{1,p}(\Omega)$ and that the estimate

$$\|y_s\|_{W^{1,p}(\Omega)} \leq C\|f_D\|_{W^{-1,p}(\Omega)}. \tag{2.1.19}$$

holds. In order to prove this, let us introduce some notation. Given $\xi \in W_0^{1,p}(\Omega)$, we define T_ξ as follows

$$\begin{aligned} T_\xi(z) &= \langle f_D\varphi_s, z \rangle + \int_\Omega \sum_{i,j=1}^N a_{ij}(x)y(x)\partial_{x_i}\varphi_s(x)\partial_{x_j}z(x) \\ &+ \int_\Omega \sum_{i,j=1}^N a_{ij}(x)\partial_{x_i}y(x)\partial_{x_j}\varphi_s(x)z(x) \\ &+ \int_{C_\rho^s} \sum_{i,j=1}^N (a_{ij}(x_s) - a_{ij}(x))\,\partial_{x_i}\xi(x)\partial_{x_j}z(x). \end{aligned}$$

It is obvious that $T_\xi \in W^{-1,p}(\Omega)$ and by using Lemma 2.1.5 we deduce the existence and uniqueness of a solution $u_\xi \in W_0^{1,p}(\Omega)$ of the variational equation

$$a_s(y_\xi, z) = T_\xi(z) \ \forall z \in W_0^{1,p'}(\Omega),$$

where $a_s(\cdot,\cdot)$ is the bilinear form associated to the operator A_s. Moreover the following estimate holds

$$\|y_\xi\|_{W_0^{1,p}(\Omega)} \leq C_0\|T_\xi\|_{W^{-1,p}(\Omega)},$$

where C_0 depends on $\|a_{ij}\|_{L^\infty(\Omega)}$, Ω and of p.

Now using this notation and taking into account that the support of φ_s is compact, equation (2.1.15) can be written in variational form as follows

$$a_s(y_s, z) = T_{y_s}(z) \ \forall z \in W_0^{1,p'}(\Omega).$$

The mapping $\xi \mapsto y_\xi$ is contractive. Indeed let us take $\xi_1, \xi_2 \in W_0^{1,p}(\Omega)$ and $y_1 = y_{\xi_1}, y_2 = y_{\xi_2}$. Then the following equality is satisfied

$$a_s(y_1 - y_2, z) = T_{\xi_1}(z) - T_{\xi_2}(z) \ \forall z \in W_0^{1,p'}(\Omega).$$

From here we deduce

$$\|y_1 - y_2\|_{W_0^{1,p}(\Omega)} \leq C_0\|T_{\xi_1} - T_{\xi_2}\|_{W^{-1,p}(\Omega)}. \tag{2.1.20}$$

We have that

$$\begin{aligned}|T_{\xi_1}(z) - T_{\xi_2}(z)| &= \Big|\int_{C^s_\rho} \sum_{i,j=1}^{N} (a_{ij}(x_s) - a_{ij}(x))\, \partial_{x_i}(\xi_1(x) - \xi_2(x))\partial_{x_j} z(x)\Big| \\ &\leq \varepsilon N \|\xi_1 - \xi_2\|_{W_0^{1,p}(\Omega)} \|z\|_{W_0^{1,p'}(\Omega)},\end{aligned} \tag{2.1.21}$$

which implies

$$\|T_{\xi_1} - T_{\xi_2}\|_{W^{-1,p}(\Omega)} \leq \varepsilon N \|\xi_1 - \xi_2\|_{W_0^{1,p}(\Omega)}. \tag{2.1.22}$$

Taking $0 < \varepsilon < \frac{1}{2N} \min\{1, 1/C_0\}$, from (2.1.20) and (2.1.22) we deduce the contractivity of the mapping $\xi \mapsto y_\xi$. Therefore there exists a unique fixed point $\hat{y}$ of this mapping. On the other hand, in $H_0^1(\Omega)$ there is also a unique fixed point, which is necessarily y_s. But $\hat{y} \in W_0^{1,p}(\Omega) \subset H_0^1(\Omega)$ is also a fixed point, and therefore $\hat{y} = y_s$.

Let us see now that the estimate (2.1.19) is satisfied. Using the continuity condition (2.1.13) like in (2.1.21) and the choice of ε, we have that

$$\Big\| \sum_{i,j=1}^{N} \partial_{x_j} \left[(a_{ij}(x_s) - a_{ij}(x)) \partial_{x_i} y_s \right] \Big\|_{W^{-1,p}(\Omega)} \leq \varepsilon N \|y_s\|_{W_0^{1,p}(\Omega)} <$$

$$< \frac{1}{2} \min\{1, 1/C_0\} \|y_s\|_{W_0^{1,p}(\Omega)}.$$

This inequality, together with (2.1.16), (2.1.17) and (2.1.18) leads to

$$\|y_s\|_{W_0^{1,p}(\Omega)} \leq \frac{1}{2} \|y_s\|_{W_0^{1,p}(\Omega)} + C\left(\|a_{ij}\|_{L^\infty(\Omega)}, \|\varphi_s\|_{W^{1,\infty}(\Omega)}, \Omega, p\right) \|f_D\|_{W^{-1,p}(\Omega)}.$$

Let us note that $\|\varphi_s\|_{W^{1,\infty}(\Omega)}$, depends on the size of the support of the function which depends on ρ, and this one depends on the modulus of continuity of the functions a_{ij} and of ε, which, as said before, only depends on Ω, $\|a_{ij}\|_{L^\infty(\Omega)}$, N and p.

Once this estimate is got, adding all the y_s up, we obtain the estimate (2.1.8):

$$\|y\|_{W_0^{1,p}(\Omega)} = \Big\| \sum_{s=1}^{\mu} y_s \Big\|_{W_0^{1,p}(\Omega)} \leq \sum_{s=1}^{\mu} \|y_s\|_{W_0^{1,p}(\Omega)} \leq \mu C \|f_D\|_{W^{-1,p}(\Omega)},$$

where the number μ of functions in the partition of the unity only depends on ρ, and therefore on Ω, $\|a_{ij}\|_{L^\infty(\Omega)}$, N, p and the modulus of continuity of the functions a_{ij}.

Let us suppose now that $p > \frac{2N}{N-2}$ if $N = 3$ or $N = 4$ and

$$\frac{2N}{N-2} \leq p \leq \frac{2N}{N-4}$$

if $N \geq 5$. In this case, all the previous arguments remain valid, except the inequality (2.1.18).Instead of inclusions $L^2(\Omega) \subset W^{-1,p}(\Omega) \subset H^{-1}(\Omega)$ we use now that $L^{\frac{2N}{N-2}}(\Omega) \subset W^{-1,p}(\Omega) \subset W^{-1,\frac{2N}{N-2}}(\Omega)$ and the fact that $u \in W_0^{1,\frac{2N}{N-2}}(\Omega)$, as well as the estimates we have just obtained to get

$$\begin{aligned}
\|\sum_{i,j=1}^{N} a_{ij}(x)\partial_{x_i}\varphi_s\partial_{x_j}y\|_{W^{-1,p}(\Omega)} &\leq \|\sum_{i,j=1}^{N} a_{ij}(x)\partial_{x_i}\varphi_s\partial_{x_j}y\|_{L^{\frac{2N}{N-2}}(\Omega)} \leq \\
\sum_{i,j=1}^{N} \|a_{ij}(x)\|_{L^\infty(\Omega)}\|\varphi_s\|_{W^{1,\infty}(\Omega)}\|\partial_{x_j}y\|_{L^{\frac{2N}{N-2}}(\Omega)} &\leq \\
C(\|a_{ij}\|_{L^\infty(\Omega)}, \|\varphi_s\|_{W^{1,\infty}(\Omega)})\|y\|_{W_0^{1,\frac{2N}{N-2}}(\Omega)} &\leq \\
C(a_{ij}, \|\varphi_s\|_{W^{1,\infty}(\Omega)}, p, N, \Omega)\|f_D\|_{W^{-1,\frac{2N}{N-2}}(\Omega)} &\leq C\|f_D\|_{W^{-1,p}(\Omega)}.
\end{aligned}$$

This process can be repeated taking p greater each time, and the result is proved for $2 \leq p < \infty$. Thus we have already proved that the mapping

$$A : W_0^{1,p}(\Omega) \longrightarrow W^{-1,p}(\Omega)$$

is an isomorphism for $p \geq 2$, therefore its adjoint operator

$$A^* : W_0^{1,p'}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$$

is also an isomorphism. This allows us conclude that the theorem is also valid for $1 < p < 2$. □

Proof of Theorem 2.1.2.

The proof is like the proof of Theorem 2.1.1, with two exceptions. The collection of open sets $\{C_\rho^s\}_{s=1}^{\mu}$ must be taken with Lipschitz boundaries. Moreover, the conditions imposed to p in the theorem imply that $L^2(\Omega) \subset W^{-1,p}(\Omega) \subset H^{-1}(\Omega)$ and there is no need to impose additional conditions to p along the proof. □

Neumann problem. Proof of Theorem 2.1.3

To make this proof we will first get estimates for a problem in the space and in the half space. We will use some of the ideas exposed in Grisvard [59, Section 2.3.2], although his methods can not be straightforward applied.

We will denote by E the fundamental solution for the operator $-\Delta + 1$.

Lemma 2.1.6 *The convolution operator by E is continuous from $W^{k,p}(\mathbb{R}^N)$ to $W^{k+2,p}(R^N)$ for every integer k.*

Proof. It is well known that for every $f \in L^p(\mathbb{R}^N)$, $E*f \in W^{2,p}(\mathbb{R}^N)$ and there exists a constant satisfying

$$\|E * f\|_{W^{2,p}(\mathbb{R}^N)} \leq C\|f\|_{L^p(\mathbb{R}^N)}. \tag{2.1.23}$$

For $k < 0$ the proof is based in two facts: the first is that every $f \in W^{k,p}(\mathbb{R}^N)$ can be written as the sum of derivatives up to the $|k|$-th order of functions f_α of $L^p(\mathbb{R}^N)$:

$$f = \sum_{0\leq|\alpha|\leq|k|} \partial^\alpha f_\alpha$$

and the norm of f in $W^{k,p}(\mathbb{R}^N)$ can be expressed in terms of the norms of the f_α in $L^p(\mathbb{R}^N)$. The second is that $\|\partial^\alpha(E * f_\alpha)\|_{W^{k+2,p}(\mathbb{R}^N)} \leq C\|E * f_\alpha\|_{W^{2,p}(\mathbb{R}^N)}$ for any multiindex α of order less or equal than $|k|$, and thanks to (2.1.23) $\|E * f_\alpha\|_{W^{2,p}(\mathbb{R}^N)} \leq C\|f_\alpha\|_{L^p(\mathbb{R}^N)}$. So we can estimate the $W^{k+2}(\mathbb{R}^N)$-norm of $E * f$ in terms of the $L^p(\mathbb{R}^N)$-norms of the f_α and therefore in terms of the $W^{k,p}(\mathbb{R}^N)$-norm of f.

If $k > 0$ we only have to take into account that for any multiindex $\beta = \alpha + \alpha_2$ with $|\alpha| = k$, $|\alpha_2| = 2$, $\|\partial^\beta(E * f)\|_{L^p(\mathbb{R}^N)} = \|\partial^{\alpha_2}(E * \partial^\alpha f)\|_{L^p(\mathbb{R}^N)}$. By the definition of the norm in $W^{2,p}$, this quantity is less or equal then $\|E * \partial^\alpha f\|_{W^{2,p}(\mathbb{R}^N)}$ and applying (2.1.23), this is less or equal than $C\|\partial^\alpha f\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{W^{k,p}(\mathbb{R}^N)}$. □

Corollary 2.1.7 *Let $\mathcal{A} = (a_{ij})$ be a positive definite matrix of real entries, $\lambda > 0$ and $f \in \left(W^{1,p'}(\mathbb{R}^N)\right)' = W^{-1,p}(\mathbb{R}^N)$. Then there exists a unique solution $y \in W^{1,p}(\mathbb{R}^N)$ of the equation*

$$-\sum_{i,j=1}^{N} \partial_{x_j}(a_{ij}\partial_{x_i}y) + \lambda y = f \text{ in } \mathbb{R}^N \tag{2.1.24}$$

Moreover, the estimate

$$\|y\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|f\|_{\left(W^{1,p'}(\mathbb{R}^N)\right)'}$$

holds for some C depending on the coefficients of the operator, N and p.

Proof. If we rename $f = f/\lambda$ and $b_{ij} = (a_{ij} + a_{ji})/(2\lambda)$, then (2.1.24) can be written

$$-\sum_{i,j=1}^{N} \partial_{x_j}(b_{ij}\partial_{x_i}y) + y = f \text{ in } \mathbb{R}^N \tag{2.1.25}$$

Since $\mathcal{B} = (b_{ij})$ is a symmetric positive definite matrix, there exists P regular such that $\mathcal{B} = PP^T$. We make the change of variable $x = \hat{x}$ and we define $\hat{y} = y \circ P$ and $\hat{f} = f \circ P$, so (2.1.25) can be written

$$-\Delta \hat{y} + \hat{y} = \hat{f} \text{ in } \mathbb{R}^N. \tag{2.1.26}$$

Since E is the fundamental solution of the operator $-\Delta + 1$, then $\hat{y} = E * \hat{f} \in W^{1,p}(\mathbb{R}^N)$ is the unique solution of (2.1.26) and

$$\|\hat{y}\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|\hat{f}\|_{(W^{1,p'}(\mathbb{R}^N))'}.$$

Uniqueness can be deduced by means of Fourier transform or taking into account the density of the space $W^{1,p}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ in $W^{1,p}(\mathbb{R}^N)$.

Undoing the change of variable, we get that $y \in W^{1,p}(\mathbb{R}^N)$ is the unique solution of (2.1.24) and

$$\|y\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|f\|_{(W^{1,p'}(\mathbb{R}^N))'},$$

where C depends on p, N y (a_{ij}). □

Now we are going to get some estimates in the half space. Let us start with problems involving only Laplace operator.

We shall introduce some notation, following Grisvard [59, pp 97–105]. For every function f defined in $\mathbb{R}^N_+$, $\tilde{f}$ is its extension by zero to the whole space.

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^N_+ \\ 0 & \text{else.} \end{cases}$$

With δ_N we denote Dirac's measure on the variable x_N and δ'_N its derivative in the distribution sense in $\mathbb{R}$. For any $s > 1/p$ and $p > 1$, the mapping

$$\gamma_N : W^{s,p}(\mathbb{R}^N_+) \to W^{s-1/p,p}(\mathbb{R}^{N-1})$$

denotes the trace operator on the x_N axis. For $g \in W^{s,p}(\mathbb{R}^{N-1})$, $s < 0$, we define

$$g \otimes \delta_N \in \left(W^{1/p'-s,p'}(\mathbb{R}^N)\right)'$$

by

$$\langle g \otimes \delta_N, u\rangle = \langle g, \gamma_N u\rangle.$$

Let $F\varphi$ stand for the partial Fourier transform of φ in $x_1, \dots, x_{N-1}$.

$$F\varphi = \frac{1}{(2\pi)^{\frac{N-1}{2}}} \int_{\mathbb{R}^{N-1}} e^{-i\xi x'} \varphi(x') dx'.$$

Lemma 2.1.8 *For $f \in \left(W^{1,p'}(\mathbb{R}^N_+)\right)'$ there exists a unique variational solution $y \in W^{1,p}(\mathbb{R}^N_+)$ of Neumann problem*

$$\begin{cases} -\Delta y + y &= f \quad \text{in } \mathbb{R}^N_+ \\ \partial_{x_N} y &= 0 \quad \text{on } \mathbb{R}^{N-1} \times \{0\}. \end{cases} \tag{2.1.27}$$

Moreover, the following estimate is satisfied:

$$\|y\|_{W^{1,p}(\mathbb{R}^N_+)} \le C \|f\|_{\left(W^{1,p'}(\mathbb{R}^N_+)\right)'},$$

where C depends on N and p.

Remark 2.1.2 *Remember that all the time we are talking about the solution of a variational problem, and that the writing of the problem as a partial differential equation is just symbolic, and allows us to keep a link in the notation with Dirichlet's case.*

Proof. Let us take a sequence of functions f_k in $\mathcal{D}(\overline{\mathbb{R}^N_+})$ converging to f in $\left(W^{1,p'}(\mathbb{R}^N_+)\right)'$ and set

$$w_k = E * \tilde{f}_k.$$

We have that $w_k \in W^{1,p}(\mathbb{R}^N)$ and

$$\|w\|_{W^{1,p}(\mathbb{R}^N)} \le C \|\tilde{f}_k\|_{W^{-1,p}(\mathbb{R}^N)} = C \|f_k\|_{\left(W^{1,p'}(\mathbb{R}^N_+)\right)'}.$$

Now let us define, for $x_N > 0$

$$y_k(x', x_N) = w_k(x', x_n) + w_k(x', -x_N).$$

Clearly in $\mathbb{R}^N_+$

$$-\Delta y_k + y_k = [-\Delta w_k(x', x_N) + w_k(x', x_N)] + [-\Delta w_k(x', -x_N) + w_k(x', -x_N)] =$$

$$= f_k + 0 = f_k \tag{2.1.28}$$

and since $w_k \in W^{2,p}(R^N_+)$ we can write

$$\partial_{x_N} y_k(x', 0) = \partial_{x_N} w_k(x', 0) - \partial_{x_N} w_k(x', 0) = 0. \tag{2.1.29}$$

Now (2.1.28) and (2.1.29) lead to

$$\int_{\mathbb{R}^N_+} (\nabla y_k \nabla z + y_k z) =< f_k, z > \quad \forall z \in W^{1,p'}(\mathbb{R}^N_+). \tag{2.1.30}$$

Moreover

$$\|y_k\|_{W^{1,p}(\mathbb{R}^N_+)} \leq \|w_k\|_{W^{1,p}(\mathbb{R}^N_+)} + \|w_k\|_{W^{1,p}(\mathbb{R}^N_-)} = \|w_k\|_{W^{1,p}(\mathbb{R}^N)}$$

and hence

$$\|y_k\|_{W^{1,p}(\mathbb{R}^N_+)} \leq C\|f_k\|_{(W^{1,p'}(\mathbb{R}^N_+))'}.$$

From the continuity of the convolution, we deduce that $w_k = E * \tilde{f}_k \to w = E * \tilde{f}$ in $W^{1,p}(\mathbb{R}^N)$, and consequently, $y_k \to y$ in $W^{1,p}(\mathbb{R}^N_+)$, with $y(x', x_N) = w(x', x_N) + w(x', -x_N)$. Now it is easy to pass to the limit in (2.1.30) to deduce that y is the variational solution of (2.1.27).

Uniqueness comes from the density of $W^{1,p}(\mathbb{R}^N_+) \cap H^1(\mathbb{R}^N_+)$ in $W^{1,p}(\mathbb{R}^N_+)$. □

We give now a key result to deal with Neumann's problem when the coefficient matrix is non symmetric. It is a result for a problem with oblique derivative. The same problem has been considered by Grisvard in [59], where he proved $W^{2,p}$-regularity of the solution for a more regular datum.

Lemma 2.1.9 *For $g \in W^{-1/p,p}(\mathbb{R}^{N-1})$ and $m_1, \ldots, m_N \in \mathbb{R}$, $m_N \neq 0$, there exists a unique variational solution $y \in W^{1,p}(\mathbb{R}^N_+)$ of the problem*

$$\left\{ \begin{array}{rcll} -\Delta y + y &=& 0 & \text{in } \mathbb{R}^N_+ \\ \displaystyle\sum_{j=1}^{N} m_j \partial_{x_j} y &=& g & \text{on } \mathbb{R}^{N-1} \times \{0\} \end{array} \right. \tag{2.1.31}$$

Moreover, the following estimate is satisfied:

$$\|y\|_{W^{1,p}(\mathbb{R}^N_+)} \leq C\|g\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}$$

for some constant C depending on N, p and the coefficients m_j.

Proof. Notice that for functions in $W^{1,p}(\mathbb{R}^N_+)$, and $1 \leq j \leq N-1$, $\partial_{x_j} y(x', 0) = (\partial_{x_j} \gamma_N y)(x') \in W^{-1/p,p}(\mathbb{R}^{N-1})$. Therefore the variational solution of (2.1.31) is the solution of the variational problem

$$\int_{R^N_+} (\nabla y \nabla z + yz) + \sum_{j=1}^{N-1} \frac{m_j}{m_N} < \partial_{x_j} y, \gamma_N z >= \frac{1}{m_N} < g, \gamma_N z > \quad \forall z \in W^{1,p'}(\mathbb{R}^N_+).$$

We are going to adapt some of the ideas in Grisvard [59]. For that purpose we take a sequence of functions $g_n \in W^{1-1/p,p}(\mathbb{R}^{N-1})$ with $g_n \to g$ in $W^{-1/p,p}(R^{N-1})$. Let us study the variational equations

$$\int_{R_+^N} (\nabla y_n \nabla z + y_n z) + \sum_{j=1}^{N-1} \frac{m_j}{m_N} < \partial_{x_j} y_n, \gamma_N z >= \frac{1}{m_N} < g_n, \gamma_N z > \quad \forall z \in W^{1,p'}(\mathbb{R}_+^N). \tag{2.1.32}$$

These equations can be written as

$$\begin{cases} -\Delta y_n + y_n &= 0 \quad \text{in } \mathbb{R}_+^N \\ \displaystyle\sum_{j=1}^{N} m_j \partial_{x_j} y_n &= g_n \quad \text{on } \mathbb{R}^{N-1} \times \{0\}. \end{cases}$$

Thanks to Grisvard [59], we know that each of these equations has a unique solution $y_n \in W^{2,p}(\mathbb{R}_+^N)$, and that it can be explicitly represented by means of Fourier transforms as

$$y_n = -E * (k_0^n \otimes \delta_N' + k_1^n \otimes \delta_N), \tag{2.1.33}$$

where

$$k_0^n = F^{-1} b F g_n,$$

$$k_1^n = F^{-1} p_- b F g_n,$$

$$b = \left(m_N p_- + \sum_{j=1}^{N-1} i m_j \xi_j \right)^{-1}$$

and

$$p_- = -i\sqrt{1 + \|\xi\|^2}.$$

We want an estimate of the $W^{1,p}(\mathbb{R}_+^N)$-norm of y_n in terms of the $W^{-1/p,p}(\mathbb{R}^{N-1})$-norm of g_n, so that we can take the limit in (2.1.32).

Lemma 2.3.2.5 in Grisvard [59] implies that

$$k_1^n \in W^{-1/p,p}(\mathbb{R}^{N-1})$$

and

$$\|k_1^n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})} \le C \|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}. \tag{2.1.34}$$

Applying Lemma 2.3.2.2 in Grisvard [59], with $s = -1/p$, we get that

$$k_1^n \otimes \delta_N \in W^{-1,p}(\mathbb{R}^N)$$

and

$$\|k_1^n \otimes \delta_N\|_{W^{-1,p}(\mathbb{R}^N)} \leq C\|k_1^n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}. \tag{2.1.35}$$

Lemma 2.1.6 implies that

$$E * (k_1^n \otimes \delta_N) \in W^{1,p}(\mathbb{R}^N)$$

and that

$$\|E * (k_1^n \otimes \delta_N)\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|k_1^n \otimes \delta_N\|_{W^{-1,p}(\mathbb{R}^N)}. \tag{2.1.36}$$

So putting together (2.1.34), (2.1.35) and (2.1.36) we have that

$$\|E * (k_1^n \otimes \delta_N)\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}. \tag{2.1.37}$$

In the same way, using Lemmas 2.3.2.5 and 2.3.2.2 in Grisvard [59] we have that

$$k_0^n \in W^{-1/p,p}(\mathbb{R}^{N-1}),$$

$$\|k_0^n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})} \leq C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})},$$

$$k_0^n \otimes \delta_N \in W^{-1,p}(\mathbb{R}^N)$$

and

$$\|k_0^n \otimes \delta_N\|_{W^{-1,p}(\mathbb{R}^N)} \leq C\|k_0^n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}. \tag{2.1.38}$$

Following again the same method than for k_1^n, we get

$$E * (k_0^n \otimes \delta_N) \in W^{1,p}(\mathbb{R}^N)$$

and

$$\|E * (k_0^n \otimes \delta_N)\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}.$$

Therefore

$$\partial_{x_N}[E * (k_0^n \otimes \delta_N)] \in L^p(\mathbb{R}^N)$$

and

$$\|\partial_{x_N}[E * (k_0^n \otimes \delta_N)]\|_{L^p(\mathbb{R}^N)} \leq C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}.$$

But

$$\partial_{x_N}[E * (k_0^n \otimes \delta_N)] = E * (k_0^n \otimes \delta_N'), \tag{2.1.39}$$

so

$$E * (k_0^n \otimes \delta_N') \in L^p(\mathbb{R}^N)$$

and

$$\|E * (k_0^n \otimes \delta_N')\|_{L^p(\mathbb{R}^N)} \leq C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}. \tag{2.1.40}$$

To see that $E * (k_0^n \otimes \delta_N') \in W^{1,p}(\mathbb{R}_+^N)$, we just have to prove that its derivatives belong to $L^p(\mathbb{R}_+^N)$. For $1 \leq j \leq N-1$ we can write

$$\partial_{x_j} k_0^n = F^{-1} i\xi_j b F g_n,$$

and then, using Lemmas 2.3.2.5 and 2.3.2.2 in Grisvard and Lemma 2.1.6 we have that

$$\partial_{x_j} k_0^n \in W^{-1/p,p}(\mathbb{R}^{N-1}),$$

$$\|\partial_{x_j} k_0^n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})} \leq C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})},$$

$$\partial_{x_j} k_0^n \otimes \delta_N \in W^{-1,p}(\mathbb{R}^N),$$

$$\|\partial_{x_j} k_0^n \otimes \delta_N\|_{W^{-1,p}(\mathbb{R}^N)} \leq C\|\partial_{x_j} k_0^n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})},$$

$$E * (\partial_{x_j} k_0^n \otimes \delta_N) \in W^{1,p}(\mathbb{R}^N) \tag{2.1.41}$$

and

$$\|E * (\partial_{x_j} k_0^n \otimes \delta_N)\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}.$$

And therefore we have that

$$\partial_{x_j} [E * (k_0^n \otimes \delta_N')] \in L^p(\mathbb{R}^N)$$

and

$$\|\partial_{x_j} [E * (k_0^n \otimes \delta_N')]\|_{L^p(\mathbb{R}^N)} \leq C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}. \tag{2.1.42}$$

To get $\partial_{x_N} [E * (k_0^n \otimes \delta_N')] \in L^p(\mathbb{R}_+^N)$ and an estimate of its norm in terms of the norm of g_n in $W^{-1/p,p}(\mathbb{R}^{N-1})$, we can write

$$\partial_{x_N} [E * (k_0^n \otimes \delta_N')] = \partial_{x_N^2}^2 [E * (k_0^n \otimes \delta_N)] = E*(k_0^n \otimes \delta_N) - \sum_{j=1}^{N-1} \partial_{x_j^2}^2 [E * (k_0^n \otimes \delta_N)] \text{ in } \mathbb{R}_+^N$$

since E is an elementary solution of $-\Delta + 1$ and $k_0^n \otimes \delta_N$ is a distribution with support on $R^{N-1} \times \{0\}$. We already know that $E * (k_0^n \otimes \delta_N) \in L^p(R^N)$ and an estimate of its norm in terms of $\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}$ (indeed, we know that it belongs to $W^{1,p}(\mathbb{R}^N)$). Taking into account (2.1.41) and writing for $1 \leq j \leq N-1$

$$\partial_{x_j^2}^2 [E * (k_0^n \otimes \delta_N)] = \partial_{x_j} \left[E * (\partial_{x_j} k_0^n \otimes \delta_N)\right],$$

we have that

$$\partial^2_{x_j^2}\left[E * (k_0^n \otimes \delta_N)\right] \in L^p(\mathbb{R}^N)$$

and

$$\|\partial^2_{x_j^2}\left[E * (k_0^n \otimes \delta_N)\right]\|_{L^p(\mathbb{R}^N)} \le C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}.$$

So finally we have that

$$\partial_{x_N}\left[E * (k_0^n \otimes \delta'_N)\right] \in L^p(\mathbb{R}^N_+)$$

and

$$\|\partial_{x_N}\left[E * (k_0^n \otimes \delta'_N)\right]\|_{L^p(\mathbb{R}^N_+)} \le C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}. \tag{2.1.43}$$

Putting together (2.1.40), (2.1.42) and (2.1.43), we have that

$$E * (k_0^n \otimes \delta'_N) \in W^{1,p}(\mathbb{R}^N_+)$$

y

$$\|E * (k_0^n \otimes \delta'_N)\|_{W^{1,p}(\mathbb{R}^N_+)} \le C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}. \tag{2.1.44}$$

Now from (2.1.33), (2.1.37) and (2.1.44), we deduce that

$$\|y_n\|_{W^{1,p}(\mathbb{R}^N_+)} \le C\|g_n\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}.$$

Now we can take y the limit of y_n in $W^{1,p}(\mathbb{R}^N_+)$, and pass to the limit in equation (2.1.32). Thus we obtain that y is a variational solution of our problem.

Uniqueness follows again from the density of $W^{1,p}(\mathbb{R}^N_+) \cap H^1(\mathbb{R}^N_+)$ in $W^{1,p}(\mathbb{R}^N_+)$. □

Corollary 2.1.10 *For $f \in \left(W^{1,p'}(\mathbb{R}^N_+)\right)'$, $g \in W^{-1/p,p}(\mathbb{R}^{N-1})$ and $m_1, \dots, m_N \in \mathbb{R}$, $m_N \neq 0$, there exists a unique variational solution $y \in W^{1,p}(\mathbb{R}^N_+)$ of the problem*

$$\begin{cases} -\Delta y + y = f & \text{in } \mathbb{R}^N_+ \\ \displaystyle\sum_{j=1}^N m_j \partial_{x_j} y = g & \text{on } \mathbb{R}^{N-1} \times \{0\}. \end{cases}$$

Moreover, the following estimate is satisfied:

$$\|y\|_{W^{1,p}(\mathbb{R}^N_+)} \le C\left(\|f\|_{(W^{1,p'}(\mathbb{R}^N_+))'} + \|g\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}\right),$$

where C depends on N, p and the coefficients m_j.

Proof. Thanks to Lemma 2.1.8,we know that there exists a unique variational solution $v \in W^{1,p}(\mathbb{R}^N_+)$ of

$$\begin{cases} -\Delta v + v &= f \quad \text{in } \mathbb{R}^N_+ \\ \partial_{x_N} v &= 0 \quad \text{on } \mathbb{R}^{N-1} \times \{0\}. \end{cases}$$

This function satisfies

$$\|v\|_{W^{1,p}(\mathbb{R}^N_+)} \leq C\|f\|_{(W^{1,p'}(\mathbb{R}^N_+))'} \tag{2.1.45}$$

Then we have that $\gamma_N v \in W^{1-1/p,p}(\mathbb{R}^{N-1})$, and for $1 \leq j \leq N-1$, $\partial_{x_j}\gamma_N v \in W^{-1/p,p}(\mathbb{R}^{N-1})$ and

$$\|\partial_{x_j}\gamma_N v\|_{W^{-1/p,p}(\mathbb{R}^{N-1})} \leq \|v\|_{W^{1,p}(\mathbb{R}^N_+)}. \tag{2.1.46}$$

Thanks to Lemma 2.1.9 we can solve the problem

$$\begin{cases} -\Delta w + w &= 0 \quad \text{in } \mathbb{R}^N_+ \\ \sum_{j=1}^{N} m_j \partial_{x_j} w &= g - \sum_{j=1}^{N-1} m_j \partial_{x_j}(\gamma_N v) \quad \text{on } \mathbb{R}^{N-1} \times \{0\} \end{cases}$$

We have that $w \in W^{1,p}(\mathbb{R}^N_+)$ and that

$$\|w\|_{W^{1,p}(\mathbb{R}^N_+)} \leq C\left(\|g\|_{W^{-1/p,p}(\mathbb{R}^{N-1})} + \|\sum_{j=1}^{N-1} m_j\partial_{x_j}\gamma_N v\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}\right).$$

Using this inequality with (2.1.46) and (2.1.45), we get

$$\|w\|_{W^{1,p}(\mathbb{R}^N_+)} \leq C\left(\|f\|_{(W^{1,p'}(\mathbb{R}^N_+))'} + \|g\|_{W^{-1/p,p}(\mathbb{R}^{N-1})}\right). \tag{2.1.47}$$

We have that $y = v + w \in W^{1,p}(\mathbb{R}^N_+)$ is the solution of our problem, and from (2.1.45) and (2.1.47) it is easily deduced that the required estimate is satisfied. □

Corollary 2.1.11 *Let $\mathcal{A} = (a_{ij})$ be a positive definite matrix of real entries, $\lambda > 0$ and $f \in (W^{1,p'}(\mathbb{R}^N_+))'$. Then, there exists a unique solution $y \in W^{1,p}(\mathbb{R}^N_+)$ of the variational equality*

$$\sum_{i,j=1}^{N} a_{ij}\int_{\mathbb{R}^N_+}\partial_{x_i}y\partial_{x_j}z + \lambda\int_{\mathbb{R}^N_+} yz = < f, z > \quad \forall z \in W^{1,p'}(\mathbb{R}^N_+). \tag{2.1.48}$$

Moreover, the estimate

$$\|y\|_{W^{1,p}(\mathbb{R}^N_+)} \leq C\|f\|_{(W^{1,p'}(\mathbb{R}^N_+))'}, \tag{2.1.49}$$

holds, where C is a constant depending only on p, m, M, λ and N.

Proof. If we call $\mathcal{B} = (b_{ij})$, $b_{ij} = (a_{ij} + a_{ji})/(2\lambda)$, and we rename $f = f/\lambda$, then our equation can formally be written

$$\left\{\begin{array}{rcll} -\sum_{i,j=1}^{N} \partial_{x_j}(b_{ij}\partial_{x_i}y) + y & = & f & \text{in } \mathbb{R}^N_+ \\ \nabla^T y \mathcal{A} n & = & 0 & \text{on } \mathbb{R}^{N-1}\times\{0\}, \end{array}\right. \tag{2.1.50}$$

where $\nu = (0,\ldots,0,1)^T$. Notice that $\mathcal{A}\nu$ does not belong to $\mathbb{R}^{N-1}\times\{0\}$.

The matrix $\mathcal{B}$ is symmetric and positive definite, so there exists a regular matrix P such that $\mathcal{B} = PP^T$. If we write $T = P^{-1}$ and make the change of variable $\hat{x} = Tx$, then (2.1.50) in transformed into

$$\left\{\begin{array}{rcll} -\Delta\hat{y} + \hat{y} & = & \hat{f} & \text{in } T\mathbb{R}^N_+ \\ \nabla^T \hat{y} T\mathcal{A} n & = & 0 & \text{on } T(\mathbb{R}^{N-1}\times\{0\}), \end{array}\right.$$

where $\hat{y} = y \circ P$ and $\hat{f} = f \circ P$. Notice again that since T is regular $T\mathcal{A}\nu \notin T(\mathbb{R}^{N-1}\times\{0\})$.

Let us take an orthogonal matrix Q such that $QT(\mathbb{R}^{N-1}\times\{0\}) = \mathbb{R}^{N-1}\times\{0\}$ and $QT\mathbb{R}^N_+ = \mathbb{R}^N_+$. If we call $\tilde{x} = Q\hat{x}$, $\tilde{y} = \hat{y}\circ Q^{-1}$ and $\tilde{f} = \hat{f}\circ Q^{-1}$, we get the equation

$$\left\{\begin{array}{rcll} -\Delta\tilde{y} + \tilde{y} & = & \tilde{f} & \text{in } \mathbb{R}^N_+ \\ \nabla^T \tilde{y} QT\mathcal{A} n & = & 0 & \text{on } \mathbb{R}^{N-1}\times\{0\}. \end{array}\right. \tag{2.1.51}$$

Again since Q is regular $QT\mathcal{A}\nu \notin \mathbb{R}^{N-1}\times\{0\}$. If we call $m = QT\mathcal{A}\nu$, this means that $m_N \neq 0$ and we are under the conditions of Corollary 2.1.10. Therefore there exists a unique variational solution $\tilde{y} \in W^{1,p}(\mathbb{R}^N_+)$ and

$$\|\tilde{y}\|_{W^{1,p}(\mathbb{R}^N_+)} \le C\|\tilde{f}\|_{(W^{1,p'}(\mathbb{R}^N_+))'}$$

Undoing the changes of variable, we get that there exists a unique variational solution $y \in W^{1,p}(\mathbb{R}^N_+)$ of (2.1.48) and it satisfies the estimate (2.1.49). □

Remark 2.1.3 *Let us note that the boundary condition of* (2.1.51) *is reduced to* $\partial_{x_N}\tilde{y} = 0$ *on* $\mathbb{R}^{N-1}\times\{0\}$ *whenever the matrix* $\mathcal{A} = (a_{ij})$ *is symmetric. In such a case Lemma 2.1.9 is not needed to establish 2.1.11, the proof being much simpler and carried out just by applying Lemma 2.1.8.*

Now we are ready to prove Theorem 2.1.3. In what follows we shall denote $f_N = f + g \circ \gamma$ and we have that $f_N \in \left(W^{1,p'}(\Omega)\right)'$. Then (2.1.9) can be written in the following way.

$$a(y,z) = \langle f_N, z\rangle \quad \forall z \in W^{1,p'}(\Omega). \tag{2.1.52}$$

We shall use a result analogous to Lemma 2.1.4; see Troianiello [87] and Stampacchia [84] for the proof.

Lemma 2.1.12 *Let us suppose $p \geq 2$. Then there exists a unique variational solution $y \in H^1(\Omega) \cap L^{p^*}(\Omega)$ satisfying the equation* (2.1.4). *Moreover, the estimate*

$$\|y\|_{L^{p^*}(\Omega)} \leq C\|f_N\|_{\left(W^{1,p'}(\Omega)\right)'}, \tag{2.1.53}$$

holds, where C is a constant which only depends on p, the dimension N, m, M and the measure of Ω. Notice that obviously also

$$\|y\|_{L^p(\Omega)} \leq C\|f_N\|_{\left(W^{1,p'}(\Omega)\right)'}.$$

Proof of Theorem 2.1.3. First let us consider the case $2 \leq p < +\infty$ if $N = 2$ and $2 \leq p \leq 2N/(N-2)$ if $N \geq 3$. Let $y \in H^1(\Omega) \cap L^p(\Omega)$ be as in Lemma 2.1.12. The plan of the proof is as follows

1. We take a collection of coordinate systems of Γ and a subdomain of Ω, as well as a partition of unity relative to this collection. Then equation (2.1.52) is studied on each of these domains.

2. A change of variables is made in order to have a problem with continuous coefficients in a rectangle. Furthermore we know that the support of the solution intersects at most one of the sides of the rectangle and it is "far away" from the others.

3. We "freeze" the coefficients, so that we have a problem with constant coefficients in a rectangle. The support of the solution may be either in the interior of the rectangle or just intersecting one side as before.

4. We extend the problem to the whole space or to the half-space and solve it.

Step 1.

Since the boundary of Ω is of class C^1, there exist (cf: Nečas [72]) numbers $\alpha > 0$, $\beta > 0$, coordinate systems $(x_{k1}, x_{k2}, \ldots, x_{kN})$, shortly (x'_k, x_{kN}), $k = 1, 2, \ldots, \Lambda$, and functions b_k of class C^1 in the $N-1$ dimensional closed cubes $|x_{ki}| \leq \alpha$, $i = 1, 2, \ldots, N-1$, in such a way that each point x in Γ may be represented at least in one of these systems like $x = (x'_k, b_k\,(x'_k))$. It is also supposed that the points (x'_k, x_{kN}) such that $x'_k \in [-\alpha, \alpha]^{N-1}$, $b_k(x'_k) < x_{kN} < b_k(x'_k) + \beta$ are in Ω, while the points (x'_k, x_{kN}) such that $x'_k \in [-\alpha, \alpha]^{N-1}$, $b_k(x'_k) - \beta < x_{kN} < b_k(x'_k)$ are out of Ω.

For each $k = 1, 2, \ldots, \Lambda$ let us denote

$$G_k = \left\{(x'_k, b_k(x'_k) + t),\ x'_k \in (-\alpha, \alpha)^{N-1},\ 0 < t < \beta\right\},$$

and let us take an open set $G_{\Lambda+1} \subset \bar{G}_{\Lambda+1} \subset \Omega$ such that $\{G_1, \ldots, G_\Lambda, G_{\Lambda+1}\}$ is a covering by open sets of the closure of Ω. We also choose $\{\psi_1, \ldots, \psi_\Lambda, \psi_{\Lambda+1}\}$ a partition of unity relative to this covering.

Taking

$$y_k = \psi_k y$$

and

$$< f_k, z > = < \psi_k f_N, z > - \int_{G_k} \sum_{i,j=1}^{N} z a_{ij} \partial_{x_i} y \partial_{x_j} \psi_k + \int_{G_k} \sum_{i,j=1}^{N} a_{ij} y \partial_{x_i} \psi_k \partial_{x_j} z \quad \forall z \in W^{1,p'}(G_k)$$

it is easy to check that y_k verifies the equation

$$\int_{G_k} \sum_{i,j=1}^{N} a_{ij} \partial_{x_i} y_k \partial_{x_j} z + \int_{G_k} y_k z = < f_k, z > \qquad \forall z \in W^{1,p'}(G_k).$$

Using Lemma 2.1.12, assumptions on p established above and arguing as in relations (2.1.16)–(2.1.18), we get that $f_k \in \left(W^{1,p'}(G_k)\right)'$.

Notice that the support of both y_k and f_k are "far away" of the part of the boundary of G_k which does not intersect Γ.

Step 2.

Now we are going to make a change of variable in order to transform the domain G_k in a rectangle. For $k = 1, 2, \ldots, \Lambda$ let us define $J_k : G_k \longrightarrow \mathcal{R} = (-\alpha, +\alpha)^{N-1} \times (0, \beta)$ by

$$y = J_k(x) = (x', -b_k(x') + x_n).$$

J_k is a C^1 diffeomorphism. The function $z_k(\tilde{x}) = y_k(\tilde{x}', b_k(\tilde{x}') + \tilde{x}_N)$ satisfies the variational equation

$$a_k(z_k, z) =< f^k_{\mathcal{R}}, z > \quad \forall z \in W^{1,p'}(\mathcal{R}),$$

where $f^k_{\mathcal{R}} \in \left(W^{1,p'}(\mathcal{R})\right)'$ is the transformed of f_k by the change of variable and

$$a_k(z_k, z) = \int_{\mathcal{R}} \nabla z_k (DJ_k)\hat{A}(DJ_k)^T \nabla z^T |\mathrm{Jac} J_k^{-1}| + \int_{\mathcal{R}} z_k z |\mathrm{Jac} J_k^{-1}|,$$

where $\hat{A}$ is the matrix (a_{ij}).

The bilinear form a_k has continuous coefficients and it is coercive in $H^1(\mathcal{R})$. We shall denote the coefficients of a_k by a^k_{ij} and a^k_0. By construction we know that $z_k \in H^1(\mathcal{R}) \cap L^p(\mathcal{R})$ and that its support intersects one of the sides of $\mathcal{R}$ and it is "far away" from the others. Let us prove that $z_k \in W^{1,p}(\mathcal{R})$ and that the estimate

$$\|z_k\|_{W^{1,p}(\mathcal{R})} \le C \|f^k_{\mathcal{R}}\|_{\left(W^{1,p'}(\mathcal{R})\right)'}. \tag{2.1.54}$$

For $G_{\Lambda+1}$ we do not need to make any change of variable. In this case the support of $u_{\Lambda+1}$ is in $G_{\Lambda+1}$.

Step 3.

This part of the proof is analogous to that of Dirichlet's case. Using (2.1.13), we take again a covering by open sets of diameter less or equal than ρ, $\left\{C^{k,s}_\rho\right\}^{\mu}_{s=1}$. These sets are squares for $k = 1, 2, \ldots, \Lambda$ or have a C^∞ boundary for $k = \Lambda + 1$. We choose a point $x_{k,s} \in C^{k,s}_\rho$. We also take a partition of unity relative to that covering $\{\varphi_{k,s}\}^{\mu}_{s=1}$. We take $z_{k,s}$ for $1 \le s \le \mu$ like in (2.1.14) and so we have that $z_{k,s}$ satisfies the variational equation

$$a_{k,s}(z_{k,s}, z) = T^N_{z_{k,s}}(z) \ \forall z \in W^{1,p'}(\mathcal{R})$$

where

$$a_{k,s}(z, v) = \sum_{i,j=1}^{N} a^k_{ij}(x_{k,s}) \int_{\mathcal{R}} \partial_{x_i} z \partial_{x_j} v + a^k_0(x_{k,s}) \int_{\mathcal{R}} zv$$

and

$$T^N_\xi(z) = < f_N, \varphi_{k,s} z > + \int_{\mathcal{R}} \sum_{i,j=1}^{n} a^k_{ij}(x) z(x) \partial_{x_i} \varphi_{k,s}(x) \partial_{x_j} z(x) +$$

$$\int_{\mathcal{R}} \sum_{i,j=1}^{n} a^k_{ij}(x) \partial_{x_i} z_k(x) \partial_{x_j} \varphi_{k,s}(x) z(x) +$$

$$\int_{C_\rho^{k,s}} \sum_{i,j=1}^{n} \left(a_{ij}^k(x_{k,s}) - a_{ij}(x)\right) \partial_{x_i}\xi(x)\partial_{x_j}z(x) +$$
$$\int_{C_\rho^{k,s}} \left(a_0^k(x_{k,s}) - a_0^k(x)\right) \xi z,$$

for any $\xi \in W^{1,p}(\mathcal{R})$. For $k = \Lambda+1$ the previous relations hold by replacing $\mathcal{R}$ for $G_{\Lambda+1}$.

Step 4.

Notice that thanks to the properties of the supports of z_k and $\varphi_{k,s}$, only two cases can appear:

- First case: the support of $z_{k,s}$ is inside $C_\rho^{k,s}$.
- Second case: the support of $z_{k,s}$ intersects one side of $C_\rho^{k,s}$ and is "far away" from the others.

Taking $E = \mathbb{R}^N$ in the first case and $E = \mathbb{R}_+^N$ in the second one, we have that $z_{k,s} \in H^1(E) \cap L^p(E)$ and satisfies the following variational equality

$$\tilde{a}_{k,s}(z_{k,s}, z) = T_{z_{k,s}}^N(z)\ \forall z \in W^{1,p'}(E),$$

where

$$\tilde{a}_{k,s}(z,v) = \sum_{i,j=1}^{N} a_{ij}^k(x_{k,s}) \int_E \partial_{x_i}z\partial_{x_j}v + a_0^k(x_{k,s}) \int_E zv.$$

Using Corollaries 2.1.7 y 2.1.11 we deduce the existence of a unique solution $z_\xi \in W^{1,p}(E)$ of

$$\tilde{a}_{k,s}(z_\xi, z) = T_\xi^N(z)\ \forall z \in W^{1,p'}(E)$$

for every $\xi \in W^{1,p}(E)$. As in the proof of Theorem 2.1.1 we can show the contractivity of the mapping $\xi \to z_\xi$ for ρ small enough. Therefore there exists a unique fixed point of this mapping, which is $z_{k,s}$. So we have $z_{k,s} \in W^{1,p}(\mathcal{R})$ and $z_{k,s}$ satisfies estimate (2.1.54).

So the proof can be concluded adding up all the $z_{k,s}$, undoing the change of variable, and adding up all the y_k.

Once again, arguing as in the proof of Theorem 2.1.1, the result can be extended for all $p > 2N/(N-2)$ and by duality to every $1 < p < 2$. □

2.2 Parabolic equations

In this section we study the regularity in $L^\tau(0,T;W^{1+\varepsilon,p}(\Omega))$, $\varepsilon \geq 0$ of the solution of a parabolic problem with Neumann boundary condition. The purpose is to deduce regularity $L^\tau(0,T;W^{1+\varepsilon,p}(\Omega))$ of the solution under minimal assumptions on the regularity of the coefficients of the main part of the operator and on the boundary of the domain. As in the elliptic case, continuous coefficients and C^1 boundary are enough for this regularity if $\varepsilon = 0$. If $\varepsilon > 0$, Hölder continuous coefficients and a $C^{1+\hat{\varepsilon}}$ boundary will be needed.

Introduction

Let Ω be an open, bounded, and connected set of $\mathbb{R}^N$. Again we will denote Γ the boundary of Ω. Let T be a positive real number. Let us take $Q = \Omega\times]0,T[$ and $\Sigma = \Gamma\times]0,T[$. We introduce the elliptic operator

$$Ay = -\sum_{i,j=1}^{N} \partial_{x_j}\left(a_{ij}(x,t)\partial_{x_i}y\right).$$

The purpose of this section is to study regularity results in $L^\tau(0,T;W^{1+\varepsilon,p}(\Omega))$ of the solution of the problem

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay &= \hat{f} \quad \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} &= \hat{g} \quad \text{on } \Sigma, \\ y(\cdot,0) &= 0 \quad \text{in } \Omega. \end{cases} \tag{2.2.1}$$

In this section, whenever it does not lead to confusion, we shall use the following shortening: $L^\tau(W^{s,p})$, $L^2(H^1)$, $W^{1,\tau}((W^{1,p})')$, $L^{\bar{k}}(L^k(\Omega))$, $L^{\bar{\sigma}}(L^\sigma(\Gamma))$, and $C(C^{0,\varepsilon}(\bar{\Omega}))$ respectively for $L^\tau(0,T;W^{s,p}(\Omega))$, $L^2(0,T;H^1(\Omega))$, $W^{1,\tau}(0,T;(W^{1,p}(\Omega))')$, $L^{\bar{k}}(0,T;L^k(\Omega))$, $L^{\bar{\sigma}}(0,T;L^\sigma(\Gamma))$ and $C([0,T];C^{0,\varepsilon}(\bar{\Omega}))$.

There exist in the literature various results related to this. To make the exposition more simple, and since most of the references are related to Dirichlet's problem, we will consider in this introduction Dirichlet's problem

$$\left\{\begin{array}{rcll}\dfrac{\partial y}{\partial t}+Ay &=& f & \text{in } Q\\ y &=& 0 & \text{on } \Sigma\\ y(0) &=& 0 & \text{in } \Omega\times\{0\}\end{array}\right. \tag{2.2.2}$$

The results we are looking for are related to maximal regularity results in the space $L^\tau(0,T;W^{1,p}(\Omega)$ ($L^\tau(W^{1,p})$-MRR to shorten):

"The mapping Λ that relates f with the solution y of the equation (2.2.2) *is continuous from $L^\tau(0,T;W^{-1,p}(\Omega))$ into $L^\tau(0,T;W_0^{1,p}(\Omega))\cap W^{1,\tau}(0,T;W^{-1,p}(\Omega))$."*

As it is explained in Theorem 2.2.1, this regularity result is closely linked to this other

"The mapping Λ that relates f with the solution y of (2.2.2) *is continuous from $L^\tau(0,T;L^p(\Omega))$ into $L^\tau(0,T;W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))\cap W^{1,\tau}(0,T;L^p(\Omega))$."*

We will refer to it as maximal regularity result in $L^\tau(W^{2,p})$ ($L^\tau(W^{2,p})$-MRR to shorten). There are some references for this kind of results:

If the boundary of Ω is of class C^2, the operator is in *non divergence* form and $a_{i,j}(x,t)\in C(\bar{Q})$, then $L^\tau(W^{2,p})$-MRR can be found in Schlag [80] or Ladyzhenskaya, Solonnikov and Ural'tseva [64] for $p=\tau$, Dore and Venni [48] or Amann [4] for $p\neq\tau$ but $a_{i,j}$ independent of time. For $a_{i,j}$ dependent of time, a $L^\tau(W^{2,p})$-MRR can be found in for Γ of class C^4 in Von Wahl [90]. Amann announces at the end of Chapter IV of [4] that other results will appear in the second volume of his monography [5]. Labbas and Moussaoui in [63] establish a $L^\tau(W^{2,p})$-MRR supposing that Γ is of class C^2, $a_{i,j}(x,t)\in C(\bar{Q})$, $\frac{\partial a_{i,j}}{\partial x_k}\in L^\infty(Q)$, y $a_{i,j}(x,t)=a_1(x)a_2(t)$ if $i=j$, $a_{i,j}=0$ else. In Cannarsa and Vespri [14] a $L^\tau(W^{2,p})$-MRR is established for $\Omega=\mathbb{R}^N$, with bounded coefficients $a_{i,j}(x,t)\in C(\bar{Q})$, $\frac{\partial a_{i,j}(x,t)}{\partial x_k}\in C(\bar{Q})$.

Let us see that a $L^\tau(W^{1,p})$-MRR can be deduced from a $L^\tau(W^{2,p})$-MRR by duality, transposition and interpolation

Theorem 2.2.1 *If the mapping Λ which associates the solution y of* (2.2.2) *to f is continuous from $L^\tau(0,T;L^p(\Omega))$ to $L^\tau(0,T;W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))\cap W^{1,\tau}(0,T;L^p(\Omega))$ then Λ is also continuous from $L^\tau(0,T;W^{-1,p}(\Omega))$ to $L^\tau(0,T;W_0^{1,p}(\Omega))\cap W^{1,\tau}(0,T;W^{-1,p}(\Omega))$.*

Proof. Let us consider the parabolic equation

$$\begin{cases} -\dfrac{\partial y}{\partial t} + Ay &= f \quad \text{in } Q \\ y &= 0 \quad \text{on } \Sigma \\ y(T) &= 0 \quad \text{in } \Omega \times \{T\} \end{cases} \tag{2.2.3}$$

From the continuity assumption on Λ, one can easily deduce that the mapping L which associates the solution y of (2.2.3) with f is continuous from $L^{\tau'}(L^{p'})$ into $L^{\tau'}(W^{2,p'} \cap W_0^{1,p'}) \cap W^{1,\tau'}(L^{p'})$. Now we suppose that f belongs to $L^{\tau}((W^{2,p'} \cap W_0^{1,p'})')$. We can define the solution to (2.2.2) by the so-called transposition method in the following way:

We say that $y \in L^{\tau}(L^p)$ is a solution of (2.2.2) (when $f \in L^{\tau}((W^{2,p'} \cap W_0^{1,p'})')$) if

$$y = L^* f \tag{2.2.4}$$

(where L^* is the adjoint operator of the operator L above defined), that is

$$\int_Q y(-\frac{\partial \varphi}{\partial t} + A\varphi) dx dt = \langle f, \varphi \rangle_{L^{\tau}((W^{2,p'} \cap W_0^{1,p'})') \times L^{\tau'}(W^{2,p'}(\Omega) \cap W_0^{1,p'})} \tag{2.2.5}$$

for all $\varphi \in L^{\tau'}(W^{2,p'} \cap W_0^{1,p'}) \cap W^{1,\tau'}(L^{p'})$.

Since L is continuous from $L^{\tau'}(L^{p'}$ to $L^{\tau'}(W^{2,p'} \cap W_0^{1,p'})$, then L^* is continuous from $L^{\tau}((W^{2,p'} \cap W_0^{1,p'})')$ to $L^{\tau}(L^p)$.

Observe that $L^{\tau}(L^p)$ may be identified with a subspace of $L^{\tau}((W^{2,p'} \cap W_0^{1,p'})')$ and that if $f \in L^{\tau}(L^p)$ then $\Lambda f = L^* f$.

Therefore L^* is a continuous operator from $L^{\tau}(L^p) + L^{\tau}((W^{2,p'} \cap W_0^{1,p'})') = L^{\tau}((W^{2,p'} \cap W_0^{1,p'})')$ into $L^{\tau}(L^p)$. It is also continuous from $L^{\tau}(L^p)$ into $L^{\tau}(W^{2,p'} \cap W_0^{1,p'})$.

Therefore L^* is a continuous operator from

$$\Big[L^{\tau}(L^p), L^{\tau}((W^{2,p'} \cap W_0^{1,p'})')\Big]_{1/2}$$

into

$$\Big[L^{\tau}(L^p), L^{\tau}(W^{2,p'} \cap W_0^{1,p'})\Big]_{1/2} = L^{\tau}(W_0^{1,p}),$$

(where $[\cdot,\cdot]_{1/2}$ is the complex interpolation functor of exponent $1/2$).

By using Triebel [85, Theorem 1.11.3] and with the identity

$$[L^{\tau'}(L^{p'}), L^{\tau'}(W^{2,p'} \cap W_0^{1,p'})]_{1/2} = L^{\tau'}(W_0^{1,p'}),$$

we obtain

$$[L^\tau(L^p), L^\tau((W^{2,p'} \cap W_0^{1,p'})')]_{1/2} = L^\tau(W^{-1,p}).$$

Therefore L^* (or Λ) is a continuous operator from $L^\tau(W^{-1,p})$ to $L^\tau(W_0^{1,p})$.

Now if y is a solution of (2.2.2) we can write

$$\langle \frac{dy}{dt}, \varphi\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} = \langle f, \varphi\rangle - \int_\Omega (\sum_{i,j=1}^N a_{i,j}\partial_{x_j} y \partial_{x_i}\varphi)dx$$

for every $\varphi \in W_0^{1,p'}(\Omega)$. Since $y \in L^\tau(W_0^{1,p})$ it follows that the vector distribution $\frac{dy}{dt}$ belongs to $L^\tau(W^{-1,p})$ and satisfies

$$\left\|\frac{dy}{dt}\right\|_{L^\tau(W^{-1,p})} \leq C\|f\|_{L^\tau(W^{-1,p})}.$$

The proof is complete. □

The aim of this section is to get a regularity result in $L^\tau(W^{1,p})$ with continuous coefficients and a C^1 boundary. Under these conditions it is impossible, to our knowledge, to obtain a result in $L^\tau(W^{2,p})$, and therefore the previous theorem is unappliable. The only similar result we have found in the literature is of Vespri [89, Theorem 3.1].

The technique we use is that of perturbation of the constant coefficient case, and we apply it directly to deduce $L^\tau(W^{1+\varepsilon,p})$ regularity.

Preliminary estimates

We suppose that $\tau \in (1,\infty)$ and $p \in (1,\infty)$ are given fixed throughout the section. We now state some hypotheses.

- The boundary Γ is of class $C^{1,\hat{\varepsilon}}$ for some $0 < \hat{\varepsilon} < 1$.
- The coefficients a_{ij} belong to $C([0,T]; C^{0,\hat{\varepsilon}}(\bar{\Omega}))$ and satisfy

$$m\|\xi\|^2 \leq \sum_{i,j=1}^N a_{ij}(x,t)\xi_i\xi_j \leq M\|\xi\|^2 \text{ for all } \xi \in \mathbb{R}^N \text{ and all } (x,t) \in Q$$

  for some $m, M > 0$.

