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All the timelike supersymmetric solutions of all ungauged $d = 4$ supergravities

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Abstract: We determine the form of all timelike supersymmetric solutions of all $N \geq 2$, $d = 4$ ungauged supergravities, coupled to vector supermultiplets for $N < 4$, using the $\mathrm{USp}(\bar{n}, \bar{n})$-symmetric formulation of Andrianopoli, D'Auria and Ferrara and the spinor-bilinears method, while preserving the global symmetries of the theories all the way.

As previously conjectured in the literature, the supersymmetric solutions are always associated to a truncation to an $N = 2$ theory that may include hypermultiplets, although fields which are eliminated in the truncations can have non-trivial values, as is required by the preservation of the global symmetry of the theories.

The solutions are determined by a number of independent functions, harmonic in transverse space, which is twice the number of vector fields of the theory ($\bar{n}$). The transverse space is flat if an only if the would-be hyperscalars of the associated $N = 2$ truncation are trivial.

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1 Introduction

The supersymmetric solutions of supergravity theories describing vacua, black holes or topological defects, play a fundamental role in the progress of superstring theory and related areas of research. It is, therefore, very important to find and study as many supersymmetric solutions as possible, a goal to which a huge effort has been devoted in the last few years.

In his pioneering work [1], Tod showed that it was possible to systematically find all the supersymmetric configurations and solutions of a given supergravity theory (pure $N = 2, d = 4$ in the case he considered, following the lead of ref. [2]) by exploiting the consistency and integrability conditions of the Killing spinor equations. He found that the supersymmetric solutions of pure $N = 2, d = 4$ supergravity fall in two classes: timelike and null. By *all the supersymmetric configurations* we mean all the field configurations that admit at least one Killing spinor, or equivalently one supercharge out of the $4N$ possible ones. The timelike supersymmetric solutions are generalizations of the Perjès-Israel-Wilson [3, 4] stationary solutions of the Einstein-Maxwell system which themselves generalize the static solutions found by Papapetrou and Majumdar [5, 6]. The solutions in the null class are examples of Brinkmann waves [7]. Tod's feat opened up the possibility of finding all the supersymmetric solutions of all the supergravity theories.

Tod [1, 8] used the Newman-Penrose formalism to find the supersymmetric solutions of the 4-dimensional pure $N = 2$ and 4 supergravity theories, so that new techniques had to be developed in order to tackle higher-dimensional cases. In ref. [9] Gauntlett et al. proposed to work with the spinor bilinears that can be constructed out of the Killing spinors. These tensors satisfy a number of algebraic and differential equations that follow from the Fierz identities and the original Killing spinor equations that their constituents satisfy and which capture enough (if not all the) information contained in them. The consistency and integrability conditions of these new equations then determine the supersymmetric configurations of the theory. In this way, in ref. [9] all the supersymmetric solutions of minimal supergravity in $d = 5$ dimensions were determined. These results were immediately extended to the Abelian gauged case [10] and later on to general matter contents and couplings [11–13] (always in the minimal $N = 2$ supergravity). The spinor-bilinear method was subsequently applied to other 4-dimensional [14]–[25], 6-dimensional [26–28], 7-dimensional [29], 11-dimensional [30–36] and, recently, to 3-dimensional [37] supergravities.

In this approach (which will be used in this article) the form of all the field configurations admitting at least one Killing spinor can be determined but (unless further work is done) no classification of the supersymmetric configurations by the number of independent Killing spinors they admit is done. A different (but fundamentally equivalent) approach based on spinorial geometry was developed in refs. [38–56]. It has advantages over the spinor-bilinear approach: using it, an exhaustive classification of the configurations with different numbers of unbroken supersymmetries can be achieved, also in higher dimensional theories where the application of the bilinear approach becomes unwieldy, by choosing convenient bases for the spinors.

Yet another approach, more adequate for finding supersymmetric solutions with special geometries or properties, exploits the fact that a Killing spinor defines a "G structure" [9, 30–36, 57]. Finally, another approach used to find the timelike supersymmetric solutions of 4-dimensional theories, and applied in particular to black holes, exploits the symmetries of the dimensionally-reduced theories which become a non-linear σ-model coupled to 3-dimensional gravity [58–64]. The main difficulty of this powerful approach resides in the reconstruction of the 4-dimensional solutions from the 3-dimensional ones.

The spinor-bilinear method that we are going to use is, we think, more adequate to find large classes of solutions preserving (as a class) the global symmetries of the theory: using it, it has been possible to find the general form of the (pure, ungauged) $N = 4, d = 4$ supergravity black holes [8, 16] written in an SO(6)-covariant form although some of them (which are singular), characterized by particular choices of the charges, preserve 1/2 of the supersymmetries instead of the generic 1/4 [65].

The spinor-bilinear method, however, becomes difficult to use for $N > 2$. For instance, in the timelike $N = 2$ case with one Killing spinor ϵ^I $(I = 1, 2)$ one can construct precisely four vector bilinears[1] $V^I{}_{J\mu} \equiv i\bar{\epsilon}^I\gamma^\mu\epsilon_J$ which can be used as a tetrad to construct the spacetime metric. For $N > 2$ we have too many vector bilinears and choosing four of them as a tetrad while preserving the U(N) invariance of the procedure seems impossible. There are several manifestations of the same problem in the whole procedure.

Another problem, one that is common to all approaches, is the necessity of treating different values of N separately due to the different field content and symmetries of each theory.

In this paper we are going to use the spinor-bilinear method to determine the general form of all the timelike supersymmetric solutions of all the $N \geq 2, d = 4$ ungauged supergravities coupled to matter vector multiplets (when these supermultiplets are available). As we will show, the main difficulties of the spinor-method problem can be solved at least to the extent that the solution allows us to determine the general form of all the timelike supersymmetric solutions. This has required a deeper study of the algebra of spinor bilinears than has been made in the literature hitherto and which has allowed us to find a way to define an SU(2) subgroup without explicitly breaking the U(N) R-symmetry of the equations. Furthermore, we are going to use the N-independent "supergravity tensor calculus" introduced in ref. [66], which allows the simultaneous study of all the $N \geq 2, d = 4$ ungauged supergravities just as one can work with tensors constructed over vector spaces of undetermined number of dimensions and obtain results valid for any d.

We have found that each timelike supersymmetric solutions is closely related to a truncation to an $N = 2$ theory determined by a U(2) subgroup of the U(N) R-symmetry group.[2] It has to be emphasized that this does not mean that each of them is just a solution of an $N = 2$ truncation since, for instance, all the vector fields are generically non-vanishing and some of them would be eliminated by a generic truncation to $N = 2$. However most (if not all) of them may be generated by duality relations from a solution of the associated $N = 2$ truncation. This process can be rather cumbersome but, in any case, our results render it unnecessary.

The construction of any timelike supersymmetric solution proceeds along the following steps:

1. We have to choose the U(2) subgroup which determines the associated $N = 2$ truncation:

   (a) Choose an x-dependent, rank-2, $N \times N$ complex antisymmetric matrix M_{IJ} satisfying $M_{I[J}M_{KL]} = 0$ (x stands for the 3 spatial coordinates). With it we can construct

$$\mathcal{J}^I{}_J \equiv 2|M|^{-2}M^{IK}M_{JK}\,, \qquad |M|^2 = M^{PQ}M_{PQ}\,,$$

[1] See appendix D.

[2] For supersymmetric black holes, this fact was conjectured in ref. [67] and earlier in ref. [68] and recently proven in the next to last of refs. [58–64].

which is a Hermitean projection operator whose trace is +2: $\mathcal{J}$ projects onto the above-mentioned U(2) subgroup.

$\mathcal{J}$ must be covariantly constant[3]

$$\mathfrak{D}\mathcal{J} \equiv d\mathcal{J} - [\mathcal{J}, \Omega] = 0\,,$$

in all cases. In practice, the imposition of this requirement may be postponed to the last stages of the construction of the supersymmetric solutions.

Parametrizing the most general matrix M_{IJ} that satisfies these requirements gives a parametrization of the most general timelike supersymmetric solutions.

(b) Given M_{IJ} and hence the covariantly-constant $\mathcal{J}^I{}_J$, we have to find three Hermitean, traceless, x-dependent $N \times N$ matrices $(\sigma^m)^I{}_J$ ($m = 1, 2, 3$), satisfying the same properties as the Pauli matrices in the subspace preserved by $\mathcal{J}$ as derived in appendix (D), to wit

$$\begin{aligned}
\sigma^m \sigma^n &= \delta^{mn}\mathcal{J} + i\varepsilon^{mnp}\sigma^p\,,\\
\mathcal{J}\sigma^m &= \sigma^m \mathcal{J} = \sigma^m\,,\\
\mathcal{J}^K{}_J \mathcal{J}^L{}_I &= \frac{1}{2}\mathcal{J}^K{}_I \mathcal{J}^L{}_J + \frac{1}{2}(\sigma^m)^K{}_I (\sigma^m)^L{}_J\,,\\
M_{K[I}(\sigma^m)^K{}_{J]} &= 0\,,\\
2|M|^{-2} M_{LI}(\sigma^m)^I{}_J M^{JK} &= (\sigma^m)^K{}_L\,.
\end{aligned}$$

It turns out that we also have to impose the constraint

$$\mathcal{J} d\sigma^m \mathcal{J} = 0\,,$$

implying that the σ-matrices are constant in the subspace preserved by the projector $\mathcal{J}$.[4]

The four matrices $\{\mathcal{J}, \sigma^m\}$ provide a basis for the U(2) subgroup of the associated $N = 2$ truncation and can be seen as generators of its R-symmetry group.

