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Fuzzy and probabilistic approaches to modelling individual choice or preference: rationality conditions and their relationships

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Preface

"The difficulty in life is the choice."
George Moore, Bending of the Bough.

Decisions, at the very end, are the essence of any intelligent being. No atom, no tree, no stone makes choices. They are exclusive domain of thinking creatures. And our lives are shaped by our decisions. For these reasons, choice is of great interest in different areas of knowledge, such as economics, psychology, sociology, philosophy, mathematics and statistics. Those disciplines are concerned with different aspects related to choice: how to regulate it, predict it, judge it, etc. In this work we are not interested in suggesting the best decision to a specific problem, nor in predicting the choices of an individual under certain circumstances. We are mainly interested in the ways in which choices can be modelled, no matter if they are correct, ethic, good, sane or fair, but just in how they can be described.

The concept of preference is closely related to that of choice. This relation is the basis of mathematical choice theory, which postulates that choices are the expression of preferences and, on the other hand, that preferences can be revealed by observing choices. In a situation where an individual has a preference of an alternative x over y, we expect that his choice between the two alternatives will be x. On the other hand, if an observer who does not know the preferences of an individual witnesses that x is chosen when

y is also available, then he can infer that the individual has a preference for x over y. In such a situation we can say that the individual acted rationally. In this framework, rationality is different from the colloquial and most philosophical use of the word. We say that an individual is rational when he has a set of preferences (or order) and chooses accordingly. In this context, there is nothing rational or irrational in preferring fish to meat, but there is something irrational in preferring fish to meat, and then ordering meat.

Choice and preference are modelled in mathematics using respectively a choice function and a preference relation. If the universe of alternatives is denoted with X, then a choice function is simply a function C that assigns to any subset S of X of available alternatives a non-empty subset $C(S) \subseteq S$ containing the alternatives that are chosen from S. A preference relation Q instead, is a function defined on the Cartesian product $X \times X$ that for any pair of alternatives x and y can take values 0 or 1. If $Q(x, y) = 1$, then x is considered at least as good as y, while if $Q(x, y) = 0$ then x is not preferred to y.

Since choice follows from a set of preferences, in order to observe a rational behaviour the preference has to be transitive. This means that if the alternative x is preferred to y and the alternative y is preferred to z, then x has to be preferred to z. It is relatively simple to establish those conditions that define a rational preference relation (e.g. transitivity), while it turned to be much harder to define rationality for choice functions. Several proposals have been made in the literature and all of them were supposed to be the *ultimate* definition of rationality. This until the so-called Arrow-Sen Theorem was proved. It establishes the equivalence of most of the proposals of the literature. All these definitions of rationality are equivalent to the fact that the choice function is coherent with the preference and that the preference is transitive. Among these equivalent conditions, the most appealing are the ones that refer to the choice function solely (called contraction/expansion conditions) and are defined independently from the preference relation revealed from the choice function. Another interesting property is the Weak Axiom of Revealed Preference (WARP), which establishes that if an individual reveals his preference for an alternative x over y, then he can not reveal also a preference for y over x.

One drawback of classical choice theory is that it does not account for those situations where choices or preferences are imprecise. For example, experiments with repeated choice situations showed that an individual can chose differently when faced with the same set of available alternatives. In theory, it is a violation of rationality, but we know that such situation is quite common. One possible cause for this "irrational" behaviour can be found in the preference that rules the choice: in fact an individual can feel different degrees of preference for different pairs of alternatives. Such a situation cannot be modelled by classical preference relations, where, for any pair of alternatives, either one is strictly preferred to the other or they are indifferent (we intentionally exclude those alternatives that are incomparable). Fuzzy preference relations turn out to be the natural solution to this problem. A fuzzy preference relation Q expresses the relation of preference between pairs of alternatives in such a way that $Q(x, y)$ can take any value in the unit interval: the closer the value to one, the stronger the degree of preference of the first alternative over the second.

Combining classical choice functions with fuzzy preference relations is only the first step towards a generalized fuzzy choice theory. Other authors, like Banerjee [9] and Georgescu [64] went even further and defined the concept of fuzzy choice function, i.e. a choice function that to any alternative x in a set S assigns a degree of choice $C(S)(x)$ that takes a value in the entire unit interval. The most general case has been proposed by Georgescu in [64]: she defined a fuzzy choice function on a family of fuzzy subsets of X and she studied the fuzzy preference relation that can be revealed from it. She also proposed a fuzzy version of most of the rationality conditions known in classical choice theory and studied the relationship between them in order to obtain a result similar to the classical Arrow-Sen Theorem.

Another way of extending classical choice theory is to assume that choice and preference have a stochastic nature. The literature on this field dates back to the seminal works of Luce [81,82] and Fishburn [47,48]. In this literature, given two alternatives x and y, the individual's choice is described with a probability distribution that assigns probability $p(x, y)$ to x and probability $p(y, x) = 1 - p(x, y)$ to y, where $p(x, y)$ indicates the probability that x will be chosen from $\{x, y\}$. Relations of this kind are usually called probabilistic relations. The same concept, extended to sets with more than two alternatives, is the key for defining probabilistic choice functions, i.e. a function that for every pair of sets $S \subseteq T$ assigns a probability $p(S, T)$ that indicates the probability that the choice from T will lie in S. The advantage of the stochastic approach is that probabilistic choice functions can be easily observed, specially in those situations where the decision makers are faced repeatedly with the same set of alternatives. Luce [82], Bandyopadhyay [5] and other authors have also proposed a stochastic version of rationality conditions known in classical choice theory.

This work is organized as follows. In Chapter 1 some basic notions are introduced. Firstly, classical choice theory and its salient results are presented in detail. The basic definitions of a choice function, a revealed preference relation and rationalization of a choice function w.r.t. a preference relation are introduced. Special attention is paid to rationality conditions and their connections. An extended version of original Arrow-Sen Theorem closes Section 1.1. In Section 1.2 fuzzy set theory is presented. The definition of a fuzzy set is introduced and fuzzy logic operators, like triangular norms and implication operators, are defined. Fuzzy preference relations and their properties close the chapter.

Chapter 2 contains our results on fuzzy choice theory. After presenting the historical background of fuzzy choice theory in Section 2.1, we present our results in Sections 2.2 and 2.3. In particular, Section 2.2 is devoted to the problem of finding sufficient and necessary conditions on a fuzzy preference relation such that it can rationalize a fuzzy choice function. In our study we tried to recover two classical results on the rationalization of choice functions: the first establishes that completeness and acyclicity of a preference relation are equivalent to its G-rationality, while the second asserts that if a preference relation is acyclic, then it can M-rationalize a choice function. Our results mimic almost perfectly those classical propositions. The case that has not been recovered has been justified with an example that shows that in the fuzzy set framework the missing implication does not hold in general. The main results of this section have been published in [89]. Section 2.3 is dedicated to the fuzzy version of Arrow-Sen Theorem. We started from some preliminary results found in the literature [64]: we proved that some of those results hold for wider families of triangular norms and also completely new results are presented. Unfortunately, in the fuzzy set framework, it is not possible to recover entirely the classical Arrow-Sen Theorem. Nevertheless, the obtained results improve considerably our understanding of the relationship between rationality conditions in fuzzy choice theory. This section contains a refined version of some preliminary results that we already presented in [92, 94].

Finally, Chapter 3 is dedicated to the study of the relationships between the probabilistic and the fuzzy approach to choice theory. After introducing the definition of probabilistic choice function and probabilistic relation, we propose a novel construction that allows to compute a fuzzy choice function from a given probabilistic choice function. This construction makes use of triangular norms and implication operators, as we already proposed in [90, 93]. A new set of rationality conditions for probabilistic choice functions is presented and we can prove that they guarantee that the fuzzy choice function obtained from the probabilistic choice function is rational. In Section 3.3 the connections between probabilistic and fuzzy preference relations are studied. In particular, we investigate how different definitions of transitivity propagate from the formalism of probabilistic relations to the one of fuzzy preference relations. The results contained in this section improve considerably our previous results published in [95]. We conclude the chapter presenting the results of a study where the techniques proposed in the thesis are applied to the problem of measuring the rationality of a group of real consumers.

Summary

The main subject of the thesis is the mathematical modelling of individual choice. In particular, we focus on two generalizations of the classical choice theory that allow to model those choice situations in which imprecision is involved. Classical choice theory is based on the idea that an individual chooses according to an inner set of preferences (a weak order between the alternatives) and that preferences can be revealed from the observation of his decisions. Choice is modelled using a choice function that to any set of alternatives associates a subset containing the chosen alternatives, while preferences are modelled using binary preference relations. When the choice is coherent with the inner preferences of the individual we say that it is rational. On the other hand, from the point of view of the analyst, only the choices of an individual can be directly observed, while preferences are most of the time unobservable. Nevertheless, preferences can be revealed from the observed decisions. If the preference relation revealed from a choice function also rationalizes it, then we speak of a normal choice function. Normality expresses the fact that choice and preference are carrying the same information, even if expressed with different formalisms. Under this condition, one can also study how interesting properties, such as transitivity, propagates from the formalism of preference relations to the one of choice functions and vice versa. This is the spirit of Arrow-Sen Theorem, a milestone in classical

choice theory that studies the relationships between different definitions of rationality proposed in the literature and proves that, under certain conditions, they are all equivalent. One drawback of classical choice theory is that it does not allow to model those situations where imprecision is involved. Several studies proved that human behaviour is seldom rational, hence a more general tool for describing choice is needed. In this sense we considered two possible generalizations:

(i) Fuzzy choice theory, in Chapter 2;

(ii) Stochastic choice theory, in Chapter 3.

The former proposal is based on the idea that choices and preferences can be modelled using fuzzy concepts. The second chapter of the thesis contains our contributions in this field: in Section 2.2 we present sufficient and necessary conditions that a fuzzy preference relation has to satisfy in order to ensure that the associated fuzzy choice function is rational. As for the classical case, it turns out that the property of acyclicty of the fuzzy preference relation is fundamental to prove these results. In Section 2.3 we study how Arrow-Sen Theorem can be generalized to the fuzzy set framework. We extend and correct previous results found in the literature and we try to reproduce the equivalences proved by Arrow and Sen in the classical case. Stochastic choice theory is another approach that allows to model choice behaviour when uncertainty is involved. It entails that assumption that both choice and preference are stochastic in nature. Concepts like preference relation and choice function have their corresponding counterpart in this theory, namely probabilistic choice function and reciprocal relation. Despite the extended literature on stochastic choice theory, it lacks a study that relates the stochastic and the fuzzy approach to choice modelling: Chapter 3 tries to fill this gap. In fact in Section 3.2 we present a novel construction, based on t-norms and an implication operators, that allows to define a fuzzy choice function from a given probabilistic choice function. The resulting fuzzy choice function is proved to be normal and its fuzzy revealed preference relation to be transitive, provided the probabilistic choice function satisfies a set of new conditions that we defined starting from past proposals found in the literature. Furthermore, in Section 3.3 we study the connections between reciprocal and fuzzy preference relations: a novel construction is proposed and under certain conditions on the t-norm it can be proved that fuzzy and probabilistic preference relations are equivalent. For these equivalent relations we study how the property of transitivity propagates from one formalism to the other and we obtain a new parametric family of upper-bound functions for cycle-transitivity of the reciprocal relation that depends on the chosen t-norm. We close the chapter with the results of an experiment with real market data, where the techniques proposed in the previous sections are used to measure the degree of rationality of a group of consumers.

Resumen

El argumento principal de esta tesis es el modelado de la elección individual. En particular, nos centramos en dos generalizaciones de la teoría de elección clásica que permiten tratar aquellas situaciones donde los datos pueden ser imprecisos. La teoría de elección clásica se fundamenta en la idea de que los individuos eligen según un conjunto de preferencias personales (un orden débil entre las alternativas) y dichas preferencias se pueden deducir mediante la observación de las decisiones. La elección se modela a través de una función de elección que a cada conjunto de alternativas disponibles asocia el subconjunto de las alternativas elegidas, mientras que la preferencia se modela a través de una relación de preferencia binaria. Cuando la elección es coherente con las preferencias internas, entonces el individuo es racional. Por otro lado, desde el punto de vista de un observador externo, solo las elecciones pueden ser observadas directamente, mientras que las preferencias están habitualmente ocultas. No obstante, las preferencias se pueden deducir observando las elecciones realizadas. Si la relación de preferencia revelada desde la función de elección racionaliza esa última, entonces hablaremos de una función de elección normal. La normalidad implica que elección y preferencia son matemáticamente equivalentes, a pesar de expresarse con distintos formalismos. Bajo esta condición, se puede estudiar cómo algunas interesantes propiedades, tales como por ejemplo la transitividad o

la completitud, se propagan desde las funciones de elección a las relaciones de preferencia y viceversa. Es este el espíritu del teorema de Arrow-Sen, un hito en la teoría de elección clásica, que estudia las relaciones entre diferentes definiciones de racionalidad propuestas en la literatura y demuestra finalmente su equivalencia bajo ciertas condiciones. Un inconveniente de la teoría clásica de la elección es que no permite tratar aquellas situaciones donde los datos son imprecisos. Diferentes estudios han probado que el comportamiento humano es a menudo irracional, de aquí la necesidad de un modelo más general. Por estas razones se consideran dos posibles generalizaciones:

(i) Teoría de elección borrosa, en el Capítulo 2;

(ii) Teoría de elección probabilística, en el Capítulo 3.

La primera propuesta se basa en la idea de que tanto las elecciones como las preferencias se puedes modelar con conceptos borrosos (fuzzy). El segundo capítulo de la tesis contiene nuestras aportaciones en este campo: en la Sección 2.2 se estudian aquellas condiciones que una relación de preferencia borrosa tiene que satisfacer para poder asegurar que de ella se puede racionalizar una función de elección borrosa. Como para el caso clásico, la propiedad de aciclicidad resulta ser fundamental. En la Sección 2.3 se ha intentado generalizar al caso borroso el Teorema de Arrow-Sen: hemos conseguido extender y mejorar resultados precedentes encontrados en la literatura, a la vez que proponer resultados del todo nuevos y originales. La teoría de elección probabilística es otra posible generalización de la teoría clásica que permite modelar el comportamiento humano cuando los datos están afectados por cierta incertidumbre. Se basa en la suposición de que tanto las elecciones como las preferencias tienen una naturaleza probabilística. Por eso, conceptos como el de función de elección o el de relación de preferencia binaria se pueden redefinir en clave probabilística. Más concretamente hablaremos de funciones de elección probabilística y de relaciones recíprocas. A pesar de la amplia literatura en teoría de elección probabilística, todavía falta un estudio que relacione el enfoque borroso con el enfoque probabilístico: el tercer capítulo de la tesis se propone llenar esa laguna. En la Sección 3.2 se presenta una construcción que, a través del uso de normas triangulares y operadores de implicación, permite expresar una función de elección borrosa como una función de elección probabilística. Si la función de elección probabilística inicial satisface ciertas condiciones, que también hemos definido a partir de las propuestas de la literatura, entonces la función de elección borrosa que se deriva es normal y la relación de preferencia que se revela de ella es transitiva. Además, en la Sección 3.3 se estudian las conexiones entre relaciones binarias borrosas y relaciones recíprocas: se propone una novedosa construcción y, bajo ciertas condiciones, se puede demostrar que la misma genera relaciones borrosas y relaciones recíprocas equivalentes. Para esas familias de relaciones equivalentes se ha estudiado como se propagan algunas importantes propiedades. En particular, se puede demostrar que la transitividad de la relación borrosa es equivalente a la transitividad de la relación recíproca con respeto a una nueva familia de funciones de límite superior, típica de las relaciones ciclo-transitivas. Cierra este capítulo un conjunto de resultados que se han obtenido con un experimento sobre datos reales, pensado para medir la racionalidad de un grupo de consumidores y que hace uso de las técnicas expuestas en las secciones anteriores.

Contents

Chapter 1

Basic notions

In this chapter some preliminary notions of choice theory, revealed preference and fuzzy logic are introduced.

1.1 Classical theory of choice and revealed preference

1.1.1 Historical background

Individual choice theory dates back to the end of the eighteenth century. The effort of both economists and psychologists was to model individual behaviour based on the observation of individuals' decisions and further generalization of the observed actions. The kind of decisions with which this theory deals are as follows: given two states, A and B, an individual chooses A in preference to B or vice versa. The economic theory of decision making, that goes under the name of consumer's choice theory, is about how to predict those decisions and since the very beginning it has been based on the concept of *utility*. In a few words, the assumption is that decision makers assign in some way a numerical value (called utility) to any of the states and then choose trying to maximize their utility.

The main criticism to the use of utility came from Samuelson [108, 109], who claimed that utility did not correspond to any directly observable phenomena. The old theory was criticized mainly from a methodological point of view, in which non-observable concepts were used. He proposed a revealed preference approach, where preferences between states can be inferred by observing actual choices of the individuals, in order "to develop the theory of consumer's behaviour freed from any vestigial traces of the utility concept" [108].

If a state B could have been chosen by a certain individual when he in fact was observed to choose another state A, it is to be presumed that he has revealed a preference for A over B. In other words, by observing that this person chooses A when B is also available, we conclude that he prefers A to B. From the point of view of the decision maker, the process runs from his preference to his choice, i.e. he chooses the alternatives that are most preferred according to his ordering, but from the point of view of the observer the process runs in the opposite direction: choices are observed first and preferences are then presumed from these observations.

Samuelson postulated that an individual should behave like a "rational *homo economicus*" [108]. This means basically two things: he can order (weakly) all the states to which he is faced and he makes his choices according to some maximization criterion. To sort all available states in a weak order, the economic man is supposed to be able to compare all pairs of states A and B (either he prefers A to B, B to A or he is indifferent) and these preferences must be transitive. Once the states are ordered, the rational economic man must make his choice in such a way as to maximize something. For example, in a travel-mode choice, and faced with the states

$\{A, B, C\} = \{airplane, bus, car\}$ an individual can choose his most preferred alternative trying to maximize (or minimize) different aspects, such as travel-time, cost, comfort or also a combination of them. With respect to previous theories of choice that were resting on a vague concept of utility, he assumed that maximization of utility only becomes specific, and therefore possibly right or wrong, when it specifies what is being maximized.

Another fundamental contribution introduced by Samuelson, which has come to be known as the Weak Axiom of Revealed Preference (WARP), establishes that if an individual reveals his preference for a state A over B, then he cannot reveal also a preference for B over A. This axiom was first presented in [108] under a different name and further developed by the same author in [110, 111] and by other authors, such as Arrow, Georgescu-Roegen, Herzberger, Hicks, Houthakker, Little, Sen and Uzawa in [1, 61, 69, 71, 72, 80, 115, 116, 126]. The axiom has been used both in its first version and in the consecutive extended formulations to prove the rationality of the decision maker.

One drawback of the approach of Samuelson, despite its great generality, is that it is confined to market choices only. In fact, in its formalization, the states faced by the decision maker are always considered to be a combination of prices and goods, subject to a budget restriction, i.e. the individual, having a fixed available budget b, has to choose a configuration of goods $x = (x_1, \dots, x_n)$ of which the prices are contained in the vector $p = (p_1, \dots, p_n)$ in such a way that $b = p \cdot x = \sum_{i=1}^{n} x_i p_i$. Nevertheless, other non-market situations should be taken into account, such as governmental resolutions, voting decisions, etc. A more general approach to choice theory was needed.

1.1.2 Classical choice theory

The first generalization of Samuelson's revealed preference theory was presented by Uzawa [126]. In his seminal paper he proposed a definition of choice function freed from any economical interpretation. In fact, he considered the choice space as a finite space of alternatives, regardless of the real nature of the alternatives, settling the foundations of what will be later called choice theory. We want to stress the fact that in this work the universe set X will be always considered as finite. Even in the literature, very little interest has been paid to the case of an infinite universe set X.

Definition 1.1 *Let X be a finite set of alternatives and $\mathcal{B} \subseteq \mathcal{P}(X)$ a family of non-empty subsets of X. The subsets of X contained in $\mathcal{B}$ are called available sets. The function $C : \mathcal{B} \to \mathcal{P}(X) \setminus \{\emptyset\}$ is called choice function. For any $S \in \mathcal{B}$ it defines a subset $C(S) \subseteq S$ containing the alternatives chosen from S under the condition that $C(S) \neq \emptyset$.*

Example 1.2 *A typical situation that can be modelled using choice functions is travel mode choice: suppose to observe an individual that has three possible ways to go and come back from its working place:*

(i) by walking (A);

(ii) by bus (B);

(iii) by car (C).

The set $X = \{A, B, C\}$ is finite and contains three alternatives. The family $\mathcal{B}$ may be composed, for example, by the following three sets:

(i) $S_1 = X$: normal day;

(ii) $S_2 = \{B, C\}$: rainy day;

(iii) $S_3 = \{A, C\}$*: bus strike day.*

The choice function $C(S)$ *indicates which are the chosen alternatives from any set* S*, i.e. which travel mode has been chosen for a specific day. The observed individual can show the following behaviour:*

(i) $C(S_1) = \{A, B\}$*: on a normal day, he goes to work by bus and he comes back by walking;*

(ii) $C(S_2) = \{B\}$*: on a rainy day, he goes and comes back by bus;*

(iii) $C(S_3) = \{A\}$*: on a day with bus strike, he goes and comes back by walking.*

Observe that the choice function C *satisfies the conditions of Definition 1.1, that the choice set* $C(S)$ *is not forced to contain only one element and that the family* $\mathcal{B}$ *does not contains all possible subsets of* X*.*

It is convenient to anticipate here a digression about the nature of the family $\mathcal{B}$ of non-empty subsets of X, even if in the literature it appeared only in a second moment. Should the family $\mathcal{B}$ contain all non-empty subsets of X or not? If it contains all subsets of X, then we are supposing that the choice function C is known on sets that sometimes have not been observed. On the other hand, if we assume that C is defined only for those sets that can be directly observed, then we reduce its descriptive power. Sen tackled the problem in Chapter 6 of [115] by choosing for his work the first option. He argued that if a property of C needs to be tested, it will fail or succeed only on the subsets of X that can be observed, thus there is no reason for assuming that the property will fail on the subsets that are not observable. Of the same opinion were also other authors, like Arrow [1, 2] and Uzawa [126]. On the other hand, other researchers preferred to include in $\mathcal{B}$ only those non-empty subsets of X on which the choice function C can be observed directly (Richter [106], Hansson [67], Suzumura [119–122]). In the rest of this chapter we will refer to conditions H and WH according to the following definition.

Definition 1.3 *Condition H is satisfied when $\mathcal{B}$ contains all non-empty subsets of X and condition WH is satisfied when $\mathcal{B}$ contains at least all the pairs and triplets of alternatives contained in X.*

The condition WH was proposed by Sen in [115]. Obviously, condition H implies condition WH, but not vice versa.

Preference Relations

Choice theory is based on the notion of revealed preference, i.e. a binary preference relation revealed from a choice function. Before presenting the details of revealed preference relations, let us recall some basic definitions and properties of binary relations.

Definition 1.4 *Let $Q : X \times X \to \{0, 1\}$ be a binary relation on X. $Q(x, y) = 1$ expresses the fact that element x is connected to element y by Q, while $Q(x, y) = 0$ indicates the lack of such connection.*

Reflexive binary relations are usually used for representing preference relations, because they fit the following interpretation: $Q(x, y) = 1$ means that the first alternative x is at least as good as the second alternative y. Obviously, a relation of this kind needs to be reflexive (any x is at least as good as itself). From now one, preference relations will always refer to reflexive binary relations.

From any preference relation Q two other binary relations can be generated:

(i) strict preference: $P_Q(x, y) = 1$ if and only if $Q(x, y) = 1$ and $Q(y, x) = 0$,

(ii) indifference: $I_Q(x, y) = 1$ if and only if $Q(x, y) = 1$ and $Q(y, x) = 1$.

The strict preference relation $P_Q(x, y)$ takes value one when the alternative x is strictly preferred to y, i.e. x is at least as good as y and the opposite does not hold. The indifference relation I_Q between alternatives x and y expresses the fact that x is at least as good as y and vice versa, y is at least as good as x.

A binary relation Q on X is:

(i) reflexive, if $Q(x, x) = 1$, for any $x \in X$;

(ii) complete, if $Q(x, y) = 1$ or $Q(y, x) = 1$, for any $x, y \in X$;

(iii) transitive, if $Q(x, y) = 1$ and $Q(y, z) = 1$ imply that $Q(x, z) = 1$, for any $x, y, x \in X$;

(iv) regular, a shorthand for the simultaneous fulfillment of completeness (implying reflexivity) and transitivity;

(v) acyclic if, for any $n \geq 2$ and $x_1, \ldots, x_n \in X$ such that $P_Q(x_1, x_2) = 1, \ldots, P_Q(x_{n-1}, x_n) = 1$ then $P_Q(x_n, x_1) = 0$.

A preference relation is called a weak order on X if it is complete and transitive.

Definition 1.5 *A preference relation Q is called quasi-transitive if the associated strict preference relation P_Q is transitive.*

Definition 1.6 *Let Q be a binary relation on X. The transitive closure of Q is the relation $\widehat{Q}$ satisfying* *(i)* $\widehat{Q}$ *is transitive;*

(ii) $Q \subseteq \widehat{Q}$*;*

(iii) *if there exists another transitive relation* Q' *such that* $Q \subseteq Q'$*, then* $\widehat{Q} \subseteq Q'$*.*

We can say that the transitive closure $\widehat{Q}$ of a relation Q is the smallest transitive relation such that $Q \subseteq \widehat{Q}$. Furthermore, it has been proved that it always exists [4]. Obviously, if Q is transitive, then it coincides with its transitive closure.

Revealed Preference Relations

Revealed preference relations are preference relations that are constructed from a given choice function C on X and that express the information about choice in the form of a pairwise comparison.

Definition 1.7 ([115]) *Let* C *be a choice function on* X*. For any* $x, y \in X$*, the relations* R_C*,* $\tilde{P}_C$ *and* $\bar{R}_C$ *can be defined as:*

(i) *revealed preference:* $R_C(x, y) = 1$ *if* $x \in C(S)$*, for some* S *containing both* x *and* y*;*

(ii) *strong revealed preference:* $\tilde{P}_C(x, y) = 1$ *if* $x \in C(S)$ *and* $y \notin C(S)$*, for some* S *containing both* x *and* y*;*

(iii) *base revealed preference [69]:* $\bar{R}_C(x, y) = 1$ *if* $x \in C(\{x, y\})$*, provided condition WH is satisfied;*

The intuition behind these definitions is consistent with the proposal of revealed preference given by Samuelson [110]: they describe the relations of preference between pairs of alternatives by observing the choice made by the individual in those sets containing the alternatives. For the revealed preference relation R_C, alternative x is preferred to y if the first is chosen when the second is also available. For the strong revealed preference relation $\tilde{P}_C(x, y)$, alternative x is preferred to y if the first is chosen when the second is available and not chosen. Finally, for the base revealed preference relation $\bar{R}_C$, alternative x is preferred to y if the first is chosen from the set $\{x, y\}$.

Example 1.8 *Considering the choice function contained in Example 1.2 on the universe set X=$\{A,B,C\}$, we can reveal the following preference relations, according to Definition 1.7:*

$$R_C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{P}_C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

while the base revealed preference relation $\bar{R}_C$ cannot be computed, since condition WH is not satisfied (the choice function is not defined for the set $\{A, B\}$).

Rationalization and Normalization

A preference relation Q on X represents the pairwise preferences of an individual on the alternatives of X. These preferences can result in an act of choice in at least two different ways: given Q and a set $S \subseteq X$ of alternatives we can construct the subset $G_Q(S)$ of S containing the alternatives of S that are the greatest w.r.t. Q (i.e. the alternatives in S that are at least as good as all the other alternatives of S, according to Q) and the subset $M_Q(S)$

that contains those alternatives that are maximal in S w.r.t. Q (i.e. the alternatives in S such that no other alternative in the set dominates them, according to Q).

Definition 1.9 ([119]) *Let Q be a preference relation on X. The set of the greatest elements of $S \in \mathcal{B}$ with respect to Q can be constructed by using*

$$G_Q(S) = \{x \in S \mid (\forall y \in S)(Q(x,y) = 1)\}, \text{ for any } S \in \mathcal{B}. \tag{1.1}$$

Similarly, the set of maximal elements of $S \in \mathcal{B}$ with respect to Q can be constructed by using

$$M_Q(S) = \{x \in S \mid (\forall y \in S)(Q(x,y) \geq Q(y,x))\}, \tag{1.2}$$

or, equivalently,

$$M_Q(S) = \{x \in S \mid (\forall y \in S)(P_Q(y,x) = 0)\}. \tag{1.3}$$

If C is a choice function on X, then it is called G-rational (resp. M-rational) if there exists a preference relation Q on X such that $C = G_Q$ (resp. $C = M_Q$). In that case, C is said to be G-rationalizable (resp. M-rationalizable) by Q and Q is called the G-rationalization (M-rationalization) of C.

Some remarks on these last definitions. First of all, for a given Q, it always holds that $G_Q(S) \subseteq M_Q(S)$, for any $S \in \mathcal{B}$. Furthermore, if Q is a complete relation, then $M_Q = G_Q$. Finally, given a preference relation Q on X, it is not always ensured that Eqs. (1.1) and (1.2) lead automatically to a choice function. In fact, Definition 1.1 of choice function requires that $C(S) \neq \emptyset$, for any $S \in \mathcal{B}$, while there can be preference relations Q that do not generate a choice function through Eqs. (1.1) or (1.2). More conditions need to be imposed on Q in order to generate a choice function using

G-rationalization or M-rationalization. The first attempt was to impose transitivity, as obvious, but later Sen [114] and Walker [129] proved that a weaker condition, i.e. acyclicity, was enough. We will recall and generalize those results in Section 2.2.

Obviously, rationalization and revealing preference are strongly related operations: in fact, the former allows to deduce choices when we know the preference preference relation, while the second converts the information observed through choices into a comparison between pairs of alternatives. When these two operations are reversible, we say that the choice function is normal.

Definition 1.10 ([119]) *A choice function C is called G-normal (resp. M-normal) if it is G-rationalizable (resp. M-rationalizable) by its own revealed preference relation R_C.*

Some results on the connections between G-rationality, G-normality, M-rationality and M-normality can be found in [119]:

Proposition 1.11 ([119]) *Let C be a choice function on X and R_C its revealed preference relation. The following statements hold, for any $S \in \mathcal{B}$:*

(i) *if C is M-rational, then it is also G-rational;*

(ii) $C(S) \subseteq G_{R_C}(S) \subseteq M_{R_C}(S)$*;*

(iii) *C is G-rational iff C is G-normal;*

(iv) *if C is M-normal then it is also G-rational from a complete preference relation Q.*

Rationality conditions

The definition of normality is crucial because it entails the equivalence of the two philosophies: the preference relation and the choice function approach. It has been massively used for constructing a definition of *rationality* for choice functions. In fact, while for preference relations it is relatively easy to speak of *rationality*, by demanding for example regularity (i.e. completeness and transitivity), it is less easy to define a similar property for choice functions. Hence, the normality of a choice function combined with the regularity of its revealed preference relation can be used as a criterion for establishing the rationality of a choice function. For several years there have been multiple attempts to establish conditions that are equivalent to the normality of a choice function and the regularity of its revealed preference relation. Many authors have proposed different definitions, conditions or axioms that were supposed to be the *ultimate* definition of rationality. The first of this conditions (Weak Axiom of Revealed Preference, a.k.a. WARP) already appeared in the work of Samuelson [108]. Houthakker [72] modified it by using the transitive closure $\widehat{R_C}$ of R_C. However, Sen [116] proved that the two conditions are equivalent provided condition H is satisfied. Other rationality conditions were proposed by Sen [115] and Richter [106], called Weak Congruence Axiom (a.k.a. WCA) and Strong Congruence Axiom (a.k.a. SCA) respectively.

Definition 1.12 *Let C be a choice function on X and let R_C be its revealed preference relation, $\tilde{P}_C$ its strong revealed preference relation and $\widehat{R_C}$ and $\widehat{\tilde{P}_C}$ their respective transitive closures. For any x, y in S and any $S \in \mathcal{B}$, the choice function C can satisfy:*

(i) Weak Axiom of Revealed Preference (WARP): if $\tilde{P}_C(x,y) = 1$, then $R_C(y,x) = 0$;

(ii) *Strong Axiom of Revealed Preference (SARP): if* $\widehat{\tilde{P}_C}(x,y) = 1$*, then* $R_C(y,x) = 0$*;*

(iii) *Weak Congruence Axiom (WCA): if* $x \in S$*,* $y \in C(S)$ *and* $R_C(x,y) = 1$*, then* $x \in C(S)$*;*

(iv) *Strong Congruence Axiom (SCA): if* $x \in S$*,* $y \in C(S)$ *and* $\widehat{R_C}(x,y) = 1$*, then* $x \in C(S)$*.*

The interpretation of these conditions is simple. WARP establishes that if an individual reveals his strict preference of x over y, then he must not reveal preference of y over x. The strong version of this axiom, SARP, substitutes the revealed strict preference relation for its transitive closure. The Weak Congruence Axiom (WCA) establishes that if an individual has chosen one alternative x from a set T where y is also contained ($R_C(x,y) = 1$), then for any other set S where both x and y are contained, if y is chosen, then x has to be chosen too. The same reasoning applies for SCA, where the revealed preference R_C is substituted by its transitive closure $\widehat{R_C}$.

Aside rationality conditions, Sen [115] saw the necessity of establishing other kinds of conditions, called consistency conditions, that control the behaviour of the choice function w.r.t. contraction and expansion of the choice set. Compared to previous axioms of revealed preference and congruence, consistency conditions make no use of the preference relations revealed from C.

Definition 1.13 *Let* C *be a choice function on* X*. For any* S, T *in* $\mathcal{B}$ *such that* $S \subseteq T$ *and any* x, y *in* S*, the choice function* C *can satisfy:*

(i) *Condition* α*: if* $x \in C(T)$*, then* $x \in C(S)$*;*

(ii) *Condition* β*: if* $x, y \in C(S)$*, then* $x \in C(T)$ *if and only if* $y \in C(T)$*;*

(iii) *Condition γ: let $\mathcal{M} \subseteq \mathcal{B}$ be a family of subsets of $\mathcal{B}$ and U the union of the elements of $\mathcal{M}$ (i.e. $U = \bigcup_{M \in \mathcal{M}} M$). If $x \in C(M)$, for all $M \in \mathcal{M}$, then $x \in C(U)$;*

(iv) *Condition δ: if $x, y \in C(S)$, then $C(T) \neq \{x\}$ and $C(T) \neq \{y\}$.*

The first condition is a contraction condition, while the last three are expansion conditions. Condition α establishes that if an alternative x is chosen from a set T and other alternatives are removed from T to obtain a smaller set S, then x still remain chosen in S. Condition β establishes that if alternatives x and y are both chosen in S, a proper subset of T, then either both x and y are chosen in T or no one of them is chosen. Condition γ establishes that if an alternative x is chosen from any set M of a family $\mathcal{M}$ of subsets of X, then it should remain chosen also from their union $U = \bigcup_{M \in \mathcal{M}} M$. Finally, condition δ establishes that if x and y are both chosen in S, a proper subset of T, then none of them can be uniquely chosen from T.