Recall the following regularity results. Assume that the boundary Γ is of class C^2. Set $\bar{a}_{ij} = a_{ij}(\bar{x}, \bar{t})$ and $\bar{A}y = -\sum_{i,j=1}^{N} \partial_{x_j}(\bar{a}_{ij}\partial_{x_i}y)$, where $(\bar{x}, \bar{t})$ is any point in $\bar{Q}$. Then the mapping that associates $\hat{f}$ with the solution y of

$$\begin{cases} \dfrac{\partial y}{\partial t} + \bar{A}y = \hat{f} & \text{in } Q, \\ \dfrac{\partial y}{\partial n_{\bar{A}}} = 0 & \text{on } \Sigma, \\ y(\cdot, 0) = 0 & \text{in } \Omega, \end{cases}$$

is continuous from $L^{\tilde{k}_1}(L^{k_1}(\Omega))$ into $L^{\tau}(W^{1+\varepsilon_k,p})$ when one of the following conditions is satisfied

$$0 < \frac{\varepsilon_k}{2} < \frac{N}{2p} + \frac{1}{\tau} + \frac{1}{2} - \frac{N}{2k_1} - \frac{1}{\tilde{k}_1}, \quad \text{if } k_1 \le p \text{ and } \tilde{k}_1 \le \tau, \tag{2.2.6}$$

$$0 < \frac{\varepsilon_k}{2} < \frac{N}{2p} + \frac{1}{2} - \frac{N}{2k_1}, \quad \text{if } k_1 \le p \text{ and } \tilde{k}_1 > \tau, \tag{2.2.7}$$

$$0 < \frac{\varepsilon_k}{2} < \frac{1}{\tau} + \frac{1}{2} - \frac{1}{\tilde{k}_1}, \quad \text{if } k_1 > p \text{ and } \tilde{k}_1 \le \tau, \tag{2.2.8}$$

$$0 < \varepsilon_k < 1, \quad \text{if } k_1 > p \text{ and } \tilde{k}_1 > \tau. \tag{2.2.9}$$

For non homogeneous boundary data, the mapping that associates $\hat{g}$ with the solution y of

$$\begin{cases} \dfrac{\partial y}{\partial t} + \bar{A}y = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial n_{\bar{A}}} = \hat{g} & \text{on } \Sigma, \\ y(\cdot, 0) = 0 & \text{in } \Omega, \end{cases}$$

is continuous from $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$ into $L^{\tau}(W^{1+\varepsilon_\sigma,p})$ when one of the following conditions is satisfied:

$$0 < \frac{\varepsilon_\sigma}{2} < \frac{N}{2p} + \frac{1}{\tau} - \frac{N-1}{2\sigma_1} - \frac{1}{\tilde{\sigma}_1}, \quad \text{if } \sigma_1 \le p \text{ and } \tilde{\sigma}_1 \le \tau, \tag{2.2.10}$$

$$0 < \frac{\varepsilon_\sigma}{2} < \frac{N}{2p} - \frac{N-1}{2\sigma_1}, \quad \text{if } \sigma_1 \le p \text{ and } \tilde{\sigma}_1 > \tau, \tag{2.2.11}$$

$$0 < \frac{\varepsilon_\sigma}{2} < \frac{1}{2p} + \frac{1}{\tau} - \frac{1}{\tilde{\sigma}_1}, \quad \text{if } \sigma_1 > p \text{ and } \tilde{\sigma}_1 \le \tau, \tag{2.2.12}$$

$$0 < \varepsilon_\sigma < \frac{1}{p}, \quad \text{if } \sigma_1 > p \text{ and } \tilde{\sigma}_1 > \tau. \tag{2.2.13}$$

The previous regularity results may be proved by using the same techniques as in [77, Prop. 3.2].

In all what follows $\varepsilon > 0$ is given fixed, strictly less than $\min(\hat{\varepsilon}, 2/\tau, 2/p)$, and less or equal than $\min(\varepsilon_\sigma, \varepsilon_k)$, where ε_σ, ε_k are chosen as in (2.2.6)–(2.2.13). We make the following hypotheses on $\tilde{k}_1$, k_1, $\tilde{\sigma}_1$, σ_1.

- The pair $(\tilde{k}_1, k_1)$ satisfies one of the conditions (2.2.6)–(2.2.9) and

$$\frac{N}{2k_1} + \frac{1}{\tilde{k}_1} < 1. \tag{2.2.14}$$

- The pair $(\tilde{\sigma}_1, \sigma_1)$ satisfies one of the conditions (2.2.10)–(2.2.13) and

$$\frac{N-1}{2\sigma_1} + \frac{1}{\tilde{\sigma}_1} < \frac{1}{2}. \tag{2.2.15}$$

Remark 2.2.1 *Conditions* (2.2.14) *and* (2.2.15) *are needed to prove Propositions 2.2.7 and 2.2.9.*

A regularity result in $L^\tau(W^{1+\varepsilon,p})$ for the linearized state equation is proved in Proposition 2.2.7. We first establish some preliminary estimates.

Proposition 2.2.2 *Assume that the boundary Γ is of class C^2. Set $\bar{a}_{ij} = a_{ij}(\bar{x}, \bar{t})$ and $\bar{A}y = -\sum_{i,j=1}^N \partial_{x_j}(\bar{a}_{ij}\partial_{x_i} y)$, where $(\bar{x}, \bar{t})$ is any point in $\bar{Q}$. Let $\hat{f}$ be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$ and $\hat{g}$ be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$. Then the weak solution y to the equation*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \bar{A}y = \hat{f} & \text{in } Q, \\ \dfrac{\partial y}{\partial n_{\bar{A}}} = \hat{g} & \text{on } \Sigma, \\ y(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{2.2.16}$$

belongs to $L^\tau(W^{1+\varepsilon,p}) \cap L^2(H^1)$, and satisfies

$$\|y\|_{L^\tau(W^{1+\varepsilon,p})\cap L^2(H^1)} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}), \tag{2.2.17}$$

where C depends on Ω, T, ε, $\tilde{k}_1$, k_1, $\tilde{\sigma}_1$, and σ_1 but is independent of the point $(\bar{x}, \bar{t})$.

Proof. The proof may be performed by using estimates on analytic semigroup as in [77, Proposition 3.2]. Observe that the conditions linking $\tilde{k}_1$, k_1, $\tilde{\sigma}_1$, and σ_1, with p, τ, ε_σ and ε_k are needed to prove the above estimate. □

Proposition 2.2.3 *Suppose that the boundary Γ is of class C^2, and define the coefficients $\bar{a}_{ij}$ as in Proposition 2.2.2. Let $\vec{f}$ be in $(L^\tau(W^{\varepsilon,q})\cap L^2(Q))^N$, with $min(p, \frac{2N}{N-2+2\varepsilon}) \leq q \leq p$. Then the weak solution y to the variational equation*

$$-\int_Q y\frac{\partial\phi}{\partial t}\,dx\,dt + \int_Q \sum_{i,j=1}^N \bar{a}_{ij}\partial_{x_i}y\partial_{x_j}\phi\,dx\,dt = \int_Q \vec{f}\cdot\nabla\phi\,dx\,dt$$

for all $\phi \in C^1(\bar{Q})$ such that $\phi(T) = 0$, belongs to $L^\tau(W^{1+\varepsilon,q}) \cap L^2(H^1)$ and satisfies

$$\|y\|_{L^\tau(W^{1+\varepsilon,q})\cap L^2(H^1)} \leq C\|\vec{f}\|_{(L^\tau(W^{\varepsilon,q})\cap L^2(Q))^N},$$

where C is independent of $(\bar{x},\bar{t}) \in \bar{Q}$ and of $q \in [min(p, \frac{2N}{N-2+2\varepsilon}), p]$.

Proof. The estimate in $L^2(H^1)$, when $\vec{f}$ belongs to $(L^2(Q))^N$ is classical. Let us prove the estimate in $L^\tau(W^{1+\varepsilon,q})$. From maximal regularity results for equations with regular coefficients, we deduce that the mapping $\vec{f} \mapsto y_{\vec{f}}$ (where $y_{\vec{f}}$ denotes the solution to the equation) is continuous from $L^\tau(W^{1,q})$ into $L^\tau(W^{2,q})$, and from $L^\tau(L^q(\Omega))$ into $L^\tau(W^{1,q})$ (see [89]). Moreover the constant in the corresponding estimates may be chosen independent of $q \in [min(p, \frac{2N}{N-2+2\varepsilon}), p]$. Since $(L^\tau(W^{2,q}), L^\tau(W^{1,q}))_{\varepsilon,q} \equiv L^\tau(W^{1+\varepsilon,q})$ (see Triebel [85], or Daners and Medina [46]), the result follows by means of real interpolation. □

Proposition 2.2.4 *Suppose that the boundary Γ is of class C^2, and define the coefficients $\bar{a}_{ij}$ as in Proposition 2.2.2. Let f be in $L^2(Q)$, and let y be the weak solution in $L^2(H^1)$ to the variational equation*

$$-\int_Q y\frac{\partial\phi}{\partial t}\,dx\,dt + \int_Q \sum_{i,j=1}^N \bar{a}_{ij}\,\partial_{x_i}y\,\partial_{x_j}\phi\,dx\,dt = \int_Q f\phi\,dx\,dt$$

for all $\phi \in C^1(\bar{Q})$ such that $\phi(T) = 0$. If $p \leq 2$, then

$$\|y\|_{L^\tau(W^{1+\varepsilon,p})\cap L^2(H^1)} \leq C\|f\|_{L^2(Q)}.$$

If $\tau \leq 2$ and $p > 2$, then

$$\|y\|_{L^\tau(W^{1+\varepsilon,q})\cap L^2(H^1)} \leq C\|f\|_{L^2(Q)},$$

with $q = \frac{2N}{N-2+2\varepsilon}$. If $\tau > 2$ y $p > 2$, then

$$\|y\|_{L^\tau(W^{1+\varepsilon,q})\cap L^2(H^1)} \leq C\|f\|_{L^2(Q)},$$

for any $q \geq 2$ satisfying $\frac{N}{4} + \frac{1}{2} < \frac{N}{2q} + \frac{1}{\tau} + \frac{1}{2} - \frac{\varepsilon}{2}$. Moreover, in the above estimates, the constants C are independent of $(\bar{x}, \bar{t}) \in \bar{Q}$.

Proof. If $p \leq 2$, using estimates on analytic semigroups, we can prove that y belongs to $L^\tau(W^{1+\varepsilon,2})$ for every $\tau \geq 2$ such that $1/2 < 1/\tau + 1/2 - \varepsilon/2$. Since $\varepsilon < 2/\tau$, y belongs to $L^\tau(W^{1+\varepsilon,2})$ for every $\tau \geq 2$. If $\tau \leq 2$ and $p > 2$, then y belongs to $L^2(W^{2,2})$. In this case, the estimate follows from Sobolev embeddings. The last case can also be treated by using estimates on analytic semigroups. □

Proposition 2.2.5 *Suppose that the boundary Γ is of class C^3, and define the coefficients $\bar{a}_{ij}$ as in Proposition 2.2.2. Let f be in $L^\tau(W^{\varepsilon,q}) \cap L^2(Q)$, with $min(p, \frac{2N}{N-2+2\varepsilon}) \leq q \leq p$. Then the weak solution y to the variational equation*

$$-\int_Q y\frac{\partial \phi}{\partial t}\, dx\, dt + \int_Q \sum_{i,j=1}^N \bar{a}_{ij}\partial_{x_i} y\, \partial_{x_j}\phi\, dx\, dt = \int_Q f\phi\, dx\, dt$$

for all $\phi \in C^1(\bar{Q})$ such that $\phi(T) = 0$, belongs to $L^\tau(W^{1+\varepsilon,\tilde{q}}) \cap L^2(H^1)$ with $\tilde{q} = \frac{Nq}{N-q}$ if $q < N$, $q = p$ if $q \geq N$, and satisfies

$$\|y\|_{L^\tau(W^{1+\varepsilon,\tilde{q}}) \cap L^2(H^1)} \leq C\|f\|_{L^\tau(W^{\varepsilon,q}) \cap L^2(Q)},$$

where C is independent of $(\bar{x}, \bar{t}) \in \bar{Q}$ and of $q \in [min(p, \frac{2N}{N-2+2\varepsilon}), p]$.

Proof. Using real interpolation, as in the proof of Proposition 2.2.3, we can first prove that

$$\|y\|_{L^\tau(W^{2+\varepsilon,q}) \cap L^2(H^1)} \leq C\|f\|_{L^\tau(W^{\varepsilon,q}) \cap L^2(Q)}.$$

We conclude with Sobolev embeddings. □

Lemma 2.2.6 *Let $\varepsilon < \tilde{\varepsilon} < \hat{\varepsilon}$. For all $q \in [min(p, \frac{2N}{N-2+2\varepsilon}), p]$, all $a \in C([0,T]; C^{0,\tilde{\varepsilon}}(\bar{\Omega}))$, all $y \in L^\tau(W^{\varepsilon,q})$, ay belongs to $L^\tau(W^{\varepsilon,q})$, and*

$$\|ay\|_{L^\tau(W^{\varepsilon,q})} \leq C\|a\|_{C([0,T]; C^{0,\tilde{\varepsilon}}(\bar{\Omega}))}\|y\|_{L^\tau(W^{\varepsilon,q})},$$

where C does not depend on $q \in [min(p, \frac{2N}{N-2+2\varepsilon}), p]$.

Proof. Using the definition of the norm in $L^\tau(W^{\varepsilon,q})$, with straightforward calculations we obtain

$$\|ay\|^\tau_{L^\tau(W^{\varepsilon,q})} = \int_0^T \left(\int_{\Omega\times\Omega} \frac{|a(x,t)y(x,t) - a(x',t)y(x',t)|^q}{|x - x'|^{n+\varepsilon q}}\, dx\, dx'\right)^{\tau/q} dt$$

$$\leq C\int_0^T\left(\int_{\Omega\times\Omega}\frac{|a(x,t)-a(x',t)|^q}{|x-x'|^{\bar\varepsilon q}}\frac{|y(x,t)|^q}{|x-x'|^{n+(\varepsilon-\bar\varepsilon)q}}\,dx\,dx'\right)^{\tau/q}dt$$

$$+C\int_0^T\left(\int_{\Omega\times\Omega}|a(x',t)|^q\frac{|y(x,t)-y(x',t)|^q}{|x-x'|^{n+\varepsilon q}}\,dx\,dx'\right)^{\tau/q}dt$$

$$\leq C\|a\|^{\tau}_{C(C^{0,\bar\varepsilon}(\bar\Omega))}\max_{\xi\in\bar\Omega}\left(\int_\Omega\frac{dx'}{|\xi-x'|^{n+(\varepsilon-\bar\varepsilon)q}}\right)^{\tau/q}\int_0^T\left(\int_\Omega|y(x,t)|^q dx\right)^{\tau/q}dt+C\|a\|^{\tau}_{C(\bar Q)}\|y\|^{\tau}_{L^\tau(W^{\varepsilon,q})}.$$

The proof is complete. □

Once stated these auxiliary estimates, we are now ready to write the needed regularity results for the study of the equations involved in the control problem. Let us start with the main result of this section.

Proposition 2.2.7 *Let a be in $L^{\tilde k_1}(L^{k_1}(\Omega))$, b be in $L^{\tilde\sigma_1}(L^{\sigma_1}(\Gamma))$, $\hat f$ be in $L^{\tilde k_1}(L^{k_1}(\Omega))$ and $\hat g$ be in $L^{\tilde\sigma_1}(L^{\sigma_1}(\Gamma))$. Then the solution y in $L^2(H^1)\cap C([0,T];L^2)$ to the equation*

$$\begin{cases}\dfrac{\partial y}{\partial t}+Ay+ay &=& \hat f & \text{in } Q,\\ \dfrac{\partial y}{\partial n_A}+by &=& \hat g & \text{on } \Sigma,\\ y(\cdot,0) &=& 0 & \text{in } \Omega,\end{cases}\tag{2.2.18}$$

satisfies the estimate

$$\|y\|_{L^\tau(W^{1+\varepsilon,p})}\leq C(\|\hat f\|_{L^{\tilde k_1}(L^{k_1}(\Omega))}+\|\hat g\|_{L^{\tilde\sigma_1}(L^{\sigma_1}(\Gamma))}),\tag{2.2.19}$$

where C only depends on Ω, T, A and an upper bound for $\|a\|_{L^{\tilde k_1}(L^{k_1}(\Omega))}+\|b\|_{L^{\tilde\sigma_1}(L^{\sigma_1}(\Gamma))}$.

Proof. Due to (2.2.14) and (2.2.15), first notice that $y\in L^\infty(Q)$ (see Casas, Raymond and Zidani [35]), and that

$$\|y\|_{L^\infty(Q)}\leq C(\|\hat f\|_{L^{\tilde k_1}(L^{k_1}(\Omega))}+\|\hat g\|_{L^{\tilde\sigma_1}(L^{\sigma_1}(\Gamma))}),\tag{2.2.20}$$

Therefore it is sufficient to consider the case where $a\equiv 0$ and $b\equiv 0$. We now suppose that we are in this case. To prove (2.2.19), when the coefficients $a_{ij}\in C([0,T];C^{\bar\varepsilon}(\bar\Omega))$, we use a technique of freezing coefficients as in Vespri [89, Theorem 3.1]. Up to Step 3, we suppose that the boundary Γ is regular.

Step 1.

First we prove an estimate in $L^\tau(W^{\varepsilon,p})$. From Ladyženskaja et al. [64, Chapter 3, Theorem 5.1], we know that the weak solution to (2.2.18) belongs to $L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$, and satisfies

$$\|y\|_{L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))} \leq C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}). \tag{2.2.21}$$

Choose $\tilde{r}$ and r, such that $\frac{\bar{\varepsilon}}{2} + \frac{1-\bar{\varepsilon}}{r} = \frac{1}{p}$, and $\frac{\bar{\varepsilon}}{2} + \frac{1-\bar{\varepsilon}}{\tilde{r}} = \frac{1}{\tau}$, where $\bar{\varepsilon}$ is an exponent strictly greater than ε. Since $\|y\|_{L^{\tilde{r}}(L^r(\Omega))} \leq C\|y\|_{L^\infty(Q)}$ and $[L^r(\Omega), W^{1,2}(\Omega)]_{\bar{\varepsilon}} \hookrightarrow W^{\varepsilon,p}(\Omega)$, from (2.2.20) and (2.2.21), and by interpolation it follows that

$$\|y\|_{L^\tau(W^{\varepsilon,p})} \leq C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}).$$

Step 2.

For any $\rho > 0$, let $0 = t_1 < t_2 < \ldots < t_k < \ldots < t_K = T$ be a regular subdivision of $[0,T]$, such that $t_k - t_{k-1} = \ell(\rho)$ and

$$\max\{\|a_{ij}(t,\cdot) - a_{ij}(t',\cdot)\|_{C^{0,\varepsilon}(\bar{\Omega})} \mid t \in [t_{k-1},t_k], t' \in [t_{k-1},t_k], 1 \leq i,j \leq N, 2 \leq k \leq K\} \leq \rho.$$

Let $\{C_\rho^s\}_{s=1}^\mu$ be a collection of open sets of class C^∞, of diameter less or equal than $\rho > 0$ such that

$$\bar{\Omega} \subset \cup_{s=1}^\mu C_\rho^s,$$

and let $\{\varphi_s\}_{s=1}^\mu$ be a partition of unity subordinate to this covering. Let ψ_k be the continuous function on $[0,T]$, affine on each interval $[t_k, t_{k+1}]$, which is equal to 1 on t_k and 0 on t_j if $j \neq k$. For a given fixed point $x_s \in C_\rho^s$, set

$$\bar{a}_{ij}^{sk} = a_{ij}(x_s, t_k) \quad \text{y} \quad y_{sk}(x,t) = \psi_k(t)\varphi_s(x)y(x,t) \text{ for } 1 \leq s \leq \mu,\ 1 \leq k \leq K. \tag{2.2.22}$$

Let us fix $1 \leq k \leq K$ and $1 \leq s \leq \mu$. For every $\xi \in L^2(H^1)$, define the operator T_ξ^{ks} by

$$\begin{aligned} T_\xi^{ks}(\phi) &= \int_Q \psi_k\varphi_s\hat{f}\phi\,dx\,dt + \int_\Sigma \psi_k\varphi_s\hat{g}\phi\,ds\,dt \\ &+ \int_Q \psi_k \sum_{i,j=1}^N a_{ij}y\partial_{x_i}\varphi_s\partial_{x_j}\phi\,dx\,dt - \int_Q \psi_k \sum_{i,j=1}^N a_{ij}\partial_{x_i}y\partial_{x_j}\varphi_s\phi\,dx\,dt \\ &+ \int_Q \varphi_s y\frac{\partial\psi_k}{\partial t}\phi\,dx\,dt + \int_{t_{k-1}}^{t_{k+1}} \int_{C_\rho^s} \sum_{i,j=1}^N (\bar{a}_{ij}^{sk} - a_{ij})\partial_{x_i}\xi\partial_{x_j}\phi\,dx\,dt, \end{aligned}$$

with the convention $t_0 = t_1 = 0$ and $t_{K+1} = t_K = T$. For every $\xi \in L^2(H^1)$, let $z(\xi)$ be the unique solution in $L^2(H^1)$ to the variational equation

$$-\int_Q z\frac{\partial\phi}{\partial t}\,dx\,dt + \int_Q \sum_{i,j=1}^{N} \bar{a}_{ij}^{sk}\partial_{x_i} z\partial_{x_j}\phi\,dx\,dt = T_\xi^{ks}(\phi) \tag{2.2.23}$$

for all $\phi \in C^1(\bar{Q})$ such that $\phi(T) = 0$. Observe that $z(y_{sk}) \equiv y_{sk}$. Let us prove that, if ρ is small enough, then the mapping $\xi \mapsto z(\xi)$ admits a fixed point in $L^\tau(W^{1+\varepsilon,p_1}) \cap L^2(H^1)$, where $p_1 = \min(p, \frac{2N}{N-2+2\varepsilon})$. Due to Lemma 2.2.6, if $\xi \in L^\tau(W^{1+\varepsilon,p_1}) \cap L^2(H^1)$, then $\sum_{i=1}^N(\bar{a}_{ij}^{st} - a_{ij})\partial_{x_i}\xi$ belongs to $L^\tau(W^{\varepsilon,p_1}) \cap L^2(Q)$ for all $1 \le j \le N$. Notice that $\psi_k\varphi_s\hat{f}$ belongs to $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, $\psi_k\varphi_s\hat{g}$ belongs to $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$. Due to step 1 and Lemma 2.2.6, $\psi_k\sum_{i=1}^N a_{ij}y\partial_{x_i}\varphi_s$ belongs to $L^\tau(W^{\varepsilon,p}) \cap L^2(Q)$ for $1 \le j \le N$. Also observe that $\psi_k\sum_{i,j=1}^N a_{ij}\partial_{x_i}y\partial_{x_j}\varphi_s$ belongs to $L^2(Q)$, and $\varphi_s y\frac{\partial\psi_k}{\partial t}$ belongs to $L^\infty(Q)$. From Propositions 2.2.2 to 2.2.4, it follows that $z(\xi)$ belongs to $L^\tau(W^{1+\varepsilon,p_1}) \cap L^2(H^1)$ for all $\xi \in L^\tau(W^{1+\varepsilon,p_1}) \cap L^2(H^1)$.

On the other hand, due to Proposition 2.2.3 and to Lemma 2.2.6, it follows that

$$\|z(\xi_1)-z(\xi_2)\|_{L^\tau(W^{1+\varepsilon,p_1})\cap L^2(H^1)} \le C\sum_{i,j=1}^N \|(\bar{a}_{ij}^{sk}-a_{ij})(\partial_{x_i}\xi_1-\partial_{x_i}\xi_2)\|_{L^\tau(W^{\varepsilon,p_1})\cap L^2(]t_{k-1},t_{k+1}[\times C_\rho^s)}$$

$$\le C\left(\max_{i,j}\|\bar{a}_{ij}^{sk} - a_{ij}(t_k,\cdot)\|_{C^{0,\tilde{\varepsilon}}(\bar{C}_\rho^s)} + \max_{i,j}\|a_{ij}(t_k,\cdot) - a_{ij}(\cdot)\|_{C([t_{k-1},t_{k+1}];C^{0,\hat{\varepsilon}}(\bar{C}_\rho^s))}\right)$$
$$\cdot\|\nabla\xi_1 - \nabla\xi_2\|_{(L^\tau(W^{\varepsilon,p_1})\cap L^2(Q))^N}$$
$$\le C(\rho^{\hat{\varepsilon}-\tilde{\varepsilon}} + \rho)\|\nabla\xi_1 - \nabla\xi_2\|_{(L^\tau(W^{\varepsilon,p_1})\cap L^2(Q))^N},$$

for some $\tilde{\varepsilon} \in]\varepsilon, \hat{\varepsilon}[$. Therefore, for ρ small enough, the mapping $\xi \to z(\xi)$ is a contraction in $L^\tau(W^{1+\varepsilon,p_1}) \cap L^2(H^1)$. Since the solution z of the equation

$$-\int_Q z\frac{\partial\phi}{\partial t}\,dx\,dt + \int_Q \sum_{i,j=1}^{N} \bar{a}_{ij}^{sk}\partial_{x_i} z\partial_{x_j}\phi\,dx\,dt = T_{y_{sk}}^{ks}(v)$$

for all $\phi \in C^1(\bar{Q})$ such that $\phi(T) = 0$, is unique in $L^2(H^1)$ and is equal to y_{sk}, this fixed point is y_{sk}. From the equality $y = \Sigma_{k=1}^K\Sigma_{s=1}^\mu y_{sk}$, it follows that y belongs to $L^\tau(W^{1+\varepsilon,p_1})$.

Paso 3.

If $p = p_1$ the proof is complete. Otherwise, we set $p_2 = \frac{Np_1}{N-p_1}$ if $p_1 < N$, and $p_2 = p$ if $p_1 \ge N$. We repeat Step 2. We want to prove that the mapping $\xi \mapsto z(\xi)$ admits a fixed point in $L^\tau(W^{1+\varepsilon,p_2}) \cap L^2(H^1)$. Due to Lemma 2.2.6, if $\xi \in L^\tau(W^{1+\varepsilon,p_2}) \cap L^2(H^1)$, then $\sum_{i=1}^N (\bar{a}_{ij}^{st} - a_{ij})\partial_{x_i}\xi$ belongs to $L^\tau(W^{\varepsilon,p_2}) \cap L^2(Q)$ for all $1 \le j \le N$. Since y belongs to $L^\tau(W^{1+\varepsilon,p_1})$, $\psi_k \sum_{i,j=1}^N a_{ij}\partial_{x_i} y \partial_{x_j}\varphi_s$ belongs to $L^\tau(W^{\varepsilon,p_1}) \cap L^2(Q)$, and due to Sobolev inequalities, $\psi_k \sum_{i=1}^N a_{ij} y \partial_{x_i}\varphi_s$ belongs to $L^\tau(W^{\varepsilon,p_2}) \cap L^2(Q)$ for $1 \le j \le N$.

As before $\psi_k\varphi_s\hat{f}$ belongs to $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, $\psi_k\varphi_s\hat{g}$ belongs to $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, and $\varphi_s y \frac{\partial\psi_k}{\partial t}$ belongs to $L^\infty(Q)$. From Propositions 2.2.2, 2.2.3 and 2.2.5,it follows that $z(\xi)$ belongs to $L^\tau(W^{1+\varepsilon,p_2}) \cap L^2(H^1)$ for all $\xi \in L^\tau(W^{1+\varepsilon,p_2}) \cap L^2(H^1)$. We conclude by proving that the mapping $\xi \mapsto z(\xi)$ is a contraction in $L^\tau(W^{1+\varepsilon,p_2}) \cap L^2(H^1)$ for the same ρ as in step 2, and that y belongs to $L^\tau(W^{1+\varepsilon,p_2})$. Repeating this argument a finite number of times, we finally prove that y belongs to $L^\tau(W^{1+\varepsilon,p})$ and that

$$\|y\|_{L^\tau(W^{1+\varepsilon,p})} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}).$$

Observe that the first iteration of Step 2 (with p_1) is different from the second one. Indeed, for the first iteration we only know that $\psi_k \sum_{i,j=1}^N a_{ij}\partial_{x_i} y \partial_{x_j}\varphi_s$ belongs to $L^2(Q)$, and we use Proposition 2.2.4. For the second iteration of Step 2, we know that $\psi_k \sum_{i,j=1}^N a_{ij}\partial_{x_i} y \partial_{x_j}\varphi_s$ belongs to $L^\tau(W^{\varepsilon,p_1}) \cap L^2(Q)$, and we use Proposition 2.2.5.

Step 4.

If the boundary Γ is of class $C^{1,\hat{\varepsilon}}$, by making a change of variable in the variational formulation of equation (2.2.18), the equation can be reduced to an equation similar to (2.2.18)but with a regular boundary. Due to steps 1-3, the corresponding solution belongs to $L^\tau(W^{1+\varepsilon,p})$. By making the reverse change of variable, we can prove that the solution to equation (2.2.18) satisfies (2.2.19). □

Suppose now that the regularity assumptions on Γ and the coefficients are replaced by

- The boundary Γ is of class C^1.
- The coefficients a_{ij} belong to $C(\bar{Q})$ and satisfy

$$m\|\xi\|^2 \le \sum_{i,j=1}^N a_{ij}(x,t)\xi_i\xi_j \le M\|\xi\|^2 \quad \text{for all } \xi \in \mathbb{R}^N \text{ and all } (x,t) \in Q$$

  for some $m, M > 0$.

In this case, we can adapt the proof of Proposition 2.2.7 to establish the following result.

Proposition 2.2.8 *Let a be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, b be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, $\hat{f}$ be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$ and $\hat{g}$ be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$. Then the solution y in $L^2(H^1)\cap C([0,T];L^2(\Omega))$ to the equation*

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + ay = \hat{f} & \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} + by = \hat{g} & \text{on } \Sigma, \\ y(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{2.2.24}$$

satisfies the estimate

$$\|y\|_{L^\tau(W^{1,p})} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}), \tag{2.2.25}$$

where C only depends on Ω, T, A and an upper bound for $\|a\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|b\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}$.

Proposition 2.2.9 *Let a be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, b be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, $\hat{f}$ be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, $\hat{g}$ be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$ and ζ be in $L^\tau(W^{1,p})$. Then the solution y to the equation*

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + ay = \hat{f}\zeta & \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} + by = \hat{g}\zeta & \text{on } \Sigma, \\ y(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{2.2.26}$$

satisfies the estimate

$$\|y\|_{L^\tau(W^{1,p})} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))})\|\zeta\|_{L^\tau(W^{1,p})}, \tag{2.2.27}$$

where C only depends on Ω, T, A and an upper bound for $\|a\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|b\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}$.

Proof. For simplicity we only treat the case where $k_1 \le p$, $\tilde{k}_1 \le \tau$, $\sigma_1 \le p$, and $\tilde{\sigma}_1 \le \tau$. The other cases can be treated in a similar way.

Notice that $\hat{f}\zeta$ belongs to $L^{\tilde{k}}(L^k)$ with $\frac{1}{\tilde{k}} = \frac{1}{\tilde{k}_1} + \frac{1}{\tau}$ and $\frac{1}{k} = \frac{1}{k_1} + \frac{N-p}{Np}$ if $p < N$, every $k < k_1$ if $p = N$, and $k = k_1$ if $p > N$. Due to condition (2.2.14) satisfied by k_1 and $\tilde{k}_1$, we can verify that

$$\frac{N}{2k} + \frac{1}{\tilde{k}} < \frac{N}{2p} + \frac{1}{\tau} + \frac{1}{2}.$$

We can also verify that $\hat{g}\zeta$ belongs to $L^{\tilde{\sigma}}(L^\sigma(\Gamma))$ with $\frac{1}{\tilde{\sigma}} = \frac{1}{\tilde{\sigma}_1} + \frac{1}{\tau}$ and $\frac{1}{k} = \frac{1}{\sigma_1} + \frac{N-p}{(N-1)p}$ if $p < N$, every $\sigma < \sigma_1$ if $p = N$, and $\sigma = \sigma_1$ if $p > N$. Due to condition (2.2.15) satisfied by σ_1 and $\tilde{\sigma}_1$, we can verify that

$$\frac{N-1}{2\sigma} + \frac{1}{\tilde{\sigma}} < \frac{N}{2p} + \frac{1}{\tau}.$$

Therefore, if $a \equiv 0$ and $b \equiv 0$ we can prove that y belongs to $L^\tau(W^{1,p})$, and that the estimate (2.2.27) is satisfied. For a in $L^{\tilde{k}_1}(L^{k_1})$ and b in $L^{\tilde{\sigma}_1}(L^{\sigma_1})$, (2.2.27) can be proved by a fixed point argument as in the end of the proof of Proposition 2.2.10. □

To deal with the adjoint state equation for control problems governed by parabolic equations, it is necessary to give it a sense. Consider the following equation.

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi = \operatorname{div} \vec{\eta} & \text{in } Q, \\ \dfrac{\partial \varphi}{\partial n_{A^*}} = -\vec{\eta}\cdot\vec{n} & \text{on } \Sigma, \\ \varphi(\cdot, T) = 0 & \text{in } \Omega, \end{cases} \tag{2.2.28}$$

where $\vec{n}$ is the outward unit normal to Γ, and $\vec{\eta}$ is supposed to be regular. (As usual A^* denotes the formal adjoint of A.) By definition, a function $\varphi \in L^1(W^{1,1})$ is a solution to (2.2.28) if, and only if,

$$\int_Q (\varphi \frac{\partial y}{\partial t} + \sum_{i,j=1}^N a_{ij}\partial_{x_j}\varphi \partial_{x_i} y)\, dx\, dt = -\int_Q \vec{\eta}\cdot\nabla y\, dx\, dt \tag{2.2.29}$$

for all $y \in C^1(\bar{Q})$ such that $y(0) = 0$. The variational equation (2.2.29) is still meaningful if $\vec{\eta}$ belongs to $L^r(Q)$ for some $r > 1$, even if the normal trace $\vec{\eta}\cdot\vec{n}$ is not defined.

For simplicity, we still continue to write the variational equation (2.2.29) in the form (2.2.28), even if the writing $\vec{\eta}\cdot\vec{n}$ may be abusive when $\vec{\eta}$ is not regular.

In the rest of the section $\tilde{k}_2$, k_2, $\tilde{\sigma}_2$, σ_2 and ν are constants satisfying

$$\frac{N}{2k_2} + \frac{1}{\tilde{k}_2} \le 1, \quad \frac{N-1}{2\sigma_2} + \frac{1}{\tilde{\sigma}_2} \le \frac{1}{2}, \quad \text{and } \nu \ge 2 \tag{2.2.30}$$

where k_1' (resp. $\tilde{k}_1'$, σ_1', $\tilde{\sigma}_1'$) is the conjugate exponent of k_1 (resp. $\tilde{k}_1$, σ_1, $\tilde{\sigma}_1$). We also suppose that$\tilde{k}_1$, k_1, $\tilde{\sigma}_1$, and σ_1 satisfy the following additional conditions

$$\tilde{k}_1 \ge \tau, \ \tilde{\sigma}_1 \ge \tau,$$

$$k_1 \ge \frac{Np'}{Np' - N + p'} \quad \text{and} \quad \sigma_1 \ge \frac{(N-1)p'}{(N-1)p' - N + p'} \quad \text{if } p' < N.$$

Proposition 2.2.10 *Let a be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, b be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, $\hat{F}$ be in $L^{\tilde{k}_2}(L^{k_2}(\Omega))$, $\vec{\eta}$ be in $(L^{\tau'}(L^{p'}))^N$, $\hat{G}$ be in $L^{\tilde{\sigma}_2}(L^{\sigma_2}(\Gamma))$ and $\hat{L}$ be in $L^\nu(\Omega)$. Then there exists a unique*

$\varphi \in L^{\tau'}(W^{1,p'}) + L^2(H^1)$ satisfying the equation

$$\left\{\begin{array}{rcll} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi + a\varphi &=& \hat{F} + \operatorname{div} \vec{\eta} & \text{in } Q, \\ \dfrac{\partial \varphi}{\partial n_{A^*}} + b\varphi &=& \hat{G} - \vec{\eta}\cdot\vec{n} & \text{on } \Sigma, \\ \varphi(\cdot, T) &=& \hat{L} & \text{in } \Omega, \end{array}\right. \tag{2.2.31}$$

and the following estimate holds

$$\|\varphi\|_{L^{\tau'}(W^{1,p'})+L^2(H^1)} \le C(\|\eta\|_{(L^{\tau'}(L^{p'}))^N} + \|\hat{F}\|_{L^{\tilde{k}_2}(L^{k_2}(\Omega))} + \|\hat{G}\|_{L^{\tilde{\sigma}_2}(L^{\sigma_2}(\Gamma))} + \|\hat{L}\|_{L^{\nu}(\Omega)}),$$

where C depends only on Ω, T, A and an upper bound for $\|a\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|b\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}$.

Moreover, if y is the solution to equation (2.2.18), the following Green formula is satisfied

$$\begin{array}{l} \displaystyle\int_Q \varphi\left(\frac{\partial y}{\partial t} + Ay + ay\right)\, dx\, dt + \int_\Sigma \varphi\left(\frac{\partial y}{\partial n_A} + by\right)\, ds\, dt = \\ \displaystyle\int_Q \hat{F} y\, dx\, dt - \int_Q \vec{\eta}\cdot\nabla y\, dx\, dt + \int_\Sigma \hat{G} y\, ds\, dt + \int_\Omega \hat{L} y(T)\, dx. \end{array} \tag{2.2.32}$$

Proof. We first consider the case where $\hat{F} \equiv 0$, $\hat{L} \equiv 0$, and $\hat{G} \equiv 0$.

If $a \equiv 0$ and $b \equiv 0$, and if the coefficients of the operator A are regular and independent of time, the existence of $\varphi \in L^{\tau'}(W^{1,p'})$ satisfying (2.2.31) can be obtained using duality techniques, interpolation and maximal regularity results as in Vespri [89, Theorem 3.3] and references therein. The passage from regular to continuous coefficients (also depending on time) for A may be performed by localization and a fixed point theorem as in [89, Theorem 3.1].

The case $a \not\equiv 0$ and $b \not\equiv 0$ may be deduced from the previous one by using a fixed point argument. Indeed, observe that if $\xi \in L^{\tau'}(W^{1,p'})$ then $\xi \in L^{\tau'}(L^{p'^*})$, $\xi_{|\Sigma} \in L^{\tau'}(L^{\beta}(\Gamma))$, where $p'^* = p'N/(N-p')$ and $\beta = ((N-1)p')/(N-p')$ if $p' < N$, p'^* and β are any real in $(1, +\infty)$ if $p' \ge N$. Since $a \in L^{\tilde{k}_1}(L^{k_1}(\Omega))$, $b \in L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, we verify that $a\xi \in L^{\tilde{r}}(L^r)$ and $b\xi_{|\Sigma} \in L^{\tilde{s}}(L^s(\Gamma))$, where $1/\tilde{r} = 1/\tilde{k}_1 + 1/\tau'$, $1/r = 1/k_1 + 1/p'^*$, $1/\tilde{s} = 1/\tilde{\sigma}_1 + 1/\tau'$ and $1/s = 1/\sigma_1 + 1/\beta$. Using (2.2.14) and (2.2.15), it follows that

$$\frac{N}{2r} + \frac{1}{\tilde{r}} < \frac{N}{2p'} + \frac{1}{\tau'} + \frac{1}{2} \quad \text{and} \quad \frac{N-1}{2s} + \frac{1}{\tilde{s}} < \frac{N}{2p'} + \frac{1}{\tau'}.$$

Suppose that $1/k_1 \ge 1/p' - 1/p'^*$ and $1/\sigma_1 \ge 1/p' - 1/\beta$. In this case, the mapping that associates the solution φ_ξ of the equation

$$-\frac{\partial \varphi_\xi}{\partial t} + A^*\varphi_\xi = \operatorname{div} \vec{\eta} - a\xi \text{ in } Q, \quad \frac{\partial \varphi_\xi}{\partial n_{A^*}} = -\vec{\eta}\cdot\vec{n} - b\xi \text{ on } \Sigma, \quad \varphi_\xi(\cdot, T) = 0 \text{ in } \Omega,$$

with ξ is affine continuous from $L^{\tau'}(W^{1,p'})$ into itself. Using this property, we can prove that $\xi \to \varphi_\xi$ is a contraction in $L^{\tau'}(0,\bar{t};W^{1,p'})$ for $\bar{t}$ small enough. The estimate in $L^{\tau'}(W^{1,p'})$ may next be deduced by a standard technique. If $1/k_1 < 1/p' - 1/p'^*$ or $1/\sigma_1 < 1/p' - 1/\beta$, the above fixed point method may be performed by replacing k_1 by $\min(k_1, (1/p' - 1/p'^*)^{-1})$, and σ_1 by $\min(\sigma_1, (1/p' - 1/\beta)^{-1})$.

Consider the case where $\hat{F}$, $\hat{L}$, and $\hat{G}$ are different from zero. The equation

$$-\frac{\partial\varphi}{\partial t} + A^*\varphi + a\varphi = \hat{F} \text{ in } Q, \quad \frac{\partial\varphi}{\partial n_{A^*}} + b\varphi = \hat{G} \text{ on } \Sigma, \quad \varphi(\cdot,T) = \hat{L} \text{ in } \Omega,$$

admits a unique solution ϕ satisfying

$$\|\varphi\|_{L^2(H^1)} \leq C(\|\hat{F}\|_{L^{k_2}(L^{k_2}(\Omega))} + \|\hat{G}\|_{L^{\sigma_2}(L^{\sigma_2}(\Gamma))} + \|\hat{L}\|_{L^\nu(\Omega)})$$

(see [64]). The Green formula is true for regular functions y, and it follows from a denseness argument. □

Chapter 3

Study of the state equations

In this chapter we will study the non linear equations that relate the control and the state in the control problems studied in the second part of the thesis.Results on existence and uniqueness of the solutions are established, and also the continuous dependence of them with respect to the control. Under extra assumptions we prove first and second order differentiability of the solution with respect to the control.

Finally we make a Taylor expansion of the state with respect to diffuse perturbations of the control. This is needed when the set of controls is not convex In this case it is not necessary to suppose differentiability conditions with respect to the control.

In this chapter, unless we specifically state another thing, Ω will denote an open bounded and conected subset of $\mathbb{R}^N$, whose boundary Γ is of class C^1.

3.1 Elliptic equations

Let A an elliptic operator of continuous coefficients of the form (2.1.1) (page 23), $p > N$, $a_0 \in L^{p/2}(\Omega)$, f a function $f : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$, $g \in L^{p-1}(\Gamma)$. Let us consider

$$U_{ad} = \{u : \Omega \to \mathbb{R} : \ u(x) \in K_\Omega(x) \text{ a.e. } x \in \Omega\},$$

where K_Ω is a measurable multimapping with non empty and closed image in $\mathcal{P}(\mathbb{R})$.

Theorem 3.1.1 *Let us suppose that $f : \Omega \times (\mathbb{R} \times \mathbb{R}) \longrightarrow \mathbb{R}$ is Carathéodory function, decreasing monotone in the second variable and such that*

E0 - for all $M \geq 0$ there exists a function $\psi_M \in L^{p/2}(\Omega)$ such that $|f(x,t,u(x)| \leq \psi_M(x)$ for a.e. $x \in \Omega$, for all $|t| \leq M$ and for all $u \in U_{ad}$.

Then, for all $u \in U_{ad}$ there exists a unique variational solution $y_u \in W^{1,p}(\Omega)$ of the problem

$$\begin{cases} Ay_u + a_0 y_u = f(x, y_u, u) & \text{in } \Omega \\ \partial_{\nu_A} y_u = g & \text{on } \Gamma. \end{cases} \tag{3.1.1}$$

and a constant $C_{U_{ad}}$ such that

$$\|y_u\|_{W^{1,p}(\Omega)} \leq C_{U_{ad}} \quad \text{for all } u \in U_{ad}.$$

Moreover, if $\{u_j\}_{j=1}^{\infty} \subset U_{ad}$ and $u_j(x) \to u(x)$ a.e. $x \in \Omega$ with $u \in U_{ad}$, then $y_{u_j} \to y_u$ in $W^{1,p}(\Omega)$.

Proof. Let us take $u \in U_{ad}$.

First we will suppose that there exists $\psi \in L^{p/2}(\Omega)$ such that $|f(x,y,u(x))| \leq \psi(x)$ for all $y \in \mathbb{R}$ and almost all $x \in \Omega$.

Let us show first that there exists a solution. Let us define $F : L^2(\Omega) \to L^2(\Omega)$ such that $F(z) = y$ if and only if

$$\begin{cases} Ay + a_0 y = f(x, z, u) & \text{in } \Omega \\ \partial_{n_A} y = 0 & \text{on } \Gamma. \end{cases}$$

Since $p > N$, there exists a solution $y_z = F(z) \in H^1(\Omega)$ and $\|F(z)\|_{H_0^1(\Omega)} \leq c\|\psi\|_{L^{p/2}(\Omega)}$. From the compact inclusion $H_0^1(\Omega) \subset L^2(\Omega)$ we have that F is a compact operator from $L^2(\Omega)$ into $L^2(\Omega)$, and due to Schauder's fixed point theorem, there exists a solution $y \in H_0^1(\Omega)$ of (3.1.1).

Uniqueness follows from the monotonicity of f in the second variable.

Let us see that the solution is bounded. Let us take $k > 0$. We define

$$y_k = \begin{cases} y - k & \text{if} \quad y > k \\ 0 & \text{if} \quad -k \leq y \leq k \\ y + k & \text{if} \quad y < -k. \end{cases}$$

We have that $y_k \in H^1(\Omega)$ for it is the composition of a function in $H^1(\Omega)$ with a Lipschitz function. Moreover y_k has the same sign than y. Using all this and that where $y_k \neq 0$, we have that the partial derivatives of y_k coincide with those of y, and that $\int_\Omega a_0 y_k y_k\,dx \leq \int_\Omega a_0 y y_k$, we have that

$$\begin{aligned}
&m\|y_k\|^2_{H^1(\Omega)} \leq a(y_k, y_k) \leq a(y, y_k) \\
&\leq a(y, y_k) - \int_\Omega (f(x, y, u(x)) - f(x, 0, u(x))y_k\,dx \\
&= \int_\Omega f(x, 0, u(x))y_k\,dx \\
&\leq \|f(x, 0, u(x))\|_{L^{p/2}(\Omega)}\|y_k\|_{L^{p/(p-2)}(\Omega)},
\end{aligned}$$

where $a(\cdot,\cdot)$ is the bilinear form associated to the operator and is defined in (2.1.10) (page 27). Using the continuous inclusion of $W^{1,p'}(\Omega)$ in $L^{p/(p-2)}(\Omega)$ we have that

$$m\|y_k\|^2_{H^1(\Omega)} \leq C\|y_k\|_{W^{1,p'}(\Omega)}.$$

Now we follow with the normal procedure. Set $A_k = \{x \in \Omega : |y(x)| > k\}$. On the right hand we have

$$\|y_k\|_{W^{1,p'}(\Omega)} \leq C|A_k|^{\frac{2-p'}{2p'}}\|y_k\|_{H^1(\Omega)},$$

then

$$\|y_k\|_{H^1(\Omega)} \leq C|A_k|^{\frac{2-p'}{2p'}}.$$

And on the left hand

$$\|y_k\|_{H^1(\Omega)} \geq \|y_k\|_{L^{\frac{2N}{N-2}}(\Omega)} = \|y_k\|_{L^{\frac{2N}{N-2}}(A_k)}$$

Take now $h > k$. In A_h, we have that $|y_k| > h - k$, and moreover $\|y_k\|_{L^{\frac{2N}{N-2}}(A_k)} \geq \|y_k\|_{L^{\frac{2N}{N-2}}(A_h)}$. Since

$$\|y_k\|_{L^{\frac{2N}{N-2}}(A_h)} = \left(\int_{A_h} |y_k|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}} \geq \left(\int_{A_h} |h-k|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}} = (h-k)|A_h|^{\frac{N-2}{2N}},$$

we have

$$(h-k)|A_h|^{\frac{N-2}{2N}} \leq c|A_k|^{\frac{2-p'}{2p'}},$$

or what is the same:

$$|A_h| \leq c\frac{|A_k|^{\frac{(2-p')N}{p'(N-2)}}}{(h-k)^{\frac{2N}{N-2}}}.$$

Now we may apply the Lemma of Kinderlehrer-Stampacchia, taking into account that $2N/(N-2) > 0$ and that the conditions imposed on p imply that $(2-p')N/(p'(N-2)) > 1$, and we have that $|A_k| = 0$ for all $k > d$, with d a constant that only depends on Ω, N, p, and $\|f(x,0,u(x))\|_{L^{p/2}(\Omega)}$. Then $y \in L^\infty(\Omega)$ and

$$\|y\|_{L^\infty(\Omega)} \leq d.$$

The regularity $W^{1,p}(\Omega)$ of y follows immediately from Theorem 2.1.3 and the inclusion $L^{p/2}(\Omega) \subset \left(W^{1,p'}(\Omega)\right)'$.

Let us suppose now that there does not exist necessarily a function ψ that bounds f independently of y, but that E0 holds. In that case we may define

$$f_j(x,y,u(x)) = \begin{cases} f(x,j,u(x)) & \text{if} \quad y > j \\ f(x,y,u(x)) & \text{if} \quad -j \leq y \leq j \\ f(x,-j,u(x)) & \text{if} \quad y < -j. \end{cases}$$

We have that f_j is decreasing monotone in the second variable and that $|f(x,y,u(x)| \leq \psi_j(x)$ for almost all $x \in \Omega$ with $\psi_j \in L^{p/2}(\Omega)$. Therefore, there exists a unique $y_j \in W^{1,p}(\Omega)$ such that

$$\begin{cases} Ay_j + a_0 y_j = f_j(x,y_j,u) & \text{in } \Omega \\ \partial_{\nu_A} y_j = g & \text{on } \Gamma. \end{cases}$$

Moreover, $\|y_j\|_{L^\infty(\Omega)} \leq d$ for all j. Thus, for $j > d$, $f_j(x,y_j,u(x)) = f(x,y_j,u(x))$ and we have that y_j is the solution of (3.1.1). From the monotonicity of f respect to y we deduce the uniqueness of the solution y_u of (3.1.1) in $W^{1,p}(\Omega)$, which implies $y_u = y_j$ for all $j > d$.

From Theorem 2.1.3 and the inclusion $L^{p/2}(\Omega) \subset \left(W^{1,p'}(\Omega)\right)'$, we get, for $M \geq \|y_u\|_{L^\infty(\Omega)}$

$$\|y_u\|_{W^{1,p}(\Omega)} \leq C\|\psi_M\|_{L^{p/2}(\Omega)}.$$

But as we have seen before, the norm in $L^\infty(\Omega)$ of y_u is bounded by a constant which only depends on Ω, N, p, and $\|f(x,0,u(x))\|_{L^{p/2}(\Omega)}$. Hence, we can find an M big enough and such that if we denote $C_{U_{ad}} = C\|\psi_M\|_{L^{p/2}(\Omega)}$, we have that

$$\|y_u\|_{W^{1,p}(\Omega)} \leq C_{U_{ad}}.$$

Let us take now $u_j(x) \to u(x)$ a.e. $x \in \Omega$. From the previous bound condition, we have that there exists $y \in W^{1,p}(\Omega)$ such that $y_{u_j} \rightharpoonup y$ weakly in $W^{1,p}(\Omega)$,and therefore $y_{u_j} \to y$ uniformly. Thus, using E0 and the dominated convergence theorem,

$f(x, y_{u_j}, u_j) \to f(x, y, u)$ in $L^{p/2}(\Omega)$ and, when we pass to the limit in the equation, necessarily $y = y_u$. Subtracting the equations that satisfy y_{u_j} and y_u and applying Theorem 2.1.1, it follows immediately that $y_{u_j} \to y_u$ in $W^{1,p}(\Omega)$. □

Theorem 3.1.2 *Suppose that*

E1 - $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ *is of class* C^1 *respect to the second and third variables,* $f(\cdot, 0, 0) \in L^{p/2}(\Omega)$, *for all* $M > 0$ *there exist a constant* $C_M > 0$ *and a function* $\psi_M \in L^{p/2}(\Omega)$ *such that*

$$\left|\frac{\partial f}{\partial y}(x,t,s)\right| \leq C_M \quad \text{and} \quad \left|\frac{\partial f}{\partial u}(x,t,s)\right| \leq \psi_M(x)$$

if $|t|, |s| \leq M$ *for a.e.* $x \in \Omega$, *and*

$$\frac{\partial f}{\partial y}(x,t,s) \leq 0$$

for all $(t, s) \in \mathbb{R}^2$ *and a.e.* $x \in \Omega$.

Then, for all $u \in L^\infty(\Omega)$ *there exists a unique solution of the state equation*

$$\left\{\begin{array}{rcll} Ay_u + a_0 y_u & = & f(x, y_u, u) & \text{in } \Omega \\ \partial_{\nu_A} y_u & = & g & \text{on } \Gamma. \end{array}\right. \tag{3.1.2}$$

and the mapping $G : L^\infty(\Omega) \to W^{1,p}(\Omega)$ *that relates the control to the state, given by* $G(u) = y_u$, *is of class* C^1. *If* $u, h \in L^\infty(\Omega)$ $y_u = G(u)$ *and* $z_h = G'(u)h$, *then* z_h *is the solution of*

$$\left\{\begin{array}{rcll} Az + a_0 z & = & \dfrac{\partial f}{\partial y}(x, y_u, u)z + \dfrac{\partial f}{\partial u}(x, y_u, u)h & \text{in } \Omega \\ \partial_{\nu_A} z & = & 0 & \text{on } \Gamma. \end{array}\right. \tag{3.1.3}$$

Proof. Observe that the assumptions in this theorem are enough to deduce for every $u \in L^\infty(\Omega)$ existence and uniqueness of a solution in $W^{1,p}(\Omega)$, y_u satisfying (3.1.1), just applying Theorem 3.1.1. Therefore, the mapping G is well defined. To check that G is of class C^1, we take

$$V(A) = \{y \in W^{1,p}(\Omega) : \; Ay + a_0 y \in L^{p/2}(\Omega), \; \partial_{n_A} y \in L^{p-1}(\Gamma)\}$$

with the norm

$$\|y\|_{V(A)} = \|y\|_{W^{1,p}(\Omega)} + \|Ay + a_0 y\|_{L^{p/2}(\Omega)} + \|\partial_{n_A} y\|_{L^{p-1}(\Gamma)}.$$

let us define now the function $F : V(A) \times L^\infty(\Omega) \to L^{p/2}(\Omega) \times L^{p-1}(\Gamma)$, $F(y,u) = (Ay + a_0 y - f(x,y,u), \partial_{n_A} y - g)$. The assumptions on f imply that F is of class C^1. Moreover $\frac{\partial F}{\partial y}(y,u)z = (Az + a_0 z - \frac{\partial f}{\partial y}(x,y,u)z, \partial_{n_A} z)$ is an isomorphism from $V(A)$ into $L^{p/2}(\Omega) \times L^{p-1}(\Gamma)$ due to Theorem 2.1.2. Taking into account that $F(y,u) = 0$ if and only if $y = G(u)$, we can apply the implicit function theorem (see for instance [15] or Zeidler [93]) to deduce that G is of class C^1 and satisfies that

$$F(G(u),u) = 0.$$

From this equality, derivating, (3.1.3) is deduced. □

Theorem 3.1.3 *Suppose that the assumptions in condition E1 of the previous theorem hold and that*

E2 - f is of class C^2 respect to the second and third variables and for all $M > 0$ there exists $\psi_M \in L^{p/2}(\Omega)$ such that

$$\left|\frac{\partial^2 f}{\partial y^2}(x,t,s)\right| + \left|\frac{\partial^2 f}{\partial u \partial y}(x,t,s)\right| + \left|\frac{\partial^2 f}{\partial u^2}(x,t,s)\right| \le \psi_M(x)$$

if $|t|, |s| \le M$ for a.e. $x \in \Omega$.

Then the mapping G is of class C^2, and if we take $h_1, h_2 \in L^\infty(\Omega)$, $z_i = G'(u)h_i$ and $z_{12} = G''(u)[h_1,h_2]$, we have

$$\left\{\begin{array}{rcll} Az_{12} + a_0 z_{12} &=& \frac{\partial f}{\partial y}(x,y_u,u)z_{12} + \frac{\partial^2 f}{\partial y^2}(x,y_u,u)z_1 z_2 + \frac{\partial^2 f}{\partial u^2}(x,y_u,u)h_1 h_2 & \\ & & + \frac{\partial^2 f}{\partial y \partial u}(x,y_u,u)(z_1 h_2 + z_2 h_1) & \text{in } \Omega \\ \partial_{\nu_A} z_{12} &=& 0 & \text{on } \Gamma. \end{array}\right. \tag{3.1.4}$$

Proof. Notice that the assumptions of this theorem are enough to deduce for every $u \in L^\infty(\Omega)$ existence and uniqueness of solution in $W^{1,p}(\Omega)$ of y_u satisfying (3.1.1), just applying Theorem 3.1.1. Therefore, the mapping G is well defined. Let us introduce again the space $V(A)$ and the mapping F just like in the proof of Theorem 3.1.2. The properties of the derivatives of f imply that F is of class C^2. Moreover, $\frac{\partial F}{\partial y}(y,u)$ is again an isomorphism from $V(A)$ into $L^{p/2}(\Omega)\times L^{p-1}(\Gamma)$. Taking into account that $F(y,u)=0$ if and only if $y=G(u)$, again we can apply the implicit function theorem to deduce that G is of class C^2 and that satisfies that

$$F(G(u),u)=0.$$

From this equality, derivating twice, (3.1.4) is deduced. □

3.2 Parabolic equations

Set T, Q, Σ y A, p, τ, k_1, $\tilde{k}_1$, σ_1, $\bar{\sigma}_1$ as in Section 2.2, with the coefficients of the operator A of class $C([0,T];C(\bar{\Omega}))$. Let us take f, g, y_0 functions, $f:Q\times\mathbb{R}\longrightarrow\mathbb{R}$, $g:\Sigma\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ y $y_0:\Omega\longrightarrow\mathbb{R}$, $y_0\in L^\infty(\Omega)\cap W^{1,p}(\Omega)$. We are going to study the parabolic equation

$$\left\{\begin{array}{rcll}\dfrac{\partial y}{\partial t}+Ay & = & f(x,t,y) & \text{in } Q,\\ \dfrac{\partial y}{\partial n_A} & = & g(s,t,y,v) & \text{on } \Sigma,\\ y(\cdot,0) & = & y_0 & \text{in } \Omega.\end{array}\right. \tag{3.2.1}$$

For every v we will denote by y_v the solution of the equation (3.2.1).