Defining the complementary projector $\tilde{\mathcal{J}} \equiv \mathbb{I}_{N\times N} - \mathcal{J}$ it is possible to separate the scalars into those corresponding to the would-be vector multiplets and hypermultiplets of the associated $N = 2$ truncation. Thus, from the scalars in the generic supergravity multiplet, described by the (pullback of the) Vielbein $P_{IJKL\,\mu} \equiv P_{[IJKL]\,\mu}$

[3]Naively one may think that it is always possible to choose a basis in U(N) space such that, for instance, $M_{12} = -M_{21} = +1$ and the rest of the components vanish, whence $\mathcal{J}$ is the identity in the corresponding 2-dimensional subspace. However, the necessary change of basis involves an, a priori, arbitrary local U(N) rotation and the theory is not really U(N) gauge-invariant even if some fields undergo field-dependent compensating U(N) transformations when one performs a global symmetry transformation and there is a U(N) gauge connection which is a composite field.

This problem was first observed by Tod in his study of the $N = 4$ theory [8] and, being unable to prove it, he conjectured that this rotation was always possible.

We have not been able to prove this hypothesis in general either. We have proven that covariant constancy is required, though, which implies in the pure $N = 4$ case studied by Tod ($\Omega^I{}_J \sim \delta^I{}_J$) as well as in the pure $N = 3$ theory ($\Omega = 0$) that $\mathcal{J}$ has to be constant.

[4]This is automatically satisfied for the projector itself $\mathcal{J} d\mathcal{J}\mathcal{J} = 0$.

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and from the scalars in the generic matter multiplet, described by $P_{i\,IJ\,\mu} \equiv P_{i\,[IJ]\,\mu}$; those in the vector multiplets are described by

$$P_{IJKL}\,\mathcal{J}^{I}{}_{[M}\mathcal{J}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]} \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^{I}{}_{[K}\mathcal{J}^{J}{}_{L]}\,,$$

and those in the hypermultiplets are described by

$$P_{IJKL}\,\mathcal{J}^{I}{}_{[M}\tilde{\mathcal{J}}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]} \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^{I}{}_{[K}\tilde{\mathcal{J}}^{J}{}_{L]}\,.$$

The discrimination between these two kinds of scalars is, however, important: those corresponding to the vector multiplets are sourced by the electric and magnetic charges and enter into the attractor mechanism while those corresponding to the hypermultiplets are not and should be frozen in supersymmetric black-hole solutions.

2. Once the choice of U(2) subgroup is made, the solutions are constructed by the following procedure:[5]

   (a) Using the symplectic functions of the scalars $\mathcal{V}_{IJ}$ (A.5), which generalize the canonical symplectic section $\mathcal{V}$ of the $N=2$ theories [69–72], we define the real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$ by

   $$\mathcal{R} + i\mathcal{I} \equiv |M|^{-2}\mathcal{V}_{IJ}M^{IJ}\,,$$

   which are U(N) singlets. No particular U(N) gauge-fixing is necessary to construct the solutions.

   (b) For the supersymmetric solutions, the components of the symplectic vector $\mathcal{I}$ are real functions satisfying the Laplace equation in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$, to be described later. This is the only differential equation that needs to be solved.

   (c) $\mathcal{R}$ can in principle be found from $\mathcal{I}$ by solving the generalization of the so-called *stabilization equations*.

   (d) The metric of the solutions has the form

   $$ds^2 \;=\; |M|^2(dt+\omega)^2 - |M|^{-2}\gamma_{\underline{mn}}dx^m dx^n\,.$$

   where

   $$|M|^{-2} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle\,,$$
   $$(d\omega)_{mn} = 2\epsilon_{mnp}\langle\, \mathcal{I} \mid \partial^p\mathcal{I}\,\rangle\,,$$

   so they can be computed directly from $\mathcal{R}$ and $\mathcal{I}$.

   The 3-dimensional transverse metric $\gamma_{\underline{mn}}$ is determined indirectly by the would-be hypers; in particular, when those scalars are frozen the metric is flat. The full condition that the 3-dimensional metric has to satisfy is that its spin-connection

[5] This procedure is completely analogous to the procedure used to build supersymmetric solutions in ungauged $N=2$ theories coupled to vector multiplets and hypermultiplets described in ref. [18].

must be related to (the pullback of) the connection of the scalar manifold, Ω in (A.9), by

$$\varpi^{mn} = i\varepsilon^{mnp}\mathrm{Tr}\,[\sigma^p\Omega]\,.$$

Observe that only the $\mathfrak{su}(2)$ part of Ω contributes to ϖ^{mn}.[6]

(e) The vector field strengths are given by

$$F = -\frac{1}{2}d(\mathcal{R}\hat{V}) - \frac{1}{2}\star(\hat{V}\wedge d\mathcal{I})\,, \qquad \hat{V} = \sqrt{2}|M|^2(dt+\omega)\,.$$

(f) The scalars corresponding to the vector multiplets in the associated $N=2$ truncation, represented by the projected Vielbeine

$$P_{IJKL}\,\mathcal{J}^I{}_{[M}\mathcal{J}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]} \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^I{}_{[K}\mathcal{J}^J{}_{L]}\,,$$

can in principle be found from $\mathcal{R}$ and $\mathcal{I}$. The Killing Spinor Identities guarantee that the equations of motion of these scalars are satisfied if the Maxwell equations and Bianchi identities are satisfied,[7] which is the case when the components of $\mathcal{I}$ are harmonic functions on the transverse space.

(g) The scalars corresponding to the hypers, described by the Vielbeine

$$P_{IJKL}\,\mathcal{J}^I{}_{[M}\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]} \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^I{}_{[K}\tilde{\mathcal{J}}^J{}_{L]}\,,$$

must be found independently by solving the supersymmetry constraints

$$\begin{aligned} P_{IJKL\,m}\,\mathcal{J}^I{}_{[M}\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}(\sigma^m)^Q{}_R &= 0\,,\\ P_{i\,IJ\,m}\,\mathcal{J}^I{}_{[K}\tilde{\mathcal{J}}^J{}_{L]}(\sigma^m)^L{}_M &= 0\,. \end{aligned}$$

The Killing Spinor Identities guarantee that their equations of motion are automatically solved.[8]

In the rest of this paper we are going to prove in full detail the above result. We are going to start by giving the generic description of all the $N\geq 2, d=4$ supergravities with vector multiplets (where available) in section 2. In section 3 we are going to present the Killing spinor equations for all these theories and we are going to find the Killing Spinor Identities that constrain the off-shell equations of motion of the bosonic fields for supersymmetric field configurations.

[6] It plays the same rôle as the $\mathfrak{su}(2)$ connection of the hyper-Kähler manifold in ref. [18] and the condition on the metric is identical to the one found in the $N=2$ case although in that case the 2×2 matrices σ^m are the standard, constant, Pauli matrices.

[7] Actually, the only independent equations of motion that need to be solved are the 0^{th} components of the Maxwell equations and Bianchi identities. Some of the other equations are just automatically satisfied for supersymmetric configurations and the rest is proportional to those 0^{th} components.

[8] This situation is completely analogous to what happens with the hyperscalars of $N=2$ theories [18].

2 Generic description of $N \geq 2, d = 4$ supergravities

We are going to study all the $N \geq 2, d = 4$ supergravities coupled to vector multiplets simultaneously, using the fact that all the supergravity multiplets and all the vector multiplets for all $N = 1, \ldots, 8$ can be written in the same generic form [66]; we only need to take into account the range of values taken by the $\mathrm{U}(N)$ R-symmetry indices, denoted by uppercase Latin letters I etc. taking on values $1, \ldots, N$, in each particular case.[9]

The generic supergravity multiplet in four dimensions is

$$\{e^a{}_\mu, \psi_{I\,\mu}, A^{IJ}{}_\mu, \chi_{IJK}, \chi^{IJKLM}, P_{IJKL\,\mu}\}\,, \quad I, J, \dots = 1, \ldots, N\,, \tag{2.1}$$

and the generic vector multiplets (labeled by $i = 1, \ldots, n$) are

$$\{A_{i\,\mu}, \lambda_{iI}, \lambda_i{}^{IJK}, P_{iIJ\,\mu}\}\,. \tag{2.2}$$

The spinor fields $\psi_{I\,\mu}, \chi_{IJK}, \chi^{IJKLM}, \lambda_{iI}, \lambda_i{}^{IJK}$ have positive chirality with the given positions of the $\mathrm{SU}(N)$ indices.

The scalars of these theories are encoded into the $2\bar{n}$-dimensional ($\bar{n} \equiv n + \frac{N(N-1)}{2}$) symplectic vectors ($\Lambda = 1, \ldots \bar{n}$) $\mathcal{V}_{IJ}$ and $\mathcal{V}_i$ whose properties are reviewed in appendix A. They appear in the bosonic sector of the theory via the pullbacks of the Vielbeine $P_{IJKL\mu}$ (supergravity multiplet) and $P_{iIJ\,\mu}$ (matter multiplets).[10] There are three instances of theories for which the scalar Vielbeine are constrained: first, when $N = 4$ the matter scalar Vielbeine are constrained by the $\mathrm{SU}(4)$ complex self-duality relation[11]

$$N = 4 :: \qquad\qquad P^{*\,iIJ} = \frac{1}{2}\varepsilon^{IJKL}\, P_{i\,KL}\,. \tag{2.3}$$

Secondly, in $N = 6$ the scalars in the supergravity multiplet are represented by one Vielbein P_{IJ} and one Vielbein P_{IJKL} related by the $\mathrm{SU}(6)$ duality relation

$$N = 6 :: \qquad\qquad P^{*\,IJ} = \frac{1}{4!}\varepsilon^{IJK_1\cdots K_4}\, P_{K_1\cdots K_4}\,, \tag{2.4}$$

and lastly the $N = 8$ case, in which the Vielbeine is constrained by the $\mathrm{SU}(8)$ complex self-duality relation

$$N = 8 :: \qquad\qquad P^{*\,I_1\cdots I_4} = \frac{1}{4!}\varepsilon^{I_1\cdots I_4 J_1\cdots J_4}\, P_{J_1\cdots J_4}\,. \tag{2.5}$$

These constraints must be taken into account in the action.