One of the most interesting results in choice theory is the so-called *Arrow-Sen Theorem*, which states the equivalence of some of the previous conditions (and other that will be listed later). It constitutes a milestone in classical choice theory, because any other attempt to define rationality after the theorem was proved needed to be compared with the definitions contained in it.

For the sake of brevity, we report here a modified version of the original Arrow-Sen Theorem, including also those equivalent conditions that were not considered by Sen in [115], but have been added later by other authors.

Theorem 1.14 *Let C be a choice function and R_C its revealed preference. If condition H is satisfied, then the joint regularity of R_C and G-normality* *of C is equivalent to any of the following conditions:*

(i) WCA;

(ii) SCA;

(iii) WARP;

(iv) SARP;

(v) $R_C = \tilde{R}_C$;

(vi) Conditions α and β.

Another result contained in [115] shows that if C is G-normal, then quasi-transitivity of the revealed preference relation R and condition δ are equivalent.

The main drawback of Arrow-Sen Theorem is that it requires condition H to be satisfied. Other authors (Richter [106], Hansson [67] and Suzumura [119]) tried to simulate a similar result dropping that condition. What they obtained can be summarized with the following:

Theorem 1.15 *Let C be a choice function on X, then*

(i) C is G-rationalizable by a regular preference relation if and only if condition SCA is satisfied;

(ii) Conditions WARP and WCA are equivalent.

The first point of Theorem 1.15 is also known as the Richter Theorem.

1.2 Introduction to approximate reasoning

In the following section we introduce the basic notions on fuzzy sets, fuzzy preference relations and the fuzzy logical operators that are typically used in the fuzzy set framework.

1.2.1 Fuzzy sets

We are used to think of mathematics, and specially of logic, as a rigid structure where sentences can only be true or false and where a body of rules rigorously establishes the conditions of how truth spreads through the sentences of the discourse. No surprise if mathematics is sometimes called *exact science*. To some extent, this is true. In classical logic, an element x belongs or does not belong to a specified set A. A sentence can only be true or false. This bivalent logic works perfectly for those situations were no imprecision is involved. Nevertheless, there exist other situations where imprecision is inevitable. Daily language gives a perfect example: expressions like "*today is really hot outside*", "*his friend is around 40 years old*", "*I will arrive around 5*" refer to some measurable attribute, like temperature, age or hour of the day, in a way that is far from being exact, but that still everyone can understand. For example, with "*very hot*" we understand a temperature that is surely higher than 10°C, presumably over 20°C and likely close to 30°C. Although these sentences are imprecise, we can still rule our course of actions based on them: "*if it is hot, I will go to the beach*" does not mean that I will check the exact temperature before deciding either to go to the beach or to stay at home. Then, also logical propagation of truth can work under an imprecise setting. Less mundane situations are likely to be affected by imprecision: there are plenty of circumstances where experts need to take decisions in a context where the lack of time or resources forces them to operate without exact data. Think on a bus driver that has to stop the vehicle in time to avoid a collision with a car riding ahead: he has no time and no way to measure exactly the distance from his bus and the car, his relative speed with respect to the obstacle, the friction of the pavement, the slope of the street, etc. Even so, he is able to operate on the breaks with the needed pressure in order to avoid the crash, thank to an internal set of rules that he learned from experience, in which the estimated and imprecise attributes of the problem (distance, relative speed, friction, slope, etc.) contribute to the (imprecise but hopefully effective) solution of the problem (pressure on the breaks).

The scientist who first formulate the idea of a *vague logic* was Zadeh, who in [134] introduced the concept of *fuzzy set*.

Definition 1.16 *Let X be a non-empty set. A fuzzy set A on X is a function $A : X \to [0,1]$. For any $x \in X$, the value of $A(x)$ is called the degree of membership of x to A.*

In analogy with the classical (or crisp) definition of set, where the membership degree of an element to a set can only be 0 or 1, he proposed to use the entire interval $[0,1]$, instead of just its extreme points. In this way, an element with null membership to A is actually out of the set, if it has membership 1, it is fully in the set, while other values of $A(x) \in]0,1[$ represent intermediate degrees of membership to A. The support of a fuzzy set A is defined as $supp(A) = \{x \in X \mid A(x) > 0\}$. In Figure 1.1 is depicted an example for the membership function of different sets: the left one is the crisp set $A = \{x$ *is a teenager*$\}$, while the right one represents the fuzzy set $B = \{x$ *is around 40 years old*$\}$. On the horizontal axis is represented the age of a person (in years) and on the vertical axis the degree of membership (between 0 and 1). Also the notion of subset of a set can be extended to the fuzzy set framework: we say that B is a fuzzy subset of A if it holds that $B(x) \leq A(x)$, for any $x \in X$.

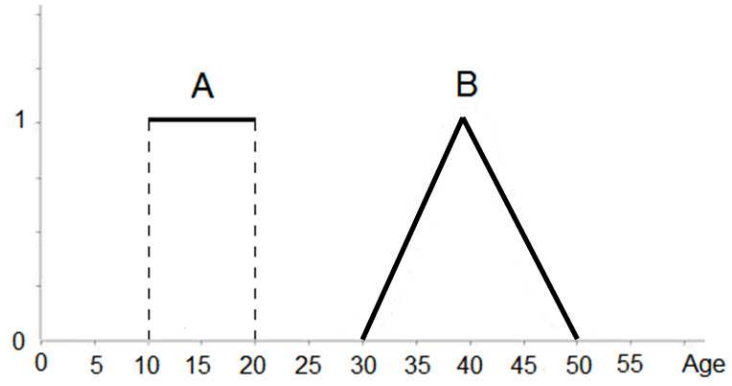


Figure 1.1: Example of a crisp and a fuzzy set

1.2.2 Fuzzy logic operators

Aside the definition of fuzzy set, the usual operators for intersection, union and negation were needed. Zadeh's first proposal was to use the minimum for intersection ($(A \cap B)(x) = \min(A(x), B(x)) = A(x) \wedge B(x)$), the maximum for the union ($(A \cup B)(x) = \max(A(x), B(x)) = A(x) \vee B(x)$) and as a negation the following operator $\neg A(x) = 1 - A(x)$. In this work we adopt the notations $a \wedge b$ and $a \vee b$ for referring, respectively, to minimum and maximum between two elements. Soon, more operators were proposed: triangular norms, triangular conorms and negation operators. Triangular norms (t-norms) were originally introduced in order to generalize the triangle inequality towards probabilistic metric spaces [112]. Nowadays, they are widely used in fuzzy set theory, specially for modelling intersection of fuzzy sets and conjunction of fuzzy statements. A standard book on triangular norms is [78]. Aware of the existence of multiple and equally valid notations for these families of operators, we decide to adopt in this work the notation proposed by Georgescu in [64].

Definition 1.17 *A triangular norm (t-norm for short) is a binary operator $*$ on $[0,1]$ that is increasing, commutative, associative and has neutral element* 1.

The three most important t-norms are the minimum, $a *_{\mathbf{M}} b = a \wedge b$, the product, $a *_{\mathbf{P}} b = a \cdot b$, and the Łukasiewicz t-norm, $a *_{\mathbf{L}} b = (a+b-1) \vee 0$. Given a t-norm $*$ and an automorphism ϕ of the unit interval, the operator $*_\phi$, defined by $a *_\phi b = \phi^{-1}(\phi(a) * \phi(b))$, is also a t-norm and it is called the ϕ-transform of $*$. A t-norm $*$ is said to have zero divisors if there exists at least one pair of values $(a,b) \in]0,1[^2$, such that $a * b = 0$. The two values a and b are called zero divisors of $*$.

Definition 1.18 *A negation operator is a unary operator $\neg$ on $[0,1]$ that is decreasing and satisfies $\neg 0 = 1$ and $\neg 1 = 0$. A negation operator is called involutive if it holds that $\neg\neg a = a$ for any $a \in [0,1]$.*

A t-norm is continuous if and only if all partial mappings are continuous. A t-norm is called left continuous if all partial mappings are left continuous. Obviously, any continuous t-norm is also left continuous. Left-continuity is crucial, since it allows to define the implication as the residuum of the conjunction. Given an implication operator, also a negation operator can be constructed.

Definition 1.19 *Let $*$ be a left-continuous t-norm. Then, for any a, b in $[0,1]$ we define*

(i) the (residual) implication operator associated to $*$ is the binary operator on $[0,1]$ defined by $a \rightarrow_* b = \bigvee\{c \in [0,1] \mid a * c \leq b\}$;

(ii) the biresiduum of $*$ is the binary operator on $[0,1]$ defined by

$$a \leftrightarrow_* b = (a \rightarrow_* b) \wedge (b \rightarrow_* a)\,;$$

(iii) the negation operator associated to $*$ *is the unary operator on* $[0,1]$ *defined by* $\neg_* a = a \rightarrow_* 0$*;*

Negation operators defined in the above way are sometimes called *residual negations*, *natural negations* or *induced negation operators* [76, 85, 86].

Good reviews of the literature on implication operators can be found in [3,40,74]. The implication, biresiduum and negation operators associated to the three t-norms $*_{\mathbf{M}}$, $*_{\mathbf{P}}$ and $*_{\mathbf{L}}$ are contained in Tables 1.1, 1.2 and 1.3.

Implication
$a \rightarrow_{\mathbf{M}} b = \begin{cases} 1 & \text{, if } a \leq b \\ b & \text{, else} \end{cases}$
$a \rightarrow_{\mathbf{P}} b = \begin{cases} 1 & \text{, if } a \leq b \\ b/a & \text{, else} \end{cases}$
$a \rightarrow_{\mathbf{L}} b = \begin{cases} 1 & \text{, if } a \leq b \\ 1 - a + b & \text{, else} \end{cases}$

Table 1.1: Implication operators associated to minimum, product and Łukasiewicz t-norms.

In this work we suppose that the t-norm is chosen first and then implication, biresiduum and negation operators are induced by the t-norm according to Definition 1.19. In no case we will work with one t-norm and an implication, biresiduum or negation operator induced from another triangular norm. For this reason we can avoid the notation of implication, biresiduum and negation operators ($\rightarrow_*$, $\leftrightarrow_*$ and $\neg_*$) where the dependence

Biresiduum
$a \leftrightarrow_{\mathbf{M}} b = \begin{cases} 1 & \text{, if } a = b \\ a \wedge b & \text{, else} \end{cases}$
$a \leftrightarrow_{\mathbf{P}} b = \frac{a \wedge b}{a \vee b}$
$a \leftrightarrow_{\mathbf{L}} b = 1- \mid a - b \mid$

Table 1.2: Biresiduum operators associated to minimum, product and Łukasiewicz t-norms.

Negation
$\neg_{\mathbf{M}} a = \begin{cases} 1 & \text{, if } a = 0 \\ 0 & \text{, else} \end{cases}$
$\neg_{\mathbf{P}} a = \begin{cases} 1 & \text{, if } a = 0 \\ 0 & \text{, else} \end{cases}$
$\neg_{\mathbf{L}} a = 1 - a$

Table 1.3: Negations associated to minimum, product and Łukasiewicz t-norms.

on the t-norm has to be made explicit, preferring the shorter notations $\rightarrow$, $\leftrightarrow$ and $\neg$.

We recall a list of properties of a t-norm $*$ and its associated operators.

Proposition 1.20 *Let $*$ be a left-continuous t-norm and let $\rightarrow$ be its associated implication operator. Then the following properties hold, for any $a, b, c \in [0, 1]$:*

Property 1: $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$*;*

Property 2: $a * (a \rightarrow b) \leq a \wedge b$*;*

Property 3: *if $*$ is continuous, then $a * (a \rightarrow b) = a \wedge b$;*

Property 4: $a \leq b \Rightarrow a \rightarrow b = 1$*;*

Property 5: *if $*$ is continuous, then $a \leq b \Leftrightarrow a \rightarrow b = 1$;*

Property 6: $b \leq a \rightarrow b$*;*

Property 7: $1 \rightarrow a = a$*;*

Property 8: $a \leq b \Rightarrow a \rightarrow c \geq b \rightarrow c$*;*

Property 9: $a \leq b \Rightarrow c \rightarrow a \leq c \rightarrow b$*;*

Property 10: $(a \rightarrow b = 0) \Rightarrow (b = 0)$*;*

Property 11: *if $*$ has no zero divisors, then $a \rightarrow 0 = 0$;*

Property 12: $(a \rightarrow b) * (b \rightarrow c) \leq a \rightarrow c$*.*

Remark 1.21 *Observe that only left-continuous t-norms guarantee the inequality $a * (a \rightarrow b) \leq b$ and therefore for these t-norms we can assure that $a * \neg a = 0$.*

A negation operator is called strong if it is strictly decreasing and involutive, i.e. if $\neg\neg a = a$ holds for any $a \in [0,1]$. The relevance of left-continuous t-norms with strong induced negations has already been stressed in the literature [75,76], specially when applied to mathematical models for fuzzy preference structures [127]. In those works the importance of the rotation invariant property of the t-norm w.r.t. the negation is highlighted: $a * b \leq c \Leftrightarrow b * \neg c \leq \neg a$, for any $a, b, c \in [0,1]$. The relevance of the t-norms that satisfy the rotation invariance property is due to the fact that the induced negation is always a strong negation.

Lemma 1.22 ([75]) *Let $*$ be a left-continuous t-norm and let $\neg$ be a strong negation operator. The following statements are equivalent:*

(i) the t-norm $*$ is rotation invariant w.r.t. $\neg$, i.e. $a*b \leq c \Leftrightarrow b*\neg c \leq \neg a$, for any $a, b, c \in [0,1]$,

(ii) the negation operator induced by $*$ equals $\neg$.

Remark 1.23 *For any left-continuous t-norm $*$ whose associated negation operator $\neg$ is involutive (i.e. $a = \neg\neg a$), it holds that $b > a \Leftrightarrow b * \neg a > 0$, for any $a, b \in [0,1]$.*

Remark 1.24 *For any rotation invariant t-norm $*$ it holds that $b > a$ if and only if $b * \neg a > 0$, for any $a, b \in [0,1]$.*

1.2.3 Fuzzy preference relations

Zadeh [134] proposed to extend the classical concept of relation between pairs of alternatives to the framework of approximate reasoning. As for the case of fuzzy sets used for describing imprecise properties of the element of an universe, also binary relations can be adapted for representing intermediate degrees of relation between alternatives. This proposal was very successful, as testifies the vast literature on fuzzy preference relations [8, 12–16, 18, 20, 21, 44, 73, 77, 79, 99, 103, 107, 117, 118]. In particular, properties like transitivity have been investigated in detail by researchers who tried to reproduce classical results on preference for the fuzzy set framework. It is impossible to refer here to all the works and authors that worked on this subject, but the interested reader can enjoy this basic literature: [22, 27, 34–36, 41–43, 51, 53, 55, 56, 100, 102, 105, 130].

A fuzzy relation Q on X is a mapping from $X \times X \to [0,1]$ such that for any two alternatives x, y in X, the value of $Q(x,y)$ stands for the degree to which x is connected to y by Q. The fuzzy relation Q can be represented as a matrix $Q = (q_{ij})_{i,j \in \{1,\dots,N\}}$, where $q_{ij} = Q(x_i, x_j)$ and $N = |X|$. In this work we are interested in fuzzy preference relations, i.e. reflexive fuzzy relations (for all $x \in X$, $Q(x,x) = 1$), where the value of $Q(x,y)$ can be interpreted as the degree to which x is considered to be at least as good as y. Given a fuzzy preference relation Q, its asymmetric part P_Q, which represents the degree of strict preference, and its symmetric part I_Q, which represents the degree of indifference, are computed as

$$P_Q(x,y) = Q(x,y) * \neg Q(y,x)\,, \tag{1.4}$$

$$I_Q(x,y) = Q(x,y) * Q(y,x)\,. \tag{1.5}$$

Other possible constructions for P_Q and I_Q have been proposed, as shown in [29, 34, 50, 52, 54, 100–102, 128], but in this work we will only consider this one, as in [64].

Other properties of fuzzy preference relations that will be mentioned in this work are the following: for any x, y and z in X, a fuzzy preference relation Q can satisfy

(i) weak completeness: if $Q(x,y) \vee Q(y,x) > 0$;

(ii) moderate completeness: if $Q(x,y) + Q(y,x) \geq 1$;

(iii) strong completeness: if $Q(x,y) \vee Q(y,x) = 1$;

(iv) $*$-transitivity: if $Q(x,y) * Q(y,z) \leq Q(x,z)$;

(v) $*$-quasi-transitivity: if $P_Q(x,y) * P_Q(y,z) \leq P_Q(x,z)$;

(vi) $*$-regularity: a shorthand for the simultaneous fulfillment of strong completeness and $*$-transitivity;

(vii) acyclicity [64]: if for any $n \geq 2$ and $x_1, x_2, \ldots, x_n \in X$:

$$\text{if } \left\{ \begin{array}{ccc} Q(x_1,x_2) & > & Q(x_2,x_1)\,, \\ Q(x_2,x_3) & > & Q(x_3,x_2)\,, \\ & \cdots & \\ Q(x_{n-1},x_n) & > & Q(x_n,x_{n-1})\,, \end{array} \right. \quad \text{then } Q(x_1,x_n) \geq Q(x_n,x_1);$$

(viii) $*$-acyclicity [65]: if, for any $n \geq 2$ and $x_1, x_2, \ldots, x_n \in X$, it holds that

$$P_Q(x_1,x_2) * P_Q(x_2,x_3) * \ldots * P_Q(x_{n-1},x_n) * Q(x_n,x_1) \leq Q(x_1,x_n)\,.$$

Obviously, strong completeness implies moderate completeness which in turn implies weak completeness. We will also prove in Proposition 2.6 that acyclicity implies $*$-acyclicity.

Given a fuzzy preference relation Q and a left-continuous t-norm $*$, we will denote its $*$-transitive closure by $\widehat{Q^*}$, i.e. $\widehat{Q^*}$ is the smallest $*$-transitive fuzzy relation such that $\widehat{Q^*}(x,y) \geq Q(x,y)$, for any $x,y \in X$. If a fuzzy preference relation Q is $*$-transitive, then $\widehat{Q^*} = Q$. Several results on the $*$-transitive closure of a fuzzy preference relation can be found in [29]. In particular, the existence of such a closure is guaranteed for any fuzzy preference relation. Furthermore, if the t-norm is the minimum and the universe X has finite cardinality N, then the $*_{\mathbf{M}}$-transitive closure of a fuzzy preference relation Q can be computed as

$$\widehat{Q^{*_{\mathbf{M}}}} = \bigcup_{k=1}^{N} Q^{(k)_{*_{\mathbf{M}}}} ,$$

where $Q^{(k)_{*_{\mathbf{M}}}} = (Q^{(k-1)_{*_{\mathbf{M}}}} \circ_{*_{\mathbf{M}}} Q)$ and $(Q \circ_{*_{\mathbf{M}}} Q')(x,z) = \sup_{y\in X}(Q(x,y) *_{\mathbf{M}} Q'(y,z))$.

Chapter 2

Fuzzy Choice Theory

2.1 The evolution of fuzzy choice modelling

In Section 1.1 the classical theory of individual choice has been presented, focusing on the concepts of choice function, revealed preference relation and rationality. All the definitions given there only allowed for crisp (or exact) interpretations: an alternative x is either chosen or not from a set S, x can only be preferred or not preferred to y. We then introduced in Section 1.2 a new formalism that allows to handle situations where vagueness is involved. We will use the adjective crisp (or exact) to refer to the classical bivalent framework, in contrast with the adjective fuzzy, which will be used to address to the multivalent situation. According to this dichotomy, we can distinguish between different proposals in the literature on choice modelling:

(i) exact preference and exact choice on crisp sets of alternatives;

(ii) fuzzy preference and exact choice on crisp sets of alternatives;

(iii) fuzzy preference and fuzzy choice on crisp sets of alternatives;

(iv) fuzzy preference and fuzzy choice on fuzzy sets of alternatives.

The first situation corresponds to the theory of choice already presented in Section 1.1.

The second case represents a first attempt to introduce vagueness in the context of choice modelling. It first appeared in a work of Orlovsky [99] and has been followed by many other authors [8, 12–16, 18, 20, 21, 44, 73, 77, 79, 103,107,117,118]. The purpose is to construct a crisp choice function from a fuzzy preference relation Q defined over a finite non-empty set of alternatives X. Again, $\mathcal{B}$ will represent a family of non-empty subsets of X. The method proposed by Orlovsky involves the use of a score function associated to a fuzzy preference relation Q, i.e. a function that to any alternative in a set $S \in \mathcal{B}$ assigns a score U_Q according to the information contained in Q:

$$\begin{aligned} U_Q : X \times \mathcal{B} &\rightarrow \mathbb{R} \\ (x, S) &\mapsto U_Q(x, S)\,. \end{aligned}$$

The choice function $C_Q(S)$ is then constructed from the score function U_Q by choosing those alternatives in S that have the best score:

$$C_Q(S) = \{x \in S \mid U_Q(x, S) \succcurlyeq_{U_Q} U_Q(y, S), \forall y \in S\}\,.$$

We say that x has better or equal score than y ($x \succcurlyeq_{U_Q} y$) if the score of x is better that the score of y (we don't use $\geq$ because there are score functions for which higher scores correspond to worse alternatives). The choice function C_Q is then called preference-based choice function. Different score functions have been proposed by Orlovsky [99] and Barret et al. [14].

Another way of defining a choice function from a given fuzzy preference relation Q is inspired by the construction of G-rational choice functions in the classical case:

$$G_Q(S) = \{x \in S \mid Q(x, y) \geq Q(y, x), \forall y \in S\}\,.$$

It is exactly the same equation used in Definition 1.9, but this time a fuzzy preference relation is used instead of a crisp one. This definition suffers from the same weakness as its crisp counterpart: there could be cases in which the generated function G_Q is not a choice function, since it assigns an empty choice set to some $S \in \mathcal{B}$ ($G_Q(S) = \emptyset$, for some $S \in \mathcal{B}$). The solution, as for the classical case, comes by imposing some conditions on Q, in particular completeness and acyclicity. Important results on the rationality of choice functions generated from a fuzzy preference relation are contained in [18,21].

The third case listed earlier (fuzzy preference and fuzzy choice between exact alternatives) has been proposed first by Banerjee [10] and further corrected and extended by Wang [131]. They develop a theory of choice where both preferences and choice functions are allowed to be fuzzy. Banerjee justifies his approach by saying that: "*if preferences are permitted to be fuzzy, it seems natural to permit the choice function to be fuzzy as well. This also tallies with experience. For instance, a decision-maker, faced with the problem of deciding whether or not to choose an alternative x from a set of alternatives A, may feel that he/she is inclined to the extent 0.8 (on, say, a scale from 0 to 1) toward choosing it. Moreover, this fuzziness of choice is, at least potentially, observable. For instance, the decision-maker in the example will be able to tell an interviewer the degree of his/her inclinations, or demonstrate these inclinations to an observer by the degree of eagerness or enthusiasm which he/she displays. Hence, while there may be problems of estimation, fuzzy choice functions are, in theory, observable*". Formally, a fuzzy choice function in the sense of Banerjee is a function $C : \mathcal{B} \to \mathcal{F}$, where $\mathcal{F}$ is a family of non-empty fuzzy subsets of X. For any $S \in \mathcal{B}$, $C(S)$ is a fuzzy set with non-empty support, where $C(S)(x)$ represents the extent to which alternative x belongs to the set of chosen alternatives from S. The fuzzy set $C(S)$ should always be included into the available set S, i.e.

$supp(C(S)) \subseteq S$ and there should always exist an element x for any $S \in \mathcal{B}$, such that $C(S)(x) > 0$. In [10] rationality conditions like WARP and SARP and other basic definitions such as normality and fuzzy revealed preference are adapted to the new, more general framework. The main result of this approach to fuzzy choice theory is a theorem on the same style as Arrow-Sen Theorem, which states equivalent conditions to the joint rationality of the fuzzy choice function and transitivity of the revealed fuzzy preference relation.

The most general definition of fuzzy choice function known in the literature is the one proposed by Georgescu [64]. In fact, she considered the case where both choices and preferences are fuzzy and are moreover defined over a family of fuzzy subsets of X. It corresponds to the fourth case listed earlier. The family $\mathcal{B}$ of sets on which the choices are performed can also contain fuzzy subsets of X. As in the crisp case, a set S in $\mathcal{B}$ is called available set.

Definition 2.1 ([64]) *Let X be a finite set of alternatives, $\mathcal{B}$ a family of non-empty fuzzy subsets of X and $\mathcal{F}$ the family of non-empty fuzzy subsets of X. The pair $(X, \mathcal{B})$ is called fuzzy choice space. A fuzzy choice function in the sense of Georgescu (a fuzzy choice function for the rest of this chapter) is a function $C : \mathcal{B} \to \mathcal{F}$ that assigns to each available fuzzy set S a fuzzy set $C(S)$ (called chosen set) in such a way that:*

(i) there is at least one alternative $x \in X$ such that $C(S)(x) > 0$;

(ii) $C(S)(x) \leq S(x)$, for any $x \in X$.

The former condition establishes that for any available fuzzy set, at least one alternative has to be chosen to some strictly positive degree. It corresponds to the condition of non-empty choice of the crisp case. The latter condition states that no element can be more *eligible* than *available* and corresponds to the crisp condition $C(S) \subseteq S$.

In this chapter, we will adhere to Definition 2.1 since it is the most general one and so the results obtained for it are trivially valid for the other cases as well, namely classical choice functions, Orlovsky choice functions or Banerjee's one. For this reason, we have to introduce the definitions of revealed preference relation and G- and M- rationalization/normalization adapted to the new framework. The following definitions involve the use of t-norms. Recall that we assume that the t-norm is chosen first and then the related operators are derived from it according to Definition 1.19.

Definition 2.2 ([64]) *Let C be a fuzzy choice function on X and $*$ a left-continuous t-norm. The fuzzy revealed preference relation R_C associated to C is defined for any $x, y \in X$ as:*

$$R_C(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y)) .$$

The value of $R_C(x, y)$ expresses the maximum degree to which x is chosen in case y is also available over all the available sets $S \in \mathcal{B}$. The associated fuzzy strict preference relation P_C and fuzzy indifference relation I_C can be computed as in Eqs. (1.4) and (1.5):

$$\begin{aligned} P_C(x, y) &= R_C(x, y) * \neg R_C(y, x) , \\ I_C(x, y) &= R_C(x, y) * R_C(y, x) . \end{aligned}$$

Georgescu [64] also generalized the definitions of base and strong revealed preference relations given in Definition 1.7 for the crisp case.

(i) $\bar{R}_C(x, y) = C(\{x, y\})(x)$, where $\{x, y\}$ is the crisp set containing x and y;

(ii) $\tilde{P}_C(x, y) = \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y) * \neg C(S)(y))$.

Observe that in order to assure that $\bar{R}_C$ is well defined, the family $\mathcal{B}$ has to contain all subsets of X with exactly two elements.

Georgescu [64] proposed two ways of rationalizing a fuzzy choice function given a fuzzy preference relation Q that generalize to the fuzzy set framework the approach based on the concept of greatest elements and the approach based on the concept of maximal elements.

Definition 2.3 ([64]) *Let $*$ be a left-continuous t-norm and Q a fuzzy preference relation on X. We define the following functions:*

$$\begin{array}{rrcl} G_Q : & \mathcal{B} & \to & \mathcal{F}(X) \\ & S & \mapsto & G_Q(S) \end{array}$$

where, for any $x \in X$ and any $S \in \mathcal{B}$,

$$G_Q(S)(x) = S(x) * \bigwedge_{y \in X} (S(y) \to Q(x, y)) \tag{2.1}$$

and

$$\begin{array}{rrcl} M_Q : & \mathcal{B} & \to & \mathcal{F}(X) \\ & S & \mapsto & M_Q(S) \end{array}$$

where, for any $x \in X$ and any $S \in \mathcal{B}$,

$$M_Q(S)(x) = S(x) * \bigwedge_{y \in X} (S(y) \to (Q(y, x) \to Q(x, y))) \, . \tag{2.2}$$

The value of $G_Q(S)(x)$ (resp. $M_Q(S)(x)$) represents the degree to which alternative x satisfies the property of being the greatest (resp. maximal) element in S w.r.t. the information contained in Q. Given a fuzzy preference relation Q, the functions G_Q and M_Q are not fuzzy choice functions in general. In fact, for some $S \in \mathcal{B}$, it may happen that $G_Q(S)(x) = 0$ or $M_Q(S)(x) = 0$, for any $x \in X$. The second condition of fuzzy choice functions, i.e. $C(S)(x) \leq S(x)$, is trivially satisfied by both G_Q and M_Q, for any $S \in \mathcal{B}$ and $x \in X$.

Definition 2.4 ([64]) *Let $*$ be a left-continuous t-norm and C a fuzzy choice function on X.*

(i) C is called G-rational if there exists a fuzzy preference relation Q such that $C(S)(x) = G_Q(S)(x)$, for any $S \in \mathcal{B}$ and $x \in X$. It is called M-rational if there exists a fuzzy preference relation Q such that $C(S)(x) = M_Q(S)(x)$, for any $S \in \mathcal{B}$ and $x \in X$. We say that Q is the G-rationalization of C (resp. M-rationalization), if C is G-rational (resp. M-rational) from the fuzzy preference relation Q. We can also say that C is G-rationalizable (resp. M-rationalizable) by Q.

(ii) C is called totally G-rational if there exists a fuzzy relation Q on X that is reflexive, $*$-transitive, weakly complete and such that $C(S)(x) = G_Q(S)(x)$, for any $S \in \mathcal{B}$ and $x \in X$.

(iii) C is called G-normal (resp. M-normal) if it is G-rationalizable (resp. M-rationalizable) by its own fuzzy revealed preference relation R_C.

Notice that a fuzzy choice function is not always G-rationalizable, as well as a fuzzy preference relation not always G-rationalizes a fuzzy choice function. Clearly, a G-normal fuzzy choice function C is always G-rational, while the converse implication is not true in general. However, for the minimum operator, both definitions are equivalent, as proved in [64].

Proposition 2.5 (Proposition 5.40 of [64]) *Let $*$ be the minimum. For any fuzzy choice function C on X, the following statements are equivalent:*

(i) C is G-rational;

(ii) C is G-normal.

2.2 Rationalization and normalization of a fuzzy choice function

The aim of this section is to unveil under which conditions a fuzzy preference relation Q rationalizes a fuzzy choice function C, i.e. which conditions on Q ensure that C is G-rational or M-rational. Among these conditions we find properties such as reflexivity, different types of completeness and two types of acyclicity. Special attention will be paid to the choice of the t-norm. In fact, older results on this subject have been proved only for particular t-norms, while we generalize them for a wider family of t-norms.

The inspiration of this section is to be found in classical choice theory, hence it is worthwhile to compare our results with those known from the classical theory (recalled here in Propositions 2.13 and 2.14 on page 41). For the case of G-rational choice functions, we have been able to recover only one implication of the original result by Sen [114], while for the case of M-rationality, the result of Walker [129] has been fully extended.

The obtained results will be used for extending the Richter Theorem to the fuzzy set framework. The Richter Theorem (here Theorem 1.15) is an important classical result establishing the equivalence between rationality and congruence of crisp choice functions. Georgescu [64] already attempted its extension to the fuzzy set framework, but she only proved one implication of the original theorem, leaving the other implication as an open problem. In this work we provide a counterexample showing that the missing implication does not hold in general in the fuzzy set framework.

The contents of this section have already been accepted for publication in [89].

2.2.1 On the acyclicity property of fuzzy preference relations

Before presenting the main results of this section we prove some interesting properties of acyclic and $*$-acyclic fuzzy preference relations.

First of all, we prove that acyclicity is a stronger condition than $*$-acyclicity.

Proposition 2.6 *Let $*$ be a left-continuous t-norm or a t-norm without zero divisors. If the fuzzy preference relation Q is acyclic, then it is also $*$-acyclic.*

Proof. Let $\{x_1, \ldots, x_n\}$ be an arbitrary subset of X.

If the inequality

$$Q(x_i, x_{i+1}) > Q(x_{i+1}, x_i)$$

is satisfied for any $i \in \{1, \ldots, n-1\}$, then the acyclicity of Q implies that $Q(x_1, x_n) \geq Q(x_n, x_1)$. It follows immediately that $Q(x_1, x_n) \geq Q(x_n, x_1) * P_Q(x_1, x_2) * \ldots * P_Q(x_{n-1}, x_n)$.

On the other hand, if there exists at least one $j \in \{1, \ldots, n-1\}$ such that $Q(x_j, x_{j+1}) \le Q(x_{j+1}, x_j)$, then

$$\begin{aligned} P_Q(x_j, x_{j+1}) &= Q(x_j, x_{j+1}) * \neg Q(x_{j+1}, x_j) \\ &= Q(x_j, x_{j+1}) * (Q(x_{j+1}, x_j) \to 0) \\ &\le Q(x_j, x_{j+1}) * (Q(x_j, x_{j+1}) \to 0). \end{aligned}$$

Now,

(i) If $*$ does not have zero divisors, then, due to Property 11 of implication operators (Proposition 1.20), it holds that $Q(x_j, x_{j+1}) \to 0 = 0$ and, hence, $P_Q(x_j, x_{j+1}) = 0$.

(ii) If $*$ is left continuous, by Property 2 of Proposition 1.20, then

$$Q(x_j, x_{j+1}) * (Q(x_j, x_{j+1}) \to 0) \le Q(x_j, x_{j+1}) \wedge 0 = 0\,,$$

and again $P_Q(x_j, x_{j+1}) = 0$.

Hence, in any case $P_Q(x_j, x_{j+1}) = 0$ and $Q(x_1, x_n) \ge Q(x_n, x_1) * P_Q(x_1, x_2) * \ldots * P_Q(x_{n-1}, x_n)$ is trivially satisfied. $\square$

The converse implication does not hold in general, regardless of the t-norm considered.

Propositions 2.7 and 2.9 shown next will play a key role in the proof of Theorems 2.15 and 2.19, the main results of this section.