Suppose that

P1 - For all $y\in\mathbb{R}$, $f(\cdot,\cdot,y)$ is measurable in Q. For almost all $(x,t)\in Q$, $f(x,t,\cdot)$ is of class C^1 in $\mathbb{R}$. The following inequalities are satisfied:

$$|f(x,t,0)|\le M_1(x,t),\qquad C_0\ge\frac{\partial f}{\partial y}(x,t,y)\ge M_1(x,t)\eta(|y|),$$

where $C_0\in\mathbb{R}$, η is a decreasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$, and $M_1\in L^{\tilde{k}_1}(L^{k_1}(\Omega))$.

For all $y,v\in\mathbb{R}$, $g(\cdot,\cdot,y,v)$ is measurable on Σ. For all $v\in\mathbb{R}$ and almost all $(s,t)\in\Sigma$, $g(s,t,\cdot,v)$ is of class C^1 in $\mathbb{R}$. For almost all $(s,t)\in\Sigma$, $g(s,t,\cdot)$ and $g'_y(s,t,\cdot)$ are continuous in $\mathbb{R}^2$. The following inequalities hold:

$$|g(s,t,0,v)|\le N_1(s,t)+|v|,\qquad C_0\ge\frac{\partial g}{\partial y}(s,t,y,v)\ge(N_1(s,t)+|v|)\eta(|y|),$$

where $N_1\in L^{\bar{\sigma}_1}(L^{\sigma_1}(\Gamma))$.

Then we have

Theorem 3.2.1 *For every $v \in L^\infty(\Sigma)$ there exists a unique $y_v \in L^\tau(W^{1,p}) \cap C_b(\bar{Q} \setminus \bar{\Omega} \times \{0\})$ solution of* (3.2.1). *Moreover, the mapping Φ, given by $\Phi(v) = y_v$ is continuous from $L^\alpha(\Sigma)$ into $L^\tau(W^{1,p}) \cap C_b(\bar{Q} \setminus \bar{\Omega} \times \{0\})$ for any $N + 1 < \alpha < \infty$.*

Proof. Taking into account Proposition 2.2.8, the proof may be performed as in Casas, Raymond and Zidani [35], or Raymond and Zidani [78, 79]. □

Giving enough differentiability assumptions on the functions involved, we can assure that Φ is differentiable.

Theorem 3.2.2 *Suppose that P1 holds and*

P2 - For a.e. $(s,t) \in \Sigma$, $g(s,t,\cdot)$ is of class C^1 and the following inequality holds.

$$\left|\frac{\partial g}{\partial v}(s,t,y,v)\right| \leq (N_1(s,t) + |v|)\eta(|y|). \tag{3.2.2}$$

Then the mapping $\Phi : L^\infty(\Sigma) \to L^\tau(W^{1,p}(\Omega))$, given by $\Phi(v) = y_v$ is of class C^1. Moreover, if v, $h \in L^\infty(\Sigma)$, $y_v = \Phi(v)$ y $z_h = \Phi'(v)h$, then z_h is the solution of

$$\left\{\begin{array}{rcll} \dfrac{\partial z_h}{\partial t} + Az_h &=& \dfrac{\partial f}{\partial y}(x,t,y_v)z_h & \text{in } Q, \\ \dfrac{\partial z_h}{\partial n_A} &=& \dfrac{\partial g}{\partial y}(s,t,y_v,v)z_h + \dfrac{\partial g}{\partial v}(s,t,y_v,v)h & \text{on } \Sigma, \\ z_h(\cdot,0) &=& 0 & \text{in } \Omega. \end{array}\right. \tag{3.2.3}$$

Proof. From the previous theorem, we have that the mapping is well defined an is continuous. We are going to act as in the elliptic case to see that it is of class C^1. For that purpose set

$$V(A) = \left\{y \in L^\tau(W^{1,p}) : \partial_t y + Ay \in L^{\tilde{k}_1}(L^{k_1}(\Omega)),\ \partial_{n_A} y \in L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma)),\ y(0) \in L^\infty(\Omega)\right\}.$$

The mapping

$$F : V(A) \times L^\infty(\Sigma) \longrightarrow L^{\tilde{k}_1}(L^{k_1}(\Omega)) \times L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma)) \times L^\infty(\Omega)$$

$$F(y,v) = (\partial_t y + Ay - f(\cdot,y),\ \partial_{n_A} y - g(\cdot,y,v),\ y(0) - y_0)$$

is of class C^1. Moreover,

$$\frac{\partial F}{\partial y}(y,v)z = (\partial_t z + Az - \frac{\partial f}{\partial y}(\cdot,y)z,\ \partial_{n_A} z - \frac{\partial g}{\partial y}(\cdot,y,v)z,\ z(0))$$

is an isomorphism from $V(A)$ into $L^{\bar{k}_1}(L^{k_1}(\Omega)) \times L^{\bar{\sigma}_1}(L^{\sigma_1}(\Gamma)) \times L^{\infty}(\Omega)$. (This follows immediately from Proposition 2.2.8 and the discussion about the exponents in the proof of Proposition 2.2.9). Since $F(y,v)=0$ if and only if $y=\Phi(v)$, we have that

$$F(\Phi(v),v)=0.$$

Applying the implicit function theorem, we obtain that Φ is of class C^1 and derivating, we get the expression (3.2.3). □

If we also make the following extra assumptions on the regularity of f and g, we can prove that the mapping that relates the state and the control is of class C^2.

P3 - For a.e. $(x,t) \in Q$, $f(x,t,\cdot)$ is of class C^2 and the following inequality holds.

$$\left|\frac{\partial^2 f}{\partial y^2}(x,t,y)\right| \leq M_1(x,t)\eta(|y|). \tag{3.2.4}$$

For a.e. $(s,t) \in \Sigma$, $g(s,t,\cdot)$ is of class C^2 and the following inequality holds

$$\left|\frac{\partial^2 g}{\partial v^2}(s,t,y,v)\right| + \left|\frac{\partial^2 g}{\partial y^2}(s,t,y,v)\right| + \left|\frac{\partial^2 g}{\partial v \partial y}(s,t,y,v)\right| \leq (N_1(s,t)+|v|)\eta(|y|), \tag{3.2.5}$$

Under these assumptions, we can prove that the mapping that relates the control and the state is of class C^2.

Theorem 3.2.3 *Suppose that P1, P2 and P3 hold. Then the mapping $\Phi : L^\infty(\Sigma) \to L^r(W^{1,p}(\Omega))$ is of class C^2. Moreover, if we take h_1, $h_2 \in L^\infty(\Sigma)$, $z_i = G'(v)h_i$ y $z_{12} = G''(v)[h_1,h_2]$, we get*

$$\left\{\begin{array}{rcll} \dfrac{\partial z_{12}}{\partial t} + Az_{12} & = & \dfrac{\partial f}{\partial y}(x,t,y_v)z_{12} + \dfrac{\partial^2 f}{\partial y^2}(x,t,y_v)z_1z_2 & \textit{in } Q, \\ \dfrac{\partial z_{12}}{\partial n_A} & = & \dfrac{\partial g}{\partial y}(s,t,y_v,v)z_{12} + \dfrac{\partial^2 g}{\partial y^2}(s,t,y_v,v)z_1z_2 + & \\ & & +\dfrac{\partial^2 g}{\partial v^2}(s,t,y_v,v)h_1h_2 + \dfrac{\partial^2 g}{\partial y \partial v}(s,t,y_v,v)(z_1h_2+z_2h_1) & \textit{on } \Sigma, \\ z_h(\cdot,0) & = & 0 & \textit{in } \Omega. \end{array}\right. \tag{3.2.6}$$

Proof. Define $V(A)$ and $F(y,v)$ as in the proof of the previous theorem. Now assumption P3 allows us assure that F is of class C^2. Since $\frac{\partial F}{\partial y}(y,v)$ is an isomorphism, the implicit function theorem lets us assure that Φ is of class C^2. Derivating twice, we obtain expression (3.2.6). □

3.3 Sensitivity of the state with respect to diffuse perturbations of the control

To establish Pontryagin's principle for the problems of page 16, we must state another kind of Taylor expansion, based on diffuse perturbations of the control. Now it is not necessary to suppose differentiability of the involved functions with respect to the control, and we only suppose that they are C^1 with respect to the state.

3.3.1 Elliptic case

Let A be the elliptic operator introduced in Section 3.1, $p > N$, $a_0 \in L^{p/2}(\Omega)$, f a function $f : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$, $g \in L^{p-1}(\Gamma)$. Let us start with the following lemma.

Lemma 3.3.1 *For all $\rho \in (0,1)$, there exists a sequence of measurable sets $E_\rho^k \subset \Omega$ such that*

$$|E_\rho^k| = \rho|\Omega|$$

and

$$\lim_{k \to 0} \frac{1}{\rho}\chi_{E_\rho^k} = 1 \text{ weakly* in } L^\infty(\Omega), \tag{3.3.1}$$

where $\chi_{E_\rho^k}$ is the characteristic function of the set E_ρ^k.

Proof. There exist two different proofs of this important lemma in the literature. A constructive one, due to Casas [22] and one by Raymond and Zidani [78] which uses Liapunov's convexity Theorem. □

Let us take

$$U_{ad} = \{u : \Omega \to \mathbb{R} : \ u(x) \in K_\Omega(x) \text{ a.e. } x \in \Omega\},$$

where K_Ω is a measurable multimapping with non empty and closed values in $\mathcal{P}(\mathbb{R})$.

Theorem 3.3.2 *Suppose that E0 (page 66) holds and that*

E3 - $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 respect to y, continuous respect to u and measurable respect to x, for all $M > 0$ there exists $C_M > 0$ such that

$$\left|\frac{\partial f}{\partial y}(x,t,u(x))\right| \le C_M$$

if $|t| \leq M$ *for all* $u \in U_{ad}$ *and a.e.* $x \in \Omega$, *and*

$$\frac{\partial f}{\partial y}(x, t, u(x)) \leq 0$$

for all $t \in \mathbb{R}$, *all* $u \in U_{ad}$ *and a.e.* $x \in \Omega$.

Then for all $\rho \in (0,1)$ *and all* $u_1, u_2 \in U_{ad}$ *there exists a measurable set* $E_\rho \subset \Omega$ *such that*

$$|E_\rho| = \rho|\Omega|,$$

and

$$y_\rho = y_1 + \rho z + r_\rho, \text{ with } \lim_{\rho \to 0} \frac{1}{\rho}\|r_\rho\|_{W^{1,p}(\Omega)} = 0, \tag{3.3.2}$$

where

$$u_\rho = \begin{cases} u_1 & \text{in } \Omega \setminus E_\rho \\ u_2 & \text{in } E_\rho, \end{cases}$$

$$y_\rho = y_{u_\rho}, \quad y_1 = y_{u_1},$$

and

$$\begin{cases} Az + a_0 z &= \dfrac{\partial f}{\partial y}(x, y_1, u_1)z + f(x, y_1, u_2) - f(x, y_1, u_1) & \text{in } \Omega \\ \partial_{n_A} z &= 0 & \text{on } \Gamma. \end{cases}$$

Proof. Set $(E_\rho^k)_k$ as in Lemma 3.3.1 and set

$$u_\rho^k = \begin{cases} u_1 & \text{in } \Omega \setminus E_\rho^k \\ u_2 & \text{in } E_\rho^k, \end{cases}$$

$$y_\rho^k = y_{u_\rho^k}$$

and

$$\xi_\rho^k = \frac{y_\rho^k - y_1}{\rho} - z.$$

We have the following equation

$$\begin{cases} A\xi_\rho^k + a_0\xi_\rho^k + a_\rho^k\xi_\rho^k &= f_\rho^k + h_\rho^k & \text{in } \Omega \\ \partial_{n_A}\xi_\rho^k &= 0 & \text{on } \Gamma, \end{cases}$$

where

$$a_\rho^k = -\int_0^1 \frac{\partial f}{\partial y}(x, y_1 + \theta(y_\rho^k - y_1), u_\rho^k)\, d\theta,$$

$$f_\rho^k = \left(\int_0^1 \frac{\partial f}{\partial y}(x, y_1 + \theta(y_\rho^k - y_1), u_\rho^k)\, d\theta - \frac{\partial f}{\partial y}(x, y_1, u_1)\right) z$$

and

$$h_\rho^k = \left(1 - \frac{1}{\rho}\chi_{E_\rho^k}\right)(f(x, y_1, u_1) - f(x, y_1, u_2)).$$

We may write $\xi_\rho^k = \xi_\rho^{k,1} + \xi_\rho^{k,2}$, where

$$\begin{cases} A\xi_\rho^{k,1} + a_0\xi_\rho^{k,1} + a_\rho^k\xi_\rho^{k,1} = f_\rho^k & \text{in } \Omega \\ \partial_{n_A}\xi_\rho^{k,1} = 0 & \text{on } \Gamma, \end{cases}$$

and

$$\begin{cases} A\xi_\rho^{k,2} + a_0\xi_\rho^{k,2} + a_\rho^k\xi_\rho^{k,2} = h_\rho^k & \text{in } \Omega \\ \partial_{n_A}\xi_\rho^{k,2} = 0 & \text{on } \Gamma. \end{cases}$$

Due to Theorem 2.1.3

$$\|\xi_\rho^{k,1}\|_{W^{1,p}(\Omega)} \le C\|f_\rho^k\|_{L^\infty(\Omega)}. \tag{3.3.3}$$

We will denote ζ_ρ^k the solution of

$$\begin{cases} A\zeta_\rho^k + a_0\zeta_\rho^k + a\zeta_\rho^k = h_\rho^k & \text{in } \Omega \\ \partial_{n_A}\zeta_\rho^k = 0 & \text{on } \Gamma, \end{cases}$$

where

$$a = -\frac{\partial f}{\partial y}(x, y_1, u_1).$$

The operator $\mathcal{T}$ that relates ζ, the solution in $W^{1,p}(\Omega)$ of

$$\begin{cases} A\zeta + a_0\zeta + a\zeta = h & \text{in } \Omega \\ \partial_{n_A}\zeta = 0 & \text{on } \Gamma, \end{cases}$$

with h is continuous from $\left(W^{1,p'}(\Omega)\right)'$ into $W^{1,p}(\Omega)$ (regularity Theorem 2.1.3). Since the injection from $L^\infty(\Omega)$ into $\left(W^{1,p'}(\Omega)\right)'$ is compact, $\mathcal{T}$ can be considered a compact operator from $L^\infty(\Omega)$ into $W^{1,p}(\Omega)$. From (3.3.1) it follows that

$$\lim_{k\to\infty} h_\rho^k = 0 \text{ weakly in } L^{p/2}(\Omega),$$

and hence

$$\lim_{k\to\infty} \|\zeta_\rho^k\|_{W^{1,p}(\Omega)} = 0.$$

So for all $\rho \in (0,1)$ there exists $k(\rho)$ such that

$$\|\zeta_\rho^{k(\rho)}\|_{W^{1,p}(\Omega)} \le \rho. \tag{3.3.4}$$

Notice that

$$\lim_{\rho\to 0} u_\rho^{k(\rho)}(x) = u_1(x) \text{ for a.e. } x \in \Omega$$

and, for Theorem 3.1.1 and the continuous injection from $W^{1,p}(\Omega)$ into $C(\bar{\Omega})$, we have that

$$\lim_{\rho\to 0} y_\rho^{k(\rho)} = y_1 \text{ in } C(\bar{\Omega}).$$

Therefore

$$\lim_{\rho\to 0} \|f_\rho^{k(\rho)}\|_{L^\infty(\Omega)} = 0, \tag{3.3.5}$$

and

$$\lim_{\rho\to 0} \|a - a_\rho^{k(\rho)}\|_{L^\infty(\Omega)} = 0. \tag{3.3.6}$$

Obviously

$$\begin{cases} A(\xi_\rho^{k(\rho),2} - \zeta_\rho^{k(\rho)}) + a_0(\xi_\rho^{k(\rho),2} - \zeta_\rho^{k(\rho)}) + a_\rho^{k(\rho)}(\xi_\rho^{k(\rho),2} - \zeta_\rho^{k(\rho)}) = (a - a_\rho^{k(\rho)})\zeta_\rho^{k(\rho)} & \text{in } \Omega \\ \partial_{n_A}(\xi_\rho^{k(\rho),2} - \zeta_\rho^{k(\rho)}) = 0 & \text{on } \Gamma, \end{cases}$$

and

$$\|\xi_\rho^{k(\rho),2} - \zeta_\rho^{k(\rho)}\|_{W^{1,p}(\Omega)} \le \|a - a_\rho^{k(\rho)}\|_{L^\infty(\Omega)} \|\zeta_\rho^{k(\rho)}\|_{W^{1,p}(\Omega)}. \tag{3.3.7}$$

If we write

$$\|\xi_\rho^{k(\rho)}\|_{W^{1,p}(\Omega)} = \|\xi_\rho^{k(\rho),1} + \xi_\rho^{k(\rho),2} - \zeta_\rho^{k(\rho)} + \zeta_\rho^{k(\rho)}\|_{W^{1,p}(\Omega)} \le$$

$$\le \|\xi_\rho^{k(\rho),1}\|_{W^{1,p}(\Omega)} + \|\xi_\rho^{k(\rho),2} - \zeta_\rho^{k(\rho)}\|_{W^{1,p}(\Omega)} + \|\zeta_\rho^{k(\rho)}\|_{W^{1,p}(\Omega)},$$

taking into account (3.3.3), (3.3.5), (3.3.4), (3.3.6) and (3.3.7), we have that

$$\lim_{\rho\to 0} \|\xi_\rho^{k(\rho)}\|_{W^{1,p}(\Omega)} = 0.$$

Let us take hence $E_\rho = E_\rho^{k(\rho)}$. We have that $r_\rho = \rho\xi_\rho^{k(\rho)}$ and (3.3.2) holds. □

3.3.2 Parabolic case

Let us suppose T, Q, Σ y A, p, τ, k_1, $\tilde{k}_1$, σ_1, $\bar{\sigma}_1$ as in Section 2.2. We will suppose some adttional regularity for the problem introduced in Section 3.2- We will suppose that the boundary Γ is of class $C^{1+\hat{\varepsilon}}$ and the coefficients of the operator A are of class $C([0,T];C^{0,\hat{\varepsilon}}(\bar{\Omega}))$, for some $0 < \hat{\varepsilon} < 1$. Set f, g, y_0 functions, $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \Sigma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ y $y_0 : \Omega \longrightarrow \mathbb{R}$, $y_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

Due to the regularity and continuity results, we are now ready to establish Taylor expansions for the state. For a proof of the following lemmas see for instance [22] or [78].

Lemma 3.3.3 *For $\rho \in (0,1)$, there exists a sequence of measurable sets $E^k_\rho \subset \Sigma$ such that*

$$|E^k_\rho| = \rho|\Sigma|$$

y

$$\lim_{k\to\infty} \frac{1}{\rho}\chi_{E^k_\rho} = 1 \text{ weakly-* in } L^\infty(\Sigma), \tag{3.3.8}$$

where $\chi_{E^k_\rho}$ is that characteristic function of the set E^k_ρ.

Remark 3.3.1 *Now, with $|E^k_\rho|$ we denote the Lebesgue measure on Σ, and not on $\mathbb{R}^N \times \mathbb{R}$, because all the measures would be zero if not*

Set

$$V_{ad} = \{v \in L^\infty(\Sigma) : \ v(s,t) \in K_\Sigma(s,t) \text{ for a.e. } (s,t) \in \Sigma\},$$

where K_Σ is a measurable multimapping with non empty, compact values in $\mathcal{P}(\mathbb{R})$

Theorem 3.3.4 *Suppose P1 holds. Then for all $\rho \in (0,1)$, and all $v_1, v_2 \in V_{ad}$, there exists a measurable set $E_\rho \subset \Sigma$ such that*

$$|E_\rho| = \rho|\Sigma|,$$

and

$$y_\rho = y_1 + \rho z + r_\rho \text{ with } \lim_{\rho\to 0} \frac{1}{\rho}\|r_\rho\|_{L^r(W^{1,p})} = 0, \tag{3.3.9}$$

where

$$v_\rho(s,t) = \begin{cases} v_1 & \text{in } \Sigma \setminus E_\rho \\ v_2 & \text{in } E_\rho \end{cases}, \quad y_\rho = y_{v_\rho}, \quad y_1 = y_{v_1},$$

and

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az = f_y'(x,t,y_1)z & \textit{in } Q, \\ \dfrac{\partial z}{\partial n_A} = g_y'(s,t,y_1,v_1)z + g(s,t,y_1,v_2) - g(s,t,y_1,v_1) & \textit{on } \Sigma, \\ z(\cdot,0) = 0 & \textit{in } \Omega\times\{0\}. \end{cases}$$

Proof. Let us prove (3.3.9). Take a sequence $(E_\rho^k)_k$ as in Lemma 3.3.3. Define

$$v_\rho^k(s,t) = \begin{cases} v_1 & \text{in } \Sigma\setminus E_\rho^k \\ v_2 & \text{in } E_\rho^k \end{cases}, \quad y_\rho^k = y_{v_\rho^k} \text{ and } \xi_\rho^k = \frac{y_\rho^k - y_1}{\rho} - z.$$

The function ξ_ρ^k satisfies equation

$$\begin{cases} \dfrac{\partial \xi_\rho^k}{\partial t} + A\xi_\rho^k + a_\rho^k\xi_\rho^k = f_\rho^k & \text{in } Q, \\ \dfrac{\partial \xi_\rho^k}{\partial n_A} + b_\rho^k\xi_\rho^k = g_\rho^k + h_\rho^k & \text{on } \Sigma, \\ \xi_\rho^k(\cdot,0) = 0 & \text{in } \Omega, \end{cases}$$

with

$$a_\rho^k(x,t) = -\int_0^1 f_y'(x,t,(y_1+\theta(y_\rho^k - y_1)))\,d\theta,$$

$$f_\rho^k = (-f_y'(x,t,y_1) - a_\rho^k)z,$$

$$b_\rho^k(s,t) = -\int_0^1 g_y'(s,t,(y_1+\theta(y_\rho^k - y_1)),v_\rho^k)\,d\theta,$$

$$g_\rho^k = (-g_y'(s,t,y_1,v_1) - b_\rho^k)z,$$

and

$$h_\rho^k = (1 - \frac{1}{\rho}\chi_{E_\rho^k})(g(s,t,y_1,v_1) - g(s,t,y_1,v_2)).$$

Denote by $\xi_\rho^{k,1}$ the solution of

$$\begin{cases} \dfrac{\partial \xi_\rho^{k,1}}{\partial t} + A\xi_\rho^{k,1} + a_\rho^k\xi_\rho^{k,1} = f_\rho^k & \text{in } Q, \\ \dfrac{\partial \xi_\rho^{k,1}}{\partial n_A} + b_\rho^k\xi_\rho^{k,1} = g_\rho^k & \text{on } \Sigma, \\ \xi_\rho^{k,1}(\cdot,0) = 0 & \text{in } \Omega, \end{cases}$$

by $\xi_\rho^{k,2}$ the solution of

$$\begin{cases} \dfrac{\partial \xi_\rho^{k,2}}{\partial t} + A\xi_\rho^{k,2} + a_\rho^k \xi_\rho^{k,2} &= 0 \quad \text{in } Q, \\ \dfrac{\partial \xi_\rho^{k,2}}{\partial n_A} + b_\rho^k \xi_\rho^{k,2} &= h_\rho^k \quad \text{on } \Sigma, \\ \xi_\rho^{k,2}(\cdot,0) &= 0 \quad \text{in } \Omega, \end{cases} \tag{3.3.10}$$

and by ζ_ρ^k the solution of

$$\begin{cases} \dfrac{\partial \zeta_\rho^k}{\partial t} + A\zeta_\rho^k + a\zeta_\rho^k &= 0 \quad \text{in } Q, \\ \dfrac{\partial \zeta_\rho^k}{\partial n_A} + b\zeta_\rho^k &= h_\rho^k \quad \text{on } \Sigma, \\ \zeta_\rho^k(\cdot,0) &= 0 \quad \text{in } \Omega, \end{cases} \tag{3.3.11}$$

where $a(x,t) = -f_y'(x,t,y_1(x,t))$, and $b(s,t) = -g_y'(s,t,y_1(s,t),v_1(s,t))$. From (3.3.10) and (3.3.11) it follows that:

$$\begin{cases} \dfrac{\partial(\xi_\rho^{k,2} - \zeta_\rho^k)}{\partial t} + A(\xi_\rho^{k,2} - \zeta_\rho^k) + a_\rho^k(\xi_\rho^{k,2} - \zeta_\rho^k) &= (a - a_\rho^k)\zeta_\rho^k \quad \text{in } Q, \\ \dfrac{\partial(\xi_\rho^{k,2} - \zeta_\rho^k)}{\partial n_A} + b_\rho^k(\xi_\rho^{k,2} - \zeta_\rho^k) &= (b - b_\rho^k)\zeta_\rho^k \quad \text{on } \Sigma, \\ (\xi_\rho^{k,2} - \zeta_\rho^k)(\cdot,0) &= 0 \quad \text{in } \Omega. \end{cases}$$

Due to Propositions 2.2.8 and 2.2.9, $\xi_\rho^{k,1}$, $\xi_\rho^{k,2}$ and ζ_ρ^k belong to $L^\tau(W^{1,p})$ and the following estimates hold:

$$\|\xi_\rho^{k,2} - \zeta_\rho^k\|_{L^\tau(W^{1,p})} \le C_1\Big(\|a - a_\rho^k\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|b - b_\rho^k\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}\Big)\|\zeta_\rho^k\|_{L^\tau(W^{1,p})}, \tag{3.3.12}$$

$$\|\xi_\rho^{k,1}\|_{L^\tau(W^{1,p})} \le C_2(\|f_\rho^k\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|g_\rho^k\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}), \tag{3.3.13}$$

where the constants C_1 and C_2 do not depend on k.

The operator $\mathcal{T}$ that relates ζ, the solution in $L^\tau(W^{1+\epsilon,p}) \cap W^{1,\tau}((W^{1,p'})')$ of

$$\begin{cases} \dfrac{\partial \zeta}{\partial t} + A\zeta + a\zeta &= 0 \quad \text{in } Q, \\ \dfrac{\partial \zeta}{\partial n_A} + b\zeta &= h \quad \text{on } \Sigma, \\ \zeta(\cdot,0) &= 0 \quad \text{in } \Omega, \end{cases} \tag{3.3.14}$$

with h, is continuous from $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$ into $L^{\tau}(W^{1+\varepsilon,p}) \cap W^{1,\tau}((W^{1,p'})')$. The continuity in $L^{\tau}(W^{1+\varepsilon,p})$ follows from Proposition 2.2.7. With equation (3.3.14) we prove that ζ belongs to $W^{1,\tau}((W^{1,p'})')$, and the corresponding estimate follows from the estimate in $L^{\tau}(W^{1+\varepsilon,p})$, in a similar way as is done at the end of the proof of Theorem 2.2.1. Since the injection from $W^{1+\varepsilon,p}(\Omega)$ in $W^{1,p}(\Omega)$ is compact, (see Grisvard [59]), then the injection from $L^{\tau}(W^{1+\varepsilon,p}) \cap W^{1,\tau}((W^{1,p'})')$ en $L^{\tau}(W^{1,p})$ is compact (see Simon, [83, Corollary 4]). So $\mathcal{T}$ can be considered a compact operator from $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$ into $L^{\tau}(W^{1,p})$. From (3.3.8) it follows that

$$\lim_{k\to\infty} h_\rho^k = 0 \text{ weakly in } L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma)),$$

and hence

$$\lim_{k\to\infty} \|\zeta_\rho^k\|_{L^{\tau}(W^{1,p})} = 0.$$

So for every $\rho \in (0,1)$, there exists $k(\rho)$ such that

$$\|\zeta_\rho^{k(\rho)}\|_{L^{\tau}(W^{1,p})} \leq \rho. \tag{3.3.15}$$

Notice that

$$\lim_{\rho\to 0} v_\rho^{k(\rho)} = v_1 \text{ in } L^{\alpha}(\Sigma) \text{ for any } \alpha < \infty.$$

Therefore, due to Theorem 3.2.1, we have that

$$\lim_{\rho\to 0} y_\rho^{k(\rho)} = y_1 \text{ in } C_b(\bar{Q} \setminus \bar{\Omega} \times \{0\}). \tag{3.3.16}$$

Relation (3.3.16) implies that

$$\lim_{\rho\to 0} f_\rho^{k(\rho)} = 0 \text{ in } L^{\tilde{k}_1}(L^{k_1}(\Omega)), \quad \lim_{\rho\to 0} g_\rho^{k(\rho)} = 0 \text{ in } L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma)), \tag{3.3.17}$$

and

$$\lim_{\rho\to 0}(a - a_\rho^{k(\rho)}) = 0 \text{ in } L^{\tilde{k}_1}(L^{k_1}(\Omega)), \quad \lim_{\rho\to 0}(b - b_\rho^{k(\rho)}) = 0 \text{ in } L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma)). \tag{3.3.18}$$

With (3.3.12), (3.3.13), (3.3.15), (3.3.17) y (3.3.18), we obtain

$$\lim_{\rho\to 0} \|\xi_\rho^{k(\rho)}\|_{L^{\tau}(W^{1,p})} = 0. \tag{3.3.19}$$

Set $E_\rho = E_\rho^{k(\rho)}$. We have that $r_\rho = \rho\xi_\rho^{k(\rho)}$. Then (3.3.9) follows from (3.3.19). □

Part II

Optimality Conditions

This part of the thesis, which is its kernel, is devoted to the study of first and second order optimality conditions for the treated control problems.

For first order conditions, two main ways exist. Deduce an Euler-Lagrange equation in case the set of controls is convex or to show that Pontryagin's Principle holds in case it is not convex.

Euler-Lagrange conditions will be deduced from general results for abstract optimization problems. Nevertheless Pontryagin's Principle requires an study more adapted to control problems. In this case the key is in doing an adequate Taylor expansion for the state, as it was done in Chapter 3, and for the functional, based in appropriate perturbations of the control. In our case we use diffuse perturbations.

We will also study in this part second order conditions for problems with a finite number of state constraints and a convex set of admissible controls First we will apply results for abstract optimization problems. In this case we just have to see that under the assumptions imposed, our control problems verify the conditions in the abstract theorems. The assumptions to be verified for a result on necessary conditions are not specially difficult. It is when we deduce sufficient conditions when the proof becomes more complicated. The abstract results are due to Casas and Tröltzsch [36]. In that paper it is also explained how to apply it to various control problems and the difficulties that appear. They remark that the regularity of the adjoint state becomes sometimes the main difficulty to deduce sufficient conditions. We must give strong enough regularity conditions on the derivatives of the functions in the objective and the restrictions to obtain a regular enough adjoint state.

Finally, we establish second order conditions that involve the Hamiltonian.

Chapter 4

Functionals involved in the control problems

In this chapter we study the functionals involved in the control problems. We establish, under adequate assumptions, properties of continuity and differentiability. The goal is to satisfy the assumptions of a theorem about optimality conditions for general optimization problems. For problems with a non convex set of admissible controls, we establish a Taylor expansion of the functional with respect to diffuse perturbations of the control. The purpose in this case is to establish optimality conditions in the form of Pontryagin's principle.

4.1 Differentiability properties

4.1.1 Elliptic case

We will suppose again that Ω is of class C^1, Γ its boundary, A an elliptic operator with continuous coefficients of the form (2.1.1) (page 23), $p > N$, $a_0 \in L^{p/2}(\Omega)$, f a function $f : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$, $g \in L^{p-1}(\Gamma)$.

Theorem 4.1.1 *Suppose that the assumption on C^1 differentiability of f E1 (page 69) holds and that $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function*

E4 - measurable in x and of class C^1 in the second and third variables and that for all

$M > 0$ there exists $\psi_M \in L^1(\Omega)$ such that $|L(x,0,0)| \leq \psi_M(x)$ for a.e. $x \in \Omega$ and

$$\left|\frac{\partial L}{\partial y}(x,y,u)\right| + \left|\frac{\partial L}{\partial u}(x,y,u)\right| \leq \psi_M(x)$$

if $|y|, |u| \leq M$ for a.e. $x \in \Omega$.

Then, the functional $J : L^\infty(\Omega) \to \mathbb{R}$, given by

$$J(u) = \int_\Omega L(x, y_u, u)dx \tag{4.1.1}$$

is of class C^1. Moreover, for all $u, h \in L^\infty(\Omega)$

$$J'(u)h = \int_\Omega \left(\frac{\partial L}{\partial u}(x, y_u, u) + \varphi_{0u}\frac{\partial f}{\partial u}(x, y_u, u)\right) h\, dx \tag{4.1.2}$$

where $y_u = G(u)$ ($G(u)$ defined as in Theorem 3.1.2) and $\varphi_{0u} \in W^{1,p'}(\Omega)$ is the unique solution of the problem

$$\left\{\begin{array}{rcll} A^*\varphi + a_0\varphi & = & \frac{\partial f}{\partial y}(x, y_u, u)\varphi + \frac{\partial L}{\partial y}(x, y_u, u) & \text{in } \Omega \\ \partial_{n_{A^*}}\varphi & = & 0 & \text{on } \Gamma, \end{array}\right. \tag{4.1.3}$$

where A^* is the adjoint operator of A

$$A^*\varphi = -\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a_{ji}(x)\frac{\partial \varphi}{\partial x_i}\right).$$

Proof. Consider the function $F_0 : C(\bar{\Omega}) \times L^\infty(\Omega) \to \mathbb{R}$ defined by

$$F_0(y, u) = \int_\Omega L(x, y(x), u(x))\, dx.$$

Due to the assumptions on L it is straight to prove that F_0 is of class C^1. Now, applying the chain rule to $J(u) = F_0(G(u), u)$ and using Theorem 3.1.2 and the fact that $W^{1,p}(\Omega) \subset C(\bar{\Omega})$ we obtain that J is of class C^1 and

$$J'(u)h = \int_\Omega \left(\frac{\partial L}{\partial y}(x, y_u, u)z_h + \frac{\partial L}{\partial u}(x, y_u, u)h\right) dx,$$

where $z_h = G'(u)h$ and is given by (3.1.3). Let us take now φ_{0u} solution of (4.1.3). The assumptions made on the derivatives of f and L and Theorem 2.1.3 assure us that $\varphi_{0u} \in W^{1,p'}(\Omega)$. We can therefore apply Green's formula and deduce (4.1.2) from the previous equality. □

Theorem 4.1.2 *Suppose that the assumptions on the differentiability on f E1 (page 69) and E2 (page 70) and on L E4 (page 87) hold. Suppose also that*

E5 - L is of class C^2 in y, u and for all $M > 0$ there exists $\psi_M \in L^1(\Omega)$, such that

$$\left|\frac{\partial^2 L}{\partial y^2}(x,y,u)\right| + \left|\frac{\partial^2 L}{\partial u \partial y}(x,y,u)\right| + \left|\frac{\partial^2 L}{\partial u^2}(x,y,u)\right| \leq \psi_M(x)$$

if $|y|, |u| \leq M$ for a.e. $x \in \Omega$.

Then, the functional $J : L^\infty(\Omega) \to \mathbb{R}$ is of class C^2 and for all $u, h_1, h_2 \in L^\infty(\Omega)$

$$J''(u)h_1h_2 =$$

$$\int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x,y_u,u)z_1z_2 + \frac{\partial^2 L}{\partial y \partial u}(x,y_u,u)(z_1h_2+z_2h_1) + \frac{\partial^2 L}{\partial u^2}(x,y_u,u)h_1h_2\right.$$

$$\left. + \varphi_{0u}\left(\frac{\partial^2 f}{\partial y^2}(x,y_u,u)z_1z_2 + \frac{\partial^2 f}{\partial y \partial u}(x,y_u,u)(z_1h_2+z_2h_1) + \frac{\partial^2 f}{\partial u^2}(x,y_u,u)h_1h_2\right)\right] dx \tag{4.1.4}$$

where $y_u = G(u)$ ($G(u)$ defined as in Theorem 3.1.2), $\varphi_{0u} \in W^{1,p'}(\Omega)$ is the unique solution of problem (4.1.3) *and $z_i = G'(u)h_i$, $i = 1, 2$.*

Proof. Consider again the function $F_0 : C(\bar{\Omega}) \times L^\infty(\Omega) \to \mathbb{R}$ defined by

$$F_0(y,u) = \int_\Omega L(x, y(x), u(x))\, dx.$$

Due to the assumptions on L it is straight to prove that F_0 is of class C^2. Now, applying the chain rule to $J(u) = F_0(G(u), u)$ and using Theorem 3.1.3 and the fact that $W^{1,p}(\Omega) \subset C(\bar{\Omega})$ we obtain that J is of class C^2 and the formula (4.1.4) for the second derivative. □

Theorem 4.1.3 *Suppose that the assumptions on C^1 differentiability of f in E1 (page 69) hold and that for all $1 \leq j \leq n_d + n_i$, $g_j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a function*

E6 - measurable in x, of class C^1 in the variable η (η denotes the variable for the gradient) and there exist a constant $C > 0$ and a function $\psi_1 \in L^{p'}(\Omega)$ such that

$$\left|\frac{\partial g_j}{\partial \eta}(x,\eta)\right| \leq C|\eta|^{p-1} + \psi_1(x)$$

for a.e. $x \in \Omega$.

Then for all $1 \le j \le n_d + n_i$, *the functional* $G_j : L^\infty(\Omega) \to \mathbb{R}$, *given by*

$$G_j(u) = \int_\Omega g_j(x, \nabla y_u(x))dx, \tag{4.1.5}$$

is of class C^1. *Moreover, for all* $u, h \in L^\infty(\Omega)$

$$G_j'(u)h = \int_\Omega \varphi_{ju} \frac{\partial f}{\partial u}(x, y_u, u)h\, dx \tag{4.1.6}$$

where $y_u = G(u)$, $\varphi_{ju} \in W^{1,p'}(\Omega)$ *is the unique solution f the problem*

$$\left\{ \begin{array}{rcll} A^*\varphi_{ju} + a_0\varphi_{ju} & = & \frac{\partial f}{\partial y}(x, y_u, u)\varphi_{ju} - \operatorname{div}\left(\frac{\partial g_j}{\partial \eta}(x, \nabla y_u)\right) & \text{in } \Omega \\ \partial_{n_{A^*}}\varphi_{ju} & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{4.1.7}$$

Proof. It is enough to consider the function of class C^1 $F_j : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$F_j(y) = \int_\Omega g_j(x, \nabla y(x))\, dx.$$

Taking into account Theorem 3.1.2, we know that $y_u \in W^{1,p}(\Omega)$. Moreover, due to assumption E6,

$$\frac{\partial g_j}{\partial \eta_i}(x, \nabla y_u) \in L^{p'}(\Omega);$$

therefore, Theorem 2.1.3 can be used to deduce that φ_{ju} is well defined and belongs to $W^{1,p'}(\Omega)$. Derivating F_j, using the chain rule and making an integration by parts, we obtain expression (4.1.6) for the derivative. □

Theorem 4.1.4 *Suppose that the assumptions on the differentiability of f E1 (page 69) and E2 (page 70) and of* g_j *E6 hold. Suppose also that*

E7 - g_j *is of class* C^2 *with respect to* η *and there exist a constant* $C > 0$ *and a function* $\psi_2 \in L^{p/(p-2)}(\Omega)$ *such that*

$$\left|\frac{\partial^2 g_j}{\partial \eta^2}(x, \eta)\right| \le C|\eta|^{p-2} + \psi_2(x) \text{ a.e. } x \in \Omega.$$

Then for all $1 \le j \le n_d + n_i$, *the functional* $G_j : L^\infty(\Omega) \to \mathbb{R}$ *is of class* C^2. *Moreover, for all* $u, h_1, h_2 \in L^\infty(\Omega)$

$$G_j''(u)h_1h_2 = \int_\Omega \Bigg[\nabla^T z_2 \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla y_u)\nabla z_1$$
$$+\varphi_{ju}\left(\frac{\partial^2 f}{\partial y^2}(x, y_u, u)z_1z_2 + \frac{\partial^2 f}{\partial y \partial u}(x, y_u, u)(z_1h_2 + z_2h_1) + \frac{\partial^2 f}{\partial u^2}(x, y_u, u)h_1h_2\right)\Bigg]\, dx \tag{4.1.8}$$

where $y_u = G(u)$, $\varphi_{ju} \in W^{1,p'}(\Omega)$ *is the unique solution of problem* (4.1.7) *and* $z_i = G'(u)h_i$, $i = 1, 2$.

Proof. The function $F_j : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$F_j(y) = \int_\Omega g_j(x, \nabla y(x))\, dx$$

is of class C^2. Derivating with the chain rule, we obtain expression (4.1.8) for the second derivative. Assumption E7 assures us that the second derivative of g_j with respect to the gradient of the state belongs to $L^{p/(p-2)}(\Omega)$, and assumption E1 assures us that the gradient of z_i is in $L^p(\Omega)$, and hence the integral is well defined. The second term of the integral must be understood as the duality product in $W^{1,p'}(\Omega)$, because, since $L^{p/2}(\Omega) \subset \left(W^{1,p'}(\Omega)\right)'$, due to E2 this is well defined. □

Remark 4.1.1 *Remember that the solution of equation* (4.1.7) *must be interpreted in the variational sense*

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ji}(x) \frac{\partial \varphi_{ku}}{\partial x_i}(x) \frac{\partial \psi}{\partial x_j}(x) + a_0(x)\varphi_{ku}(x)\psi(x) \right) dx = \int_\Omega \frac{\partial f}{\partial y}(x, y_u, u)\varphi_{ku}(x)\psi(x)\, dx$$

$$+ \sum_{j=1}^N \int_\Omega \frac{\partial g_k}{\partial \eta_j}(x, \nabla y_u) \frac{\partial \psi}{\partial x_j}(x)\, dx$$

for all $\psi \in W^{1,p}(\Omega)$.

4.1.2 Parabolic case

Set Ω, Γ, T, Q, Σ and A, p, τ, k_1, $\tilde{k}_1$, σ_1, $\tilde{\sigma}_1$ as in Section 2.2, with the boundary Γ of class C^1 and the coefficients of the operator A of class $C([0,T]; C(\bar{\Omega}))$. Set f, g, y_0 functions, $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \Sigma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $F : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $G : \Sigma \times \mathbb{R} \longrightarrow \mathbb{R}$ and $y_0 : \Omega \longrightarrow \mathbb{R}$, $y_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. Take k_2, $\tilde{k}_2$, σ_2, $\tilde{\sigma}_2$ and ν as in section 2.2.

To show that the functional

$$J(v) = \int_0^T \int_\Omega F(x, t, y_v)\, dx\, dt + \int_0^T \int_\Gamma G(s, t, y_v, v)\, ds\, dt + \int_\Omega L(x, y_v(x, T))\, dx$$

is of class C^1, we will use the following assumption.

P4 - for all $y \in \mathbb{R}$, $F(\cdot,\cdot,y)$ is measurable in Q. For a.e. $(x,t) \in Q$, $F(x,t,\cdot)$ is of class C^1 in $\mathbb{R}$. The following estimates hold:

$$|F(x,t,0)| \leq M_2(x,t), \qquad \left|\frac{\partial F}{\partial y}(x,t,y)\right| \leq M_2(x,t)\eta(|y|),$$

where $M_2 \in L^{\tilde{k}_2}(L^{k_2}(\Omega))$.

For all $y, v \in \mathbb{R}$, $G(\cdot,y,v)$ is measurable in Σ. for all $v \in \mathbb{R}$ and a.e $(s,t) \in \Sigma$, $G(s,t,\cdot,v)$ is of class C^1 en $\mathbb{R}$. For a.e. $(s,t) \in \Sigma$, $G(s,t,\cdot)$ and $G'_y(s,t,\cdot)$ are continuous in $\mathbb{R}^2$. The following estimates hold:

$$|G(s,t,0,v)| \leq N_2(s,t) + |v|, \qquad \left|\frac{\partial G}{\partial y}(s,t,y,v)\right| \leq (N_2(s,t) + |v|)\eta(|y|),$$

where $N_2 \in L^{\tilde{\sigma}_2}(L^{\sigma_2}(\Gamma))$.

For all $y \in \mathbb{R}$, $L(\cdot,y)$ is measurable in Ω. For a.e. $x \in \Omega$, $L(x,\cdot)$ is of class C^1 in $\mathbb{R}$. The following estimates hold:

$$|L(x,y)| \leq M_3(x), \qquad \left|\frac{\partial L}{\partial y}(x,y)\right| \leq M_4(x)\eta(|y|),$$

where $M_3(x) \in L^1(\Omega)$ and $M_4 \in L^{\nu}(\Omega)$.

P5 - $G(s,t,v,\cdot)$ is of class C^1 en $\mathbb{R}$. The following estimate holds:

$$\left|\frac{\partial G}{\partial v}(s,t,y,v)\right| \leq (N_2^*(s,t) + |v|)\eta(|y|),$$

where $N_2^* \in L^1(\Sigma)$.

Theorem 4.1.5 *Suppose that the assumptions on f and g, P1 and P2 and the assumptions on F, G and L P4 and P5 hold. Then the functional $J : L^\infty(\Sigma) \to \mathbb{R}$ is of class C^1. Moreover, for all v, $h \in L^\infty(\Sigma)$*

$$J'(v)h = \int_\Sigma \left(\frac{\partial G}{\partial v}(s,t,y_v,v) + \varphi_{0v}\frac{\partial g}{\partial v}(s,t,y_v,v)\right) h\, ds\, dt,$$

where $y_v = \Phi(v)$ is the solution of the equation (3.2.1), *$\varphi_{0v} \in L^{\tau'}(W^{1,p'}) + L^2(H^1)$ is the* *unique solution of the problem*

$$\left\{\begin{array}{rcll} -\dfrac{\partial\varphi}{\partial t}+A^*\varphi-\dfrac{\partial f}{\partial y}(x,t,y_v)\varphi &=& \dfrac{\partial F}{\partial y}(x,t,y_v) & \text{in } Q,\\[2ex] \dfrac{\partial\varphi}{\partial n_{A^*}}-\dfrac{\partial g}{\partial y}(s,t,y_v,v)\varphi &=& \dfrac{\partial G}{\partial y}(s,t,y_v,v) & \text{on } \Sigma,\\[2ex] \varphi(\cdot,T) &=& \dfrac{\partial L}{\partial y}(x,y(T)) & \text{in } \Omega, \end{array}\right. \tag{4.1.9}$$

Proof. Consider the function $F_0 : L^r(W^{1,p}(\Omega)) \times L^\infty(\Sigma) \to \mathbb{R}$ defined by

$$F_0(y,v)=\int_0^T\int_\Omega F(x,t,y)\;dx\;dt+\int_0^T\int_\Gamma G(s,t,y,v)\;ds\;dt+\int_\Omega L(x,y(x,T))\;dx$$

Due to the assumptions on F, G and L it is straight to prove that F_0 is of class C^1. Now, applying the chain rule to $J(v) = F_0(\Phi(v),v)$ and using Theorem 3.2.2 we have that J is of class C^1 and

$$J'(v)h=\int_0^T\int_\Omega\frac{\partial F}{\partial y}(x,t,y)z_h\;dx\;dt+\int_0^T\int_\Gamma\frac{\partial G}{\partial y}(s,t,y,v)z_h\;ds\;dt+$$

$$\int_0^T\int_\Gamma\frac{\partial G}{\partial v}(s,t,y,v)h\;ds\;dt+\int_\Omega\frac{\partial L}{\partial y}(x,y(x,T))z_h(T)\;dx$$

where $z_h = \Phi'(v)h$ and is given by (3.2.3). Let us take now φ_{0v} solution of (4.1.9). The assumptions made on the derivatives of f, g, F, G and L and Proposition 2.2.10 assure us that $\varphi_{0u} \in L^{r'}(W^{1,p'}(\Omega)) + L^2(H^1)$ and that we can apply Green's formula to deduce the expression for the derivative from the previous inequality. □

To get a twice differentiable functional, we will suppose that

P6 - $F(x,t,y)$ is of class C^2 en y and there exists $\psi_1 \in L^1(Q)$ such that

$$\left|\frac{\partial^2 F}{\partial y^2}(x,t,y)\right| \le \psi_1(x,t)\eta(|y|)$$

for a.e. $(x,t) \in Q$.

$G(s,t,y,v)$ is of class C^2 in y and in v and there exists $\psi_2 \in L^1(\Sigma)$ such that

$$\left|\frac{\partial^2 G}{\partial y^2}(s,t,y,v)\right|+\left|\frac{\partial^2 G}{\partial y\partial v}(s,t,y,v)\right|+\left|\frac{\partial^2 G}{\partial v^2}(s,t,y,v)\right| \le (\psi_2(s,t)+|v|)\eta(|y|)$$

for a.e. $(s,t) \in \Sigma$.

$L(x,y)$ is of class C^2 in y and there exists $\psi_3 \in L^1(\Omega)$ such that

$$\left|\frac{\partial^2 L}{\partial y^2}(x,y)\right| \leq \psi_3(x)\eta(|y|)$$

for a.e. $x \in \Omega$.

Theorem 4.1.6 *Suppose that P1–P6 hold. Then, the functional $J : L^\infty(\Sigma) \to \mathbb{R}$ is of class C^2. Moreover for all v, h_1, $h_2 \in L^\infty(\Sigma)$*

$$J''(v)h_1h_2 =$$

$$= \int_\Sigma \left(\frac{\partial^2 G}{\partial y^2}(s,t,y_v,v)z_1z_2 + \frac{\partial^2 G}{\partial y \partial v}(s,t,y_v,v)(z_1h_2 + z_2h_1) + \frac{\partial^2 G}{\partial v^2}(s,t,y_v,v)h_1h_2\right) ds\,dt +$$

$$+ \int_\Sigma \varphi_{0v} \left(\frac{\partial^2 g}{\partial y^2}(s,t,y_v,v)z_1z_2 + \frac{\partial^2 g}{\partial y \partial v}(s,t,y_v,v)(z_1h_2 + z_2h_1) + \frac{\partial^2 g}{\partial v^2}(s,t,y_v,v)h_1h_2\right) ds\,dt,$$

where y_v is the solution of the equation (3.2.1), *φ_{0v} is the solution of* (4.1.9) *and z_i is the solution of* (3.2.3) *respectively for h_i, $i \in \{1,2\}$.*

Proof. Consider F_0 as in the proof of the previous result. Due to the assumptions on f, g, F, G and L we have that F_0 is of class C^2. Applying the chain rule and Theorem 3.2.3, we obtain that J is of class C^2 and the expression for its second derivative. □

Finally we are going to state adequate differentiability conditions for the constraints. In Problem $(\mathbf{P_p})$ of page 16 we define

$$C = \left\{\vec{f} \in L^\tau(L^p)^N : \int_0^T \zeta_j\left(\int_\Omega g_j(x,t,\vec{f})dx\right) dt = 0 \text{ if } 1 \leq j \leq n_i,\right.$$

$$\left.\int_0^T \zeta_j\left(\int_\Omega g_j(x,t,\vec{f})dx\right) dt \leq 0 \text{ if } n_i + 1 \leq j \leq n_i + n_d\right\},$$

where $\zeta_j : \mathbb{R} \to \mathbb{R}$ and $g_j : Q \times \mathbb{R}^N \to \mathbb{R}$ are functions.

Example 4.1.1 *If we had an inequality constraint with $\zeta_1(s) = s^{\tau/p} - \delta/T$ and $g_1(x,t,f) = |f - g_d(x,t)|^p$, with $\delta \in \mathbb{R}$ and $g_d \in L^\tau(L^p)^N$ given, the constraint would be*

$$\int_0^T \left(\int_\Omega |\nabla y - g_d(x,t)|^p dx\right)^{\frac{\tau}{p}} dt \leq \delta,$$

i.e., $C = \bar{B}_{L^\tau(L^p)}(g_d,\delta)$.

We are interested in differentiability properties of

$$G_j(v) = \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\nabla_x y_v) dx \right) dt.$$

Suppose that

P7 - $\zeta_j(s)$ is C^1 and $g_j(x,t,\eta)$ is of class C^1 in η and there exist a constant $C > 0$ and a function $\psi \in L^{\tau'}(L^{p'})$ such that

$$|\zeta_j'(s)| \le C|s|^{\frac{\tau}{p}-1} \quad \text{and} \quad \left|\frac{\partial g_j}{\partial \eta}(x,t,\eta)\right| \le C|\eta|^{p-1} + \psi(x,t)$$

for a.e. $(x,t) \in Q$.

The we have the following result.

Theorem 4.1.7 *Suppose that P1, P2 and P7 hold. Then for all j, the functional $G_j : L^\infty(\Sigma) \to \mathbb{R}$ is of class C^1. Moreover, for all v, $h \in L^\infty(\Sigma)$*

$$G_j'(v)h = \int_\Sigma \varphi_{jv} \frac{\partial g}{\partial v}(s,t,y_v,v) ds\, dt,$$

where y_v is the solution of equation (3.2.1), $\varphi_{jv} \in L^{\tau'}(W^{1,p'}) + L^2(H^1)$ is the unique solution of the problem

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi - \dfrac{\partial f}{\partial y}(x,t,y_v)\varphi = -\operatorname{div} \zeta_j' \left(\displaystyle\int_\Omega g_j(x,t,\nabla_x y_v) dx \right) \dfrac{\partial g_j}{\partial \eta}(s,t,\nabla_x y_v(x,t)) & \text{in } Q, \\ \dfrac{\partial \varphi}{\partial n_{A^*}} - \dfrac{\partial g}{\partial y}(s,t,y_v,v)\varphi = \operatorname{div} \zeta_j' \left(\displaystyle\int_\Omega g_j(x,t,\nabla_x(y_v)) dx \right) \dfrac{\partial g_j}{\partial \eta}(s,t,\nabla_x y_v(x,t)) \cdot \vec{n} & \text{on } \Sigma, \\ \varphi(\cdot,T) = \dfrac{\partial L}{\partial y}(x,\bar{y}(T)) & \text{in } \Omega. \end{cases} \tag{4.1.10}$$

Proof. Consider the function of class C^1, $F_j : L^\tau(W^{1,p}(\Omega)) \to \mathbb{R}$ defined by

$$F_j(y) = \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\nabla_x y) dx \right) dt.$$

So we have that $G_j = F_j \circ \Phi$, and due to the chain rule, G_j is of class C^1.

Now, taking into account Theorem 3.2.2, we can assure that $y_v \in L^\tau(W^{1,p}(\Omega))$, and due to P7, we have that

$$\zeta_j'\left(\int_\Omega g_j(x,t,\nabla_x y_v)dx\right)\frac{\partial g_j}{\partial \eta}(s,t,\nabla_x y_v(x,t)) \in L^{\tau'}(L^{p'})^N.$$

Therefore, we can use Proposition 2.2.10 to deduce that φ_{jv} is well defined and belongs to $L^{\tau'}(W^{1,p'}(\Omega))+L^2(H^1)$. Derivating F_j, using the chain rule and making an integration by parts, we obtain the expression for the second derivative of $G_j(v)$. □

Example 4.1.2 *Let us resume Example 4.1.1, with $g_d = 0$ to simplify the writing. In this case*

$$F_1(y) = \int_0^T \left(\int_\Omega |\nabla_x y|^p dx\right)^{\frac{\tau}{p}} dt - \delta$$

and

$$F_1'(y)z = \int_0^T \left[\left(\int_\Omega |\nabla_x y|^p dx\right)^{\frac{\tau}{p}-1} \int_\Omega |\nabla_x y|^{p-2}\nabla_x y\,\nabla_x z dx\right] dt.$$

To prove that the constraints are of class C^2, we make the following assumption.

P8 - $\zeta_j(s)$ is C^2 and $g_j(x,t,\eta)$ is of class C^2 in η and there exist a constant $C > 0$ and a function $\psi \in L^{\tau/(\tau-2)}(L^{p/(p-2)})$ such that

$$|\zeta_j''(s)| \le C|s|^{\frac{\tau}{p}-2} \quad \text{and} \quad \left|\frac{\partial^2 g_j}{\partial \eta^2}(x,t,\eta)\right| \le C|\eta|^{p-2} + \psi(x,t)$$

for a.e. $(x,t) \in Q$.

Now we can state the following result.

Theorem 4.1.8 *Suppose that P1, P2, P3, P7 and P8 hold. For all j, the functional $G_j : L^\infty(\Sigma) \to \mathbb{R}$ is of class C^2. Moreover, for all v, h_1, $h_2 \in L^\infty(\Sigma)$*

$$G_j''(v)h_1h_2 = \int_0^T \left[\zeta_j''\left(\int_\Omega g_j(x,t,\nabla_x y_v)dx\right)\int_\Omega \frac{\partial g_j}{\partial \eta}\nabla_x z_1\,dx \int_\Omega \frac{\partial g_j}{\partial \eta}\nabla_x z_2\,dx\right]dt +$$

$$\int_0^T \left[\zeta_j'\left(\int_\Omega g_j(x,t,\nabla_x y_v)dx\right)\int_\Omega \nabla_x^T z_1 \frac{\partial^2 g_j}{\partial \eta^2}(x,t,\nabla_x y_v)\nabla_x z_2 dx\right]dt +$$

$$+\int_{\Sigma}\varphi_{jv}\left(\frac{\partial^2 g}{\partial y^2}(s,t,y_v,v)z_1z_2+\frac{\partial^2 g}{\partial y\partial v}(s,t,y_v,v)(z_1h_2+z_2h_1)+\frac{\partial^2 g}{\partial v^2}(s,t,y_v,v)h_1h_2\right)ds\,dt,$$

where y_v is the solution of the equation (3.2.1), $\varphi_{jv}\in L^{r'}(W^{1,p'})+L^2(H^1)$ is the solution of (4.1.10) and z_i is the solution of (3.2.3) respectively for h_i, $i\in\{1,2\}$.

Proof. The function $F_j : L^r(W^{1,p}(\Omega))\to\mathbb{R}$ defined by

$$F_j(y)=\int_0^T\zeta_j\left(\int_\Omega g_j(x,t,\nabla_x y)dx\right)dt$$

is of class C^2. Derivating and using the chain rule, we obtain the expression for the second derivative of $G_j(v)$. The assumptions made assure us that the integral is well defined. □

Example 4.1.3 *Resume examples 4.1.1 and 4.1.2. We have that*

$$F_1''(y)z_1z_2=\int_0^T\left[\left(\int_\Omega|\nabla_x y|^p dx\right)^{\frac{r}{p}-2}\int_\Omega|\nabla_x y|^{p-2}\nabla_x y\nabla_x z_1 dx\int_\Omega|\nabla_x y|^{p-2}\nabla_x y\nabla_x z_2 dx\right]dt+$$

$$\int_0^T\left[\left(\int_\Omega|\nabla_x y|^p dx\right)^{\frac{r}{p}-1}\int_\Omega\nabla_x^T z_2\left(|\nabla_x y|^{p-4}\nabla_x y\nabla_x^T y+|\nabla_x y|^{p-2}I_N\right)\nabla_x z_1 dx\right]dt,$$

where I_N is the identity matrix $N\times N$.