[9] This formalism is taken from ref. [66], but adapted to the notations of ref. [16]. Furthermore, throughout this paper we use the convention that the only fields and terms that should be considered are those whose number of antisymmetric $\mathrm{SU}(N)$ indices is correct, i.e. objects with more than N antisymmetric indices are zero and terms with Levi-Cività symbols $\epsilon^{I_1\cdots I_M}$ should only be considered when M equals the N of the supergravity theory under consideration. There are also constraints on the generic fields for specific values of N that we are going to review.

[10] The Vielbeine $P_{ij\,\mu}$ either vanish identically or depend on $P_{IJKL\mu}$ and $P_{iIJ\,\mu}$, depending on the specific value of N. Thus, they are not needed as independent variables to construct the theories.

[11] In order to highlight the fact that an equation holds for a specific N only, we write a numerical variation of the token "$N = 4 ::$" to the left of the equation.

The graviphotons $A^{IJ}{}_{\mu}$ do not appear directly in the theory, rather they only appear through the "dressed" vectors, which are defined by

$$A^{\Lambda}{}_{\mu} \equiv \frac{1}{2} f^{\Lambda}{}_{IJ} A^{IJ}{}_{\mu} + f^{\Lambda}{}_{i} A^{i}{}_{\mu} \,. \tag{2.6}$$

The action for the bosonic fields is

$$\begin{aligned} S = \int d^4x \sqrt{|g|} \Big[& R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^{\Sigma}{}_{\mu\nu} \\ & + \frac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu} P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu} P_{iIJ}{}^{\mu} \Big] \,, \end{aligned} \tag{2.7}$$

where $\mathcal{N}_{\Lambda\Sigma}$ is the generalization of the $N=2$ period matrix, defined in eq. (A.11), and where the parameters α_1, α_2 are equal to 1 in all cases except for $N = 4, 6$ and 8 as one needs to take into account the above constraints on the Vielbeine: $\alpha_2 = 1/2$ for $N = 4$, $\alpha_1 + \alpha_2 = 1$ for $N = 6$ (the simplest choice being $\alpha_2 = 0$) and $\alpha_1 = 1/2$ for $N = 8$. The action is good enough to compute the Einstein and Maxwell equations, but not the scalars' equations of motion in the cases in which the scalar Vielbeine are constrained: these constraints have to be properly dealt with and the resulting equations of motion are given below.

The supersymmetry transformations of the bosonic fields can be written in the form

$$\delta_\epsilon e^{a}{}_{\mu} = -i\bar{\psi}_{I\mu}\gamma^{a}\epsilon^{I} - i\bar{\psi}^{I}{}_{\mu}\gamma^{a}\epsilon_{I} \,, \tag{2.8}$$

$$\begin{aligned} \delta_\epsilon A^{\Lambda}{}_{\mu} = {} & f^{\Lambda}{}_{IJ}\bar{\psi}^{I}{}_{\mu}\epsilon^{J} + f^{*\Lambda IJ}\bar{\psi}_{I\mu}\epsilon_{J} - \frac{i}{2}(f^{\Lambda}{}_{i}\bar{\lambda}^{iI}\gamma_{\mu}\epsilon_{I} + f^{*\,\Lambda i}\bar{\lambda}_{iI}\gamma_{\mu}\epsilon^{I}) \\ & - \frac{i}{4}(f^{\Lambda}{}_{IJ}\bar{\chi}^{IJK}\gamma_{\mu}\epsilon_{K} + f^{*\Lambda IJ}\bar{\chi}_{IJK}\gamma_{\mu}\epsilon^{K}) \,, \end{aligned} \tag{2.9}$$

$$(U^{-1}\delta_\epsilon U)_{IJKL} = 4\bar{\chi}_{[IJK}\epsilon_{L]} + \bar{\chi}_{IJKLM}\epsilon^{M} \,, \tag{2.10}$$

$$(U^{-1}\delta_\epsilon U)_{iIJ} = 2\bar{\lambda}_{i[I}\epsilon_{J]} + \frac{1}{2}\bar{\lambda}_{iIJK}\epsilon^{K} \,, \tag{2.11}$$

where U is the $\mathrm{Usp}(\bar{n},\bar{n})$ matrix describing the scalars, defined in eq. (A.2). Those of the fermionic fields can be put in the form

$$\delta_\epsilon \psi_{I\mu} = \mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^{\nu}\epsilon^{J} \,, \tag{2.12}$$

$$\delta_\epsilon \chi_{IJK} = -\frac{3i}{2}\not{T}_{[IJ}{}^{+}\epsilon_{K]} + i\not{P}_{IJKL}\epsilon^{L} \,, \tag{2.13}$$

$$\delta_\epsilon \lambda_{iI} = -\frac{i}{2}\not{T}_{i}{}^{+}\epsilon_{I} + i\not{P}_{iIJ}\epsilon^{J} \,, \tag{2.14}$$

$$\delta_\epsilon \chi_{IJKLM} = -5i\not{P}_{[IJKL}\epsilon_{M]} + \frac{i}{2}\varepsilon_{IJKLMN}\not{T}^{-}\epsilon^{N} + \frac{i}{4}\varepsilon_{IJKLMNOP}\not{T}^{NO-}\epsilon^{P} \,, \tag{2.15}$$

$$\delta_\epsilon \lambda_{iIJK} = -3i\not{P}_{i[IJ}\epsilon_{K]} + \frac{i}{2}\varepsilon_{IJKL}\not{T}_{i}{}^{-}\epsilon^{L} + \frac{i}{4}\varepsilon_{IJKLMN}\not{T}^{LM-}\epsilon_{N} \,, \tag{2.16}$$

where we have defined the graviphoton and matter vector field strengths

$$T_{IJ}{}^{+}{}_{\mu\nu} = 2if^{\Lambda}{}_{IJ}\ \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\ F^{\Sigma+}{}_{\mu\nu} \,, \quad T_{i}{}^{+}{}_{\mu\nu} = 2if^{\Lambda}{}_{i}\ \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\ F^{\Sigma+}{}_{\mu\nu} \,, \tag{2.17}$$

and where

$$\mathfrak{D}_\mu \epsilon_I \equiv \nabla_\mu \epsilon_I - \epsilon_J \Omega_\mu{}^{J}{}_{I} \,, \tag{2.18}$$

Ω being the pullback of the connection on the scalar manifold, defined in appendix A.

We stress that, according to our conventions, the terms with ε-symbols should only be considered when the value of N equals its rank. Furthermore, when $N = 4, 6$ or 8 eqs. (2.15) and (2.16) depend on the first three supersymmetry rules, whereas for $N = 2$ they are equations for non-existing fields: therefore, eqs. (2.15) and (2.16) only need to be considered in the cases $N = 3$ and 5, and then only the first term on the l.h.s. is non-vanishing.

For convenience, we denote the Bianchi identities for the vector field strengths by

$$\mathcal{B}^{\Lambda\,\mu} \equiv \nabla_\nu \star F^{\Lambda\,\nu\mu} . \tag{2.19}$$

and the bosonic equations of motion by

$$\begin{aligned}
\mathcal{E}_a{}^\mu &\equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,, \quad \mathcal{E}^{IJKL} \equiv -\frac{1}{2\sqrt{|g|}}\left(\frac{\delta S}{\delta U}U\right)^{IJKL} = -\frac{1}{2\sqrt{|g|}}P^{*\,IJKL\,A}\frac{\delta S}{\delta\phi^A}\,,\\
\mathcal{E}_\Lambda{}^\mu &\equiv \frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^\Lambda{}_\mu}\,, \qquad \mathcal{E}^{iIJ} \equiv -\frac{1}{2\sqrt{|g|}}\left(\frac{\delta S}{\delta U}U\right)^{iIJ} = -\frac{1}{2\sqrt{|g|}}P^{*\,iIJ\,A}\frac{\delta S}{\delta\phi^A}\,,
\end{aligned} \tag{2.20}$$

where $P^{*\,IJKL\,A}$ and $P^{*\,iIJ\,A}$ are the inverse Vielbeine and ϕ^A are the physical fields of the theory.