Proposition 2.7 *If a fuzzy preference relation Q on X satisfies at least one of the following sets of hypotheses:*

Hypo. 1 *Q is weakly complete and $*$-acyclic, with $*$ a t-norm without zero divisors;*

Hypo. 2 *Q is moderately complete and $*$-acyclic with $*$ any t-norm,* *then*

$$\bigvee_{x\in X}\bigwedge_{y\in X} Q(x,y) > 0\,. \tag{2.3}$$

Proof. Since the proofs for both sets of hypotheses are similar, we merged them into a single proof. Those steps in the proof that differ for the two sets of hypotheses will be treated separately and labelled Hypo. 1 and Hypo. 2. Suppose, by absurdum, that $\bigvee_{x\in X}\bigwedge_{y\in X} Q(x,y) = 0$, then for any $x \in X$, it holds that $\bigwedge_{y\in X} Q(x,y) = 0$. This means that for any $x \in X$, there exists at least one $x_1 \in X$ such that $Q(x,x_1) = 0$. Since Q is reflexive, it holds that $x_1 \neq x$. Consider such an element $x_1 \in X$. By the preceding reasoning, there exists $x_2 \neq x_1 \in X$ such that $Q(x_1,x_2) = 0$. Since Q is at least weakly complete, it holds that $Q(x_2,x_1) > 0$. Thus,

$$Q(x_1,x_2) = 0\,, \tag{2.4}$$

$$Q(x_2,x_1) > 0\,. \tag{2.5}$$

Similarly, there exists an element $x_3 \neq x_2$ such that $Q(x_2,x_3) = 0$ and $Q(x_3,x_2) > 0$. Thus,

$$Q(x_2,x_3) = 0\,, \tag{2.6}$$

$$Q(x_3,x_2) > 0\,. \tag{2.7}$$

and $x_3 \neq x_1$, since otherwise $Q(x_1,x_2) = Q(x_3,x_2) > 0$ contradicts Eq. (2.4). By induction, suppose that we have a set $\{x_1,\ldots,x_n\} \subseteq X$ such that:

(i) $Q(x_{i+1},x_i) > Q(x_i,x_{i+1}) = 0$ for $i \in \{1,\ldots,n-1\}$;

(ii) $x_i \neq x_j$ for $i \neq j$.

Since $Q(x_i,x_{i+1}) = 0$, it holds that $\neg Q(x_i,x_{i+1}) = 1$ and so

$$P_Q(x_{i+1},x_i) = Q(x_{i+1},x_i) * \neg Q(x_i,x_{i+1}) = Q(x_{i+1},x_i)\,.$$

Also, if Q is moderately complete, then $Q(x_i, x_{i+1}) = 0$ implies $Q(x_{i+1}, x_i) = 1$. Then we can add two conditions that follow from the hypothesis of induction:

(iii) $P_Q(x_{i+1}, x_i) = Q(x_{i+1}, x_i)$ for $i \in \{1, \ldots, n-1\}$,

(iv) $Q(x_{i+1}, x_i) = 1$, for $i \in \{1, \ldots, n-1\}$; but only if using Hypo. 2.

Consider x_n. Since $\bigvee_{x\in X} \bigwedge_{y\in X} Q(x, y) = 0$, there exists an element $x_{n+1} \neq x_n$ such that $Q(x_n, x_{n+1}) = 0$ and $Q(x_{n+1}, x_n) > 0$. Thus

$$Q(x_n, x_{n+1}) = 0\,, \tag{2.8}$$

$$Q(x_{n+1}, x_n) > 0\,. \tag{2.9}$$

The element x_{n+1} cannot be equal to x_{n-1}, otherwise Eq. (2.9) contradicts point (i) of the induction. Moreover, also $x_{n+1} \neq x_j$ for $j \leq n-2$. To prove this, observe that due to $*$-acyclicity, for any $j \leq n-2$,

$$P_Q(x_n, x_{n-1}) * P_Q(x_{n-1}, x_{n-2}) * \cdots * P_Q(x_{j+1}, x_j) * Q(x_j, x_n) \leq Q(x_n, x_j)\,,$$

or, equivalently,

$$Q(x_n, x_{n-1}) * Q(x_{n-1}, x_{n-2}) * \cdots * Q(x_{j+1}, x_j) * Q(x_j, x_n) \leq Q(x_n, x_j)\,.$$

Now, if we suppose that $x_{n+1} = x_j$, then we have

$$\begin{aligned} 0 \;=\; & Q(x_n, x_{n+1}) = Q(x_n, x_j) \geq \\ & Q(x_n, x_{n-1}) * Q(x_{n-1}, x_{n-2}) * \cdots * Q(x_{j+1}, x_j) * Q(x_j, x_n)\,, \end{aligned}$$

but this will lead to a contradiction.

Hypo. 1 The contradiction follows from the fact that $*$ has no zero divisors and according to what stated in (i), all the elements in the right-hand side of the inequality are strictly positive.

Hypo. 2 According to what stated in (iv), the above inequality becomes

$$Q(x_j, x_n) \leq Q(x_n, x_j)\,.$$

However, if $x_{n+1} = x_j$, this contradicts Eqs. (2.8) and (2.9).

Hence, we have shown that $x_{n+1} \neq x_i$, for any $i \in \{1, 2, \ldots, n\}$, and this holds for any n. Since X is finite, this leads to a contradiction. □

In the previous proposition, both sets of hypotheses act as sufficient conditions. The first set contains the weakest completeness condition; its drawback is that a restriction on the t-norm has to be imposed. On the other hand, the second set of hypotheses does not contain restriction on the t-norm, forcing the completeness condition to be strengthened. As a direct consequence of Propositions 2.6 and 2.7, a new sufficient condition can be obtained.

Corollary 2.8 *If a fuzzy preference relation Q on a finite set X is weakly complete and acyclic, then Eq. (2.3) holds.*

The second set of results in this section is based on another property, similar to Eq. (2.3), presented in the following proposition. It will be essential in the proof of Theorem 2.19 of Subsection 2.2.2.

Proposition 2.9 *Let $*$ be a left-continuous t-norm. If a fuzzy preference relation Q on X is $*$-acyclic, then*

$$\bigvee_{x \in X} \bigwedge_{y \in X} (Q(y, x) \rightarrow Q(x, y)) > 0\,. \tag{2.10}$$

Proof. The proof of this proposition follows the same steps as the proof of Proposition 2.7, *mutatis mutandis*, and is not reported here. The interested reader can easily work it out by himself, by following the next steps:

1. By absurdum, suppose that Eq. (2.10) is not satisfied.
2. Consider an alternative $x_1 \in X$ for which Eq. (2.10) does not hold.
3. Using Properties 4 and 10 of Proposition 1.20, find another alternative $x_2 \neq x_1$ such that
   (i) $Q(x_2, x_1) \to Q(x_1, x_2) = 0$;
   (ii) $Q(x_2, x_1) > Q(x_1, x_2) = 0$.
4. Repeat step 3 for alternative x_2, in order to find a new alternative $x_3 \neq x_2 \neq x_1 \neq x_3$.
5. The basis of the induction: construct a set of alternatives $\{x_1, \ldots, x_n\}$ contained in X such that
   (i) $Q(x_{i+1}, x_i) \to Q(x_i, x_{i+1}) = 0$, for any $i \in \{1, 2, \ldots, n-1\}$;
   (ii) $Q(x_{i+1}, x_i) > Q(x_i, x_{i+1}) = 0$, for any $i \in \{1, 2, \ldots, n-1\}$;
   (iii) $x_i \neq x_j$, for any $i \neq j$;
   (iv) $P_Q(x_i, x_{i+1}) = Q(x_i, x_{i+1})$, for any $i \in \{1, 2, \ldots, n-1\}$.
6. The inductive step: consider the last element x_n and prove that under the hypothesis of the induction, there exists a new alternative x_{n+1} such that $x_{n+1} \neq x_i$, for any $i \in \{1, 2, \ldots, n\}$. (Hint: use $*$-acyclicity of Q)
7. The absurdum: the induction leads to an infinite sequence of different alternatives that do not satisfy Eq. (2.10). However, X is finite, hence the contradiction. □

We conclude this subsection recalling a useful result of Wang [130] and two properties of t-norms with zero divisors.

Proposition 2.10 ([130]) *If a fuzzy preference relation Q is acyclic on X, then it is acyclic on any $S \subseteq X$.*

The same result can be shown easily for $*$-acyclic relations.

Lemma 2.11 *Let $*$ be a t-norm. If a fuzzy preference relation Q is $*$-acyclic on X, then it is $*$-acyclic on any $S \subseteq X$.*

Lemma 2.12 *For any pair (a, b) of zero divisors of a t-norm $*$ (i.e. $a * b = 0$), it holds that*

(i) $a \to 0 \geq b$ *and* $b \to 0 \geq a$*;*

(ii) $a \to c \geq b$ *for any* $c \in [0, 1]$.

Proof. The first point follows from the definition of implication operator, while the second one follows from Property 9 of implication operators (Proposition 1.20) and the previous point. □

2.2.2 Sufficient conditions for the G-rationality and the M-rationality of a fuzzy choice function

In this subsection we present sufficient conditions on a fuzzy preference relation Q to ensure that G_Q and M_Q are fuzzy choice functions. They are inspired by the two classical results of Sen [114] and Walker [129].

Proposition 2.13 ([114]) *Let Q be a crisp complete preference relation on X. Then Q is acyclic if and only if the set of greatest elements $G_Q(S) = \{x \in S \mid Q(x, y) = 1, \forall y \in S\}$ is not empty for any crisp subset S of X.*

Proposition 2.14 ([129]) *Let Q be a crisp acyclic preference relation on X, then the set of maximal elements $M_Q(S) = \{x \in S \mid P_Q(y, x) = 0, \forall y \in S\}$ is non empty for any crisp subset S of X.*

G-rationality. Sufficient conditions to ensure that a fuzzy preference relation G-rationalizes a fuzzy choice function were already presented in [64]. However, those results are proved only for the t-norm of the minimum and did not involve any completeness condition. Moreover, we showed that they are incorrect [91]. The following theorem is a twofold improvement: on the one hand it uses $*$-acyclicity instead of the more demanding $*$-transitivity and on the other hand it is proved for a general continuous t-norm $*$ instead of only for the minimum.

Theorem 2.15 *Let $*$ be a continuous t-norm. If a fuzzy preference relation Q on X satisfies at least one of the following sets of hypotheses:*

Hypo. 1 *Q is weakly complete and $*$-acyclic and the t-norm $*$ has no zero divisors;*

Hypo. 2 *Q is moderately complete and $*$-acyclic;*

then G_Q is a fuzzy choice function.

Proof. To prove that G_Q is a fuzzy choice function, we need to show that:

(i) $G_Q(S)(x) \leq S(x)$, for any $S \in \mathcal{B}$ and $x \in X$;

(ii) there exists at least one $x \in X$ such that $G_Q(S)(x) > 0$, for any $S \in \mathcal{B}$.

The first point follows from the definition of G and the monotonicity of the t-norm. The second one is less trivial. To prove it, consider a generic available fuzzy set $S \in \mathcal{B}$ and let N denote the cardinality of X. Recall the expression of the function G:

$$G_Q(S)(x) = S(x) * \bigwedge_{y \in X} (S(y) \to Q(x,y)) \,.$$

This equation can be expanded in the following way:

$$\begin{aligned} G_Q(S)(x) = & \left[S(x) * (S(x_1) \to Q(x,x_1))\right] \\ & \wedge\left[S(x) * (S(x_2) \to Q(x,x_2))\right] \\ & \wedge \ldots \wedge \left[S(x) * (S(x_N) \to Q(x,x_N))\right]. \end{aligned}$$

For a better understanding of the proof, we propose a more graphical representation of the problem. Consider the following matrix:

$$\begin{pmatrix} S(x_1) * (S(x_1) \to Q(x_1,x_1)) & \ldots & S(x_1) * (S(x_N) \to Q(x_1,x_N)) \\ S(x_2) * (S(x_1) \to Q(x_2,x_1)) & \ldots & S(x_2) * (S(x_N) \to Q(x_2,x_N)) \\ \vdots & & \vdots \\ S(x_N) * (S(x_1) \to Q(x_N,x_1)) & \ldots & S(x_N) * (S(x_N) \to Q(x_N,x_N)) \end{pmatrix} . \tag{2.11}$$

The value of $G_Q(S)(x_i)$ is the minimum of the values taken by the elements in the i-th row of Matrix (2.11). There exists an alternative x such that $G_Q(S)(x) > 0$ if and only if there exists a row in which all elements are strictly positive. To further simplify the graphical representation, we propose the following notations:

+ if the corresponding value in the matrix is strictly positive;

? if the corresponding value in the matrix has not been identified yet;

0 if the corresponding value in the matrix is zero.

As said before, we are looking for a row without zeros, so, as soon as an element is detected to be zero, the corresponding row becomes irrelevant and for simplicity will be filled with zeros. For the moment, since no assumption has been made yet, the $N \times N$ matrix is filled with **?**.

Let us take the first step forward. Denote by $\Sigma_0 = \{x \in X \mid S(x) > 0\}$ the support of the fuzzy set S. It is obvious that if an alternative x does not belong to Σ_0, then

$$S(x) * (S(y) \to Q(x,y)) = 0 * (S(y) \to Q(x,y)) = 0\,,$$

for any $y \in X$. Therefore, those rows in the matrix corresponding to alternatives in $X \setminus \Sigma_0$ are filled with zeros. For any $x \in \Sigma_0$ and $y \notin \Sigma_0$, it holds that

$$S(x) * (S(y) \to Q(x,y)) = S(x) * (0 \to Q(x,y)) = S(x) > 0\,.$$

These first results are graphically described in Matrix (2.12), where we have reordered the elements of X (and the corresponding rows and columns of the matrix) in such a way that the elements not belonging to Σ_0 come first. Such kind of reordering does not affect the correctness of the proof and will be done several times along the proof.

$$\begin{array}{c} X \setminus \Sigma_0 \left[\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}} \right. \\ \Sigma_0 \left[\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}} \right. \end{array} \begin{pmatrix} \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ + & \dots & + & \mathbf{?} & \dots & \mathbf{?} \\ \vdots & & \vdots & \vdots & & \vdots \\ + & \dots & + & \mathbf{?} & \dots & \mathbf{?} \end{pmatrix} \begin{matrix} \\ \end{matrix} \qquad \underbrace{\qquad}_{X \setminus \Sigma_0} \; \underbrace{\qquad}_{\Sigma_0} \tag{2.12}$$

We can now focus only on the sub-matrix corresponding to $\Sigma_0 \times \Sigma_0$. The proof will conclude if we find an element in Σ_0 such that $S(x) * (S(y) \to$

$Q(x, y)) > 0$ for any $y \in \Sigma_0$. It follows from Lemma 2.11 that if Q is $*$-acyclic on X, so it is on any subset of X. Also properties like weak completeness, moderate completeness and reflexivity are preserved when a subset of X is considered, so if Q satisfies Hypo. 1 in X then it satisfies it in any subset, in particular in Σ_0. The same can be said if it satisfies Hypo. 2. Hence, using Proposition 2.7, we know that

$$\bigvee_{x\in\Sigma_0} \bigwedge_{y\in\Sigma_0} Q(x, y) > 0\,.$$

Therefore, there exists an element $x_0 \in \Sigma_0$ such that

$$\bigwedge_{y\in\Sigma_0} Q(x_0, y) = \bigvee_{x\in\Sigma_0} \bigwedge_{y\in\Sigma_0} Q(x, y) > 0\,.$$

In other words, the observed x_0 is such that $Q(x_0, y) > 0$ for any $y \in \Sigma_0$. Let y_0 be an alternative in Σ_0 in which the minimum of $\bigwedge_{y\in\Sigma_0} (S(y) \to Q(x_0, y))$ is reached. If

$$S(x_0) * (S(y_0) \to Q(x_0, y_0)) > 0\,,$$

then the proof is complete, since in that case all the elements in the row of x_0 are strictly positive and $G_Q(S)(x_0) > 0$. We distinguish two cases:

Case 1 If $S(y_0) \leq Q(x_0, y_0)$, then $S(y_0) \to Q(x_0, y_0) = 1$ and $G_Q(S)(x_0) = S(x_0) > 0$.

Case 2 Assume $S(y_0) > Q(x_0, y_0)$.

Case 2.a If $S(y_0) \leq S(x_0)$, then using Properties 3 and 8 of Proposition 1.20 we obtain

$$\begin{aligned} S(x_0) * (S(y_0) \to Q(x_0, y_0)) &\geq S(x_0) * (S(x_0) \to Q(x_0, y_0)) \\ &= S(x_0) \wedge Q(x_0, y_0) > 0\,, \end{aligned}$$

and therefore $G_Q(S)(x_0) > 0$.

Case 2.b Assume that $S(y_0) > S(x_0)$. Since $Q(x_0, y_0) > 0$, using Property 6 of implication operators (Proposition 1.20), we obtain $S(y_0) \to Q(x_0, y_0) \geq Q(x_0, y_0) > 0$. Thus, $S(x_0) * (S(y_0) \to Q(x_0, y_0)) = 0$ holds only if $S(x_0)$ and $S(y_0) \to Q(x_0, y_0)$ are zero divisors of the t-norm.

In Cases 1 and 2.a the theorem is proved, since we have been able to exhibit one alternative x_0 such that $G_Q(S)(x_0) > 0$. It only remains to investigate what happens in Case 2.b, i.e. when

$$S(y_0) > S(x_0) > 0; \tag{2.13}$$

$$S(y_0) > Q(x_0, y_0) > 0; \tag{2.14}$$

and

$$S(x_0) \text{ and } S(y_0) \to Q(x_0, y_0) \text{ are zero divisors of the t-norm}. \tag{2.15}$$

If Hypo. 1 is assumed the proof is finished since in that case $*$ has no zero divisors. We will then focus only on Hypo. 2.

The first row of Matrix (2.16) contains the element $S(x_0) * (S(y_0) \to Q(x_0, y_0))$, which according to Eq. (2.15) is zero and then the entire row can be filled with zeros.

$$\Sigma_0 \left[\begin{array}{l} x_0 \,[\\ \\ \\ \\ \end{array} \right. \underbrace{\begin{pmatrix} \mathbf{0} & \cdots\cdots & \mathbf{0} \\ \mathbf{?} & \cdots\cdots & \mathbf{?} \\ \vdots & & \vdots \\ \mathbf{?} & \cdots\cdots & \mathbf{?} \end{pmatrix}}_{\Sigma_0} \tag{2.16}$$

Then we have to look for another element in Σ_0 such that $G_Q(S)$ is strictly positive in it. Let us introduce the subset Σ_1 of Σ_0 containing the elements z that satisfy one of the following conditions:

(i) $S(z) \geq S(y_0)$ or

(ii) $S(z)$ and $S(y_0) \to Q(x_0, y_0)$ are not zero divisors of the t-norm.

It holds that $\Sigma_1 \neq \emptyset$, since at least alternative y_0 belongs to it, but $x_0 \notin \Sigma_1$. We will prove that there exists an element x in Σ_1 such that the corresponding row of Matrix (2.11) is made up of strictly positive values (and therefore $G_Q(S)(x) > 0$). To prove this, we focus on the rows of Matrix (2.16) corresponding to alternatives in Σ_1 and we get the sub-matrix:

$$\Sigma_1 \left[\underbrace{\begin{pmatrix} ? & \dots & ? \\ \vdots & & \vdots \\ ? & \dots & ? \end{pmatrix}}_{\Sigma_0} \right. . \tag{2.17}$$

First of all, let us prove that

$$S(x) * (S(y) \to Q(x, y)) > 0$$

for any $x \in \Sigma_1$ and $y \in \Sigma_0 \backslash \Sigma_1$. To prove this, observe that since $y \in \Sigma_0 \backslash \Sigma_1$, $S(y)$ and $S(y_0) \to Q(x_0, y_0)$ are zero divisors of the t-norm. Then, using Lemma 2.12 (ii), we obtain

$$S(y) \to Q(x, y) \geq S(y_0) \to Q(x_0, y_0) \, .$$

Using this inequality, since $x \in \Sigma_1$, we get that

(i) either $S(x) \geq S(y_0)$ and thus, using Property 3 of implication operators (Proposition 1.20),

$$\begin{aligned} S(x) * (S(y) \to Q(x, y)) &\geq S(y_0) * (S(y_0) \to Q(x_0, y_0)) \\ &= S(y_0) \wedge Q(x_0, y_0) > 0 \, , \end{aligned}$$

(ii) or $S(x)$ and $S(y_0) \to Q(x_0, y_0)$ are not zero divisors of the t-norm and

$$S(x) * (S(y) \to Q(x, y)) \geq S(x) * (S(y_0) \to Q(x_0, y_0)) > 0\,.$$

In any case we have proven that $S(x) * (S(y) \to Q(x, y)) > 0$, for any $x \in \Sigma_1$ and $y \in \Sigma_0 \setminus \Sigma_1$. Thus Matrix (2.17) becomes:

$$\Sigma_1 \left[\begin{pmatrix} + & + & \dots & + & ? & \dots & ? \\ + & + & \dots & + & ? & \dots & ? \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ + & + & \dots & + & ? & \dots & ? \end{pmatrix} \right. \begin{matrix} \\ \\ \\ \end{matrix}\,. \qquad \underbrace{}_{\Sigma_0 \setminus \Sigma_1} \quad \underbrace{}_{\Sigma_1}$$

So we can focus on the sub-matrix corresponding to $\Sigma_1 \times \Sigma_1$. If all the elements of one row of this matrix are strictly positive, the proof is complete. The relation Q restricted to Σ_1 is still reflexive, weakly (or moderately) complete and due to Lemma 2.11, $*$-acyclic. Hence, we can reproduce the same reasoning that was followed for the set Σ_0:

1. Choose one alternative x_1 in Σ_1 such that

   $$\bigwedge_{y \in \Sigma_1} Q(x_1, y) > 0\,.$$

   The existence of such x_1 is guaranteed by Proposition 2.7. Let y_1 be an alternative such that

   $$S(x_1) * (S(y_1) \to Q(x_1, y_1)) = \bigwedge_{y \in \Sigma_1} S(x_1) * (S(y) \to Q(x_1, y))\,.$$

2. The only case in which $S(x_1) * (S(y_1) \to Q(x_1, y_1)) = 0$ is when the following three conditions hold:

(i) $S(y_1) > S(x_1)$;

(ii) $S(y_1) > Q(x_1, y_1) > 0$;

(iii) $S(x_1)$ and $S(y_1) \to Q(x_1, y_1)$ are zero divisors of the t-norm.

If one of these conditions is not satisfied, then we have found a row (the one corresponding to x_1) of which all the elements are strictly positive and the theorem is proved.

3. If all of these conditions are satisfied simultaneously, define the subset Σ_2 as the subset of Σ_1 containing the elements z such that one of the following conditions is satisfied:

   (i) $S(z) \geq S(y_1)$ or

   (ii) $S(z)$ and $S(y_1) \to Q(x_1, y_1)$ are not zero divisors of the t-norm.

4. The set Σ_2 is not empty since $y_1 \in \Sigma_2$ and for any $x \in \Sigma_2$ and $y \in \Sigma_1 \setminus \Sigma_2$ it holds that

$$S(x) * (S(y) \to Q(x, y)) > 0\,.$$

5. The relation Q restricted to Σ_2 is still reflexive, weakly (or moderately) complete and, due to Lemma 2.11, $*$-acyclic.

This process eventually comes to an end either when an alternative x_i in some subset Σ_i of X satisfies that all the elements in its row are positive, or when the last set of type Σ_n contains only one alternative and the corresponding matrix is made up by one element:

$$S(x_n) * (S(x_n) \to Q(x_n, x_n)) = S(x_n) \wedge Q(x_n, x_n) = S(x_n) > 0\,,$$

where the first equality follows from Property 3 of implication operators (Proposition 1.20) and the second one from the reflexivity of Q. $\square$

Let us analyze the above theorem. First of all, consider the two sets of conditions Hypo. 1 and Hypo. 2. It is interesting to know whether these hypotheses can be weakened. In that sense, Example 2.16 is a double counterexample. On the one hand, it shows that if we remove the condition regarding zero divisors in Hypo. 1, the theorem no longer holds. On the other hand, it shows that if we relax the completeness condition in Hypo. 2, the condition is no longer sufficient either. Example 2.17 shows that the continuity condition of Theorem 2.15 cannot be relaxed to left continuity. Finally, Example 2.18 shows that the classical result on G-rationality (acyclicity $\Leftrightarrow$ G-rationality, here recalled in Proposition 2.13), is no longer valid in the fuzzy setting, since G-rationality does not imply acyclicity.

Example 2.16 *Consider the fuzzy preference relation Q on the set $X = \{x, y, z\}$:*

$$Q = \begin{pmatrix} 1 & 0.1 & 0 \\ 0 & 1 & 0.9 \\ 0.1 & 0 & 1 \end{pmatrix}.$$

The fuzzy preference relation Q is weakly complete, but not moderately complete. It is also $*_{\mathbf{L}}$-acyclic (w.r.t. the Łukasiewicz t-norm, which has zero divisors), but not $*$-acyclic w.r.t. to t-norms without zero divisors. So it almost satisfies Hypo. 1 and Hypo. 2, but it does not satisfy any of them. To show that the associated function G_Q is not a fuzzy choice function, consider the available set S such that $S(x) = S(y) = S(z) = 1$. It is immediately clear that $G_Q(S)(x) = G_Q(S)(y) = G_Q(S)(z) = 0$. This example shows that the sufficient conditions in Theorem 2.15 cannot easily be weakened.

We also want to stress the importance of the hypothesis of continuity of the t-norm in Theorem 2.15. It would be interesting to weaken this condition, but the following example shows that left continuity is not enough to ensure that the function G_Q is a fuzzy choice function.

Example 2.17 *Consider the following t-norm, which is a transform of the nilpotent minimum t-norm, and its implication operator:*

$$a *_{\varphi nM} b = \begin{cases} 0 & \text{, if } a^2 + b^2 \leq 1 \\ a \wedge b & \text{, otherwise.} \end{cases}$$

$$a \rightarrow_{\varphi nM} b = \begin{cases} 1 & \text{, if } a \leq b \\ \sqrt{1-a^2} \vee b & \text{, otherwise.} \end{cases}$$

This is a left-continuous t-norm that is not continuous. It is immediately seen that the fuzzy preference relation Q on the set $X = \{x, y, z\}$:

$$Q = \begin{pmatrix} 1 & 0.6 & 0.6 \\ 0.5 & 1 & 0.6 \\ 0.5 & 0.6 & 1 \end{pmatrix}$$

is moderately complete and $*_{\varphi nM}$-acyclic. For the available fuzzy set defined by $S(x) = 0.1$ and $S(y) = S(z) = 0.7$, the value of $G_Q(S)$ is everywhere equal to zero. Hence, this Q does not generate a fuzzy choice function, proving that the continuity condition in Hypo. 2 cannot be weakened to left continuity.

The classical result of Sen on crisp rationality states that acyclicity and G-rationality are equivalent for a complete preference relation. Theorem 2.15 shows that one implication is valid in the fuzzy setting, while the following example shows that the converse does not hold in general.

Example 2.18 *Consider the following fuzzy preference relation Q on the set $X = \{x, y, z\}$:*

$$Q = \begin{pmatrix} 1 & 0 & 0.8 \\ 0.9 & 1 & 0 \\ 0.7 & 0.8 & 1 \end{pmatrix}$$

Let the t-norm $*_{\mathbf{M}}$ *be the minimum operator and* $\rightarrow$ *be the associated implication operator. The fuzzy preference relation* Q *is weakly complete, but neither acyclic nor* $*_{\mathbf{M}}$*-acyclic. Nevertheless, it generates a fuzzy choice function through the function* G_Q*. To prove this, consider an arbitrary available set* S *such that* $S(x) = a$*,* $S(y) = b$ *and* $S(z) = c$ *for some* $a, b, c \in [0, 1]$*. Then*

$$\begin{aligned} G_Q(S)(x) &= a *_{\mathbf{M}} (b \rightarrow_{\mathbf{M}} 0) *_{\mathbf{M}} (c \rightarrow_{\mathbf{M}} 0.8) \\ G_Q(S)(y) &= b *_{\mathbf{M}} (a \rightarrow_{\mathbf{M}} 0.9) *_{\mathbf{M}} (c \rightarrow_{\mathbf{M}} 0) \\ G_Q(S)(z) &= c *_{\mathbf{M}} (a \rightarrow_{\mathbf{M}} 0.7) *_{\mathbf{M}} (b \rightarrow_{\mathbf{M}} 0.8) . \end{aligned}$$

It is easy to see that the only combination of real numbers a, b, c *in the unit interval such that* $G_Q(S)(x)$*,* $G_Q(S)(y)$ *and* $G_Q(S)(z)$ *are simultaneously equal to zero is* $a = b = c = 0$*. Recall that the family of available sets* $\mathcal{B}$ *contains only fuzzy sets with non-empty support, hence the set just described is not contained in* $\mathcal{B}$*. Then* $G_Q(S)$ *always takes at least one strictly positive value, provided that* $S \in \mathcal{B}$*. In this way we have proved that neither acyclicity, nor* $*$*-acyclicity are necessary conditions on* Q *to generate a fuzzy choice function.*

M-rationality. The case of M-rationality remains to be analyzed. In classical choice theory the result of Walker [129] (here recalled in Proposition 2.14) states the conditions for a crisp preference relation in order to be M-rational. It only needs to be acyclic and, compared to the case of G-rationality, it does not require any kind of completeness. Another difference is that the result of Walker is not a characterization, in the sense that only one implication is proved (acyclicity implies M-rationality), since acyclicity is not a necessary condition. The following theorem extends that result to the fuzzy set framework.

Theorem 2.19 *Let $*$ be a continuous t-norm. If a fuzzy preference relation Q on X is $*$-acyclic, then M_Q is a fuzzy choice function.*

Proof. The proof of this theorem has the same structure as the proof of Theorem 2.15. It is based on Proposition 2.9 instead of Proposition 2.7, but essentially follows the same reasoning and the same steps. For this reason we prefer not to report it here, leaving to the interested reader the possibility to work it out for himself. □

Since acyclicity is a stronger condition than $*$-acyclicity, the following corollary is immediate.

Corollary 2.20 *If a fuzzy preference relation Q on X is acyclic, then M_Q is a fuzzy choice function.*

2.2.3 On the existence of a fuzzy version of the Richter Theorem

In this subsection we consider the extension to the fuzzy set framework of Richter Theorem. Its classical form is the following:

Theorem 2.21 ([106]) *A (crisp) choice function is G-rationalizable by a regular (i.e. complete and transitive) preference relation if and only if it is congruous, i.e. it satisfies Strong Congruence Axiom (SCA).*

We are interested in establishing a fuzzy version of this classical theorem. The definition of $*$-congruous fuzzy choice function is given next.

Definition 2.22 ([64]) *Let $*$ be a left-continuous t-norm. A fuzzy choice function is $*$-congruous if it satisfies the Strong Fuzzy Congruence Axiom*

(SFCA), i.e. for any $S \in \mathcal{B}$ and $x, y \in X$, it holds that

$$\widehat{R_C^*}(x,y) * C(S)(y) * S(x) \leq C(S)(x)\,, \tag{2.18}$$

where $\widehat{R_C^*}$ is the $*$-transitive closure of the fuzzy revealed preference relation R_C from C.

This axiom expresses that the degree of choice of an alternative x from a fuzzy set S ($C(S)(x)$) is controlled by the degree to which it belongs to S (i.e. $S(x)$), by the degree of choice of any other element y (i.e. $C(S)(y)$) and by the relation that exists between x and this other element y (i.e. $\widehat{R_C^*}(x,y)$). Intuitively, it expresses that if x is available, y is chosen and x is at least as good as y, then x should be chosen too.

Georgescu proved in [63] and [64] that one implication of Theorem 2.21 can be recovered in the fuzzy set framework:

Proposition 2.23 ([63]) *Let $*$ be a left-continuous t-norm. If a fuzzy choice function is G-rationalizable by a complete and $*$-transitive fuzzy preference relation, then it is $*$-congruous.*

The same author left as an open problem the study of the converse implication, in order to completely emulate the result of Richter. This subsection is dedicated to that problem: we provide an example to prove that the converse implication does not hold in general. Indeed, we provide a $*$-congruous fuzzy choice function that is G-rational, but that it is not totally G-rational.

Example 2.24 *Consider the following fuzzy preference relation Q on the set $X = \{x, y, z\}$*

$$Q = \begin{pmatrix} 1 & 0.1 & 0.1 \\ 1 & 1 & 0.1 \\ 0.4 & 0.9 & 1 \end{pmatrix}.$$

Let the t-norm $*$ be the minimum operator. The fuzzy relation Q is weakly complete and reflexive. Clearly, Q is not min-transitive, since

$$Q(z, y) \wedge Q(y, x) = 0.9 \wedge 1 = 0.9 \not\leq 0.4 = Q(z, x)\,.$$

Thus if a fuzzy choice function C is derived from it, then it will be G-rational, but not totally rational. On the other hand, it is easy to prove that it is acyclic. Since Q is a weakly complete, reflexive and acyclic fuzzy preference relation and $*$ is the minimum t-norm, it follows from Theorem 2.15 that G_Q is a fuzzy choice function. Let us call this fuzzy choice function C for brevity. Furthermore, due to Proposition 2.5, we know that it is not only G-rational, but also G-normal. Then, the fuzzy revealed preference relation R_C from C is given by $R_C = Q$. Let us verify whether the fuzzy choice function C is $*$-congruous, i.e. satisfies axiom SFCA. For doing this, we first compute the transitive closure $\widehat{R_C^*}$ of R_C, using the construction proposed in [29]:

$$\widehat{R_C^*} = \bigcup_{k=1}^{3} R_C^{(k)} = \begin{pmatrix} 1 & 0.1 & 0.1 \\ 1 & 1 & 0.1 \\ 0.9 & 0.9 & 1 \end{pmatrix}.$$

In order for SFCA to be satisfied, it should hold for any pair of alternatives (a, b) in X^2 and any $S \in \mathcal{F}(X)$ that

$$\widehat{R_C^*}(a, b) \wedge C(S)(b) \wedge S(a) \leq C(S)(a)\,. \tag{2.19}$$

Consider first the case $a = b$. Then SFCA becomes, for any $a \in X$,

$$\widehat{R_C^*}(a, a) \wedge C(S)(a) \wedge S(a) \leq C(S)(a)\,,$$

which is trivially satisfied for any $S \in \mathcal{F}(X)$, since, for any $a \in \{x, y, z\}$, it holds that $\widehat{R_C^*}(a, a) = 1$ and $C(S)(a) \leq S(a)$. Next, consider the case $a \neq b$.