4.2 Sensitivity of the functionals with respect to diffuse perturbations

4.2.1 Elliptic case

Take again Ω of class C^1; Γ its boundary; A an elliptic operator of continuous coefficients of the form (2.1.1) (page 23); $p>N$; $a_0\in L^{p/2}(\Omega)$; $f:\Omega\times\mathbb{R}^2\longrightarrow\mathbb{R}$; $g:\Gamma\to\mathbb{R}$, $g\in L^{p-1}(\Gamma)$; $L:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g_j:\Omega\times\mathbb{R}^N\to\mathbb{R}$ for $1\le j\le n_i+n_e$.

To establish Pontryagin's principle for Problem $(\mathbf{P_e})$ of page 16, we must establish another kind of Taylor expansions, based on diffuse perturbations of the control. Now we need not suppose differentiability of the involved functions with respect to the control.

Theorem 4.2.1 *Suppose that the assumptions on f E0 (page 66), E3 (page 74) and on g_j E6(page 89) hold. Suppose also that $L:\Omega\times(\mathbb{R}\times\mathbb{R})\to\mathbb{R}$ is a*

E8 - Carathéodory function, of class C^1 in the second variable and for all $M > 0$ there exists $\psi_M \in L^1(\Omega)$ such that $|L(x, 0, u(x))| \leq \psi_M(x)$ for all $u \in U_{ad}$ and a.e. $x \in \Omega$ and

$$\left|\frac{\partial L}{\partial y}(x, y, u(x))\right| \leq \psi_M(x)$$

if $|y| \leq M$ for all $u \in U_{ad}$ and a.e. $x \in \Omega$.

For every $\rho \in (0,1)$ and every $u_1, u_2 \in U_{ad}$ let us take E_ρ, u_ρ, y_ρ and z as in Theorema 3.3.2.

Then for every $\rho \in (0,1)$ and every $u_1, u_2 \in U_{ad}$ we have that

$$J(u_\rho) = J(u_1) + \rho\Delta J + o(\rho)$$

and

$$G_j(u_\rho) = G_j(u_1) + \rho\Delta G_j + o(\rho) \text{ for } 1 \leq j \leq n_i + n_d,$$

where

$$\Delta J = \int_\Omega \frac{\partial L}{\partial y}(x, y_1, u_1) z \, dx + \int_\Omega (L(x, y_1, u_2) - L(x, y_1, u_1)) \, dx$$

and

$$\Delta G_j = \int_\Omega \frac{\partial g_j}{\partial \eta}(x, \nabla y_1) \nabla z \, dx$$

for $1 \leq j \leq n_i + n_d$.

Remark 4.2.1 *Notice that $\Delta G_j \neq G_j'(u_1)u_2$, because $z \neq G'(u_1)u_2$.*

Proof. Using the definitions of E_ρ, u_ρ, y_ρ y z given in Theorem 3.3.2 we have that

$$\frac{J(u_\rho) - J(u_1)}{\rho} - \Delta J = \int_\Omega \left(\int_0^1 \frac{\partial L}{\partial y}(x, y_1 + \theta(y_\rho - y_1), u_\rho) d\theta - \frac{\partial L}{\partial y}(x, y_1, u_1)\right) z \, dx -$$

$$- \int_\Omega \left(1 - \frac{1}{\rho}\chi_{E_\rho}\right) (L(x, y_1, u_2) - L(x, y_1, u_1)) \, dx$$

and due to Lemma 3.3.1 this quantity converges to 0.

Also

$$\frac{G_j(u_\rho) - G_j(u_1)}{\rho} - \Delta G_j = \int_\Omega \left(\int_0^1 \frac{g_j}{\partial \eta}(x, \nabla y_1 + \theta(\nabla y_\rho - \nabla y_1)) d\theta - \frac{g_j}{\partial \eta}(x, \nabla y_1)\right) \nabla z \, dx$$

and due to the growing properties imposed on $g_j(\eta)$ in E6 and (3.3.2), this quantity converges to 0. The proof is complete. □

4.2.2 Parabolic case

Set Ω, Γ, T, Q, Σ and A, p, τ, k_1, $\tilde{k}_1$, σ_1, $\tilde{\sigma}_1$ as in Section 2.2, with the boundary Γ of class $C^{1+\hat{\varepsilon}}$ and the coefficients of the operator A of class $C([0,T]; C^{0,\hat{\varepsilon}}(\bar{\Omega}))$, for some $0 < \hat{\varepsilon} < 1$. Set f, g, y_0 functions, $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \Sigma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $F : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $G : \Sigma \times \mathbb{R} \longrightarrow \mathbb{R}$ and $y_0 : \Omega \longrightarrow \mathbb{R}$, $y_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. Set k_2, $\tilde{k}_2$, σ_2, $\tilde{\sigma}_2$ and ν as in Section 2.2.

Consider problem $(\mathbf{P_p})$ of page 16.

Theorem 4.2.2 *Suppose that assumptions P1 and P4 hold. For all $\rho \in (0,1)$ and all $v_1, v_2 \in V_{ad}$ let us take E_ρ, v_ρ, y_ρ and z as in Theorema 3.3.4.*

Then, for all $\rho \in (0,1)$, and all $v_1, v_2 \in V_{ad}$ we have that

$$J(v_\rho) = J(v_1) + \rho \Delta J + o(\rho), \tag{4.2.1}$$

where

$$\Delta J = \int_Q F'_y(\cdot, y_1) z \, dx \; dt + \int_\Sigma G'_y(\cdot, y_1, v_1) z ds \; dt + \int_\Omega L'_y(\cdot, y_1(\cdot, T)) z(\cdot, T) \; dx$$
$$+ \int_\Sigma (G(s,t,y_1,v_2) - G(s,t,y_1,v_1)) \; ds \; dt.$$

Proof. Using the definitions of E_ρ, u_ρ, y_ρ and z given in Theorem 3.3.4 we have that

$$\frac{J(v_\rho) - J(v_1)}{\rho} - \Delta J = \int_Q \left(\int_0^1 F'_y(x,t,y_1 + \theta(y_\rho - y_1)) d\theta - F'_y(x,t,y_1) \right) z \, dx dt +$$
$$+ \int_\Sigma \left(\int_0^1 G'_y(s,t,y_1 + \theta(y_\rho - y_1), v_\rho) d\theta - G'_y(s,t,y_1,v_1) \right) z \, ds dt +$$
$$+ \int_\Omega \left(\int_0^1 L'_y(x, y_1 + \theta(y_\rho - y_1)) d\theta - L'_y(x, y_1) \right) z \, dx -$$
$$\int_\Sigma \left(1 - \frac{1}{\rho} \right) \chi_{E_\rho} \left(G(s,t,y_1,u_2) - G(s,t(y_1,u_1)\right) ds dt.$$

Due to Lemma 3.3.3 we can take limits and verify (4.2.1). □

Chapter 5

Pontryagin's principle

The main result of this chapter is a Pontryagin for problems ($\mathbf{P_e}$) (page 16) and ($\mathbf{P_p}$) (page 16). In the last years there has been a growing interest in Pontryagin principles for control problems governed by partial differential equations wit pointwise or integral state constraints. Among others, we can cite Casas [22], Fattorini [52, 54], Bei Hu and Yong [60], Li and Yong [65], Raymond and Zidani [78], Casas, Raymond and Zidani [35].

The proofs of Theorems 5.1.1 and 5.2.1 are based in Ekeland's variational principle. To obtain an approximate Pontryagin principle corresponding to the optimality conditions deduced from Ekeland's variational principle, we use the method of diffuse perturbations, as in the articles of Raymond and Zidani [78] or Casas, Raymond and Zidani [35].

5.1 Elliptic case

Consider problem ($\mathbf{P_e}$) of page 16. Let us take again Ω of class C^1; Γ its boundary; A an elliptic operator of continuous coefficients of the form (2.1.1) (page 23); $p > N$; $a_0 \in L^{p/2}(\Omega)$; $f : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$; $g : \Gamma \to \mathbb{R}$, $g \in L^{p-1}(\Gamma)$; $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ for $1 \leq j \leq n_i + n_e$.

Define the Hamiltonian $H : \Omega \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ as

$$H(x, y, u, \varphi, \nu) = \nu L(x, y, u) + \varphi f(x, y, u).$$

Pontryagin's principle holds

Theorem 5.1.1 *Let $\bar{u}$ be a solution of* $(\mathbf{P_e})$. *Suppose that the assumptions on f E0 (page 66) and E3 (page 74), on g_j E6 (page 89) and on L E8 (page 98) hold. Then there exist real numbers $\bar{\nu}$, $\bar{\lambda}_j$, $j = 1, \dots, n_d + n_i$ not all zero and functions $\bar{y} \in W^{1,p}(\Omega)$, $\bar{\varphi} \in W^{1,p'}(\Omega)$ such that*

$$\bar{\lambda}_j \geq 0 \quad n_i + 1 \leq j \leq n_i + n_d, \qquad \bar{\lambda}_j \int_\Omega g_j(x, \nabla\bar{y}(x))\, dx = 0, \tag{5.1.1}$$

$$\begin{cases} A\bar{y} + a_0\bar{y} = f(x, \bar{y}(x), \bar{u}(x)) & \text{in } \Omega \\ \partial_{n_A}\bar{y} = 0 & \text{on } \Gamma, \end{cases} \tag{5.1.2}$$

$$\begin{cases} A^*\bar{\varphi} + a_0\bar{\varphi} = \dfrac{\partial f}{\partial y}(x, \bar{y}, \bar{u})\bar{\varphi} + \bar{\nu}\dfrac{\partial L}{\partial y}(x, \bar{y}, \bar{u}) - \displaystyle\sum_{j=1}^{n_i+n_d} \bar{\lambda}_j \mathrm{div}\left(\dfrac{\partial g_j}{\partial \eta}(x, \nabla\bar{y})\right) & \text{in } \Omega \\ \partial_{n_{A^*}}\bar{\varphi} = 0 & \text{on } \Gamma, \end{cases} \tag{5.1.3}$$

and for a.e. $x \in \Omega$,

$$H(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x), \bar{\nu}) = \min_{k \in K_\Omega(x)} H(x, \bar{y}(x), k, \bar{\varphi}(x), \bar{\nu}).$$

Proof. We define Ekeland's distance on the set U_{ad} as

$$d_E(u_1, u_2) = |\{x \in \Omega : \ u_1(x) \neq u_2(x)\}|.$$

We have that (U_{ad}, d_E) is a complete metric space and that convergence in (U_{ad}, d_E) implies pointwise convergence in Ω.

Let us define the penalized functional

$$J_n(u) = \left\{ \left[\left(J(u) - J(\bar{u}) + \frac{1}{n^2} \right)^+ \right]^2 + \sum_{j=1}^{n_i} G_j(u)^2 + \sum_{j=n_i+1}^{n_i+n_d} \left(G_j(u)^+\right)^2 \right\}^{\frac{1}{2}},$$

where for all $a \in \mathbb{R}$

$$a^+ = \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a \leq 0. \end{cases}$$

Consider the problem

$$(\mathrm{P_n}) \begin{cases} \min_{u \in U_{ad}} J_n(u). \end{cases}$$

The solution of our original problem $\bar{u}$ is a $\frac{1}{n^2}$-solution of (P_n). J_n is continuous for Ekeland's distance, so, due to Ekeland's variational principle [50], there exists $u_n \in U_{ad}$ such that

$$d_E(u_n, \bar{u}) \leq \frac{1}{n}$$

and

$$J_n(u_n) \leq J_n(u) + \frac{1}{n} d_E(u, u_n) \text{ for all } u \in U_{ad}. \tag{5.1.4}$$

Take any $u \in U_{ad}$. Due to Theorems 3.3.2 and 4.2.1, for all $\rho \in (0,1)$ there exists a measurable set $E_\rho \subset \Omega$ such that

$$|E_\rho| = \rho|\Omega|,$$

$$y_\rho = y_n + \rho z_n + r_\rho, \text{ with } \lim_{\rho \to 0} \frac{1}{\rho} \|r_\rho\|_{W^{1,p}(\Omega)} = 0,$$

$$J(u_\rho) = J(u_n) + \rho \Delta J^n + o(\rho) \tag{5.1.5}$$

and

$$G_j(u_\rho) = G_j(u_n) + \rho \Delta G_j^n + o(\rho) \text{ for } 1 \leq j \leq n_i + n_d,$$

where

$$u_\rho = \begin{cases} u_n & \text{in } \Omega \setminus E_\rho \\ u & \text{in } E_\rho, \end{cases}$$

$$y_\rho = y_{u_\rho},$$

$$\begin{cases} Az_n + a_0 z_n = \dfrac{\partial f}{\partial y}(x, y_n, u_n) z_n + f(x, y_n, u) - f(x, y_n, u_n) & \text{in } \Omega \\ \partial_{n_A} z_n = 0 & \text{on } \Gamma, \end{cases}$$

$$\Delta J^n = \int_\Omega \frac{\partial L}{\partial y}(x, y_n, u_n) z_n \, dx + \int_\Omega (L(x, y_n, u) - L(x, y_n, u_n)) \, dx$$

and

$$\Delta G_j^n = \int_\Omega \frac{\partial g_j}{\partial \eta}(x, \nabla y_n) \nabla z_n \, dx$$

for $1 \leq j \leq n_i + n_d$.

Due to (5.1.4)

$$J_n(u_n) \leq J_n(u_\rho) + \frac{1}{n} d_E(u_\rho, u_n).$$

But

$$d_E(u_\rho, u_n) = |E_\rho| = \rho|\Omega|,$$

and thus

$$\frac{J_n(u_n) - J_n(u_\rho)}{\rho} \leq \frac{1}{n}|\Omega|.$$

We are going to take limits when ρ tends to zero this expression to obtain an integral approximate Pontryagin principle.

$$\frac{J_n(u_n) - J_n(u_\rho)}{\rho} = \frac{J_n^2(u_n) - J_n^2(u_\rho)}{\rho(J_n(u_n) + J_n(u_\rho))} =$$

$$= \frac{\left[\left(J(u_n) - J(\bar{u}) + \frac{1}{n^2}\right)^+\right]^2 - \left[\left(J(u_\rho) - J(\bar{u}) + \frac{1}{n^2}\right)^+\right]^2}{\rho(J_n(u_n) + J_n(u_\rho))} +$$

$$+ \frac{\sum_{j=1}^{n_i} \left(G_j(u_n)^2 - G_j(u_\rho)^2\right)}{\rho(J_n(u_n) + J_n(u_\rho))} + \frac{\sum_{j=n_i+1}^{n_i+n_d} \left((G_j(u_n)^+)^2 - (G_j(u_\rho)^+)^2\right)}{\rho(J_n(u_n) + J_n(u_\rho))}$$

Let us see what happens when $\rho \to 0$ term by term.

$$A^\rho = \frac{\left[\left(J(u_n) - J(\bar{u}) + \frac{1}{n^2}\right)^+\right]^2 - \left[\left(J(u_\rho) - J(\bar{u}) + \frac{1}{n^2}\right)^+\right]^2}{\rho(J_n(u_n) + J_n(u_\rho))} = A_1^\rho \cdot A_2^\rho,$$

where

$$A_1^\rho = \frac{\left(J(u_n) - J(\bar{u}) + \frac{1}{n^2}\right)^+ - \left(J(u_\rho) - J(\bar{u}) + \frac{1}{n^2}\right)^+}{\rho},$$

and

$$A_2^\rho = \frac{\left(J(u_n) - J(\bar{u}) + \frac{1}{n^2}\right)^+ + \left(J(u_\rho) - J(\bar{u}) + \frac{1}{n^2}\right)^+}{J_n(u_n) + J_n(u_\rho)}$$

Due to the continuity of J we have that

$$\lim_{\rho \to 0} A_2^\rho = \frac{\left(J(u_n) - J(\bar{u}) + \frac{1}{n^2}\right)^+}{J_n(u_n)}$$

We will call this quantity ν^n. To take the limit in A_1^ρ we have to take into account the sign of $J(u_n) - J(\bar{u}) + \frac{1}{n^2}$. If $J(u_n) - J(\bar{u}) + \frac{1}{n^2} > 0$, then for all ρ small enough we have that $J(u_\rho) - J(\bar{u}) + \frac{1}{n^2} > 0$ and hence

$$A_1^\rho = \frac{J(u_n) - J(u_\rho)}{\rho};$$

due to (5.1.5), this quantity converges to $-\Delta J^n$. If $J(u_n) - J(\bar{u}) + \frac{1}{n^2} \leq 0$ then $\nu^n = 0$. Moreover, for all ρ we have that $|A_1^\rho|$ is uniformly bounded: We know that for any pair of real numbers t_1 and t_2 we have that $|t_1^+ - t_2^+| \leq |t_1 - t_2|$. Therefore, and using (5.1.5) we have that

$$|A_1^\rho| \leq \frac{|J(u_n) - J(u_\rho)|}{\rho} \leq |\Delta J^n| + \frac{|o(\rho)|}{\rho},$$

and therefore $|A_1^\rho|$ is bounded independently of ρ. So in any case we can write

$$\lim_{\rho\to 0} A^\rho = -\nu^n \Delta J^n.$$

Secondly, for $1 \leq j \leq n_i$, we have that

$$\frac{G_j(u_n)^2 - G_j(u_\rho)^2}{\rho(J_n(u_n) + J_n(u_\rho))} = \frac{G_j(u_n) - G_j(u_\rho)}{\rho} \cdot \frac{G_j(u_n) + G_j(u_\rho)}{J_n(u_n) + J_n(u_\rho)}$$

and this quantity converges to $-\lambda_j^n \Delta G_j^n$, where

$$\lambda_j^n = \frac{G_j(u_n)}{J_n(u_n)}$$

In a similar way, if $n_i + 1 \leq j \leq n_i + n_d$ we can assure that

$$\lim_{\rho\to 0} \frac{(G_j(u_n)^+)^2 - (G_j(u_\rho)^+)^2}{\rho(J_n(u_n) + J_n(u_\rho))} = -\lambda_j^n \Delta G_j^n,$$

being in this case

$$\lambda_j^n = \frac{G_j(u_n)^+}{J_n(u_n)}$$

So we have that

$$\lim_{\rho\to 0} \frac{J_n(u_n) - J_n(u_\rho)}{\rho} = -\nu^n \Delta J^n - \sum_{j=1}^{n_i+n_d} \lambda_j^n \Delta G_j^n,$$

and hence

$$-\nu^n \Delta J^n - \sum_{j=1}^{n_i+n_d} \lambda_j^n \Delta G_j^n \leq \frac{1}{n}|\Omega|.$$

If we write the first term explicitly we have that

$$-\nu^n \Delta J^n - \sum_{j=1}^{n_i+n_d} \lambda_j^n \Delta G_j^n = -\int_\Omega \nu^n \frac{\partial L}{\partial y}(x, y_n, u_n) z_n \, dx - \int_\Omega \nu^n \left(L(x, y_n, u) - L(x, y_n, u_n)\right) dx -$$

$$-\sum_{j=1}^{n_i+n_d} \int_\Omega \lambda_j^n \frac{\partial g_j}{\partial \eta}(x, \nabla y_n) \nabla z_n \, dx \leq \frac{1}{n}|\Omega|.$$

Let us take φ_n the approximate adjoint state, which satisfies the equation

$$\left\{\begin{array}{rcll} A^*\varphi_n + a_0\varphi_n & = & \dfrac{\partial f}{\partial y}(x, y_n, u_n)\varphi_n + & \\ & & \nu^n \dfrac{\partial L}{\partial y}(x, y_n, u_n) - \displaystyle\sum_{j=1}^{n_i+n_d} \lambda_j^n \mathrm{div}\left(\dfrac{\partial g_j}{\partial \eta}(x, \nabla y_n)\right) & \text{in } \Omega \\ \partial_{n_{A^*}}\varphi_n & = & 0 & \text{on } \Gamma. \end{array}\right.$$

Integrating by parts and using the definition of z_n, we obtain

$$-\int_\Omega \varphi_n \left(f(x, y_n, u) - f(x, y_n, u_n)\right) dx - \int_\Omega \nu^n \left(L(x, y_n, u) - L(x, y_n, u_n)\right) \leq \frac{1}{n}|\Omega|.$$

And therefore, we have an approximate Pontryagin principle in integral form:

$$\int_\Omega (\nu^n L(x, y_n, u_n) + \varphi_n f(x, y_n, u_n))\, dx \leq \int_\Omega (\nu^n L(x, y_n, u) + \varphi_n f(x, y_n, u))\, dx + \frac{1}{n}|\Omega|$$

for all $u \in U_{ad}$.

Now, since

$$\nu^{n2} + \sum_{j=1}^{n_i+n_d} \lambda_j^{n2} = 1,$$

we can take subsequences that converge to real numbers $\bar\nu$ and $\bar\lambda_j$, $1 \leq j \leq n_i + n_d$, obviously not all zero. These satisfy (5.1.1). We also have that $u_n \to \bar u$ pointwise, and therefore, due to Theorem 3.1.1 $y_n \to \bar y$ in $W^{1,p}(\Omega)$, and therefore uniformly, so $\varphi_n \to \bar\varphi$ in $W^{1,p'}(\Omega)$, and we can take the limit to obtain Pontryagin's principle in integral form:

$$\int_\Omega (\bar\nu L(x, \bar y, \bar u) + \bar\varphi f(x, \bar y, \bar u))\, dx \leq \int_\Omega (\bar\nu L(x, \bar y, u) + \bar\varphi f(x, \bar y, u))\, dx$$

for all $u \in U_{ad}$.

The pointwise form of Pontryagin's principle is deduced now as in [78, page 1875] □

Some extensions

In the same way we can prove Pontryagin's principle for boundary control. Consider the problem

$$
(\mathrm{P}'_e)\begin{cases}\text{Minimize } J(v)=\displaystyle\int_\Gamma \ell(s,y_v,v)ds,\\ v\in V_{ad}=\{v:\Gamma\to\mathbb{R}:\ v(s)\in K_\Gamma(s)\ \text{a.e. } s\in\Gamma\},\\ \displaystyle\int_\Omega g_j(x,\nabla y_u(x))\,dx=0,\ 1\le j\le n_i,\\ \displaystyle\int_\Omega g_j(x,\nabla y_u(x))\,dx\le 0,\ n_i+1\le j\le n_i+n_d,\end{cases}
$$

where

$$
\begin{cases} Ay_v+a_0y &= f \quad \text{in } \Omega\\ \partial_{n_A}y_u &= g(s,y_v,v) \quad \text{on } \Gamma,\end{cases}
$$

and K_Γ is a measurable multimapping with non empty and closed values in $\mathcal{P}(\mathbb{R})$.

Let us define the boundary Hamiltonian $H:\Gamma\times\mathbb{R}^4\to\mathbb{R}$, as

$$
H(s,y,v,\varphi,\nu)=\nu\ell(s,y,v)+\varphi g(s,y,v).
$$

Theorem 5.1.2 *Let $\bar v$ be a solution of (P'_e). Suppose that $f\in L^{p/2}(\Omega)$; $g:\Gamma\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a measurable function on Γ, of class C^1 in the second variable, continuous in the third variable and for all $M>0$ there exist $\psi_M\in L^{p-1}(\Gamma)$ and $C_M>0$ such that $|g(s,0,v(s))|\le\psi_M(s)$,*

$$
-C_M\le\frac{\partial g}{\partial y}(s,t,v(s))\le 0
$$

for all $|t|\le M$, $v\in V_{ad}$ and a.e. $s\in\Gamma$; $\ell:\Gamma\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a measurable function on Γ, of class C^1 in the second variable, continuous in the third variable and for all $M>0$ there exists $\psi_M\in L^1(\Gamma)$ such that $|\ell(s,0,v(s))|\le\psi_M(s)$,

$$
|\frac{\partial \ell}{\partial y}(s,t,v(s))|\le\psi_M(s)
$$

for all $|t|\le M$, $v\in V_{ad}$ and a.e. $s\in\Gamma$. Suppose also that E6 (page 89) holds.

Then there exist real numbers $\bar\nu$, $\bar\lambda_j$, $j=1,\dots,n_d+n_i$ not all zero and functions $\bar y\in W^{1,p}(\Omega)$, $\bar\varphi\in W^{1,p'}(\Omega)$ such that

$$
\bar\lambda_j\ge 0\quad n_i+1\le j\le n_i+n_d,\qquad \bar\lambda_j\int_\Omega g_j(x,\nabla\bar y(x))\,dx=0,
$$

$$\begin{cases} A\bar{y} + a_0\bar{y} &= f \qquad \text{in } \Omega \\ \partial_{n_A}\bar{y} &= g(s, y_v, v) \quad \text{on } \Gamma, \end{cases}$$

$$\begin{cases} A^*\bar{\varphi} + a_0\bar{\varphi} &= -\sum_{j=1}^{n_i+n_d} \bar{\lambda}_j \mathrm{div}\left(\frac{\partial g_j}{\partial \eta}(x, \nabla\bar{y})\right) \quad \text{in } \Omega \\ \partial_{n_{A^*}}\bar{\varphi} &= \frac{\partial g}{\partial y}(s, \bar{y}, \bar{v})\bar{\varphi} + \bar{\nu}\frac{\partial \ell}{\partial y}(s, \bar{y}, \bar{v}) \quad \text{on } \Gamma, \end{cases}$$

and for a.e. $s \in \Gamma$,

$$H(s, \bar{y}(s), \bar{v}(s), \bar{\varphi}(s), \bar{\nu}) = \min_{k \in K_\Gamma(s)} H(s, \bar{y}(s), k, \bar{\varphi}(s), \bar{\nu}).$$

5.2 Parabolic case

Set Ω, Γ, T, Q, Σ and A, p, τ, k_1, $\bar{k}_1$, σ_1, $\bar{\sigma}_1$ as in Section 2.2, with the boundary Γ of class $C^{1+\hat{\varepsilon}}$ and the coefficients of the operator A of class $C([0,T]; C^{0,\hat{\varepsilon}}(\bar{\Omega}))$, for some $0 < \hat{\varepsilon} < 1$. Set f, g, y_0 functions, $f : Q\times\mathbb{R} \longrightarrow \mathbb{R}$, $g : \Sigma\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, $F : Q\times\mathbb{R} \longrightarrow \mathbb{R}$, $G : \Sigma \times \mathbb{R} \longrightarrow \mathbb{R}$ and $y_0 : \Omega \longrightarrow \mathbb{R}$, $y_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. Set k_2, $\bar{k}_2$, σ_2, $\bar{\sigma}_2$ and ν as in Section 2.2.

Consider problem $(\mathbf{P_p})$ in page 16. For the parabolic problem we are not going to consider only the case with a finite number of gradient state constraints, but we will deal with the more general constraint

$$\nabla_x y \in C,$$

where C is a closed, convex with non empty interior subset of $(L^\tau(0,T; L^p(\Omega)))^N$.

Let us define the boundary Hamiltonian as

$$H_\Sigma(s, t, y, v, \varphi, \nu) = \nu G(s, t, y, v) + \varphi g(s, t, y, v)$$

for all $(s, t, y, v, \varphi, \nu) \in \Gamma \times [0, T] \times \mathbb{R}^4$.

In the following theorem, we establish Pontryagin's principle.

Theorem 5.2.1 *Suppose that P1 and P4 hold. If* $\bar{v}$ *is a solution of the control problem* $(\mathbf{P_p})$ *in page 16, then there exist* $\bar{\varphi} \in L^{\tau'}(W^{1,p'}) + L^2(H^1)$, $\bar{\nu} \in \mathbb{R}^+$, *and* $\bar{f} \in (L^{\tau'}(L^{p'}))^N$, *such that*

$$(\vec{f}, \bar{\nu}) \neq (0,0), \tag{5.2.1}$$

$$\int_Q (z - \nabla_x \bar{y})\vec{f} \leq 0 \quad \textit{for all } z \in C, \tag{5.2.2}$$

$$\left\{\begin{aligned}
-\frac{\partial \bar{\varphi}}{\partial t} + A^*\bar{\varphi} - \frac{\partial f}{\partial y}(x,t,\bar{y})\bar{\varphi} &= \bar{\nu}\frac{\partial F}{\partial y}(x,t,\bar{y}) + \operatorname{div}\vec{f} && \textit{in } Q,\\
\frac{\partial \bar{\varphi}}{\partial n_{A^*}} - \frac{\partial g}{\partial y}(s,t,\bar{y},\bar{v})\bar{\varphi} &= \bar{\nu}\frac{\partial G}{\partial y}(s,t,\bar{y},\bar{v}) - \vec{f}\cdot\vec{n} && \textit{on } \Sigma,\\
\bar{\varphi}(\cdot,T) &= \bar{\nu}\frac{\partial L}{\partial y}(x,\bar{y}(T)) && \textit{in } \Omega,
\end{aligned}\right. \tag{5.2.3}$$

and

$$H_\Sigma(s,t,\bar{y}(s,t),\bar{v}(s,t),\bar{\varphi}(s,t),\bar{\nu}) = \min_{v \in K_\Sigma(s,t)} H_\Sigma(s,t,\bar{y}(s,t),v,\bar{\varphi}(s,t),\bar{\nu}) \tag{5.2.4}$$

for a.e. (s,t) *in* Σ.

Proof. Let us define Ekeland's distance in V_{ad}:

$$d_E(v_1,v_2) = |\{(s,t): \; v_1(s,t) \neq v_2(s,t)\}|.$$

The space (V_{ad}, d_E) is a complete metric space, and convergence in (V_{ad}, d_E) implies convergence in $L^\alpha(\Sigma)$ for any $\alpha < \infty$. Consider the penalized functional

$$J_n(v) = \left\{ \left[\left(J(v) - J(\bar{v}) + \frac{1}{n^2} \right)^+ \right]^2 + d_C(\nabla_x y_v)^2 \right\}^{1/2},$$

where $d_C(\cdot)$ is the distance in $(L^\tau(L^p))^N$ to the set C, defined by

$$d_C(z) = \inf_{\varphi \in C} ||z - \varphi||_{(L^\tau(L^p))^N}.$$

The functional $d_C(\cdot)$ is Lipschitz, convex and Gâteaux differentiable for all $z \notin C$, and in those points

$$||\nabla d_C(z)||_{(L^{\tau'}(L^{p'}))^N} = 1.$$

Consider the problem

$$(P_n): \min_{v \in V_{ad}} J_n(v).$$

With such an election, $\bar{v}$ is a $\frac{1}{n^2}$-solution of (P_n). Theorem 3.2.1 and assumption P4 imply that $J_n(v)$ is continuous for Ekeland's distance. So, due to Ekeland's variational principle, there exists $v_n \in V_{ad}$ such that

$$d_E(v_n, \bar{v}) \leq \frac{1}{n} \quad \text{and} \quad J_n(v_n) \leq J_n(v) + \frac{1}{n} d_E(v, v_n) \quad \text{for all } v \in V_{ad}. \tag{5.2.5}$$

Take $v \in V_{ad}$. Due to Theorems 3.3.4 and 4.2.2, for all $\rho \in (0,1)$, there exists a measurable set $E_\rho \subset \Sigma$ such that

$$|E_\rho| = \rho|\Sigma|, \tag{5.2.6}$$

$$y_\rho = y_n + \rho z_n + r_\rho \text{ with } \lim_{\rho \to 0} \frac{1}{\rho} \|r_\rho\|_{L^\tau(W^{1,p})} = 0, \tag{5.2.7}$$

and

$$J(v_\rho) = J(v_n) + \rho \Delta J^N + o(\rho), \tag{5.2.8}$$

where

$$v_\rho(s,t) = \begin{cases} v_n & \text{in } \Sigma \setminus E_\rho \\ v & \text{in } E_\rho \end{cases}, \quad y_\rho = y_{v_\rho},$$

$$\begin{cases} \dfrac{\partial z_n}{\partial t} + A z_n - \dfrac{\partial f}{\partial y}(x,t,y_n) z_n = 0 & \text{in } Q, \\ \dfrac{\partial z_n}{\partial n_A} - \dfrac{\partial g}{\partial y}(s,t,y_n,v_n) z_n = g(s,t,y_n,v) - g(s,t,y_n,v_n) & \text{on } \Sigma, \\ z_n(\cdot,0) = 0 & \text{in } \Omega, \end{cases}$$

and

$$\Delta J^N = \int_Q \frac{\partial F}{\partial y}(\cdot, y_n) z_n dx\, dt + \int_\Sigma \frac{\partial G}{\partial y}(\cdot, y_n, v_n) z_n ds\, dt + \int_\Omega \frac{\partial L}{\partial y}(\cdot, y_n(\cdot,T)) z_n(\cdot,T)\, dx$$
$$+ \int_\Sigma (G(\cdot, y_n, v) - G(\cdot, y_n, v_n))\, ds\, dt.$$

Relations (5.2.5) and (5.2.6) imply that

$$\frac{J_n(v_n) - J_n(v_\rho)}{\rho} \leq \frac{1}{n}|\Sigma|. \tag{5.2.9}$$

We have
$$\frac{J_n(v_n)-J_n(v_\rho)}{\rho}=\frac{J_n^2(v_n)-J_n^2(v_\rho)}{\rho\,(J_n(v_n)+J_n(v_\rho))}$$
$$=\frac{\left[\left(J(v_n)-J(\bar v)+\frac{1}{n^2}\right)^+\right]^2-\left[\left(J(v_\rho)-J(\bar v)+\frac{1}{n^2}\right)^+\right]^2}{\rho\,(J_n(v_n)+J_n(v_\rho))}+$$
$$\frac{d_C(\nabla y_n)^2-d_C(\nabla y_\rho)^2}{\rho\,(J_n(v_n)+J_n(v_\rho))}.$$
From (5.2.8) it follows that
$$\lim_{\rho\to 0}\frac{\left[\left(J(v_n)-J(\bar v)+\frac{1}{n^2}\right)^+\right]^2-\left[\left(J(v_\rho)-J(\bar v)+\frac{1}{n^2}\right)^+\right]^2}{\rho\,(J_n(v_n)+J_n(v_\rho))}=-\nu_n\Delta J^N, \tag{5.2.10}$$
with
$$\nu_n=\frac{\left(J(v_n)-J(\bar v)+\frac{1}{n^2}\right)^+}{J_n(v_n)}.$$
With (5.2.7), and the properties of the distance function $d_C(\cdot)$, we may write
$$\lim_{\rho\to 0}\frac{d_C(\nabla y_n)^2-d_C(\nabla y_\rho)^2}{\rho\,(J_n(v_n)+J_n(v_\rho))}=\lim_{\rho\to 0}\frac{d_C(\nabla y_n)-d_C(\nabla y_\rho)}{\rho}\frac{d_C(\nabla y_n)+d_C(\nabla y_\rho)}{(J_n(v_n)+J_n(v_\rho))}=$$
$$=\int_Q \vec{f}_n\cdot\nabla z_n\,dx\,dt, \tag{5.2.11}$$
where
$$\vec{f}_n=\begin{cases}\dfrac{d_C(\nabla y_n)}{J_n(v_n)}\nabla d_C(\nabla y_n) & \text{if } \nabla y_n\notin C,\\ 0 & \text{if not no.}\end{cases}$$

To deduce an approximate Pontryagin principle, we introduce the approximate adjoint equation. Due to the assumptions made on the derivatives of the functions that are involved in the problem and to the regularity result of Proposition 2.2.10, there exists a unique $\varphi_n\in L^{\tau'}(W^{1,p'})+L^2(H^1)$ satisfying
$$\begin{aligned}-\frac{\partial\varphi_n}{\partial t}+A^*\varphi_n-\frac{\partial f}{\partial y}(x,t,y_n)\varphi_n &= \nu_n\frac{\partial F}{\partial y}(x,t,y_n)+\operatorname{div}\vec{f}_n && \text{in } Q,\\ \frac{\partial\varphi_n}{\partial n_{A^*}}-\frac{\partial g}{\partial y}(s,t,y_n,v_n)\varphi_n &= \nu_n\frac{\partial G}{\partial y}(s,t,y_n,v_n)-\vec{f}_n\cdot\vec{n} && \text{on } \Sigma,\\ \varphi_n(\cdot,T) &= \nu_n\frac{\partial L}{\partial y}(\cdot,y_n(T)) && \text{in } \Omega.\end{aligned}$$

With Green's formula (2.2.32) of Proposition 2.2.10 we have that

$$\int_Q \nu_n \frac{\partial F}{\partial y}(x,t,y_n) z_n\, dx\, dt - \int_Q \vec{f}\cdot\nabla z_n\, dx\, dt + \int_\Sigma \nu_n \frac{\partial G}{\partial y}(s,t,y_n,v_n)\, ds\, dt +$$
$$\int_\Omega \nu_n \frac{\partial L}{\partial y}(x, y_n(T))\, dx =$$
$$= \int_Q \varphi_n \left(\frac{\partial z_n}{\partial t} + Az_n - \frac{\partial f}{\partial y}(x,t,y_n)z_n\right)\, dx\, dt +$$
$$+ \int_\Sigma \varphi_n \left(\frac{\partial z_n}{\partial n_A} - \frac{\partial g}{\partial y}(s,t,y_n,v_n)z_n\right)\, ds\, dt$$
$$= \int_\Sigma \varphi_n \left(g(s,t,y_n,v) - g(s,t,y_n,v_n)\right)\, ds\, dt.$$

Taking the limit when ρ tends to zero in (5.2.9), with (5.2.10), (5.2.11) and the previous Green formula, we obtain the approximate Pontryagin principle:

$$\int_\Sigma (\nu_n G(s,t,y_n,v_n) + \varphi_n g(s,t,y_n,v_n))\, ds\, dt \le$$
$$\int_\Sigma (\nu_n G(s,t,y_n,v) + \varphi_n g(s,t,y_n,v))\, ds\, dt + \frac{1}{n}|\Sigma| \quad \text{for all } v \in V_{ad}. \tag{5.2.12}$$

Notice that $\nu_n^2 + \|\vec{f}_n\|^2_{(L^{\tau'}(L^{p'}))^N} = 1$. Thus there exists subsequences, still indexed by n, such that $(\nu_n)_n$ converges to ν, and $(\vec{f}_n)_n$ converges weakly to $\vec{f}$ in $(L^{\tau'}(L^{p'}))^N$. If $\nu > 0$ then (5.2.1) holds. Otherwise, using that $\lim_{n\to\infty} \|\vec{f}_n\|^2_{(L^{\tau'}(L^{p'}))^N} = 1$, and that the interior of C is non empty, we can show that $\vec{f} \neq 0$ in a standard way (see [78], for instance). We know that there exists a ball $B_{L^\tau(L^p)^N}(\vec{z}, \rho) \subset C$, with $\rho > 0$. Take $\vec{z}_n \in B_{L^\tau(L^p)^N}(\vec{0}, \rho)$ such that

$$\int_Q \vec{z}_n \cdot \vec{f}_n dx\, dt = \frac{1}{2}\rho + \|\vec{f}_n\|_{L^\tau(L^p)^N}.$$

Since $\vec{z} + \vec{z}_n \in C$, from the definition of $\vec{f}_n$ and the definition of subdifferential in the sense of convex analysis (see for instance [45]), we have that

$$\int_Q \vec{f}_n \cdot (\vec{z} + \vec{z}_n - \nabla y_n) dx\, dt \le 0.$$

Taking the limit we obtain that

$$\frac{1}{2}\rho + \int_Q \vec{f} \cdot (\vec{z} - \nabla y_n) \le 0,$$

which proves $\vec{f} \neq 0$.

Condition (5.2.2) holds due to the definition of subdifferential of the convex functional $d_C(\cdot)$.

With (5.2.5), we can show that $(y_n)_n$ converges to $\bar{y}$ in $C_b(\bar{Q} \setminus \bar{\Omega} \times \{0\})$. With the assumptions made and with Proposition 2.2.10, we prove that $(\varphi_n)_n$ converges in $L^{\tau'}(W^{1,p'}) + L^2(H^1)$ to the solution $\bar{\varphi}$ of (5.2.3).

Taking into account the convergence results for $(y_n)_n$, $(v_n)_n$, $(\varphi_n)_n$, $(\nu_n)_n$, we can pass to the limit in (5.2.12) when n tends to infinity, and obtain an integral form of Pontryagin's principle.

$$\int_\Sigma (\bar{\nu} G(s,t,\bar{y},\bar{v}) + \bar{\varphi} g(s,t,\bar{y},\bar{v}))\ ds\, dt \leq \int_\Sigma (\bar{\nu} G(s,t,\bar{y},v) + \bar{\varphi} g(s,t,\bar{y},v))\ ds\, dt.$$

for all $v \in V_{ad}$.

Pointwise Pontryagin's principle can be now deduced as in [78, page 1875]. The proof is complete. □

Some extensions

In this section we have only treated of bounded boundary controls. The treatment of unbounded controls can also be done as in [78], but this implies some technical difficulties. We refer to [78] for such extensions. All the results could be performed for distributed controls, with no important changes in the proofs.

To illustrate these remarks, consider the control problem corresponding to:

- the state equation:

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + f(x,t,y,u) &= 0 \quad \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} + g(s,t,y,v) &= 0 \quad \text{on } \Sigma, \\ y(\cdot,0) &= y_0 \quad \text{in } \Omega, \end{cases} \tag{5.2.13}$$

  with $u \in U_{ad} \subset L^q(Q)$, $v \in V_{ad} \subset L^\sigma(\Sigma)$, $q > N/2+1$ and $\sigma > N+1$. The control sets U_{ad} and V_{ad} are defined as follows.

$$U_{ad} = \{u \in L^q(\Sigma) :\ u(x,t) \in K_Q(x,t) \text{ for almost all } (x,t) \in Q\},$$

$$V_{ad} = \{v \in L^\sigma(\Sigma) :\ v(s,t) \in K_\Sigma(s,t) \text{ for almost all } (s,t) \in \Sigma\},$$

where K_Q and K_Σ are measurable multimapping with nonempty compact values in $\mathcal{P}(\mathbb{R})$.

- the cost functional:

$$\begin{aligned} J(y_{uv}, u, v) &= \int_0^T \int_\Omega F(x,t,y_{uv},u)\, dx\, dt + \int_0^T \int_\Gamma G(s,t,y_{uv},v)\, ds\, dt \\ &+ \int_\Omega L(x, y_{uv}(x,T))\, dx, \end{aligned} \tag{5.2.14}$$

- the state constraint:

$$\int_0^T \left(\int_\Omega |\nabla_x y - g_d|^p dx \right)^{\tau/p} dt \leq \delta, \tag{5.2.15}$$

where g_d is a given function in $(L^\tau(L^p))^N$.

We define the distributed and the boundary Hamiltonian function by

$$H_Q(x,t,y,u,\varphi,\nu) = \nu F(x,t,y,u) - \varphi f(x,t,y,u)$$

for every $(x,t,y,u,\varphi,\nu) \in \Omega \times [0,T] \times \mathbb{R}^4$,

$$H_\Sigma(s,t,y,v,\varphi,\nu) = \nu G(s,t,y,v) - \varphi g(s,t,y,v)$$

for every $(s,t,y,v,\varphi,\nu) \in \Gamma \times [0,T] \times \mathbb{R}^4$. With some modifications on the assumptions P1 and P4 on f, g, F and G (we should suppose that f y F depend on the control u and give the adquate growing conditions on u), we can prove the following result.

Theorem 5.2.2 *If $(\bar{y}, \bar{u}, \bar{v})$ is a solution to the control problem, then there exists $\bar{\varphi} \in L^{\tau'}(W^{1,p'})$, $\bar{\nu} \in \mathbb{R}^+$, $\bar{\mu} \in \mathbb{R}^+$ such that*

$$(\bar{\nu}, \bar{\mu}) \neq (0,0), \tag{5.2.16}$$

$$\bar{\mu}\left(\int_0^T (|\nabla_x \bar{y} - g_d|^p dx)^{\tau/p}\, dt - \delta \right) = 0, \tag{5.2.17}$$

$$\left\{ \begin{array}{rcll} -\dfrac{\partial \bar{\varphi}}{\partial t} + A\bar{\varphi} + f'_y(x,t,\bar{y},\bar{u})\bar{\varphi} &=& \bar{\nu} F'_y(x,t,\bar{y},\bar{u}) + \bar{\mu}\, \mathrm{div}\, \vec{f} & \text{in } Q, \\ \dfrac{\partial \bar{\varphi}}{\partial n_A} + g'_y(s,t,\bar{y},\bar{v})\bar{\varphi} &=& \bar{\nu} G'_y(s,t,\bar{y},\bar{v}) - \bar{\mu} \vec{f} \cdot \vec{n} & \text{on } \Sigma, \\ \bar{\varphi}(\cdot, T) &=& \bar{\nu} L'_y(x, \bar{y}(T)) & \text{in } \Omega, \end{array} \right. \tag{5.2.18}$$

where

$$\vec{f} = \left(\int_\Omega |\nabla_x \bar{y} - g_d|^p \, dx \right)^{\frac{r}{p}-1} \left(|\nabla_x \bar{y} - g_d|^{p-2} (\nabla_x \bar{y} - g_d) \right),$$

$$H_Q(x,t,\bar{y}(x,t),\bar{u}(x,t),\bar{\varphi}(x,t),\bar{\nu}) = \min_{u \in K_Q(x,t)} H_Q(x,t,\bar{y}(x,t),u,\bar{\varphi}(x,t),\bar{\nu})$$

for a.e. (x,t) *en* Q, *y*

$$H_\Sigma(s,t,\bar{y}(s,t),\bar{v}(s,t),\bar{\varphi}(s,t),\bar{\nu}) = \min_{v \in K_\Sigma(s,t)} H_\Sigma(s,t,\bar{y}(s,t),v,\bar{\varphi}(s,t),\bar{\nu})$$

for a.e. (s,t) *en* Σ.

Chapter 6

First and second order conditions

In this chapter we state first and second order conditions for the studied control problems. Similar theorems for problems with a finite number of pointwise or integral constraints on the state have been studied for instance in [37]. The same theorems can not be directly applied for problems with an infinite number of state constraints (for instance $|y(x)| \leq \delta$ in $\bar{\Omega}$). In [10] first order conditions for this kind of problems can be found.

6.1 Conditions for abstract optimization problems

In this section we introduce some results about optimality conditions for abstract optimization problems that have been obtained by Casas and Tröltzsch [36].

Let us take $(X, \mathcal{B}, \mu)$ a measure space. Consider the following optimization problem

$$\textbf{(Q)} \begin{cases} \text{Minimize } J(u) \\ u \in U_{ad} = \{u \in L^\infty(X) : u_a(x) \leq u(x) \leq u_b(x) \quad \text{for a.e. } x \in X\}, \\ G_j(u) = 0, \; 1 \leq j \leq n_i, \\ G_j(u) \leq 0, \; n_i + 1 \leq j \leq n_i + n_d \end{cases}$$

where $u_a, u_b \in L^\infty(X)$ and $J, G_j : L^\infty(X) \longrightarrow \mathbb{R}$ are given functions, $1 \leq j \leq n_i + n_d$. Moreover, for $u \in L^\infty(X)$ and $\lambda = (\lambda_j)_{j=1}^{n_i+n_d} \in \mathbb{R}^{n_i+n_d}$ let us define the Lagrangian of the problem as

$$\mathcal{L}(u, \lambda) = J(u) + \sum_{j=1}^{n_i+n_d} \lambda_j G_j(u)$$

First order necessary conditions

Suppose that $\bar{u}$ is a local solution of **(Q)**, i.e., there exists a real number $\rho > 0$ such that for all admissible point of **(Q)**, with $\|u - \bar{u}\|_{L^\infty(X)} < \rho$, we have that $J(\bar{u}) \leq J(u)$.

Under this assumption, we can deduce first order necessary optimality conditions satisfied by $\bar{u}$. For a proof see, for instance, Clarke [44]).

Theorem 6.1.1 *Suppose that J and $\{G_j\}_{j=1}^{n_i+n_d}$ are of class C^1 in a neighborhood of $\bar{u}$. Then there exist real numbers λ_0, $\{\bar{\lambda}_j\}_{j=1}^{n_i+n_d}$ not all zero such that*

$$\bar{\lambda}_j \geq 0, \quad n_i + 1 \leq j \leq n_i + n_d, \; \bar{\lambda}_j = 0 \text{ if } G_j(\bar{u}) < 0; \tag{6.1.1}$$

$$\langle \lambda_0 J'(\bar{u}) + \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j G_j'(\bar{u}), u - \bar{u} \rangle \geq 0 \quad \text{for all } u_a \leq u \leq u_b. \tag{6.1.2}$$

Obviously, if $\lambda_0 = 0$, equation (6.1.2) does not give us much information. In this case, it is said that the optimality conditions are in non qualified form. Under extra assumptions, we can assure that $\lambda_0 \neq 0$ (and therefore, rescaling, that $\lambda_0 = 1$). In finite dimension it is typical to impose the condition of independence of the gradients of the active constraints. This condition must be a bit stronger in problems with an infinite number of constraints (the bound conditions on u). We will establish the following regularity assumptions that grants the qualification of the optimality conditions. Take

$$I_0 = \{j \leq n_i + n_d \,|\, G_j(\bar{u}) = 0\}$$

the set of indexes corresponding to the active constraints. We will also denote the set of non active constraints with

$$I_- = \{j \leq n_i + n_d \,|\, G_j(\bar{u}) < 0\}.$$

For all $\varepsilon > 0$, we denote

$$X_\varepsilon = \{x \in X : u_a(x) + \varepsilon \leq \bar{u}(x) \leq u_b(x) - \varepsilon\}.$$

We make the following regularity assumption

$$\left\{ \begin{array}{l} \exists \varepsilon_{\bar{u}} > 0 \text{ and } \{h_j\}_{j \in I_0} \subset L^\infty(X), \text{ with } \operatorname{supp} h_j \subset X_{\varepsilon_{\bar{u}}}, \text{ such that} \\ G_i'(\bar{u}) h_j = \delta_{ij}, \quad i, j \in I_0, \end{array} \right. \tag{6.1.3}$$

We have that

Theorem 6.1.2 *Suppose that* (6.1.3) *and the assumptions of Theorem 6.1.1 hold. Then the conclusions of that theorem remain valid with* $\lambda_0 = 1$.

Proof. Suppose $\lambda_0 = 0$. Take $\rho > 0$ small enough in such a way that $u_0 = \bar{u} - \rho \sum_{i>n_i, i\in I_0} h_j$ belongs to U_{ad}. Using the regularity assumption (6.1.3)

$$\langle G_j'(\bar{u}), (u_0 - \bar{u})\rangle = \begin{cases} 0 & \text{if } j \leq n_i \\ -\rho & \text{if } j > n_i \text{ and } j \in I_0. \end{cases}$$

Moreover, we know that if $j > n_i$ then $\bar{\lambda}_j \geq 0$, and that if $j \in I_-$, then $\bar{\lambda}_j = 0$. Therefore, using (6.1.2) and these considerations, we have that

$$0 \leq \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j G_j'(\bar{u})(u_0 - \bar{u}) = \sum_{j>n_i, j\in I_0} \bar{\lambda}_j G_j'(\bar{u})(u_0 - \bar{u}) = -\sum_{j>n_i, j\in I_0} \bar{\lambda}_j \rho \leq 0,$$

Thus, if $j > n_i$ then $\bar{\lambda}_j = 0$.

Suppose now that $j \leq n_i$, and take a $\rho > 0$ small enough in such a way that $u_{j-} = \bar{u} - \rho h_j$ and $u_{j+} = \bar{u} + \rho h_j$ belong to U_{ad}. We have that for $i \leq n_i$

$$G_i'(\bar{u})(u_{j-} - \bar{u}) = -\rho\delta_{ij}$$

and

$$G_i'(\bar{u})(u_{j+} - \bar{u}) = \rho\delta_{ij}.$$

Hence

$$0 \leq \sum_{i=1}^{n_i+n_d} \bar{\lambda}_i G_i'(\bar{u})(u_{j-} - \bar{u}) = -\rho\bar{\lambda}_j$$

and

$$0 \leq \sum_{i=1}^{n_i+n_d} \bar{\lambda}_i G_i'(\bar{u})(u_{j+} - \bar{u}) = \rho\bar{\lambda}_j,$$

and we have that $\bar{\lambda}_j = 0$.

We have shown that $\bar{\lambda}_j = 0$ for $1 \leq j \leq n_i + n_d$. This contradicts that fact that not all the multipliers are zero, so $\lambda_0 \neq 0$, and rescaling we can take $\lambda_0 = 1$. □

Notice that we can write (6.1.2) as

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})(u - \bar{u}) \geq 0 \text{ for all } u_a \leq u \leq u_b. \tag{6.1.4}$$

Second order necessary conditions

We summarize in this section the main results for optimization problems of [36].

Since we want to give second order optimality conditions useful for the study of the control problems $(\mathbf{P_e})$ of page 16 and $(\mathbf{P_p})$ of page 16, we need to take into account the two-norm discrepancy; for this topic see Ioffe [61] and Maurer [69]. We will have to impose additional conditions on the functionals J and G_j.

(A1) There exist functions $\phi, \psi_j \in L^2(X)$, $1 \le j \le n_i + n_d$, such that for all $h \in L^\infty(X)$

$$J'(\bar{u})h = \int_X \phi(x)h(x)dx \quad \text{and} \quad G_j'(\bar{u})h = \int_X \psi_j(x)h(x)dx, \; 1 \le j \le n_i + n_d. \tag{6.1.5}$$

(A2) If $\{h_k\}_{k=1}^\infty \subset L^\infty(X)$ is bounded, $h \in L^\infty(X)$ and $h_k(x) \to h(x)$ for a.e. in X, then

$$[J''(\bar{u}) + \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j G_j''(\bar{u})]h_k^2 \to [J''(\bar{u}) + \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j G_j''(\bar{u})]h^2. \tag{6.1.6}$$

If we define

$$d(x) = \phi(x) + \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j \psi_j(x), \tag{6.1.7}$$

then

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})h = [J'(\bar{u}) + \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j G_j'(\bar{u})]h = \int_X d(x)h(x)dx \quad \forall h \in L^\infty(X). \tag{6.1.8}$$

From (6.1.4) we deduce that

$$d(x) = \begin{cases} 0 & \text{for a.e. } x \in X \text{ such that } u_a(x) < \bar{u}(x) < u_b(x), \\ \ge 0 & \text{for a.e. } x \in X \text{ such that } \bar{u}(x) = u_a(x), \\ \le 0 & \text{for a.e. } x \in X \text{ such that } \bar{u}(x) = u_b(x). \end{cases} \tag{6.1.9}$$

Associated with d we define

$$X^0 = \{x \in X : |d(x)| > 0\}. \tag{6.1.10}$$

Given $\{\bar{\lambda}_j\}_{j=1}^{n_i+n_d}$ by Theorem 6.1.2 we define

$$C_{\bar{u}}^0 = \{h \in L^\infty(X) \;\text{ satisfying }\; (6.1.12) \;\text{ and }\; h(x) = 0 \;\text{ a.e. }\; x \in X^0\}, \tag{6.1.11}$$

with

$$\begin{cases} G_j'(\bar{u})h = 0 \text{ if } (j \le n_i) \text{ or } (j > n_i,\ G_j(\bar{u}) = 0 \text{ and } \bar{\lambda}_j > 0); \\ G_j'(\bar{u})h \le 0 \ \text{ if } j > n_i,\ G_j(\bar{u}) = 0 \text{ and } \bar{\lambda}_j = 0; \\ h(x) = \begin{cases} \ge 0 & \text{if } \bar{u}(x) = u_a(x); \\ \le 0 & \text{if } \bar{u}(x) = u_b(x). \end{cases} \end{cases} \tag{6.1.12}$$

In the following theorem we state second order necessary optimality conditions.

Theorem 6.1.3 *Suppose that* (6.1.3), **(A1)** *and* **(A2)** *hold,* $\{\bar{\lambda}_j\}_{j=1}^{n_i+n_d}$ *are the Lagrange multipliers satisfying* (6.1.1) *and* (6.1.2), *with* $\bar{\lambda}_0 = 1$, *and* J *and* $\{G_j\}_{j=1}^{n_i+n_d}$ *are of class* C^2 *in a neighborhood of* $\bar{u}$. *Then the following inequality holds:*

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 \ge 0 \quad \forall h \in C_{\bar{u}}^0. \tag{6.1.13}$$

With a slightly stronger assumption than **(A2)** we can prove a slightly stronger necessary condition than that in Theorem 6.1.3. To do this, let us first introduce the set

$$C_{\bar{u},L^2(X)}^0 = \{h \in L^2(X) \ \text{ satisfying } \ (6.1.12) \ \text{ and } \ h(x) = 0 \ \text{ a.e. } \ x \in X^0\}, \tag{6.1.14}$$

We have the following property

Lemma 6.1.4 *Suppose that* **(A1)** *and the regularity assumption* (6.1.3) *hold. Then*

$$C_{\bar{u},L^2(X)}^0 = \bar{C}_{\bar{u}}^0,$$

where $\bar{C}_{\bar{u}}^0$ *denotes the closure of* $C_{\bar{u}}^0$ *in* $L^2(X)$.

Proof. That $C_{\bar{u}}^0 \subset C_{\bar{u},L^2(X)}^0$ is straight. Moreover, $C_{\bar{u},L^2(X)}^0$ is closed, which leads us to conclude that $\bar{C}_{\bar{u}}^0 \subset C_{\bar{u},L^2(X)}^0$.

To see that $C_{\bar{u},L^2(X)}^0 \subset \bar{C}_{\bar{u}}^0$ let us take $h \in C_{\bar{u},L^2(X)}^0$. We are going to build a sequence $\{h_k\}_{k=1}^{\infty} \subset C_{\bar{u}}^0$ that converges to h in $L^2(X)$. Set

$$\hat{h}_k(x) = \begin{cases} k & \text{if} \quad h(x) \ge k \\ h(x) & \text{if} \quad -k \le h(x) \le k \\ -k & \text{if} \quad h(x) \le -k. \end{cases}$$

Obviously

$$\lim_{k\to\infty} \hat{h}_k = h \text{ in } L^2(X).$$

For $j \in I_0$, take

$$\alpha_{kj} = G_j'(\bar{u})\hat{h}_k - G_j'(\bar{u})h.$$

We have that for all j

$$\lim_{k\to\infty} \alpha_{kj} = 0.$$

Due to the regularity assumption, we know that there exist $\varepsilon_{\bar{u}} > 0$ and $\{\bar{h}_j\}_{j\in I_0} \subset L^\infty(X)$, with $\operatorname{supp}\bar{h}_j \subset X_{\varepsilon_{\bar{u}}}$, such that $G_i'(\bar{u})\bar{h}_j = \delta_{ij},\ \ i,j \in I_0$.