The explicit forms of the Einstein and Maxwell equations are

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + \frac{1}{12}\alpha_1\left[P^{*\,IJKL}{}_{(\mu|}P_{IJKL\,|\nu)} - \frac{1}{2}g_{\mu\nu}P^{*\,IJKL}{}_\rho P_{IJKL}{}^\rho\right]\\
&\quad +\alpha_2 P^{*\,iIJ}{}_{(\mu|}P_{iIJ\,|\nu)} - \frac{1}{2}g_{\mu\nu}P^{*\,iIJ}{}_\rho P_{iIJ}{}^\rho + 8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_\mu{}^\rho F^{\Sigma\,-}{}_{\nu\rho}\,,
\end{aligned} \tag{2.21}$$

$$\mathcal{E}_\Lambda{}^\mu = \nabla_\nu \star \tilde{F}_\Lambda{}^{\nu\mu}\,, \tag{2.22}$$

where we have defined the dual vector field strength $\tilde{F}_\Lambda$ by

$$\tilde{F}_{\Lambda\,\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta^\star F^\Lambda{}_{\mu\nu}} = 2\Re\mathrm{e}(\mathcal{N}_{\Lambda\Sigma}F^{\Sigma\,+}) = \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^\Sigma{}_{\mu\nu} + \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\star F^\Sigma{}_{\mu\nu}\,. \tag{2.23}$$

Using eqs. (A.29) and (A.30) and taking into account the constraints satisfied by the Vielbeine in the cases $N = 4, 6$ and 8, we find that the scalar equations of motion take the following forms, slightly different for each value of N:

$N = 2$::

$$\mathcal{E}^{iIJ} = \mathfrak{D}^\mu P^{*\,iIJ}{}_\mu + 2T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu} + P^{*\,iIJ\,A}P^{*\,jk}{}_A T_j{}^+{}_{\mu\nu}T_k{}^{+\,\mu\nu}\,. \tag{2.24}$$

$N = 3$::

$$\mathcal{E}^{iIJ} = \mathfrak{D}^\mu P^{*\,iIJ}{}_\mu + 2T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu}\,. \tag{2.25}$$

$N = 4$::

$$\mathcal{E}^{IJKL} = \mathfrak{D}^\mu P^{*\,IJKL}{}_\mu + 6T^{[IJ|-}{}_{\mu\nu}T^{|KL]-\,\mu\nu} + P^{*\,IJKL\,A}P^{*\,ij}{}_A T_i{}^+{}_{\mu\nu}T_j{}^{+\,\mu\nu}\,, \tag{2.26}$$

$$\mathcal{E}^{iIJ} = \mathfrak{D}^\mu P^{*\,iIJ}{}_\mu + T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu} + \frac{1}{2}\varepsilon^{IJKL}T_i{}^+{}_{\mu\nu}T_{KL}{}^{+\,\mu\nu}\,. \tag{2.27}$$

$N = 5$::

$$\mathcal{E}^{IJKL} = \mathfrak{D}^{\mu} P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu} T^{|KL]-\,\mu\nu} \,. \tag{2.28}$$

$N = 6$::

$$\mathcal{E}^{IJKL} = \mathfrak{D}^{\mu} P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu} T^{|KL]-\,\mu\nu} + \varepsilon^{IJKLMN} T^{+}{}_{\mu\nu} T_{MN}{}^{+\,\mu\nu} \,. \tag{2.29}$$

$N = 8$::

$$\mathcal{E}^{IJKL} = \mathfrak{D}^{\mu} P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu} T^{|KL]-\,\mu\nu} + \frac{1}{4}\varepsilon^{IJKLMNPQ} T_{MN}{}^{+}{}_{\mu\nu} T_{PQ}{}^{+\,\mu\nu} \,. \tag{2.30}$$

3 Generic $N \geq 2, d = 4$ Killing spinor equations and identities

The Killing spinor equations are

$$\mathfrak{D}_{\mu}\epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^{\nu}\epsilon^{J} = 0\,, \tag{3.1}$$

$$\not{P}_{IJKL}\epsilon^{L} - \frac{3}{2}\not{T}_{[IJ}{}^{+}\epsilon_{K]} = 0\,, \tag{3.2}$$

$$\not{P}_{i\,IJ}\epsilon^{J} - \frac{1}{2}\not{T}_{i}{}^{+}\epsilon_{I} = 0\,, \tag{3.3}$$

$$N = 5 \quad :: \qquad \not{P}_{[IJKL}\epsilon_{M]} = 0\,, \tag{3.4}$$

$$N = 3 \quad :: \qquad \not{P}_{i\,[IJ}\epsilon_{K]} = 0\,, \tag{3.5}$$

where, as indicated by the notation, the last two KSEs should only be considered for $N = 5$ and $N = 3$, respectively.

From the bosonic supersymmetry transformation rules we immediately find using the algorithm of refs. [73, 74]

$$\mathcal{E}_{a}{}^{\mu}\gamma^{a}\epsilon^{I} - 4i\mathcal{E}_{\Lambda}{}^{\mu} f^{*\,\Lambda\,IJ}\epsilon_{J} = 0\,, \tag{3.6}$$

$$\mathcal{E}_{\Lambda}{}^{\mu} f^{*\,\Lambda\,[IJ}\gamma_{\mu}\epsilon^{K]} - \frac{i}{3!}\mathcal{E}^{IJKL}\epsilon_{L} = 0\,, \tag{3.7}$$

$$\mathcal{E}_{\Lambda}{}^{\mu} f^{*\,\Lambda\,i}\gamma_{\mu}\epsilon^{I} - \frac{i}{2}\mathcal{E}^{i\,IJ}\epsilon_{J} = 0\,, \tag{3.8}$$

$$N = 5 \quad :: \qquad \mathcal{E}^{[IJKL}\epsilon^{M]} = 0\,, \tag{3.9}$$

$$N = 3 \quad :: \qquad \mathcal{E}^{i\,[IJ}\epsilon^{K]} = 0\,. \tag{3.10}$$

In these equations it is implicitly assumed that the Bianchi identities are satisfied, i.e. $\mathcal{B}^{\Lambda\,\mu} = 0$. It is, however, convenient not to make use of this assumption as to preserve the manifest electric-magnetic duality of the formalism. We can, and will, introduce the Bianchi identities into these equations by the replacement

$$\mathcal{E}_{\Lambda}{}^{\mu} f^{\Lambda} \;\longrightarrow\; \langle\, \mathcal{E} \mid \mathcal{V}\,\rangle\,, \tag{3.11}$$

where $\mathcal{E}$ is the symplectic vector containing the Maxwell equations and Bianchi identities.

We can start to derive consequences from these identities in terms of the spinor bilinears defined and studied in appendix D and in this paper we will only study the case in which the vector bilinear, $V^{a} = i\bar{\epsilon}^{I}\gamma^{a}\epsilon_{I}$, is timelike ($V^{2} = V^{a}V_{a} = 2|M|^{2} > 0$).

3.1 Timelike case

It is convenient to work with flat indices and use a Vierbein basis in which $e^0 \equiv \frac{1}{\sqrt{2}}|M|^{-1}V_\mu dx^\mu$. Acting with $i\bar{\epsilon}_I$ and $\bar{\epsilon}^K\gamma^\nu$ on the first KSI eq. (3.6) we get,

$$V^b\mathcal{E}_b{}^a + 4\langle\,\mathcal{E}^a \mid \mathcal{V}^{*\,IJ}\,\rangle M_{IJ} = 0\,, \tag{3.12}$$

$$\mathcal{E}_c{}^a(g^{cb}M^{KI} + \Phi^{KI\,cb}) + 4\langle\,\mathcal{E}^a \mid \mathcal{V}^{*\,JI}\,\rangle V^K{}_J{}^b = 0\,, \tag{3.13}$$

respectively. Multiplying the second identity with M_{KI} we obtain

$$|M|^2\mathcal{E}^{ab} + 2\langle\,\mathcal{E}^a \mid \mathcal{V}^{*\,IJ}\,\rangle M_{IJ}V^b = 0\,. \tag{3.14}$$

The symmetry and reality of the Einstein equation imply, firstly

$$\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0\,, \tag{3.15}$$

so all components of the Einstein equations but $\mathcal{E}^{00}$ are automatically and identically satisfied;[12] secondly[13]

$$\mathcal{E}^{00} = -2\sqrt{2}|M|\langle\,\mathcal{E}^0 \mid \mathcal{R}\,\rangle\,, \tag{3.16}$$

where we have defined the U(N)-neutral real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$ by

$$|M|^{-2}M^{IJ}\mathcal{V}_{IJ} \equiv \mathcal{V} = \mathcal{R} + i\mathcal{I}\,, \tag{3.17}$$

whence the remaining component of the Einstein equations is satisfied if the 0^{th} component of the Maxwell equations and Bianchi identities are satisfied. Thirdly and finally

$$\langle\,\mathcal{E}^m \mid \mathcal{R}\,\rangle = 0\,, \tag{3.18}$$

$$\langle\,\mathcal{E}^a \mid \mathcal{I}\,\rangle = 0\,. \tag{3.19}$$

Acting with $i\bar{\epsilon}_L$ and $\bar{\epsilon}^L\gamma^\nu$ on eq. (3.7), which is only to be considered for $N \geq 3$, we obtain

$$\langle\,\mathcal{E}^a \mid \mathcal{V}^{*\,[IJ}\,\rangle V^{K]}{}_{L\ a} - \frac{1}{3!}\varepsilon^{IJKM}M_{ML} = 0\,, \tag{3.20}$$

$$\langle\,\mathcal{E}^a \mid \mathcal{V}^{*\,[IJ}\,\rangle(-\delta^b{}_a M^{K]L} + \Phi^{K]L\ b}{}_a) - \frac{1}{3!}\varepsilon^{IJKM}V^{L\,b}{}_M = 0\,. \tag{3.21}$$

Multiplying eq. (3.20) by $2M^{NL}|M|^{-2}$ and antisymmetrizing the four free indices we get

$$\langle\,\mathcal{E}^a \mid \mathcal{V}^{*\,[IJ}\,\rangle\frac{M^{KL]}}{|M|} - \frac{1}{\sqrt{2}\cdot 3!}\delta^a{}_0\varepsilon^{M[IJK}\mathcal{J}^{L]}{}_M = 0\,. \tag{3.22}$$

Setting $K = L$ in eq. (3.20), using the antisymmetric part of eq. (3.13) and taking into account eq. (3.16), we get

$$\langle\,\mathcal{E}^m \mid \mathcal{V}^{*\,IJ}\,\rangle = 0\,, \tag{3.23}$$

[12] As explained in ref. [75] this poses strong constraints on the sources of the solutions because having supersymmetry unbroken everywhere implies that the KSIs should be identically (i.e. not up to δ-function terms) satisfied everywhere.