In the table below, all possible combinations are considered; the third column represents the value of the left-hand side of Eq. (2.19), while the last column represents the value of the right-hand side of the same equation. As long as the value in the third column is smaller than or equal to the corresponding value in the fourth column, then condition SFCA is satisfied. It is easily verified that this is indeed the case:

Case a	b	$\widehat{R_C^*}(a,b) \wedge C(S)(b) \wedge S(a)$	$C(S)(a)$
x	y	$0.1 \wedge S(x) \wedge S(y)$	$S(x) \wedge 0.1$
x	z	$0.1 \wedge S(x) \wedge S(z)$	$S(x) \wedge 0.1$
y	x	$S(x) \wedge S(y) \wedge (S(z) \to 0.1)$	$S(y) \wedge (S(z) \to 0.1)$
y	z	$0.1 \wedge S(z) \wedge S(y)$	$S(y) \wedge (S(z) \to 0.1)$
z	x	$0.1 \wedge S(x) \wedge S(z)$	$S(z) \wedge 0.4$
z	y	$0.1 \wedge S(y) \wedge S(z)$	$S(z) \wedge 0.4$

Thus, we have shown that there exists a fuzzy choice function C that is G-rational (but not totally rational) and satisfies SFCA, while being obtained from an acyclic and complete fuzzy preference relation Q.

2.3 A fuzzy version of the Arrow-Sen Theorem

In this section we continue the study of rationality conditions and their relationship for the case of fuzzy choice functions. In particular, the aim of this section is to prove an extended version of Arrow-Sen Theorem that holds for the fuzzy set framework. An attempt in this sense has already been done in [64] or in [133], but the results presented there were proved under quite strong conditions. Here we propose a set of more general results regarding the connections between rationality conditions of a fuzzy choice function.

A significant property for fuzzy choice functions is G- or M-normality, which corresponds to the possibility of constructing the fuzzy choice function starting from a fuzzy preference relation and vice versa, revealing a fuzzy preference relation from a fuzzy choice function: when these two processes are reversible, we can speak of a normal choice function. The advantage of having such a strong connection between fuzzy choice functions and fuzzy preference relations is that we can pass interesting properties from one definition to the other. For example, we know that the $*$-transitivity of the fuzzy preference relation is a good property when we speak of the rationality of a decision maker, but what can be considered a good property for a choice function? Several answers to the previous question have been proposed by different authors in different disciplines, at least in the classical case (see [1, 106, 108, 113, 119]). All of them seem to be acceptable and reliable, but which one is the correct one? The answer is given by the so-called Arrow-Sen Theorem (here Theorem 1.14), which states that under suitable hypotheses, several of those proposals are equivalent. This section tries to extend those equivalences to the class of fuzzy choice functions. In particular it generalizes the results contained in Chapters 5 and 6 of [64]. The main results of this section have already been accepted for publication [94].

2.3.1 Rationality conditions for fuzzy choice functions and domain properties

We already recalled in Section 1.1 that the classical version of Arrow-Sen Theorem was proved under the hypothesis that the family $\mathcal{B}$ contains all non-empty subsets of X (condition H). Similar conditions are proposed in [64] for the fuzzy choice framework:

(i) *Condition H1*: All $S \in \mathcal{B}$ and $C(S)$ are normal fuzzy sets, i.e. for any $S \in \mathcal{B}$ there exists $x \in X$ such that $C(S)(x) = 1$;

(ii) *Condition H2*: $\mathcal{B}$ contains all crisp non-empty subsets of X.

However, these conditions are very restrictive and should be weakened. In the classical theory, Sen already noticed that condition H can be weakened to condition WH [115]. So, following Sen's intuition, we propose the following:

(i) *Condition WH1*: For any $S \in \mathcal{B}$, there exists an alternative $x \in X$ such that $S(x) > 0$ and $C(S)(x) = S(x)$;

(ii) *Condition WH2*: $\mathcal{B}$ contains the crisp sets $\{x\}$, $\{x, y\}$ and $\{x, y, z\}$ for any $x, y, z \in X$.

Given a left-continuous t-norm $*$, the degree of inclusion between two fuzzy sets is defined as follows (see [64]), for any $S, T \in \mathcal{F}(X)$:

$$I(S,T) = \bigwedge_{x \in X} (S(x) \to T(x)) \,.$$

The degree of inclusion is $*$-transitive, i.e. for any S, T, U in $\mathcal{B}$ it holds that:

$$I(S,T) * I(T,U) \leq I(S,U) \,.$$

Definition 2.25 *Let $C : \mathcal{B} \to \mathcal{F}(X)$ be a fuzzy choice function on $\mathcal{B}$. It can satisfy the following conditions, for any $S, T \in \mathcal{B}$ and any $x, y \in X$:*

(i) Weak Fuzzy Congruence Axiom (WFCA)*: $R_C(x, y) * S(x) * C(S)(y) \leq C(S)(x)$;*

(ii) Strong Fuzzy Congruence Axiom (SFCA)*:* $\widehat{R^*_C}(x,y) * S(x) * C(S)(y) \leq C(S)(x)$*;*

(iii) Weak Axiom of Fuzzy Revealed Preference (WAFRP)*:* $\tilde{P}_C(x,y) \leq \neg R_C(y,x)$*;*

(iv) Strong Axiom of Fuzzy Revealed Preference (SAFRP)*:* $\widehat{\tilde{P}^*_C}(x,y) \leq \neg R_C(y,x)$*;*

(v) Condition Fα*:* $I(S,T) * S(x) * C(T)(x) \leq C(S)(x)$*;*

(vi) Condition Fβ*:* $I(S,T) * C(S)(x) * C(S)(y) \leq C(T)(x) \leftrightarrow C(T)(y)$*;*

(vii) Condition Fδ*: for any crisp sets* S *and* T *in* $\mathcal{B}$ *and* $x \neq y \in X$*, it holds that:*

$$I(S,T) * C(S)(x) * C(S)(y) \leq \neg \left(C(T)(x) * \bigwedge_{t \neq x} (\neg C(T)(t)) \right).$$

Let us state a direct lemma that will be helpful in the rest of this section.

Lemma 2.26 *Let* $*$ *be a t-norm. For any fuzzy relation* R*, the fuzzy relation* R' *defined by* $R'(x,y) = R(y,x) \to R(x,y)$ *is reflexive and strongly complete.*

Proof. It holds that $R'(x,x) = R(x,x) \to R(x,x) = 1 \to 1 = 1$. We now prove that R' is strongly complete. If $R(x,y) \geq R(y,x)$, then $R'(x,y) = R(y,x) \to R(x,y) = 1$. If $R(x,y) < R(y,x)$, then $R'(y,x) = R(x,y) \to R(y,x) = 1$. Hence, $R'(x,y) \vee R'(y,x) = 1$, i.e. R' is strongly complete. $\square$

We recall some useful results from Chapter 5 of [64] about G-rationality, M-rationality and fuzzy revealed preference relations. They have been proved under the condition that the t-norm is continuous, while, to fit our framework, they should stand also for left-continuous t-norms. We checked the proofs and they can be extended to the case of left-continuous t-norms without any modification, so we report them here without proof.

Lemma 2.27 ([64]) $\tilde{P}_C \subseteq R_C$.

Lemma 2.28 ([64]) *If a fuzzy preference relation Q on X is strongly complete, then $G_Q = M_Q$.*

Lemma 2.29 (Lemma 5.36 of [64]) *Any M-rational fuzzy choice function C is G-rational.*

Remark 2.30 *The proof of Lemma 2.29 is constructive. If the fuzzy choice function C is M-rationalizable by the fuzzy preference relation Q, then C is G-rationalizable by the fuzzy preference relation $Q'(x,y) = Q(y,x) \to Q(x,y)$ and it holds that $C(S)(x) = G_{Q'}(S)(x) = M_Q(S)(x)$, for any $x \in X$ and any $S \in \mathcal{B}$.*

Remark 2.31 *Recall the result of Proposition 2.5: if the t-norm is the minimum, then a fuzzy choice function C is G-rational if and only if it is G-normal.*

The following result is a generalization of Lemma 5.26 of [64], where conditions H1 and H2 have been substituted by the more general conditions WH1 and WH2.

Lemma 2.32 *Let C be a fuzzy choice function and $*$ be a left-continuous t-norm. If condition WH2 is satisfied, then it holds that:*

(i) $\bar{R}_C \subseteq R_C$*;*

(ii) if also WH1 is satisfied, then $\bar{R}_C$ *and* R_C *are reflexive and strongly complete.*

Proof. No proof is given since the one given in Lemma 5.26 of [64] is still valid. □

2.3.2 G-rationality, M- rationality and normality

In this section we investigate the connections between the four rationality conditions (G-rationality/normality and M-rationality/normality) for a fuzzy choice function. First of all, we recall those implications that are trivial or already proved: obviously, if the fuzzy choice function C is G-normal (M-normal, resp.), then it is G-rational (M-rational, resp.). Furthermore, by Lemma 2.29, if C is M-rational, then it is also G-rational and, by Proposition 2.5, if the t-norm is the minimum, then G-rationality and G-normality are equivalent. Figure 2.1 depicts these connections in the case that the t-norm is the minimum.

In this subsection we generalize the above results to left-continuous t-norms. We also prove new implications previously unknown even in the case of the minimum.

We start with an auxiliary result.

Lemma 2.33 *Let* C *be a fuzzy choice function and* $*$ *be a left-continuous t-norm. If* C *is G-rationalizable by a fuzzy preference relation* Q *and* R_C *is the fuzzy preference relation revealed from* C*, then* $R_C(x, y) \leq Q(x, y)$*, for any* $x, y \in X$*.*

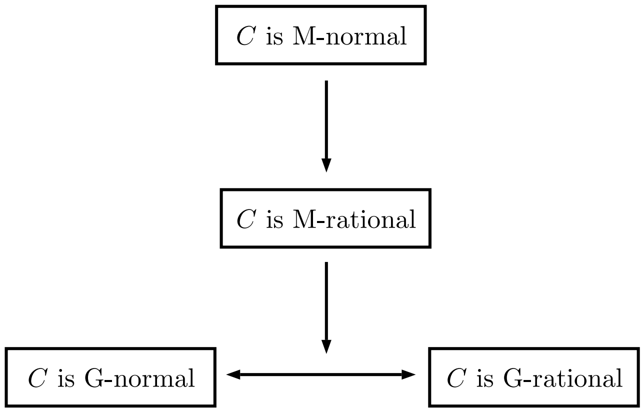


Figure 2.1: G-/M-rationality and normality. Implications known for the minimum t-norm under conditions H1 and H2.

Proof. Recall that the fuzzy preference relation revealed from C is given by

$$R_C(x,y) = \bigvee_{S\in\mathcal{B}} (S(y) * C(S)(x))\,. \tag{2.20}$$

Since C is G-rational, there exists a fuzzy preference relation Q such that for any $S \in \mathcal{B}$ and $x \in X$, the fuzzy choice function C can be written as

$$C(S)(x) = G_Q(S)(x) = S(x) * \bigwedge_{y\in X} (S(y) \to Q(x,y))\,. \tag{2.21}$$

Substituting $C(S)(x)$ in Eq. (2.20) with its expression in Eq. (2.21), we obtain:

$$R_C(x,y) = \bigvee_{S\in\mathcal{B}} (S(y) * [S(x) * \bigwedge_{k\in X} (S(k) \to Q(x,k))])\,. \tag{2.22}$$

Consider only the last part of Eq. (2.22) and recall Property 2 Proposition 1.20: for any $S \in \mathcal{B}$, it holds that

$$\begin{aligned} S(y) * S(x) * \bigwedge_{k\in X} (S(k) \to Q(x,k)) &\leq S(x) * S(y) * (S(y) \to Q(x,y)) \\ &\leq S(x) * (S(y) \wedge Q(x,y)) \leq Q(x,y)\,. \end{aligned}$$

Hence, we have proved that $R_C(x,y) \leq Q(x,y)$, for any $x, y \in X$. □

The following result proves that Proposition 2.5 can be generalized to the case of left-continuous t-norms, provided condition WH2 is satisfied.

Lemma 2.34 *Let C be a fuzzy choice function and $*$ be a left-continuous t-norm. If condition WH2 is satisfied, then the following statements are equivalent:*

(i) C is G-rational;

(ii) C is G-normal.

Proof. We already know that G-normality implies G-rationality. In order to prove that G-rationality implies G-normality, it should be shown that the fuzzy preference relation Q that rationalizes C coincides with the fuzzy preference relation R_C revealed from C. We already know from Lemma 2.33 that $R_C(x,y) \leq Q(x,y)$, for any $x, y \in X$. Let us prove the converse inequality. We start again from the definition of the fuzzy revealed preference relation and consider in particular the crisp set $\{x,y\}$ that belongs to $\mathcal{B}$, according to WH2:

$$
\begin{aligned}
R_C(x,y) &= \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y)) \\
&\geq C(\{x,y\})(x) * \{x,y\}(y) = C(\{x,y\})(x) * 1 \\
&= \{x,y\}(x) * \bigwedge_{k \in X} (\{x,y\}(k) \to Q(x,k)) \\
&= 1 * [(\{x,y\}(x) \to Q(x,x)) \wedge (\{x,y\}(y) \to Q(x,y))] \\
&= Q(x,x) \wedge Q(x,y) = Q(x,y)\,.
\end{aligned}
$$

The last equality follows from the reflexivity of Q. □

Lemma 2.35 *Let C be a fuzzy choice function and $*$ be a left-continuous t-norm. If condition WH2 is satisfied, then the following statements are equivalent:*

(i) C is M-rational;

(ii) C is M-normal.

Proof. We already know that M-normality implies M-rationality. In order to prove the other implication, we know from Lemma 2.29 and Remark 2.30 that if C is M-rationalizable by a fuzzy preference relation Q, then C is also G-rationalizable by $Q'(x, y) = Q(y, x) \to Q(x, y)$. According to Lemma 2.26, the relation Q' is strongly complete. By Lemma 2.34 we know that if C is G-rational, then it is also G-normal, which implies that the fuzzy preference relation R_C revealed from C is Q', therefore it is strongly complete. From Lemma 2.28 we know that if Q' is strongly complete, then $G_{Q'} = M_{Q'}$. In short, we have that:

(i) $M_Q = G_{Q'}$ (Lemma 2.29);

(ii) $Q' = R_C$ (Lemma 2.34);

(iii) $G_{Q'} = M_{Q'}$ (Lemma 2.28);

and we can conclude that $C = M_{R_C}$, i.e. C is M-normal. $\square$

Proposition 2.36 *Let $*$ be a left-continuous t-norm. If conditions WH1 and WH2 are satisfied, then any G-rational fuzzy choice function C is also M-normal.*

Proof. It follows from Lemma 2.34 that G-rationality implies G-normality. On the other hand, according to Lemma 2.32, the fuzzy preference relation R_C is strongly complete, and therefore $G_{R_C} = M_{R_C}$, as stated in Lemma 2.28. Then the fuzzy choice function is M-rational. Lemma 2.35 ensures that it is also M-normal. □

We can finally state the main result of this subsection.

Theorem 2.37 *Let C be a fuzzy choice function and $*$ be a left-continuous t-norm. If conditions WH1 and WH2 are satisfied, then the following statements are equivalent:*

(i) C is M-rational;

(ii) C is G-rational;

(iii) C is M-normal;

(iv) C is G-normal.

Proof. The equivalence between G-rationality and G-normality and the equivalence between M-rationality and M-normality are the results of Lemmas 2.34 and 2.35, respectively. Furthermore, if C is M-rational, Lemma 2.29 ensures that it is also G-rational. Finally, the converse implication follows from Proposition 2.36. □

The results obtained in this subsection are summarized in Figure 2.2, that, compared with Figure 2.1 shows the achieved improvement: now the four conditions are proved to be equivalent, for all left-continuous t-norms instead that for only the minimum and under conditions WH1 and WH2, that are weaker than conditions H1 and H2.

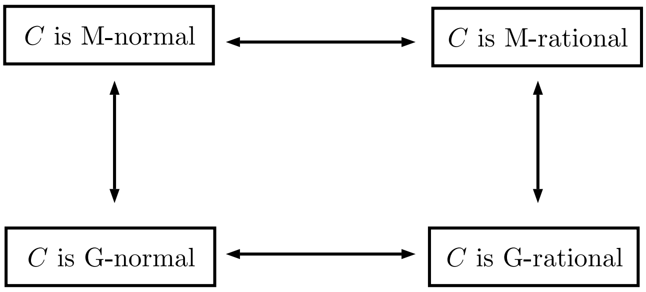


Figure 2.2: G-/M-rationality and normality. Implications proved for any left-continuous t-norm under conditions WH1 and WH2.

2.3.3 Fuzzy revealed preference and fuzzy congruence axioms

This section is dedicated to the four conditions WAFRP, SAFRP, WFCA and SFCA. They are equivalent in the classical case, as we recalled in Theorem 1.14. In [64] the equivalences are proved, but only for the Łukasiewicz t-norm and under conditions H1 and H2. Here we extend those results to a wider family of t-norms and replacing conditions H1 and H2 by their weaker versions WH1 and WH2. Let us start by recalling those implications that are trivial or already proved:

Lemma 2.38 *If the t-norm $*$ is left-continuous, then any fuzzy choice function C satisfying SAFRP also satisfies WAFRP and any fuzzy choice function C satisfying SFCA also satisfies WFCA.*

The proof follows from the definition of $*$-transitive closure.

Lemma 2.39 ([64]) *Let C be a fuzzy choice function. If the t-norm $*$ is continuous and conditions H1 and H2 are satisfied, then*

(i) it holds that

(a) if C satisfies WFCA, then it also satisfies SFCA;

(b) if R_C is $*$-regular and C is G-normal, then C satisfies WFCA.

(ii) If the t-norm is the minimum, then WFCA implies that R_C is $*$-regular and C is G-normal.

(iii) If the t-norm is the Łukasiewicz t-norm, then the following statements are equivalent:

(a) C satisfies WFCA;

(b) C satisfies WAFRP;

(c) C satisfies SAFRP.

Proof. (i) (a) See Proposition 6.1 in [64].
(i) (b) and (ii) See Proposition 6.7 (2) in [64].
(iii) See Theorem 6.7 (4) in [64]. □

Remark 2.40 *We know that both implications in (i) hold also for the case of left-continuous t-norms. Furthermore, the implication in (i) (a) holds if we relax conditions H1 and H2 and just impose conditions WH1 and WH2. We do not report the proofs of these generalized results since the ones given in [64] still hold without changes.*

The implication in (ii) of Lemma 2.39 does not hold if we replace condition H1 by condition WH1 as the following counterexample shows.

Example 2.41 *Let $*$ be the minimum t-norm and consider the set $X = \{x, y, z\}$, $\mathcal{B} = 2^X \setminus \{\emptyset\} \cup S$, where S is a fuzzy set with membership degrees given in the following table. Let C be the fuzzy choice function defined as follows:*

	S	$\{x\}$	$\{y\}$	$\{z\}$	$\{x,y\}$	$\{x,z\}$	$\{y,z\}$	X
$T(x)$	0.9	1	0	0	1	1	0	1
$C(T)(x)$	0.8	1	0	0	1	1	0	1
$T(y)$	0.5	0	1	0	1	0	1	1
$C(T)(y)$	0.5	0	1	0	0.5	0	1	0.5
$T(z)$	0.5	0	0	1	0	1	1	1
$C(T)(z)$	0.5	0	0	1	0	0.5	0.5	0.5

The fuzzy revealed preference relation R_C is given by

R_C	x	y	z
x	1	1	1
y	0.5	1	1
z	0.5	0.5	1

Then C satisfies conditions WH1, WH2 and WFCA, but it does not satisfy H1 and it is not G-normal:

$$\begin{aligned} G_{R_C}(S)(x) &= S(x) \wedge ((S(y) \to R_C(x,y)) \wedge (S(z) \to R_C(x,z))) \\ &= 0.9 \wedge 1 \wedge 1 = 0.9\,, \end{aligned}$$

but $C(S)(x) = 0.8$.

This example shows that the implication proved by Georgescu and recalled in Lemma 2.39 (ii) cannot be obtained under weaker conditions. If $C(S)$ is not a normal fuzzy set for all $S \in \mathcal{B}$, WFCA is not enough to guarantee that the fuzzy choice function C is G-normal. Therefore the connection between G-normality and congruence axioms cannot be easily generalized.

We now present our results on the connections between the four axioms WARFP, SAFRP, WFCA and SFCA. We can prove that conditions H1 and H2 can be replaced by WH1 and WH2. Also we prove that the Łukasiewicz t-norm in case (iii) of Lemma 2.39 can be substituted by a left-continuous t-norm that induces a strong negation operator.

Proposition 2.42 *If the t-norm $*$ is left continuous, then any fuzzy choice function C satisfying WFCA also satisfies WAFRP.*

Proof. Suppose that WAFRP does not hold. Then there exist two alternatives $x, y \in X$ such that:

$$\tilde{P}_C(x,y) > \neg R_C(y,x)\,. \tag{2.23}$$

Recall the property of induced negation operators: $a \leq \neg b \Leftrightarrow a * b = 0$ and so $a > \neg b \Leftrightarrow a*b > 0$. This implies in Eq. (2.23) that $\tilde{P}_C(x,y)*R_C(y,x) > 0$. Using the definition of $\tilde{P}_C$ and the property $a * (\bigvee_{i\in I} a_i) = \bigvee_{i\in I}(a * a_i)$, we have:

$$\begin{aligned} 0 &< \tilde{P}_C(x,y) * R_C(y,x) = R_C(y,x) * \bigvee_{S\in\mathcal{B}} (C(S)(x) * S(y) * \neg C(S)(y)) \\ &= \bigvee_{S\in\mathcal{B}} (C(S)(x) * S(y) * \neg C(S)(y) * R_C(y,x))\,. \end{aligned} \tag{2.24}$$

Equation (2.24) implies that there exists at least one available fuzzy set S_0 in $\mathcal{B}$ such that

$$C(S_0)(x) * S_0(y) * \neg C(S_0)(y) * R_C(y,x) > 0\,. \tag{2.25}$$

Using the properties $a \leq \neg b \Leftrightarrow a * b = 0$ and $a \leq \neg\neg a$ of induced negation operators in Eq. (2.25), we finally have

$$C(S_0)(x) * S_0(y) * R_C(y,x) > \neg\neg C(S_0)(y) \geq C(S_0)(y)\,,$$

which contradicts condition WFCA. □

Proposition 2.43 *Let $*$ be a left-continuous t-norm. If the t-norm induces a strong negation, then any fuzzy choice function C satisfying WAFRP also satisfies WFCA.*

Proof. Suppose that WFCA does not hold. Then there exist an available fuzzy set $S_0 \in \mathcal{B}$ and $x, y \in X$ such that

$$R_C(y,x) * C(S_0)(x) * S_0(y) > C(S_0)(y)\,. \tag{2.26}$$

Since the t-norm $*$ satisfies $a = \neg\neg a$, we know by Remark 1.23 that $a > b$ implies $a * \neg b > 0$, hence Eq. (2.26) becomes

$$R_C(y,x) * C(S_0)(x) * S_0(y) * \neg C(S_0)(y) > 0\,. \tag{2.27}$$

Consider now the following inequality:

$$\begin{aligned}\tilde{P}_C(x,y) * R_C(y,x) &= \bigvee_{S \in \mathcal{B}} (C(S)(x) * S(y) * \neg C(S)(y) * R_C(y,x)) \\ &\geq (C(S_0)(x) * S_0(y) * \neg C(S_0)(y) * R_C(y,x)). \end{aligned} \tag{2.28}$$

Combining Eqs. (2.27) and (2.28) we have that $\tilde{P}_C(x,y) * R_C(y,x) > 0$ and then $\tilde{P}_C(x,y) > \neg R_C(y,x)$, which contradicts WAFRP. □

Remark 2.44 *Combining Propositions 2.42 and 2.43 we have that conditions WAFRP and WFCA are equivalent, provided the t-norm is left continuous and induces a strong negation operator. The equivalence holds for any rotation-invariant t-norm and in particular for any ϕ-transform of the Łukasiewicz t-norm.*

In the following proposition, we study the connection between the conditions WAFRP and SAFRP.

Proposition 2.45 *Let $*$ be a left-continuous t-norm. If conditions WH1 and WH2 are satisfied, then any fuzzy choice function C satisfying WAFRP also satisfies SAFRP.*

Proof. We will use two properties that we recall here for convenience:

(i) $\tilde{P}_C(x,y) \leq R_C(x,y)$, for any $x, y \in X$ (Lemma 2.27);

(ii) $\neg(a \vee b) = \neg a \wedge \neg b$.

To prove that WAFRP implies SAFRP, it suffices to show that the fuzzy relation $\tilde{P}_C$ is $*$-transitive. Consider three alternatives $x, y, z \in X$ and let T be the crisp set $\{x, y, z\}$. Condition WH2 ensures that T belongs to $\mathcal{B}$. We want to prove that

$$\tilde{P}_C(x,y) * \tilde{P}_C(y,z) \leq \tilde{P}_C(x,z)\,. \tag{2.29}$$

Observe that

$$\begin{aligned} \tilde{P}_C(x,y) &\leq \neg R_C(y,x) = \neg\big(\bigvee_{S\in\mathcal{B}} (C(S)(y) * S(x))\big) \\ &= \bigwedge_{S\in\mathcal{B}} \neg(C(S)(y) * S(x)) \leq \neg(C(T)(y) * T(x)) \\ &= \neg C(T)(y)\,, \end{aligned} \tag{2.30}$$

where the last equality follows from the fact that T is a crisp set and therefore $T(x) = 1$. Analogously, we can prove that $\tilde{P}_C(y,z) \leq \neg C(T)(z)$. Now, since WH1 holds and T is crisp, $C(T)$ is a normal fuzzy set and at least one among $C(T)(x)$, $C(T)(y)$ and $C(T)(z)$ has to be equal to one.

(i) If $C(T)(y) = 1$, then $\neg C(T)(y) = 0$ and $\tilde{P}_C(x,y) = 0$.

(ii) Analogously, if $C(T)(z) = 1$, then $\neg C(T)(z) = 0$ and $\tilde{P}_C(y,z) = 0$.

In both cases it holds that, $\tilde{P}_C(x,y) * \tilde{P}_C(y,z) = 0$ and Eq. (2.29) holds trivially. Next, assume that $C(T)(x) = 1$. Consider the second part of Eq. (2.29), $\tilde{P}_C(x,z)$. By definition of $\tilde{P}_C$, we know that

$$\begin{aligned} \tilde{P}_C(x,z) &= \bigvee_{S\in\mathcal{B}} (C(S)(x) * S(z) * \neg C(S)(z)) \\ &\geq C(T)(x) * T(z) * \neg C(T)(z) \\ &= C(T)(x) * \neg C(T)(z) = \neg C(T)(z)\,. \end{aligned}$$

Now consider Eq. (2.30),

$$\tilde{P}_C(x,y) * \tilde{P}_C(y,z) \leq \neg C(T)(y) * \neg C(T)(z) \leq \neg C(T)(z) \leq \tilde{P}_C(x,z)\,,$$

and Eq. (2.29) holds. □

To conclude, let us summarize the connections proved between fuzzy revealed preference axioms and congruence axioms.

Theorem 2.46 *Let C be a fuzzy choice function and $*$ be a left-continuous t-norm. If conditions WH1 and WH2 are satisfied, then*

(i) C satisfies WAFRP if and only if it satisfies SAFRP.

(ii) C satisfies WFCA if and only if it satisfies SFCA.

(iii) If C satisfies WFCA, then it also satisfies WAFRP.

(iv) If the t-norm induces a strong negation operator and C satisfies WAFRP, then it also satisfies WFCA. In this case the four conditions are equivalent.

Proof. Combine Propositions 2.42, 2.43 and 2.45 with the implications already proved in [64]. □

Corollary 2.47 *Let $*$ be a rotation-invariant t-norm and C be a fuzzy choice function. If conditions WH1 and WH2 are satisfied, then the following statements are equivalent:*

(i) C satisfies WFCA;

(ii) C satisfies WAFRP;

(iii) C satisfies SFCA;

(iv) C satisfies SAFRP.

The results obtained in this subsection are summarized in Figure 2.3, where a solid line indicates that the implication holds for any left-continuous t-norm, while a dashed line indicates that the implication holds only for left-continuous t-norms that induce a strong negation operator.

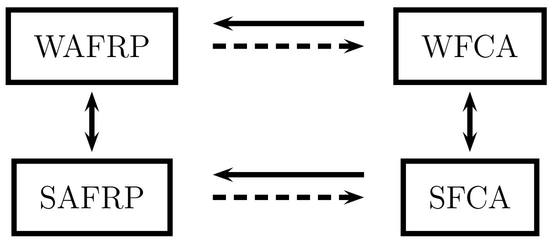


Figure 2.3: Relationship between Axioms of fuzzy revealed preference and Axioms of Fuzzy Congruence, under conditions WH1 and WH2.

2.3.4 Contraction/expansion conditions and rationality

The Arrow-Sen Theorem also connects the axioms of revealed preference and the axioms of congruence with expansion/contraction conditions. In particular, it proves that in the crisp case conditions α and β together are equivalent to the joint normality of the choice function together with the regularity of the revealed preference relation, which in turn is equivalent to the congruence and revealed preferred axioms. In this subsection we study the contraction/expansion conditions Fα and Fβ in relation with the property of rationality of a fuzzy choice function. We first recall two propositions proved in [64].

Proposition 2.48 ([64]) *Let C be a fuzzy choice function and $*$ be a continuous t-norm. If conditions H1 and H2 are satisfied and C is G-normal, then it satisfies condition Fα.*

Proposition 2.49 ([64]) *Let C be a fuzzy choice function and let $*$ be the minimum. If conditions H1 and H2 are satisfied and C satisfies conditions Fα and Fβ, then WFCA holds.*

Remark 2.50 *The two preceding results are proved imposing conditions H1 and H2. Nevertheless we realized that these conditions can be weakened. In particular, Proposition 2.48 still holds if they are removed, while for Proposition 2.49 condition WH2 is sufficient. They also have been proved assuming that the t-norm is continuous, although left continuity is sufficient. We do not present here the proofs since the ones given in [64] are still valid.*

The following result only holds if condition H1 is satisfied. Condition WH1 is not sufficient, as we will illustrate later in Example 2.52.

Proposition 2.51 *Let $*$ be a left-continuous t-norm. If condition H1 is satisfied, then any fuzzy choice function C satisfying condition WFCA also satisfies condition Fα.*

Proof. Suppose that Fα does not hold. Then there exist two available fuzzy sets $S, T \in \mathcal{B}$ and an alternative $x \in X$ such that:

$$I(S,T) * S(x) * C(T)(x) > C(S)(x)\,. \tag{2.31}$$

Equation (2.31) implies that:

(i) $I(S,T) > C(S)(x)$;

(ii) $S(x) > C(S)(x)$.

Since

$$I(S,T) = \bigwedge_{t \in X} (S(t) \to T(t)) \leq S(t) \to T(t), \text{ for any } t \in X\,,$$

it follows from (i) that $S(t) \to T(t) > C(S)(x)$, for any $t \in X$. From (ii), we can infer that $1 \geq S(x) > C(S)(x)$ implies that $C(S)(x) < 1$. Since condition H1 holds, there exists $y \in X$ such that $C(S)(y) = S(y) = 1$. Using the definition of R_C, we know that $R_C(x,y) = \bigvee_{M \in \mathcal{B}} (C(M)(x) * M(y))$. Then $R_C(x,y) \geq C(M)(x) * M(y)$ for all $M \in \mathcal{B}$. In particular,

$$R_C(x,y) \geq C(T)(x) * T(y)\,.$$

Now consider the value of $T(y)$:

(i) If $T(y) = 1$, then $R_C(x,y) \geq C(T)(x) * T(y) = C(T)(x)$ and hence

$$\begin{aligned} R_C(x,y) * C(S)(y) * S(x) &= R_C(x,y) * 1 * S(x) \\ &\geq C(T)(x) * S(x) \\ &\geq I(S,T) * C(T)(x) * S(x) > C(S)(x)\,, \end{aligned}$$

by Eq. (2.31), but this contradicts WFCA.

(ii) If $T(y) < 1$, then $I(S,T) \leq S(y) \to T(y) = 1 \to T(y) = T(y)$ and hence

$$\begin{aligned} R_C(x,y) * C(S)(y) * S(x) &= R_C(x,y) * 1 * S(x) \\ &\geq C(T)(x) * T(y) * S(x) \\ &\geq I(S,T) * C(T)(x) * S(x) > C(S)(x)\,, \end{aligned}$$

by Eq. (2.31), but also this contradicts WFCA. □

As previously pointed out, condition H1 is essential in Proposition 2.51. If only condition WH1 is satisfied, the implication of Proposition 2.51 does not hold as we illustrate next.

Example 2.52 *It suffices to consider the same family $\mathcal{B}$ and the same fuzzy choice function as in Example 2.41. It was proved there that C satisfies WFCA and that condition H1 is not satisfied. Let $X = \{x, y, z\}$ and S be the non-normal fuzzy set as in Example 2.41. Since $X \in \mathcal{B}$ and it is crisp, the membership degree of all the elements is 1 and therefore, $S(t) \to X(t) = 1$ for all $t \in X$. Therefore,*

$$I(S,X) = \bigwedge_{x \in X} (S(t) \to X(t)) = 1\,.$$

Now, if we consider the alternative $x \in X$ and we verify condition $F\alpha$ for the fuzzy sets S and X in $\mathcal{B}$, we have that

$$I(S,X) * S(x) * C(X)(x) = 1 * 0.9 * 1 = 0.9 > 0.8 = C(S)(x)\,.$$

Therefore, C does not satisfy condition $F\alpha$, although it satisfies WFCA.