Take

$$h_k = \hat{h}_k - \sum_{j\in I_0} \alpha_{kj}\bar{h}_j.$$

Obviously, for the considerations about the limits of $\hat{h}_k$ and α_{jk} we have that

$$\lim_{k\to\infty} h_k = h \text{ in } L^2(X).$$

Let us see that $h_k \in C_{\bar{u}}^0$.

First, notice that $h(x) = 0$ a.e. in X^0. Given $x \in X$, for $j \in I_0$, if $\bar{h}_j(x) \neq 0$, then $x \in X_{\varepsilon_{\bar{u}}}$. Therefore $u_a(x) < \bar{u}(x) < u_b(x)$, and due to (6.1.9), $d(x) = 0$. Then $x \notin X^0$. So in X^0, $\bar{h}_j = 0$. Due to the definition of $\hat{h}_k$ we have then that $h_k(x) = 0$ a.e. in X^0.

Secondly, for $i \in I_0$

$$G_i'(\bar{u})h_k = G_i'(\bar{u})\hat{h}_k - \sum_{j\in I_0} \alpha_{kj}G_i'(\bar{u})\bar{h}_j = G_i'(\bar{u})\hat{h}_k - \alpha_{ki} = G_i'(\bar{u})h.$$

Using now that h satisfies the relations $G_i'(\bar{u})h = 0$ if $j \le n_i$ or $j > n_i$, $G_i(\bar{u}) = 0$, $\bar{\lambda}_i > 0$ and $G_i'(\bar{u})h \le 0$ if $j > n_i$, $G_i(\bar{u}) = 0$, $\bar{\lambda}_i = 0$ from (6.2.8), we deduce from the equality $G_i'(\bar{u})h_k = G_i'(\bar{u})h$ that h_k also satisfies them.

Finally we have to check the sign condition. Since $\operatorname{supp}\bar{h}_j \subset X_{\varepsilon_{\bar{u}}}$, then $\bar{h}_j(x) = 0$ whenever $\bar{u}(x) = u_a(x)$ or $\bar{u}(x) = u_b(x)$. Consequently, the sign of $\hat{h}_k(x)$ is the same as the sign of $h_k(x)$ if $\bar{u}(x) = u_a(x)$ or $\bar{u}(x) = u_b(x)$. Finally it is enough to notice that the sign of $\hat{h}_k(x)$ is equal to the sign of $h(x)$ for every $x \in X$ and that $h \in C^0_{u,L^2(X)}$ to conclude that h_k satisfies the sign condition. So $h_k \in C_{\bar{u}}^0$ and the proof is complete. □

Let us introduce now the following assumption, slightly stronger than **(A2)**.

(A2') $\dfrac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})$ is bilinear and continuous in $L^2(X)$.

Then we can prove

Theorem 6.1.5 *Suppose that* (6.1.3), **(A1)** *and* **(A2')** *hold,* $\{\bar{\lambda}_j\}_{j=1}^{n_i+n_d}$ *are the Lagrange multipliers satisfying* (6.1.1) *and* (6.1.2) *with* $\bar{\lambda}_0 = 1$*, and* J *and* $\{G_j\}_{j=1}^{n_i+n_d}$ *are of class* C^2 *in a neighborhood of* $\bar{u}$*. Then the following inequality is satisfied*

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 \geq 0 \quad \forall h \in C^0_{\bar{u},L^2(X)}. \tag{6.1.15}$$

Proof. Take $h \in C^0_{\bar{u},L^2(X)}$. Due to Lemma 6.1.4 we can find a sequence $\{h_k\}_{k=1}^{\infty} \subset C^0_{\bar{u}}$ such that $h_k \to h$ in $L^2(X)$. Noting that **(A2')** implies **(A2)** and using Theorem 6.1.3 we have that

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h_k^2 \geq 0$$

for all k. Due to assumption **(A2')**, we can take the limit and obtain

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 \geq 0.$$

The proof is complete. □

Second order sufficient conditions

Now $\bar{u}$ is a given admissible element for problem **(Q)** that satisfies the first order necessary conditions. Motivated again by the considerations about the two norm discrepancy, we must make some assumptions that involve the norms in $L^\infty(X)$ and $L^2(X)$.

(A3) There exists a positive number $\rho > 0$ such that J and $\{G_j\}_{j=1}^{n_i+n_d}$ are of class C^2 in the ball of $L^\infty(X)$, $B_{L^\infty(X)}(\bar{u},\rho)$ and for all $\delta > 0$ there exists $\varepsilon \in (0,\rho)$ such that for all $u \in B_{L^\infty(X)}(\bar{u},\rho)$, $\|v-\bar{u}\|_{L^\infty(X)} < \varepsilon$, $h, h_1, h_2 \in L^\infty(X)$ and $1 \le j \le n_i + n_d$ we have that

$$\left\{\begin{array}{l} \left|\left[\frac{\partial^2\mathcal{L}}{\partial u^2}(v,\bar{\lambda}) - \frac{\partial^2\mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})\right] h^2\right| \le \delta\|h\|^2_{L^2(X)}, \\ \\ |J'(u)h| \le M_{0,1}\|h\|_{L^2(X)}, \quad |G'_j(u)h| \le M_{j,1}\|h\|_{L^2(X)}, \\ \\ |J''(u)h_1h_2| \le M_{0,2}\|h_1\|_{L^2(X)}\|h_2\|_{L^2(X)}, \\ \\ |G''_j(u)h_1h_2| \le M_{j,2}\|h_1\|_{L^2(X)}\|h_2\|_{L^2(X)}, \end{array}\right. \tag{6.1.16}$$

Analogously to (6.1.10) and (6.1.11) we define for all $\tau > 0$

$$X^\tau = \{x \in X : |d(x)| > \tau\} \tag{6.1.17}$$

and

$$C^\tau_{\bar{u}} = \{h \in L^\infty(X) \;\text{ that satisfy }\; (6.1.12) \;\text{ and }\; h(x) = 0 \;\text{ a.e. }\; x \in X^\tau\}. \tag{6.1.18}$$

The following theorem gives us second order sufficient conditions for **(Q)**.

Theorem 6.1.6 *Let $\bar{u}$ be an admissible for problem* **(Q)** *that satisfies first order necessary conditions, and let us suppose that assumptions* (6.1.3); **(A1)** *and* **(A3)** *hold. Suppose also that*

$$\frac{\partial^2\mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})h^2 \ge \delta\|h\|^2_{L^2(X)} \quad \forall h \in C^\tau_{\bar{u}} \tag{6.1.19}$$

for given $\delta > 0$ and $\tau > 0$. Then there exist $\varepsilon > 0$ and $\alpha > 0$ such that $J(\bar{u}) + \alpha\|u - \bar{u}\|^2_{L^2(X)} \le J(u)$ for every admissible point u for(**Q**)*, with $\|u-\bar{u}\|_{L^\infty(X)} < \varepsilon$.*

Remark 6.1.1 *If* **(A1)** *and the regularity assumption* (6.1.3) *hold, we can prove, just like in Lemma 6.1.4, that*

$$C^{\tau}_{\bar{u},L^2(X)} = \bar{C}^{\tau}_{\bar{u}},$$

where

$$C^{\tau}_{\bar{u},L^2(X)} = \{h \in L^2(X) \ \ \textit{satisfying} \ \ (6.1.12) \ \ \textit{and} \ \ h(x) = 0 \ \ \textit{a.e.} \ \ x \in X^{\tau}\}$$

and $\bar{C}^{\tau}_{\bar{u}}$ denotes the closure of $C^{\tau}_{\bar{u}}$ in $L^2(X)$.

Notice also that assumption **(A3)** *implies* **(A2')**. *Therefore, if the assumptions* (6.1.3), **(A1)**, **(A3)** *and* (6.1.19)*hold for given $\delta > 0$ and $\tau > 0$, then condition* (6.1.19) *holds not only for the functions of $C^{\tau}_{\bar{u}}$, but for all the functions of $C^{\tau}_{\bar{u},L^2(X)}$:*

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 \geq \delta \|h\|^2_{L^2(X)} \quad \forall h \in C^{\tau}_{\bar{u},L^2(X)},$$

which is a condition that, a priori, seems stronger.

6.2 Elliptic case

Take again Ω of class C^1; Γ its boundary; A an elliptic operator of continuous coefficients of the form (2.1.1) (page 23); $p > N$; $a_0 \in L^{p/2}(\Omega)$; $f : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$; $g : \Gamma \to \mathbb{R}$, $g \in L^{p-1}(\Gamma)$; $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ for $1 \leq j \leq n_i + n_e$.

Moreover, we will suppose that the set of admissible controls is of the form

$$U_{ad} = \{u \in L^\infty(\Omega) : u_a(x) \leq u(x) \leq u_b(x) \text{ for a.e. } x \in \Omega\},$$

where $u_a, u_b \in L^\infty(\Omega)$. With the notation of Chapter 1 we have $K_\Omega(x) = [u_a(x), u_b(x)]$. We will use the same notation as in Section 6.1. In this case $X = \Omega$. Now $J(u)$ is defined as in (4.1.1) and $G_j(u)$ is defined as in (4.1.5).

$$J(u) = \int_\Omega L(x, y_u, u)dx,$$

$$G_j(u) = \int_\Omega g_j(x, \nabla y_u(x))dx.$$

The Lagrangian of the problem is given in this case by

$$\mathcal{L}(u, \lambda) = \int_\Omega L(x, y_u, u)dx + \sum_{j=1}^{n_i+n_d} \lambda_j \int_\Omega g_j(x, \nabla y_u(x))dx.$$

It is interesting to introduce again

$$F_j(y) = \int_\Omega g_j(x, \nabla y(x))dx.$$

Observe that

$$F_j'(y) = -\text{div}\,\frac{\partial g}{\partial \eta}(x, \nabla y)$$

and $G_j = F_j \circ G$, where $G(u) = y_u$.

We are going to formulate a regularity assumption analogous to (6.1.3). For $\varepsilon > 0$, set

$$\Omega_\varepsilon = \{x \in \Omega : u_a(x) + \varepsilon \le \bar{u}(x) \le u_b(x) - \varepsilon\}$$

Lemma 6.2.1 *Given $\bar{u}$ an element of U_{ad}, the following two conditions are equivalent:*

(1) *there exists $\varepsilon_{\bar{u}} > 0$ and functions $\{h_j\}_{j\in I_0} \subset L^\infty(\Omega)$ with* supp $h_j \subset \Omega_{\varepsilon_{\bar{u}}}$ *such that $G_i'(\bar{u})h_j = \delta_{ij}$ for $i, j \in I_0$;*

(2) *there exists $\varepsilon_{\bar{u}} > 0$ such that*

$$\text{the family } \{\bar{\varphi}_i \frac{\partial f}{\partial u}(x, \bar{y}, \bar{u})\}_{i\in I_0} \text{ is linearly independent in } L^1(\Omega_{\varepsilon_{\bar{u}}}), \tag{6.2.1}$$

where $\bar{y} = G(\bar{u})$ and $\bar{\varphi}_i = \varphi_{i\bar{u}}$ is the solution of (4.1.7) for $u = \bar{u}$.

Proof. Let us remain the expresion for $G_i'(\bar{u})h$, given in (4.1.6),

$$G_j'(\bar{u})h = \int_\Omega \bar{\varphi}_j \frac{\partial f}{\partial u}(x, \bar{y}, \bar{u})h\,dx.$$

Let us prove first that (1) implies (2). Suppose that $G_i'(\bar{u})h_j = \delta_{ij}$ and $\{\bar{\varphi}_i\frac{\partial f}{\partial u}(x, \bar{y}, \bar{u})\}_{i\in I_0}$ are not linearly independent. Then there exist numbres $\{\alpha_i\}_{i\in I_0}$, not all zero, such that $\sum_{i\in I_0} \alpha_i\bar{\varphi}_i\frac{\partial f}{\partial u}(x, \bar{y}, \bar{u}) = 0$ for a.e. $x \in \Omega_{\varepsilon_{\bar{u}}}$. Suppose that $\alpha_j \neq 0$. On one hand

$$\int_\Omega \left(\sum_{i\in I_0} \alpha_i\bar{\varphi}_i \frac{\partial f}{\partial u}(x, \bar{y}, \bar{u})\right) h_j dx = \int_\Omega 0 h_j dx = 0$$

and on the other hand

$$\int_\Omega \left(\sum_{i\in I_0} \alpha_i\bar{\varphi}_i \frac{\partial f}{\partial u}(x, \bar{y}, \bar{u})\right) h_j dx = \sum_{i\in I_0} \alpha_i G_i'(\bar{u})h_j dx = \alpha_j.$$

Both identities imply that $\alpha_j = 0$, which is a contradiction with our assumption $\alpha_j = 0$. Therefore $\{\bar\varphi_i \frac{\partial f}{\partial u}(x, \bar y, \bar u)\}_{i\in I_0}$ are linearly independent.

Let us see now that (2) implies (1). From the linear independence of $\{\bar\varphi_i \frac{\partial f}{\partial u}(x, \bar y, \bar u)\}_{i\in I_0}$ it follows that the functional $T : L^\infty(\Omega_{\varepsilon_{\bar u}}) \to \mathbb{R}^{|I_0|}$ that maps every h to

$$Th = \left(\int_\Omega \bar\varphi_i \frac{\partial f}{\partial u}(x, \bar y, \bar u) h\, dx\right)_{i\in I_0}$$

is surjective. Indeed, if T were not surjective, then there exists $\alpha \in \mathbb{R}^{|I_0|}$, $\alpha \neq 0$ such that

$$\alpha \cdot Th = 0 \text{ para todo } h \in L^\infty(\Omega_{\varepsilon_{\bar u}}),$$

which implies that

$$\sum_{j\in I_0} \bar\varphi_j \frac{\partial f}{\partial u}(x, \bar y, \bar u) = 0 \text{ para c.t.p. } x \in \Omega_{\varepsilon_{\bar u}},$$

which contradicts (2). So for every j, there exists a $h_j \in L^\infty(\Omega_{\varepsilon_{\bar u}})$ such that Th_j is the vector whose j-th component is 1 and the others are zeroes. The proof is complete. □

First order necessary conditions

First order necessary conditions satisfied by a local solution of $(\mathbf{P_e})$ can be deduced from Theorem 6.1.2 with the aid of Theorems 4.1.1 and 4.1.3.

Theorem 6.2.2 *Suppose that $\bar u$ is a local solution for problem* $(\mathbf{P_e})$. *Suppose that the assumptions on f, L and g_j established in E1 (page 69), E4 (page 87) and E6 (page 89) hold. Suppose also that* (6.2.1) *holds. Then there exist real numbers $\bar\lambda_j$, $j = 1, \ldots, n_d + n_i$ and functions $\bar y \in W^{1,p}(\Omega)$, $\bar\varphi \in W^{1,p'}(\Omega)$ such that*

$$\bar\lambda_j \geq 0 \quad n_i + 1 \leq j \leq n_i + n_d, \qquad \bar\lambda_j \int_\Omega g_j(x, \nabla\bar y(x))\, dx = 0, \tag{6.2.2}$$

$$\left\{\begin{array}{rcll} A\bar y + a_0 \bar y &=& f(x, \bar y(x), \bar u(x)) & \text{in } \Omega \\ \partial_{n_A}\bar y &=& 0 & \text{on } \Gamma, \end{array}\right. \tag{6.2.3}$$

$$\left\{\begin{array}{rcll} A^*\bar\varphi + a_0\bar\varphi &=& \dfrac{\partial f}{\partial y}(x, \bar y, \bar u)\bar\varphi + \dfrac{\partial L}{\partial y}(x, \bar y, \bar u) - \displaystyle\sum_{j=1}^{n_i+n_d} \bar\lambda_j \mathrm{div}\left(\frac{\partial g_j}{\partial \eta}(x, \nabla\bar y)\right) & \text{in } \Omega \\ \partial_{n_{A^*}}\bar\varphi &=& 0 & \text{on } \Gamma, \end{array}\right. \tag{6.2.4}$$

and

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u},\bar{\lambda})(u-\bar{u}) = \int_\Omega \left(\frac{\partial L}{\partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial f}{\partial u}(x,\bar{y},\bar{u})\right)(u-\bar{u})dx \geq 0 \quad \text{for all } u \in U_{ad}. \tag{6.2.5}$$

Moreover, if $\bar{\varphi}_0 = \varphi_{0\bar{u}}$ *and* $\bar{\varphi}_j = \varphi_{j\bar{u}}$ *for* $1 \leq j \leq n_i + n_d$ *are the solutions of* (4.1.3) *and* (4.1.7) *respectively, for* $u = \bar{u}$ *, then*

$$\bar{\varphi} = \bar{\varphi}_0 + \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j \bar{\varphi}_j. \tag{6.2.6}$$

Proof. The assumptions made, Theorems 4.1.1 and 4.1.3 and Lemma 6.2.1 allow us to figure out the expression

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u},\bar{\lambda})(u-\bar{u}) = \int_\Omega \left(\frac{\partial L}{\partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial f}{\partial u}(x,\bar{y},\bar{u})\right)(u-\bar{u})dx.$$

Now we can apply directly Theorem 6.1.2 to deduce conditions(6.2.2)–(6.2.5). □

Let us see now an example of a sufficient condition to check the regularity condition (6.2.1)

Lemma 6.2.3 *Let us suppose that there exist* $\varepsilon_{\bar{u}} > 0$ *and an open, nonempty set* $A_{\varepsilon_{\bar{u}}} \subset \Omega_{\varepsilon_{\bar{u}}}$ *such that*

$$\frac{\partial f}{\partial u}(x,\bar{y}(x),\bar{u}(x)) \neq 0 \text{ en } A_{\varepsilon_{\bar{u}}}$$

and $\{F_j'(\bar{y})\}_{j\in I_0}$ *are linearly indepenedent in* $\left(W^{1,p'}(A_{\varepsilon_{\bar{u}}})\right)'$. *Then the regularity condition* (6.2.1) *holds.*

Proof. What we want to prove is the oinear independence of $\left\{\bar{\varphi}_i\frac{\partial f}{\partial u}(x,\bar{y},\bar{u})\right\}_{i\in I_0}$.

Suppose that $\left\{\bar{\varphi}_i\frac{\partial f}{\partial u}(x,\bar{y},\bar{u})\right\}_{i\in I_0}$ are not linearly independent in $L^1(\Omega_{\varepsilon_{\bar{u}}})$. Then there exist real numbers $\{\alpha_i\}_{i\in I_0}$ not all zero such that

$$\sum_{i\in I_0} \alpha_i \bar{\varphi}_i(x)\frac{\partial f}{\partial u}(x,\bar{y},\bar{u}) = 0$$

for a.e. $x \in \Omega_{\varepsilon_{\bar{u}}}$. Since $|A_{\varepsilon_{\bar{u}}}| > 0$ and $\frac{\partial f}{\partial u}(x,\bar{y},\bar{u}) \neq 0$ in $A_{\varepsilon_{\bar{u}}}$, then for a.e. $x \in A_{\varepsilon_{\bar{u}}}$

$$\sum_{i\in I_0} \alpha_i \bar{\varphi}_i(x) = 0.$$

Taking into account that $\bar{\varphi}_i$ is the solution of

$$\begin{cases} A^*\bar{\varphi}_i + a_0\bar{\varphi}_i &= \dfrac{\partial f}{\partial y}(x, y_u, u)\bar{\varphi}_i - \operatorname{div}\left(\dfrac{\partial g_i}{\partial \eta}(x, \nabla\bar{y})\right) \quad \text{in } \Omega \\ \partial_{n_{A^*}}\varphi_i &= 0 \quad \text{on } \Gamma \end{cases}$$

the expression

$$F_i'(\bar{y}) = -\operatorname{div}\frac{\partial g_i}{\partial \eta}(x, \nabla\bar{y}),$$

and that $A_{\varepsilon_{\bar{u}}}$ is open, we obtain that

$$\sum_{i\in I_0}\alpha_i F_i'(\bar{y}) = 0 \text{ in } A_{\varepsilon_{\bar{u}}}$$

with not all the $\{\alpha_i\}_{i\in I_0}$ zero. This contradicts the assumptions.The proof is complete. □

Second order necessary conditions

Taking into account Theorems 4.1.2 and 4.1.4 we can show that the assumptions for Theorem 6.1.3 hold for problem $(\mathbf{P_e})$. Moreover, in this case, given $\bar{u} \in U_{ad}$, we can identify

$$d(x) = \frac{\partial L}{\partial u}(x, \bar{y}(x), \bar{u}(x)) + \bar{\varphi}(x)\frac{\partial f}{\partial u}(x, \bar{y}(x), \bar{u}(x)),$$

where $\bar{y}$ is given by (6.2.3) and $\bar{\varphi}$ is given by (6.2.4). We introduce

$$\Omega^0 = \{x \in \Omega : |d(x)| > 0\}. \tag{6.2.7}$$

$$C_{\bar{u}}^0 = \{h \in L^\infty(\Omega) \ \text{ satisfying } \ (6.2.8) \ \text{ and } \ h(x) = 0 \ \text{ a.e. } \ x \in \Omega^0\},$$

and

$$C_{\bar{u},L^2(\Omega)}^0 = \{h \in L^2(\Omega) \ \text{ satisfying } \ (6.2.8) \ \text{ and } \ h(x) = 0 \ \text{ a.e. } \ x \in \Omega^0\},$$

where

$$\begin{cases} \int_\Omega \bar\varphi_j \frac{\partial f}{\partial u}(x,\bar y,\bar u)h\,dx = 0 \text{ if } (j \le n_i) \text{ or } (j > n_i, \int_\Omega g_j(x,\nabla \bar y)dx = 0 \text{ and } \bar\lambda_j > 0) \\ \int_\Omega \bar\varphi_j \frac{\partial f}{\partial u}(x,\bar y,\bar u)h\,dx \le 0 \text{ if } n_i + 1 \le j \le n_d + n_i \text{ and } \int_\Omega g_j(x,\nabla \bar y)dx = 0 \text{ and } \bar\lambda_j = 0 \\ h(x) \ge 0 \text{ if } \bar u(x) = u_a(x) \\ h(x) \le 0 \text{ if } \bar u(x) = u_b(x). \end{cases} \tag{6.2.8}$$

The second derivative of the Lagrangian is given in this case by the expression

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\lambda)h^2 = \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x,\bar y,\bar u) + \bar\varphi \frac{\partial^2 f}{\partial y^2}(x,\bar y,\bar u)\right) z_h^2\,dx +$$

$$2\int_\Omega \left(\frac{\partial^2 L}{\partial y \partial u}(x,\bar y,\bar u) + \bar\varphi \frac{\partial^2 f}{\partial y \partial u}(x,\bar y,\bar u)\right) h z_h\,dx +$$

$$\int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,\bar y,\bar u) + \bar\varphi \frac{\partial^2 f}{\partial u^2}(x,\bar y,\bar u)\right) h^2\,dx +$$

$$\sum_{j=1}^{n_d+n_i} \bar\lambda_j \int_\Omega \nabla^T z_h \frac{\partial^2 g_j}{\partial \eta^2}(x,\nabla \bar y)\nabla z_h\,dx$$

Now it is necessary some more regularity for some of the second derivatives of f and L. We are going to suppose that f and L are of class C^2 with respect to the second and third variables and there exists $\varepsilon > 0$ such that for all $M > 0$ there exist $\psi_M^1 \in L^{1+\varepsilon}(\Omega)$ and $\psi_M^2 \in L^{p/2+\bar\varepsilon}(\Omega)$, $\bar\varepsilon = p^2\varepsilon/(4-2(p-2)\varepsilon)$ such that

$$\left|\frac{\partial^2 L}{\partial u \partial y}(x,y,u)\right| + \left|\frac{\partial^2 L}{\partial u^2}(x,y,u)\right| \le \psi_M^1(x) \tag{6.2.9}$$

and

$$\left|\frac{\partial^2 f}{\partial u \partial y}(x,t,s)\right| + \left|\frac{\partial^2 f}{\partial u^2}(x,t,s)\right| \le \psi_M^2(x) \tag{6.2.10}$$

if $|y|, |u| \le M$ for a.e. $x \in \Omega$. So we obtain the following theorem.

Theorem 6.2.4 *Suppose that $\bar{u}$ is a local solution of problem* $(\mathbf{P_e})$ *and that the assumptions on f, L and g_j established in E1 (page 69), E2 (page 70), E4 (page 87), E5 (page 89), E6 (page 89), E7 (page 90),* (6.2.9) *and* (6.2.10) *hold. Suppose also that* (6.2.1) *holds. Then*

$$
\begin{aligned}
&\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})h^2 = \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\right) z_h^2\, dx + \\
&2\int_\Omega \left(\frac{\partial^2 L}{\partial y \partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y \partial u}(x,\bar{y},\bar{u})\right) h z_h\, dx + \\
&\int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(x,\bar{y},\bar{u})\right) h^2\, dx + \\
&\sum_{j=1}^{n_d+n_i} \bar{\lambda}_j \int_\Omega \nabla^T z_h \frac{\partial^2 g_j}{\partial \eta^2}(x,\nabla\bar{y})\nabla z_h\, dx \geq 0
\end{aligned}
\qquad (6.2.11)
$$

for all $h \in C_{\bar{u}}^0$, where z_h is given by

$$
\begin{cases}
Az_h + a_0 z_h = \dfrac{\partial f}{\partial y}(x,\bar{y},\bar{u})z_h + \dfrac{\partial f}{\partial u}(x,\bar{y},\bar{u})h & \text{in } \Omega \\
\partial_{\nu_A} z_h = 0 & \text{on } \Gamma.
\end{cases}
$$

Proof. Notice that we can apply Theorem 6.2.2 to deduce the existence of the Lagrange multipliers. Now, due to Theorem 6.1.3, we only have to verify that **(A1)** and **(A2)** hold. In our case, assumption **(A1)** (see page 120), holds with

$$
\phi = \frac{\partial L}{\partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}_0\frac{\partial f}{\partial u}(x,\bar{y},\bar{u})
$$

and

$$
\psi_j = \bar{\varphi}_j\frac{\partial f}{\partial u}(x,\bar{y},\bar{u}).
$$

From the expression for the second derivatives of J and G_j and the properties imposed to the derivatives of f, L and g_j, it follows that **(A2)** holds. In fact, take $\{h_k\}_{k=1}^{\infty} \subset L^\infty(\Omega)$, bounded in $L^\infty(\Omega)$ and pointwise convergent to h. We want to check that

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})h_k^2 = \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\right) z_{h_k}^2\, dx +$$

$$2\int_\Omega \left(\frac{\partial^2 L}{\partial y \partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y \partial u}(x,\bar{y},\bar{u})\right) h_k z_{h_k}\, dx +$$

$$\int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(x,\bar{y},\bar{u})\right) h_k^2\, dx +$$

$$\sum_{j=1}^{n_d+n_i} \bar{\lambda}_j \int_\Omega \nabla^T z_{h_k} \frac{\partial^2 g_j}{\partial \eta^2}(x,\nabla\bar{y})\nabla z_{h_k}\, dx$$

converges to

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})h^2 = \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\right) z_h^2\, dx +$$

$$2\int_\Omega \left(\frac{\partial^2 L}{\partial y \partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y \partial u}(x,\bar{y},\bar{u})\right) h z_h\, dx +$$

$$\int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(x,\bar{y},\bar{u})\right) h^2\, dx +$$

$$\sum_{j=1}^{n_d+n_i} \bar{\lambda}_j \int_\Omega \nabla^T z_h \frac{\partial^2 g_j}{\partial \eta^2}(x,\nabla\bar{y})\nabla z_h\, dx,$$

where

$$\left\{\begin{array}{rcll} Az_{h_k} + a_0 z_{h_k} &=& \frac{\partial f}{\partial y}(x,\bar{y},\bar{u})z_{h_k} + \frac{\partial f}{\partial u}(x,\bar{y},\bar{u})h_k & \text{in } \Omega \\ \partial_{\nu_A} z_{h_k} &=& 0 & \text{on } \Gamma. \end{array}\right.$$

We can do this term by term. First, let us remark that $h_k \to h$ in $L^q(\Omega)$ for all $q < \infty$, which implies that $z_{h_k} \to z_h$ in $W^{1,p}(\Omega)$.

So, using Hölder's inequality and the assumptions on the second derivatives, we have that

$$\int_\Omega \left|\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\right| |z_{h_k}^2 - z_h^2|\, dx \leq$$

$$\leq \|\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\|_{L^1(\Omega)}\|z_{h_k} + z_h\|_{L^\infty(\Omega)}\|z_{h_k} - z_h\|_{L^\infty(\Omega)}.$$

The first two factors are bounded and the last converges to zero.

$$\int_\Omega \left|\frac{\partial^2 L}{\partial y \partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y \partial u}(x,\bar{y},\bar{u})\right| |h_k z_{h_k} - h z_h|\, dx \leq$$

$$\|\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\|_{L^1(\Omega)}\|h_k\|_{L^\infty(\Omega)}\|z_{h_k} - z_h\|_{L^\infty(\Omega)} +$$

$$+\|\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\|_{L^{1+\epsilon}(\Omega)}\|z_h\|_{L^\infty(\Omega)}\|h_k - h\|_{L^{(1+\epsilon)/\epsilon}(\Omega)}.$$

In each term, the first two factors are bounded and the last one converges to zero. Here we see the need for the new regularity assumption for some second derivatives of f and L, because we do not have uniform convergence for the h_k.

$$\int_\Omega \left|\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(x,\bar{y},\bar{u})\right| |{h_k}^2 - h^2|\, dx \leq$$

$$\leq \|\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\|_{L^{1+\epsilon}(\Omega)}\|h_k + h\|_{L^\infty(\Omega)}\|h_k - h\|_{L^{(1+\epsilon)/\epsilon}(\Omega)}.$$

The first two factors are bounded and the last one converges to zero. Finally

$$\int_\Omega \left|\nabla^T z_{h_k}\frac{\partial^2 g_j}{\partial \eta^2}(x,\nabla\bar{y})\nabla z_{h_k} \nabla^T z_h \frac{\partial^2 g_j}{\partial \eta^2}(x,\nabla\bar{y})\nabla z_h\right| dx =$$

$$= \int_\Omega \left|\nabla^T(z_{h_k} + z_h)\frac{\partial^2 g_j}{\partial \eta^2}(x,\nabla\bar{y})\nabla(z_{h_k} - z_h)\right| dx \leq$$

$$\leq \|\nabla(z_{h_k} + z_h)\|_{L^p(\Omega)}\|\frac{\partial^2 g_j}{\partial \eta^2}(x,\nabla\bar{y})\|_{L^{p/(p-2)}(\Omega)}\|\nabla(z_{h_k} - z_h)\|_{L^p(\Omega)}.$$

Again the first two factors are bounded and the last one converges to zero. Therefore, assumption **(A2)** holds. □

To prove an analogous result to Theorem 6.1.5 we have to give conditions for the derivatives of f, L and g_j for the second derivative of the Lagrangian to be bilinear and continuous on $L^2(\Omega)$. Like before, we want to check that

$$\frac{\partial^2 \mathcal{L}}{\partial u}(\bar{u},\bar{\lambda})h_k^2 \to \frac{\partial^2 \mathcal{L}}{\partial u}(\bar{u},\bar{\lambda})h^2,$$

but now we only have that $h_k \to h$ in $L^2(\Omega)$. Looking at the proof of the previous result, one of the first things we see is that to prove the convergence of

$$\int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(x,\bar{y},\bar{u})\right) h_k^2\, dx$$

to

$$\int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(x,\bar{y},\bar{u})\right) h^2\, dx$$

it is necessary that

$$\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(x,\bar{y},\bar{u}) \in L^\infty(\Omega).$$

Notice that we will need the adjoint state to be bounded, and therefore it is be necessary to impose also conditions on the first derivatives of f, L and g_j.

Another question that comes up is that of the regularity of z_h and its gradient. We have that $L^2(\Omega) \subset (W^{1,q'}(\Omega))'$ for all $q < \infty$ if $N = 2$ and for all $q \leq 2N/(N-2)$ if $N \geq 3$. Therefore, the maximal regularity we can expect for z_h is $z_h \in W^{1,q}(\Omega)$, depending on the regularity of the first derivatives of f (cf. page 69 for the equation of z_h and page 27 for the regularity result). Moreover, for $N = 3$, we have that $q = 2N/(N-2) = 6$, which is greater than N, and hence $z_h \in L^\infty(\Omega)$, but if $N > 4$ then z_h does not have to be a bounded function. Considering all these things, we are going to introduce the following assumptions on the functions that intervene in the problem, taking into account that they could be slightly weakened for the cases $N = 2$ and $N = 3$.

E8

- f is of class C^2 with respect to the second and third variables,

  $$\frac{\partial f}{\partial y}(x,t,s) \leq 0$$

  and for all $M > 0$ there exists a constant $C_M > 0$ such that

  $$\left|\frac{\partial f}{\partial y}(x,t,s)\right| + \left|\frac{\partial f}{\partial u}(x,t,s)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x,t,s)\right| + \left|\frac{\partial^2 f}{\partial y \partial u}(x,t,s)\right| + \left|\frac{\partial^2 f}{\partial u^2}(x,t,s)\right| \leq C_M$$

  if $|t|, |s| \leq M$ for a.e. $x \in \Omega$.

- $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory, of class C^2 in the second and third variables, $|L(x,0,0)| \in L^{p/2}(\Omega)$, and for all $M > 0$ there exist a constant $C_M > 0$ and functions $\psi_M \in L^{p/2}(\Omega)$ and $\psi_M^* \in L^{\max\{p/2,2\}}(\Omega)$ such that

  $$\left|\frac{\partial L}{\partial y}(x,y,u)\right| \leq \psi_M(x),$$

$$\left|\frac{\partial L}{\partial u}(x,y,u)\right| \leq \psi_M^*(x)$$

and

$$\left|\frac{\partial^2 L}{\partial y^2}(x,y,u)\right| + \left|\frac{\partial^2 L}{\partial y \partial u}(x,y,u)\right| + \left|\frac{\partial^2 L}{\partial u^2}(x,y,u)\right| \leq C_M$$

if $|y|, |u| \leq M$ for a.e. $x \in \Omega$,

- for all $1 \leq j \leq n_d + n_i$, $g_j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is measurable in x, of class C^2 in the variable η and there exist exponents $r \in [1, \infty)$ and $s > N$ a constant $C > 0$, a function $\psi_1 \in L^s(\Omega)$ such that

$$\left|\frac{\partial g_j}{\partial \eta}(x,\eta)\right| \leq C|\eta|^r + \psi_1(x)$$

and

$$\left|\frac{\partial^2 g_j}{\partial \eta^2}(x,\eta)\right| \leq C(1 + |\eta|^r).$$

Under this assumptions we can write the following necessary condition.

Theorem 6.2.5 *Suppose that $\bar{u}$ is a local solution of problem $(\mathbf{P_e})$ and that the assumptions on f, L and g_j established in* **E8** *hold. Suppose also that the regularity assumption* (6.2.1) *holds. Then*

$$\begin{aligned}\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})h^2 = &\int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\right) z_h^2\, dx + \\ &2\int_\Omega \left(\frac{\partial^2 L}{\partial y \partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y \partial u}(x,\bar{y},\bar{u})\right) h z_h\, dx + \\ &\int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(x,\bar{y},\bar{u})\right) h^2\, dx + \\ &\sum_{j=1}^{n_d+n_i} \bar{\lambda}_j \int_\Omega \nabla^T z_h \frac{\partial^2 g_j}{\partial \eta^2}(x,\nabla \bar{y}) \nabla z_h\, dx \geq 0\end{aligned} \tag{6.2.12}$$

for all $h \in C^0_{\bar{u},L^2(\Omega)}$.

Proof. Assumption **E8** implies that $\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})$ is bilinear and continuous in $L^2(\Omega)$. So we can apply Theorem 6.1.5 and deduce that the inequality (6.2.12) is true for all $h \in C^0_{\bar{u},L^2(\Omega)}$. □

Second order sufficient conditions

Clearly, we are going to apply here Theorem 6.1.6. Let us see that our problem satisfies the assumptions of this Theorem. The main difficulty appears when we prove that **(A3)** holds. To do that it is necessary to prove enough regularity for the adjoint state. We need that it is in $L^\infty(\Omega)$. To achieve this regularity we need to suppose more regularity for the derivatives of f, L and g_j. Again we are going to suppose that **(E8)** holds. Analogously to what we did in the abstract case, given $\bar{u}$ an admissible control, we introduce

$$\Omega^\tau = \{x \in \Omega : |d(x)| > \tau\}.$$

Theorem 6.2.6 *Let $\bar{u}$ be an admissible control for problem* $(\mathbf{P_e})$ *satisfying the regularity assumption* (6.2.1), **(E8)** *and such that there exist real numbers $\bar{\lambda}_j$, $j = 1, \ldots, n_d + n_i$ and function $\bar{y} \in W^{1,p}(\Omega)$, $\bar{\varphi} \in W^{1,p'}(\Omega)$ satisfying* (6.2.2), (6.2.3), (6.2.4) *and* (6.2.5). *Suppose also that*

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 = & \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x, \bar{y}, \bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x, \bar{y}, \bar{u}) \right) z_h^2 dx \\
& +2 \int_\Omega \left(\frac{\partial^2 L}{\partial y \partial u}(x, \bar{y}, \bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y \partial u}(x, \bar{y}, \bar{u}) \right) h z_h dx \\
& + \int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x, \bar{y}, \bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(x, \bar{y}, \bar{u}) \right) h^2 \, dx + \\
& \sum_{j=1}^{n_d+n_i} \bar{\lambda}_j \int_\Omega \nabla^T z_h \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla \bar{y}) \nabla z_h \, dx \geq \delta \|h\|^2_{L^2(\Omega)}
\end{aligned} \tag{6.2.13}$$

for all $h \in L^\infty(\Omega)$ satisfying (6.2.8) *and $h(x) = 0$ for a.e. $x \in \Omega^\tau$ and given $\delta > 0$ and $\tau > 0$. Then there exist $\varepsilon > 0$ and $\alpha > 0$ such that $J(\bar{u}) + \alpha\|u - \bar{u}\|^2_{L^2(\Omega)} \leq J(u)$ for all admissible control u with $\|u - \bar{u}\|_{L^\infty(\Omega)} < \varepsilon$.*

Proof. Notice first that the new conditions introduces on the first derivatives of f, L and g_j imply that the adjoint state belongs to $W^{1,p}(\Omega)$ for all $p > N$ and therefore the adjoint state belongs to $L^\infty(\Omega)$.

We are going to prove that **(A3)** holds. Let $\bar{u}$ an admissible control satisfying first order necessary conditions (6.2.2)–(6.2.5). Given $v \in L^\infty(\Omega)$, we will denote $\varphi_v = \varphi_{0v} + \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j \varphi_{jv}$, where φ_{0v} and φ_{jv} are the solutions of (4.1.3) and (4.1.7) for $u = v$, respectively. Take $h \in L^\infty(\Omega)$ and $\delta > 0$.

Let us verify the first inequality in (6.1.16). In fact, we will establish that

$$\left| \left[\frac{\partial^2 \mathcal{L}}{\partial u^2}(v, \bar{\lambda}) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda}) \right] h^2 \right| \leq$$

$$\int_\Omega \left| \frac{\partial^2 L}{\partial u^2}(x, y_v, v) + \varphi_v \frac{\partial^2 f}{\partial u^2}(x, y_v, v) - \frac{\partial^2 L}{\partial u^2}(x, \bar{y}, \bar{u}) - \bar{\varphi} \frac{\partial^2 f}{\partial u^2}(x, \bar{y}, \bar{u}) \right| h^2 \, dx +$$

$$\int_\Omega \left| \left(\frac{\partial^2 L}{\partial y \partial u}(x, y_v, v) + \varphi_v \frac{\partial^2 f}{\partial y \partial u}(x, y_v, v) \right) z_h - \left(\frac{\partial^2 L}{\partial y \partial u}(x, \bar{y}, \bar{u}) + \bar{\varphi} \frac{\partial^2 f}{\partial y \partial u}(x, \bar{y}, \bar{u}) \right) \bar{z}_h \right| |h|$$

$$+ \int_\Omega \left| \left(\frac{\partial^2 L}{\partial y^2}(x, y_v, v) + \varphi_v \frac{\partial^2 f}{\partial y^2}(x, y_v, v) \right) z_h^2 - \left(\frac{\partial^2 L}{\partial y^2}(x, \bar{y}, \bar{u}) + \bar{\varphi} \frac{\partial^2 f}{\partial y^2}(x, \bar{y}, \bar{u}) \right) \bar{z}_h^2 \right| \, dx +$$

$$\sum_{j=1}^{n_i+ni} |\bar{\lambda}_j| \int_\Omega \left| \nabla^T z_h \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla y_v) \nabla z_h - \nabla^T \bar{z}_h \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla \bar{y}) \nabla \bar{z}_h \right| \, dx \leq \delta \|h\|^2_{L^2(\Omega)} \quad (6.2.14)$$

supposing that $\|v - \bar{u}\|_{L^\infty(\Omega)} < \varepsilon$ with ε small enough, where

$$\left\{ \begin{array}{rcll} A\bar{z}_h + a_0 \bar{z}_h &=& \frac{\partial f}{\partial y}(x, \bar{y}, \bar{u}) \bar{z}_h + \frac{\partial f}{\partial u}(x, \bar{y}, \bar{u}) h & \text{en } \Omega \\ \partial_{n_A} \bar{z}_h &=& 0 & \text{on } \Gamma. \end{array} \right. \quad (6.2.15)$$

$$\left\{ \begin{array}{rcll} Az_h + a_0 z_h &=& \frac{\partial f}{\partial y}(x, y_v, v) z_h + \frac{\partial f}{\partial u}(x, y_v, v) h & \text{in } \Omega \\ \partial_{n_A} z_h &=& 0 & \text{on } \Gamma. \end{array} \right. \quad (6.2.16)$$

We can work with each term in a separate way. Let us remark the fact that the main tools to prove (6.2.14) are the continuity of the functional G, the C^2 regularity of f and g_j $j = 0, 1, \ldots, n_i + n_d$ and the assumptions on the regularity of the derivatives of f, L and g_j.

Given $\bar{\delta} > 0$, for the first term in the left of (6.2.14) we can establish that

$$\left\|\frac{\partial^2 L}{\partial u^2}(x,y_v,v)+\varphi_v\frac{\partial^2 f}{\partial u^2}(x,y_v,v)-\frac{\partial^2 L}{\partial u^2}(x,\bar y,\bar u)-\bar\varphi\frac{\partial^2 f}{\partial u^2}(x,\bar y,\bar u)\right\|_{L^\infty(\Omega)}<\tilde\delta$$

supposing that $\|v-\bar u\|_{L^\infty(\Omega)}$ is small enough: this is a direct consequence of the continuous dependence of φ_v with respect to v in the norm of $L^\infty(\Omega)$, that can be obtained from Proposition 2.1.3.

For the second term of (6.2.14), Hölder's inequality leads us to

$$\int_\Omega\left|\left(\frac{\partial^2 L}{\partial y\partial u}(x,y_v,v)+\varphi_v\frac{\partial^2 f}{\partial y\partial u}(x,y_v,v)\right)z_h-\left(\frac{\partial^2 L}{\partial y\partial u}(x,\bar y,\bar u)+\bar\varphi\frac{\partial^2 f}{\partial y\partial u}(x,\bar y,\bar u)\right)\bar z_h\right||h|$$

$$\begin{aligned}\le \|h\|_{L^2(\Omega)}\Bigg(&\left\|\frac{\partial^2 L}{\partial y\partial u}(x,y_v,v)-\frac{\partial^2 L}{\partial y\partial u}(x,\bar y,\bar u)\right\|_{L^\infty(\Omega)}\|z_h\|_{L^2(\Omega)}\\
+&\left\|\frac{\partial^2 L}{\partial y\partial u}(x,\bar y,\bar u)\right\|_{L^\infty(\Omega)}\|z_h-\bar z_h\|_{L^2(\Omega)}\\
+&\left\|\varphi_v\frac{\partial^2 f}{\partial y\partial u}(x,y_v,v)-\bar\varphi\frac{\partial^2 f}{\partial y\partial u}(x,\bar y,\bar u)\right\|_{L^\infty(\Omega)}\|z_h\|_{L^2(\Omega)}\\
+&\left\|\bar\varphi\frac{\partial^2 f}{\partial y\partial u}(x,\bar y,\bar u)\right\|_{L^\infty(\Omega)}\|z_h-\bar z_h\|_{L^2(\Omega)}\Bigg)\end{aligned}$$

The argument is completed taking into account the estimates

$$\|z_h\|_{L^2(\Omega)}+\|\bar z_h\|_{L^2(\Omega)}\le C_1\|h\|_{L^2(\Omega)}\quad\text{and}\tag{6.2.17}$$

$$\|z_h-\bar z_h\|_{L^2(\Omega)}\le\tilde\delta\|h\|_{L^2(\Omega)},\tag{6.2.18}$$

when $\|v-\bar u\|_{L^\infty(\Omega)}$ is small.

Following the same sketch we have

$$\int_\Omega\left|\left(\frac{\partial^2 L}{\partial y^2}(x,y_v,v)+\varphi_v\frac{\partial^2 f}{\partial y^2}(x,y_v,v)\right)z_h^2-\left(\frac{\partial^2 L}{\partial y^2}(x,\bar y,\bar u)+\bar\varphi\frac{\partial^2 f}{\partial y^2}(x,\bar y,\bar u)\right)\bar z_h^2\right|\,dx\le$$

$$\le\left\|\frac{\partial^2 L}{\partial y^2}(x,y_v,v)-\frac{\partial^2 L}{\partial y^2}(x,\bar y,\bar u)\right\|_{L^\infty(\Omega)}\|z_h\|^2_{L^2(\Omega)}$$

$$+\left\|\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u})\right\|_{L^\infty(\Omega)} \|z_h-\bar{z}_h\|_{L^2(\Omega)}\|z_h+\bar{z}_h\|_{L^2(\Omega)}$$

$$+\left\|\varphi_v\frac{\partial^2 f}{\partial y^2}(x,y_v,v)-\bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\right\|_{L^\infty(\Omega)} \|z_h\|^2_{L^2(\Omega)}$$

$$+\left\|\bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y},\bar{u})\right\|_{L^\infty(\Omega)} \|z_h-\bar{z}_h\|_{L^2(\Omega)}\|z_h+\bar{z}_h\|_{L^2(\Omega)},$$

which, together with (6.2.17)-(6.2.18) allows us to deal with the third term of (6.2.14).

Let us study the last term decomposing it as follows and using again Hölder's inequality.

$$\int_\Omega \left|\nabla^T z_h\frac{\partial^2 g_j}{\partial\eta^2}(x,\nabla y_v)\nabla z_h-\nabla^T\bar{z}_h\frac{\partial^2 g_j}{\partial\eta^2}(x,\nabla\bar{y})\nabla\bar{z}_h\right| dx \leq$$

$$\leq \int_\Omega\left|\nabla^T z_h\left(\frac{\partial^2 g_j}{\partial\eta^2}(x,\nabla y_v)-\frac{\partial^2 g_j}{\partial\eta^2}(x,\nabla\bar{y})\right)\nabla z_h\right| dx$$

$$+\int_\Omega\left|(\nabla^T z_h-\nabla^T\bar{z}_h)\frac{\partial^2 g_j}{\partial\eta^2}(\nabla\bar{y})(\nabla z_h+\nabla\bar{z}_h)\right| dx \leq$$

$$\leq \|\nabla z_h\|^2_{L^p(\Omega)^N}\left\|\frac{\partial^2 g_j}{\partial\eta^2}(x,\nabla y_v)-\frac{\partial^2 g_j}{\partial\eta^2}(x,\nabla\bar{y})\right\|_{L^q(\Omega)^{N^2}}$$

$$+\|\nabla z_h-\nabla\bar{z}_h\|_{L^p(\Omega)^N}\|\nabla z_h+\nabla\bar{z}_h\|_{L^p(\Omega)^N}\left\|\frac{\partial^2 g_j}{\partial\eta^2}(x,\nabla\bar{y})\right\|_{L^q(\Omega)^{N^2}}$$

with $p=2N/(N-2)$ (if $N>2$), $p=3$ (if $N=1$ or 2) and $q=pp'/(p-p')$ (q is in this case the conjugate exponent of $p/2$).

Exponent p has been chosen in such a way that $L^2(\Omega)\subset(W^{1,p'}(\Omega))'$. Thus, using Proposition 2.1.3, we have that

$$\|\nabla z_h\|_{L^p(\Omega)}+\|\nabla\bar{z}_h\|_{L^p(\Omega)}\leq C_2\|h\|_{L^2(\Omega)}. \tag{6.2.19}$$

when $\|v-\bar{u}\|_{L^\infty(\Omega)}$ is bounded. Moreover, in this case subtracting the equations (6.2.15) and (6.2.16) and using Theorem 2.1.3 again, we can deduce that

$$\|\nabla z_h-\nabla\bar{z}_h\|_{L^p(\Omega)}\leq\tilde{\delta}\|h\|_{L^2(\Omega)}.$$

Finally, we can deduce that

$$\left\| \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla \bar{y}) \right\|_{L^q(\Omega)^{N^2}} < \tilde{\delta}$$

for $\|v - \bar{u}\|_{L^\infty(\Omega)}$ small enough, uniformly with respect to v. Let us show this in detail: due to the continuity of the functional G and using the regularity $L^p(\Omega)$ of the gradient of the state and the assumption made on the second derivatives of g_j, fixed $\bar{q} > q$, there exists a positive constant C_3 such that for any admissible control v

$$\|\nabla y_v\|_{L^{r\bar{q}}(\Omega)} + \|\nabla \bar{y}\|_{L^{r\bar{q}}(\Omega)} + \left\| \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla y_v) \right\|_{L^{\bar{q}}(\Omega)^{N^2}} + \left\| \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla \bar{y}) \right\|_{L^{\bar{q}}(\Omega)^{N^2}} \leq C_3,$$

being r the exponent introduced in the assumptions of the theorem. Given $M > 0$, let us introduce the following sets $E_1^M = \{x \in \Omega : \|\nabla y_v(x)\| \geq M\}$ and $E_2^M = \{x \in \Omega : \|\nabla \bar{y}(x)\| \geq M\}$. Clearly E_1^M and E_2^M depend on v and $\bar{u}$, respectively, but we will not remark this. here it is important to remark the trivial inequality

$$m(E_1^M) \leq \frac{1}{M} \int_\Omega \|\nabla y_v(x)\| dx \leq \frac{C_4}{M}.$$

The same reasoning is valid for E_2^M.

Due to the regularity of g_j, the second order derivatives are uniformly continuous in the ball of $\mathbb{R}^N$ centered in the origin and with radius M. Hence, there exists $\epsilon_1 > 0$ such that for $\|\eta - \bar{\eta}\|_{\mathbb{R}^N} \leq \epsilon_1$ with $\|\eta\|_{\mathbb{R}^N}, \|\bar{\eta}\|_{\mathbb{R}^N} \leq M$, we have that

$$\left\| \frac{\partial^2 g_j}{\partial \eta^2}(x, \eta) - \frac{\partial^2 g_j}{\partial \eta^2}(x, \bar{\eta}) \right\|_{\mathbb{R}^{N^2}} < \left(\frac{\tilde{\delta}}{4m(\Omega)} \right)^{1/q}$$

Using again the continuity of the functional G, there exists $\epsilon_2 > 0$ such that when $\|v - \bar{u}\|_{L^\infty(\Omega)} \leq \epsilon_2$, then

$$\int_\Omega \|\nabla y_v(x) - \nabla \bar{y}(x)\| dx \leq \epsilon_1 \frac{C_4}{M}.$$

Let us introduce now another set $E_3^M = \{x \in \Omega : \|\nabla y_v(x) - \nabla \bar{y}(x)\| > \epsilon_1\}$. Arguing as before, we may deduce that

$$\epsilon_1 m(E_3^M) \leq \int_\Omega \|\nabla y_v(x) - \nabla \bar{y}(x)\| dx.$$

Particularly, the last two relations imply that $m(E_3^M) \leq \frac{C_4}{M}$. Combining the previous estimates and using Hölder's inequality with $s = \bar{q}/q$, we obtain that

$$\int_\Omega \left\| \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla \bar{y}) \right\|^q dx \leq \int_{E_1^M} \left\| \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla \bar{y}) \right\|^q dx +$$

$$\int_{E_2^M} \left\| \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla \bar{y}) \right\|^q dx + \int_{E_3^M} \left\| \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla \bar{y}) \right\|^q dx +$$

$$\int_{\Omega \setminus (E_1^M \cup E_2^M \cup E_M^3)} \left\| \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla \bar{y}) \right\|^q dx \leq$$

$$\frac{\tilde{\delta}}{4} + \left(\sum_{j=1}^{3} m(E_j^M)^{1/s'} \right) \left(\int_\Omega \left\| \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(x, \nabla \bar{y}) \right\|^{\bar{q}} dx \right)^{1/s}$$

$$\leq \frac{\tilde{\delta}}{4} + 3 \left(\frac{C_4}{M} \right)^{1/s'} 2^{q+1/s} C_3^q$$

This term on the right can be taken less that $\tilde{\delta}$, if M is large enough.

For all these considerations, we can assure that the first condition on the continuity of the second derivative of the Lagrangian in (6.1.16) holds. The rest of the conditions follows easily from the properties of the functions f, L and g_j, $j = 0, 1, \ldots, n_i + n_d$. □

Some extensions

Analogous results can be proved for the boundary control problem $(\mathrm{P_s})'$ described in page 107. Let us take now

$$K_\Gamma(x) = [v_a(x), v_b(x)],$$

where $v_a, v_b \in L^\infty(\Gamma)$. The Lagrangian associated to this problem is

$$\mathcal{L}(v, \lambda) = \int_\Gamma \ell(x, y_v, v) dx + \sum_{j=1}^{n_i+n_d} \lambda_j \int_\Omega g_j(x, \nabla y_v(x)) dx.$$

Remember that

$$F_j(y) = \int_\Omega g_j(x, \nabla y(x)) dx.$$

We establish now a regularity assumption analogous to (6.2.1). Given $\bar{v} \in V_{ad}$, for $\varepsilon > 0$, set

$$\Gamma_\varepsilon = \{x \in \Gamma : v_a(x) + \varepsilon \leq \bar{v}(x) \leq v_b(x) - \varepsilon\}.$$

Given a control $\bar{v}$, we will say that it satisfies the regularity condition if there exists $\varepsilon_{\bar{v}} > 0$ such that

$$\text{the family } \{\bar{\varphi}_i \frac{\partial g}{\partial v}(s, \bar{y}, \bar{v})\}_{i \in I_0} \text{ is linearly independent in } L^1(\Gamma_{\varepsilon_{\bar{v}}}), \tag{6.2.20}$$

where $\bar{y}$ is the associated state to $\bar{v}$ and φ_i is the unique solution of

$$\begin{cases} A^*\bar{\varphi} + a_0\bar{\varphi} &= -\mathrm{div}\left(\frac{\partial g_j}{\partial \eta}(x, \nabla\bar{y})\right) \quad \text{in } \Omega \\ \partial_{n_{A^*}}\bar{\varphi} &= \frac{\partial g}{\partial y}(s, \bar{y}, \bar{v})\bar{\varphi} \quad \text{on } \Gamma. \end{cases}$$

Suppose that

- $g : \Gamma \times \mathbb{R} \times \mathbb{R}$ is measurable on Γ and of class C^1 with respect to the second and third variables, $g(\cdot, 0, 0) \in L^{p-1}(\Gamma)$, for all $M > 0$ there exist $C_M > 0$ and $\psi_M \in L^{p-1}(\Gamma)$ such that

$$\left|\frac{\partial g}{\partial y}(x, y, v)\right| \leq C_M \quad \text{and} \quad \left|\frac{\partial g}{\partial v}(x, y, v)\right| \leq \psi_M(x)$$

  for all $(y, v) \in \mathbb{R}^2$ and a.e. $x \in \Gamma$ and

$$\frac{\partial g}{\partial y}(x, y, v) \leq 0.$$

- $\ell : \Gamma \times \mathbb{R} \times \mathbb{R}$ is measurable on Γ and of class C^1 with respect to the second and third variables for all $M > 0$ there exists $\psi_M \in L^1(\Gamma)$ such that

$$\left|\frac{\partial \ell}{\partial y}(x, y, v)\right| + \left|\frac{\partial \ell}{\partial v}(x, y, v)\right| \leq \psi_M(x)$$

and the differentiability conditions on the g_j established in E6 (page 89) hold.