[13] The imaginary part of the equation $\langle\,\mathcal{E}^0 \mid \mathcal{I}\,\rangle = 0$ is related to the absence of sources of NUT charge in globally supersymmetric solutions [75].

and

$$\mathcal{E}^{IJKM}M_{KM} = -2\sqrt{2}|M|(\delta^{IJ}{}_{KL} - |M|^{-2}M^{IJ}M_{KL})\langle\,\mathcal{E}^0 \mid \mathcal{V}^{*\,KL}\,\rangle\,. \tag{3.24}$$

This implies that the projections

$$\mathcal{E}^{MNPQ}\,\mathcal{J}^{[I}{}_M\mathcal{J}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^{L]}{}_Q\,, \tag{3.25}$$

which should be understood as the equations of motion of the scalars that would correspond to the vector multiplets scalars in the associated $N = 2$ truncations, are satisfied if the 0^{th} component of the Maxwell equations and Bianchi identities are. From eq. (3.22) we can derive

$$\mathcal{E}^{MNPQ}\,\mathcal{J}^{[I}{}_M\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^{L]}{}_Q = 0\,, \tag{3.26}$$

whence the projections that would correspond to the hypers are automatically satisfied.

From eq. (3.8) we get

$$\langle\,\mathcal{E}^a \mid \mathcal{V}^{*\,i}\,\rangle + \frac{1}{2\sqrt{2}}\delta^a{}_0\mathcal{E}^{iIJ}\frac{M_{IJ}}{|M|} = 0\,, \tag{3.27}$$

$$\langle\,\mathcal{E}^a \mid \mathcal{V}^{*\,i}\,\rangle M^{KI} - \frac{1}{4}\mathcal{E}^{i[I|J}V^{|K]}{}_J{}^a = 0\,. \tag{3.28}$$

The first of these equations states first of all that

$$\langle\,\mathcal{E}^m \mid \mathcal{V}^{*\,i}\,\rangle = 0\,, \tag{3.29}$$

which, combined with eqs. (3.23) implies by means of the completeness relation eq. (A.14) that

$$\mathcal{E}^m = 0\,. \tag{3.30}$$

Therefore, the only component of the Maxwell equations and Bianchi identities that are not automatically satisfied due to supersymmetry, are $\mathcal{E}^0$; secondly, for the projections onto equations of motion of scalars in $N = 2$ vector multiplets

$$\mathcal{E}^{i\,KL}\mathcal{J}^I{}_K\mathcal{J}^J{}_L = -2\sqrt{2}\frac{M^{IJ}}{|M|}\langle\,\mathcal{E}^a \mid \mathcal{V}^{*\,i}\,\rangle\,. \tag{3.31}$$

Contracting the second of these equations with $V_a|M|^{-2}$ we get

$$\langle\,\mathcal{E}^a \mid \mathcal{V}^{*\,i}\,\rangle\frac{M^{IJ}}{|M|} - \frac{1}{2\sqrt{2}}\delta^a{}_0\mathcal{E}^{iK[I}\mathcal{J}^{J]}{}_K = 0\,, \tag{3.32}$$

from which we get for the projections onto equations of motion of scalars in $N = 2$ hypermultiplets

$$\mathcal{E}^{i\,KL}\mathcal{J}^I{}_{[K}\tilde{\mathcal{J}}^J{}_{L]} = 0\,. \tag{3.33}$$

For the special cases $N = 5$ and 3 we can define the SU(N) duals of the scalar equations of motion:

$$\tilde{\mathcal{E}}_I \equiv \frac{1}{4!}\varepsilon_{IJKLM}\mathcal{E}^{JKLM}\,, \qquad \tilde{\mathcal{E}}^i{}_I \equiv \frac{1}{2}\varepsilon_{IJK}\mathcal{E}^{iJK}\,, \tag{3.34}$$

and we can rewrite eqs. (3.9) and (3.10) in a more useful form:

$$\tilde{\mathcal{E}}_I \mathcal{J}^I{}_J = 0\,, \tag{3.35}$$

$$\tilde{\mathcal{E}}^i{}_I \mathcal{J}^I{}_J = 0\,. \tag{3.36}$$

Thus, in all cases the Einstein equations $\mathcal{E}^{0m}, \mathcal{E}^{mn}$, the Maxwell equations and Bianchi identities $\mathcal{E}^m$ and the scalar equations $\mathcal{E}^{i\,KL}\mathcal{J}^I{}_{[K}\tilde{\mathcal{J}}^J{}_{L]}$ and $\mathcal{E}^{MNPQ}\,\mathcal{J}^{[I}{}_M\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^{L]}{}_Q$ are automatically satisfied; the Einstein equation $\mathcal{E}^{00}$ and the scalar equations $\mathcal{E}^{i\,KL}\mathcal{J}^I{}_{[K}\mathcal{J}^J{}_{L]}$ and $\mathcal{E}^{MNPQ}\,\mathcal{J}^{[I}{}_M\mathcal{J}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^{L]}{}_Q$ are satisfied if the 0^{th} component of the Maxwell equations and Bianchi identities $\mathcal{E}^0$ are satisfied. To check that all the scalar equations of motion are, therefore, satisfied if $\mathcal{E}^0$ are, it is convenient to make a detailed analysis case by case.

$N = 2$:: As mentioned before, eq. (3.27) relates the complete scalar equations of motion to the 0^{th} component of the Maxwell equations an Bianchi identities. Therefore, we only need to solve $\mathcal{E}^0 = 0$.

$N = 3$:: The KSIs eqs. (3.32) and (3.36) can be combined into

$$\tilde{\mathcal{E}}^i{}_I = -2\sqrt{2}\frac{\tilde{M}_I}{|M|}\langle\,\mathcal{E}^0 \mid \mathcal{V}^{*\,i}\,\rangle\,, \tag{3.37}$$

and we conclude that, as in the $N = 2$ case, the only equation that needs to be solved is $\mathcal{E}^0 = 0$.

$N = 4$:: As mentioned before, eq. (3.22) relates the complete scalar equation $\mathcal{E}^{IJKL}$ to $\mathcal{E}^0$ because in the $N = 4$ case $\mathcal{E}^{IJKL} = \varepsilon^{IJKL}\mathcal{E}$, where $\mathcal{E}$ is the equation of motion of the complex scalar parametrizing $\mathrm{Sl}(2,\mathbb{R})/\mathrm{SO}(2)$. More explicitly, we have

$$\mathcal{E} = -\sqrt{2}\frac{\tilde{M}_{IJ}}{|\tilde{M}|}\langle\,\mathcal{E}^0 \mid \mathcal{V}^{*\,IJ}\,\rangle\,. \tag{3.38}$$

From eq. (3.32) and its SU(4) dual, using the $N = 4$ constraint $\mathcal{E}^{iIJ} = \frac{1}{2}\varepsilon^{IJKL}\mathcal{E}_{iKL}$ we arrive at the $N = 4$-specific KSI

$$\mathcal{E}_{iIJ} = -2\sqrt{2}\left\{\frac{\tilde{M}_{IJ}}{|\tilde{M}|}\langle\,\mathcal{E}^0 \mid \mathcal{V}^{*\,i}\,\rangle + \frac{M_{IJ}}{|M|}\langle\,\mathcal{E}^0 \mid \mathcal{V}_i\,\rangle\right\}\,, \tag{3.39}$$

which guarantees that, as in the foregoing cases, the matter scalar equations of motion are satisfied if $\mathcal{E}^0 = 0$ is satisfied.

$N = 5$:: In this case we have to consider the SU(5) dual of eqs. (3.22) and (3.35) which can be combined into the single identity

$$\tilde{\mathcal{E}}_I = -\sqrt{2}\frac{\tilde{M}_{IJK}}{|M|}\langle\,\mathcal{E}^0 \mid \mathcal{V}^{*\,JK}\,\rangle\,, \tag{3.40}$$

which leads us to the same conclusion as in the previous cases.

$N=6$:: In this case we have to consider the KSIs (3.22) involving $\mathcal{E}^{IJKL}$ and (3.32), involving $\mathcal{E}^{IJ}$ plus the constraint relating these equations of motion: $\mathcal{E}^{IJKL} = \frac{1}{2}\varepsilon^{IJKLMN}\mathcal{E}_{MN}$. Expressing both KSIs in terms of $\mathcal{E}^{IJ}$ only, we can combine them into

$$\mathcal{E}^{IJ} = -2\sqrt{2}\frac{M^{IJ}}{|M|}\langle\, \mathcal{E}^0 \mid \mathcal{V}^* \,\rangle - \sqrt{2}\frac{\tilde{M}^{IJKL}}{|M|}\langle\, \mathcal{E}^0 \mid \mathcal{V}_{KL} \,\rangle\,, \tag{3.41}$$

which brings us to the same conclusion as before.

$N=8$:: The KSI (3.22) plus the constraint $\mathcal{E}^{IJKL} = \frac{1}{4!}\varepsilon^{IJKLMNPQ}\mathcal{E}_{MNPQ}$ result in the KSI

$$\mathcal{E}^{IJKL} = 12\sqrt{2}\left\{\frac{M^{[IJ|}}{|M|}\langle\, \mathcal{E}^0 \mid \mathcal{V}^{*\,|KL]} \,\rangle + \frac{1}{12}\frac{\tilde{M}^{IJKLMN}}{|M|}\langle\, \mathcal{E}^0 \mid \mathcal{V}_{MN} \,\rangle\right\}. \tag{3.42}$$

In all cases the equations of motion of the scalars are automatically satisfied if the 0^{th} component of the Maxwell equations and Bianchi identities are. This will simplify the task of finding supersymmetric solutions enormously as there is only one independent symplectic vector of equations $\mathcal{E}^0$. On the other hand, to check consistency, we have to check that all the supersymmetric configurations satisfy the above KSIs.