Proposition 2.53 *Let $*$ be a left-continuous t-norm. If conditions WH1 and WH2 are satisfied, then any fuzzy choice function C that is G-rational and whose fuzzy revealed preference relation R_C is $*$-transitive satisfies condition $F\beta$.*

Proof. Recall that if WH1 and WH2 are satisfied, then, by Lemma 2.32 (ii), we have that R_C is strongly complete and reflexive, while, thanks to Lemma 2.34 we have that C is also G-normal. Since R_C is also $*$-transitive, R_C is also $*$-regular. Now, since C is G-normal and R_C is $*$-regular, by Lemma 2.39 (i) (b), C satisfies WFCA. Consider arbitrary $x, y \in X$ and $S, T \in \mathcal{B}$. If $C(T)(x) = C(T)(y)$, then $C(T)(x) \leftrightarrow C(T)(y) = 1$ and there is nothing to prove. So, suppose that they are different. Without loss of generality, we can assume that $C(T)(x) > C(T)(y)$. Consider the following chain of inequalities:

$$\begin{aligned}
&(C(S)(x) * C(S)(y) * C(T)(x)) \to C(T)(y) \\
=& (C(S)(x) * S(y) * \bigwedge_{k\in X} [S(k) \to R_C(y,k)] * C(T)(x)) \to C(T)(y) \\
\geq& (S(x) * \bigwedge_{k\in X} [S(k) \to R_C(x,k)] * S(y) * (S(x) \to R_C(y,x)) * C(T)(x)) \to C(T)(y) \\
\geq& (S(y) * (S(x) \wedge R_C(y,x)) * \bigwedge_{k\in X} [S(k) \to R_C(x,k)] * C(T)(x)) \to C(T)(y) \\
=& S(y) \to (((S(x) \wedge R_C(y,x)) * \bigwedge_{k\in X} [S(k) \to R_C(x,k)] * C(T)(x)) \to C(T)(y)) \\
\geq& S(y) \to ((R_C(y,x) * C(T)(x)) \to C(T)(y)) \,.
\end{aligned} \tag{2.32}$$

Since WFCA is satisfied and using the properties of t-norms and implication operators, it follows that

$$\begin{aligned}
R_C(y,x) * C(T)(x) * T(y) &\leq C(T)(y) \Leftrightarrow T(y) \\
&\leq (R_C(y,x) * C(T)(x)) \to C(T)(y) \,.
\end{aligned}$$

Returning to the previous chain of inequalites, we can write

$$\begin{aligned}
&(C(S)(x) * C(S)(y) * C(T)(x)) \to C(T)(y) \\
\geq\ & S(y) \to ((R_C(y,x) * C(T)(x)) \to C(T)(y)) \\
\geq\ & S(y) \to T(y) \geq \bigwedge_{k\in X} (S(k) \to T(k)) = I(S,T) \,,
\end{aligned} \tag{2.33}$$

and therefore,

$$(C(S)(x) * C(S)(y) * C(T)(x)) \to C(T)(y) \geq I(S,T)\,,$$

or, equivalently,

$$I(S,T) * C(S)(x) * C(S)(y) \leq C(T)(x) \to C(T)(y) = C(T)(x) \leftrightarrow C(T)(y)\,,$$

when $C(T)(x) > C(T)(y)$. Hence, condition Fβ holds. □

Proposition 2.54 *Let the t-norm $*$ be the minimum. If condition WH2 is satisfied, then any fuzzy choice function C satisfying WFCA also satisfies condition Fβ.*

Proof. Let us write condition Fβ, for arbitrary $S, T \in \mathcal{B}$ and $x, y \in X$:

$$\begin{aligned} & I(S,T) \wedge C(S)(x) \wedge C(S)(y) \leq C(T)(x) \leftrightarrow C(T)(y) \\ = \; & [C(T)(x) \to C(T)(y)] \wedge [C(T)(y) \to C(T)(x)]\,. \end{aligned}$$

If $C(T)(x) = C(T)(y)$, then $C(T)(x) \leftrightarrow C(T)(y) = 1$ and Fβ is trivially satisfied. Suppose for example that $C(T)(x) > C(T)(y)$, then

$$C(T)(x) \leftrightarrow C(T)(y) = C(T)(x) \to C(T)(y)\,.$$

Since WFCA is satisfied and using the properties of t-norms and implication operators, it follows that

$$R_C(y,x) \wedge T(y) \leq C(T)(x) \to C(T)(y)\,.$$

Since condition WH2 is satisfied, all crisp doubletons are contained in $\mathcal{B}$. Consider the value of $I(\{x,y\}, M)$ for a generic available set $M \in \mathcal{B}$ and two alternatives $x, y \in X$:

$$\begin{aligned} I(\{x,y\}, M) &= \bigwedge_{z \in X} (\{x,y\}(z) \to M(z)) \\ &= [1 \to M(x)] \wedge [1 \to M(y)] = M(x) \wedge M(y)\,. \end{aligned}$$

By the $*$-transitivity of the degree of inclusion it follows that:

$$\begin{aligned} C(T)(x) \to C(T)(y) &\geq R_C(y,x) \wedge T(y) \\ &\geq R_C(y,x) \wedge T(y) \wedge T(x) \\ &= R_C(y,x) \wedge I(\{x,y\},T) \\ &\geq I(\{x,y\},S) \wedge I(S,T) \wedge R_C(y,x) \\ &= I(S,T) \wedge S(x) \wedge S(y) \wedge R_C(y,x) \\ &\geq I(S,T) \wedge S(x) \wedge S(y) \wedge C(S)(y) \wedge S(x) \\ &\geq I(S,T) \wedge [S(x) \wedge C(S)(x)] \wedge [S(y) \wedge C(S)(y)] \\ &= I(S,T) \wedge C(S)(x) \wedge C(S)(y)\,. \end{aligned}$$

Hence, condition Fβ holds. □

Before stating the main result of this subsection, let us prove a result on the fuzzy revealed preference relations $\bar{R}_C$ and R_C.

Proposition 2.55 *Let $*$ be a left-continuous t-norm. If conditions WH1 and WH2 are satisfied, then any fuzzy choice function C satisfying WFCA also satisfies $\bar{R}_C = R_C$.*

Proof. Recall that, according to Lemma 2.32, $\bar{R}_C(x,y) \leq R_C(x,y)$ for any $x, y \in X$. Thus, it suffices to prove that $\bar{R}_C(x,y) \geq R_C(x,y)$, for any $x, y \in X$. Consider arbitrary alternatives $x, y \in X$ and condition WFCA: for the alternatives x, y and the crisp set $\{x,y\}$, which is contained in $\mathcal{B}$ by condition WH2, we have that:

$$R_C(x,y) * C(\{x,y\})(y) * \{x,y\}(x) \leq C(\{x,y\})(x)\,. \tag{2.34}$$

It follows from condition WH1 that either $C(\{x,y\})(x)$ or $C(\{x,y\})(y)$ is equal to one.

(i) If $C(\{x,y\})(y)=1$, then Eq. (2.34) becomes $R_C(x,y) \leq C(\{x,y\})(x)$. According to the definition of $\bar{R}_C$, we have that

$$R_C(x,y) \leq C(\{x,y\})(x) = \bar{R}_C(x,y)\,.$$

(ii) If $C(\{x,y\})(x) = 1 = \bar{R}_C(x,y)$, then we immediately obtain that $R_C(x,y) \leq \bar{R}_C(x,y)$. □

Combining the propositions proved in Subsection 2.3.4, we can state the following:

Theorem 2.56 *Let C be a fuzzy choice function and $*$ be a left-continuous t-norm. If conditions WH1 and WH2 are satisfied, then the following statements hold:*

(i) If C is G-rational, then condition $F\alpha$ is satisfied;

(ii) If C is G-rational and R_C is $*$-transitive, then condition $F\beta$ is satisfied;

(iii) If C satisfies condition WFCA, then $\bar{R}_C = R_C$;

(iv) If also condition H1 is satisfied and C satisfies condition WFCA, then condition $F\alpha$ holds;

(v) If $*$ is the minimum and C satisfies condition WFCA, then C satisfies condition $F\beta$;

(vi) If $*$ is the minimum and C satisfies conditions $F\alpha$ and $F\beta$, then C satisfies WFCA.

Moreover, we can exhibit two counterexamples that show that the conditions on the t-norm in (v) and (vi) of Theorem 2.56 cannot be weakened.

Example 2.57 *The result proved in Proposition 2.54 (item (v) of Theorem 2.56) for the minimum cannot be extended to any other t-norm. For any t-norm different from the minimum there exist* $a, b \in]0,1[$ *such that* $a * b < a \wedge b$. *We can provide a fuzzy choice function that respects condition WFCA, but not* $F\beta$. *Assume without loss of generality that* $a = a \wedge b$ *and consider the following example, with* $X = \{x, y, z\}$*:*

	$S(x)$	$C(S)(x)$	$S(y)$	$C(S)(y)$	$S(z)$	$C(S)(z)$
$\{x\}$	1	1	0	0	0	0
$\{y\}$	0	0	1	1	0	0
$\{z\}$	0	0	0	0	1	1
$\{x,y\}$	1	1	1	a	0	0
$\{x,z\}$	1	1	0	0	1	1
$\{y,z\}$	0	0	1	a	1	1
$\{x,y,z\}$	1	1	1	a	1	1
T_1	1	1	b	a	$a*b$	$a*b$
T_2	1	1	b	$a*b$	$a*b$	$a*b$

with fuzzy revealed preference relation R_C *given by*

R_C	x	y	z
x	1	1	1
y	a	1	a
z	1	1	1

For the sets T_1 *and* T_2 *and alternatives* x *and* y, *condition* $F\beta$ *does not hold:*

$$\begin{aligned} I(T_1,T_2) * C(T_1)(x) * C(T_1)(y) &= 1 * 1 * a \\ &= a > a * b = 1 \leftrightarrow (a * b) \\ &= C(T_2)(x) \leftrightarrow C(T_2)(y)\,. \end{aligned}$$

On the other hand, the fuzzy choice function C satisfies condition WFCA (the computations are not reported here). So, we have found a case in which condition $F\beta$ does not hold, even if WFCA holds.

Example 2.58 *We know that if the t-norm is the minimum and conditions WH1 and WH2 are satisfied, then a fuzzy choice function satisfying conditions $F\alpha$ and $F\beta$ also satisfies WFCA. We show next that a similar result cannot be proved when the t-norm is different from the minimum. Let $a, b \in]0,1[$ be such that $a * b < a \wedge b \leq b$. Consider the following fuzzy choice function C:*

	$\{x\}$	$\{y\}$	$\{z\}$	$\{x,y\}$	$\{x,z\}$	$\{y,z\}$	$\{x,y,z\}$
$S(x)$	1	0	0	1	1	0	1
$C(S)(x)$	1	0	0	1	$a*b$	0	$a*b$
$S(y)$	0	1	0	1	0	1	1
$C(S)(y)$	0	1	0	a	0	a	a
$S(z)$	0	0	1	0	1	1	1
$C(S)(z)$	0	0	1	0	1	1	1

Clearly C satisfies conditions WH1 and WH2 (in fact, it even satisfies H1 and H2), since $\mathcal{B} = 2^X \setminus \{\emptyset\}$. Then, although we do not report all the computations, conditions $F\alpha$ and $F\beta$ are satisfied by the fuzzy choice function C. Now, let us check condition WFCA. First of all, the fuzzy revealed preference relation R_C is given by:

R_C	x	y	z
x	1	1	$a*b$
y	a	1	a
z	1	1	1

Consider now the set $\{x, y, z\}$ and the alternatives x and y. Computing

$$R_C(x, y) * \{x, y, z\}(x) * C(\{x, y, z\})(y) = 1 * 1 * a > a * b = C(\{x, y, z\})(x)\,,$$

it follows that condition WFCA does not hold.

Some of the results obtained in this section are summarized in Figure 2.4, where a solid line indicates that the implication holds for any left-continuous t-norm, while a dotted line indicates that the implication holds only for the t-norm of the minimum. Recall that all these implications hold under the additional conditions WH1 and WH2, while those implications that needs the stronger condition H1 are labelled accordingly.

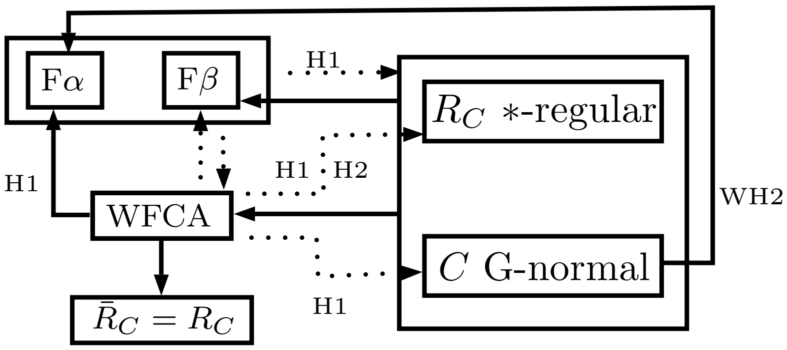


Figure 2.4: Relationship between contraction/expansion conditions.

2.3.5 ∗-quasi-transitivity and condition Fδ

Sen [115] has shown that in the crisp case, for a normal choice function, condition δ is equivalent to the quasi-transitivity of R_C, i.e. equivalent to the transitivity of the associated strict preference relation P_{R_C}. The aim of this subsection is to investigate whether the result of Sen can be extended to the fuzzy set framework. A first attempt can be found in [64], but there only the minimum is considered. We can prove that one implication holds for any left-continuous t-norm, while the converse implication holds for any t-norm without zero divisors.

Theorem 2.59 *Let $*$ be a left-continuous t-norm. If conditions WH1 and WH2 are satisfied, then for any G-rational fuzzy choice function C it holds that:*

(i) If C satisfies $F\delta$, then the fuzzy revealed preference relation R_C is $*$-quasi-transitive.

(ii) If the t-norm has no zero divisors and R_C is $*$-quasi-transitive, then C satisfies $F\delta$.

Proof. Recall that, by Lemma 2.34, C is G-normal. We start by proving the first part of the theorem. Consider three arbitrary alternatives x, y and z in X. We want to prove that R_C is $*$-quasi-transitive, i.e.

$$P_C(x,y) * P_C(y,z) \leq P_C(x,z) = R_C(x,z) * \neg R_C(z,x) \,. \tag{2.35}$$

Since conditions WH1 and WH2 are satisfied, by Lemma 2.32, we know that R_C is reflexive and strongly complete. Since C is G-normal and all the crisp triplets are contained in $\mathcal{B}$, we have that:

$$\begin{aligned} C(\{x,y,z\})(x) &= \{x,y,z\}(x) * \bigwedge_{t\in X} (\{x,y,z\}(t) \to R_C(x,t)) \\ &= 1 * [(1 \to R_C(x,y)) \wedge (1 \to R_C(x,z))] \\ &= R_C(x,y) \wedge R_C(x,z) \,. \end{aligned}$$

Similarly,

(i) $C(\{x,y,z\})(y) = R_C(y,x) \wedge R_C(y,z)$ and

(ii) $C(\{x, y, z\})(z) = R_C(z, x) \wedge R_C(z, y)$.

By condition WH1, we know that at least one among the aforementioned $C(\{x, y, z\})(x)$, $C(\{x, y, z\})(y)$ and $C(\{x, y, z\})(z)$ has to be equal to one. Let us consider the three cases:

(i) If $C(\{x, y, z\})(y) = 1$, then $R_C(y, x) = R_C(y, z) = 1$. Hence

$$P_C(x, y) = R_C(x, y) * \neg R_C(y, x) = R_C(x, y) * \neg 1 = 0\,,$$

and Eq. (2.35) is trivially satisfied.

(ii) If $C(\{x, y, z\})(z) = 1$, then $R_C(z, x) = R_C(z, y) = 1$. Hence

$$P_C(y, z) = R_C(y, z) * \neg R_C(z, y) = R_C(y, z) * \neg 1 = 0\,,$$

, and Eq. (2.35) is trivially satisfied.

(iii) If $C(\{x, y, z\})(x) = 1$, then $R_C(x, y) = R_C(x, z) = 1$. Then Eq. (2.35) becomes $P_C(x, y) * P_C(y, z) \leq \neg R_C(z, x)$, or, equivalently,

$$P_C(x, y) * P_C(y, z) * R_C(z, x) \leq 0\,. \tag{2.36}$$

Since $a \leq \neg b \Leftrightarrow a * b = 0$, using again conditions WH1 and WH2 and the fact that C is G-normal, we can conclude that $C(\{x, z\})(x) = R_C(x, z) = 1$ and $C(\{x, z\})(z) = R_C(z, x)$. Now, using condition F$\delta$ and some properties of negation operators, we have the following chain of inequalities

$$\begin{aligned}
R_C(z,x) &= R_C(x,z) * R_C(z,x) \\
&= C(\{x,z\})(x) * C([x,z])(z) \\
&= I(\{x,z\},\{x,y,z\}) * C(\{x,z\})(x) * C(\{x,z\})(z) \\
&\leq \neg\left(C(\{x,y,z\})(x) * \bigwedge_{t\neq x} \neg C(\{x,y,z\})(t)\right) \\
&= \neg\left(\bigwedge_{t\neq x} \neg C(\{x,y,z\})(t)\right) \\
&= \neg(\neg C(\{x,y,z\})(y) \wedge \neg C(\{x,y,z\})(z)) \\
&= \neg(\neg(R_C(y,x) \wedge R_C(y,z)) \wedge \neg(R_C(z,x) \wedge R_C(z,y))) \\
&\leq \neg(\neg R_C(y,x) \wedge \neg R_C(z,y)) \\
&\leq \neg(\neg R_C(y,x) * \neg R_C(z,y))\,.
\end{aligned}$$

So, it has been proved that $R_C(z,x) \leq \neg(\neg R_C(y,x) * \neg R_C(z,y))$. Using this fact in Eq. (2.36), and the fact that $P_C(x,y) \leq \neg R_C(y,x)$ and $P_C(y,z) \leq \neg R_C(z,y)$, we have:

$$\begin{aligned}
& P_C(x,y) * P_C(y,z) * R_C(z,x) \\
\leq\ & (\neg R_C(y,x) * \neg R_C(z,y)) * \neg(\neg R_C(y,x) * \neg R_C(z,y)) = 0\,.
\end{aligned}$$

Hence we have proved that also Eq. (2.36) holds and this concludes the first part of the proof.

Now we can prove the second part of the theorem and we need the assumption that $*$ has no zero divisors. For such a t-norm, the induced negation operator takes the form

$$\neg a = \begin{cases} 1 & \text{, if } a = 0 \\ 0 & \text{, otherwise.} \end{cases} \tag{2.37}$$

Now suppose that condition Fδ does not hold for some $S = \{a_1, \dots, a_n\}$ and $T = \{b_1, \dots, b_m\}$, $x \neq y \in X$, i.e.

$$I(S,T) * C(S)(x) * C(S)(y) > \neg(C(T)(x) * \bigwedge_{t \neq x} (\neg C(T)(t))) \,. \qquad (2.38)$$

By Eq. (2.38) we have that

(i) $I(S,T) = 1$, since S and T are crisp sets and $I(S,T) > 0$;

(ii) $x \in S = \{a_1, \dots, a_n\}$, since $S(x) \geq C(S)(x) > 0$ and S is a crisp set;

(iii) $y \in S = \{a_1, \dots, a_n\}$, since $S(y) \geq C(S)(y) > 0$ and S is a crisp set;

(iv) $x, y \in T$, by the previous three points;

(v) $R_C(y,x) > 0$, since $R_C(y,x) \geq C(S)(y) * S(x) \geq C(S)(x) * C(S)(y) > 0$;

(vi) $P_C(x,y) = 0$, since $P_C(x,y) = R_C(x,y) * \neg R_C(y,x)$ and $\neg R_C(y,x) = 0$;

(vii) $C(T)(t) = 0$, for any $t \neq x$, indeed

$$\begin{aligned}
& 1 > \neg(C(T)(x) * \bigwedge_{t \neq x} (\neg C(T)(t))) \\
\Rightarrow\ & 0 < C(T)(x) * \bigwedge_{t \neq x} (\neg C(T)(t)) \\
\Rightarrow\ & 0 < \bigwedge_{t \neq x} (\neg C(T)(t)) \\
\Rightarrow\ & 0 < \neg C(T)(t) \\
\Rightarrow\ & 0 = C(T)(t) \,.
\end{aligned}$$

Using the last point and the fact that C is G-normal, we can write the following chain of equalities:

$$\begin{aligned} 0 &= C(T)(y) = T(y) * \bigwedge_{k\in X} (T(k) \to R_C(y,k)) \\ &= 1 * \bigwedge_{k\in T} (1 \to R_C(y,k)) \\ &= \bigwedge_{k\in T} R_C(y,k)\,. \end{aligned}$$

This means that there exists at least one alternative $b_i \in T$ such that $R_C(y, b_i) = 0$. From the fifth point of the previous list, $b_i \neq x$, since $R_C(y,x) > 0$. By Lemma 2.32, R_C is reflexive and strongly complete, hence $b_i \neq y$ and $R_C(b_i, y) = 1$. Hence, we can write:

$$P_C(b_i, y) = R_C(b_i, y) * \neg R_C(y, b_i) = 1 * \neg 0 = 1\,.$$

Following the same reasoning for the alternative b_i as we have done for y, we can find another $b_j \in T$ such that $R_C(b_i, b_j) = 0$. We know from the previous point that $R_C(b_i, y) = 1$, hence $b_j \neq y$ and $b_j \neq b_i$, because of the reflexivity of R_C. As before, we can prove that $P_C(b_j, b_i) = R_C(b_j, b_i) * \neg R_C(b_i, b_j) = 1$. Now we want to prove that $b_j \neq x$: suppose $b_j = x$ and consider the $*$-quasi-transitivity of R_C:

$$P_C(x, b_i) * P_C(b_i, y) = P_C(b_j, b_i) * P_C(b_i, y) = 1 * 1 = 1 \nleq P_C(x, y) = 0\,.$$

This contradicts the $*$-quasi-transitivity of R_C, hence $b_j \neq x$.
Applying induction, we can find an infinite family of different alternatives $\{b_i, b_j, \ldots, b_m, \ldots\} \subseteq T$, but this contradicts the finiteness of X. $\square$

We now provide a counterexample that shows that the restriction to a t-norm without zero divisors in Theorem 2.59 cannot be weakened. First we need an auxiliary result.

Lemma 2.60 *If $*$ is a t-norm with zero divisors, then there exists a value $a \in]0,1[$ such that $a = \neg\neg a$.*

Proof. Consider a zero divisor c of $*$. If $c = \neg\neg c$, then take $a = c$. Otherwise, since $*$ is a left-continuous t-norm, it holds that $c * \neg c = 0$, and $c < \neg\neg c$. Now define $b = \neg\neg c$, then $b > c$. It holds that $b * \neg c = \neg\neg c * \neg c = 0$. So $\neg c \leq \neg b$. On the other hand, $0 = \neg b * b \geq \neg b * c$ and this implies $\neg b \leq \neg c$. Therefore, $\neg b = \neg c$. Taking into account the definition of b, we get $\neg\neg\neg c = \neg c$. Then we can take $a = \neg c$ and thus $a = \neg\neg a$. $\square$

Example 2.61 *We show that the second part of Theorem 2.59 does not hold if the restriction on the t-norm is removed. Consider a t-norm $*$ with zero divisors. Take a value $a \in]0,1[$ such that $a = \neg\neg a$. Such a value exists by Lemma 2.60. Consider the set of alternatives $X = \{x, y, z\}$, the set of available sets $\mathcal{B} = 2^X \setminus \{\emptyset\}$ and the fuzzy choice function $C : \mathcal{B} \to \mathcal{F}(X)$ defined as follows:*

	$\{x\}$	$\{y\}$	$\{z\}$	$\{x,y\}$	$\{x,z\}$	$\{y,z\}$	$\{x,y,z\}$
$S(x)$	1	0	0	1	1	0	1
$C(S)(x)$	1	0	0	1	1	0	1
$S(y)$	0	1	0	1	0	1	1
$C(S)(y)$	0	1	0	1	0	a	a
$S(z)$	0	0	1	0	1	1	1
$C(S)(z)$	0	0	1	0	$\neg a$	1	$\neg a$

with fuzzy revealed preference relation R_C given by

R_C	x	y	z
x	1	1	1
y	1	1	a
z	$\neg a$	1	1

P_C	x	y	z
x	0	0	a
y	0	0	0
z	0	$\neg a$	0

Then C is G-normal and WH1 and WH2 hold. Also the associated fuzzy preference relation R_C is $*$-quasi-transitive, but condition Fδ does not hold. Consider the sets $\{x, y\}$ and $\{x, y, z\}$. It follows that

$$
\begin{aligned}
& I(\{x,y\},\{x,y,z\}) * C(\{x,y\})(x) * C(\{x,y\})(y) \\
= \; & 1 * 1 * 1 = 1 > \neg a \vee a = \neg(1 * (\neg a \wedge \neg\neg a)) \\
= \; & \neg(C(\{x,y,z\})(x) * \bigwedge_{t \neq x} (\neg C(\{x,y,z\})(t)))\,.
\end{aligned}
$$

Hence, Fδ does not hold.

The results obtained in this subsection are summarized in Figure 2.5, where a solid line indicates that the implication holds for any left-continuous t-norm, while a dashed line indicates that the implication holds only for t-norms without zero divisors. Recall that all these implications hold under the additional conditions WH1 and WH2 and assuming that C is G-normal.

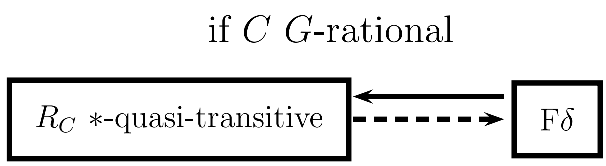


Figure 2.5: Relationship between $*$-quasi transitivity of R_C and condition Fδ when C is G-rational.

Chapter 3

Preference modelling under uncertainty

The classical theory of consumer behaviour assumes either a deterministic utility function or derives it from the prior assumption of a deterministic preference ordering. We already recalled in Chapter 1 how this economic theory gave way to a more general theory of individual choice without the concept of utility, mainly thanks to the works of Arrow [1, 2], Houthakker [72], Richter [106] and Sen [115]. However, that theory still assumes that both choice and preference are deterministic. We then presented in Chapter 2 a new approach to choice theory where the notions of choice and preference were allowed to be fuzzy. In the present chapter we consider another generalization of classical choice theory which assumes that individual choices and preferences are stochastic in nature.

After recalling the basic definitions of probabilistic choice functions and probabilistic relations in Section 3.1, we study in Sections 3.2 and 3.3 the connections that can be proved between the stochastic and the fuzzy formalisms of choice, focusing specially on the transitivity property. Section 3.4 concludes the chapter with an experiment set up to measure the rationality of a group of consumers and which makes use of the theoretical construction presented in Sections 3.2 and 3.3.

3.1 Probabilistic choice and preference

Experimental evidence suggests that the observed choices of individuals are often stochastic in nature, in contrast with the assumptions of classical choice theory. In response to this, a large literature has been developed, with contributions coming from economists as well as psychologists, which deals with decision makers with stochastic preference and/or stochastic choice behaviour [5–7, 11, 19, 23, 25, 26, 28, 45–49, 59–62, 66, 70, 81–84, 96–98, 104]. The first proposals of Luce [81, 82] were meant to find a set of general assumptions that would allow the development of a mathematical model for interpreting and understanding choice behaviours in a stochastic setting. He started with two basic assumptions of individual choice behaviour:

(i) it is probabilistic;

(ii) the probability of choosing an option from one set of alternatives is related to the probability of choosing the same option from a larger set of alternatives.

The first assumption sets the basis for a rich literature on stochastic choice and preference. In this literature, given a set of alternatives $\{x, y\}$, the individual's choice may be described with a probability distribution that assigns probability $p(x, y)$ to x and probability $p(y, x) = 1 - p(x, y)$ to y, where $p(x, y)$ indicates the probability that x will be chosen from $\{x, y\}$. Obviously, $0 \leq p(x, y) \leq 1$ and $p(x, y) + p(y, x) = 1$. Relations of this kind are usually called probabilistic relations. The same concept, extended to sets with more than two alternatives, constitutes the key for defining a probabilistic choice function.

3.1.1 Probabilistic relations

Probabilistic relations (also known as reciprocal or ipsodual relations) are defined in the following way.

Definition 3.1 *Given a finite set of alternatives X, a probabilistic relation p is a mapping $p : X \times X \to [0,1]$ such that $p(x,y) + p(y,x) = 1$ for any pair of alternatives x and y in X.*

The interpretation of a probabilistic relation is the following:

(i) $p(x,y) = 1$ expresses that alternative x is totally preferred to y;

(ii) $p(x,y) = 0$ expresses that alternative y is totally preferred to x;

(iii) $p(x,y) = \frac{1}{2}$ expresses that alternatives x and y are indifferent;

(iv) $p(x,y) \in]\frac{1}{2}, 1[$ expresses that alternative x is preferred to y to some degree;

(v) $p(x,y) \in]0, \frac{1}{2}[$ expresses that alternative y is preferred to x to some degree;

Compared to the deterministic preference relations of classical choice theory, probabilistic relations allow to express intermediate degrees of preference. Formally, they are similar to fuzzy preference relations defined in Section 1.2, since both of them take values in the unit interval, but they have a completely different interpretation. In fact, the probabilistic relation $p(x,y)$ between two alternatives x and y carries a bipolar semantic, meaning that the left-half of the unit interval ($[0, \frac{1}{2}[$) represents preference of y over x, the right-half of the unit interval ($]\frac{1}{2}, 1]$) represents the preference of x over y and the central value ($\{\frac{1}{2}\}$) represents indifference. The value of a fuzzy preference relation $Q(x, y)$ between x and y only represents the connection between the ordered pair (x, y), while for understanding the complete relationship standing between the two alternatives, both $Q(x, y)$ and $Q(y, x)$ need to be known.

There are several definitions of transitivity for probabilistic relations (see among others [17, 46, 57, 68, 84, 123–125]). Some of them are particular types of g-stochastic transitivity.

Definition 3.2 *Let g be a commutative increasing $[\frac{1}{2}, 1]^2 \to [\frac{1}{2}, 1]$ mapping. A probabilistic relation p defined on X is stochastic transitive with respect to g (g-stochastic transitive, for short) if, for any x, y, z in X, it holds that*

$$p(x, y) \geq \frac{1}{2} \text{ and } p(y, z) \geq \frac{1}{2} \text{ imply } p(x, z) \geq g(p(x, y), p(y, z)) .$$

The most important types of g-stochastic transitivity are the following:

(i) strong stochastic transitivity, if $g(a, b) = a \vee b$;

(ii) moderate stochastic transitivity, if $g(a, b) = a \wedge b$;

(iii) weak stochastic transitivity, if $g(a, b) = \frac{1}{2}$;

(iv) λ-stochastic transitivity, if $g(a, b) = \lambda \cdot (a \vee b) + (1 - \lambda) \cdot (a \wedge b)$, with $\lambda \in [0, 1]$.

Strong stochastic transitivity implies λ-transitivity, which implies moderate stochastic transitivity, which in turn implies weak stochastic transitivity.

Despite the fact that g-stochastic transitivity of probabilistic relations and $*$-transitivity of fuzzy preference relations belong to different contexts, there are several attempts in the literature to unify these two definitions (see for example [17, 30, 123]). Both types of transitivity are particular cases of cycle-transitivity, a concept introduced by De Schuymer et al. in [38].

Before recalling the definition of cycle-transitivity, let us fix some useful notation. Let x, y, z be three alternatives in X and p be a probabilistic relation defined on X. We denote

$$\begin{aligned} \alpha_{xyz} &= \min(p(x,y), p(y,z), p(z,x))\,, \\ \beta_{xyz} &= \mathrm{median}(p(x,y), p(y,z), p(z,x))\,, \\ \gamma_{xyz} &= \max(p(x,y), p(y,z), p(z,x))\,. \end{aligned} \tag{3.1}$$

Let us remark that $\alpha_{xyz} \leq \beta_{xyz} \leq \gamma_{xyz}$, $\alpha_{xyz} = \alpha_{yzx} = \alpha_{zxy} = 1 - \gamma_{yxz}$, $\beta_{xyz} = \beta_{yzx} = \beta_{zxy} = 1 - \beta_{yxz}$ and $\gamma_{xyz} = \gamma_{yzx} = \gamma_{zxy} = 1 - \alpha_{yxz}$. Finally, we denote with Δ the following subset of $[0,1]^3$: $\Delta = \{(a,b,c) \in [0,1]^3 \mid a \leq b \leq c\}$.

Definition 3.3 ([33]) *A function $U : \Delta \to \mathbb{R}$ is called an upper bound function if, for any $\alpha, \beta, \gamma \in \Delta$ it holds that*

(i) $U(0,0,1) \geq 0$*;*

(ii) $U(0,1,1) \geq 1$*;*

(iii) $U(\alpha,\beta,\gamma) + U(1-\gamma, 1-\beta, 1-\alpha) \geq 1$.

To any upper bound function U we can associate another function $L : \Delta \to \mathbb{R}$ called lower bound function defined by

$$L(\alpha,\beta,\gamma) = 1 - U(1-\gamma, 1-\beta, 1-\alpha)\,.$$

The function L is called dual lower bound function of U. If $L(\alpha,\beta,\gamma) = U(\alpha,\beta,\gamma)$ for any (α,β,γ) in Δ, U is called a self-dual upper bound function.

Now we can define cycle-transitivity.

Definition 3.4 ([33]) *Let p be a probabilistic relation defined on X. We say that p is cycle-transitive with respect to the upper bound function U if, for any x, y, z in X, it holds that*

$$L(\alpha_{xyz},\beta_{xyz},\gamma_{xyz}) \leq \alpha_{xyz}+\beta_{xyz}+\gamma_{xyz}-1 \leq U(\alpha_{xyz},\beta_{xyz},\gamma_{xyz})\,.$$

Recall that if Q is cycle-transitive with respect to U_1 and $U_1(\alpha,\beta,\gamma) \leq U_2(\alpha,\beta,\gamma)$ for any (α,β,γ) in Δ, then Q is cycle-transitive with respect to U_2.

De Baets et al. [33] proved that, under certain conditions on g, the g-stochastic transitivity of a probabilistic relation is a special case of cycle-transitivity.

Proposition 3.5 ([33]) *Let $g : [\frac{1}{2},1]^2 \to [\frac{1}{2},1]$ be a commutative, increasing function such that $g(\frac{1}{2},x) \leq x$, for any $x \in [\frac{1}{2},1]$. A probabilistic relation p defined on X is g-stochastic transitive if and only if it is cycle-transitive with respect to the upper bound function U_g defined as*

$$U_g(\alpha,\beta,\gamma) = \begin{cases} \frac{1}{2} & \text{, if } \alpha \geq \frac{1}{2}, \\ 2 & \text{, if } \beta < \frac{1}{2}, \\ \beta+\gamma-g(\beta,\gamma) & \text{, else.} \end{cases}$$

Other interesting results on the properties of cycle-transitivity can be found in [30–33, 37–39].

3.1.2 Probabilistic choice functions

A probabilistic choice function (also called stochastic choice function) is an extension of the concept of probabilistic relation to sets with cardinality greater than two. It first appeared in [82] as a probabilistic version of a classical choice function.