Theorem 6.2.7 *Suppose that $\bar{v}$ is a local solution of $(\mathrm{P}_e)'$. Suppose also that (6.2.20) holds. Then there exist real numbers $\bar{\lambda}_j$, $j = 1, \ldots, n_d + n_i$ and functions $\bar{y} \in W^{1,p}(\Omega)$, $\bar{\varphi} \in W^{1,p'}(\Omega)$ such that*

$$\bar{\lambda}_j \geq 0 \quad n_i + 1 \leq j \leq n_i + n_d, \qquad \bar{\lambda}_j \int_\Omega g_j(x, \nabla\bar{y}(x))\, dx = 0, \tag{6.2.21}$$

$$\begin{cases} A\bar{y} + a_0\bar{y} &= f \quad \text{in } \Omega \\ \partial_{n_A}\bar{y} &= g(s, y_v, v) \quad \text{on } \Gamma, \end{cases} \tag{6.2.22}$$

$$\left\{\begin{array}{rcll} A^*\bar\varphi + a_0\bar\varphi &=& -\sum_{j=1}^{n_i+n_d}\bar\lambda_j \mathrm{div}\left(\frac{\partial g_j}{\partial\eta}(x,\nabla\bar y)\right) & \text{in } \Omega \\ \partial_{n_{A^*}}\bar\varphi &=& \frac{\partial g}{\partial y}(s,\bar y,\bar v)\bar\varphi + \bar\nu\frac{\partial \ell}{\partial y}(s,\bar y,\bar v) & \text{on } \Gamma, \end{array}\right. \tag{6.2.23}$$

and

$$\frac{\partial \mathcal{L}}{\partial v}(\bar v,\bar\lambda)(v-\bar v) = \int_\Gamma\left(\frac{\partial \ell}{\partial v}(s,\bar y,\bar u) + \bar\varphi\frac{\partial f}{\partial v}(s,\bar y,\bar s)\right)(v-\bar v)ds \geq 0 \quad \text{for all } v\in V_{ad}. \tag{6.2.24}$$

Set

$$d(s) = \frac{\partial \ell}{\partial v}(s,\bar y,\bar u) + \bar\varphi\frac{\partial f}{\partial v}(s,\bar y,\bar s)$$

and

$$\Gamma^0 = \{s\in\Gamma : |d(s)| > 0\}.$$

The second derivative of the Lagrangian is given in this case by

$$\frac{\partial^2\mathcal{L}}{\partial v^2}(\bar v,\bar\lambda)h^2 = \int_\Gamma\left(\frac{\partial^2\ell}{\partial y^2}(s,\bar y,\bar v) + \bar\varphi\frac{\partial^2 g}{\partial y^2}(s,\bar y,\bar v)\right)z_h^2\,ds +$$

$$2\int_\Gamma\left(\frac{\partial^2\ell}{\partial y\partial v}(s,\bar y,\bar v) + \bar\varphi\frac{\partial^2 g}{\partial y\partial v}(s,\bar y,\bar v)\right)hz_h\,ds +$$

$$\int_\Gamma\left(\frac{\partial^2\ell}{\partial v^2}(s,\bar y,\bar v) + \bar\varphi\frac{\partial^2 g}{\partial v^2}(s,\bar y,\bar v)\right)h^2\,ds +$$

$$\sum_{j=1}^{n_d+n_i}\bar\lambda_j\int_\Omega \nabla^T z_h\frac{\partial^2 g_j}{\partial\eta^2}(x,\nabla\bar y)\nabla z_h\,dx,$$

where $h\in L^\infty(\Gamma)$ and z_h is the solution of

$$\left\{\begin{array}{rcll} Az_h + a_0 z_h &=& 0 & \text{in } \Omega \\ \partial_{n_A} z_h &=& \frac{\partial g}{\partial y}(s,\bar y,\bar v)z_h + \frac{\partial g}{\partial v}(s,\bar y,\bar v)h & \text{on } \Gamma. \end{array}\right.$$

Suppose that the C^1 differentiability conditions on g and ℓ previously established and on g_j established in E6 hold. Also suppose that condition E7 about the second derivatives of g_j holds and that g and ℓ are of class C^2 with respect to the second and third variables and that for all $M > 0$ there exist $\varepsilon, \tilde{\varepsilon} > 0$ and functions $\psi_M^1 \in L^1(\Gamma)$, $\psi_M^{1,\varepsilon}(\Gamma) \in L^{1+\varepsilon}(\Gamma)$, $\psi_M^2 \in L^{p-1}(\Gamma)$ and $\psi_M^{2,\tilde{\varepsilon}}(\Gamma) \in L^{p-1+\tilde{\varepsilon}}(\Gamma)$ such that

$$\left|\frac{\partial^2 \ell}{\partial y^2}(s,y,v)\right| \leq \psi_M^1(s), \qquad \left|\frac{\partial^2 \ell}{\partial y \partial v}(s,y,v)\right| + \left|\frac{\partial^2 \ell}{\partial v^2}(s,y,v)\right| \leq \psi_M^{1,\varepsilon}(s),$$

$$\left|\frac{\partial^2 g}{\partial y^2}(s,y,v)\right| \leq \psi_M^2(s) \quad \text{and} \quad \left|\frac{\partial^2 g}{\partial y \partial v}(s,y,v)\right| + \left|\frac{\partial^2 g}{\partial v^2}(s,y,v)\right| \leq \psi_M^{2,\tilde{\varepsilon}}(s)$$

if $|y|$, $|v| \leq M$ for a.e. $s \in \Gamma$. Then we can state second order necessary conditions.

Theorem 6.2.8 *Suppose that $\bar{v}$ is a local solution of $(\mathrm{P_e})'$. Suppose also that* (6.2.20) *holds. Then*

$$\frac{\partial^2 \mathcal{L}}{\partial v^2}(\bar{v}, \bar{\lambda})h^2 \geq 0$$

for all $h \in L^\infty(\Gamma)$ such that $h(s) = 0$ for a.e. $s \in \Gamma^0$ and

$$\begin{cases} \displaystyle\int_\Gamma \bar{\varphi}_j \frac{\partial g}{\partial u}(s,\bar{y},\bar{v})h\,ds = 0 \text{ if } (j \leq n_i) \text{ or } (j > n_i, \int_\Omega g_j(x,\nabla\bar{y}) = 0 \text{ and } \bar{\lambda}_j > 0) \\ \displaystyle\int_\Gamma \bar{\varphi}_j \frac{\partial f}{\partial v}(s,\bar{y},\bar{v})h\,ds \leq 0 \text{ if } n_i + 1 \leq j \leq n_d + n_i, \int_\Omega g_j(x,\nabla\bar{y}) = 0,\ \bar{\lambda}_j = 0 \\ h(s) \geq 0 \text{ if } \bar{v}(s) = v_a(s) \\ h(s) \leq 0 \text{ if } \bar{v}(s) = v_b(s). \end{cases} \tag{6.2.25}$$

To establish sufficient conditions we have to introduce

$$\Gamma^\tau = \{s \in \Gamma : |d(s)| > \tau\}.$$

Again the assumptions made on the functions that intervene in the problem are stronger, in order to make the trace of the adjoint a bounded function.

- g is of class C^2 with respect to the second and third variables,

$$\frac{\partial g}{\partial y}(s,y,v) \leq 0$$

and for all $M > 0$ there exists a constant $C_M > 0$ such that

$$\left|\frac{\partial g}{\partial y}(s,y,v)\right| + \left|\frac{\partial g}{\partial v}(s,y,v)\right| + \left|\frac{\partial^2 g}{\partial y^2}(s,y,v)\right| + \left|\frac{\partial^2 g}{\partial y \partial v}(s,y,v)\right| + \left|\frac{\partial^2 g}{\partial v^2}(s,y,v)\right| \leq C_M$$

if $|y|, |v| \leq M$ for a.e. $s \in \Gamma$.

- $\ell : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory, of class C^2 in the second and third variables, $|\ell(s,0,0)| \in L^{p-1}(\Gamma)$ and for all $M > 0$ there exist a constant $C_M > 0$ and a function $\psi_M \in L^{p-1}(\Gamma)$ such that

$$\left|\frac{\partial \ell}{\partial y}(s,y,v)\right| + \left|\frac{\partial \ell}{\partial v}(s,y,v)\right| \leq \psi_M(s)$$

and

$$\left|\frac{\partial^2 \ell}{\partial y^2}(s,y,v)\right| + \left|\frac{\partial^2 \ell}{\partial y \partial v}(s,y,v)\right| + \left|\frac{\partial^2 \ell}{\partial v^2}(s,y,v)\right| \leq C_M$$

if $|y|, |v| \leq M$ for a.e. $s \in \Gamma$,

- for all $1 \leq j \leq n_d + n_i$, $g_j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is measurable in x, of class C^2 in the variable η and there exist exponents $r \in [1, \infty)$ and $s > N$, a constant $C > 0$, a function $\psi_1 \in L^s(\Omega)$ such that

$$\left|\frac{\partial g_j}{\partial \eta}(x,\eta)\right| \leq C|\eta|^r + \psi_1(x)$$

and

$$\left|\frac{\partial^2 g_j}{\partial \eta^2}(x,\eta)\right| \leq C(1 + |\eta|^r).$$

Then

Theorem 6.2.9 *Let $\bar{v}$ be an admissible control for problem $(\mathrm{P}_e)'$ that satisfies the regularity assumption (6.2.20) and such that there exist real numbers $\bar{\lambda}_j$, $j = 1, \ldots, n_d + n_i$ and functions $\bar{y} \in W^{1,p}(\Omega)$, $\bar{\varphi} \in W^{1,p'}(\Omega)$ satisfying (6.2.21), (6.2.22), (6.2.23) and (6.2.24). Suppose also that*

$$\frac{\partial^2 \mathcal{L}}{\partial v^2}(\bar{v}, \bar{\lambda})h^2 \geq \delta \|h\|^2_{L^2(\Omega)}$$

for all $h \in L^\infty(\Gamma)$ satisfying (6.2.25) and $h(s) = 0$ for a.e. $s \in \Gamma^\tau$ and given $\delta > 0$ and $\tau > 0$. Then there exist $\varepsilon > 0$ and $\alpha > 0$ such that $J(\bar{v}) + \alpha\|v - \bar{v}\|^2_{L^2(\Gamma)} \leq J(v)$ for all admissible control v with $\|v - \bar{v}\|_{L^\infty(\Gamma)} < \varepsilon$.

6.3 Parabolic case

Set Ω, Γ, T, Q, Σ and A, p, τ, k_1, $\tilde{k}_1$, σ_1, $\bar{\sigma}_1$ as in Section 2.2, with the boundary Γ of class C^1 and the coefficients of the operator A of class $C([0,T];C(\bar{\Omega}))$. Set f, g, y_0 functions, $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \Sigma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $F : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $G : \Sigma \times \mathbb{R} \longrightarrow \mathbb{R}$ and $y_0 : \Omega \longrightarrow \mathbb{R}$, $y_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. Take k_2, $\tilde{k}_2$, σ_2, $\bar{\sigma}_2$ and ν as in Section 2.2.

Consider the problem $(\mathbf{P_p})$ of page 16. Suppose that the set of admissible controls is of the form

$$V_{ad} = \{v \in L^\infty(\Sigma) : \ v_a(s,t) \le v(s,t) \le v_b(s,t) \text{ a.e. } (s,t) \in \Sigma\},$$

where v_a, $v_b \in L^\infty(\Sigma)$. This election corresponds to the case of taking

$$K_\Sigma(s,t) = [v_a(s,t), v_b(s,t)].$$

Just like in Section 4.1.2, we will consider

$$C = \left\{ \vec{f} \in L^\tau(L^p)^N : \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\vec{f})dx \right) dt = 0 \text{ if } 1 \le j \le n_i, \right.$$

$$\left. \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\vec{f})dx \right) dt \le 0 \text{ if } n_i + 1 \le j \le n_i + n_d \right\},$$

where $\zeta_j : \mathbb{R} \to \mathbb{R}$ and $g_j : Q \times \mathbb{R}^N \to \mathbb{R}$ are functions. We are going to adapt for problem $(\mathbf{P_p})$ the abstract Theorems given in the beginning of the chapter. In this case

$$J(v) = \int_0^T \int_\Omega F(x,t,y_v)\, dx\, dt + \int_0^T \int_\Gamma G(s,t,y_v,v)\, ds\, dt + \int_\Omega L(x, y_v(x,T))\, dx$$

and

$$G_j(v) = \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\nabla_x y_v)dx \right) dt.$$

The Lagrangian of this problem is given by

$$\mathcal{L}(v,\lambda) = \int_0^T \int_\Omega F(x,t,y_v)\, dx\, dt + \int_0^T \int_\Gamma G(s,t,y_v,v)\, ds\, dt + \int_\Omega L(x, y_v(x,T))\, dx +$$

$$\sum_{j=1}^{n_i+n_d} \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\nabla_x y_v)dx \right) dt.$$

Remember also that

$$F_j(y) = \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\nabla_x y)dx \right) dt,$$

and that its derivative is given by

$$F_j'(y) = -\operatorname{div} \zeta_j' \left(\int_\Omega g_j(x,t,\nabla_x y)dx \right) \frac{\partial g_j}{\partial \eta}(s,t,\nabla_x y).$$

We are going to establish a regularity assumption analogous to (6.1.3). For $\varepsilon > 0$, set

$$\Sigma_\varepsilon = \{(s,t) \in \Sigma : v_a(s,t) + \varepsilon \le \bar{v}(s,t) \le v_b(s,t) - \varepsilon\}$$

Lemma 6.3.1 *Given $\bar{v}$ an element of V_{ad}, the followong two conditions are equivalent:*

1. *there exists $\varepsilon_{\bar{v}} > 0$ and functions $\{h_j\}_{j\in I_0} \subset L^\infty(\Omega)$ with $\operatorname{supp} h_j \subset \Sigma_{\varepsilon_{\bar{v}}}$ such that $G_i'(\bar{v})h_j = \delta_{ij}$ for $i,j \in I_0$;*

2. *there exists $\varepsilon_{\bar{v}} > 0$ such that*

$$\text{the family } \{\bar{\varphi}_i \frac{\partial g}{\partial v}(s,t,\bar{y},\bar{v})\}_{i\in I_0} \text{ is linearly independente in } L^1(\Sigma_{\varepsilon_{\bar{v}}}), \tag{6.3.1}$$

where $\bar{y} = G(\bar{u})$ and $\bar{\varphi}_i = \varphi_{i\bar{v}}$ is the solution of (4.1.10) for $v = \bar{v}$.

Proof. The proof is completely anlogous to that of Lemma 6.2.1. □

First order necessary conditions

First order necessary conditions satisfied by $\bar{v}$ can be deduced from the abstract Theorem 6.1.2 with the aid of Theorems 4.1.5 and 4.1.7.

Theorem 6.3.2 *Suppose that f and g satisfy assumptions P1 and P2, that F, G and L satisfy P4 and P5 and that the ζ_j and the g_j satisfy P7. Suppose also that (6.3.1) holds. Then there exist real numbers $\bar{\lambda}_j$, $j = 1,\dots,n_d+n_i$ and functions $\bar{y} \in L^\tau(W^{1,p}(\Omega))$ and $\bar{\varphi} \in L^{\tau'}(W^{1,p'}(\Omega)) + L^2(H^1)$ such that*

$$\bar{\lambda}_j \ge 0 \quad n_i + 1 \le j \le n_i + n_d, \qquad \bar{\lambda}_j \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\nabla_x \bar{y})dx \right) dt = 0, \tag{6.3.2}$$

$$\left\{ \begin{array}{rcll} \dfrac{\partial \bar{y}}{\partial t} + A\bar{y} &=& f(x,t,\bar{y}) & \text{in } Q, \\ \dfrac{\partial \bar{y}}{\partial n_A} &=& g(s,t,\bar{y},\bar{v}) & \text{on } \Sigma, \\ \bar{y}(\cdot,0) &=& w & \text{in } \Omega, \end{array} \right. \tag{6.3.3}$$

$$\begin{cases} -\dfrac{\partial\bar{\varphi}}{\partial t} + A^*\bar{\varphi} - \dfrac{\partial f}{\partial y}(x,t,\bar{y})\bar{\varphi} = -\displaystyle\sum_{j=1}^{n_d+n_i} \lambda_j \mathrm{div}\, \zeta_j'\left(\int_\Omega g_j(x,t,\nabla_x\bar{y})dx\right)\dfrac{\partial g_j}{\partial\eta}(s,t,\nabla_x\bar{y}) + \dfrac{\partial F}{\partial y}(x,t,\bar{y}) & \text{in } Q, \\ \dfrac{\partial\bar{\varphi}}{\partial n_{A^*}} - \dfrac{\partial g}{\partial y}(s,t,\bar{y},\bar{v})\bar{\varphi} = \displaystyle\sum_{j=1}^{n_d+n_i} \lambda_j\zeta_j'\left(\int_\Omega g_j(x,t,\nabla_x\bar{y})dx\right)\dfrac{\partial g_j}{\partial\eta}(s,t,\nabla_x\bar{y})\cdot\vec{n} + \dfrac{\partial G}{\partial y}(s,t,\bar{y},\bar{v}) & \text{on } \Sigma, \\ \bar{\varphi}(\cdot,T) = \dfrac{\partial L}{\partial y}(x,\bar{y}(T)) & \text{in } \Omega, \end{cases} \tag{6.3.4}$$

$$\frac{\partial\mathcal{L}}{\partial v}(\bar{v},\bar{\lambda})(v-\bar{v}) = \int_\Sigma \left(\frac{\partial G}{\partial v}(s,t,\bar{y},\bar{v}) + \bar{\varphi}\frac{\partial g}{\partial v}(s,t,\bar{y},\bar{v})\right)(v-\bar{v})ds\,dt \geq 0 \;\forall\; v_a \leq v \leq v_b. \tag{6.3.5}$$

Moreover,

$$\bar{\varphi} = \varphi_{0\bar{v}} + \sum_{j=1}^{n_i+n_d} \lambda_j\varphi_{j\bar{v}},$$

where $\varphi_{0\bar{v}}$ *and* $\varphi_{j\bar{v}}$ *for* $1 \leq j \leq n_i + n_d$ *are the solutions of* (4.1.9) *and* (4.1.10) *for* $v = \bar{v}$.

Proof. We apply Theorems 4.1.5 and 4.1.7 to calculate the expression of the derivative of the Lagrangian, and deduce expression (6.3.5) as a direct application of Theorem 6.1.2 and Lemma 6.3.1. □

Again we can give a sufficient condition to check the regularity condition (6.3.1).

Lemma 6.3.3 *Suppose that there exist* $\varepsilon_{\bar{v}} > 0$ *and an open nonempty set (relative to the topology of* Σ*)* $A_{\varepsilon_{\bar{v}}} \subset \Sigma_{\varepsilon_{\bar{v}}}$ *such that*

$$\frac{\partial g}{\partial u}(s,t\bar{y}(s,t),\bar{v}(s,t)) \neq 0 \text{ in } A_{\varepsilon_{\bar{v}}}$$

and $\{F_j'(\bar{y})\}_{j\in I_0}$ *are linearly independent in* $L^{\tau'}\left((W^{1,\bar{p}'}(A_{\varepsilon_{\bar{v}}}))'\right)$. *Then condition* (6.3.1) *holds.*

Proof. The proof is analogous to that of the elliptic case. □

Second order necessary conditions

Taking into account Theorems 4.1.5 and 4.1.7, we can prove that the assumptions of Theorem 6.1.3 are satisfied by problem $(\mathbf{P_p})$. In this case we can identify

$$d(s,t) = \frac{\partial G}{\partial v}(s,t,\bar{y}(s,t),\bar{v}(s,t)) + \bar{\varphi}(s,t)\frac{\partial g}{\partial v}(s,t,\bar{y}(s,t),\bar{v}(s,t)),$$

where $\bar{y}$ is given by (6.3.3) and $\bar{\varphi}$ is given by (6.3.4). Let us introduce

$$\Sigma^0 = \{(s,t) \in \Sigma : |d(s,t)| > 0\}.$$

Again it is necessary some more regularity for the second derivatives of g and G. So, besides P3 and P6, we will suppose that there exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\bar{\varepsilon}_2 > 0$, $\psi_M^1 \in L^{1+\varepsilon_1}(\Sigma)$ and $\psi_M^2 \in L^{\bar{\sigma}_2+\bar{\varepsilon}_2}(L^{\sigma_2+\varepsilon_2}(\Gamma))$, such that

$$\left|\frac{\partial^2 G}{\partial v^2}(s,t,y,v)\right| + \left|\frac{\partial^2 G}{\partial v \partial y}(s,t,y,v)\right| \le \psi_M^1(s,t) \tag{6.3.6}$$

and

$$\left|\frac{\partial^2 g}{\partial v^2}(s,t,y,v)\right| + \left|\frac{\partial^2 g}{\partial v \partial y}(s,t,y,v)\right| \le \psi_M^2(s,t) \tag{6.3.7}$$

if $|y|$, $|v| \le M$ for a.e. $(s,t) \in \Sigma$.

So we obtain

Theorem 6.3.4 *Suppose that $\bar{v}$ is a local solution for problem $(\mathbf{P_p})$ and that P1–P8, (6.3.6) and (6.3.7) hold. Suppose also that the regularity assumption (6.3.1) holds. Then*

$$\frac{\partial^2 \mathcal{L}}{\partial v^2}(\bar{v},\bar{\lambda})h^2 = \int_\Sigma \left(\frac{\partial^2 G}{\partial y^2}(s,t,\bar{y},\bar{v}) + \bar{\varphi}\frac{\partial^2 g}{\partial y^2}(s,t,\bar{y},\bar{v})\right) z_h^2 \, ds\, dt +$$

$$2\int_\Sigma \left(\frac{\partial^2 G}{\partial y \partial v}(s,t,\bar{y},\bar{v}) + \bar{\varphi}\frac{\partial^2 g}{\partial y \partial v}(s,t,\bar{y},\bar{v})\right) h z_h \, ds\, dt +$$

$$\int_\Sigma \left(\frac{\partial^2 G}{\partial v^2}(s,t,\bar{y},\bar{v}) + \bar{\varphi}\frac{\partial^2 g}{\partial v^2}(s,t,\bar{y},\bar{v})\right) h^2 \, ds\, dt +$$

$$\sum_{j=1}^{n_d+n_i} \bar{\lambda}_j \left\{\int_0^T \left[\zeta_j''\left(\int_\Omega g_j(x,t,\nabla_x\bar{y})dx\right) \int_\Omega \frac{\partial g_j}{\partial \eta}(x,t,\nabla_x\bar{y})\nabla_x z_h \, dx \int_\Omega \frac{\partial g_j}{\partial \eta}(x,t,\nabla_x\bar{y})\nabla_x z_h \, dx\right] dt +\right.$$

$$\left.\int_0^T \left[\zeta_j'\left(\int_\Omega g_j(x,t,\nabla_x\bar{y})dx\right) \int_\Omega \nabla_x^T z_h \frac{\partial^2 g_j}{\partial \eta^2}(x,t,\nabla_x\bar{y})\nabla_x z_h dx\right] dt\right\} \ge 0 \tag{6.3.8}$$

for all $h \in L^\infty(\Sigma)$ *such that* $h(s,t) = 0$ *for a.e.* $(s,t) \in \Sigma^0$ *and*

$$\left\{\begin{array}{l} \displaystyle\int_\Sigma \bar\varphi_j \frac{\partial g}{\partial v}(s,t,\bar y,\bar v) h\, ds\, dt = 0 \text{ if } (j \le n_i) \text{ or } \left(j > n_i, \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\bar f) dx\right) dt = 0 \text{ and } \bar\lambda_j > 0\right) \\ \displaystyle\int_\Sigma \bar\varphi_j \frac{\partial g}{\partial v}(s,t,\bar y,\bar v) h\, ds\, dt \le 0 \text{ if } n_i + 1 \le j \le n_d + n_i,\ \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\bar f) dx\right) dt = 0,\ \bar\lambda_j = 0 \\ h(s,t) \ge 0 \text{ if } \bar v(s,t) = v_a(s,t) \\ h(s,t) \le 0 \text{ if } \bar v(s,t) = v_b(s,t), \end{array}\right. \tag{6.3.9}$$

where z_h *is given by*

$$\left\{\begin{array}{rcll} \dfrac{\partial z_h}{\partial t} + Az_h &=& \dfrac{\partial f}{\partial y}(x,t,\bar y) z_h & \text{in } Q, \\ \dfrac{\partial z_h}{\partial n_A} &=& \dfrac{\partial g}{\partial y}(s,t,\bar y,\bar v) z_h + \dfrac{\partial g}{\partial v}(s,t,\bar y,\bar v) h & \text{on } \Sigma, \\ z_h(\cdot,0) &=& 0 & \text{in } \Omega. \end{array}\right.$$

Proof. Notice first that we can apply Theorem 6.3.2 to deduce the existence of the Lagrange multipliers. Now, due to Theorem 6.1.3, we only have to verify that **(A1)** and **(A2)** hold. In our case, the assumption **(A1)** (see page 120), is satisfied with

$$\phi = \frac{\partial G}{\partial v}(s,t,\bar y,\bar v) + \bar\varphi_0 \frac{\partial g}{\partial v}(s,t,\bar y,\bar v)$$

and

$$\psi_j = \bar\varphi_j \frac{\partial g}{\partial v}(s,t,\bar y,\bar v).$$

From the expression for the second derivatives of J and G_j and from the properties imposed to the derivatives of g, G and g_j, it follows that **(A2)** holds:. Take $\{h_k\}_{k=1}^\infty \subset$

$L^\infty(\Sigma)$, bounded in $L^\infty(\Sigma)$ and pointwise convergent to h. We want to to check that

$$\int_\Sigma \left(\frac{\partial^2 G}{\partial y^2}(s,t,\bar{y},\bar{v}) + \bar{\varphi}\frac{\partial^2 g}{\partial y^2}(s,t,\bar{y},\bar{v})\right) z_{h_k}^2\,ds\,dt +$$

$$2\int_\Sigma \left(\frac{\partial^2 G}{\partial y \partial v}(s,t,\bar{y},\bar{v}) + \bar{\varphi}\frac{\partial^2 g}{\partial y\partial v}(s,t,\bar{y},\bar{v})\right) h_k z_{h_k}\,ds\,dt +$$

$$\int_\Sigma \left(\frac{\partial^2 G}{\partial v^2}(s,t,\bar{y},\bar{v}) + \bar{\varphi}\frac{\partial^2 g}{\partial v^2}(s,t,\bar{y},\bar{v})\right) h_k^2\,ds\,dt +$$

$$\sum_{j=1}^{n_d+n_i} \bar{\lambda}_j \left\{ \int_0^T \left[\zeta_j''\left(\int_\Omega g_j(x,t,\nabla_x\bar{y})dx\right) \int_\Omega \frac{\partial g_j}{\partial \eta}(x,t,\nabla_x\bar{y})\nabla_x z_{h_k}\,dx \int_\Omega \frac{\partial g_j}{\partial \eta}(x,t,\nabla_x\bar{y})\nabla_x z_{h_k}\,dx\right] dt + \right.$$

$$\left. \int_0^T \left[\zeta_j'\left(\int_\Omega g_j(x,t,\nabla_x\bar{y})dx\right) \int_\Omega \nabla_x^T z_{h_k}\frac{\partial^2 g_j}{\partial \eta^2}(x,t,\nabla_x\bar{y})\nabla_x z_{h_k} dx\right] dt \right\},$$

where z_{h_k} is given by

$$\left\{\begin{array}{rcll} \dfrac{\partial z_{h_k}}{\partial t} + Az_{h_k} &=& \dfrac{\partial f}{\partial y}(x,t,\bar{y})z_{h_k} & \text{in } Q,\\ \dfrac{\partial z_{h_k}}{\partial n_A} &=& \dfrac{\partial g}{\partial y}(s,t,\bar{y},\bar{v})z_{h_k} + \dfrac{\partial g}{\partial v}(s,t,\bar{y},\bar{v})h_k & \text{on } \Sigma,\\ z_{h_k}(\cdot,0) &=& 0 & \text{in } \Omega, \end{array}\right.$$

converges to

$$\int_\Sigma \left(\frac{\partial^2 G}{\partial y^2}(s,t,\bar y,\bar v) + \bar\varphi \frac{\partial^2 g}{\partial y^2}(s,t,\bar y,\bar v) \right) z_h^2 \, ds\, dt +$$

$$2\int_\Sigma \left(\frac{\partial^2 G}{\partial y \partial v}(s,t,\bar y,\bar v) + \bar\varphi \frac{\partial^2 g}{\partial y \partial v}(s,t,\bar y,\bar v) \right) h z_h \, ds\, dt +$$

$$\int_\Sigma \left(\frac{\partial^2 G}{\partial v^2}(s,t,\bar y,\bar v) + \bar\varphi \frac{\partial^2 g}{\partial v^2}(s,t,\bar y,\bar v) \right) h^2 \, ds\, dt +$$

$$\sum_{j=1}^{n_d+n_i} \bar\lambda_j \left\{ \int_0^T \left[\zeta_j'' \left(\int_\Omega g_j(x,t,\nabla_x \bar y) dx \right) \int_\Omega \frac{\partial g_j}{\partial \eta}(x,t,\nabla_x \bar y) \nabla_x z_h \, dx \int_\Omega \frac{\partial g_j}{\partial \eta}(x,t,\nabla_x \bar y) \nabla_x z_h \, dx \right] dt + \right.$$

$$\left. \int_0^T \left[\zeta_j' \left(\int_\Omega g_j(x,t,\nabla_x \bar y) dx \right) \int_\Omega \nabla_x^T z_h \frac{\partial^2 g_j}{\partial \eta^2}(x,t,\nabla_x \bar y) \nabla_x z_h dx \right] dt \right\}$$

We can do this term by term. First, let us remark that $h_k \to h$ in $L^q(\Sigma)$ for all $q < \infty$, which implies that $z_{h_k} \to z_h$ in $L^r(W^{1,p}(\Omega))$.

The "lines" 1, 2, 3 and 5 can be treated just like in the elliptic case. Let us check that

$$\left| \int_0^T \left[\zeta_j'' \left(\int_\Omega g_j(x,t,\nabla_x \bar y) dx \right) \int_\Omega \frac{\partial g_j}{\partial \eta}(x,t,\nabla_x \bar y) \nabla_x z_{h_k} \, dx \int_\Omega \frac{\partial g_j}{\partial \eta}(x,t,\nabla_x \bar y) \nabla_x z_{h_k} \, dx \right] dt - \right.$$
$$\left. \int_0^T \left[\zeta_j'' \left(\int_\Omega g_j(x,t,\nabla_x \bar y) dx \right) \int_\Omega \frac{\partial g_j}{\partial \eta}(x,t,\nabla_x \bar y) \nabla_x z_h \, dx \int_\Omega \frac{\partial g_j}{\partial \eta}(x,t,\nabla_x \bar y) \nabla_x z_h \, dx \right] dt \right|$$

converges to zero. To simplify the writing, we will suppose without loss of generality that in P7 we have that

$$\left| \frac{\partial g_j}{\partial \eta}(x,t,\eta) \right| \le C|\eta|^{p-1}.$$

So, supposing that $g(x,t,0) = 0$, we will have that

$$|g_j(x,t,\eta)| \le C|\eta|^p.$$

I will not write now the dependence of $(x,t,\nabla_x \bar y)$ in g_j and its derivative because of lack of space in the line and because this cannot lead to confusion. We have, applying P8 and Hölder's inequality,

$$\left| \int_0^T \left(\zeta_j'' \left(\int_\Omega g_j dx \right) \int_\Omega \frac{\partial g_j}{\partial \eta} (\nabla_x z_{h_k} + \nabla_x z_h) \, dx \int_\Omega \frac{\partial g_j}{\partial \eta} (\nabla_x z_{h_k} - \nabla_x z_h) \, dx \right) dt \right| \le$$

$$\int_0^T \left(\left| \int_\Omega g_j dx \right|^{\frac{\tau}{p}-2} \| \frac{\partial g_j}{\partial \eta} \|^2_{L^{p'}(\Omega)} \| \nabla_x z_{h_k} + \nabla_x z_h \|_{L^p(\Omega)} \| \nabla_x z_{h_k} - \nabla_x z_h \|_{L^p(\Omega)} \right) dt \leq$$

$$\int_0^T \left(\left(\int_\Omega |\nabla_x \bar{y}|^p \, dx \right)^{\frac{\tau-2p}{p}} \left(\int_\Omega |\nabla_x \bar{y}|^{(p-1)p'} \, dx \right)^{\frac{2}{p'}} \| \nabla_x z_{h_k} + \nabla_x z_h \|_{L^p(\Omega)} \| \nabla_x z_{h_k} - \nabla_x z_h \|_{L^p(\Omega)} \right) dt \leq$$

$$\int_0^T \left(\left(\int_\Omega |\nabla_x \bar{y}|^p \, dx \right)^{\frac{\tau-2p}{p}\frac{\tau}{\tau-2}} \left(\int_\Omega |\nabla_x \bar{y}|^p \, dx \right)^{\frac{2p-2}{p}\frac{\tau}{\tau-2}} \right) dt \cdot$$

$$\| \nabla_x z_{h_k} + \nabla_x z_h \|_{L^\tau(L^p(\Omega))} \| \nabla_x z_{h_k} - \nabla_x z_h \|_{L^\tau(L^p(\Omega))} \leq$$

$$\int_0^T \left(\int_\Omega |\nabla_x \bar{y}|^p \, dx \right)^{\frac{\tau}{p}} dt \cdot \| \nabla_x z_{h_k} + \nabla_x z_h \|_{L^\tau(L^p(\Omega))} \| \nabla_x z_{h_k} - \nabla_x z_h \|_{L^\tau(L^p(\Omega))}$$

The regularity of $\bar{y}$, z_{h_k} and z_h, together with the convergence of z_{h_k} previously indicated, assure us that the first two factors are bounded and the last one converges to zero.

Thus we have that assumption **(A2)** holds and the result is therefore a direct consequence of Theorem 6.1.3. □

Remark 6.3.1 *Now we cannot, as in the elliptic case, give sufficient conditions for the second derivative of the Lagrangian to be bilinear and continuous in $L^2(\Sigma)$. This is because we can not achieve regularity enough for the adjoint state. See the remarks given now for sufficient conditions.*

Sufficient conditions

To prove an analogous result for the parabolic case is still an open problem. The main difficulty is the regularity of the adjoint state. In this case of the trace of the adjoint state. It is compulsory to show that it belongs to $L^\infty(\Sigma)$ and it depends continuously on the data. This problem is pointed by Raymond and Tröltzsch in [76]. They show that if the adjoint state is given by an equation with a second member –the part that corresponds to the multiplier– is a Lebesgue, then it is possible to prove in some case that the adjoint state is bounded. Nevertheless if the multiplier is a measure, this is not possible (cf. Theorem 4.3 and section 7.3 of [76]). In our case the multipliers is an element of $L^{\tau'}((W^{1,p})')$. It is not in a Lebesgue space and it is a measure. We cannot prove that its trace is bounded.

Chapter 7

Second order conditions involving the Hamiltonian

7.1 Introduction

We will consider in this chapter problems $(\mathbf{P_e})$ and $(\mathbf{P_p})$, taking a convex set of admissible controls. In these two problems, under adequate assumptions, we have seen that a Pontryagin principle holds. The aim of this chapter is to give second order conditions that involve the Hamiltonian of the problem. Necessary conditions appear in a natural way, and they are nothing but corollaries of the analogous result for real valued real functions. The difficulty appears when we deduct sufficient conditions. With the aid of a condition on the Hamiltonian, we can deduce analogous conditions to the ones in finite dimension.

Second order conditions imposed in Theorem 6.2.6 differ in an important detail from the second order conditions given for problems with a finite number of control constraints. For these problems of finite type, it is sufficient that the Lagrangian is positive definite for all $h \in C_{\bar{u}}^0$. There exist examples (see for instance Dunn [49] or Casas and Tröltzsch [36]) that prove that this condition generally is not sufficient for problems with an infinite number of constraints.

Bonnans and Zidani in [12] prove that this condition is sufficient if the second derivative of the Lagrangian is a Legendre form. Letus remind what this means. We say that a quadratic form Q on a Hilbert space X, is of Legendre if it is weakly lower semicontinuous, and for every sequence $\{x_k\} \subset X$ that converges weakly $x_k \rightharpoonup x$ and such that

$Q(x_k) \to Q(x)$, we have that $x_k \to x$ strongly. In this case, we can follow the same sketch of the proof as in finite dimension.

7.2 Elliptic case

Consider problem $(\mathbf{P_e})$, where we take

$$K_\Omega(x) = [u_a(x), u_b(x)].$$

We take again Ω of class C^1; Γ its boundary; A an elliptic operator of continuous coefficients of the form (2.1.1) (page 23); $p > N$; $a_0 \in L^{p/2}(\Omega)$; $f : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$; $g : \Gamma \to \mathbb{R}$, $g \in L^{p-1}(\Gamma)$; $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ for $1 \le j \le n_i + n_e$.

Remember that the Hamiltonian of the problem is given by

$$H(x, y, u, \varphi) = L(x, y, u) + \varphi f(x, y, u).$$

In this chapter we are going to give sufficient conditions for the multiplier ν that goes with L to be 1, and therefore we are not going to write it explicitly in the Hamiltonian.

It is interesting to write some of the derivatives of H and observe its relation with the derivatives of the Lagrangian.

$$H_u(x, y, u, \varphi) = \frac{\partial L}{\partial u}(x, y, u) + \varphi \frac{\partial f}{\partial u}(x, y, u),$$

$$H_{uu}(x, y, u, \varphi) = \frac{\partial^2 L}{\partial u^2}(x, y, u) + \varphi \frac{\partial^2 f}{\partial u^2}(x, y, u),$$

$$H_{uy}(x, y, u, \varphi) = \frac{\partial^2 L}{\partial u \partial y}(x, y, u) + \varphi \frac{\partial^2 f}{\partial u \partial y}(x, y, u)$$

and

$$H_{yy}(x, y, u, \varphi) = \frac{\partial^2 L}{\partial y^2}(x, y, u) + \varphi \frac{\partial^2 f}{\partial y^2}(x, y, u).$$

Given $\bar{u} \in U_{ad}$, $\bar{\lambda}_j$, $j = 1, \dots, n_d + n_i$ and functions $\bar{y} \in W^{1,p}(\Omega)$, $\bar{\varphi} \in W^{1,p'}(\Omega)$ satisfying (6.2.2), (6.2.3) and (6.2.4), if we denote

$$\bar{H}_u(x) = H_u(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)),$$

$$\bar{H}_{uu}(x) = H_{uu}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)),$$

$$\bar{H}_{yu}(x) = H_{yu}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x))$$

and

$$\bar{H}_{yy}(x) = H_{yy}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)),$$

then

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})h = \int_\Omega \bar{H}_u(x)h(x)\,dx$$

and

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 = \int_\Omega \bar{H}_{uu}(x)h^2(x)\,dx + 2\int_\Omega \bar{H}_{yu}(x)h(x)z_h(x)\,dx + \int_\Omega \bar{H}_{yy}(x)z_h^2(x)\,dx +$$

$$+ \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j \int_\Omega \nabla^T z_h \frac{\partial^2 g_j}{\partial \eta^2} \nabla z_h\,dx.$$

where z_h is given by (3.1.3) and $\mathcal{L}(u, \lambda)$ is the Lagrangian of the problem, defined in Section 6.2, page 125.

First order necessary conditions

The first thing we are going to do is writing first order conditions in qualified form.

Theorem 7.2.1 *Let $\bar{u}$ a local solution of* $(\mathbf{P_e})$ *and suppose that the assumptions on f, L and g E1 (page 69), E4 (page 87) and E6 (page 89) and the regularity assumption* (6.2.1) *hold. Then there exist real numbers $\bar{\lambda}_j$, $j = 1, \ldots, n_d + n_i$ and functions $\bar{y} \in W^{1,p}(\Omega)$, $\bar{\varphi} \in W^{1,p'}(\Omega)$ such that* (6.2.2), (6.2.3), (6.2.4) *are satisfied and*

$$H_u(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x))(k - \bar{u}(x)) \geq 0$$

for all $u_a(x) \leq k \leq u_b(x)$ and a.e. $x \in \Omega$.

Proof. Set

$$H_\nu(x, y, u, \varphi) = \nu L(x, y, u) + \varphi f(x, y, u).$$

Notice first that the conditions of Theorem 5.1.1 are satisfied, and therefore Pontryagin's principle holds.

$$H_{\bar{\nu}u}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)) = \min_{k \in K_\Omega(x)} H_{\bar{\nu}}(x, \bar{y}(x), k, \bar{\varphi}(x)) \text{ para c.t.p. } x \in \Omega.$$

Due to the differentiability conditions on L and f we have that

$$\frac{\partial H_{\bar{\nu}u}}{\partial u}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x))(k - \bar{u}(x)) \geq 0 \text{ para todo } u_a(x) \leq k \leq u_b(x) \text{ y c.t.p. } x \in \Omega.$$

Let us denote

$$\mathcal{L}_\nu(u,\lambda) = \nu J(u) + \sum_{j=1}^{n_i+n_d} \lambda_j G_j(u).$$

We have that

$$\frac{\partial \mathcal{L}_{\bar\nu}}{\partial u}(\bar u,\bar\lambda)(u-\bar u) = \int_\Omega \frac{\partial H_{\bar\nu}}{\partial u}(x,\bar y(x),\bar u(x),\bar\varphi(x))(u-\bar u(x))dx \geq 0 \text{ para todo } u \in U_{ad}.$$

But, as we saw in Theorem 6.1.2, the regularity assumption implies that $\bar\nu$ must be different from zero, because if not we would get a contradiction. Rescaling, we can take $\bar\nu = 1$. The proof is complete just observing that $H(x,y,u,\varphi) = H_1(x,y,u,\varphi)$. □

Second order necessary conditions

To establish second order necessary conditions, we need not establish now extra assumptions on the regularity of some of the derivatives of f and L, as we did in (6.2.9) and (6.2.10).

Remember that Ω^0, defined as is the previous chapter (page 129), is

$$\Omega^0 = \{x \in \Omega : |d(x)| > 0\},$$

where

$$d(x) = \frac{\partial L}{\partial u}(x,\bar y(x),\bar u(x)) + \bar\varphi(x)\frac{\partial f}{\partial u}(x,\bar y(x),\bar u(x)).$$

Notice that $d(x) = \bar H_u(x)$.

Theorem 7.2.2 *Let $\bar u$ be again a local solution for problem* ($\mathbf{P_e}$) *(page 16). Suppose that the assumptions on f, L and g_j established in E1 (page 69), E2 (page 70), E4 (page 87), E5 (page 89), E6 (page 89) and E7 (page 90) and the regularity assumption* (6.2.1) *hold. Then*

$$H_{uu}(x,\bar y(x),\bar u(x),\bar\varphi(x)) \geq 0 \text{ for a.e. } x \in \Omega\setminus\Omega^0. \tag{7.2.1}$$

Proof. Again Pontryagin's minimum principle holds, and since H is C^2 with respect to u, the second order necessary condition for one variable problems is written in this case

$$H_{uu}(x,\bar y(x),\bar u(x),\bar\varphi(x)) \geq 0 \text{ for a.e. } x \in \Omega\setminus\Omega^0.$$

This is, where $H_u(x,\bar y(x),\bar u(x),\bar\varphi(x)) = 0$, the second derivative is greater or equal than 0. Condition (7.2.1) is complementary information to (6.2.11). □

An analogous result to this one for control problems governed by ordinary differential equations can be found in Warga [91].

Second order sufficient conditions

In the following Theorem we give an additional condition on the Hamiltonian for the positivity condition of the Lagrangian analogous to the condition in finite dimension to be sufficient. Remember that

$$C^0_{\bar{u},L^2(\Omega)} = \{h \in L^2(\Omega) \ \text{ satisfying } \ (6.2.8) \ \text{ and } \ h(x) = 0 \ \text{ a.e. } \ x \in \Omega^0\}$$

and

$$\Omega^\tau = \{x \in \Omega : |d(x)| > \tau\}.$$

To establish the following result, we must also suppose that the assumptions on the derivatives of f, L and g_j established in page 134, assumption **E8**, hold.

Theorem 7.2.3 *Let $\bar{u}$ be an admissible control for problem* ($\mathbf{P_e}$) *that satisfies the regularity assumption* (6.2.1)*and such that there exist real numbers $\bar{\lambda}_j$, $j = 1, \ldots, n_d + n_i$ and functions $\bar{y} \in W^{1,p}(\Omega)$, $\bar{\varphi} \in W^{1,p'}(\Omega)$ satisfying* (6.2.2), (6.2.3), (6.2.4) *and* (6.2.5). *Suppose also that there exist $\omega > 0$, $\tau > 0$ such that*

$$\begin{cases} H_{uu}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)) \geq \omega \ \text{for a.e. } x \in \Omega \setminus \Omega^\tau \\ \\ \dfrac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 > 0 \ \text{for all } h \in C^0_{\bar{u},L^2(\Omega)}. \end{cases} \tag{7.2.2}$$

Then there exist $\varepsilon > 0$ and $\alpha > 0$ such that $J(\bar{u}) + \alpha\|u - \bar{u}\|^2_{L^2(\Omega)} \leq J(u)$ for all admissible control u with $\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \varepsilon$.

Proof. Let us suppose that the result is false. Then there exists a sequence $\{u_k\}$ of admissible controls with $u_k \to u$ in $L^\infty(\Omega)$ such that

$$J(\bar{u}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Omega)} > J(u_k). \tag{7.2.3}$$

Since u_k is admissible, we have that

$$G_j(u_k) = 0 \text{ if } 1 \leq j \leq n_i$$

and

$$G_j(u_k) \leq 0 \text{ if } n_i + 1 \leq j \leq n_i + n_d.$$

Since $\bar{\lambda}_j \geq 0$ if $n_i + 1 \leq j \leq n_i + n_d$, we have that

$$\bar{\lambda}_j G_j(u_k) \leq 0 \text{ for } 1 \leq j \leq n_i + n_d.$$

On the other hand $\bar{\lambda}_j G_j(\bar{u}) = 0$. Hence

$$\mathcal{L}(\bar{u}, \bar{\lambda}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Omega)} > \mathcal{L}(u_k, \bar{\lambda}). \tag{7.2.4}$$

Set $\delta_k = \|u_k - \bar{u}\|_{L^2(\Omega)}$ and

$$h_k = \frac{u_k - \bar{u}}{\delta_k}.$$

The norm $\|h_k\|_{L^2(\Omega)} = 1$, so there exists a subsequence of $\{h_k\}$, which will be denoted in the same way, and $h \in L^2(\Omega)$ such that $h_k \rightharpoonup h$ weakly in $L^2(\Omega)$. Moreover, h satisfies the sign condition in (6.2.8), because the h_k satisfy it, and the set of functions that satisfy the sign condition in (6.2.8) is convex and closed, and thus weakly closed. Also

$$\mathcal{L}(u_k, \bar{\lambda}) = \mathcal{L}(\bar{u}, \bar{\lambda}) + \delta_k \frac{\partial \mathcal{L}}{\partial u}(v_k, \bar{\lambda})h_k,$$

where v_k is an intermediate point between u and u_k. Since $\delta_k > 0$ and using (7.2.4), we have that

$$\frac{\partial \mathcal{L}}{\partial u}(v_k, \bar{\lambda})h_k < \frac{1}{k}\|u_k - \bar{u}\|_{L^2(\Omega)}.$$

This expression explicitly is

$$\int_\Omega \left(\frac{\partial L}{\partial u}(x, y_k, v_k) + \varphi_k \frac{\partial f}{\partial u}(x, y_k, v_k)\right) h_k < \frac{1}{k}\|u_k - \bar{u}\|_{L^2(\Omega)}, \tag{7.2.5}$$

where y_k and φ_k are respectively the state and adjoint state associated to v_k. The regularity Theorems, the conditions imposed on g_j and the uniform convergence $v_k \to \bar{u}$ implies the uniform convergence $y_k \to \bar{y}$ and the convergence in $L^2(\Omega)$, $\varphi_k \to \bar{\varphi}$. Moreover, the conditions imposed on L implies the convergence in $L^2(\Omega)$ of its derivative with respect to u. Therefore, the weak convergence $h_k \rightharpoonup h$ in $L^2(\Omega)$ is enough to take the limit in (7.2.5) and obtain

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})h \leq 0. \tag{7.2.6}$$

But since we have supposed that $\bar{u}$ satisfies (6.2.5), and $h_k = (u_k - \bar{u})/\delta_k$, with $\delta_k > 0$ and $u_k \in U_{ad}$

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})h_k \geq 0.$$

Taking the limit we obtain

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})h \geq 0. \tag{7.2.7}$$

So, from (7.2.6) and (7.2.7) we have that

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})h = 0. \tag{7.2.8}$$

Since h satisfies the sign condition, this is only possible if $h \in C^0_{\bar{u},L^2(\Omega)}$. let us see this in detail. Let us check that

$$G_j'(\bar{u})h = 0 \text{ if } \begin{cases} j \leq n_i \\ \text{or} \\ j > n_i,\ G_j(\bar{u}) = 0,\ \bar{\lambda}_j > 0, \end{cases}$$

and

$$G_j'(\bar{u})h \leq 0 \text{ if } j > n_i,\ G_j(\bar{u}) = 0,\ \bar{\lambda}_j = 0.$$

If $j \leq n_i$, then $G_j(u_k) = G_j(\bar{u} + \delta_k h_k) = 0$ and $G_j(\bar{u}) = 0$. Therefore

$$0 = \frac{G_j(\bar{u} + \delta_k h_k) - G_j(\bar{u})}{\delta_k},$$

and taking the limit we obtain

$$G_j'(\bar{u})h = 0.$$

If $j > n_i$ and $G_j(\bar{u}) = 0$, we have that $G_j(u_k) = G_j(\bar{u} + \delta_k h_k) \leq 0$. So

$$0 \geq \frac{G_j(\bar{u} + \delta_k h_k) - G_j(\bar{u})}{\delta_k},$$

and taking the limit we obtain

$$G_j'(\bar{u})h \leq 0.$$

It only remains to see what happens when $\bar{\lambda}_j > 0$. Taking into account (7.2.3) and that $\delta_k = ||u_k - \bar{u}||_{L^2(\Omega)}$, we get

$$\frac{\delta_k}{k} \geq \frac{J(u_k) - J(\bar{u})}{\delta_k}.$$

Since $\delta_k \to 0$, taking the limit we obtain

$$0 \geq J'(\bar{u})h.$$

Using now (7.2.8) and the expression for the derivative of the Lagrangian, we have that

$$0 = J'(\bar{u})h + \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j G_j'(\bar{u})h.$$

Taking into account that if $j \leq n_i$ we have just proved that $G_j'(\bar{u})h = 0$, and that if $G_j(\bar{u}) < 0$, then $\bar{\lambda}_j = 0$, if we denote

$$I_1 = \{j : n_i < j < n_i + n_d;\ G_j(\bar{u}) = 0;\ \bar{\lambda}_j > 0\},$$

we have that

$$0 = J'(\bar{u})h + \sum_{j\in I_1} \bar{\lambda}_j G_j'(\bar{u})h.$$

So

$$0 \leq -J'(\bar{u})h = \sum_{j\in I_1} \bar{\lambda}_j G_j'(\bar{u})h \leq 0.$$

Thus, if $j \in I_1$ necessarily $G_j'(\bar{u})h = 0$. To finish checking that $h \in C^0_{\bar{u},L^2(\Omega)}$ we must prove that $h(x) = 0$ in a.e. Ω^0. Since h satisfies the sign condition, in a.e. in Ω^0 we have that $d(x)h(x) \geq 0$. If there existed a set $A \subset \Omega^0$, with $|A| > 0$, such that $|h(x)| > 0$ in A, then

$$\int_\Omega d(x)h(x)\,dx > 0,$$

but

$$\int_\Omega d(x)h(x)\,dx = \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})h = 0.$$

Therefore $h(x) = 0$ in a.e. Ω^0 and $h \in C^0_{\bar{u},L^2(\Omega)}$. So, due to the assumption of the Theorem, we have that

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 > 0 \text{ if } h \neq 0. \tag{7.2.9}$$

On the other hand

$$\mathcal{L}(u_k, \bar{\lambda}) = \mathcal{L}(\bar{u}, \bar{\lambda}) + \delta_k \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})h_k + \frac{\delta_k^2}{2}\frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k, \bar{\lambda})h_k^2, \tag{7.2.10}$$

where w_k is an intermediate point between u_k and $\bar{u}$.

Now, taking into account the considerations made before about the relations between the derivatives of the Lagrangian and of the Hamiltonian, we may write

$$\delta_k \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})h_k + \frac{\delta_k^2}{2}\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h_k^2 = \delta_k \int_\Omega \bar{H}_u(x)h_k(x)\,dx + \frac{\delta_k^2}{2}\int_\Omega \bar{H}_{uu}(x)h_k^2(x)\,dx +$$

$$+\frac{\delta_k^2}{2}\left[\int_\Omega \bar{H}_{yy}(x)z_{h_k}^2(x)\,dx+2\int_\Omega \bar{H}_{yu}(x)h_k(x)z_{h_k}(x)\,dx+\sum_{j=1}^{n_i+n_d}\bar{\lambda}_j\int_\Omega \nabla z_{h_k}\frac{\partial^2 g_j}{\partial\eta^2}\nabla z_{h_k}\,dx\right].$$

Taking into account that $\bar{H}_u(x)=0$ en $\Omega\setminus\Omega^0$

$$A=\delta_k\int_\Omega \bar{H}_u(x)h_k(x)\,dx+\frac{\delta_k^2}{2}\int_\Omega \bar{H}_{uu}(x)h_k^2(x)\,dx=\delta_k\int_{\Omega^0\setminus\Omega^\tau}\bar{H}_u(x)h_k(x)\,dx+$$

$$+\delta_k\int_{\Omega^\tau}\bar{H}_u(x)h_k(x)\,dx+\frac{\delta_k^2}{2}\int_{\Omega^\tau}\bar{H}_{uu}(x)h_k^2(x)\,dx+\frac{\delta_k^2}{2}\int_{\Omega\setminus\Omega^\tau}\bar{H}_{uu}(x)h_k^2(x)\,dx.$$

Using now that $\bar{H}_u(x)h_k(x)\geq 0$ for a.e. $x\in\Omega$, that in Ω^τ we have that $\bar{H}_u(x)\geq\tau$,

$$A\geq\delta_k\tau\int_{\Omega^\tau}|h_k(x)|\,dx+\frac{\delta_k^2}{2}\int_{\Omega^\tau}\bar{H}_{uu}(x)h_k^2(x)\,dx+\frac{\delta_k^2}{2}\int_{\Omega\setminus\Omega^\tau}H_{uu}(x)h_k^2(x)\,dx.$$

Since $\|\delta_k h_k\|_{L^\infty(\Omega)}=\|u_k-\bar{u}\|_{L^\infty(\Omega)}<\varepsilon$, then for a.e. $x\in\Omega$, $\delta_k|h_k(x)|\leq\varepsilon$. Therefore

$$\frac{\delta_k^2h_k^2(x)}{\varepsilon}\leq\delta_k|h_k(x)|.$$

Hence

$$A\geq\frac{\delta_k^2}{2}\int_{\Omega^\tau}\left(\frac{2\tau}{\varepsilon}+\bar{H}_{uu}(x)\right)h_k^2(x)\,dx+\frac{\delta_k^2}{2}\int_{\Omega\setminus\Omega^\tau}\bar{H}_{uu}(x)h_k^2(x)\,dx.$$

Now, from (7.2.4), (7.2.10) and taking into account the previous considerations, we have that

$$\frac{\delta_k^2}{k}>\delta_k\frac{\partial\mathcal{L}}{\partial u}(\bar{u},\bar{\lambda})h_k+\frac{\delta_k^2}{2}\frac{\partial^2\mathcal{L}}{\partial u^2}(w_k,\bar{\lambda})h_k^2=\delta_k\frac{\partial\mathcal{L}}{\partial u}(\bar{u},\bar{\lambda})h_k+\frac{\delta_k^2}{2}\frac{\partial^2\mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})h_k^2+$$

$$+\frac{\delta_k^2}{k}\left[\frac{\partial^2\mathcal{L}}{\partial u^2}(w_k,\bar{\lambda})h_k^2-\frac{\partial^2\mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})h_k^2\right],$$

$$\frac{\delta_k^2}{2}>\frac{\delta_k^2}{2}\int_{\Omega^\tau}\left(\frac{2\tau}{\varepsilon}+\bar{H}_{uu}(x)\right)h_k^2(x)\,dx+\frac{\delta_k^2}{2}\int_{\Omega\setminus\Omega^\tau}\bar{H}_{uu}(x)h_k^2(x)dx+$$

$$+\frac{\delta_k^2}{2}\left[\int_\Omega \bar{H}_{yy}(x)z_{h_k}^2(x)\,dx+2\int_\Omega \bar{H}_{yu}(x)h_k(x)z_{h_k}(x)\,dx+\sum_{j=1}^{n_i+n_d}\bar{\lambda}_j\int_\Omega \nabla^T z_{h_k}\frac{\partial^2 g_j}{\partial\eta^2}\nabla z_{h_k}\,dx\right]$$

$$+\frac{\delta_k^2}{2}\left[\frac{\partial^2\mathcal{L}}{\partial u^2}(w_k,\bar{\lambda})h_k^2-\frac{\partial^2\mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})h_k^2\right].\qquad(7.2.11)$$

Let us divide now by $\delta_k^2/2$. Taking into account the assumptions made on the second derivatives of the functions, there exists a constant $C_H>0$ such that $\bar{H}_{uu}(x)\geq -C_H$ for a.e. $x\in\Omega$. So, taking ε small enough, we have that

$$\frac{2\tau}{\varepsilon}+\bar{H}_{uu}(x)\geq\frac{2\tau}{\varepsilon}-C_H>0\text{ a.e. }x\in\Omega.$$

So

$$\liminf_{k\to\infty} \int_{\Omega^\tau} \left(\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x)\right) h_k^2 dx \geq \int_{\Omega^\tau} \left(\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x)\right) h^2 dx.$$

Moreover, in $\Omega \setminus \Omega^\tau$, $\bar{H}_{uu}(x) > \omega > 0$, and then

$$\liminf_{k\to\infty} \int_{\Omega\setminus\Omega^\tau} \bar{H}_{uu}(x) h_k^2 dx \geq \int_{\Omega\setminus\Omega^\tau} \bar{H}_{uu}(x) h^2 dx.$$

Taking into account that **(A3)** holds, we can take the lower limit in (7.2.11) and obtain

$$0 \geq \int_{\Omega^\tau} \left(\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x)\right) h^2(x)\, dx + \int_{\Omega\setminus\Omega^\tau} \bar{H}_{uu}(x) h^2 dx +$$

$$+ \int_\Omega \bar{H}_{yy}(x) z_h^2(x)\, dx + 2\int_\Omega \bar{H}_{yu}(x) h(x) z_h(x)\, dx + \sum_{j=1}^{n_i+n_d} \bar{\lambda}_j \int_\Omega \nabla^T z_h \frac{\partial^2 g_j}{\partial \eta^2} \nabla z_h\, dx.$$

Therefore

$$0 \geq \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda}) h^2$$

and from (7.2.9) and this, we obtain that $h = 0$.

So in the expression where we take lower limit, we can actually take the limit. Since all the terms converge to zero, but at most

$$\int_{\Omega^\tau} \left(\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x)\right) h_k^2(x)\, dx + \int_{\Omega\setminus\Omega^\tau} \bar{H}_{uu}(x) h_k^2(x) dx,$$

we have that this also converges to zero. But

$$\min\left\{\omega, \frac{2\tau}{\varepsilon} - C_H\right\} \int_\Omega h_k^2(x) dx \leq \int_{\Omega^\tau} \left(\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x)\right) h_k^2(x)\, dx + \int_{\Omega\setminus\Omega^\tau} \bar{H}_{uu}(x) h_k^2(x) dx.$$

Therefore,

$$\lim_{k\to\infty} \|h_k\|_{L^2(\Omega)} = 0.$$

But $\|h_k\|_{L^2(\Omega)} = 1$. So we have achieved a contradiction. So the theorem is true. □

Remark 7.2.1 *If we impose the condition* (7.2.2) *for a.e.* $x \in \Omega$, *we will obtain during the proof that the second derivative of the Lagrangian is a quadratic Legendre form for the sequence* $\{h_k\}$.