4 $N \geq 2, d=4$ Killing spinor equations for the bilinears

The supersymmetry rules in section (3) induce differential relations between the spinor-bilinears, defined in section (D), and the various supergravity fields. As such, these relations contain the local information of the supersymmetric configurations and the solutions and are therefore the starting point in the deductive reconstruction process of the supergravity fields from the KSEs. We start this process by enumerating said differential relations.

From eq. (3.1) we get

$$\mathfrak{D}_\mu M_{IJ} - 2iT_{K[I|}{}^{+}{}_{\mu\nu}V^{K}{}_{|J]}{}^{\nu} = 0\,, \tag{4.1}$$

$$\mathfrak{D}_\mu V^{I}{}_{J\,\nu} + i\left\{\left[M^{IK}T_{JK}{}^{+}{}_{\mu\nu} - \text{h.c.}\right] - \left[\Phi^{IK}{}_{(\mu|}{}^{\rho}T_{KJ}{}^{+}{}_{|\nu)\rho} - \text{h.c.}\right]\right\} = 0\,. \tag{4.2}$$

From eq. (3.2) we get

$$M^{KL}P_{KLIJ\mu} + 6iT_{[IJ|}{}^{+}{}_{\mu\nu}V^{K}{}_{|K]}{}^{\nu} = 0\,, \tag{4.3}$$

$$P_{IJKL}\cdot V^{L}{}_{M} - \frac{3i}{2}T_{[IJ}{}^{+}\cdot\Phi_{K]M} = 0\,. \tag{4.4}$$

From eq. (3.3) we get

$$M^{IJ}P_{iIJ\mu} + 2iT_i{}^{+}{}_{\mu\nu}V^{\nu} = 0\,, \tag{4.5}$$

$$P_{iIJ}\cdot V^{J}{}_{K} - \frac{i}{2}T_i{}^{+}\cdot\Phi_{IK} = 0\,. \tag{4.6}$$

From eq. (3.4), which is only to be considered for $N=5$, we obtain

$$N=5 :: \qquad P_{[IJKL}\cdot V^{N}{}_{M]} = 0\,, \tag{4.7}$$

$$N=5 :: \qquad P_{[IJKL|\,\mu}M_{|M]N} = 0\,. \tag{4.8}$$

The last equation can be written as

$$N=5 :: \qquad \tilde{P}^{I}{}_{\mu}\, \mathcal{J}_{I}{}^{J} = 0\,, \tag{4.9}$$

where we have used the dual Vielbein $\tilde{P}^{I}{}_{\mu} = \frac{1}{4!}\varepsilon^{IJKLM}P_{JKLM\,\mu}$.

As was said before, in the case of $N=3$ we must also take into account eq. (3.5), which leads to

$$N=3 :: \qquad P_{i[IJ}\cdot V^{L}{}_{K]} = 0\,, \tag{4.10}$$

$$N=3 :: \qquad P_{i[IJ|\,\mu}M_{|K]L} = 0\,. \tag{4.11}$$

As in the $N=5$ case, we can use the dual Vielbein $\tilde{P}^{iI}{}_{\mu} = \frac{1}{2}\varepsilon^{IJK}P_{iJK\,\mu}$ to rewrite the last equations as

$$N=3 :: \qquad \tilde{P}^{iI}{}_{\mu}\, \mathcal{J}_{I}{}^{J} = 0\,. \tag{4.12}$$

4.1 First consequences

Having enumerated the differential relations, we start the analysis by expanding eq. (4.3), as to obtain

$$M^{KL}P_{KLIJ\mu} + 2iT_{IJ}{}^{+}{}_{\mu\nu}V^{\nu} + 4iT_{K[I|}{}^{+}{}_{\mu\nu}V^{K}{}_{|J]}{}^{\nu} = 0\,. \tag{4.13}$$

Substituting eq. (4.1) in the last term, we get

$$C_{IJ}{}^{+}{}_{\mu} \equiv V^{\nu}T_{IJ}{}^{+}{}_{\nu\mu} = -\frac{i}{2}M^{KL}P_{KLIJ\mu} - i\mathfrak{D}_{\mu}M_{IJ}\,, \tag{4.14}$$

from which we can find $T_{IJ}{}^{+}$ by means of the following relation that holds in the timelike case

$$T_{IJ}{}^{+} = V^{-2}[\hat{V}\wedge C_{IJ}{}^{+} + i\star(\hat{V}\wedge C_{IJ}{}^{+})]\,. \tag{4.15}$$

Likewise from eq. (4.5) we deduce

$$C_{i}{}^{+}{}_{\mu} \equiv V^{\nu}T_{i}{}^{+}{}_{\nu\mu} = -\frac{i}{2}M^{IJ}P_{iIJ\mu} \quad\longrightarrow\quad T_{i}{}^{+} = V^{-2}[\hat{V}\wedge C_{i}{}^{+} + i\star(\hat{V}\wedge C_{i}{}^{+})]\,. \tag{4.16}$$

Eqs. (4.14), (4.16) and (A.20) can then be used to find the complete field strengths, i.e.

$$\begin{aligned} C^{\Lambda+}{}_{\mu} \equiv V^{\nu}F^{\Lambda+}{}_{\nu\mu} &= \frac{i}{2}f^{*\Lambda IJ}C_{IJ}{}^{+}{}_{\mu} + if^{*\Lambda i}C_{i}{}^{+}{}_{\mu} \\ &= \frac{1}{4}M^{IJ}f^{*\Lambda KL}P_{IJKL\mu} + \frac{1}{2}M^{IJ}f^{*\Lambda i}P_{iIJ\mu} + \frac{1}{2}f^{*\Lambda IJ}\mathfrak{D}_{\mu}M_{IJ} \\ &= \frac{1}{2}M^{IJ}\mathfrak{D}_{\mu}f^{\Lambda}{}_{IJ} + \frac{1}{2}f^{*\Lambda IJ}\mathfrak{D}_{\mu}M_{IJ}\,, \end{aligned} \tag{4.17}$$

and

$$F^{\Lambda+} = V^{-2}[\hat{V}\wedge C^{\Lambda+} + i\star(\hat{V}\wedge C^{\Lambda+})]\,. \tag{4.18}$$

The trace over I over J in eq. (4.2) gives

$$\nabla_{\mu}V_{\nu} + i\left[M^{IJ}T_{IJ}{}^{+}{}_{\mu\nu} - \text{c.c.}\right] = 0\,, \tag{4.19}$$

which implies that V^μ is always a Killing vector

$$\nabla_{(\mu} V_{\nu)} = 0\,, \tag{4.20}$$

and that, had we been dealing with the null case ($M_{IJ} = 0$), it would have been covariantly constant.

Considering the equations involving the Vielbeine for each value of N, we can derive the general result

$$V^\mu P_{IJKL\mu} = V^\mu P_{iIJ\mu} = 0\,. \tag{4.21}$$

The first of these equations together with the expression for $T_{IJ}{}^{+}{}_{\mu\nu}V^\nu$, eq. (4.15), implies

$$V^\mu \mathfrak{D}_\mu M_{IJ} = 0\,. \tag{4.22}$$

4.2 Timelike case

We define the time coordinate t by

$$V^\mu \partial_\mu \equiv \sqrt{2}\partial_t\,, \tag{4.23}$$

which implies that all the fields are (covariantly) time-independent. Taking into account that $V^2 = 2|M|^2$ and the above choice of coordinate, $\hat{V}$ must take the form

$$\hat{V} \equiv V_\mu dx^\mu = \sqrt{2}|M|^2(dt + \omega) \tag{4.24}$$

where $\omega = \omega_{\underline{m}} dx^m$ is a time-independent 1-form to be determined. We can use the 1-form $\hat{V}$ to construct the 0^{th} component of a Vielbein basis $\{e^a\}$

$$e^0 \equiv \frac{1}{\sqrt{2}}|M|^{-1}\hat{V}\,. \tag{4.25}$$

The other three 1-forms of the basis $\{e^1, e^2, e^3\}$ will be chosen arbitrarily.[14] In general none of the remaining vector bilinears is an exact 1-form: with the available information we can only say that the 4-dimensional metric takes the form

$$ds^2 \;=\; |M|^2(dt + \omega)^2 - |M|^{-2}\gamma_{\underline{mn}} dx^m dx^n\,, \tag{4.26}$$

where the 3-dimensional metric $\gamma_{\underline{mn}}$ also has to be determined. The 1-forms $\hat{V}^m$ defined in eq. (D.26) can be taken as Dreibeine for the metric $\gamma_{\underline{mn}}$. We are going to derive from eq. (4.2), which contains a great deal of information, equations for $\hat{V}$, $\hat{V}^m$ and the matrices $(\sigma^m)^I{}_J$, defined in eq. (D.27), that will determine ω and $\gamma_{\underline{mn}}$.