Definition 3.6 *Let X be a finite set of alternatives and let $\mathcal{B}$ be the family of all subsets of X. A probabilistic choice function p on $\mathcal{B}$ is a function that for any $S \in \mathcal{B}$ specifies exactly one finitely additive probability measure over the family of all subsets of S. Given a set S and $S \subseteq T$, we denote with $p(S,T)$ the probability that the choice from the set of alternatives T will lie in S.*

Obviously, when the set T contains only two alternatives, i.e. $T = \{x, y\}$, the probabilistic choice function $p(x, \{x, y\})$ coincides with the probabilistic relation $p(x, y)$, hence, for brevity, we will denote it $p(x, y)$.

The probabilistic choice function p of a set $S \subseteq T$ is completely determined by its values $p(x,T)$, in the sense that $p(S,T) = \sum_{x \in S} p(x,T)$.

3.1.3 Rationality conditions for probabilistic choice functions

A large literature exists on rationality conditions for probabilistic choice functions. The most known and intuitively most compelling of these conditions is usually called regularity condition RC (RG in [28], NC in [26]), which simply postulates that the probability of choosing S from a set of alternatives T cannot increase if the set T is expanded to a larger set $Z \supset T$.

Definition 3.7 ([28]) *A probabilistic choice function p on X satisfies the*

Regularity Condition (RC) if, for any S, T, Z in $\mathcal{B}$ such that $S \subseteq T \subseteq Z$, it holds that $p(S,T) \geq p(S,Z)$.

Regularity Condition can be interpreted as an adaptation to the stochastic framework of Condition α proposed by Sen [115] and reported here in Definition 1.13.

Another well-known rationality condition for probabilistic choice functions has been proposed by Luce in [82] and is usually referred as Luce's Axiom of Choice. Formally, it is composed of two parts:

Definition 3.8 ([82]) *Let p be a probabilistic choice function on X. It is said that p satisfies Luce's Axiom of Choice if the following conditions hold: for any $S \in \mathcal{B}$ and $x \in S$:*

Part 1 *If $p(x,y) \neq 0$, for any $y \in S$, then*

$$p(x,X) = p(x,S)p(S,X)\,. \tag{3.2}$$

Part 2 *If $p(x,y) = 0$, for some y in S, then*

$$p(S,X) = p(S \setminus \{x\}, X \setminus \{x\})\,. \tag{3.3}$$

Part 2 is the least restrictive assumption: those alternatives $x \in S$ that are never chosen in pairwise comparisons with other alternatives in S can be deleted from S without affecting the choice probabilities. Part 1 states that the probability of selecting alternative x from the universe X is equal to the probability of selecting x from a set S multiplied by the probability of selecting S from X.

One consequence of Luce's Choice Axiom is that the probabilistic choice function p satisfies the so-called constant ratio rule: for any x, y in S, it holds that

$$\frac{p(x,S)}{p(y,S)} = \frac{p(x,X)}{p(y,X)}\,.$$

This condition trivially implies that, for any $S \in \mathcal{B}$,

$$\frac{p(x,y)}{p(y,x)} = \frac{p(x,S)}{p(y,S)}\,.$$

Another condition for the rationality of the probabilistic choice function has been proposed by Bandyopadhyay et al. in [5]. It is inspired by the Weak Axiom of Revealed Preference (WARP, see Definition 1.12) of classical choice theory.

Definition 3.9 *A probabilistic choice function p on X satisfies the Weak Axiom of Stochastic Revealed Preference (WASRP, for short) if, for any S, T in $\mathcal{B}$ and any A such that $A \subseteq S \cap T$, it holds that*

$$p(A,T) - p(A,S) \leq p(S \setminus T, S)\,.$$

The intuition behind this condition is the following: at the beginning S is the set of available alternatives and $A \subseteq S$. The probability that the choice from S will lie in A is $p(A,S)$. Then the set of available alternatives changes from S to T and A is also contained in T. If the new choice probability $p(A,T)$ is greater than $p(A,S)$, then it is reasonable to argue that this increase occurs only because the move eliminates some alternatives that were present in S and are no longer available in T. But then the increase $P(A,T) - P(A,S)$ should not exceed the initial probability of choosing a subset in $S \setminus T$. Hence $p(A,T) - p(A,S)$ cannot be greater than $p(S \setminus T, S)$.

The connection between conditions RC and WASRP has been studied in [5–7, 28]. It is proved that WASRP necessarily implies RC, regardless of the domain of the probabilistic choice function. It has also been showed that if the domain is not complete ($\mathcal{B} \neq 2^X \setminus \{\emptyset\}$), then there exist examples of probabilistic choice functions that satisfy RC, but fail to satisfy WASRP. However, the most interesting result is that, on a complete domain, WASRP and RC are equivalent.

Finally, a definition of rationalizable probabilistic choice function is needed for a comparison with the classical concept of G-rational choice function.

Definition 3.10 ([28]) *Let $\mathbb{O}$ be the set of all weak orders over X (i.e. the set of all reflexive, complete and transitive binary relations on X). For any non-empty subsets S and T of X, let $g(S,T)$ be the set of all $J \in \mathbb{O}$, such that J has a unique greatest element in S and this unique greatest element in S belongs to T. A probabilistic choice function p on X is rationalizable in terms of stochastic orderings if there exists a finitely additive probability measure π defined over the class of all subsets of $\mathbb{O}$ such that, for any $S \subseteq T$ in $\mathcal{B}$, $p(S,T) = \pi(g(T,S))$.*

It is known that a probabilistic choice function that is rationalizable in terms of stochastic orderings necessarily satisfies RC and WASRP, though neither WASRP nor RC necessarily implies rationalizability in terms of stochastic orderings. In particular, Bandyopadhyay et al. [5] showed that a probabilistic choice function that is rationalizable in terms of stochastic orderings must satisfy WASRP and that the converse implication is not necessarily true. Block and Marschak [19] showed that rationalizability of a probabilistic choice function in terms of stochastic orderings implies RC and that the converse is not necessarily true when the cardinality of X is greater than or equal to four.

3.2 A common framework for probabilistic and fuzzy choice

According to Fishburn [48] there exist at least three ways of representing choices in a mathematical way:

(i) binary preference relations;

(ii) choice functions;

(iii) probabilistic choice functions.

All of them are legitimate and prove to be appropriate in different circumstances. The relationships between them have been studied in the literature. In particular, the connections between binary relations and choice functions have been analyzed in the classical theory of choice, reported here in Section 1.1 and the connections between binary preference relations and probabilistic choice functions have been studied, among others, by Fishburn [47] and Luce [84]. Fishburn already addressed in [48] the lack of results on the connections between choice functions and probabilistic choice functions and hence he proposed a set of conditions that should be satisfied by the probabilistic choice function in order for the associated choice function to be rational.

The same situation appears in the framework of fuzzy choice theory. In the last years the results of classical choice theory have been extended to the fuzzy framework, laying bare the connections between fuzzy preference relations and fuzzy choice functions (see, amongst others, [10, 64, 88, 89, 91, 92, 131, 132]). Surprisingly, while the connection between fuzzy preference relations and fuzzy choice functions has been studied in depth, there appears to be no literature on the comparison between fuzzy choice functions and probabilistic choice functions. We recently approached this problem in [90, 93], where we proved some preliminary results.

The approach used by Fishburn in [48] was to construct a choice function from a given probabilistic choice function and then to find suitable conditions on the latter in order to ensure that the derived choice function is rational. One of the constructions proposed by Fishburn is the following:

Definition 3.11 ([48]) *Given the probabilistic choice function p, a fuzzy choice function C_p can be defined in the following way:*

$$C_p(S)(x) = \frac{p(x,S)}{\max_{y\in S} p(y,S)}\,. \tag{3.4}$$

Remark 3.12 *Observe that Eq. (3.4) can also be written using the implication operator derived from the product t-norm ($a \to_{\mathbf{P}} b = (b/a) \wedge 1$), i.e.*

$$C_p(S)(x) = \frac{p(x,S)}{\max_{y\in S}(p(y,S))} = \bigwedge_{y\in S} (p(y,S) \to_{\mathbf{P}} p(x,S))\,. \tag{3.5}$$

Inspired by the observation in Remark 3.12, we propose a novel construction of fuzzy choice functions (in the sense of Banerjee) from a given probabilistic choice function:

Definition 3.13 ([90]) *Let $*$ be a left-continuous t-norm and let p be a probabilistic choice function on X. A fuzzy choice function C_p can be constructed using the following formula, for any $S \in \mathcal{B}$ and any $x \in S$:*

$$C_p(S)(x) = \bigwedge_{y\in S} (p(y,S) \to p(x,S))\,. \tag{3.6}$$

Let us list some properties of the fuzzy choice function C_p derived from p, which hold independently from the chosen t-norm:

(i) C_p is actually a fuzzy choice function, since, for any $S \in \mathcal{B}$, there exists at least one x in X, such that $C_p(S)(x) > 0$;

(ii) C_p satisfies condition H1: in fact, for any $S \in \mathcal{B}$, there exists an element x such that $C_p(S)(x) = 1$; in particular, the element x is the one in S for which $p(x, S)$ is the greatest;

(iii) if the choice probability is defined on $\mathcal{B} = 2^X \setminus \{\emptyset\}$, then C_p is automatically defined on the same family of sets, and hence condition H2 holds.

After introducing the construction of a fuzzy choice function C_p from a given probabilistic relation p, we turn to the problem of finding suitable conditions on p that can ensure the rationality of C_p. For this reason we turn to the two known conditions presented in Definitions 3.8 and 3.9: Luce's Axiom of Choice and WASRP.

Let us start with Luce's Axiom of Choice, in particular, its first part expressed by Eq. (3.2): for any $S \in \mathcal{B}$ and $x \in X$, it holds that

$$p(x, X) = p(x, S)p(S, X)\,.$$

This implies immediately that, for any $S \in \mathcal{B}$ and $x \in X$, it holds that

$$p(x, X) \geq p(x, S)p(S, X)\,. \tag{3.7}$$

It is easy to prove that, for any $S \subseteq T \subseteq Z \in \mathcal{B}$, Eq. (3.7) implies

$$p(S, Z) \geq p(S, T)p(T, Z)\,. \tag{3.8}$$

We can rewrite Eq. (3.8) using the t-norm of the product $*_{\mathbf{P}}$:

$$p(S,Z) \geq p(S,T) *_{\mathbf{P}} p(T,Z)\,,$$

or, equivalently, using Property 1 of implication operators ($a * b \leq c \Leftrightarrow a \leq b \rightarrow c$),

$$p(T,Z) \rightarrow_{\mathbf{P}} p(S,Z) \geq p(S,T)\,. \tag{3.9}$$

On the other hand, consider condition WASRP: for any T and Z in $\mathcal{B}$ it holds

$$p(S,T) - p(S,Z) \leq p(Z \setminus T, Z), \text{ for any } S \subseteq T \cap Z\,. \tag{3.10}$$

In particular, if we choose S such that $S \subseteq T \subseteq Z$, we can rewrite Eq. (3.10) as

$$p(S,T) - p(S,Z) \leq p(Z \setminus T, Z) = 1 - p(T,Z)\,.$$

Rewriting this last equation as

$$p(S,T) \leq 1 + p(S,Z) - p(T,Z)\,,$$

we notice that the same can be stated using the implication operator derived from the Łukasiewicz t-norm ($a \rightarrow_{\mathbf{L}} b = (1 - a + b) \wedge 1$):

$$p(T,Z) \rightarrow_{\mathbf{L}} p(S,Z) \geq p(S,T)\,. \tag{3.11}$$

Now, comparing Eq. (3.9) and Eq. (3.11), a clear pattern is visible: it seems that a rationality condition can be proposed using implication operators in such a way that, for a given left-continuous t-norm $*$ and associated implication operator $\rightarrow$ and for any $S \subseteq T \subseteq Z$ in $\mathcal{B}$, it holds that

$$p(S,T) \leq p(T,Z) \rightarrow p(S,Z)\,. \tag{3.12}$$

Equation (3.12) is hardly interpretable, so we look for an equivalent statement with a more appealing formulation. Recall that Luce's Axiom of Choice is equivalent to the constant ratio rule, i.e.

$$\frac{p(x,S)}{p(y,S)} = \frac{p(x,X)}{p(y,X)} \,. \tag{3.13}$$

Equation (3.13) can be rewritten using the biresiduum operator of the product t-norm ($a \leftrightarrow_{\mathbf{P}} b = \frac{a \wedge b}{a \vee b}$):

$$p(x,S) \leftrightarrow_{\mathbf{P}} p(y,S) = p(x,X) \leftrightarrow_{\mathbf{P}} p(y,X) \,. \tag{3.14}$$

From Eq. (3.14) we can derive the following more general condition, holding for any left-continuous t-norm:

$$p(x,S) \leftrightarrow p(y,S) \leq p(x,X) \leftrightarrow p(y,X) \,. \tag{3.15}$$

In particular, since $\{x,y\}$ is the smallest set in $\mathcal{B}$ containing both x and y, Eq. (3.15) implies that, for any $S \in \mathcal{B}$, it holds that

$$p(x,y) \leftrightarrow p(y,x) \leq p(x,S) \leftrightarrow p(y,S) \,. \tag{3.16}$$

The interpretation of Eq. (3.16) is quite simple: observe that the operator $\leftrightarrow$ applied to a pair of values a and b expresses the degree to which a and b can be considered to be equal. Obviously, if $a = b$, then $a \leftrightarrow b = 1$, while $a \leftrightarrow b = 0$ if $a = 0$ and $b = 1$, or vice versa. Then, Eq. (3.16) can be interpreted in the following way: the degree to which two alternatives x and y are equally probable to be chosen is minimal when there are no more alternatives that can be chosen. If more elements are added to the set of possible choices, the probabilities of choosing x or y can only become more similar.

We can now state two new sets of rationality conditions for a probabilistic choice function based on Eq. (3.16).

Definition 3.14 *Let $*$ be a left-continuous t-norm. A probabilistic choice function p on X satisfies the Weak Scalability Condition (WSC) if, for any S, T in $\mathcal{B}$ such that $S \subseteq T$ and x, y in S, it holds that*

$$p(x, S) \leftrightarrow p(y, S) \leq p(x, T) \leftrightarrow p(y, T)\,.$$

A probabilistic choice function p satisfies the Strong Scalability Condition (SSC) if, for any S, T in $\mathcal{B}$ such that $S \subseteq T$ and x, y in S, it holds that

$$p(x, S) \leftrightarrow p(y, S) = p(x, T) \leftrightarrow p(y, T)\,.$$

If regularity condition RC of Definition 3.7 can be interpreted as a probabilistic version of Condition α of Sen [115], we can observe that the scalability condition, either in its weak or in its strong version, can play the role of Condition β of Sen in the probabilistic setting.

We propose two last rationality conditions inspired by the Weak Congruence Axiom (WCA) of classical choice theory.

Definition 3.15 *A probabilistic choice function p on X satisfies the Weak Stochastic Congruence Axiom (WSCA) if for any $S \in \mathcal{B}$ and $x, y \in S$ such that $p(x, S) > p(y, S)$, it holds that $p(x, T) \geq p(y, T)$ for any other set T containing x and y.*

It satisfies the Strong Stochastic Congruence Axiom (SSCA) if for any $S \in \mathcal{B}$ and $x, y \in S$ such that $p(x, S) > p(y, S)$, it holds that either $p(x, T) > p(y, T)$ or $p(x, T) = p(y, T) = 0$, for any other set T containing x and y.

The interpretation of these two conditions is the following: for condition WSCA, if there is at least one set S where an alternative x is strictly more probable to be chosen than another alternative y, then x has to be considered at least as good as y in all other sets that contain both alternatives. In other words, there cannot exist two sets S and T such that x is strictly preferred to y in S and y is strictly preferred to x in the other set T. Condition SSCA is stronger: it establishes that if there exists one set S where an alternative x is strictly more probable to be chosen than another alternative y, then x is always strictly preferred to y, for any T that contains both x and y, unless both of them are unlikely to be chosen ($p(x,T) = p(y,T) = 0$). In other words, it implies the same as WSCA, i.e. that if x is strictly preferred to y in one set S, then there cannot exist another set T where y is strictly preferred to x. Furthermore, it adds the restriction that if in one set T the two alternatives x and y happen to be equally probable, this is due to the fact that the other alternatives in $T \setminus \{x, y\}$ are considered to be much better compared to x and y that they extinguish the probability of choosing x or y.

Example 3.16 *Consider the set X containing only three alternatives x, y and z. If we know, for example, that $p(x,y) = 0.7$ and $p(y,x) = 0.3$, then according to WSCA it can happen in X that $p(x,X) = 0.2$, $p(y,X) = 0.2$ and $p(z,X) = 0.6$. This means that adding alternative z to the initial set $\{x,y\}$ has decreased the probability of choosing x and y and furthermore we have lost the strict preference of x over y. Nevertheless, x and y are still likely to be chosen. This situation would contradict SSCA instead. In fact, according to SSCA, if x and y have to be equally probable in T then their probabilities have to vanish, so the only possibility would be $p(x,X) = p(y,X) = 0$ and $p(z,X) = 1$.*

Remark 3.17 *It is immediate to prove that if p is a probabilistic choice function satisfying SSCA, then it will also satisfy WSCA.*

3.2.1 On the rationality of a fuzzy choice function constructed from a probabilistic choice function

We know that in classical choice theory Conditions α and β are equivalent to the joint G-normality of the choice function and regularity of the revealed preference relation, that in turn is equivalent to the Weak Congruence Axiom (WCA). In Section 2.3 we studied the fuzzy version of those statements, proving that conditions Fα and Fβ imply G-normality of a fuzzy choice function and $*$-regularity of the fuzzy revealed preference relation R_C, provided the t-norm $*$ is the minimum (Theorem 2.56 (vi) and Lemma 2.39 (ii)). We also proved that condition WFCA implies the $*$-transitivity of the fuzzy revealed preference relation, provided the t-norm is the minimum. Analogously, we study the consequences of assuming regularity and scalability of the probabilistic choice function for the constructed fuzzy choice function.

Let us prove first two auxiliary results.

Lemma 3.18 *Let $*$ be a left-continuous t-norm. If the probabilistic choice function p satisfies SSC and WSCA, then for any x, y in X and $S \supseteq \{x, y\}$ it holds that*

$$p(x,y) \to p(y,x) = p(x,S) \to p(y,S)\,. \tag{3.17}$$

Proof. Consider an arbitrary pair of alternatives x and y and $S \in \mathcal{B}$ such that $\{x, y\} \subseteq S$. For $p(x, y)$ and $p(y, x)$, we only have three possibilities:

(i) If $p(x, y) < p(y, x)$, by WSCA, we have that $p(x, S) \leq p(y, S)$ and then trivially

$$p(x,y) \to p(y,x) = 1 = p(x,S) \to p(y,S)\,.$$

(ii) If $p(x, y) = p(y, x)$, then $p(x, y) \leftrightarrow p(y, x) = 1$ and by SSC we have that

$$1 = p(x, y) \leftrightarrow p(y, x) = p(x, S) \leftrightarrow p(y, S) \, .$$

Hence $p(x, S) = p(y, S)$ and Eq. (3.17) is trivially satisfied.

(iii) If $p(x, y) > p(y, x)$, by WSCA, we know that $p(x, S) \geq p(y, S)$ and furthermore, using SSC, we have that

$$\begin{aligned} 1 &> p(x, y) \to p(y, x) \\ &= p(x, y) \leftrightarrow p(y, x) = p(x, S) \leftrightarrow p(y, S) \, . \end{aligned}$$

Hence $p(x, S) \neq p(y, S)$, and in particular, by WSCA, $p(x, S) > p(y, S)$. Finally, using SSC, we have that

$$\begin{aligned} p(x, y) \to p(y, x) &= p(x, y) \leftrightarrow p(y, x) \\ &= p(x, S) \leftrightarrow p(y, S) = p(x, S) \to p(y, S) \, . \end{aligned}$$

In all three cases we proved that Eq. (3.17) is satisfied. □

Lemma 3.19 *Let $*$ be a left-continuous t-norm. If the probabilistic choice function p satisfies WSC and SSCA, then for any x, y in X and $S \supseteq \{x, y\}$ it holds that*

$$p(x, y) \to p(y, x) \leq p(x, S) \to p(y, S) \, . \tag{3.18}$$

Proof. Consider an arbitrary pair of alternatives x and y and $S \in \mathcal{B}$ such that $\{x, y\} \subseteq S$. For $p(x, y)$ and $p(y, x)$ we only have three possibilities:

(i) If $p(x, y) < p(y, x)$, by SSCA, we have that either $p(x, S) < p(y, S)$ or $p(x, S) = p(y, S) = 0$. In both cases $p(x, S) \to p(y, S) = 1 = p(x, y) \to p(y, x)$.

(ii) If $p(x,y) = p(y,x)$, then by WSC we have that

$$1 = p(x,y) \leftrightarrow p(y,x) \leq p(x,S) \leftrightarrow p(y,S)\,.$$

Hence $p(x,S) = p(y,S)$ and Eq. (3.18) is trivially satisfied.

(iii) If $p(x,y) > p(y,x)$, by SSCA, we have that either $p(x,S) > p(y,S)$ or $p(x,S) = p(y,S) = 0$. In the second case $p(x,S) \to p(y,S) = 1$ and Eq. (3.18) is trivially satisfied. In the first case, using WSC, we have that

$$\begin{aligned} p(x,y) \to p(y,x) &= p(x,y) \leftrightarrow p(y,x) \\ &\leq p(x,S) \leftrightarrow p(y,S) = p(x,S) \to p(y,S)\,. \end{aligned}$$

In all three cases we proved that Eq. (3.18) is satisfied. □

The next result shows that the fuzzy preference relation revealed from C_p can be written as a function of the probabilistic relation p, provided certain conditions are satisfied.

Proposition 3.20 *Let $*$ be a left-continuous t-norm and p a probabilistic choice function on X. If one of the following sets of hypotheses hold*

Hypo. A *p satisfies conditions SSC and WSCA;*

Hypo. B *p satisfies conditions WSC, SSCA, RC and the t-norm $*$ is the minimum,*

then the fuzzy preference relation R_{C_p} revealed from C_p can be written as:

$$R_{C_p}(x,y) = p(y,x) \to p(x,y)\,. \tag{3.19}$$

Proof. Consider an arbitrary pair of alternatives x and y such that $x \neq y$ and recall the formula for computing the fuzzy revealed preference relation of a fuzzy choice function:

$$R_{C_p}(x,y) = \bigvee_{\{S \in \mathcal{B} | x,y \in S\}} C_p(S)(x)\,. \tag{3.20}$$

Applying Eq. (3.6) of Definition 3.13 to Eq. (3.20) we obtain:

$$R_{C_p}(x,y) = \bigvee_{\{S \in \mathcal{B} | x,y \in S\}} \left(\bigwedge_{k \in S} p(k,S) \to p(x,S) \right)\,. \tag{3.21}$$

The supremum in Eq. (3.21) is computed over the sets S in $\mathcal{B}$ such that $x, y \in S$. This family of subsets of $\mathcal{B}$ can be split into $\{x, y\} \cup S_{xy}$, where $S_{xy} = \{S \in \mathcal{B} \mid x, y \in S \text{ and } |S| \geq 3\}$ and then Eq. (3.21) becomes:

$$R_{C_p}(x,y) = (p(y,x) \to p(x,y)) \vee \bigvee_{S \in S_{xy}} \left(\bigwedge_{k \in S} (p(k,S) \to p(x,S)) \right)\,. \tag{3.22}$$

If $p(x, y) \geq p(y, x)$, then $p(y, x) \to p(x, y) = 1$ and hence $R_{C_p}(x, y) = 1 = p(y, x) \to p(x, y)$. Consider then the case $p(x, y) < p(y, x)$. Equation (3.22) implies that $R_{C_p}(x, y) \geq p(y, x) \to p(x, y)$. We next prove the opposite inequality, i.e. $R_{C_p}(x, y) \leq p(y, x) \to p(x, y)$, that, by Eq. (3.22), can also be proved by showing that

$$\bigvee_{S \in S_{xy}} \left(\bigwedge_{k \in S} (p(k,S) \to p(x,S)) \right) \leq p(y,x) \to p(x,y)$$

If **Hypo. A** is satisfied, then using Lemma 3.18 we can write the right-hand part of Eq. (3.22) as:

$$\begin{aligned}\bigvee_{S\in S_{xy}}\left(\bigwedge_{k\in S}(p(k,S)\to p(x,S))\right) &\le \bigvee_{S\in S_{xy}}(p(y,S)\to p(x,S))\\ &= \bigvee_{S\in S_{xy}}(p(y,x)\to p(x,y))\\ &= p(y,x)\to p(x,y)\,.\end{aligned}$$

Hence $R_{C_p}(x,y)\le p(y,x)\to p(y,x)$.

If **Hypo. B** is satisfied and $p(y,x)>p(x,y)$, then the set S_{xy} can be split into two subsets according to SSCA:

(i) $S_{xy}^{>}=\{S\in S_{xy}\mid p(y,S)>p(x,S)\}$;

(ii) $S_{xy}^{=}=\{S\in S_{xy}\mid p(y,S)=p(x,S)=0\}$.

We will prove that in both cases $\bigwedge_{k\in S}(p(k,S)\to p(x,S))\le p(y,x)\to p(x,y)$, for any $S\in S_{xy}$. Assume $S\in S_{xy}^{>}$, then by Lemma 3.19 and the fact that the t-norm is the minimum, we have that

$$p(x,y)=p(y,x)\to_{\mathbf{M}} p(x,y)\le p(y,S)\to_{\mathbf{M}} p(x,S)=p(x,S)\,.$$

Hence $p(x,y)\le p(x,S)$, but by RC we know that $p(x,y)\ge p(x,S)$, so $p(x,y)=p(x,S)$ and the previous chain of inequalities implies $p(y,x)\to_{\mathbf{M}} p(x,y)=p(y,S)\to_{\mathbf{M}} p(x,S)$.
If $S\in S_{xy}^{=}$, then there exists at least one alternative $z\in S$ such that $p(z,S)>0$ and hence $p(z,S)\to_{\mathbf{M}} p(x,S)=0$. Hence, in Eq. (3.22) the part $\bigwedge_{k\in S_{xy}^{=}}(p(k,S)\to p(x,S))$ becomes zero.

In both cases we proved that the right-hand part of Eq. (3.22) either is equal to zero or is equal to $p(y,x)\to p(x,y)$, hence $R_{C_p}(x,y)=p(y,x)\to p(x,y)$. □

Proposition 3.20 shows that the fuzzy preference relation R_{C_p} revealed from C_p can be written as a function of the reciprocal relation derived from p. In this way, not only the choice functions are connected, but also their respective preference relations.

Remark 3.21 *Combining Proposition 3.20 and Lemma 3.18 we immediately, have that if p satisfies SSC and WSCA, then $R_{C_p}(x, y) = p(y, S) \rightarrow p(x, S)$, for any $x, y \in S$ and any $S \in \mathcal{B}$ such that $\{x, y\} \subseteq S$.*

We can now present the two main results of this section. They establish under which conditions the fuzzy choice function C_p constructed from the probabilistic choice function p is G-normal and the associated fuzzy revealed preference relation is $*$-transitive.

Proposition 3.22 *Let $*$ be a left-continuous t-norm and let p be a probabilistic choice function on X. If one of the following sets of hypotheses hold*

Hypo. A *p satisfies conditions SSC and WSCA;*

Hypo. B *p satisfies conditions WSC, SSCA, RC and the t-norm $*$ is the minimum,*

then the associated fuzzy choice function C_p is G-normal.

Proof. To prove that C_p is G-normal we need to show that it can be G-rationalized by its own fuzzy preference relation R_{C_P}, i.e. $C_p(S)(x) = G_{R_{C_p}}(S)(x)$, for any $S \in \mathcal{B}$ and any $x \in S$. Consider an arbitrary $x \in X$ and $S \in \mathcal{B}$ such that $x \in S$.

If **Hypo. A** is satisfied, then, using Proposition 3.20 and Lemma 3.18, we can prove that

$$\begin{aligned} G_{R_{C_p}}(S)(x) &= \bigwedge_{y\in S} R_{C_p}(x,y) \\ &= \bigwedge_{y\in S} (p(y,x)\to p(x,y)) \\ &= \bigwedge_{y\in S} (p(y,S)\to p(x,S)) = C_p(S)(x)\,. \end{aligned}$$

If **Hypo. B** is satisfied, we first prove that $G_{R_{C_p}}(S)(x) \geq C_p(S)(x)$. By definition of the fuzzy revealed preference relation R_{C_p}, we have that

$$R_{C_p}(x,y) = \bigvee_{\{T\in\mathcal{B}\mid x,y\in T\}} C_p(T)(x) \geq C_p(S)(x)\,. \tag{3.23}$$

Also, by definition of $G_{R_{C_p}}$ and Eq. (3.23), we have that

$$G_{R_{C_p}}(S)(x) = \bigwedge_{y\in S} R_{C_p}(x,y) \geq \bigwedge_{y\in S} C_p(S)(x) = C_p(S)(x)\,,$$

hence $G_{R_{C_p}}(S)(x) \geq C_p(S)(x)$.

Using Proposition 3.20 and Lemma 3.19, we can also prove the opposite inequality:

$$\begin{aligned} G_{R_{C_p}}(S)(x) &= \bigwedge_{y\in S} R_{C_p}(x,y) \\ &= \bigwedge_{y\in S} p(y,x)\to_{\mathbf{M}} p(x,y) \\ &\leq \bigwedge_{y\in S} p(y,S)\to_{\mathbf{M}} p(x,S) = C_p(S)(x)\,. \end{aligned}$$

Combining $C_p(S)(x) \leq G_{R_{C_p}}(S)(x)$ and $C_p(S)(x) \geq G_{R_{C_p}}(S)(x)$, we then proved that $C_p(S)(x) = G_{R_{C_p}}(S)(x)$. □

Proposition 3.23 *Let $*$ be a left-continuous t-norm and p a probabilistic choice function on X. If one of the following sets of hypotheses hold*

Hypo. A *p satisfies conditions SSC and WSCA;*

Hypo. B *p satisfies conditions WSC, SSCA, RC and the t-norm $*$ is the minimum,*

then the fuzzy revealed preference relation R_{C_p} is $*$-transitive.

Proof. Let x, y, z be three arbitrary alternatives in X. We want to prove that

$$R_{C_p}(x,y) * R_{C_p}(y,z) \leq R_{C_p}(x,z)\,. \qquad (3.24)$$

If **Hypo. A** is satisfied, using Proposition 3.20 and Remark 3.21, we know that $R_{C_p}(x,z) = p(z,x) \to p(x,z) = p(z,S) \to p(x,S)$, $R_{C_p}(x,y) = p(y,S) \to p(x,S)$ and $R_{C_p}(y,z) = p(z,S) \to p(y,S)$, for any S containing x, y and z. Using Property 12 of implication operators ($(a \to b) * (b \to c) \leq a \to c$) we have that

$$\begin{aligned} R_{C_p}(x,z) &= p(z,S) \to p(x,S) \\ &\geq (p(z,S) \to p(y,S)) * (p(y,S) \to p(x,S)) \\ &= R_{C_p}(y,z) * R_{C_p}(x,y)\,, \end{aligned}$$

i.e. is exactly Eq. (3.24).

If **Hypo. B** is satisfied, by Proposition 3.20, we know that $R_{C_p}(x,z) = p(z,x) \to p(x,z)$. Hence, if $p(z,x) \leq p(x,z)$, we immediately have that $R_{C_p}(x,z) = 1$ and then Eq. (3.24) is trivially satisfied. Let us consider the case in which $p(z,x) > p(x,z)$ and $S = \{x,y,z\}$. By SSCA, there are two possibilities:

(i) $p(x,S) = p(z,S) = 0$ or

(ii) $p(z, S) > p(x, S)$.

In the first case we have that $p(y, S) = 1$. By Lemma 3.19 and Proposition 3.20, we have that

$$R_{C_p}(x, y) = p(y, x) \to_{\mathbf{M}} p(x, y) \leq p(y, S) \to_{\mathbf{M}} p(x, S) = 1 \to_{\mathbf{M}} 0 = 0$$

and then Eq. (3.24) is trivially satisfied.

In the second case, when $p(z, S) > p(x, S)$, suppose by absurdum that Eq. (3.24) is not satisfied, i.e.

$$R_{C_p}(x, z) < R_{C_p}(x, y) *_{\mathbf{M}} R_{C_p}(y, z) \,.$$

Using Proposition 3.20, Lemma 3.19 and Property 12 of implication operators we have that

$$\begin{aligned} p(z, x) \to_{\mathbf{M}} p(x, z) &= R_{C_p}(x, z) < R_{C_p}(x, y) *_{\mathbf{M}} R_{C_p}(y, z) \\ &= (p(y, x) \to_{\mathbf{M}} p(x, y)) *_{\mathbf{M}} (p(z, y) \to_{\mathbf{M}} p(y, z)) \\ &\leq (p(y, S) \to_{\mathbf{M}} p(x, S)) *_{\mathbf{M}} (p(z, S) \to_{\mathbf{M}} p(y, S)) \\ &\leq p(z, S) \to_{\mathbf{M}} p(x, S) \,. \end{aligned}$$

Since the t-norm is the minimum, we have that

$$p(x, z) = p(z, x) \to_{\mathbf{M}} p(x, z) < p(z, S) \to_{\mathbf{M}} p(x, S) = p(x, S) \,,$$

but this contradicts RC, i.e. $p(x, z) \geq p(x, S)$. □

We can finally state the main result of this section.

Theorem 3.24 *Let $*$ be a left-continuous t-norm and p a probabilistic choice function on X. If one of the following sets of hypotheses hold*

Hypo. A *p satisfies conditions SSC and WSCA;*

Hypo. B *p satisfies conditions WSC, SSCA, RC and the t-norm $*$ is the minimum,*

then the fuzzy choice function C_p is G-normal and its fuzzy revealed preference relation R_{C_p} is $*$-transitive.

Let us stress the importance of the results proved in this section. First of all, the possibility of constructing a fuzzy choice function from a given probabilistic choice function is fundamental, especially in a practical context. In fact, while fuzzy choice functions have been widely studied, it is still not clear how they can be observed in the real world. We agree with Banerjee [10] when he says that "*[...] there may be problems of estimation, but fuzzy choice functions are, in theory, observable*". With the proposed method we can rely on probabilistic choice functions, which are much easier to observe: they can be estimated by using the frequency of choice of an element in a set of alternatives, in a data set of repeated observations of choices on the same set of alternatives. Furthermore, if the observed probabilistic choice function satisfies certain conditions, then Theorem 3.24 ensures that the associated fuzzy choice function is G-normal and that its fuzzy revealed preference relation is $*$-transitive.