7.3 Parabolic case

Set Ω, Γ, T, Q, Σ and A, p, τ, k_1, $\tilde{k}_1$, σ_1, $\tilde{\sigma}_1$ as in Section 2.2, with the boundary Γ of class C^1 and the coefficients of the operator A of class $C([0,T];C(\bar{\Omega}))$. Set f, g, y_0 functions, $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \Sigma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $F : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $G : \Sigma \times \mathbb{R} \longrightarrow \mathbb{R}$ and $y_0 : \Omega \longrightarrow \mathbb{R}$, $y_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. Take k_2, $\tilde{k}_2$, σ_2, $\tilde{\sigma}_2$ and ν as in Section 2.2.

Consider problem $(\mathbf{P_p})$ of page 16. We will suppose that the set of admissible controls is of the form

$$V_{ad} = \{v \in L^\infty(\Sigma) : \ v_a(s,t) \leq v(s,t) \leq v_b(s,t) \text{ a.e. } (s,t) \in \Sigma\},$$

where v_a, $v_b \in L^\infty(\Sigma)$. This election corresponds to the case of taking

$$K_\Sigma(s,t) = [v_a(s,t), v_b(s,t)].$$

Just like in Section 4.1.2, we will consider

$$C = \left\{ \vec{f} \in L^\tau(L^p)^N : \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\vec{f}) dx \right) dt = 0 \text{ if } 1 \leq j \leq n_i, \right.$$

$$\left. \int_0^T \zeta_j \left(\int_\Omega g_j(x,t,\vec{f}) dx \right) dt \leq 0 \text{ if } n_i + 1 \leq j \leq n_i + n_d \right\},$$

where $\zeta_j : \mathbb{R} \to \mathbb{R}$ and $g_j : Q \times \mathbb{R}^N \to \mathbb{R}$ are functions.

The Hamiltonian of the problem is given by

$$H(s,t,y,v,\varphi) = G(s,t,y,v) + \varphi g(s,t,y,v).$$

We write it it in this way and not like in page 108 because we are going to give sufficient conditions for $\bar{\nu} = 1$. Now

$$H_v(s,t,y,v,\varphi) = \frac{\partial G}{\partial v}(s,t,y,v) + \varphi \frac{\partial g}{\partial v}(s,t,y,v).$$

Given $v \in V_{ad}$, real numbers $\bar{\lambda}_j$, $j = 1, \dots, n_d + n_i$ and functions $\bar{y} \in L^\tau(W^{1,p}(\Omega))$ and $\bar{\varphi} \in L^{\tau'}(W^{1,p'}(\Omega))$ satisfying (6.3.2)–(6.3.4), then

$$\frac{\partial \mathcal{L}}{\partial v}(\bar{v}, \bar{\lambda})h = \int_\Sigma H_v(s,t,\bar{y}(s,t), \bar{v}(s,t), \bar{\varphi}(s,t))h(s,t)\, ds\, dt,$$

where $\mathcal{L}(v,\lambda)$ is the Lagrangian of the problem defined in Section 6.3, page 146.

First order necessary conditions

The first thing we are going to do is writing first order conditions in qualified form.

Theorem 7.3.1 *Suppose that f and g satisfy assumptions P1 and P2, that F, G and L satisfy P4 and P5 and that ζ_j and g_j satisfy P7. Suppose also that* (6.3.1) *holds. Then there exist real numbers $\bar\lambda_j$, $j = 1, \dots, n_d + n_i$ and functions $\bar y \in L^\tau(W^{1,p}(\Omega))$ and $\bar\varphi \in L^{\tau'}(W^{1,p'}(\Omega)) + L^2(H^1)$ such that* (6.3.2)–(6.3.4) *hold and*

$$H_v(s,t,\bar y(s,t),\bar u(s,t),\bar\varphi(s,t))(k - \bar v(s,t)) \geq 0$$

for all $v_a(x) \leq k \leq v_b(x)$ and a.e. $(s,t) \in \Sigma$.

Proof. The proof is completely analogous to that of the elliptic case. If we define

$$H_\Sigma(s,t,y,v,\varphi,\nu) = \nu G(s,t,y,v) + \varphi g(s,t,y,v),$$

due to Pontryagin's principle, proved in Theorem 5.2.1,

$$H_\Sigma(s,t,\bar y(s,t),\bar v(s,t),\bar\varphi(s,t),\bar\nu) = \min_{v \in K_\Sigma(s,t)} H_\Sigma(s,t,\bar y(s,t),v,\bar\varphi(s,t),\bar\nu)$$

Due to the differentiability conditions imposed now, we have that

$$H_{\Sigma v}(s,t,\bar y(s,t),\bar u(s,t),\bar\varphi(s,t),\bar\nu)(k - \bar v(s,t)) \geq 0$$

for all $v_a(x) \leq k \leq v_b(x)$ and a.e. $(s,t) \in \Sigma$. If we denote

$$\bar{\mathcal{L}}(v,\lambda,\nu) = \nu J(v) + \sum_{j=1}^{n_i+n_d} \lambda_j G_j(v),$$

then

$$\frac{\partial \mathcal{L}}{\partial v}(\bar v,\bar\lambda,\bar\nu)(v - \bar v) = \int_\Sigma H_{\Sigma v}(s,t,\bar y(s,t),\bar u(s,t),\bar\varphi(s,t),\bar nu)(v - \bar v(s,t)) \geq 0$$

for all $v \in V_{ad}$. But, as it was seen in Theorem 6.1.2, the regularity assumption implies that $\bar\nu$ must be different from zero, because if not, we would get a contradiction. Rescaling we can taker $\bar\nu = 1$. The proof is completed just noticing that $H(s,t,y,v,\varphi) = H_\Sigma(s,t,y,v,\varphi,1)$. □

Second order necessary conditions

To establish second order necessary conditions, it is not necessary to state extra assumptions on the second derivatives of G and g as we did in (6.3.6) and (6.3.7).

Remember that

$$\Sigma^0 = \{(s,t) \in \Sigma : |d(s,t)| > 0\},$$

where

$$d(s,t) = \frac{\partial G}{\partial v}(s,t,\bar{y}(s,t),\bar{v}(s,t)) + \bar{\varphi}(s,t)\frac{\partial g}{\partial v}(s,t,\bar{y}(s,t),\bar{v}(s,t)),$$

Notice that $d(s,t) = H_v(s,t,\bar{y}(s,t),\bar{u}(s,t),\bar{\varphi}(s,t))$.

Theorem 7.3.2 *Suppose that $\bar{v}$ is a local solution for problem* ($\mathbf{P_p}$) *and that P1–P8 hold. Then*

$$H_{vv}(s,t,\bar{y}(s,t),\bar{v}(s,t),\bar{\varphi}(s,t)) \geq 0 \textit{ for a.e. } (s,t) \in \Sigma \setminus \Sigma^0.$$

Proof. Again Pontryagin's principle is satisfied, and since H is C^2 with respect to v, the second order necessary conditions for one variable problems is written in this case as

$$H_{vv}(s,t,\bar{y}(s,t),\bar{v}(s,t),\bar{\varphi}(s,t)) \geq 0 \text{ for a.e. } (s,t) \in \Sigma \setminus \Sigma^0.$$

This is, where the first derivative is zero, the second derivative is greater or equal than zero. □

Sufficient conditions

We have the same problem as in page 153. We cannot grant that the adjoint state has a bounded trace.

Part III

Numerical Analysis

The last part of this thesis is devoted to the numerical analysis of a control problem. Chapter 8 is dedicated to the study of the uniform convergence of the finite element method applied to the study of semilinear equations. In Chapter 9 we study a problem with pointwise state constraints. This problem is different from the problem studied in Chapter 4 because now we have an infinite number of state constraints.

Chapter 8

Uniform convergence of the F.E.M. for semilinear equations

This chapter is dedicated to the study of the approximation of the solution of a semilinear equation with the finite element method. Concretely, we study the uniform convergence of the discrete approximations to the solution of the equation. A similar study is carried out in Ciarlet [43] for linear equations. Ciarlet studies a Dirichlet problem and uses triangulations of non negative type. We will also study Neumann's problem and, in some case, we do not use triangulations of non negative type.

The first section describes the common elements to both Dirichlet and Neumann problems, and the discretization. In the second section we give results for Dirichlet's problem and in the third one for Neumann's problem.

8.1 Discretization

Let Ω be a convex subset of R^N, $N = 2$ or $N = 3$, Γ its boundary and A an operator of the form

$$Ay = -\sum_{i,j=1}^{N} \partial_{x_j} \left[a_{ij}\partial_{x_i} y\right],$$

where $a_{i,j} \in C^{0,1}(\bar{\Omega})$ and such that there exist $m, M > 0$ such that

$$m\|\xi\|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq M\|\xi\|^2 \ \forall \xi \in \mathbb{R}^N \text{ y } \forall x \in \Omega.$$

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory function, monotone decreasing in the second variable, with $f(\cdot, 0) \in L^{p/2}(\Omega)$ and satisfying the following local Lipschitz condition For all $M > 0$ there exists $\phi_M \in L^2(\Omega)$ such that

$$|f(x, y_1) - f(x, y_2)| \leq |\phi_M(x)||y_1 - y_2| \text{ for a.e. } x \in \Omega \tag{8.1.1}$$

if $|y_1|$, $|y_2| < M$.

To make a numeric approximation, we take a family of triangulations on $\bar{\Omega}$, $\{\mathcal{T}_h\}_{h>0}$. To each element $T \in \mathcal{T}_h$ let us associate two parameters: $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the greatest ball included in T. We will suppose that $h = \max_{T\in\mathcal{T}_h} \rho(T)$ converges to zero. We will make the following assumptions on the triangulation:

- Regularity assumption: there exists $\sigma > 0$ such that $\frac{\rho(T)}{\sigma(T)} \leq \sigma \quad \forall T \in \mathcal{T}_h$ and $h > 0$.
- Inverse assumption: there exists $\rho > 0$ such that $\frac{h}{\rho(T)} \leq \rho \quad \forall T \in \mathcal{T}_h$ and $h > 0$.
- Set $\bar{\Omega}_h = \cup_{T\in\mathcal{T}_h} T$, Ω_h its interior and Γ_h its boundary. Then we will suppose that the vertexes of $\mathcal{T}_h$ placed on the boundary of Γ_h are points of Γ.

Consider the spaces

$$V_h = \left\{y_h \in C(\bar{\Omega}) \cap H_0^1(\Omega) : \; y_{h|_T} \in P_1(T) \quad \forall T \in \mathcal{T}_h \; y_h = 0 \text{ in } \Omega \setminus \Omega_h\right\}$$

and

$$W_h = \left\{y_h \in C(\bar{\Omega}_h) : \; y_{h|_T} \in P_1(T) \quad \forall T \in \mathcal{T}_h\right\},$$

where $P_1(T)$ is the space of polynomies of degree 1 on T. V_h is a vector subspace of $W_0^{1,p}(\Omega)$ and W_h is a subspace of $W^{1,p}(\Omega)$.

We will use Lagrange interpolation operator

$$\Pi_h : \; C(\bar{\Omega}) \longrightarrow W_h$$

being $\Pi_h z$ the unique element in W_h such that $\Pi_h z(x_i) = z(x_i)$ for all x_i node of the triangulation.

8.2 Dirichlet case

We will also introduce $f_2 \in W^{-1,p}(\Omega)$. We want to to study the uniform approximation by the finite element method of the solution of the equation

$$\begin{cases} Ay = f(\cdot, y) + f_2 & \text{in } \Omega \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{8.2.1}$$

For every h, let us define $y_h \in V_h$ as the unique element that satisfies

$$\sum_{i,j=1}^{N} \int_\Omega a_{i,j}(x)\partial_{x_i}y_h(x)\partial_{x_j}z_h(x)dx = \int_\Omega f(x, y_h(x))z_h dx + \langle f_2, z_h\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p}(\Omega)} \quad \forall z_h \in V_h. \tag{8.2.2}$$

Lemma 8.2.1 *Equation* (8.2.2) *has a unique solution.*

Proof. Let N_h be the dimension of V_h. To prove the lemma, we will write the equation of the form

$$A_h y = F(y) + b$$

where A_h is an $N_h \times N_h$ positive definite matrix, $F : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ is locally Lipschitz, of constant, say, L, and satisfies that

$$\langle F(y_1) - F(y_2), y_1 - y_2\rangle \le 0 \text{ for all } y_1,\, y_2 \in \mathbb{R}^{N_h}$$

and b is a vector of $\mathbb{R}^{N_h}$. Without loss of generality, we will suppose that $F(0) = 0$. We truncate F by

$$F_M(y) = \begin{cases} F(y) & \text{if } \|F(y)\| \le M \\ M\dfrac{F(y)}{\|F(y)\|} & \text{if } \|F(y)\| \ge M. \end{cases}$$

We have that the mapping that to every $z \in \mathbb{R}^{N_h}$ associates y_z such that $A_h(y_z) = F_M(z) + b$ satisfies that $\|y_z\| \le (M + \|b\|)/\alpha$, where α is the smallest eigenvalue of A_h. So, applying Brauer's fixed point Theorem, we have that there exists y_M that solves $A_h y_M = F_M(y_M) + b$. Moreover

$$\alpha\|y_M\|^2 \le y_M^T A_h y_M = (F_M(y_M), y_M) + (b, y_M) \le \|b\|\|y_M\|,$$

and hence

$$\|y_M\| \le \frac{\|b\|}{\alpha}.$$

Therefore y_M is bounded independently of M. Since F is Lipschitz on the ball $\bar{B}(0, \frac{\|b\|}{\alpha})$,

$$F(y_M) \leq \frac{L\|b\|}{\alpha} \text{ for all } M > 0$$

and if we take $M \geq L\|b\|/\alpha$, $F(y_M) = F_M(y_M)$, and we will have found a solution to our equation. Uniqueness follows from the monotonicity of F. □

Our purpose is to show that $y_h \to y$ in $L^\infty(\Omega)$. We will start studying the linear case, supposing a regular enough solution. Next we will apply these results to the study of a semilinear equation, also with regular solution. Finally, we will study the interesting case, in which the maximal regularity for the state is $W_0^{1,p}(\Omega)$.

Linear case. $y \in H^2(\Omega)$

Suppose that $f(\cdot, y) \equiv 0$ and that $f_2 = g \in L^2(\Omega)$. There exists a unique function $y \in H^2(\Omega) \cap H_0^1(\Omega)$ (cf. Grisvard [59]) that satisfies

$$\left\{\begin{array}{rcl} Ay & = & g \text{ in } \Omega \\ y & = & 0 \text{ on } \Gamma. \end{array}\right. \tag{8.2.3}$$

We also have that there exists a constant $C > 0$ such that

$$\|y\|_{H^2(\Omega)} \leq C\|g\|_{L^2(\Omega)}. \tag{8.2.4}$$

We can formulate problem (8.2.3) variationally as

$$\left\{\begin{array}{l} \text{Find } y \in H_0^1(\Omega) \text{ such that} \\ a(y, z) = (g, z)\ \forall z \in H_0^1(\Omega). \end{array}\right. \tag{8.2.5}$$

The approximate problem can be formulated as

$$\left\{\begin{array}{l} \text{Find } y_h \in V_h \text{ such that} \\ a(y_h, z_h) = (g, z_h)\ \forall z_h \in V_h. \end{array}\right. \tag{8.2.6}$$

The following lemma is known as Aubin-Nitsche Lemma; see for instance Ciarlet [43, Theorem 19.1] or Raviart-Thomas [74, Theorem 5.2-1].

Lemma 8.2.2 *Let y and y_h be the solutions of problems* (8.2.5) *and* (8.2.6) *respectively. Then there exists a constant $C > 0$ independent of h such that*

$$\|y - y_h\|_{L^2(\Omega)} \leq Ch^2\|g\|_{L^2(\Omega)}.$$

Proof. Let us see that there exists a constant $C > 0$ independent of h such that $\forall \psi \in L^2(\Omega)$ we have:

$$\int_\Omega \psi(y - y_h)dx \leq Ch^2 \|\psi\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

Take $\psi \in L^2(\Omega)$ and let $z_\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique element that satisfies

$$\begin{cases} A^* z_\psi &= \psi \text{ in } \Omega \\ z_\psi &= 0 \text{ on } \Gamma, \end{cases} \tag{8.2.7}$$

where A^* is the adjoint operator of A.

Just like before, we know that there exists a constant $C > 0$ independent of h such that

$$\|z_\psi\|_{H^2(\Omega)} \leq C \|\psi\|_{L^2(\Omega)}. \tag{8.2.8}$$

The variational formulation of (8.2.7) is written:

$$\begin{cases} \text{Find } z_\psi \in H_0^1(\Omega) \text{ such that} \\ a(z, z_\psi) = (\psi, z) \; \forall z \in H_0^1(\Omega). \end{cases} \tag{8.2.9}$$

and it can be approximated by

$$\begin{cases} \text{Find } z_{\psi,h} \in V_h \text{ such that} \\ a(z_h, z_{\psi,h}) = (\psi, z_h) \; \forall z_h \in V_h. \end{cases} \tag{8.2.10}$$

So, using (8.2.9), (8.2.5) and (8.2.6), the continuity of the bilinear form a on $H^1(\Omega)$, the usual estimates for finite elements (see for instance Raviart-Thomas [74, Theorem 5.2-1, equation (5.2-20)]), and the estimates (8.2.4) and (8.2.8), we obtain:

$$\begin{aligned} (\psi, y - y_h) &= a(y - y_h, z_\psi) && = \\ &= a(y - y_h, z_\psi - z_{\psi,h}) && \leq \\ &\leq C\|y - y_h\|_{H^1(\Omega)} \|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} && \leq \\ &\leq Ch^2 \|y\|_{H^2(\Omega)} \|z_\psi\|_{H^2(\Omega)} && \leq \\ &\leq Ch^2 \|g\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)}. \end{aligned}$$

Therefore

$$\|y - y_h\|_{L^2(\Omega)} = \sup_{\|\psi\|_{L^2(\Omega)} \leq 1} (\psi, y - y_h) \leq Ch^2 \|g\|_{L^2(\Omega)},$$

and the proof is complete. □

Now we are going to give an error estimate in the norm of $L^\infty(\Omega)$. Due to the assumptions made $y \in C(\bar{\Omega})$, and therefore $y - y_h \in C(\bar{\Omega})$.

We will use the following lemma (see Ciarlet [43, Theorem 16.1]), which gives us the interpolation error:

Lemma 8.2.3 *Set $m \geq 0$, $k \geq 0$, and $p, q \in [1, \infty]$. If we have the embeddings*

$$W^{k+1,p}(T) \hookrightarrow C^0(T)$$
$$W^{k+1,p}(T) \hookrightarrow W^{m,q}(T)$$

then there exists a constant $C > 0$ independent of h such that

$$\|y - \Pi_T y\|_{W^{m,q}(T)} \leq C h^{N\left(\frac{1}{q} - \frac{1}{p}\right) + k + 1 - m} \|y\|_{W^{k+1,p}(T)},$$

where $\Pi_T y$ is the restriction to the element T of $\Pi_h y$.

The following inequality, whose proof can be found in Ciarlet [43, Theorem 17.2], which gives us the equivalence constant between two Sobolev norms in a finite dimensional space:

$$\|y_h\|_{W^{m,q}(\Omega_h)} \leq C \frac{1}{h^{N \max\left\{0, \frac{1}{p} - \frac{1}{q}\right\}} h^{m-l}} \|y_h\|_{W^{l,p}(\Omega_h)} \; \forall v_h \in V_h, \text{ if } l \leq m, \tag{8.2.11}$$

being $C > 0$ independent of h.

We have now the main result of this section (Ciarlet [43, Theorem 19.3]):

Theorem 8.2.4 *Let y and y_h be the solutions of problems* (8.2.5) *and* (8.2.6) *respectively. Then there exists a constant $C > 0$ independent of h such that*

$$\|y - y_h\|_{L^\infty(\Omega_h)} \leq C h^{2 - \frac{N}{2}} \|y\|_{H^2(\Omega)}.$$

Proof. We have that

$$\|y - y_h\|_{L^\infty(\Omega_h)} \leq \|y - \Pi_h y\|_{L^\infty(\Omega_h)} + \|\Pi_h y - y_h\|_{L^\infty(\Omega_h)}. \tag{8.2.12}$$

Due to Lemma 8.2.3, taking $m = 0$, $q = \infty$, $k = 1$ and $p = 2$, we have that

$$\|y - \Pi_h y\|_{L^\infty(\Omega_h)} \leq C h^{2 - \frac{N}{2}} \|y\|_{H^2(\Omega)}. \tag{8.2.13}$$

Applying (8.2.11) we have that

$$\|\Pi_h y - y_h\|_{L^\infty(\Omega_h)} \leq Ch^{-\frac{N}{2}} \|\Pi_h y - y_h\|_{L^2(\Omega_h)}. \tag{8.2.14}$$

Again due to Lemma 8.2.3, taking $m = 0$, $q = 2$, $k = 1$ and $p = 2$, we gets

$$\|\Pi_h y - y\|_{L^2(\Omega_h)} \leq Ch^2 \|y\|_{H^2(\Omega)}, \tag{8.2.15}$$

and due to Lemma 8.2.2

$$\|y - y_h\|_{L^2(\Omega_h)} \leq \|y - y_h\|_{L^2(\Omega)} \leq Ch^2 \|y\|_{H^2(\Omega)}. \tag{8.2.16}$$

From (8.2.15) and (8.2.16) it follows that

$$\|\Pi_h y - y_h\|_{L^2(\Omega_h)} \leq \|\Pi_h y - y\|_{L^2(\Omega_h)} + \|y - y_h\|_{L^2(\Omega_h)}, \leq Ch^2 \|y\|_{H^2(\Omega)}.$$

This, together with (8.2.14) implies that

$$\|\Pi_h y - y_h\|_{L^\infty(\Omega_h)} \leq Ch^{2-\frac{N}{2}} \|y\|_{H^2(\Omega)},$$

which together with (8.2.13) and with (8.2.12) complete the proof of the theorem. □

Semilinear case. $y \in H^2(\Omega)$

Suppose now that $f_2 \equiv 0$. We will also suppose that there exists a function $\phi \in L^2(\Omega)$ such that

$$|f(x, t_1) - f(x, t_2)| \leq |\phi(x)|\, |t_1 - t_2| \quad \forall t_1, t_2 \in \mathbb{R}, \quad \text{a.e. } x \in \Omega. \tag{8.2.17}$$

This restrictive condition of global type will be relaxed later to one of local type. We are going to suppose that $f(\cdot, 0) \in L^2(\Omega)$. So

$$|f(x,t)| \leq |f(x,t) - f(x,0)| + |f(x,0)| \leq |\phi(x)|\, |t| + |f(x,0)|$$

and this way we have that for any real number $M > 0$ there exists a function $\varphi_M(x) = \phi(x)\, M + f(x,0) \in L^2(\Omega)$ such that if $|t| \leq M$ then $|f(x,t)| \leq |\varphi_M(x)|$. Combinig the technique of Theorem 3.1.1 with regularity results in Grisvard [59], under this two conditions we can deduce now that the equation

$$\left\{ \begin{array}{rcl} Ay & = & f(x,y) \text{ in } \Omega \\ y & = & 0 \text{ on } \Gamma, \end{array} \right. \tag{8.2.18}$$

has a unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$.

Let us see now the error estimates of the finite element method in the norms of $H^1(\Omega)$, $L^2(\Omega)$ y $L^\infty(\Omega)$.
Equation (8.2.18) can be formulated variationally as

$$\begin{cases} \text{Find } y \in H_0^1(\Omega) \text{ such that} \\ a(y, z) = (f(x, y), z) \ \forall z \in H_0^1(\Omega), \end{cases} \tag{8.2.19}$$

and it can be approximated by

$$\begin{cases} \text{Find } y_h \in V_h \text{ such that} \\ a(y_h, z_h) = (f(x, y_h), z_h) \ \forall z_h \in V_h. \end{cases} \tag{8.2.20}$$

The following result is a generalization for semilinear equations of the known Céa's Lemma (cf Céa [39, Proposition 3.1])

Lemma 8.2.5 *Let y and y_h be solutions of the variational problems* (8.2.19) *and* (8.2.20) *respectively. Then there exists a constant $C > 0$ independent of h such that*

$$\|y - y_h\|_{H^1(\Omega)} \leq C\|y - \Pi_h y\|_{H^1(\Omega)}.$$

Proof. The result is a consequence of the $H_0^1(\Omega)$-ellipticity of a, the monotonicity of f in the second variable, the Lipschitz condition imposed on f and the continuous embedding from $H^1(\Omega)$ in $L^4(\Omega)$:

$$\begin{aligned}
&\|y - y_h\|^2_{H^1(\Omega)} \leq Ca(y - y_h, y - y_h) \leq \\
&\leq Ca(y - y_h, y - y_h) - (f(\cdot, y) - f(\cdot, y_h), y - y_h) = \\
&= Ca(y - y_h, y - z_h) - (f(\cdot, y) - f(\cdot, y_h), y - z_h) \leq \\
&\leq C\left\{\|y - y_h\|_{H^1(\Omega)}\|y - z_h\|_{H^1(\Omega)} + \|\phi\|_{L^2(\Omega)}\|y - y_h\|_{L^4(\Omega)}\|y - z_h\|_{L^4(\Omega)}\right\} \leq \\
&\leq C\left\{\|y - y_h\|_{H^1(\Omega)}\|y - z_h\|_{H^1(\Omega)} + \|\phi\|_{L^2(\Omega)}\|y - y_h\|_{H^1(\Omega)}\|y - z_h\|_{H^1(\Omega)}\right\} \leq \\
&\leq C\|y - y_h\|_{H^1(\Omega)}\|y - z_h\|_{H^1(\Omega)} \text{ for all } z_h \in V_h.
\end{aligned}$$

Dividing by $\|y - y_h\|_{H^1(\Omega)}$ and taking $z_h = \Pi_h y$ we achieve to the desired result. □

Now we have the following lemma.

Lemma 8.2.6 *Let y and y_h be solutions of the variational problems* (8.2.19) *and* (8.2.20) *respectively. Then there exists a constant $C > 0$ independent of h such that*

$$\|y - y_h\|_{H^1(\Omega)} \leq Ch\|y\|_{H^2(\Omega)}.$$

Proof. Using Lemma 8.2.5, the inequality

$$\|y\|_{H^1(\Omega\setminus\Omega_h)} \leq Ch\|y\|_{H^2(\Omega)}$$

(cf. Raviart-Thomas [74, Lemma 5.2-3]) and Lemma 8.2.3 with $m = 1$, $q = 2$, $k = 1$ and $p = 2$, we have that

$$\|y - y_h\|_{H^1(\Omega)} \leq C\|y - \Pi_h y\|_{H^1(\Omega)} \leq C\left(\|y\|_{H^1(\Omega\setminus\Omega_h)} + \|y - y_h\|_{H^1(\Omega_h)}\right) \leq Ch\|y\|_{H^2(\Omega)},$$

and the proof is complete. □

To obtain the error estimate in $L^2(\Omega)$ let us introduce the function

$$\alpha(x) = \begin{cases} \dfrac{f(x, y_h(x)) - f(x, y(x))}{y(x) - y_h(x)} & \text{if } y(x) \neq y_h(x) \\ 0 & \text{in other case.} \end{cases} \tag{8.2.21}$$

Notice that $\alpha(x) \geq 0$.

We have again that for all $\psi \in L^2(\Omega)$ there exists a unique $z_\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$\begin{cases} A^* z_\psi + \alpha(x) z_\psi = \psi & \text{in } \Omega \\ z_\psi = 0 & \text{on } \Gamma. \end{cases}$$

Since $\|\alpha\|_{L^2(\Omega)} \leq \|\phi\|_{L^2(\Omega)}$, there exists a constant $C > 0$ independent of α such that $\|z_\psi\|_{H^2(\Omega)} \leq C\|\psi\|_{L^2(\Omega)}$.

This problem can be formulated variationally as

$$a(z, z_\psi) + (\alpha z_\psi, z) = (\psi, z) \ \forall z \in H_0^1(\Omega), \tag{8.2.22}$$

and it can be approximated by

$$a(z_h, z_{\psi,h}) + (\alpha z_{\psi,h}, z_h) = (\psi, z_h) \ \forall z_h \in V_h. \tag{8.2.23}$$

We are going to apply a very similar technique to that of the linear case to find an error estimate $y - y_h$ in $L^2(\Omega)$.

Lemma 8.2.7 *Let y and y_h be solutions of the variational problems* (8.2.19) *and* (8.2.20) *respectively. Then there exists a constant $C > 0$ independent of h such that*

$$\|y - y_h\|_{L^2(\Omega)} \leq Ch^2 \|y\|_{H^2(\Omega)}.$$

Proof. Take any $\psi \in L^2(\Omega)$. Using (8.2.22), the definition of $\alpha(x)$, (8.2.19) and (8.2.20), the continuity of a, Lipschitz's condition (8.2.17), and Sobolev's and Hölder's inequalities as in the previous proof, we have

$$\begin{aligned}
&(\psi, y - y_h) = a(y - y_h, z_\psi) + (\alpha z_\psi, y - y_h) = \\
&= a(y - y_h, z_\psi - z_{\psi,h}) + a(y - y_h, z_{\psi_h}) + (\alpha z_\psi, y - y_h) = \\
&= a(y - y_h, z_\psi - z_{\psi,h}) + \int_\Omega (f(x, y) - f(x, y_h)) z_{\psi,h} \, dx + \\
&+ \int_\Omega \frac{f(x, y_h) - f(x, y)}{y - y_h} z_\psi (y - y_h) \, dx = \\
&= a(y - y_h, z_\psi - z_{\psi,h}) + \int_\Omega (f(x, y_h) - f(x, y))(z_\psi - z_{\psi,h}) \, dx \leq \\
&\leq C \|y - y_h\|_{H^1(\Omega)} \|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} + \int_\Omega |\phi(x)| \, |y - y_h| \, |z_\psi - z_{\psi,h}| \, dx \leq \\
&\leq C \left\{ \|y - y_h\|_{H^1(\Omega)} \|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} + \|\phi\|_{L^2(\Omega)} \|y - y_h\|_{H^1(\Omega)} \|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} \right\} \leq \\
&\leq C \|y - y_h\|_{H^1(\Omega)} \|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} \leq Ch \|y\|_{H^2(\Omega)} h \|z_\psi\|_{H^2(\Omega)} \leq \\
&\leq Ch^2 \|y\|_{H^2(\Omega)} \|\psi\|_{L^2(\Omega)},
\end{aligned}$$

where the last estimates follow from Lemma 8.2.6 and the usual estimates for finite elements. Thus

$$\|y - y_h\|_{L^2(\Omega)} = \sup_{\|\psi\|_{L^2(\Omega)} \leq 1} (\psi, y - y_h) \leq Ch^2 \|y\|_{H^2(\Omega)},$$

and the proof is complete. □

Finally, we have only to repeat the proof of Theorem 8.2.4 to obtain an identical result for the semilinear case:

Theorem 8.2.8 *Let y and y_h be solutions of the variational problems* (8.2.19) *and* (8.2.20) *respectively. Then there exists a constant $C > 0$, independent of h such that*

$$\|y - y_h\|_{L^\infty(\Omega_h)} \leq Ch^{2 - \frac{N}{2}} \|y\|_{H^2(\Omega)}.$$

Let us see now how we can obtain the same results with less restrictive conditions on the growing of f in the second variable.

Theorem 8.2.9 *Suppose that* (8.1.1) *holds and that* $f(x,0) \in L^2(\Omega)$. *Then the conclusions of Lemmas 8.2.6 and 8.2.7 and of Theorem 8.2.8 remain valid.*

Proof. Notice first that this condition also implies that for all $M > 0$ there exists $\varphi_M(x) = \phi_M(x)\,M + f(x,0) \in L^2(\Omega)$ such that $|f(x,t)| \leq \varphi_M(x)$ for every $|t| \leq M$, and thus we are in the same conditions as before with respect to the existence, uniqueness and regularity of the solution. We have that $y \in C(\bar{\Omega})$. Set $M = \|y\|_{L^\infty(\Omega)} + 1$ and

$$f_M(x,t) = \begin{cases} f(x,-M) & \text{if } t < -M \\ f(x,t) & \text{if } |t| \leq M \\ f(x,M) & \text{if } t > M. \end{cases}$$

We have that for all $x \in \Omega$, $f_M(x, y(x)) \equiv f(x, y(x))$. And therefore we have that

$$\begin{cases} Ay &= f_M(x,y) \text{ in } \Omega \\ y &= 0 \text{ on } \Gamma. \end{cases}$$

Take y_h^M the solution of the discrete variational problem

$$\begin{array}{l} \text{Find } y_h^M \in V_h \text{ such that} \\ a(y_h^M, z_h) = (f_M(x, y_h^M), z_h)\ \forall z_h \in V_h. \end{array}$$

From Theorem 8.2.8 we have that

$$\|y - y_h^M\|_{L^\infty(\Omega_h))} \leq C h^{2-\frac{N}{2}} \|y\|_{H^2(\Omega)},$$

therefore for all h less than a certain h_0 we have that $\|y - y_h^M\|_{L^\infty(\Omega_h))} \leq 1$, and then $\|y_h^M\|_{L^\infty(\Omega_h)} \leq \|y\|_{L^\infty(\Omega)} + 1 = M$, which implies that $f_M(x, y_h^M) = f(x, y_h^M)$ and consecuentemente y_h^M is the solution of the problem (8.2.20) and the desired estimates hold.
□

Case $y \in W_0^{1,p}(\Omega)$, $p > N$

Suppose now that we are in the extreme case: $f(\cdot,0) \in L^{p/2}(\Omega)$, $f_2 \in W^{-1,p}(\Omega)$ and the local Lipschitz condition (8.1.1) holds. As before, we will start supposing that the global condition (8.2.17) holds. In this case, with Stampacchia's truncature method and using the regularity results (2.1.1) for a C^1 boundary and (2.1.2) in the general case (remember that a convex domain is always Lipschitz), we can assure that $y \in W_0^{1,p}(\Omega)$ for $p > N$, p close to N, supposing the coefficients $a_{i,j} \in C(\bar{\Omega})$.

Using the convergence of the finite element method in the norm of $H^1(\Omega)$ we can prove the uniform convergence for $N = 2$. To achieve the same result for $N = 3$ we must use triangulations of non negative type, as it is done in Ciarlet y Raviart [42] for the linear case. In the last case, it is only necessary that the coefficients $a_{i,j}$ are in $L^\infty(\Omega)$ (supposing we know the $W^{1,p}(\Omega)$-regularity of the solution, because, as we have seen, this assumption is not enough to prove this regularity for y). Let us state first four lemmas.

Lemma 8.2.10 *For all $y \in W^{1,p}(\Omega)$, $p > N$*

$$\lim_{h\to 0} \|y - \Pi_h y\|_{W^{1,p}(\Omega)} = 0.$$

Proof. Due to Lemma 8.2.3, Π_h is continuous on $W^{1,p}(\Omega)$ with norm bounded independently of h: Indeed let us take $y \in W^{1,p}(\Omega)$. Then,

$$\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)} \le C\|y\|_{W^{1,p}(\Omega)}$$

and therefore

$$\|\Pi_h y\|_{W^{1,p}(\Omega)} = \|\Pi_h y\|_{W^{1,p}(\Omega_h)} \le (1 + C)\|y\|_{W^{1,p}(\Omega)}.$$

Take $y \in W^{2,p}(\Omega)$. Also directly from Lemma 8.2.3 we have that

$$\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)} \le Ch\|y\|_{W^{2,p}(\Omega)}. \tag{8.2.24}$$

The result follows by a demsity argument: Take $y \in W^{1,p}(\Omega)$, $N < p < \infty$. From the density of $W^{2,p}(\Omega)$ in $W^{1,p}(\Omega)$ we have that, given a $\varepsilon > 0$, there exists $y_\varepsilon \in W^{2,p}(\Omega)$ such that $\|y - y_\varepsilon\|_{W^{1,p}(\Omega_h)} \le \|y - y_\varepsilon\|_{W^{1,p}(\Omega)} \le \frac{1}{3(1+C)}\varepsilon \le \frac{1}{3}\varepsilon$. Due to the continuity of Π_h shown above, we also have that $\|\Pi_h y - \Pi_h y_\varepsilon\|_{W^{1,p}(\Omega_h)} \le \frac{1}{3}\varepsilon$. From (8.2.24) we deduce the existence $h_0 > 0$, depending on ε, such that for all $h \le h_0$, $\|y_\varepsilon - \Pi_h y_\varepsilon\|_{W^{1,p}(\Omega_h)} \le \frac{1}{3}\varepsilon$. And the result follows from the triangular inequality:

$$\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)} \le \|y - y_\varepsilon\|_{W^{1,p}(\Omega_h)} + \|y_\varepsilon - \Pi_h y_\varepsilon\|_{W^{1,p}(\Omega_h)} + \|\Pi_h y - \Pi_h y_\varepsilon\|_{W^{1,p}(\Omega_h)} \le$$
$$\le \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon.$$

And therefore the limit is zero. To complete the proof, we just have to observe that, since $|\Omega \setminus \Omega_h| \to 0$,

$$\|y - \Pi_h y\|_{W^{1,p}(\Omega\setminus\Omega_h)} = \|y\|_{W^{1,p}(\Omega\setminus\Omega_h)} \to 0.$$

The proof is complete. □

Lemma 8.2.11 *Let y and y_h be respectively the solution of equations* (8.2.1) *and* (8.2.2). *Then*

$$\lim_{h\to 0} \|y - y_h\|_{H^1(\Omega)} = 0.$$

Proof. Due to Cea's Lemma 8.2.5, the previous result and the embedding $W^{1,p}(\Omega) \subset H^1(\Omega)$, we have that

$$\lim_{h\to 0} \|y - y_h\|_{H^1(\Omega)} \leq \lim_{h\to 0} C\|y - \Pi_h y\|_{H^1(\Omega)} = 0.$$

□

Remark 8.2.1 *For the previous result it is only needed continuous coefficients, or even only bounded, supposing we know the regularity $W^{1,p}(\Omega)$ of the solution.*

A convergence result in $L^2(\Omega)$ can also be proved.

Lemma 8.2.12 *Suppose that the coefficients $a_{i,j} \in C^{0,1}(\bar{\Omega})$, and let y and y_h respectively the solutions of equations* (8.2.1) *and* (8.2.2). *Then*

$$\lim_{h\to 0} \frac{\|y - y_h\|_{L^2(\Omega)}}{h} = 0.$$

Proof. Take $\psi \in L^2(\Omega)$. Following exactly the proof of Lemma 8.2.7 we obtain

$$(\psi, y - y_h) \leq C\|y - y_h\|_{H^1(\Omega)}\|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} \leq Ch\|y - y_h\|_{H^1(\Omega)}\|\psi\|_{L^2(\Omega)}.$$

So

$$\frac{1}{h}\|y - y_h\|_{L^2(\Omega)} \leq C\|y - y_h\|_{H^1(\Omega)}$$

and applying Lemma 8.2.11 we obtain the desired limit. □

Lemma 8.2.13 *Let $y \in W^{1,p}(\Omega)$ with $p > N$. Then*

$$\lim_{h\to 0} \frac{\|y - \Pi_h y\|_{L^p(\Omega_h)}}{h} = 0.$$

Proof. For the proof we take advantage of $\Pi_h y \in W^{1,p}(\Omega_h)$, we use the interpolation lemma 8.2.3 and obtain that

$$\|y - \Pi_h y\|_{L^p(\Omega_h)} = \|y - \Pi_h y - \Pi_h(y - \Pi_h y)\|_{L^p(\Omega_h)} \leq Ch\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)}$$

and the result follows dividing by h and applying Lemma 8.2.10. □

Now we can prove uniform convergence, at least in dimension 2.

Theorem 8.2.14 *Suppose $N = 2$ and the coefficients $a_{i,j} \in C^{0,1}(\bar{\Omega})$. Let y and y_h be respectively the solution of equations* (8.2.1) *and* (8.2.2)*. Then*

$$\lim_{h\to 0} \|y - y_h\|_{L^\infty(\Omega)} = 0.$$

Proof. If we apply the triangular inequality, Lemma 8.2.3, the inequality (8.2.11) of equivalence between two Sobolev norms in a finite dimensional space, and that $N = 2$ we obtain that

$$\begin{aligned}
\|y - y_h\|_{L^\infty(\Omega_h)} &\leq \|y - \Pi_h y\|_{L^\infty(\Omega_h)} + \|\Pi_h y - y_h\|_{L^\infty(\Omega_h)} \leq \\
&\leq C\left[h^{1-\frac{N}{p}}\|y\|_{W^{1,p}(\Omega)} + h^{-\frac{N}{2}}\|\Pi_h y - y_h\|_{L^2(\Omega_h)}\right] \leq \\
&\leq C\left[h^{1-\frac{N}{p}}\|y\|_{W^{1,p}(\Omega)} + \frac{\|\Pi_h y - y\|_{L^2(\Omega_h)}}{h} + \frac{\|y - y_h\|_{L^2(\Omega_h)}}{h}\right].
\end{aligned}$$

Since $p > N$, Lemma 8.2.12 and the continuous embedding $L^p(\Omega) \in H^2(\Omega)$ this quantity converges to zero.

Notice that since $y \in C(\bar{\Omega})$, $\|y\|_{L^\infty(\Omega\setminus\Omega_h)}$ tends to zero when h decreases, so the proof is complete. □

To give a result in dimension 3 or simply for continuous coefficients, we must make two extra assumptions:

(H1) Function ϕ given in (8.2.17) belongs to an space $L^r(\Omega)$ with $r > 2$

(H2) The triangulation is of non negative type:

Denote b_i, $1 \leq i \leq n$ y b_i, $n \leq i \leq n+m$ the vertexes of $\mathcal{T}_h$ that belong to Ω and to Γ respectively, and set w_i, $1 \leq i \leq n+m$ the functions of W_h satisfying

$$w_i(b_j) = \delta_{ij}, \quad 1 \leq i, j \leq n+m,$$

i.e., the functions w_i, $1 \le i \le n$ or w_i, $1 \le i \le n+m$, form a basis of V_h or W_h. Set $\tilde{a}_{ij} = a(w_j, w_i)$, $1 \le i \le n$, $1 \le j \le n+m$. We will say that the discrete problem (8.2.20) is **of non negative type** (or that the triangulation $\mathcal{T}_h$ is of non negative type) if the matrix $\tilde{A} = (\tilde{a}_{ij})$ is irreducibly diagonally dominant and the relations

$$\begin{aligned} &\tilde{a}_{ij} \le 0 \quad \text{for } i \ne j,\ 1 \le i \le n,\ 1 \le j \le n+m, \\ &\sum_{j=1}^{n+m} \tilde{a}_{ij} \ge 0 \quad 1 \le i \le n \end{aligned}$$

hold.

Following Ciarlet [43, Theorem 21.4], we have that for $p > N$, taking $a_{i,j} \in L^\infty(\Omega)$, if y_h is the solution of the discrete problem

$$a(y_h, z_h) = \langle g, z_h \rangle \quad \text{for all } z_h \in V_h,$$

with $g \in W^{-1,p}(\Omega_h)$, then the discrete maximum principle holds:

$$\|y_h\|_{L^\infty(\Omega_h)} \le C\|g\|_{W^{-1,p}(\Omega_h)} \tag{8.2.25}$$

for discretizations of non negative type.

Using this principle we have:

Theorem 8.2.15 *Suppose that the coefficients $a_{i,j} \in L^\infty(\Omega)$, and let y be y_h respectively the solution of the equations* (8.2.1) *and* (8.2.2). *Then, if the triangulation is of non negative type,*

$$\|y - y_h\|_{L^\infty(\Omega_h)} \le Ch\|y\|_{W^{2,p}(\Omega)} \text{ if } y \in W^{2,p}(\Omega),\ p > 2N \tag{8.2.26}$$

and

$$\lim_{h\to 0} \|y - y_h\|_{L^\infty(\Omega)} = 0 \text{ if } y \in W^{1,p}(\Omega),\ p > N. \tag{8.2.27}$$

Proof. Notice first that in order to have the solution in $W^{1,p}(\Omega)$ it is sufficient that the coefficients $a_{i,j} \in C(\bar{\Omega})$ and in $W^{2,p}(\Omega)$ it is sufficient that the coefficients are in $C^{0,1}(\bar{\Omega})$ and that $f(\cdot, y)$ and f_2 are in $L^p(\Omega)$. Let $y \in W_0^{1,p}(\Omega)$ and $y_h \in V_h$ be solutions of the problems (8.2.19) and (8.2.20) respectively (variational formulation for (8.2.1) and a short writing for (8.2.2) respectively). We have that $y_h - \Pi_h y$ is the unique element of V_h that satisfies

$$a(y_h - \Pi_h y, z_h) = a(y - \Pi_h y, z_h) + (f(x, y_h) - f(x, y), z_h)\ \forall z_h \in V_h. \tag{8.2.28}$$

Let us study the norm of the operator

$$T : W_0^{1,p'}(\Omega_h) \longrightarrow \mathbb{R}$$

that relates every $z \in W_0^{1,p'}(\Omega_h)$ to $Tz = a(y - \Pi_h y, z) + (f(x, y_h) - f(x, y), z)$.

Due to Hölder's inequality, we know that

$$a(y - \Pi_h y, z) \leq C\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)}\|z\|_{W_0^{1,p'}(\Omega_h)} \quad \forall z \in W_0^{1,p'}(\Omega_h),$$

where p' is the conjugate exponent of p. We have that $W^{1,p}(\Omega_h) \hookrightarrow H^1(\Omega_h) \hookrightarrow L^6(\Omega_h)$. If we also have that $p \leq 3 + \varepsilon$, with ε small enough, then $W^{1,p'}(\Omega_h) \hookrightarrow L^s(\Omega_h)$, with $s < 3$, as close to 3 as we precise. So s can be chosen in such a way that

$$\frac{1}{r} + \frac{1}{6} + \frac{1}{s} = 1.$$

So, using Hölder's inequality and Cèa's generalized lemma (Lemma 8.2.5),

$$\begin{aligned}
\left|\int_{\Omega_h} (f(x, y_h) - f(x, y))z\,dx\right| &\leq \int_{\Omega_h} |\phi(x)|\,|y - y_h|\,|z|\,dx &\leq \\
&\leq \|\phi\|_{L^r(\Omega)}\|y - y_h\|_{L^6(\Omega_h)}\|z\|_{L^s(\Omega_h)} &\leq \\
&\leq C\|y - y_h\|_{H^1(\Omega_h)}\|z\|_{W^{1,p'}(\Omega_h)} &\leq \\
&\leq C\|y - \Pi_h y\|_{H^1(\Omega_h)}\|z\|_{W^{1,p'}(\Omega_h)} &\leq \\
&\leq C\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)}\|z\|_{W^{1,p'}(\Omega_h)}.
\end{aligned}$$

Therefore

$$\|T\|_{W^{-1,p}(\Omega_h)} \leq C\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)}$$

But, applying maximum principle (8.2.25) if $3 < p \leq 3 + \varepsilon$ to equation (8.2.28) we have that there exists a constant $C > 0$ independent of h such that

$$\|y_h - \Pi_h y\|_{L^\infty(\Omega_h)} \leq C\|T\|_{W^{-1,p}(\Omega_h)} \leq C\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)},$$

and using that $W^{1,p}(\Omega_h) \hookrightarrow L^\infty(\Omega_h)$, we get to:

$$\begin{aligned}
\|y - y_h\|_{L^\infty(\Omega_h)} &\leq \|y - \Pi_h y\|_{L^\infty(\Omega_h)} + \|y_h - \Pi_h y\|_{L^\infty(\Omega_h)} \leq \\
&\leq C\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)}.
\end{aligned} \tag{8.2.29}$$

If $y \in W^{2,p}(\Omega)$, applying Lemma 8.2.3 we have that

$$\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)} \leq Ch\|y\|_{W^{2,p}(\Omega)}, \tag{8.2.30}$$

and we can deduce (8.2.26). If $p > 3+\varepsilon$ the result follows from the continuous inclusion $W^{2,p}(\Omega) \hookrightarrow W^{2,3+\varepsilon}(\Omega)$.

The limit

$$\lim_{h\to 0} \|y - y_h\|_{L^\infty(\Omega_h)} = 0 \text{ if } y \in W^{1,p}(\Omega)$$

follows from (8.2.29) and Lemma 8.2.10 if $p < \infty$. If $y \in W^{1,\infty}(\Omega)$ we just have to notice that it is also in $W^{1,p}(\Omega)$ for all $p < \infty$.

To proof (8.2.27) we just make the same than at the end of the previous proof: since $y \in L^\infty(\Omega)$, $\|y\|_{L^\infty(\Omega\setminus\Omega_h)}$ tends to zero when h decreases. □

8.3 Neumann case

We will suppose for Neumann's problem that Γ is polygonal or polyhedrical. In this case $\Omega_h = \Omega$. Consider now $a_0 \in L^{\frac{Np}{N+p}}(\Omega)$, $a_0 \geq 0$, $a_0 \not\equiv 0$ in Ω, $f_2 \in \left(W^{1,p'}(\Omega)\right)'$ and $v \in L^\infty(\Gamma)$. We want to study the uniform approximation by the finite element method of the solution of the equation

$$\left\{\begin{array}{rcll} Ay + a_0 y &=& f(\cdot, y) + f_2 & \text{in } \Omega \\ \partial_{n_A} y &=& v & \text{on } \Gamma. \end{array}\right. \tag{8.3.1}$$

For each h, let us define $y_h \in W_h$ as the unique element that satisfies

$$\begin{gathered}\sum_{i,j=1}^{N} \int_\Omega a_{i,j}(x)\partial_{x_i} y_h(x)\partial_{x_j} z_h(x)dx + \int_\Omega a_0(x) y_h(x) z_h(x) dx = \\ \int_\Omega f(x, y_h(x)) z_h dx + \langle f_2, z_h\rangle_{(W^{1,p'}(\Omega))' \times W^{1,p'}(\Omega)} + \int_\Gamma v(s) z_h(s) ds \quad \forall z_h \in W_h.\end{gathered} \tag{8.3.2}$$

Lemma 8.3.1 *Equation* (8.3.2) *has a unique solution.*

Proof. The proof is identical to the one made for equation (8.2.2). □

Our objective is to show that $y_h \to y$ in $L^\infty(\Omega)$. We will get advantage of these results in next chapter to study a control problem, where v will stand for the control. Generally $v \notin H^{\frac{1}{2}}(\Gamma)$ and therefore it is nonsense to study the regular case. With Stampacchia's truncature method [84] and a regularity Theorem due to Dauge [47], we can prove, as in Theorem 3.1.1 that $y \in W^{1,p}(\Omega)$.

We will start with the well known convergence result for the finite elements method

Lemma 8.3.2 *Let y and y_h be respectively the solutions of the equations* (8.3.1) *and* (8.3.2). *Then*

$$\lim_{h\to 0} \|y - y_h\|_{H^1(\Omega)} = 0.$$

We also have a result in $L^2(\Omega)$.

Lemma 8.3.3 *Suppose that the coefficients $a_{i,j} \in C^{0,1}(\bar{\Omega})$, and let y and y_h respectively the solutions of equations* (8.3.1) *and* (8.3.2). *Then*

$$\lim_{h\to 0} \frac{\|y - y_h\|_{L^2(\Omega)}}{h} = 0.$$

Proof. For every $\psi \in L^2(\Omega)$ there exists a unique $z_\psi \in H^2(\Omega)$ satisfying

$$\begin{cases} A^* z_\psi + a_0 z_\psi + \alpha(x) z_\psi &= \psi \quad \text{in } \Omega \\ \partial_{n_{A^*}} z_\psi &= 0 \quad \text{on } \Gamma, \end{cases} \tag{8.3.3}$$

with $\alpha(x)$ defined as in (8.2.21). Since $\|\alpha\|_{L^2(\Omega)} \leq \|\phi\|_{L^2(\Omega)}$, there exists a constant $C > 0$ that does not depend neither on h nor on α such that $\|z_\psi\|_{H^2(\Omega)} \leq C\|\psi\|_{L^2(\Omega)}$.

Now we can continue as in the proof of Lemma 8.2.12, and apply the previous lemma.
□

Now we can proof, exactly in the same way than in Theorem 8.2.14 the uniform convergence, at least in dimension 2.

Theorem 8.3.4 *Suppose that $N = 2$ and the coefficients $a_{i,j} \in C^{0,1}(\bar{\Omega})$. Let y and y_h be respectively the solutions of the equations* (8.3.1) *and* (8.3.2). *Then*

$$\lim_{h\to 0} \|y - y_h\|_{L^\infty(\Omega)} = 0.$$

To prove a result about uniform convergence for $N = 3$ or simply for continuous coefficients, we must suppose again that $\phi \in L^r(\Omega)$, $r > 2$ and that the triangulation is of non negative type. For Neumann's problem, we define a triangulation of non negative type as follows. Denote b_i, $1 \leq i \leq n+m$ the vertexes of $\mathcal{T}_h$ that belong to $\bar{\Omega}$ and set w_i, $1 \leq i \leq n+m$ the functions of W_h satisfying

$$w_i(b_j) = \delta_{ij}, \quad 1 \leq i, j \leq n+m,$$

i.e., the functions w_i, $1 \leq i \leq n+m$, form a basis of W_h. Set $\tilde{a}_{ij} = a(w_j, w_i)$, $1 \leq i \leq n+m$, $1 \leq j \leq n+m$. We will say that the discrete problem (8.3.2) is **of non negative**

type (or that the triangulation $\mathcal{T}_h$ is of non negative type) if the matrix $\hat{A} = (\tilde{a}_{ij})$ is irreducibly diagonally dominant and the relations

$$\tilde{a}_{ij} \leq 0 \quad \text{for } i \neq j,\ 1 \leq i \leq n+m,\ 1 \leq j \leq n+m,$$
$$\sum_{j=1}^{n+m} \tilde{a}_{ij} \geq 0 \quad 1 \leq i \leq n+m$$

hold. In this case, the discrete maximum principles is satisfied. If $y_h \in W_h$ is the solution of the discrete problem

$$a(y_h, z_h) = \langle g, z_h \rangle_{(W^{1,p'}(\Omega))' \times W^{1,p'}(\Omega)} + \langle v, \gamma z_h \rangle_{W^{-\frac{1}{p},p}(\Gamma) \times W^{\frac{1}{p},p'}(\Gamma)} \quad \text{for all } z_h \in W_h,$$

with $g \in \left(W^{1,p'}(\Omega)\right)'$ then the discrete maximum principle holds:

$$\|y_h\|_{L^\infty(\Omega)} \leq C\left(\|g\|_{(W^{1,p'}(\Omega))'} + \|v\|_{W^{-\frac{1}{p},p}(\Gamma)}\right). \tag{8.3.4}$$

Theorem 8.3.5 *Suppose that the coefficients $a_{i,j} \in L^\infty(\Omega)$, and let y and y_h be respectively the solutions of the equations* (8.3.1) *and* (8.3.2). *Then, if the triangulation is of non negative type,*

$$\|y - y_h\|_{L^\infty(\Omega_h)} \leq Ch\|y\|_{W^{2,p}(\Omega)} \text{ if } y \in W^{2,p}(\Omega),\ p > 2N \tag{8.3.5}$$

and

$$\lim_{h\to 0} \|y - y_h\|_{L^\infty(\Omega)} = 0 \text{ if } y \in W^{1,p}(\Omega),\ p > N. \tag{8.3.6}$$

Proof. The proof is identical to that of Dirichlet's case. □

Chapter 9

Convergence of the F.E.M. for control problems

This chapter is dedicated to the study of the discretizations of a control problem. In the first section we study a distributed problem governed by a semilinear equation with Dirichlet boundary conditions and in the second section a boundary control governed by an equation with Neumann boundary conditions.

9.1 Dirichlet case

Consider the sets, operators and spaces described in Section 8.1.

Let K a convex, weakly-* closed, bounded and non empty subset of $L^\infty(\Omega)$; $p > N$;$f(\cdot,y) = f_1(\cdot,y) + f_2(\cdot)$, where $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function, monotone decreasing in the second variable, with $f_1(\cdot,0) \in L^{p/2}(\Omega)$ and satisfying the local Lipschitz condition (8.1.1) and $f_2 \in W^{-1,p}(\Omega)$; $L : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ a Carathéodory function, convex in the third variable and that satisfies that for all $M > 0$ there exists $\psi_M \in L^1(\Omega)$ such that $|L(x,y,u)| \leq \psi_M(x)$ for a.e. $x \in \Omega$, for all $|y|$, $|u| \leq M$. Set $g : \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function. Let us formulate the optimal control problem

$$(P_\delta) \begin{cases} \min J(u) = \displaystyle\int_\Omega L\left(x, y_u(x), u(x)\right) dx \\ u \in K \quad g\left(x, y_u(x)\right) \leq \delta \quad \forall x \in \bar{\Omega}, \end{cases} \tag{9.1.1}$$

where

$$\left\{\begin{array}{rcll} Ay_u & = & f(x,y_u)+u & \text{in } \Omega \\ y_u & = & 0 & \text{on } \Gamma. \end{array}\right. \tag{9.1.2}$$

Aplying the same techniques than in Theorem 3.1.1 we have the following results.

Theorem 9.1.1 *For every $u \in K$ there exists a unique $y_u \in W_0^{1,p}(\Omega)$ solution of (9.1.2). Moreover, there exists a constant C_K, which only depends on a bound for K, such that $\|y_u\|_{W^{1,p}(\Omega)} \leq C_K$ for all $u \in K$. Finally, if $u_j \to u$ weakly-* in $L^\infty(\Omega)$ then $y_{u_j} \to y_u$ strongly in $W^{1,p}(\Omega)$.*

Theorem 9.1.2 *If $f_1(\cdot,0) \in L^2(\Omega)$ and $f_2 \equiv 0$, then for every $u \in K$ there exists a unique $y_u \in H^2(\Omega) \cap H_0^1(\Omega)$ solution of (9.1.2). Moreover, there exists a constant C_K, which only depends on a bound for K, such that $\|y_u\|_{H^2(\Omega)} \leq C_K$ for every $u \in K$. Finally, if $u_j \to u$ weakly-* in $L^\infty(\Omega)$ then $y_{u_j} \to y_u$ strongly in $H^2(\Omega)$.*

The following result appears in Casas [18].