Using the decompositions (D.28), (D.21) and the expression for the graviphotons field strengths, eq. (4.15), in eq. (4.2) we get

$$d\hat{V} + |M|^{-2}\left\{\hat{V} \wedge d|M|^2 + i\star\left[\hat{V} \wedge (M^{IJ}\mathfrak{D}M_{IJ} - M_{IJ}\mathfrak{D}M^{IJ})\right]\right\} = 0\,, \tag{4.27}$$

[14] It is worth stressing the differences with the procedure followed in the $N = 2$ case in ref. [18]: in the $N = 2$ case one can use the well-known constant Pauli matrices and construct $\{e^1, e^2, e^3\}$ decomposing the vector bilinear $V^I{}_{J\mu}$ with respect to $\{\sigma^1, \sigma^2, \sigma^3\}$. In the general case there are a priori no constant $N \times N$ Pauli matrices available and we are forced to choose $\{e^1, e^2, e^3\}$ first, and then use them to construct the $N \times N$ Pauli matrices, which generically will be non-constant: see appendix D for more detail.

$$d\hat{V}^m + \frac{1}{2}\mathrm{Tr}\,(\sigma^m \mathfrak{D}\sigma^n) \wedge \hat{V}^n = 0\,, \tag{4.28}$$

$$\mathfrak{D}_m\sigma^n + \mathfrak{D}_n\sigma^m = 0\,, \tag{4.29}$$

$$\varepsilon_{mnp}\left[\mathfrak{D}_n\sigma^p + \frac{1}{2}\mathrm{Tr}\,(\sigma^p\mathfrak{D}_n\sigma^q)\,\sigma^q\right] - i\,(\mathfrak{D}_m\mathcal{J}\mathcal{J} - \mathcal{J}\mathfrak{D}_m\mathcal{J}) = 0\,, \tag{4.30}$$

$$\mathfrak{D}_m\mathcal{J}^I{}_J + 2i|M|^{-2}\varepsilon_{mnp}\left[\mathfrak{D}_n M_{JK}(\sigma^p)^K{}_L M^{LI} - \mathrm{h.c.}\right] = 0\,. \tag{4.31}$$

Observe that, even though the σ-matrices bear indices m, n and p, these indices are not tangent space indices and the covariant derivatives acting on them is the $\mathrm{U}(N)$ connection Ω only.

If we act with $\mathcal{J}^I{}_L$ on eq. (4.1) and use the expression for the graviphoton field strengths eq. (4.15) and the trace of eq. (4.29), we get $\mathcal{J}\mathfrak{D}\mathcal{J} = 0$, which together with its Hermitean conjugate imply the very important condition

$$\mathfrak{D}\mathcal{J} \;=\; 0\,. \tag{4.32}$$

This equation does not imply that it is possible to choose a gauge in which $d\mathcal{J} = 0$ because the theories we are considering are only invariant under global $\mathrm{U}(N)$ transformations and not under *arbitrary* gauge transformations (the connection Ω is a composite field). Nevertheless, observe that $\mathcal{J}$ is constant in the $\mathrm{U}(2)$ directions of the Killing spinors:

$$\mathcal{J}d\mathcal{J}\mathcal{J} = 0\,, \tag{4.33}$$

as follows from its idempotency $\mathcal{J}^2 = \mathcal{J}$. On the other hand, this condition will allow us to relate consistently each supersymmetric configuration to a truncation to an $N = 2$ theory with vector supermultiplets and hypermultiplets: $\mathcal{J}$ projects the $\mathrm{U}(N)$ space onto an $\mathrm{U}(2)$ subspace, which defines the associated $N = 2$ truncation. Using $\mathcal{J}$ we are going to be able to project the scalar Vielbeine P_{IJKL} and $P_{i\,IJ}$ onto scalar Vielbeine belonging to the vector supermultiplets or the hypermultiplets of the truncation.

The integrability condition of $\mathfrak{D}\mathcal{J} = 0$ is

$$[R(\Omega), \mathcal{J}] = 0\,, \tag{4.34}$$

which restricts the holonomy of the pullback of the connection of the scalar manifold to the group generated by the $\mathrm{U}(N)$ subalgebra that commutes with $\mathcal{J}$; this group is $\mathrm{U}(2)\otimes\mathrm{U}(N-2)$, the first factor being generated by $\{\mathcal{J}, \sigma^1, \sigma^2, \sigma^3\}$.

Since $R(\Omega)$ can be expressed in terms of the scalar Vielbeine using eq. (A.33), the above condition is a condition on the Vielbeine. Below, we are going to derive several conditions for the Vielbeine that will ensure that the above condition is satisfied.

Another important consequence of the condition $\mathfrak{D}\mathcal{J} = 0$ is

$$\mathfrak{D}M^{IJ} = |M|^{-2}M^{IJ}M_{KL}\mathfrak{D}M^{KL}\,, \tag{4.35}$$

which leads to relations such as

$$\mathfrak{D}M^{[IJ}\mathfrak{D}M^{K]L} = 0\,, \tag{4.36}$$

and solves eq. (4.31).

Let us continue by analyzing eq. (4.27): taking the exterior derivative of $\hat{V}$ in eq. (4.24) and comparing it with eq. (4.27) we find that

$$d\omega = \frac{i}{\sqrt{2}|M|^4} \star \left[(M^{IJ}\mathfrak{D}M_{IJ} - M_{IJ}\mathfrak{D}M^{IJ}) \wedge \hat{V} \right] , \tag{4.37}$$

which can be rewritten as an equation in the background of the 3-dimensional spatial metric:

$$(d\omega)_{mn} = -\frac{i}{|M|^4}\varepsilon_{mnp}(M^{IJ}\mathfrak{D}_p M_{IJ} - M_{IJ}\mathfrak{D}_p M^{IJ}) . \tag{4.38}$$

Using the symplectic vectors $\mathcal{R}$ and $\mathcal{I}$ defined in eq. (3.17) and the constraint $M^{[IJ}M^{K]L} = 0$, eq. (D.8), we find that

$$M^{IJ}\mathfrak{D}_m M_{IJ} - M_{IJ}\mathfrak{D}_m M^{IJ} = 2i|M|^4 \langle \mathcal{I} \mid \partial_m \mathcal{I} \rangle , \tag{4.39}$$

and then we can rewrite the equation for ω in terms of $\mathcal{I}$

$$(d\omega)_{mn} = 2\epsilon_{mnp}\langle \mathcal{I} \mid \partial^p \mathcal{I} \rangle , \tag{4.40}$$

and $|M|$ in terms of $\mathcal{R}$ and $\mathcal{I}$

$$|M|^{-2} = \langle \mathcal{R} \mid \mathcal{I} \rangle , \tag{4.41}$$

which are identical to the ones obtained in refs. [17, 75] for $N = 2$ theories coupled to vector multiplets and with the same integrability condition, namely

$$\langle \mathcal{I} \mid \nabla^2_{(3)} \mathcal{I} \rangle = 0 . \tag{4.42}$$

Let us now move on to eq. (4.28): it can be interpreted as Cartan's first structure equation for a torsionless connection $\varpi^{mn} = -\varpi^{nm}$ on the 3-dimensional space

$$d\hat{V}^m - \varpi^{mn} \wedge \hat{V}^n = 0 , \tag{4.43}$$

where the connection can be read off and is

$$\varpi^{mn} = -\frac{1}{2}\mathrm{Tr}\,[\sigma^m \mathfrak{D}\sigma^n] = i\varepsilon^{mnp}\mathrm{Tr}\,[\sigma^p \Omega] - \frac{1}{2}\mathrm{Tr}\,[\sigma^m d\sigma^n] . \tag{4.44}$$

This equation relates the spin connection of the 3-dimensional transverse space to the pullback of the connection of the scalar manifold. This spin connection is constrained by eq. (4.29): multiplying by σ^p and taking the trace, we find that

$$\varpi_{(mn)p} = 0 , \quad \Rightarrow \quad \varpi_{mnp} = \varpi_{[mnp]} , \tag{4.45}$$

which is a gauge condition associated to our choices.

Defining a new covariant derivative $\hat{\mathfrak{D}} = \mathfrak{D} + \varpi$, where ϖ^{mn} acts on the upper m, n indices of the σ matrices[15] we can rewrite now eqs. (4.29) and (4.30) in the combined form

$$\hat{\mathfrak{D}}_m \sigma^n = 0 . \tag{4.46}$$

[15] Explicitly, $\hat{\mathfrak{D}}_m \sigma^n \equiv \mathfrak{D}_m \sigma^n - \varpi_m{}^{np}\sigma^p$. We do not distinguish between upper and lower flat 3-dimensional indices.

The integrability condition of this equation relates the curvature 2-form of ϖ^{mn} to an $\mathfrak{su}(2)$ projection the curvature of the pullback of the connection of the scalar manifold Ω:

$$R^{mn}(\varpi) = i\varepsilon^{mnp}\,\mathrm{Tr}\,[\sigma^p R(\Omega)]\,. \tag{4.47}$$

If we compute the curvature $R^{mn}(\varpi)$ using eq. (4.44) we find on the r.h.s. the extra term

$$i\varepsilon^{mnp}\,\mathrm{Tr}\,[\mathcal{J}d\sigma^p\mathcal{J}\wedge\Omega]\,, \tag{4.48}$$

which must vanish for consistency. We are going to impose the condition

$$\mathcal{J}d\sigma^p\mathcal{J} = 0\,, \tag{4.49}$$

which says that the σ^m matrices are constant in the U(2) directions of the Killing spinors, just as $\mathcal{J}$. We have not found a better proof of this condition, but we shall see that it is the simplest condition that solves the KSEs.