The estimation of probabilistic choice functions for big sets of alternatives can be problematic. In fact, even for a set X with few alternatives, the number of subsets for which the probabilistic choice function needs to be known is $2^{|X|} - 1$. Sets with more than 10 alternatives are already critical. Furthermore, the number of subsets S of X for which the probabilistic choice function p can be directly observed is usually reduced. Then condition H2 (p is known for any subset S of X) is hardly satisfied. Also weaker conditions on the domain, like WH2 (p is known for any pair and triplet of alternatives of X), are seldom satisfied. In fact, it is more likely to know

p for big subsets of X instead of for small sets, as WH2 would require. Thus, in the worst (and most probable) case, we can only observe p on a reduced number of subsets of X and most of them contain more than 2 or 3 alternatives. In this situation, the conditions of rationality involved in Theorem 3.24 can be used to reconstruct artificially the missing information on the subsets of X that are not directly observed. In this way, we obtain a double benefit: we can complete the information on the non-observed sets and we are sure that the conditions needed for Theorem 3.24 are trivially satisfied.

3.3 Probabilistic and fuzzy preference relations

This section investigates the connections between fuzzy preference relations and probabilistic relations. In fact, both of them are used to formalize the pairwise comparison of alternatives and both of them allow to express certain degree of imprecision. We first propose a way to compute a fuzzy preference relation as a function of a given probabilistic relation using an expression similar to the one contained in Proposition 3.20, which makes use of an implication operator derived from a t-norm. Provided the t-norm satisfies certain conditions, we can prove that the process can be reversed, i.e. the original probabilistic relation can be computed as a function of the fuzzy preference relation. Once the interchangeability of the probabilistic and the fuzzy preference relations is proved, we then study which properties pass from one formalism to another. In particular, we focus on the transitivity property. The main result of this section is Theorem 3.34, which links the $*$-transitivity of a fuzzy preference relation to the cycle-transitivity of the associated probabilistic relation, provided that the two relations are equivalent w.r.t. our construction. Some of our preliminary results in this field can be found in [95] and the results presented here extend the ones contained in that paper to a more general framework. In particular, the contents of [95] correspond to the special case of the results presented in this section when the t-norm is the Łukasiewicz t-norm.

3.3.1 Interchangeable probabilistic and fuzzy preference relations

In Proposition 3.20 we proved that under certain conditions a fuzzy preference relation R_{C_p} can be expressed as a function of a probabilistic relation p: let $*$ be a left-continuous t-norm and $\rightarrow$ its associated implication operator, then, for any $x, y \in X$, it holds that

$$R_{C_p}(x,y) = p(y,x) \rightarrow p(x,y) \,.$$

The same formula, extrapolated from the context of Section 3.1, can be used for generating a fuzzy preference relation Q_p^* from a given probabilistic relation p on X.

Definition 3.25 *Let $*$ be a left-continuous t-norm. Given a probabilistic relation p on X, we define the fuzzy preference relation Q_p^* for any $x, y \in X$ as:*

$$Q_p^*(x,y) = p(y,x) \rightarrow p(x,y) \,. \tag{3.25}$$

Remark 3.26 *Since $p(x,y) + p(y,x) = 1$, for any $x, y \in X$, it holds that $Q_p^*(x,y) \vee Q_p^*(y,x) = 1$, i.e. Q_p^* is strongly complete. In particular, using*

Property 4 of implication operators ($a \leq b \Leftrightarrow a \to b = 1$), for any pair of $x, y \in X$, it holds that

(i) $p(y,x) \leq p(x,y)$ if and only if $Q_p^*(x,y) = 1$;

(ii) $p(x,y) = p(y,x) = \frac{1}{2}$ if and only if $Q_p^*(x,y) = Q_p^*(y,x) = 1$.

To reverse the construction presented in Definition 3.25, observe that given the fuzzy preference relation Q_p^*, it holds that, for any $x, y \in X$:

$$\begin{aligned} Q_p^*(x,y) &= p(y,x) \to p(x,y)\,, \\ Q_p^*(y,x) &= p(x,y) \to p(y,x)\,. \end{aligned}$$

Hence, by taking the minimum of $Q_p^*(x,y)$ and $Q_p^*(y,x)$ and considering Definition 1.19 of the biresiduum operator ($a \leftrightarrow b = (a \to b) \wedge (b \to a)$), we have that

$$\begin{aligned} Q_p^*(x,y) \wedge Q_p^*(y,x) &= (p(y,x) \to p(x,y)) \wedge (p(x,y) \to p(y,x)) \\ &= p(x,y) \leftrightarrow p(y,x)\,. \end{aligned} \tag{3.26}$$

Recalling that $p(y,x) = 1 - p(x,y)$, Eq. (3.26) can be written as

$$Q_p^*(x,y) \wedge Q_p^*(y,x) = p(x,y) \leftrightarrow (1 - p(x,y))\,. \tag{3.27}$$

For the case of the three t-norms $*_\mathbf{M}$, $*_\mathbf{P}$ and $*_\mathbf{L}$, considering the corresponding expressions of the biresiduum operators contained in Table 1.1, Eq. (3.26) takes the following form:

(i) If the t-norm is the minimum, then, either $Q_p^{*_\mathbf{M}}(x,y) = Q_p^{*_\mathbf{M}}(y,x) = 1$ and hence $p(x,y) = p(y,x) = \frac{1}{2}$, or

$$\begin{aligned} p(x,y) \wedge p(y,x) &= Q_p^{*_\mathbf{M}}(x,y) \wedge Q_p^{*_\mathbf{M}}(y,x)\,, \\ p(x,y) \vee p(y,x) &= 1 - (Q_p^{*_\mathbf{M}}(x,y) \wedge Q_p^{*_\mathbf{M}}(y,x))\,. \end{aligned}$$

(ii) If the t-norm is the product, then

$$\begin{aligned} p(x,y) \wedge p(y,x) &= \frac{Q_p^{*\mathbf{P}}(x,y) \wedge Q_p^{*\mathbf{P}}(y,x)}{1 + (Q_p^{*\mathbf{P}}(x,y) \wedge Q_p^{*\mathbf{P}}(y,x))}\,, \\ p(x,y) \vee p(y,x) &= \frac{1}{1 + (Q_p^{*\mathbf{P}}(x,y) \wedge Q_p^{*\mathbf{P}}(y,x))}\,. \end{aligned}$$

(iii) If the t-norm is the Łukasiewicz t-norm (see [95]),

$$\begin{aligned} p(x,y) &= \frac{1 + Q_p^{*\mathbf{L}}(x,y) - Q_p^{*\mathbf{L}}(y,x)}{2}\,, \\ p(y,x) &= \frac{1 + Q_p^{*\mathbf{L}}(y,x) - Q_p^{*\mathbf{L}}(x,y)}{2}\,. \end{aligned}$$

It would be interesting to prove for which t-norms Eq. (3.27) has a solution and the solution is unique for a given pair of values $Q_p^*(x,y)$ and $Q_p^*(y,x)$. Let us restate the problem in a more concise form: we want to prove under which conditions there exists a unique solution to the following problem:

$$\begin{cases} a \in [0,1]\,, \\ c \in C = \{s \in [0,1] \mid \exists a \in [0,1] \text{ such that } s = a \to (1-a)\}\,, \\ c = a \to (1-a)\,. \end{cases}$$

Recall that the biresiduum operator is defined by $a \leftrightarrow (1-a) = (a \to (1-a)) \wedge ((1-a) \to a)$. At least one of the two values $(a \to (1-a))$ and $((1-a) \to a)$ has to be equal to one by Property 4 of implication operators. Then, for convenience, we can further reformulate the problem as follows: given $c \in C$, prove under which conditions is there a unique $a \in [\frac{1}{2}, 1]$ such that the equation $c = a \to (1-a)$ is satisfied. To solve this problem we must find the conditions that ensure that the function $f : [\frac{1}{2}, 1] \to C$, $f(a) = a \to (1-a)$ is a bijection, i.e. it is strictly monotone, $f(\frac{1}{2}) = 1$ and $f(1) = 0$. One solution is obtained by demanding continuity of the t-norm. Let us recall a result by Jenei [76] on continuous t-norms.

Lemma 3.27 ([76]) *Let $*$ be a continuous t-norm and $\to$ its associated implication operator. If $a \to b = c < 1$, then $a * c = b$.*

If $*$ is a continuous t-norm, we can prove that the function f is a bijection.

Proposition 3.28 *Let $*$ be a continuous t-norm and $\to$ its associated implication operator. The function $f : [\frac{1}{2}, 1] \to [0, 1]$ such that $f(a) = a \to (1 - a)$ is strictly decreasing and $f(\frac{1}{2}) = 1$ and $f(1) = 0$.*

Proof. For any t-norm it holds that $f(\frac{1}{2}) = \frac{1}{2} \to \frac{1}{2} = 1$ since $\frac{1}{2} * 1 = \frac{1}{2}$. On the other hand, also $f(1) = 0$ holds for any t-norm since $1 * 0 = 0$ and $1 * a = a$ for any $a > 0$. Then $f(1) = 0$. To prove monotonicity, consider $a \leq b$ in $[\frac{1}{2}, 1]$. Recall that the implication operator is decreasing in the first component and increasing in the second (Properties 8 and 9 of implication operators). We also have that $a \leq b$ implies $1 - b \leq 1 - a$, then

$$f(a) = a \to (1 - a) \geq b \to (1 - a) \geq b \to (1 - b) = f(b)\,.$$

Hence, if $a \leq b$ it follows that $f(a) \geq f(b)$, proving that the function f is decreasing.
To prove strictness of f, let us suppose by absurdum that there exist $a < b$ in $[\frac{1}{2}, 1]$ such that $f(a) = f(b)$, or, equivalently, that

$$a \to (1 - a) = b \to (1 - b) = c\,.$$

By Lemma 3.27, we have that

$$a * c = 1 - a\,, \tag{3.28}$$
$$b * c = 1 - b\,. \tag{3.29}$$

By construction $1 - a > 1 - b$ and from Eqs. (3.28) and (3.29) it follows that

$$b * c = 1 - b < 1 - a = a * c\,,$$

which implies $b * c < a * c$. But this contradicts the monotonicity of $*$, in fact if $a \leq b$ then $a * c \leq b * c$. $\square$

We proved that if the t-norm $*$ is continuous, then f is invertible. This solves our initial problem: knowing that $p(x, y)$ and $p(y, x)$ can be expressed as a function of $Q_p^*(x, y)$ and $Q_p^*(y, x)$ using Eq. (3.27), we can ensure that there exists a unique value $p(x, y) \in [0, 1]$ that satisfies

$$Q_p^*(x, y) \wedge Q_p^*(y, x) = p(x, y) \leftrightarrow p(y, x) .$$

As an open problem it remains to prove whether the condition of continuity of the t-norm can be weakened. For example, we know that left continuity is not sufficient, as shown in the following example.

Example 3.29 *In [78] it is proved that a function* $T : [0, 1]^2 \to [0, 1]$ *defined as*

$$T(a, b) = \begin{cases} v(a, b) & , \textit{if } (a, b) \in A^2 \\ a \wedge b & , \textit{else,} \end{cases}$$

where A *is a subinterval of the half-open unit interval* $[0, 1[$ *and* $v : A^2 \to A$ *is an operation such that, for any* $a, b, c \in [0, 1]$, *it holds:*

(i) $v(a, b) = v(b, a)$;

(ii) $v(a, v(b, c)) = v(v(a, b), c)$;

(iii) $v(a, b) \leq v(a, c)$, *if* $b \leq c$;

(iv) $v(a, b) \leq a \wedge b$,

then T *is a t-norm.*

Consider the following construction: let $A = [0, 0.8]$ *and* $*_A : [0,1]^2 \to [0,1]$ *such that*

$$a *_A b = \begin{cases} 0 & , \textit{ if } (a,b) \in A^2 \\ a \wedge b & , \textit{ else.} \end{cases}$$

The function $v(a,b) = 0$*, for any* $(a,b) \in A^2$ *satisfies conditions (i)-(iv) and hence* $*_A$ *is a t-norm. Furthermore, it is easy to prove that it is also left-continuous, but not continuous. The function* $f_A(a) = a \to_A (1-a)$ *is defined as*

$$f_A(a) = \begin{cases} 0.8 & , \textit{ if } a \in [0.5, 0.8] \\ 1-a & , \textit{ else,} \end{cases}$$

The function $f_A(a)$ *is clearly not invertible.*

3.3.2 Transitivity of interchangeable probabilistic and fuzzy preference relations

In this subsection we study how the transitivity property propagates from fuzzy preference relations to probabilistic relations and vice versa, when the construction presented in Eq. (3.25) is used to connect the two relations with a generic continuous t-norm. The main result is a new family of upper bound functions for cycle-transitivity of the probabilistic relation p that are connected to the $*$-transitivity of the associated fuzzy preference relation Q_p^*. Some preliminary results that we obtained for the case of the t-norm of Łukasiewicz can be found in [95].

Recall that given a probabilistic relation p on X, for any triplet (x, y, z), we define $\alpha_{xyz} \leq \beta_{xyz} \leq \gamma_{xyz}$ as follows:

$$\begin{aligned}\alpha_{xyz} &= \min(p(x,y), p(y,z), p(z,x))\,,\\ \beta_{xyz} &= \mathrm{median}(p(x,y), p(y,z), p(z,x))\,,\\ \gamma_{xyz} &= \max(p(x,y), p(y,z), p(z,x))\,.\end{aligned}$$

Whenever the dependence of α_{xyz}, β_{xyz} and γ_{xyz} from the triplet (x, y, z) is obvious, we will switch to the lighter notation α, β and γ.

Let us start with a preliminary result.

Lemma 3.30 *Let $*_1$ be a left-continuous t-norm and let $*_2$ be a t-norm. Given a probabilistic relation p on X and its associated fuzzy preference relation $Q_p^{*_1}$, if $Q_p^{*_1}$ is $*_2$-transitive, then there are only two possibilities, for any $x, y, z \in X$:*

(i) $\alpha = \beta = \gamma = \frac{1}{2}$*;*

(ii) $\alpha < \frac{1}{2} < \gamma$.

Proof. Recall that if $Q_p^{*_1}$ has been generated from p, then it is strongly complete. Let x, y and z be three alternatives in X and consider the corresponding $\alpha \leq \beta \leq \gamma$. If $\alpha \geq \frac{1}{2}$, then $\gamma \geq \beta \geq \alpha \geq \frac{1}{2}$. This means that $p(x,y) = p(y,z) = p(z,x) \geq \frac{1}{2}$, which, by the construction of $Q_p^{*_1}$, implies $Q_p^{*_1}(x,y) = Q_p^{*_1}(y,z) = Q_p^{*_1}(z,x) = 1$. Since $Q_p^{*_1}$ is $*_2$-transitive, we have

$$\begin{aligned}Q_p^{*_1}(x,z) &\geq Q_p^{*_1}(x,y) *_2 Q_p^{*_1}(y,z) = 1 *_2 1 = 1\,,\\ Q_p^{*_1}(y,x) &\geq Q_p^{*_1}(y,z) *_2 Q_p^{*_1}(z,x) = 1 *_2 1 = 1\,,\\ Q_p^{*_1}(z,y) &\geq Q_p^{*_1}(z,x) *_2 Q_p^{*_1}(x,y) = 1 *_2 1 = 1\,.\end{aligned}$$

Therefore, $1 = Q_p^{*_1}(x,y) = Q_p^{*_1}(y,x) = Q_p^{*_1}(y,z) = Q_p^{*_1}(z,y) = Q_p^{*_1}(x,z) = Q_p^{*_1}(z,x)$ and hence $p(x,y) = p(y,x) = p(y,z) = p(z,y) = p(x,z) =$

$p(x,z) = \frac{1}{2}$. This proves that if $\alpha \geq \frac{1}{2}$, then $\alpha = \beta = \gamma = \frac{1}{2}$. If we suppose that $\gamma \leq \frac{1}{2}$, then $\frac{1}{2} \geq \gamma \geq \beta \geq \alpha$. Equivalently, $\frac{1}{2} \leq (1-\gamma) \leq (1-\beta) \leq (1-\alpha)$. Using a similar reasoning as before, we can prove that $p(x,y) = p(y,z) = p(z,x) = \frac{1}{2}$. Summarizing, if one of the two conditions $\alpha \geq \frac{1}{2}$ or $\gamma \leq \frac{1}{2}$ is satisfied, then $\alpha = \beta = \gamma = \frac{1}{2}$. The only remaining possibility is that $\alpha < \frac{1}{2} < \gamma$. □

We can now prove the connection between the $*_2$-transitivity of the fuzzy preference relation $Q_p^{*_1}$ and the cycle-transitivity of p.

Proposition 3.31 *Let $*_1$ be a left-continuous t-norm and $*_2$ be a t-norm. Given a probabilistic relation p on X and its associated fuzzy preference relation $Q_p^{*_1}$, if $Q_p^{*_1}$ is $*_2$-transitive, then, for any $x, y, z \in X$, it holds that:*

$$\gamma \to_1 (1-\gamma) \geq ((1-\alpha) \to_1 \alpha) *_2 ((1-\beta) \to_1 \beta)\ . \tag{3.30}$$

Proof. Consider arbitrary $x, y, z \in X$. By Lemma 3.30 and $*_2$-transitivity of $Q_p^{*_1}$, there exist only two possibilities: either $\alpha = \beta = \gamma = \frac{1}{2}$ or $\alpha < \frac{1}{2} < \gamma$. In the former case, Eq. (3.30) is trivially satisfied, since

$$\frac{1}{2} \to_1 \left(1 - \frac{1}{2}\right) = 1 \geq \left(\left(1 - \frac{1}{2}\right) \to_1 \frac{1}{2}\right) *_2 \left(\left(1 - \frac{1}{2}\right) \to_1 \frac{1}{2}\right) = 1 *_2 1 = 1\,.$$

Next, consider the case $\alpha < \frac{1}{2} < \gamma$. Suppose, w.l.o.g., that $\gamma = p(x,y) > \frac{1}{2}$. Hence, by construction of $Q_p^{*_1}$, we have that

$$\begin{aligned} Q_p^{*_1}(x,y) &= 1\,, \\ Q_p^{*_1}(y,x) &= \gamma \to_1 (1-\gamma)\,. \end{aligned}$$

For α and β there exist only two possibilities:

Case A: $\alpha = p(z,x)$ and $\beta = p(y,z)$;

Case B: $\alpha = p(y,z)$ and $\beta = p(z,x)$.

In Case A we have that

$$\begin{aligned} Q_p^{*_1}(y,z) &= (1-\beta) \to_1 \beta \text{ and } Q_p^{*_1}(z,y) = \beta \to_1 (1-\beta)\,, && (3.31)\\ Q_p^{*_1}(z,x) &= (1-\alpha) \to_1 \alpha \text{ and } Q_p^{*_1}(x,z) = 1\,, && (3.32) \end{aligned}$$

while in Case B we have

$$\begin{aligned} Q_p^{*_1}(y,z) &= (1-\alpha) \to_1 \alpha \text{ and } Q_p^{*_1}(z,y) = 1\,, && (3.33)\\ Q_p^{*_1}(z,x) &= (1-\beta) \to_1 \beta \text{ and } Q_p^{*_1}(x,z) = \beta \to_1 (1-\beta)\,. && (3.34) \end{aligned}$$

Since $Q_p^{*_1}$ is $*_2$-transitive, the following condition is satisfied:

$$Q_p^{*_1}(y,x) \geq Q_p^{*_1}(y,z) *_2 Q_p^{*_1}(z,x)\,.$$

Now, using Eqs. (3.31) and (3.32), for Case A, we obtain:

$$\gamma \to_1 (1-\gamma) \geq ((1-\beta) \to_1 \beta) *_2 ((1-\alpha) \to_1 \alpha)\,.$$

Analogously, using Eqs. (3.33) and (3.34) for Case B we have that

$$\gamma \to_1 (1-\gamma) \geq ((1-\beta) \to_1 \beta) *_2 ((1-\alpha) \to_1 \alpha)\,.$$

This concludes the proof. □

The next step is to prove the converse implication of Proposition 3.31, but first we need a result similar to Lemma 3.30.

Lemma 3.32 *Let $*_1$ be a left-continuous t-norm and $*_2$ be a t-norm. Given a probabilistic relation p on X and its associated fuzzy preference relation $Q_p^{*_1}$, if the condition*

$$\gamma \to_1 (1-\gamma) \geq ((1-\alpha) \to_1 \alpha) *_2 ((1-\beta) \to_1 \beta) \qquad (3.35)$$

is satisfied for any x, y, z in X, then there are only two possibilities, for α, β and γ:

(i) $\alpha = \beta = \gamma = \frac{1}{2}$;

(ii) $\alpha < \frac{1}{2} < \gamma$.

Proof. Let x, y and z be three alternatives in X and consider the corresponding $\alpha_{xyz} \leq \beta_{xyz} \leq \gamma_{xyz}$.
If $\alpha_{xyz} \geq \frac{1}{2}$, then $\gamma_{xyz} \geq \beta_{xyz} \geq \alpha_{xyz} \geq \frac{1}{2}$ and, consequently, $\frac{1}{2} \geq 1 - \alpha_{xyz} \geq 1 - \beta_{xyz} \geq 1 - \gamma_{xyz}$. Recall that if $*_1$ is left continuous, then, for any $a, b \in [0,1]$, $a \leq b$ is equivalent to $a \rightarrow_1 b = 1$. From Eq. (3.35) it follows

$$\gamma_{xyz} \rightarrow_1 (1-\gamma_{xyz}) \geq ((1-\alpha_{xyz}) \rightarrow_1 \alpha_{xyz}) *_2 ((1-\beta_{xyz}) \rightarrow_1 \beta_{xyz}) = 1 *_2 1 = 1\,.$$

Hence $\gamma_{xyz} \rightarrow_1 (1-\gamma_{xyz}) = 1$ or, equivalently, $\gamma_{xyz} \leq (1-\gamma_{xyz})$, i.e. $\gamma_{xyz} \leq \frac{1}{2}$. Combining this with $\gamma_{xyz} \geq \beta_{xyz} \geq \alpha_{xyz} \geq \frac{1}{2}$, we have that $\alpha_{xyz} = \beta_{xyz} = \gamma_{xyz} = \frac{1}{2}$.
If $\gamma_{xyz} \leq \frac{1}{2}$, then $\alpha_{xyz} \leq \beta_{xyz} \leq \gamma_{xyz} \leq \frac{1}{2}$ and, consequently, $\frac{1}{2} \leq 1 - \gamma_{xyz} \leq 1 - \beta_{xyz} \leq 1 - \alpha_{xyz}$. Since Eq. (3.35) holds for any x, y, z in X, then, for $\alpha_{yxz}, \beta_{yxz}, \gamma_{yxz}$, we have that

$$\gamma_{yxz} \rightarrow_1 (1-\gamma_{yxz}) \geq ((1-\alpha_{yxz}) \rightarrow_1 \alpha_{yxz}) *_2 ((1-\beta_{yxz}) \rightarrow_1 \beta_{yxz})\,. \quad (3.36)$$

Recall that $\alpha_{xyz} = 1 - \gamma_{yxz}$, $\beta_{xyz} = 1 - \beta_{yxz}$ and $\gamma_{xyz} = 1 - \gamma_{yxz}$. Combining this with $\alpha_{xyz} \leq \beta_{xyz} \leq \gamma_{xyz} \leq \frac{1}{2}$, we obtain

$$1 - \alpha_{yxz} \leq 1 - \beta_{yxz} \leq 1 - \gamma_{yxz} \leq \frac{1}{2}\,,$$

which, applied to Eq. (3.36) gives

$$\gamma_{yxz} \rightarrow_1 (1-\gamma_{yxz}) \geq ((1-\alpha_{yxz}) \rightarrow_1 \alpha_{yxz}) *_2 ((1-\beta_{yxz}) \rightarrow_1 \beta_{yxz}) = 1 *_2 1\,.$$

This implies that $\gamma_{yxz} \rightarrow_1 (1 - \gamma_{yxz}) = 1$ and hence $\gamma_{yxz} \leq 1 - \gamma_{yxz}$ or, equivalently, $\gamma_{yxz} \leq \frac{1}{2}$. Since $\gamma_{yxz} = 1-\alpha_{xyz}$, we can conclude that $1-\alpha_{xyz} \leq \frac{1}{2}$ or, equivalently, $\alpha_{xyz} \geq \frac{1}{2}$, that, combined with $\alpha_{xyz} \leq \beta_{xyz} \leq \gamma_{xyz} \leq \frac{1}{2}$ gives us $\alpha_{xyz} = \beta_{xyz} = \gamma_{xyz} = \frac{1}{2}$.
The only remaining possibility is that $\alpha_{xyz} < \frac{1}{2} < \gamma_{xyz}$. $\square$

We can now prove the converse implication of Proposition 3.31.

Proposition 3.33 *Let $*_1$ be a continuous t-norm and $*_2$ be a t-norm. Given a probabilistic relation p on X and its associated fuzzy preference relation $Q_p^{*_1}$, if the condition*

$$\gamma \rightarrow_1 (1 - \gamma) \geq ((1 - \alpha) \rightarrow_1 \alpha) *_2 ((1 - \beta) \rightarrow_1 \beta) \tag{3.37}$$

is satisfied for any x, y, z in X, then $Q_p^{*_1}$ is $*_2$-transitive.

Proof. Consider three alternatives x, y and z in X. We want to prove that

$$Q_p^{*_1}(x, y) *_2 Q_p^{*_1}(y, z) \leq Q_p^{*_1}(x, z) \tag{3.38}$$

By Lemma 3.32, we know that there are only two possibilities for α, β and γ:

(i) either $\alpha = \beta = \gamma = \frac{1}{2}$,

(ii) or $\alpha < \frac{1}{2} < \gamma$.

In the first case, the associated fuzzy preference relation $Q_p^{*_1}$ is everywhere equal to one for the triplet x, y and z and hence Eq. (3.38) is trivially satisfied. Next, suppose that x, y and z are such that $\alpha < \frac{1}{2} < \gamma$. For $p(z, x)$ there are three possibilities:

Case A $p(z, x) = \gamma$;

Case B $p(z,x) = \beta$;

Case C $p(z,x) = \alpha$.

We start with Case A: if $p(z,x) = \gamma > \frac{1}{2}$, then by Eq. (3.37) we have,

$$
\begin{aligned}
& \gamma \rightarrow (1-\gamma) \geq ((1-\alpha) \rightarrow_1 \alpha) *_2 ((1-\beta) \rightarrow_1 \beta) \\
\Leftrightarrow \quad & p(z,x) \rightarrow_1 p(x,z) \geq (p(y,x) \rightarrow_1 p(x,y)) *_2 (p(z,y) \rightarrow_1 p(y,z)) \\
\Leftrightarrow \quad & Q_p^{*_1}(x,z) \geq Q_p^{*_1}(x,y) *_2 Q_p^{*_1}(y,z)\,,
\end{aligned}
$$

i.e., exactly Eq. (3.38).

Consider now Case C: if $p(z,x) = \alpha < \frac{1}{2}$, then $p(x,z) = 1-\alpha > \frac{1}{2}$. It follows that $Q_p^{*_1}(x,z) = p(z,x) \rightarrow_1 p(x,z) = 1$ and hence Eq. (3.38) is trivially satisfied.

Finally, let us study Case B. If $p(z,x) = \beta$ and $\beta \leq \frac{1}{2}$, then $p(x,z) = 1-\beta \geq \frac{1}{2}$. Hence $Q_p^{*_1}(x,z) = p(z,x) \rightarrow_1 p(x,z) = 1$ and Eq. (3.38) is trivially satisfied. If $p(z,x) = \beta$ and $\beta > \frac{1}{2}$, then $Q_p^{*_1}(x,z) = \beta \rightarrow_1 (1-\beta) < 1$ and $(1-\beta) \rightarrow_1 \beta = 1$. This last condition implies that Eq. (3.37) becomes

$$
\gamma \rightarrow_1 (1-\gamma) \geq (1-\alpha) \rightarrow_1 \alpha\,.
$$

Since $f(a) = a \rightarrow_1 (1-a)$ is a decreasing function, then $\gamma \leq (1-\alpha)$, which, combined with $\gamma \geq \beta$ gives us $1-\alpha \geq \beta$. Hence, using again the monotonicity of f, we have that

$$
f(\beta) = \beta \rightarrow_1 (1-\beta) \geq (1-\alpha) \rightarrow_1 \alpha = f(1-\alpha)\,. \tag{3.39}
$$

If $\alpha = p(y,z)$, then Eq. (3.39) becomes

$$\begin{aligned} Q_p^{*_1}(x,z) &= p(z,x) \rightarrow_1 p(x,z) \\ &= \beta \rightarrow_1 (1-\beta) \\ &\geq (1-\alpha) \rightarrow_1 \alpha \\ &= p(z,y) \rightarrow_q p(y,z) = Q_p^{*_1}(y,z) \\ &\geq Q_p^{*_1}(x,y) *_2 Q_p^{*_1}(y,z)\,, \end{aligned}$$

i.e. , exactly Eq. (3.38).
If $\alpha = p(x,y)$, then Eq. (3.39) becomes

$$\begin{aligned} Q_p^{*_1}(x,z) &= p(z,x) \rightarrow_1 p(x,z) \\ &= \beta \rightarrow_1 (1-\beta) \\ &\geq (1-\alpha) \rightarrow_1 \alpha \\ &= p(y,x) \rightarrow_q p(x,y) = Q_p^{*_1}(x,y) \\ &\geq Q_p^{*_1}(x,y) *_2 Q_p^{*_1}(y,z)\,, \end{aligned}$$

i.e. , exactly Eq. (3.38). □

We can finally state the main result of this section.

Theorem 3.34 *Let $*_1$ be a continuous t-norm and $*_2$ be a t-norm. Given a probabilistic relation p on X and its associated fuzzy preference relation $Q_p^{*_1}$, $Q_p^{*_1}$ is $*_2$-transitive if and only if the condition*

$$\gamma \rightarrow_1 (1-\gamma) \geq ((1-\alpha) \rightarrow_1 \alpha) *_2 ((1-\beta) \rightarrow_1 \beta) \tag{3.40}$$

is satisfied for any x, y, z in X by p.

In the particular case in which $*_1$ and $*_2$ are the same continuous t-norm, we can easily prove the following corollary.

Corollary 3.35 *Let $*$ be a continuous t-norm. Given a probabilistic relation p on X and its associated fuzzy preference relation Q_p^*, then Q_p^* is $*$-transitive if and only if for p it holds that, for any $x, y, z \in X$:*

$$\gamma \to (1-\gamma) \geq ((1-\alpha) \to \alpha) * ((1-\beta) \to \beta) \ .$$

The importance of the result contained in Theorem 3.34 is double: on the one hand it proposes a new family of upper bound functions for cycle-transitivity of probabilistic relations. In fact, according to Property 1 of implication operators, Eq. (3.37) can also be written as:

$$\alpha + \beta + \gamma - 1 \leq \alpha + \beta - \gamma *_1 ((1-\alpha) \to_1 \alpha) *_2 ((1-\beta) \to_1 \beta) \ .$$

The associated upper bound function is then:

$$U_{*_2}^{*_1}(\alpha, \beta, \gamma) = \alpha + \beta - \gamma *_1 ((1-\alpha) \to_1 \alpha) *_2 ((1-\beta) \to_1 \beta) \ ,$$

which depends on the chosen t-norms $*_1$ and $*_2$. On the other hand, already known types of cycle-transitivity can be rewritten using the expression contained in Eq. (3.37) for the appropriate choice of $*_1$ and $*_2$. Those situations in which an upper bound function can be expressed using Eq. (3.37) can be contextualized in our framework, where the probabilistic relation p and the fuzzy preference relation $Q_p^{*_1}$ are connected by the construction $Q_p^{*_1}(x, y) = p(y, x) \to_1 p(x, y)$ and $Q_p^{*_1}$ is $*_2$-transitive.

We propose now the explicit expression of the upper bound functions $U_{*_2}^{*_1}$ for all combinations of the t-norms $*_1$ and $*_2$ in $\{*_{\mathbf{M}}, *_{\mathbf{P}}, *_{\mathbf{L}}\}$. Before that, let us prove an auxiliary result.

Lemma 3.36 *Let $*$ be a continuous t-norm and $\to$ its associated implication operator. Given $a \in [0, \frac{1}{2}]$ and $b \in [\frac{1}{2}, 1]$, it holds that $1 - b \geq a$ is equivalent to $b \to (1-b) \geq (1-a) \to a$.*

Proof. The condition $1 - b \geq a$ is equivalent to $1 - a \geq b$. Since the function $f(c) = c \to (1 - c)$ is srtictly decreasing, the proof follows. $\square$

From Lemma 3.36 it easily follows that in Eq. (3.37), if $\beta \geq \frac{1}{2}$, then

$$\begin{aligned}
& \gamma \to_1 (1-\gamma) \geq ((1-\alpha) \to_1 \alpha) *_2 ((1-\beta) \to_1 \beta) \\
\Leftrightarrow\ & \gamma \to_1 (1-\gamma) \geq ((1-\alpha) \to_1 \alpha) *_2 1 \\
\Leftrightarrow\ & \gamma \to_1 (1-\gamma) \geq ((1-\alpha) \to_1 \alpha) \\
\Leftrightarrow\ & (1-\gamma) \geq \alpha \\
\Leftrightarrow\ & \alpha + \beta + \gamma - 1 \leq \beta \,.
\end{aligned}$$

With this result we can present the upper bound functions $U^{*_1}_{*_2}$ for all possible combinations of $*_1$ and $*_2$ in $\{*_\mathbf{M}, *_\mathbf{P}, *_\mathbf{L}\}$ in a more concise form:

$$U^{*_1}_{*_2}(\alpha,\beta,\gamma) = \begin{cases} \beta & \text{, if } \beta \geq \frac{1}{2}\,, \\ g^{*_1}_{*_2}(\alpha,\beta,\gamma) & \text{, else,} \end{cases}$$

where $g^{*_1}_{*_2}$ is given by the corresponding expression in the following table:

$g^{*_1}_{*_2}$		$*_2$		
		$*_\mathbf{M}$	$*_\mathbf{P}$	$*_\mathbf{L}$
$*_1$	$*_\mathbf{M}$	β	$\alpha+\beta-\alpha\beta$	$\alpha+\beta$
	$*_\mathbf{P}$	β	$\alpha+\beta-\frac{\alpha\beta\gamma}{(1-\alpha)(1-\beta)}$	$\alpha+\beta-\gamma\left[\left(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}-1\right)\vee 0\right]$
	$*_\mathbf{L}$	β	$\alpha+\beta-2\alpha\beta$	$\frac{1}{2}$

The expressions of $g^{*_1}_{*_2}$ presented in the table have been computed using Eq. (3.40) and the corresponding t-norms $*_1$ and $*_2$. We outline here the steps that lead to those expressions. Recall that $U^{*_1}_{*_2} = g^{*_1}_{*_2}$ when $\beta > \frac{1}{2}$.