Theorem 9.1.3 *There exists a number $\delta_0 \in \mathbb{R}$ such that problem (P_δ) has at least one solution for every $\delta \geq \delta_0$, and (P_δ) has no admissible controls for $\delta < \delta_0$.*

Proof. From the regularity results and taking into account that K is bounded in $L^\infty(\Omega)$ we deduce that there exists a constant C such that $\|y_u\|_{L^\infty(\Omega)} \leq C$ for every $u \in K$. Let M and m be the respectively the supremum and the infimum of g in $\bar{\Omega} \times [-C, C]$. Then it is obvious that (P_δ) does not have admissible contols for $\delta < m$ and all the elements of K are admissible controls for $\delta \geq M$. Let δ_0 be the infimum of the values δ for which (P_δ) has admissible controls. Then $m \leq \delta_0 \leq M$ and (P_δ) has not admissible controls for $\delta < \delta_0$. Let us prove that there exists at least and admissible control for (P_{δ_0}). Let $\{\delta_j\}$ be a decreasing sequence converging to δ_0 and $\{u_j\} \subset K$ a sequence of controls such that every $\{u_j\}$ is admissible for (P_{δ_j}). Since K is bounded, we can take subsequence, which will dented in the same way, weakly-* convergent in $L^\infty(\Omega)$ to an element $u_0 \in K$. Due to the continuity result, we have that the states $\{y_{u_j}\}$ converge uniformly to y_{u_0} and hence

$$g(x, y_{u_0}(x)) = \lim_{j\to\infty} g(x, y_{u_j}(x)) \leq \lim_{j\to\infty} \delta_j = \delta_0 \text{ for all } x \in \bar{\Omega}.$$

Therefor u_0 is an admissible control for (P_{δ_0}).

To conclude the proof, we must establish the existence of an optimal control for every $\delta \geq \delta_0$. Let $\{u_k\} \subset K$ be a minimizing sequence for (P_δ), this is $J(u_k) \to \inf(P_\delta)$. We can take a subsequence, denoted again in the same way, which converges weakly-* in $L^\infty(\Omega)$ to an element $\bar{u} \in K$. Using an reasoning similar to the one in the previous paragraph, we can check that $g(x, y_{\bar{u}}(x)) \leq \delta$ for every $x \in \bar{\Omega}$. So $\bar{u}$ is an admissible control for problem (P_δ). Let us check that $J(\bar{u}) = \inf(P_\delta)$. To do that we use Mazur's Theorem (see, for instance, Ekeland and Temam [51]): given $1 < p < \infty$ there exists a sequence of convex combinations $\{v_k\}_{k\in\mathbb{N}}$,

$$v_k = \sum_{j=k}^{n(k)} \lambda_{k,j} u_j, \text{ con } \sum_{j=k}^{n(k)} \lambda_{k,j} = 1 \text{ y } \lambda_{k,j} \geq 0,$$

such that $v_k \to \bar{u}$ strongly in $L^p(\Omega)$. Then, using the convexity of L with respect tothe third variable, the dominated convergence theorem and that L is dominated by a function of $L^1(\Omega)$, we get

$$J(\bar{u}) = \lim_{k\to\infty} \int_\Omega L(x, y_{\bar{u}}(x), v_k(x))dx \leq \limsup_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} \int_\Omega L(x, y_{\bar{u}}(x), u_j(x))dx \leq$$

$$\limsup_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} J(u_j) + \limsup_{k\to\infty} \int_\Omega \sum_{j=k}^{n(k)} \lambda_{k,j} |L(x, y_{u_j}(x), u_j(x)) - L(x, y_{\bar{u}}(x), u_j(x))| dx =$$

$$\inf(P_\delta) + \limsup_{k\to\infty} \int_\Omega \sum_{j=k}^{n(k)} \lambda_{k,j} |L(x, y_{u_j}(x), u_j(x)) - L(x, y_{\bar{u}}(x), u_j(x))| dx,$$

where we have used the convergence $J(u_k) \to \inf(P_\delta)$. To check that the second summand of the previous expression tends to zero, we just have to notice that for every fixed x, the function $L(x, \cdot, \cdot)$ is uniformly continuous on bounded sets of $\mathbb{R}^2$, that the sequences $\{y_{u_j}(x)\}$ and $\{u_j(x)\}$ are uniformly bounded an that $y_{u_j}(x) \to y_{\bar{u}}(x)$ when $j \to \infty$. Therefore

$$\lim_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} |L(x, y_{u_j}(x), u_j(x)) - L(x, y_{\bar{u}}(x), u_j(x))| = 0 \text{ for a.e. } x \in \Omega.$$

Using again the dominated convergence theorem we deduce that

$$\limsup_{k\to\infty} \int_\Omega \sum_{j=k}^{n(k)} \lambda_{k,j} |L(x, y_{u_j}(x), u_j(x)) - L(x, y_{\bar{u}}(x), u_j(x))| dx = 0,$$

and the rpoof is complete. □

In this section our aim is to study the convergence of the discretizations of this problem. For the study of the convergence of the control problem, it is necessary to study the state equation. In this case, since we have pointwise constraints, we must establish the uniform convergence of the approximations of the state.

Let us consider the space

$$U_h = \{u_h \in L^\infty(\Omega) : u_{h|_T} \in P_0(T) \quad \forall T \in \mathcal{T}_h\}.$$

For all $u_h \in U_h$ we will denote by $y_h(u_h)$ the unique element in V_h that satisfies

$$\sum_{i,j=1}^{N} \int_\Omega a_{i,j}(x)\partial_{x_i} y_h(x)\partial_{x_j} z_h(x)dx = \int_\Omega (f(x, y_h(x)) + u_h)z_h dx \quad \forall z_h \in V_h, \tag{9.1.3}$$

where we understand that $\int_\Omega f_2 z_h\, dx$ denotes $\langle f_2, z_h\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p}(\Omega)}$.

For every $h > 0$ we take K_h a convex, closed, bounded and non empty subset of U_h in such a way that $\{K_h\}$ constitutes an internal approximation of K in the following sense

1. For all $u \in K$ there exists $u_h \in K_h$ with $u_h \to u$ in $L^1(\Omega)$.
2. If $u_h \in K_h$ and $u_h \to u$ weakly-* in $L^\infty(\Omega)$, then $u \in K$.
3. The $\{K_h\}$ are uniformly bounded in $L^\infty(\Omega)$.

Let us formulate the following finite dimensional problem.

$$(P_{\delta h}) \begin{cases} \min J_h(u_h) = \displaystyle\int_\Omega L\,(x, y_h(u_h)(x), u_h(x))\, dx \\ u_h \in K_h \quad g\,(x_j, y_h(u_h)(x_j)) \le \delta \quad \forall j \in I_h, \end{cases} \tag{9.1.4}$$

where $\{x_j\}_{j=1}^{n(h)}$ is the set of vertexes of $\mathcal{T}_h$, I_h is the set of indexes corresponding to the interior vertexes.

It is the purpose of this chapter to show that the solutions of the discrete problems converge to the solution of the continuous problem. To do that, it is necessary to prove the fact that if $u_h \to u$ weakly-* in $L^\infty(\Omega)$, then $y_h(u_h) \to y_u$ uniformly in Ω.

Observe that we are not exactly in the case of the previous chapter, because what we proved there is that $y_h = y_h(u)$ converges to y_u.

The technique to prove this is different depending on whether we have a regular state or not, or if the triangulation is of non negative type or not. We are going to state different theorems, in which we can see that, under different assumptions each time, we can achieve the desired conclusion.

Theorem 9.1.4 *Suppose now that $a_{i,j} \in C^{0,1}(\bar{\Omega})$ and that $f_2 \equiv 0$. Moreover, we will suppose that for all $M > 0$ there exists a function $\phi_M \in L^2(\Omega)$ in such a way that the local Lipschitz condition* (8.1.1) *holds. Suppose also that $f_1(\cdot, 0) \in L^2(\Omega)$. For all $h > 0$ set $u_h \in K_h$, so that $u_h \to u$ weakly-* in $L^\infty(\Omega)$. Then*

$$\lim_{h \to 0} \|y_h(u_h) - y_u\|_{L^\infty(\Omega)} = 0. \tag{9.1.5}$$

Proof. The assumptions made assure us that the state is regular enough. Observe that, since the K_h are unformly bounded in $L^\infty(\Omega)$ (assumption 3 on the K_h, page 196), there exists a constant C such that

$$\|y_{u_h}\|_{H^2(\Omega)} \leq C \text{ for all } u_h \in K_h \text{ and for all } h > 0. \tag{9.1.6}$$

This is the classical case. We have error estimates. Let us write

$$\|y_h(u_h) - y_u\|_{L^\infty(\Omega)} \leq \|y_h(u_h) - y_{u_h}\|_{L^\infty(\Omega)} + \|y_{u_h} - y_u\|_{L^\infty(\Omega)}.$$

From Theorem 9.1.1 it follows that the second summand converges to zero.

For the first one, if we fix h, due to Theorem 8.2.8, we have that $\|y_h(u_h) - y_{u_h}\|_{L^\infty(\Omega)} \leq Ch^{2-\frac{N}{2}}\|y_{u_h}\|_{H^2(\Omega)}$. Due to this and to (9.1.6) we have that the first summand tends to zer, and the proof is complete. □

To prove analogous results in the case where the states are not regular enough, we are going to introduce the following result.

Lemma 9.1.5 *For all $h > 0$, all $u \in K$ and all $u_h \in K_h$ there exists $C > 0$ independent of h such that*

$$\|y_h(u_h) - y_h(u)\|_{H_0^1(\Omega)} \leq C\|u - u_h\|_{H^{-1}(\Omega)}.$$

Proof. From the monotonicity of f and the $H_0^1(\Omega)$ ellipticity of $a(\cdot,\cdot)$, we hace that

$$m\|y_h(u_h) - y_h(u)\|^2_{H_0^1(\Omega)} \leq a(y_h(u_h) - y_h(u), y_h(u_h) - y_h(u)) =$$

$$(f(x, y_h(u_h)) - f(x, y_h(u)), y_h(u_h) - y_h(u)) + (u - u_h, y_h(u_h) - y_h(u)) \leq$$

$$\leq (u - u_h, y_h(u_h) - y_h(u)) \leq \|u - u_h\|_{H^{-1}(\Omega)} \|y_h(u_h) - y_h(u)\|_{H_0^1(\Omega)}.$$

Therefore

$$\|y_h(u_h) - y_h(u)\|_{H_0^1(\Omega)} \leq \frac{1}{m} \|u - u_h\|_{H^{-1}(\Omega)}.$$

□

Theorem 9.1.6 *Suppose that the coefficients* $a_{i,j} \in C(\bar{\Omega})$, $f_1(\cdot, 0) \in L^{p/2}(\Omega)$, $f_2 \in W^{-1,p}(\Omega)$ *and for all* $M > 0$ *there exists a function* $\phi_M \in L^r(\Omega)$, $r > 2$ *in such a way that the local Lipschitz condition* (8.1.1). *Let us also suppose that triangulation is of non negative type. For all* $h > 0$ *set* $u_h \in K_h$, *such that* $u_h \to u$ *weakly-* in* $L^\infty(\Omega)$. *Then*

$$\lim_{h \to 0} \|y_h(u_h) - y_u\|_{L^\infty(\Omega)} = 0. \tag{9.1.7}$$

Proof. Now the adjoint state belongs to $W^{1,p}(\Omega)$ and we do not have error estimates, just a convergence result.

In this case, we may write

$$\|y_h(u_h) - y_u\|_{L^\infty(\Omega)} \leq \|y_h(u_h) - y_h(u)\|_{L^\infty(\Omega)} + \|y_h(u) - y_u\|_{L^\infty(\Omega)}.$$

The second summand converges to zero as a consequence of Theorem 8.2.15.

We know that $y_h(u_h) - y_h(u)$ solves the discrete problem

$$a(y_h(u_h) - y_h(u), z_h) = (f(x, y_h(u_h)) + u_h - f(x, y_h(u)) - u, z_h) \quad \forall z_h \in V_h.$$

In this case we can apply the discrete maximum principle (8.2.25), and we get

$$\|y_h(u_h) - y_h(u)\|_{L^\infty(\Omega)} \leq C \|f(x, y_h(u)) + u - f(x, y_h(u_h)) - u_h\|_{W^{-1,p}(\Omega)} \leq$$

$$\leq C \left(\|f(x, y_h(u)) - f(x, y_h(u_h))\|_{W^{-1,p}(\Omega)} + \|u - u_h\|_{W^{-1,p}(\Omega)} \right).$$

In the second summand, the weak-* convergence in $L^\infty(\Omega)$ of the u_h implies the strong convergence in $W^{-1,p}(\Omega)$.

On the other side

$$\|f(x, y_h(u)) - f(x, y_h(u_h))\|_{W^{-1,p}(\Omega)} \leq \|\phi\|_{L^r(\Omega)} \|y_h(u) - y_h(u_h)\|_{H_0^1(\Omega)}.$$

Due to Lemma 9.1.5

$$\|y_h(u_h) - y_h(u)\|_{H_0^1(\Omega)} \leq \frac{1}{m} \|u - u_h\|_{H^{-1}(\Omega)}.$$

The weak-* convergence of the u_h implies the strong convergence in $H^{-1}(\Omega)$. Therefore the states converge uniformly. □

We are going to state now four lemmas anlogous to Lemmas 8.2.10–8.2.13

Lemma 9.1.7 *For all $h > 0$ let $u_h \in K_h$, such that $u_h \to u$ weakly-* in $L^\infty(\Omega)$. Then*

$$\lim_{h\to 0} \|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega_h)} = 0.$$

Proof. We can bound $\|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega_h)}$ as

$$\|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega_h)} \le \|y_{u_h} - y_u\|_{W^{1,p}(\Omega_h)} + \|y_u - \Pi_h y_u\|_{W^{1,p}(\Omega_h)} + \|\Pi_h y_u - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega_h)}.$$

The first summand converges to zero due to Theorem 9.1.1. The second one due to Lemma 8.2.10. The third one, due to the continuity of Π_h (proved at the beginning of the proof of Lemma 8.2.10), can be bounded by a constant that multiplies $\|y_{u_h} - y_u\|_{W^{1,p}(\Omega_h)}$, which agains converges to zero. □

Lemma 9.1.8 *For all $h > 0$ let $u_h \in K_h$ be such that $u_h \to u$ weakly-* in $L^\infty(\Omega)$. Then*

$$\lim_{h\to 0} \|y_h(u_h) - y_{u_h}\|_{H^1(\Omega)} = 0.$$

Proof. We can bound $\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)}$ as

$$\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)} \le \|y_{u_h} - y_u)\|_{H^1(\Omega)} + \|y_u - y_h(u)\|_{H^1(\Omega)} + \|y_h(u) - y_h(u_h)\|_{H^1(\Omega)}.$$

The first summand converges to zero due to Theorem 9.1.1. The second one due to Lemma 8.2.11 and the third one, due to Lema 9.1.5, can be bounded by $\|u - u_h\|_{H^{-1}(\Omega)}$. Weak-* convergence of the u_h implies strong convergence in $H^{-1}(\Omega)$. □

Lemma 9.1.9 *Suppose $N = 2$, the coefficients $a_{i,j} \in C^{0,1}(\bar\Omega)$, $f(\cdot, 0) \in L^{p/2}(\Omega)$, $f_2 \in W^{-1,p}(\Omega)$ and for all $M > 0$ there exists a function $\phi_M \in L^2(\Omega)$ in such a way that the local Lipschitz condition* (8.1.1) *holds. For all $h > 0$ set $u_h \in K_h$, such that $u_h \to u$ weakly-* in $L^\infty(\Omega)$. Then*

$$\lim_{h\to 0} \frac{\|y_h(u_h) - y_{u_h}\|_{L^2(\Omega)}}{h} = 0.$$

Proof. Since $y_h(u_h)$ and y_{u_h} are the continuos and discrete states associated to the same control, following exactly the proof of Lemma 8.2.7 we obtain that for every $\psi \in L^2(\Omega)$

$$(\psi, y_{u_h} - y_h(u_h)) \leq Ch\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)}\|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} \leq Ch\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)}\|\psi\|_{L^2(\Omega)},$$

where let us remember that z_ψ is the solution of problem (8.2.7) introduced in page 177 and $z_{\psi h}$ is the solution of (8.2.10). So

$$\frac{1}{h}\|y_{u_h} - y_h(u_h)\|_{L^2(\Omega)} \leq C\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)}$$

and we can apply the previous lemma. The proof is complete. □

Lemma 9.1.10 *For every $h > 0$ let $u_h \in K_h$ be such that $u_h \to u$ weakly-* in $L^\infty(\Omega)$. Then*

$$\lim_{h\to 0} \frac{\|y_{u_h} - \Pi_h y_{u_h}\|_{L^2(\Omega_h)}}{h} = 0.$$

Proof. For the proof we use that $\Pi_h y_{u_h} \in W^{1,p}(\Omega_h)$, we use the interpolation lemma 8.2.3 and we obtain that

$$\|y_{u_h} - \Pi_h y_{u_h}\|_{L^p(\Omega_h)} = \|y_{u_h} - \Pi_h y_{u_h} - \Pi_h(y_{u_h} - \Pi_h y_{u_h})\|_{L^p(\Omega_h)} \leq Ch\|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega_h)}$$

and the result is obtained dividing by h and applying Lemma 9.1.7. □

Theorem 9.1.11 *Suppose $N = 2$, the coefficients $a_{i,j} \in C^{0,1}(\bar{\Omega})$, $f(\cdot, 0) \in L^{p/2}(\Omega)$, $f_2 \in W^{-1,p}(\Omega)$ and for all $M > 0$ there exists a function $\phi_M \in L^2(\Omega)$ in such a way that the local Lipschitz condition* (8.1.1) *holds. For all $h > 0$ set $u_h \in K_h$, such that $u_h \to u$ weakly-* in $L^\infty(\Omega)$. Then*

$$\lim_{h\to 0} \|y_h(u_h) - y_u\|_{L^\infty(\Omega)} = 0. \tag{9.1.8}$$

Proof. We have

$$\|y_h(u_h) - y_u\|_{L^\infty(\Omega_h)} \leq \|y_h(u_h) - y_{u_h}\|_{L^\infty(\Omega_h)} + \|y_{u_h} - y_u\|_{L^\infty(\Omega_h)}.$$

The second summand converges to zero due to Theorem 9.1.1. The first one can again be bounded by the triangular inequality with

$$\|y_h(u_h) - y_{u_h}\|_{L^\infty(\Omega_h)} \leq \|y_h(u_h) - \Pi_h y_{u_h}\|_{L^\infty(\Omega_h)} + \|\Pi_h y_{u_h} - y_{u_h}\|_{L^\infty(\Omega_h)}.$$

Due to Lemma (8.2.3), we can estimate the second summand:

$$\|\Pi_h y_{u_h} - y_{u_h}\|_{L^\infty(\Omega)} \leq Ch^{1-\frac{N}{p}}\|y_{u_h}\|_{W^{1,p}(\Omega)}.$$

Since the $\{u_h\{$ is uniformly bounded, due to Theorem 9.1.1 $\{y_{u_h}\}$ is also in bounded in $W^{1,p}(\Omega)$. So this second summand converges to zero. To estimate $\|y_h(u_h)-\Pi_h y_{u_h}\|_{L^\infty(\Omega_h)}$, let us take into account (8.2.11), which gives us the equivalence between two Sobolev norms in finite dimensional spaces and we obtain, taking into account that $N = 2$ and applying again the triangular inequality

$$\|y_h(u_h) - \Pi_h y_{u_h}\|_{L^\infty(\Omega_h)} \leq \frac{C}{h}\|y_h(u_h) - \Pi_h y_{u_h}\|_{L^2(\Omega_h)} \leq$$

$$C\left(\frac{\|y_h(u_h) - y_{u_h}\|_{L^2(\Omega)}}{h} + \frac{\|y_{u_h} - \Pi_h y_{u_h}\|_{L^2(\Omega_h)}}{h}\right).$$

Now we can apply Lemmas 9.1.9 and 9.1.10 and deduce that this quantity converges to zero. So we have proved that

$$\lim_{h\to 0}\|y_h(u) - y_u\|_{L^\infty(\Omega_h)} = 0.$$

Notice that since $y_u \in C(\bar{\Omega}) \cap H_0^1(\Omega)$, $\|y_u\|_{L^\infty(\Omega\setminus\Omega_h)}$ tends to zero when h decreases. The proof is complete. □

We are now ready to prove that the discrete optimal controls converge to the solution of the problem. One of the key assumptions to prove the convergence of the discretizations is the weak stability on the left.

Definition 9.1.1 *We will say that control problem (P_δ) is weakly stable on the left at δ if*

$$\lim_{\delta' \nearrow \delta} \inf(P_{\delta'}) = \inf(P_\delta).$$

Notice that weak stability on the right

$$\lim_{\delta' \searrow \delta} \inf(P_{\delta'}) = \inf(P_\delta) \tag{9.1.9}$$

is always true: Take u_δ a solution of (P_δ). Since K is bounded, we can deduce the existence of a sequence $\{\delta_j\}$ such that $\delta_j \searrow \delta$ when $j \to \infty$ and $\lim_{j\to\infty} u_{\delta_j} = \bar{u}$ weakly-* in $L^\infty(\Omega)$ for some $\bar{u} \in K$, being u_{δ_j} a solution of (P_{δ_j}). If y_j and $\bar{y}$ are the associated states to u_{δ_j} and $\bar{u}$ respectively, we have that $y_j \to \bar{y}$ uniformly in $\bar{\Omega}$. Therefore $\bar{u}$ is an admissible control for (P_δ). Now, using the convexity in the third variable of L and the admissibility of u_δ for each $(P_{\delta'})$, with $\delta' > \delta$, we obtain

$$\inf(P_\delta) \le J(\bar{u}) \le \liminf_{j\to\infty} J(u_{\delta_j}) = \lim_{\delta' \searrow \delta} \inf(P_{\delta'}) \le J(u_\delta) = \inf(P_\delta),$$

which proves (9.1.9).

Therefore, weak stability on the left assures us that $\inf(P_\delta)$ is a continuous function in δ.

There are problems not weakly stable on the left. Let us see two examples of problems not weakly stable on the left. The first one is in finite dimension and will help us to illustrate geometrically that the lack of weak stability on the left implies that the problem is ill posed numerically.

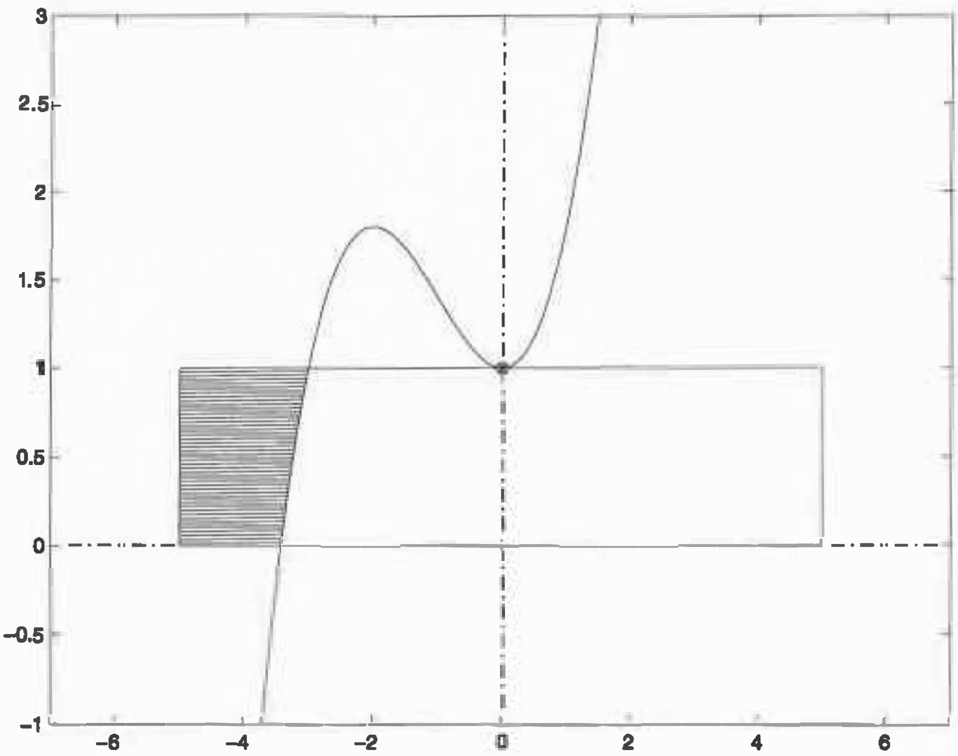
Example 9.1.1 *Consider the problem*

$$(Q_\delta)\begin{cases} \text{Minimize } x^2 + (y-1)^2 \\ -5 \le x \le 5 \\ 0 \le y \le 1 \\ \frac{1}{5}x^3 + \frac{3}{5}x^2 - y + 2 \le \delta. \end{cases}$$

Problem (Q_δ) is not weakly stable on the left for $\delta = 1$. In fact, $\inf(P_1) = 0$, reaching the solution at the point $(0,1)$. If we take $\delta' < 1$, then $1 \ge y > (1/5)x^3 + (3/5)x^2 + 1 = x^2((1/5)x + 3/5) + 1$, and therefore we have that $x + 3 < 0$, or what is the same $x < -3$. From here we deduce that

$$\lim_{\delta' \nearrow 1} \inf(P_{\delta'}) \ge 9 > \inf(P_1).$$

Observe that the problem id that for $\delta = 1$ the admissible region has an isolated point, and it is the point where the minimum is attained.



Next we introduce a control problem not weakly stable on the left.

Example 9.1.2 *Take $\Omega = B(0,1)$ in $\mathbb{R}^n$ and Γ its boundary. Given $u \in L^\infty(\Omega)$ consider the partial differential equation*

$$\begin{cases} -\Delta y_u = u & \text{in } \Omega \\ y_u = 0 & \text{on } \Gamma. \end{cases}$$

Set

$$z(x) = 2\left(1 - \|x\|^2\right).$$

it is clear that z satisfies the partial differential equation

$$\begin{cases} -\Delta z = 4n & \text{in } \Omega \\ z = 0 & \text{on } \Gamma, \end{cases}$$

and $z(0) = 2$.

Set

$$g(t) = \begin{cases} t & \text{if } t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}$$

Let us state the following control problem

$$(P_\delta)\begin{cases} \min J(u) = \displaystyle\int_\Omega (u-4n)^2\,dx \\ u \in L^\infty(\Omega)\ g(y_u) \leq \delta \text{ in } \Omega. \end{cases}$$

Let us see that our example is not weakly stable on the left for $\delta = 1$.

The solution to (P_δ) *is attained by taking* $u_1 = 4n$;*then* $y_{u_1} = z$, *and we have that* $g(y_{u_1}) \leq 1 \leq \delta$ *and* $J(u) = 0$.

Take $\delta' < 1$. *Let* $u_{\delta'}$ *and* $y_{\delta'} = y_{u_{\delta'}}$ *be such that they solve* $(P_{\delta'})$. *Necessarily* $y_{\delta'}(0) < 1$ *and therefore* $1 \leq \|y_{\delta'} - z\|_{L^\infty(\Omega)}$ *since both* $y_{\delta'}(x)$ *and* $z(x)$ *are continuous functions. Moreover* $y_{\delta'} - z$ *solves the problem*

$$\begin{cases} -\Delta\,(y_{\delta'} - z) &= u_{\delta'} - 4n \quad \text{in } \Omega \\ y_{\delta'} - z &= 0 \quad \text{on } \Gamma, \end{cases}$$

and we obtain the inequality

$$1 \leq \|y_{\delta'} - z\|_{L^\infty(\Omega)} \leq C\|y_{\delta'} - z\|_{H^2(\Omega)} \leq C\|u_{\delta'} - 4n\|_{L^2(\Omega)} = C\sqrt{J(u_{\delta'})}.$$

where C *is a constant that does not depend on* δ'. *Therefore, for all* $\delta' < 1$

$$\inf(P_{\delta'}) \geq \frac{1}{C^2} > 0$$

and it is impossible to have weak stability on the left.

Nevertheless, almost all the problems are weakly stable on the left.

Theorem 9.1.12 *Take* δ_0 *as in Theorem 9.1.3. Then, for all* $\delta > \delta_0$ *but at most a numerable set, problem* (P_δ) *is weakly stable on the left.*

Proof. Let δ_0 be the number obtained in Theorem 9.1.3. If we define $\varphi : [\delta_0, +\infty) \to \mathbb{R}$ with $\varphi(\delta) = \inf(P_\delta)$, then φ is a monotone decreasing function, and therefore it is continuous at every point of $[\delta_0, +\infty)$ but at most is a countable number of them. But, as we have already seen, weak stability on the left is equivalent to the continuity of φ in δ, and that proves the Theorem. □

For weakly stable on the left problems, we have the following result. Casas [17] gives a proof for this result in the case of a regular state. The key is to prove that the states converge uniformly.

Definition 9.1.2 *Given a family of elements* $\{u_h\}_{h>0}$, *with* $u_h \in K_h$ *for every* $h > 0$, *we will say that* u *is an accumulation point of* $\{u_h\}_{h>0}$ *if there exists a subsequence* $\{u_{h_k}\}_{k=1}^{\infty}$, *with* $h_k \to 0$ *such that* $u_{h_k} \to u$ *weakly-* in* $L^\infty(\Omega)$.

Obviously, form the definition of the K_h, for every non empty family different there exist accumulation points, and these ones belong to K. Due to the convexity of L with respect to the third variable, we have the follwong result.

Lemma 9.1.13 *Let* $\{u_{h_k}\}_{k=1}^{\infty}$ *be sequence with* $h_k \to 0$, $u_{h_k} \to u$ *weakly-* in* $L^\infty(\Omega)$. *Then*

$$J(u) \leq \liminf_{k\to\infty} J_{h_k}(u_{h_k}).$$

Proof. We know that there exists a sequence v_{h_k} of finite convexe combinations of u_{h_k} thet converges strongly to u in $L^p(\Omega)$ for some $p \in (1,\infty)$:

$$v_{h_k} = \sum_{j=k}^{n(k)} \lambda_{k,j} u_{h_j},$$

con $\lambda_{k,j} \geq 0$, $\sum_{j=k}^{n(k)} \lambda_{k,j} = 1$, $\lim_{k\to\infty} v_{h_k} = u$ in $L^p(\Omega)$.

So we can write

$$J(u) = \int_\Omega L(x, y_u, u)dx = \lim_{k\to\infty} \int_{\Omega_{h_k}} L(x, y_u, v_{h_k})dx \leq$$

$$\leq \liminf_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} \int_{\Omega_{h_k}} L(x, y_u, u_{h_j})dx \leq$$

$$\leq \limsup_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} \int_{\Omega_{h_k}} (L(x, y_u, u_{h_j}) - L(x, y_{h_j}(u_{h_j}), u_{h_j}))dx +$$

$$+ \liminf_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} \int_{\Omega_{h_j}} L(x, y_{h_j}(u_{h_j}), u_{h_j})dx.$$

The second summand is $\liminf_{k\to\infty} J_{h_k}(u_{h_k})$. Just like at the end of the proof of Theorem 9.1.3 we get that

$$\lim_{k\to\infty} \int_{\Omega_{h_k}} |L(x, y_u, u_{h_k}) - L(x, y_{h_k}(u_{h_k}), u_{h_k})|dx = 0.$$

From here it follows that

$$\lim_{k\to\infty}\sum_{j=k}^{n(k)}\lambda_{k,j}\int_{\Omega_{h_k}}|L(x,y_u,u_{h_j})-L(x,y_{h_j}(u_{h_j}),u_{h_j})|dx=0.$$

The proof is complete. □

Theorem 9.1.14 *Let δ_0 be as in Theorem 9.1.3 and $\delta > \delta_0$. If (P_δ) is weakly stable on the left, then there exists $h_0 > 0$ such that $(P_{\delta h})$ has at least a solution u_h for $h \le h_0$. Moreover, each accumulation point u of $\{u_h\}_{h\le h_0}$ is solution of (P_δ). Finally*

$$\lim_{h\to 0} J_h(u_h) = \inf(P_\delta). \tag{9.1.10}$$

Proof. Since every K_h is compact and J_h is continuous, the existence of a solution of $(P_{\delta h})$ will be established if we prove that the set of admissible controls for $(P_{\delta h})$ is not empty. To do that take $u_0 \in K$ an admissible control for problem (P_{δ_0}) and take $u_{0h} \in K_h$ in such a way that $u_{0h} \to u_0$ a.e. $x \in \Omega$. Since $u_{0h} \to u$ in every $L^p(\Omega)$, $1 \le p < \infty$, then, due to the previous theorems, $y_h(u_{0h}) \to y_{u_0}$ uniformly in $\bar\Omega$. Since $g(x, y_{u_0}(x)) \le \delta_0$ for every $x \in \Omega$, we can deduce from the uniform convergence and the relation $\delta > \delta_0$ the existence of a $h_0 > 0$ such that $g(x, y_h(u_{0h})) \le \delta$ for all $x \in \bar\Omega$ and each $h \le h_0$. So we conclude that (P_{δ_h}) has a solution for every $h \le h_0$.

Now let $u_{\delta h}$ be a solution of $(P_{\delta h})$, $h \le h_0$, whose associated state will be denoted y_{δ_h}. Since $\{u_{\delta h}\}_{h\le h_0} \subset K$ and K is bounded, we can extract a subsequence $\{u_{\delta h_k}\}$ such that $h_k \to 0$ and $u_{\delta h_k} \to \bar u$ weakly-* in $L^\infty(\Omega)$ for some $\bar u \in K$. Let us prove that $\bar u$ is a solution of (P_δ). Let $\bar y$ be the associate state to $\bar u$. Since $y_{\delta h_k} \to \bar y$ uniformly in Ω and $g(x_j, y_{\delta h_k}(x_j)) \le \delta$ for each node of the triangulation, we deduce that $g(x, \bar y(x)) \le \delta$ for every $x \in \bar\Omega$, and therefore $\bar u$ is admissible control for (P_δ).

Let us take $\delta' \in (\delta_0, \delta)$ and let $u_{\delta'}$ be a solution of (P'_δ). For every $h \le h_0$ let us take $u_{\delta' h} \in K_h$ such that $u_{\delta' h} \to u_{\delta'}$ a.e. in Ω. From the uniform convergence $y_h(u_{\delta' h}) \to y_{u_{\delta'}}$ and the relation $g(x, y_{\delta'}(x)) \le \delta' < \delta$ for every $x \in \Omega$, we deduce the existence of $h_{\delta'} > 0$ such that $g(x, y_h(u_{\delta' h})(x)) \le \delta$ for all $x \in \Omega$ and all $h \le h_{\delta'}$, this is, $u_{\delta' h}$ is an admissible control for $(P_{\delta h})$ always that $h \le h_{\delta'}$. From here we obtain that $J_{h_k}(u_{\delta h_k}) \le J_{h_k}(u_{\delta' h_k})$ for each k big enough. Using now Lemma 9.1.13 it follows that

$$J(\bar u) \le \liminf_{k\to\infty} J_{h_k}(u_{\delta h_k}) \le \liminf_{k\to\infty} J_{h_k}(u_{\delta' h_k}) = J(u_{\delta'}) = \inf(P_{\delta'}).$$

Finally the stability on the left condition allow us to conclude

$$\inf(P_\delta) \le J(\bar{u}) \le \lim_{\delta' \nearrow \delta} (\inf(P_{\delta'})) = \inf(P_\delta),$$

which, together with the admissibility of $\bar{u}$ for (P_δ) proves that $\bar{u}$ is a solution of (P_δ). The rest of the theorem is immediate. □

Remark 9.1.1 *If the solution of the problem is unique, we have that all the sequence converges weakly-* to the solution of the problem.*

Theorem 9.1.15 *Let us suppose that the assumptions of the previous theorem apply and that L is of class C^2 in the third variable and that there exists $\alpha > 0$ such that*

$$\frac{\partial^2 L}{\partial u^2}(x, y, u) \ge \alpha > 0 \text{ for a.e. } x \in \Omega \text{ and all } y, u \in \mathbb{R}.$$

For every $h \le h_0$ let u_h be a solution of $(P_{\delta h})$ and let $\bar{u}$ be an accumulation point point of $\{u_h\}$ with $u_{h_k} \to \bar{u}$ weakly- in $L^\infty(\Omega)$. Then*

$$\lim_{k\to\infty} \|\bar{u} - u_{h_k}\|_{L^2(\Omega)} = 0.$$

Proof. On one hand

$$\int_\Omega (L(x, y_{h_k}(u_{h_k}), u_{h_k}) - L(x, \bar{y}, \bar{u}))dx = (J_{h_k}(u_{h_k}) - J(\bar{u})) + \int_{\Omega\setminus\Omega_{h_k}} L(x, y_{h_k}(u_{h_k}), u_{h_k})dx.$$

The first summand converges to zero due to the previous theorem and the second one because $\{L(x, y_{h_k}, u_{h_k})\}$ is dominated by a function $\psi_M \in L^1(\Omega)$. So

$$\lim_{k\to\infty} \int_\Omega (L(x, y_{h_k}(u_{h_k}), u_{h_k}) - L(x, \bar{y}, \bar{u}))dx = 0. \tag{9.1.11}$$

On the other hando

$$\int_\Omega (L(x, y_{h_k}(u_{h_k}), u_{h_k}) - L(x, \bar{y}, \bar{u}))dx = \int_\Omega (L(x, y_{h_k}(u_{h_k}), u_{h_k}) - L(x, \bar{y}, u_{h_k}))dx + \\ + \int_\Omega (L(x, \bar{y}, u_{h_k}) - L(x, \bar{y}, \bar{u}))dx. \tag{9.1.12}$$

As in the proof of Theorem 9.1.3

$$\lim_{k\to\infty} \int_\Omega (L(x, y_{h_k}(u_{h_k}), u_{h_k}) - L(x, \bar{y}, u_{h_k}))dx = 0. \tag{9.1.13}$$

As a consequence of (9.1.11)–(9.1.13) we have that

$$\lim_{k\to\infty}\int_\Omega (L(x,\bar y,u_{h_k}) - L(x,\bar y,\bar u))dx = 0. \tag{9.1.14}$$

Making now a Taylor expansion of order two we obtain that

$$\int_\Omega (L(x,\bar y,u_{h_k})-L(x,\bar y,\bar u))dx = \int_\Omega \frac{\partial L}{\partial u}(x,\bar y,\bar u)(u_{h_k}-\bar u)dx+\frac{1}{2}\int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar y,v_k)(u_{h_k}-\bar u)^2dx,$$

where v_k is an intermediate point between u_{h_k} and $\bar u$. Since u_{h_k} converges weakly-* to $\bar u$, the first summand converges to zero:

$$\lim_{k\to\infty}\int_\Omega \frac{\partial L}{\partial u}(x,\bar y,\bar u)(u_{h_k}-\bar u)dx = 0. \tag{9.1.15}$$

Finally we have that

$$\frac{1}{2}\int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar y,v_k)(u_{h_k}-\bar u)^2dx \geq \frac{\alpha}{2}\|\bar u - u_{h_k}\|^2_{L^2(\Omega)}.$$

Therefore we can write

$$\frac{\alpha}{2}\|\bar u - u_{h_k}\|^2_{L^2(\Omega)} \leq \int_\Omega (L(x,\bar y,u_{h_k}) - L(x,\bar y,\bar u))dx - \int_\Omega \frac{\partial L}{\partial u}(x,\bar y,\bar u)(u_{h_k}-\bar u)dx$$

which converges to zero due to (9.1.14) and (9.1.15). So $\|\bar u - u_{h_k}\|_{L^2(\Omega)}$ converges to zero and the proof is complete. □

9.2 Neumann case

Consider the sets, operators, and spaces described in Sections 8.1 and 8.3. We will denote Γ the boundary of Ω, and we will suppose that it is polygonal o polyhedric . Consider also $a_0 \in L^{\frac{Np}{N+p}}(\Omega)$, $a_0 \geq 0$, $a_0 \not\equiv 0$ in Ω, $p > N$.

Let K a convex, weakly-* closed, bounded and non empty subset of $L^\infty(\Gamma)$, $\ell : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ a Carathéodory function, convex in the third variable and that satisfies that for all $M > 0$ there exists $\psi_M \in L^1(\Gamma)$ such that $|\ell(s,y,v)| \leq \psi_M(s)$ for a.e. $s \in \Gamma$, for all $|y|$, $|v| \leq M$. Let $g : \bar\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Let us formulate the optimal control problem

$$(PN_\delta)\begin{cases} \min J(u) = \int_\Gamma \ell\,(s, y_u(s), u(s))\,ds \\ u \in K \quad g\,(x, y_u(x)) \leq \delta \quad \forall x \in \bar\Omega, \end{cases} \tag{9.2.1}$$

where

$$\begin{cases} Ay + a_0 y &= f(\cdot, y) + f_2 \quad \text{in } \Omega \\ \partial_{n_A} y &= u \quad \text{on } \Gamma. \end{cases} \tag{9.2.2}$$

Aplying the same techniques than in Theorem 3.1.1 and using the regularity results in Dauge [47] we have the following result.

Theorem 9.2.1 *Para cada $u \in K$ existe una única $y_u \in W^{1,p}(\Omega)$ solución de* (9.2.2)*. Además existe una constante C_K, que sólo depende una cota para K, tal que $\|y_u\|_{W^{1,p}(\Omega)} \leq C_K$ para todo $u \in K$. Finalmente si $u_j \to u$ *débilmente en $L^\infty(\Gamma)$ entonces $y_{u_j} \to y_u$ fuertemente en $W^{1,p}(\Omega)$.*

Análogamente al caso Dirichlet, se tiene el siguiente resultado sobre existencia de solución.

Theorem 9.2.2 *Existe un número $\delta_0 \in \mathbb{R}$ de forma que que el problema (PN_δ) posee al menos una solución para cada $\delta \geq \delta_0$, mientras que (PN_δ) no posee controles admisibles para $\delta < \delta_0$.*

Consider now the space U_h of elements u of $L^\infty(\Gamma)$ in such a way that every side (face if $N = 3$) of an element T of $\mathcal{T}_h$ that is on Γ, u is constant.

For every $u_h \in U_h$, let us define $y_h(u_h) \in W_h$ as the unique element that satisfies

$$\sum_{i,j=1}^{N} \int_\Omega a_{i,j}(x) \partial_{x_i} y_h(u_h)(x) \partial_{x_j} z_h(x) dx + \int_\Omega a_0(x) y_h(u_h)(x) z_h(x) dx =$$

$$\int_\Omega f(x, y_h(u_h)(x)) z_h dx + \langle f_2, z_h \rangle_{(W^{1,p'}(\Omega))' \times W^{1,p'}(\Omega)} + \int_\Gamma u_h(s) y_h(u_h)(s) ds \quad \forall z_h \in W_h, \tag{9.2.3}$$

Lemma 9.2.3 *Equation* (9.2.3) *has a unique solution.*

The discrete control problem is formulated then as

$$(PN_{\delta h}) \begin{cases} \min J_h(u_h) = \displaystyle\int_\Gamma \ell\left(s, y_h(u_h)(s), u_h(s)\right) ds \\ u_h \in K_h \quad g\left(x_j, y_h(u_h)(x_j)\right) \leq \delta \quad \forall j \in I_h, \end{cases} \tag{9.2.4}$$

where $\{x_j\}_{j=1}^{n(h)}$ is the set of vertexes of $\mathcal{T}_h$, I_h is the set of indexes corresponding to the interior vertexes.

We are going to state now the convergence result for our problem. The proofs are very similar to those of Dirichlet's case.

Lemma 9.2.4 *Para todo $h > 0$, todo $u \in K$ y todo $u_h \in K_h$ existe $C > 0$ independiente de h tal que*

$$\|y_h(u_h) - y_h(u)\|_{H^1(\Omega)} \leq C\|u - u_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Proof. De la monotonía de f, la $H^1(\Omega)$ elipticidad de $a(\cdot,\cdot)$ y la continuidad de la traza en hu tenemos que

$$m\|y_h(u_h) - y_h(u)\|^2_{H^1(\Omega)} \leq a(y_h(u_h) - y_h(u), y_h(u_h) - y_h(u)) =$$

$$(f(x, y_h(u_h)) - f(x, y_h(u)), y_h(u_h) - y_h(u)) + \int_\Gamma (u - u_h)(y_h(u_h) - y_h(u))ds \leq$$

$$\leq \int_\Gamma (u - u_h)(y_h(u_h) - y_h(u))ds \leq \|u - u_h\|_{H^{-\frac{1}{2}}(\Gamma)}\|y_h(u_h) - y_h(u)\|_{H^1(\Omega)}.$$

Por lo tanto

$$\|y_h(u_h) - y_h(u)\|_{H^1(\Omega)} \leq \frac{1}{m}\|u - u_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

□

Theorem 9.2.5 *Suppose that there exists a function $\phi_M \in L^r(\Omega)$, $r > 2$ in such a way that the local Lipschitz condition* (8.1.1) *holds. Suppose also that the triangulation is of non negative type. For all $h > 0$ set $u_h \in K_h$, such that $u_h \to u$ weakly-* in $L^\infty(\Gamma)$. Then*

$$\lim_{h\to 0} \|y_h(u_h) - y_u\|_{L^\infty(\Omega)} = 0. \tag{9.2.5}$$

Proof. The state belongs to $W^{1,p}(\Omega)$ and we have not error estimates, just a convergence result.

In this case we write

$$\|y_h(u_h) - y_u\|_{L^\infty(\Omega)} \leq \|y_h(u_h) - y_h(u)\|_{L^\infty(\Omega)} + \|y_h(u) - y_u\|_{L^\infty(\Omega)}.$$

The second summand converges to zero as a consequence of Theorem 8.3.5.

We know that $y_h(u_h) - y_h(u)$ solves the discrete problem

$$a(y_h(u_h) - y_h(u), z_h) = (f(x, y_h(u_h)) - f(x, y_h(u)), z_h) + \int_\Gamma (u_h - u)z_h ds \quad \forall z_h \in W_h,$$

where

$$a(y, z) = \sum_{i,j=1}^N \int_\Omega a_{i,j}(x)\partial_{x_i}y(x)\partial_{x_j}z(x)dx + \int_\Omega a_0(x)y(x)z(x)dx.$$

In this case we can apply the discrete maximum principle (8.3.4), and hence

$$\|y_h(u_h) - y_h(u)\|_{L^\infty(\Omega)} \le C\|f(x, y_h(u)) - f(x, y_h(u_h))\|_{(W^{1,p'}(\Omega))'} + \|u_h - u\|_{W^{-\frac{1}{p},p}(\Gamma)}.$$

On one hand, the weak-* convergence of the u_h implies the strong convergence in $W^{-\frac{1}{p},p}(\Gamma)$.

On the other hand

$$\|f(x, y_h(u)) - f(x, y_h(u_h)\|_{(W^{1,p'}(\Omega))'} \le \|\phi\|_{L^r(\Omega)}\|y_h(u) - y_h(u_h)\|_{H^1(\Omega)}.$$

Due to Lemma 9.2.4

$$\|y_h(u_h) - y_h(u)\|_{H^1_0(\Omega)} \le \frac{1}{m}\|u - u_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Weak-* convergence of the u_h implies strong convergence in $H^{-\frac{1}{2}}(\Gamma)$. Therefore, the states converge uniformly. □

Vamos a dar ahora cuatro lemas análogos a los Lemas 8.2.10–8.2.13 y a los Lemas 9.1.7–9.1.10.

Lemma 9.2.6 *Para todo $h > 0$ sea $u_h \in K_h$, tales que $u_h \to u$ *-débilmente en $L^\infty(\Gamma)$. Entonces*

$$\lim_{h\to 0} \|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega)} = 0.$$

Proof. Podemos acotar $\|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega)}$ como

$$\|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega)} \le \|y_{u_h} - y_u\|_{W^{1,p}(\Omega)} + \|y_u - \Pi_h y_u\|_{W^{1,p}(\Omega)} + \|\Pi_h y_u - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega)}.$$

El primer sumando converge hacia cero por continuidad. El segundo en virtud del Lema 8.2.10. El tercero, gracias a la continuidad de Π_h (demostrada al principio de la prueba del Lema 8.2.10), lo podemos acotar por una constante que multiplica a $\|y_{u_h} - y_u\|_{W^{1,p}(\Omega)}$, que converge hacia cero por continuidad. □

Lemma 9.2.7 *Para todo $h > 0$ sea $u_h \in K_h$, tales que $u_h \to u$ *-débilmente en $L^\infty(\Gamma)$. Entonces*

$$\lim_{h\to 0} \|y_h(u_h) - y_{u_h}\|_{H^1(\Omega)} = 0.$$

Proof. Podemos acotar $\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)}$ como

$$\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)} \leq \|y_{u_h} - y_u)\|_{H^1(\Omega)} + \|y_u - y_h(u)\|_{H^1(\Omega)} + \|y_h(u) - y_h(u_h)\|_{H^1(\Omega)}.$$

El primer sumando converge hacia cero por continuidad. El segundo en virtud del lema 8.2.11 y el tercero, gracias al Lema 9.2.4, lo podemos acotar por $\|u - u_h\|_{H^{-\frac{1}{2}}(\Gamma)}$. La convergencia *-débil de los u_h implica la convergencia fuerte en $H^{-\frac{1}{2}(\Gamma)}$. □

Lemma 9.2.8 *Supongamos $N = 2$ y que los coeficientes $a_{i,j} \in C^{0,1}(\bar{\Omega})$. Para todo $h > 0$ sea $u_h \in K_h$, tales que $u_h \to u$ *-débilmente en $L^\infty(\Gamma)$. Entonces*

$$\lim_{h\to 0} \frac{\|y_h(u_h) - y_{u_h}\|_{L^2(\Omega)}}{h} = 0.$$

Proof. Como $y_h(u_h)$ y y_{u_h} son los estados discreto y continuo asociados al mismo control, siguiendo exactamente la demostración del lema 8.3.3 se obtiene que para todo $\psi \in L^2(\Omega)$

$$(\psi, y_{u_h} - y_h(u_h)) \leq Ch\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)}\|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} \leq Ch\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)}\|\psi\|_{L^2(\Omega)},$$

donde recordemos que z_ψ es la solución del problema (8.3.3) introducido en la página 190. Así

$$\frac{1}{h}\|y_{u_h} - y_h(u_h)\|_{L^2(\Omega)} \leq C\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)}$$

y podemos aplicar le lema anterior. La prueba está completa. □

Lemma 9.2.9 *Para todo $h > 0$ sea $u_h \in K_h$, tales que $u_h \to u$ *-débilmente en $L^\infty(\Gamma)$. Entonces*

$$\lim_{h\to 0} \frac{\|y_{u_h} - \Pi_h y_{u_h}\|_{L^2(\Omega)}}{h} = 0.$$

Proof. Para la demostración aprovechamos que $\Pi_h y_{u_h} \in W^{1,p}(\Omega)$ usamos el lema de interpolación 8.2.3 y se tiene que

$$\|y_{u_h} - \Pi_h y_{u_h}\|_{L^p(\Omega)} = \|y_{u_h} - \Pi_h y_{u_h} - \Pi_h(y_{u_h} - \Pi_h y_{u_h})\|_{L^p(\Omega)} \leq Ch\|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega)}$$

y el resultado se obtiene dividiendo por h y aplicando el Lema 9.2.6. □

Theorem 9.2.10 *Suppose $N = 2$ and the coefficients $a_{i,j} \in C^{0,1}(\bar{\Omega})$. For all $h > 0$ set $u_h \in K_h$, such that $u_h \to u$ weakly-* in $L^\infty(\Gamma)$. Then*

$$\lim_{h\to 0} \|y_h(u_h) - y_u\|_{L^\infty(\Omega)} = 0. \tag{9.2.6}$$

Proof. Due to the triangular inequality, we have

$$\|y_h(u_h) - y_u\|_{L^\infty(\Omega)} \le \|y_h(u_h) - y_{u_h}\|_{L^\infty(\Omega)} + \|y_{u_h} - y_u\|_{L^\infty(\Omega)}.$$

The second summand converges to zero due to the continuity. The first one can again be bounded by the triangular inequality with

$$\|y_h(u_h) - y_{u_h}\|_{L^\infty(\Omega)} \le \|y_h(u_h) - \Pi_h y_{u_h}\|_{L^\infty(\Omega)} + \|\Pi_h y_{u_h} - y_{u_h}\|_{L^\infty(\Omega)}.$$

Due to Lemma (8.2.3), we can bound the second summand:

$$\|\Pi_h y_{u_h} - y_{u_h}\|_{L^\infty(\Omega)} \le C h^{1-\frac{N}{p}} \|y_{u_h}\|_{W^{1,p}(\Omega)}.$$

Since the u_h converge, they are uniformly bounded, and therefore y_{u_h} is also in bounded in $W^{1,p}(\Omega)$. So this second summand converges to zero. To estimate $\|y_h(u_h) - \Pi_h y_{u_h}\|_{L^\infty(\Omega)}$, let us take into account (8.2.11), which gives us the equivalence between two Sobolev norms in finite dimensional spaces and we obtain, taking into account that $N = 2$ and applying again the triangular inequality

$$\|y_h(u_h) - \Pi_h y_{u_h}\|_{L^\infty(\Omega)} \le \frac{C}{h} \|y_h(u_h) - \Pi_h y_{u_h}\|_{L^2(\Omega)} \le$$

$$C\left(\frac{\|y_h(u_h) - y_{u_h}\|_{L^2(\Omega)}}{h} + \frac{\|y_{u_h} - \Pi_h y_{u_h}\|_{L^2(\Omega)}}{h}\right).$$

Ahora podemos aplicar los Lemas 9.2.8 y 9.2.9 y deducir que esta cantidad converge hacia cero. The proof is complete. □

Finally, using again the concept of weak stability on the left, we can prove that the solutions of the discrete problems converge to the solutions of the continuous problem.

Definition 9.2.1 *Diremos que el problema de control (PN_δ) es débilmente estable por la izquierda en δ si*

$$\lim_{\delta' \nearrow \delta} \inf(PN_{\delta'}) = \inf(PN_\delta).$$

Definition 9.2.2 *Given a family of elements $\{u_h\}_{h>0}$, with $u_h \in K_h$ for every $h > 0$, we will say that u is an accumulation point of $\{u_h\}_{h>0}$ if there exists a subsequence $\{u_{h_k}\}_{k=1}^{\infty}$, with $h_k \to 0$ such that $u_{h_k} \to u$ weakly-* in $L^\infty(\Gamma)$.*

Theorem 9.2.11 *If (PN_δ) is weakly stable on the left, then there exists $h_0 > 0$ such that $(PN_{\delta h})$ has at least a solution u_h for $h \le h_0$. Moreover, each accumulation point u of $\{u_h\}_{h\le h_0}$ is solution of (PN_δ). Finally*

$$\lim_{h\to 0} J_h(u_h) = \inf(PN_\delta).$$

Proof. Since every K_h is compact and J_h is continuous, the existence of a solution of $(PN_{\delta h})$ will be established if we prove that the set of admissible controls for $(PN_{\delta h})$ is not empty. To do that take $u_0 \in K$ an admissible control for problem (PN_{δ_0}) and take $u_{0h} \in K_h$ in such a way that $u_{0h} \to u_0$ a.e. $x \in \Gamma$. Since $u_{0h} \to u$ in every $L^p(\Gamma)$, $1 \le p < \infty$, then, due to the previous theorems, $y_h(u_{0h}) \to y_{u_0}$ uniformly in $\bar\Omega$. Since $g(x, y_{u_0}(x)) \le \delta_0$ for every $x \in \Omega$, we can deduce from the uniform convergence and the relation $\delta > \delta_0$ the existence of a $h_0 > 0$ such that $g(x, y_h(u_{0h})) \le \delta$ for all $x \in \bar\Omega$ and each $h \le h_0$. So we conclude that $(PN_{\delta h})$ has a solution for every $h \le h_0$.

Now let $u_{\delta h}$ be a solution of $(PN_{\delta h})$, $h \le h_0$, whose associated state will be denoted y_{δ_h}. Since $\{u_{\delta h}\}_{h\le h_0} \subset K$ and K is bounded, we can extract a subsequence $\{u_{\delta h_k}\}$ such that $h_k \to 0$ and $u_{\delta h_k} \to \bar u$ weakly-* in $L^\infty(\Gamma)$ for some $\bar u \in K$. Let us prove that $\bar u$ is a solution of (PN_δ). Let $\bar y$ be the associate state to $\bar u$. Since $y_{\delta h_k} \to \bar y$ uniformly in Ω and $g(x_j, y_{\delta h_k}(x_j)) \le \delta$ for each node of the triangulation, we deduce that $g(x, \bar y(x)) \le \delta$ for every $x \in \bar\Omega$, and therefore $\bar u$ is admissible control for (PN_δ).

Let us take $\delta' \in (\delta_0, \delta)$ and let $u_{\delta'}$ be a solution of (PN_δ'). For every $h \le h_0$ let us take $u_{\delta' h} \in K_h$ such that $u_{\delta' h} \to u_{\delta'}$ a.e. in Γ. From the uniform convergence $y_h(u_{\delta' h}) \to y_{u_{\delta'}}$ and the relation $g(x, y_{\delta'}(x)) \le \delta' \le \delta$ for every $x \in \Omega$, we deduce the existence of $h_{\delta'} > 0$ such that $g(x, y_h(u_{\delta' h})(x)) \le \delta$ for all $x \in \Omega$ and all $h \le h_{\delta'}$, this is, $u_{\delta' h}$ is an admissible control for $(PN_{\delta h})$ always that $h \le h_{\delta'}$. From here we obtain that $J_{h_k}(u_{\delta h_k}) \le J_{h_k}(u_{\delta' h_k})$ for each k big enough. Using now the convexity of L with respect to the third component it follows that

$$J(\bar u) \le \liminf_{k\to\infty} J_{h_k}(u_{\delta h_k}) \le \liminf_{k\to\infty} J_{h_k}(u_{\delta' h_k}) = J(u_{\delta'}) = \inf(PN_\delta').$$

Finally, the admissibility of $\bar u$ for (PN_δ) and the stability on the left condition allow us to conclude

$$\inf(PN_\delta) \le J(\bar u) \le \lim_{\delta' \nearrow \delta} (\inf(PN_{\delta'})) = \inf(PN_\delta),$$

what proves that $\bar{u}$ is a solution of (PN_δ). The rest of the theorem is immediate. □

Análogamente al caso distribuido, podemos enunciar el siguiente resultado.

Theorem 9.2.12 *Supongamos que se cumplen las hipótesis del teorema anterior y que además ℓ es de clase C^2 en la tercera variable y existe $\alpha > 0$ tal que*

$$\frac{\partial^2 \ell}{\partial u^2}(s, y, u) \geq \alpha > 0 \text{ para c.t.p. } s \in \Gamma \text{ y todo } y, u \in \mathbb{R}.$$

*Para cada $h \leq h_0$ sea u_h una solución de (PN_h) y sea $\bar{u}$ un punto de acumulación de $\{u_h\}$ con $u_{h_k} \to \bar{u}$ *débilmente en $L^\infty(\Gamma)$. Entonces*

$$\lim_{k\to\infty} \|\bar{u} - u_{h_k}\|_{L^2(\Omega)} = 0.$$

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