If $*_2 = *_\mathbf{M}$, then the expression of $g^{*_1}_{*_\mathbf{M}}$ is always equal to β, in fact Eq. (3.40) becomes

$$\gamma \to_1 (1-\gamma) \geq ((1-\alpha) \to_1 \alpha) \wedge ((1-\beta) \to_1 \beta)\ . \tag{3.41}$$

Since $f(a) = a \to (1-a)$ is a decreasing function and $1-\alpha \geq 1-\beta$, $f(1-\beta) \geq f(1-\alpha)$. Hence Eq. (3.41) becomes

$$\gamma \to_1 (1-\gamma) \geq (1-\alpha) \to_1 \alpha\,,$$

that, again for the monotonicity of f, implies that

$$\begin{aligned} & \gamma \leq 1-\alpha \\ \Leftrightarrow\quad & \alpha+\gamma-1 \leq 0 \\ \Leftrightarrow\quad & \alpha+\beta+\gamma-1 \leq \beta\,, \end{aligned}$$

which leads to the upper-bound function $U^{*_1}_{*_\mathbf{M}}$.

If $*_2 = *_\mathbf{P}$, then Eq. (3.40) takes the following form:

$$\gamma \to_1 (1-\gamma) \geq ((1-\alpha) \to_1 \alpha)\,((1-\beta) \to_1 \beta)\ . \tag{3.42}$$

(i) if $*_1 = *_\mathbf{M}$, Eq. (3.42) becomes $1-\gamma \geq \alpha\beta$ and hence

$$U^{*_\mathbf{P}}_{*_\mathbf{M}}(\alpha,\beta,\gamma) = \alpha+\beta-\alpha\beta\,.$$

(ii) if $*_1 = *_\mathbf{P}$, Eq. (3.42) becomes $1-\gamma \geq \gamma\left(\frac{\alpha}{1-\alpha}\frac{\beta}{1-\beta}\right)$ and hence

$$U^{*_\mathbf{P}}_{*_\mathbf{P}}(\alpha,\beta,\gamma) = \alpha+\beta-\frac{\alpha\beta\gamma}{(1-\alpha)(1-\beta)}\,.$$

(iii) if $*_1 = *_\mathbf{L}$, Eq. (3.42) becomes $2(1-\gamma) \geq 4\alpha\beta$ and hence

$$U^{*_\mathbf{P}}_{*_\mathbf{L}}(\alpha,\beta,\gamma) = \alpha+\beta-2\alpha\beta\,.$$

If $*_2 = *_\mathbf{L}$, then Eq. (3.40) takes the following form:

$$\gamma \to_1 (1-\gamma) \geq [((1-\alpha) \to_1 \alpha) + ((1-\beta) \to_1 \beta) - 1] \vee 0\,. \tag{3.43}$$

(i) if $*_1 = *_{\mathbf{M}}$, Eq. (3.43) becomes $1-\gamma \geq (\alpha+\beta-1) \vee 0$, i.e. $1-\gamma \geq 0$, but this condition is always satisfied, hence $U^{*_{\mathbf{L}}}_{*_{\mathbf{M}}}(\alpha,\beta,\gamma)$ is trivial ($U^{*_{\mathbf{L}}}_{*_{\mathbf{M}}}(\alpha,\beta,\gamma) = \alpha+\beta$).

(ii) if $*_1 = *_{\mathbf{P}}$, Eq. (3.43) becomes $1-\gamma \geq \gamma\left[\left(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}-1\right)\vee 0\right]$ and hence

$$U^{*_{\mathbf{L}}}_{*_{\mathbf{P}}}(\alpha,\beta,\gamma) = \alpha+\beta-\gamma\left[\left(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}-1\right)\vee 0\right].$$

(iii) if $*_1 = *_{\mathbf{L}}$, Eq. (3.43) becomes $2(1-\gamma) \geq (2\alpha+2\beta-1)\vee 0$ and hence

$$U^{*_{\mathbf{L}}}_{*_{\mathbf{L}}}(\alpha,\beta,\gamma) = \frac{1}{2}.$$

3.4 An experiment for measuring rationality of consumers

In this section we propose an experiment with real data to test the theoretical results proved in the previous sections of this chapter. In particular, we will use the construction of fuzzy choice functions from a given probabilistic choice function described in Definition 3.13. Also rationality conditions of probabilistic choice functions will be used to reconstruct the missing information on sets of alternatives where choices have not been observed directly. This section is based on [87]. The dataset used in the experiment contains the recorded purchases of several clients in a super market where the products are divided into different classes. The aim of the experiment is to measure the rationality of the consumers by observing their purchases along a period of time.

3.4.1 Measuring transitivity and rationality

Past works on fuzzy decision theory [24, 64, 65] proposed a new way of dealing with the problem of rationality, namely assigning to a decision maker a degree of rationality based on the degree to which he satisfies the transitivity property. The possibility of assigning a degree of rationality is really attractive since it allows to distinguish those decision-makers that are seriously irrational from others that are mainly coherent in their behaviour. A formal and detailed presentation on fuzzy rationality measures can be found in [24], while we rely on the construction of transitivity degrees proposed in Chapter 9 of [64].

Definition 3.37 *The degree of $*$-transitivity of a fuzzy preference relation Q w.r.t. a t-norm $*$ is given by:*

$$T^*(Q) = \bigwedge_{x,y,z \in X} ((Q(x,y) * Q(y,z)) \to Q(x,z)) \ . \tag{3.44}$$

Remark 3.38 *The degree of $*$-transitivity obeys the same natural order of t-norms: given two t-norms such that $x *_1 y \geq x *_2 y$, for any $x, y \in [0,1]$, then it is immediate to prove that $T^{*_1}(Q) \leq T^{*_2}(Q)$, for the same reason for which $*_1$-transitivity implies $*_2$-transitivity.*

3.4.2 The data set

The data set used in our research is courtesy of Ta-Feng (`aiia.iis.sinica.edu.tw`), a Chinese retailer warehouse that sells a wide range of merchandise, from food and grocery to office supplies and furniture. The data set contains shopping records collected in a time span of four months, from November 2000 to February 2001. Each record consists of four attributes:

Customer ID, Product ID, Class ID and amount of purchase (other attributes like Date, Age, Price, etc. have been removed to reduce the data set). There are a total of 817741 records, representing the purchases of 32266 different clients, who bought in a basket of 23812 possible different products, divided into 2012 classes. All classes are pairwise disjoint and contain from 1 to 275 different products each.

3.4.3 The experiment

Part I The first experiment focuses on the behaviour inside classes, independently of the customers. We look at the data base as a unique choice situation, in which the 23812 different products belonging to 2012 separated classes are chosen. Then we study the rationality of the corresponding choice function, where X is the set of products and $\mathcal{B}$ is the family of classes of products. We followed these steps:

Step 1 Construct a probabilistic choice function for each one of the 2012 different classes. The probability of a product x of being chosen from class A ($p(x, A)$) is estimated using the total amount of purchased product x, divided by the total amount of purchased products in the class A. The results are 2012 different probabilistic choice functions.

Step 2 For each class $A \in \mathcal{B}$, three reciprocal relations ($p_A^{*\mathbf{M}}$, $p_A^{*\mathbf{P}}$, $p_A^{*\mathbf{L}}$) are computed from the corresponding probabilistic choice function p, using the strong scalability condition (SSC) of Definition 3.14:

(i) $p_A^{*\mathbf{M}}$, with minimum t-norm,

(ii) $p_A^{*\mathbf{P}}$, with product t-norm,

(iii) $p_A^{*\mathbf{L}}$, with Łukasiewicz t-norm.

Step 3 Transform all reciprocal relations into fuzzy revealed preference relations using Eq. (3.19) and the corresponding t-norm. The results are, for any $A \in \mathcal{B}$, three fuzzy preference relations $R_A^{*\mathbf{M}}$, $R_A^{*\mathbf{P}}$ and $R_A^{*\mathbf{L}}$.

Step 4 Compute the degree of $*$-transitivity of fuzzy preference relations $R_A^{*\mathbf{M}}$, $R_A^{*\mathbf{P}}$ and $R_A^{*\mathbf{L}}$, for any class of products $A \in \mathcal{B}$. For more insight, we compute three degrees of $*$-transitivity for any fuzzy preference relation, using T^{*_M}, T^{*_P} or T^{*_L}.

The output consists of nine vectors of the same size as $\mathcal{B}$, containing the rationality degree of each class, according to the t-norm used in its construction and the t-norm used for measuring the degree of $*$-transitivity.

Part II The second experiment is oriented to the customers. We retrace the steps of the previous experiment, but this time we apply that procedure to the different customers of the store, instead of on the classes of products. The rationality of each client is measured only in those classes of products in which he purchased at least once, using again three constructions (using the three t-norms) and each of them being measured with three rationality measures ($T^{*\mathbf{M}}$, $T^{*\mathbf{P}}$ or $T^{*\mathbf{L}}$). For any customer, we have different measurements, one for each class he visited. The overall rationality of each customer is aggregated from its rationality degree on the classes using two operators: for simplicity we used only minimum and arithmetic mean.

Results and comments The transitivity degrees $T^{*\mathbf{M}}$, $T^{*\mathbf{P}}$ and $T^{*\mathbf{L}}$ of the relations $R_A^{*\mathbf{M}}$, $R_A^{*\mathbf{P}}$ and $R_A^{*\mathbf{L}}$ are denoted with the notation indicated in the next table.

		Measure of $*$-transitivity		
		Minimum	Product	Łukasiewicz
Construction of p	Minimum	MM	MP	ML
	Product	PM	PP	PL
	Łukasiewicz	LM	LP	LL

We already know from Proposition 3.23 that if R_{C_p} is the fuzzy preference relation revealed from the fuzzy choice function C_p and C_p has been constructed from the probabilistic choice function p using the t-norm $*_1$, then R_{C_p} is $*_1$-transitive. Furthermore, if the t-norms $*_1$ and $*_2$ are such that $a *_1 b \geq a *_2 b$, for any $a, b \in [0, 1]$, then a fuzzy preference relation Q that is $*_1$-transitive it is also $*_2$-transitive. For these reasons, we have that the transitivity degrees MM, MP, ML, PP, PL and LL are always equal to 1.

Hence, in both experiments, we can focus just on:

PM the probabilistic choice function constructed with the product t-norm and rationality measured with $*_{\mathbf{M}}$-transitivity;

LM the probabilistic choice function constructed with the Łukasiewicz t-norm and rationality measured with $*_{\mathbf{M}}$-transitivity;

LP the probabilistic choice function constructed with the Łukasiewicz t-norm and rationality measured with $*_{\mathbf{P}}$-transitivity.

Part I In the first part of the experiment, we obtained the following results: we started with 2012 classes, 601 of which contained less than 3 elements and have been immediately excluded from the study, since they are trivially rational. Of the remaining 1411 classes of products, we observed that in:

53 of them the behaviour of the customers is completely rational, under any construction and w.r.t. all transitivity measures (3.7%).

1358 of them an irrational behaviour is detected (96.3%). The rationality measures associated to these classes are depicted in Figure 3.1.

It can be concluded that the choice of a specific t-norm in the construction of the probabilistic choice function strongly affects the resulting model. If the construction is made with the minimum t-norm, we are automatically imposing the strongest type of transitivity, when is known that the human behaviour is seldom rational [58]. In contrast, the constructions with the product and Łukasiewicz t-norms allow to detect non-rational behaviour. The classes of products with rationality measure less than 1 are more than 95% of the total number of classes. It may be important for the analyst to recognize those classes where rationality is not satisfied.

Part II In the second part of the experiment, we obtained the following results: we started with 32266 customers, 17832 of which has bought less than 3 different products and have been immediately excluded from the study, since they are trivially rational. Of the remaining 14434 customers, we observed that:

11040 of them have a perfectly rational behaviour, under any construction and w.r.t. all transitivity measures (76.5%).

3394 of them show an irrational behaviour (23.5%). The rationality measures associated to these customers are depicted in Figures 3.2 and 3.3.

From this part of the experiment it can be concluded that less than 25% of the observed customers show an irrational behaviour. Again, the choice

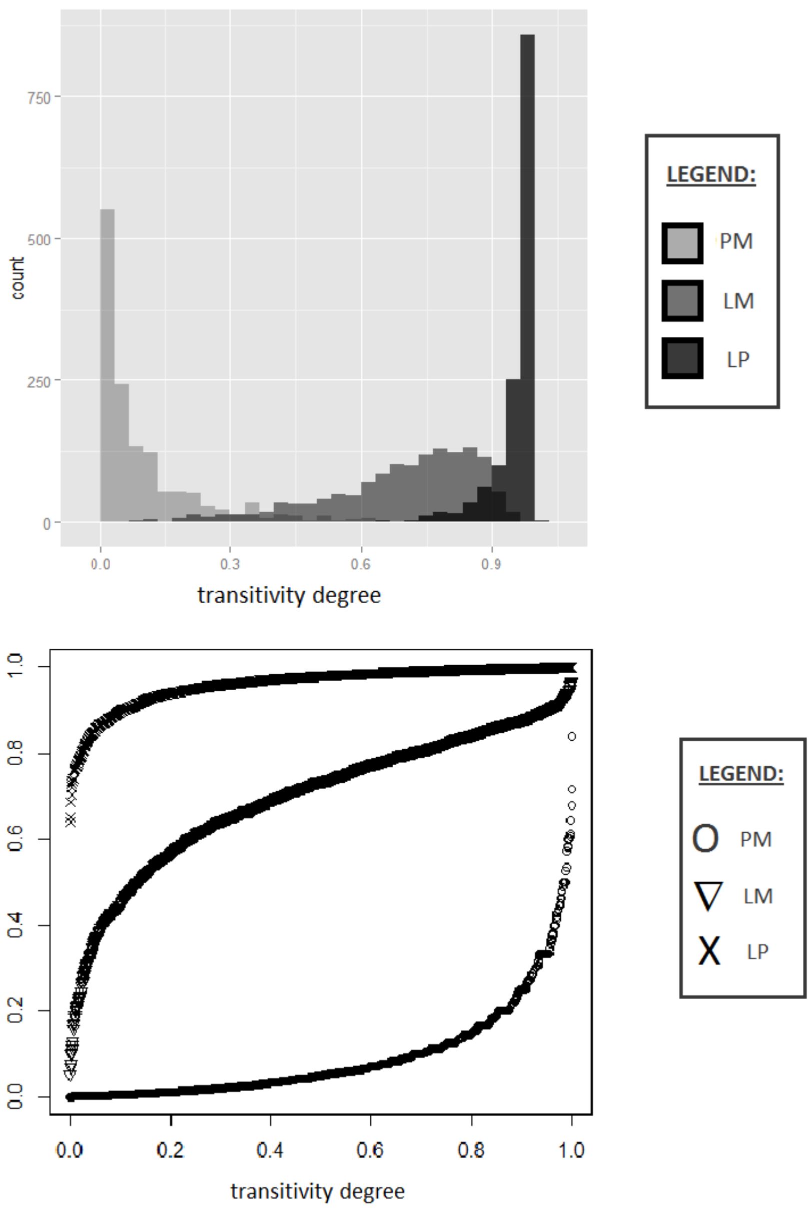


Figure 3.1: Transitivity degrees for Experiment 1

of the t-norm in the construction of the probabilistic choice function affects the resulting model. Using the minimum t-norm forces the description of the behaviour of the customers to be strongly rational, while other t-norms allow for a more realistic description. In Figure 3.2 we can appreciate that smaller t-norms are associated to higher values of the degree of transitivity. For example, if we consider the construction with Łukasiewicz t-norm and we compare the values of the degree of transitivity w.r.t. the minimum and the product t-norm (LM and LP), we observe that the values of LP are always greater than the values of LM. Also the t-norm used in the costruction of p affects the degree of transitivity. If fact, if we consider the degree of $*_{\mathbf{M}}$-transitivity for the construction with the product or the Łukasiewicz t-norm (PM and LM), we can appreciate the the value of PM are always smaller than the values of LM. It seems that the weaker the t-norm used in the construction, the more rational choice becomes.

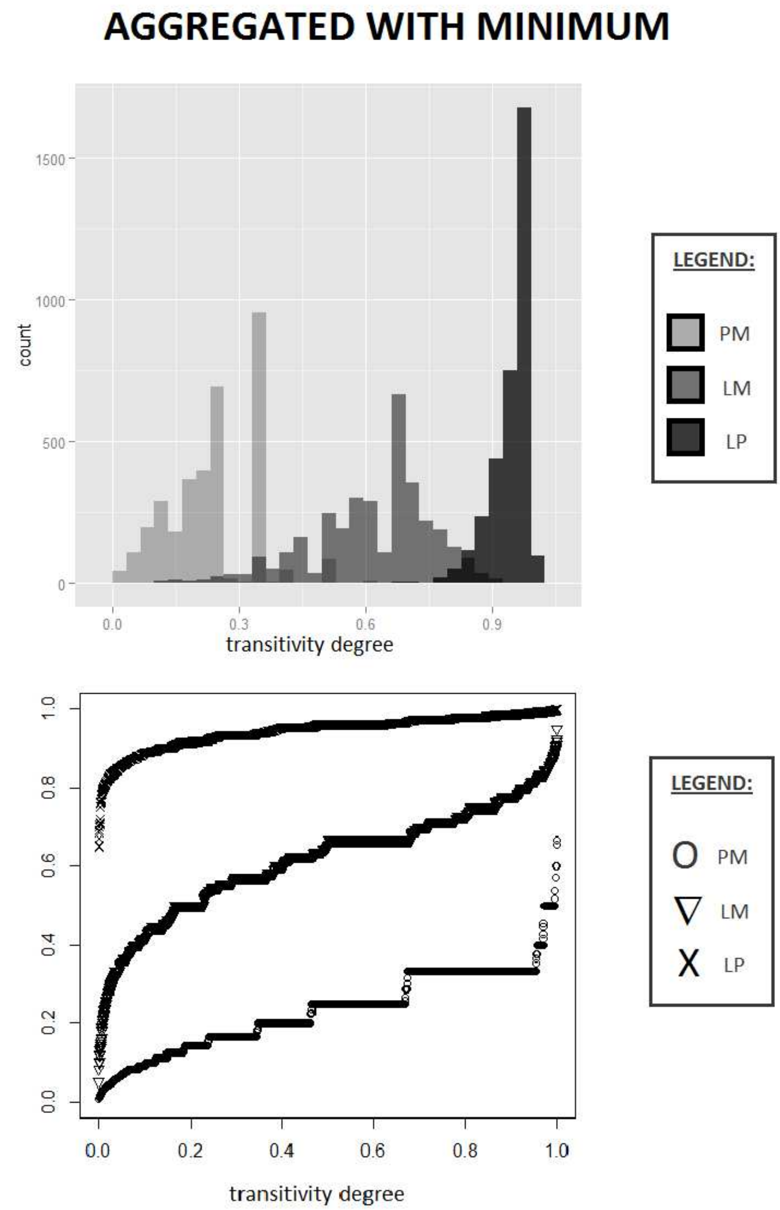


Figure 3.2: Transitivity degrees for Experiment 2. Aggregation with the minimum.

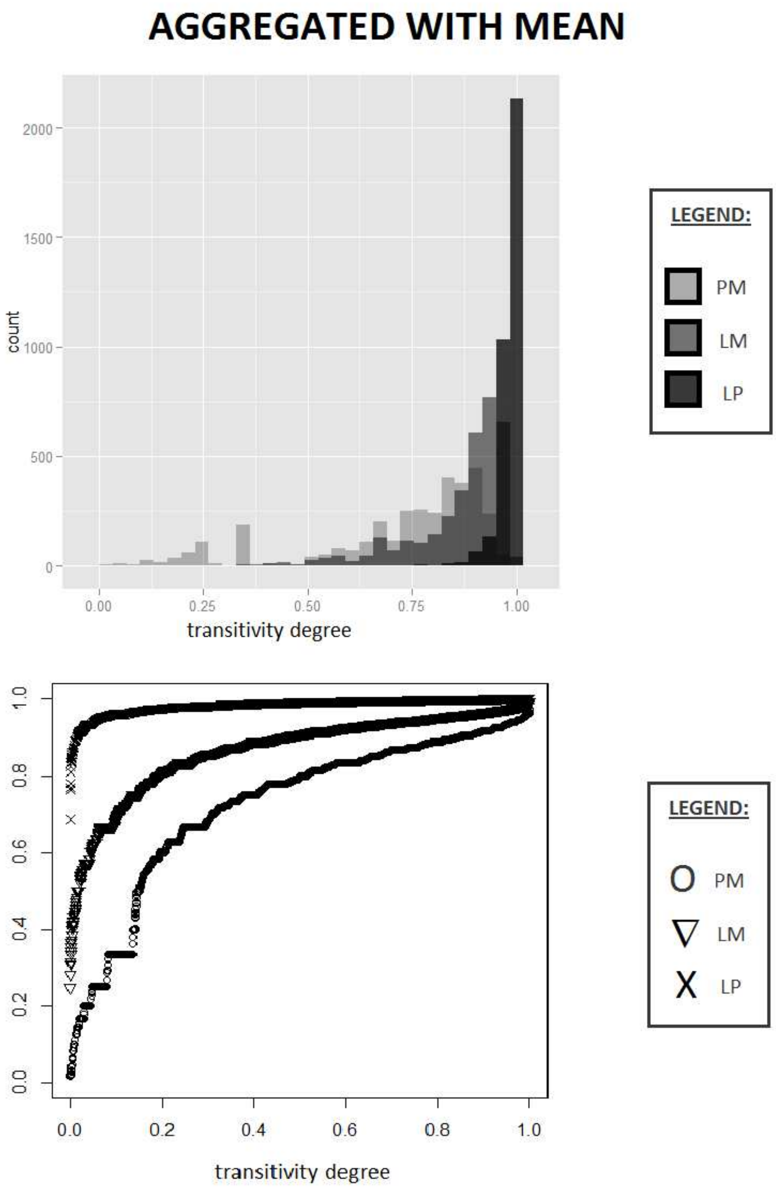


Figure 3.3: Transitivity degrees for Experiment 2. Aggregation with the mean.

Chapter 4

Conclusions

Different approaches to choice theory have been studied. We tried to mimic different well-known results of classical choice theory in a framework where choice and preference are modelled using fuzzy set theory or probability theory.

In Section 2.2, sufficient conditions for generating fuzzy choice functions through G_Q and M_Q have been unveiled. Existing results have been improved on several aspects. First of all, $*$-transitivity conditions have been replaced by weaker conditions, such as acyclicity and $*$-acyclicity. In the study of both crisp and fuzzy preference relations, acyclicity is one of the weakest possible rationality conditions, in the sense that asking for a weaker condition would mean losing any kind of coherence. This leads us to consider that this hypothesis cannot be weakened further in the context of sufficient conditions. On the other hand, since we are operating in the fuzzy setting, t-norms are naturally involved. We have checked whether different t-norms lead to different results. We have proved that left-continuity or the absence of zero divisors play a crucial role. We have also discussed the connection between the results known in the crisp case and the new conditions we have provided. We have checked that in the fuzzy setting we cannot expect results that are fully analogous to those in the crisp case. This last fact becomes clear when looking for a fuzzy version of the classical Richter Theorem: the characterization of rationality of crisp choice functions seems to be no longer valid in the fuzzy setting.

In Section 2.3 we studied the relationship between fuzzy rationality conditions, trying to find the most accurate generalization of the Arrow-Sen Theorem possible. To summarize the results obtained in this work, we present the scheme of Figure 4.1 in which the rationality conditions are presented in separated boxes. An arrow going from one box to another indicates that the former condition implies the latter. Different line styles of the arrows are used to indicate the t-norms for which the implication holds:

(i) solid lines indicate that the t-norm is left continuous;

(ii) dashed lines with pattern - - - indicate that the t-norm has no zero divisors;

(iii) dashed lines with pattern -·- indicate that the t-norm is rotation invariant;

(iv) dotted lines indicate that the t-norm is the minimum.

Additional conditions like H1, WH1 and WH2 are depicted along the arrows when needed.

In Section 3.2 we proposed a new construction that links probabilistic and fuzzy choice functions making use of t-norms and implication operators. We also observed that well-known rationality conditions in probabilistic choice theory such as Luce's Axiom of Choice or WASRP obey a common

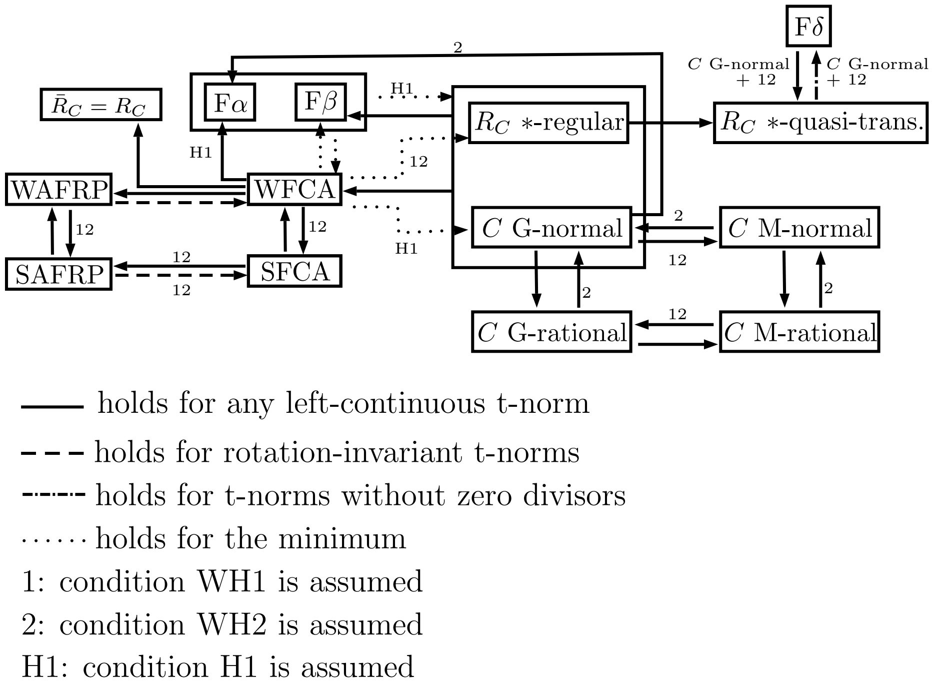


Figure 4.1: Summary of the results obtained.

pattern and we have generalized them to what we have called scalability conditions. Those new conditions, together with a probabilistic version of the Weak Congruence Axiom, helped us proving that the proposed construction of fuzzy choice functions from probabilistic choice functions is consistent: in fact, it holds that the generated fuzzy choice function is G-normal and that the associated fuzzy revealed preference relation is ∗-transitive.

In Section 3.3 we studied the connections between probabilistic and fuzzy preference relations. Any probabilistic relation can be associated with a strongly complete fuzzy relation using the construction described in Eq. (3.25). This construction depends on the implication operator derived from a t-norm and, under suitable conditions (continuity of the t-norm), it can be reversed. In those cases, the fuzzy preference relation uniquely characterizes the probabilistic relation from which it has been generated. This strong connection between probabilistic and fuzzy choice functions can be of great interest in the study of relations that use different scales: in fact, probabilistic relations use a bipolar scale, while fuzzy preference relations use a unipolar scale. In those cases in which the probabilistic and the fuzzy preference relations are uniquely characterized, we studied how transitivity propagates from one relation to the other. Even if transitivity is defined differently for the two types of relations, we have been able to prove that $*$-transitivity of the fuzzy preference relation is equivalent to a new type of cycle-transitivity of the probabilistic relation.

Finally, in Section 3.4, we applied the techniques presented in Chapter 3 to measure the rationality (in terms of degrees of $*$-transitivity) of a group of consumers of a super-market. The study shows that the hypothesis of rationality of the decision makers that is assumed in many economical and statistical choice models is sometimes not satisfied. Furthermore, we observed that the aggregated behaviour of the consumers is even less rational than the behaviour of the single individuals.

Chapter 5

Conclusiones

En esta tesis se han estudiado distintos enfoques de la teoría de la elección individual. En particular se han intentado emular varios resultados de la teoría clásica en el marco de unas teorías más generales, donde elección y preferencia se modelan a través de conceptos probabilísticos o borrosos (difusos). El Capítulo 2 está dedicado a la teoría de elección borrosa. En particular, en la Sección 2.2 se han estudiado aquellas condiciones que una relación de preferencia borrosa Q tiene que satisfacer para poder asegurar que de ella se pueda racionalizar una función de elección borrosa a través de las construcciones G_Q y M_Q presentadas en la Definición 2.3. En comparación con los resultados encontrados en la literatura correspondiente, hemos logrado rebajar esas condiciones. En particular, la condición de $*$-transitividad ha sido substituida por condiciones más débiles como la de aciclicidad o la de $*$-aciclicidad. Tanto en los estudios clásicos como en el caso difuso, la propiedad de aciclicidad es de las más débiles posibles, ya que pedir una condición más débil conllevaría una pérdida total de coherencia. Por otro lado, como acostumbra a ocurrir en los estudios en entornos difusos, es importante establecer para qué familias de t-normas se pueden probar los resultados. En este sentido hemos comprobado que propiedades como la continuidad o la existencia de divisores de cero juegan un papel fundamental a la hora de probar la posibilidad de generar una función de elección borrosa a partir de una relación de preferencia borrosa. Además, hemos comparado los resultados obtenidos en el caso difuso con sus homólogos en el caso clásico: para el caso de la función G_Q solo una implicación del caso clásico se ha podido recuperar, pero eso ha permitido demostrar que un resultado análogo al teorema de Richter no es válido en general en el caso borroso (Ejemplo 2.24). Para la función M_Q, el resultado clásico de Walker (Proposición 2.14) ha sido recuperado completamente en el entorno borroso. En la Sección 2.3 hemos estudiado las relaciones entre distintas condiciones de racionalidad que se han propuesto para el caso de funciones de elección borrosa, en el intento de emular el famoso teorema de Arrow-Sen de la teoría clásica. Para resumir los resultados obtenidos en este apartado, presentamos el esquema de la Figura 5.1, donde las condiciones de racionalidad se presentan en caja separadas. Una flecha de una caja a otra indica que la condición contenida en la primera caja implica la condición contenida en la segunda. Se han usado diferentes estilos de línea para marcar aquellas implicaciones que valen bajo diferentes t-normas:

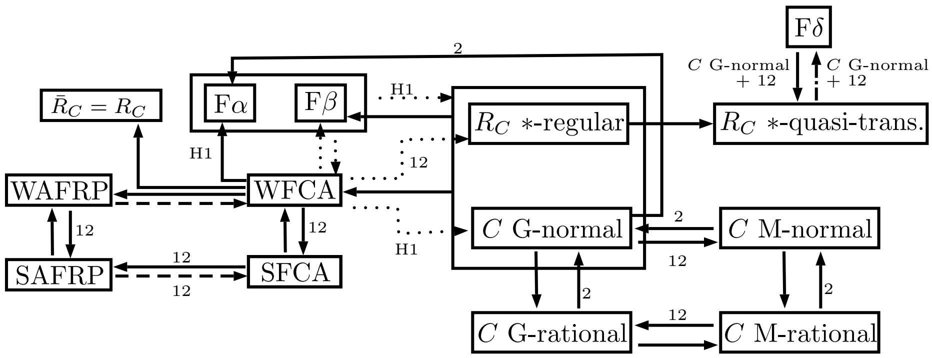
(i) una línea continua indica que el resultado es válido para cualquier t-norma continua por la izquierda;

(ii) una línea discontinua con patrón - - - indica que el resultado es válido para cualquier t-norma sin divisores de cero;

(iii) una línea discontinua con patrón -·- indica que el resultado es válido para cualquier t-norma que es invariante con respeto a rotación;

(iv) una línea punteada indica que el resultado es válido para la t-norma del mínimo.

Ulteriores condiciones, tales como H1, WH1 y WH2 se representarán al lado de las flechas para aquellos resultados que las necesiten.



—— vale para toda t-norma continua por la izquierda

– – – vale para toda t-norma invariante por rotación

-·-·-· vale para toda t-norma sin divisores de cero

…… vale para la t-norma del mínimo

1: se asume la condición WH1

2: se asume la condición WH2

H1: se asume la condición H1

Figure 5.1: Resumen de los resultados obtenidos.

El Capítulo 3 está dedicado a la teoría de elección probabilística. En particular, en la Sección 3.2 se presenta una construcción que, a través del uso de t-normas y operadores de implicación, permite expresar una función de elección probabilística como una función de elección borrosa. Si la función de elección probabilística inicial satisface ciertas condiciones, que también hemos definido a partir de las propuestas de la literatura, entonces la función de elección borrosa que se deriva es normal y la relación de preferencia que se revela de ella es transitiva. Estas condiciones se inspiran en varias propuesta encontradas en la literatura, como por ejemplo el Axioma de elección de Luce o la condición WASRP propuesta por Bandyopadhyay et al. en [5]. Hemos comprobado que las dos condiciones se puede formalizar de forma parecida, usando una expresión común que depende de la t-norma. A partir de esta observación hemos definido dos nuevas condiciones (WSC y SSC), que también dependen de la t-norma, y que generalizan las condiciones propuestas en la literatura. Estas nuevas condiciones, junto con una versión probabilística del Axioma de Congruencia Débil, permiten demostrar que la función de elección definida a partir de un función de elección borrosa es G-normal y que, además, la relación de preferencia que se revela de ella es $*$-transitiva. La Sección 3.3 está dedicada a comparar las relaciones de preferencia borrosas con las relaciones de preferencia probabilística. Ambas relaciones permiten expresar la preferencia a través de una función multivaluada, pero responden a dos semánticas completamente diferentes: las relaciones borrosas son univariadas, mientras las relaciones probabilísticas son bivariadas. Esto implica que, a pesar de formalizarse de una forma parecida, su significado es diferente y no pueden ser intercambiadas. Lo que proponemos en este trabajo es una construcción que permite expresar una relación en función de la otra y que además respecta las diferencias semánticas entre ellas. Bajo ciertas condiciones (continuidad de la t-norma) las relaciones construidas de esta manera son equivalentes. Gracias a esa equivalencia se puede estudiar cómo ciertas propiedades, como por ejemplo la transitividad, se propagan entre una definición y la otra. Hemos descubierto así una nueva familia de funciones de límite superior, típicamente utilizadas para describir las relaciones probabilísticas ciclo-transitivas. Finalmente, en la Sección 3.4, se recoge un conjunto de resultados que se han obtenido con un experimento sobre datos reales, pensado para poder medir la racionalidad de un grupo de consumidores y que hace uso de las técnicas expuestas en las secciones anteriores.

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