Using eq. (A.33) we can rewrite eq. (4.47) in a form that can be compared directly with the SU(2) curvature and quaternionic structures of the quaternionic-Kähler manifold in which the scalars of $N = 2$ hypermultiplets live. Then eq. (4.47) relates the curvature of the spatial 3-dimensional metric γ with the SU(2) curvature of the hyperscalars, completely analogous to what happens in the $N = 2$ case with hypermultiplets [18]. To find the projections of the scalar Vielbeine that correspond to the hyperscalars in the associated $N = 2$ truncation defined by $\mathcal{J}$, we first use eqs. (4.47) and (A.33) to write the Ricci tensor of γ as

$$R(\gamma)_{mn} = -\frac{i}{N-2}\varepsilon^{npq}(\sigma^q)^I{}_J[P^{*\,JKLM}{}_{[m|}P_{IKLM\,|p]} + 2P^{*\,i\,JK}{}_{[m|}P_{i\,IK\,|p]}]\,. \tag{4.50}$$

Further identities are needed: using the decompositions (D.28), (D.21) and the time-independence of the scalars eq. (4.21) in eqs. (4.4) and (4.6), together with the expressions for the supergravity and matter vector field strengths eqs. (4.14)–(4.16), we get the following constraints on the scalar Vielbeine:

$$\left[P_{IJKL\,m} - 3|M|^{-2}M^{PQ}P_{PQ[IJ|\,m}M_{|K]L}\right](\sigma^m)^L{}_M = 0\,, \tag{4.51}$$

$$P_{i\,MN\,m}\left(\delta^{MN}{}_{IJ} - \mathcal{J}^M{}_{[I}\mathcal{J}^N{}_{J]}\right)(\sigma^m)^J{}_K = 0\,, \tag{4.52}$$

which can be rewritten in the form[16]

$$P_{IJKL\,m}\,\mathcal{J}^I{}_{[M}\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}(\sigma^m)^Q{}_R = 0\,, \tag{4.53}$$

$$P_{i\,IJ\,m}\,\mathcal{J}^I{}_{[K}\tilde{\mathcal{J}}^J{}_{L]}(\sigma^m)^L{}_M = 0\,. \tag{4.54}$$

Using them in the above equation, the Ricci tensor of γ takes the form[17]

$$\begin{aligned} R(\gamma)_{mn} = -\frac{1}{N-2}[&P_{IJKL\,(m|}\mathcal{J}^I{}_M\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_Q P^{*\,MNPQ}{}_{|n)} \\ &+ 2P_{i\,IJ\,(m|}\mathcal{J}^I{}_M\tilde{\mathcal{J}}^J{}_N P^{*\,i\,MN}{}_{|n)}]\,. \end{aligned} \tag{4.55}$$

[16] These equations should be compared with the conditions that supersymmetry imposes on the pullbacks of the quaternionic Vielbeine in $N = 2$ theories [18].

[17] For $N = 2$ the r.h.s. vanishes identically, as the formalism used only takes into account vector multiplets.

The hyperscalar Vielbeine in the associated $N = 2$ truncation are clearly identified in this expression. The conditions for a flat 3-dimensional metric, or said differently the *no-hypers* conditions, are therefore

$$P_{IJKL}\, \mathcal{J}^{I}{}_{[M}\tilde{\mathcal{J}}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]} = 0\,, \tag{4.56}$$

$$P_{i\,IJ}\, \mathcal{J}^{I}{}_{[M}\tilde{\mathcal{J}}^{J}{}_{N]} = 0\,. \tag{4.57}$$

5 Solving the KSEs

We have thus far obtained the following necessary conditions for a field configuration to admit at least one Killing spinor and to lie in the timelike class of solutions:

1. All the fields are time-independent and related to a complex, antisymmetric matrix M^{IJ} satisfying $M^{[IJ}M^{K]L} = 0$, from which we must construct the covariantly constant projection $\mathcal{J}^{I}{}_{J}$, and to generalized Pauli matrices $(\sigma^m)^I{}_J$ which must satisfy eqs. (D.30)–(D.37) and (4.49).

2. The scalars have to satisfy eqs. (4.53) and (4.54); in the special cases of $N = 3$ and 5 they further need to satisfy eqs. (4.9) and (4.12).

3. The vector field strengths are given in terms of the scalars and the matrix M^{IJ} by eqs. (4.14)–(4.16).[18]

4. The spacetime metric is of conforma-stationary form, eq. (4.26), where

   (a) The 1-form ω is related to the matrix M^{IJ}, the scalar fields (through the pullback of the scalar connection) and the 3-dimensional transverse metric γ_{mn} through eq. (4.40).

   (b) The 3-dimensional metric is related to the scalars and the generalized Pauli matrices by eq. (4.44) which relates its spin connection to an SU(2) projection of the pullback of the connection of the scalar manifold.

We are going to see that these necessary conditions are also sufficient: let us start by plugging our result for T_i eq. (4.16) into eq. (3.3), leading to

$$P_{iKL\,m}\gamma^m \left[\delta_I{}^K\epsilon^L - \frac{i}{\sqrt{2}}|M|^{-1}M^{KL}\gamma^0\epsilon_I\right] = 0\,. \tag{5.1}$$

Decomposing now

$$P_{iKL\,m} = P_{iMN\,m}\mathcal{J}^M{}_{[K}\mathcal{J}^N{}_{L]} + P_{iMN\,m}(\delta^{MN}{}_{KL} - \mathcal{J}^M{}_{[K}\mathcal{J}^N{}_{L]})\,, \tag{5.2}$$

we get

$$P_{iMN\,m}|M|^{-1}M^{MN}\gamma^m\left[|M|^{-1}M_{IL}\epsilon^L - \frac{i}{\sqrt{2}}\gamma^0\epsilon_I\right] + P_{iMN\,m}(\delta^{MN}{}_{KL} - \mathcal{J}^M{}_{[K}\mathcal{J}^N{}_{L]})\gamma^m\epsilon^L = 0\,. \tag{5.3}$$

Each of the two terms has to vanish separately because they depend on independent components of $P_{iIJ\,m}$. The first term can vanish in two different ways:

[18] Simpler expressions for the vector field strengths will be given in the next section.

1. $P_{iMN\,m}M^{MN} = 0$ (vanishing matter vector field strengths T_i (4.16)). In this case, the generic way to make the second term to vanish is to impose[19]

$$\Pi^{m\pm\,I}{}_J\epsilon^J \equiv \frac{1}{2}[\delta^I{}_J \pm \gamma^{0(m)}(\sigma^{(m)})^I{}_J]\epsilon^J = 0\,, \tag{5.4}$$

for each value of m for which $P_{iIJ\,m} \neq 0$ and then use eq. (4.54). The consistency of this condition for a given m requires[20]

$$(\delta^I{}_J - \mathcal{J}^I{}_J)\epsilon^J = 0\,, \tag{5.5}$$

which reduces the number of unbroken supersymmetries to just two (i.e. eight real independent supercharges), out of which only one half (i.e. $1/N$) survives the projection eq. (5.4) for one given value of m. If we have to impose another projector of the same kind, the number of unbroken supersymmetries is lowered by another factor of $1/2$. In the generic case we will have to impose all three projectors and the supersymmetry preserved is just one (i.e. $1/(4N)$ of the total).

If eq. (5.5) is satisfied and $P_{iMN\,m}(\delta^{MN}{}_{KL} - \mathcal{J}^M{}_{[K}\mathcal{J}^N{}_{L]})\mathcal{J}^L{}_J = 0$ (which is identical to the "no-hypers" condition eq. (4.57), we do not need to impose eq. (5.4), which is associated to the hypermultiplets in the associated $N = 2$ truncation. It is clear that the projected scalar Vielbeine $P_{iMN\,m}\mathcal{J}^M{}_{[K}\mathcal{J}^N{}_{L]}$ correspond to the complex scalar of the vector multiplets of the $N = 2$ truncation.

2. If $P_{iMN\,m}M^{MN} \neq 0$ then we have to impose

$$\epsilon_I + i\sqrt{2}|M|^{-1}M_{IJ}\gamma^0\epsilon^J = 0\,, \tag{5.6}$$

which is consistent only if eq. (5.5) is satisfied, which means that, generically, $1/(2N)$ of the total amount of available supercharges are preserved by this condition.

The second term vanishes when we impose again the generic condition eq. (5.4), which is compatible with eq. (5.6), and use eq. (4.54). Again, if eq. (4.57) is satisfied, the condition eq. (5.4) is unnecessary.

In the case of $N = 3$ supergravity we have to consider the KSE eq. (3.5), which is readily seen to be solved by the condition eq. (4.9). Observe that this condition automatically implies the "no-hypers" condition, in agreement with the absence of hypermultiplets in the truncations from $N = 3$ to $N = 2$. Therefore, in $N = 3$ supergravity the only projector that ever needs to be imposed on the Killing spinors is eq. (5.6).

Let us then consider the KSE eq. (3.2). Substituting our result for T_{IJ}, eqs. (4.14) and (4.15), we can immediately write it as

$$\begin{aligned}&\left[P_{IJKM\,m} - 3\left(|M|^{-2}M^{MN}P_{MN[IJ|\,m}M_{|K]L} + 2|M|^{-2}\mathfrak{D}_m M_{[IJ}M_{K]L}\right)\right]\gamma^m\epsilon^L + \\ &+3\left(|M|^{-2}M^{MN}P_{MN[IJ|\,m} + 2|M|^{-2}\mathfrak{D}_m M_{[IJ|}\right)\gamma^m\left(|M|^{-1}M_{|K]L}\epsilon^L - \frac{i}{\sqrt{2}}\gamma^0\epsilon_{|K]}\right) = 0\,.\end{aligned} \tag{5.7}$$

Again, we can distinguish two different cases:

[19] Compare this equation with eq. (4.35) of ref. [18].

[20] These projectors satisfy $(\Pi^{m\pm})^2 = \Pi^{m\pm} - \frac{1}{4}(1 - \mathcal{J})$ and $[\Pi^{m\pm}, \Pi^{n\pm}] = 0$.

1. $M^{MN}P_{MNIJ\,m} + 2\mathfrak{D}_m M_{IJ} = 0$, which implies the vanishing of the vector field strengths (4.14) in the graviton supermultiplet. In this case, the equation can generically be solved by imposing the projector eq. (5.4) on the Killing spinors and using the constraint eq. (4.53). If $P_{IJKM\,m}\mathcal{J}^M{}_L = 0$, equivalent in this case to the "no-hypers" condition eq. (4.56), then the condition eq. (5.5) suffices.

2. $M^{MN}P_{MNIJ\,m}+2\mathfrak{D}_m M_{IJ} \neq 0$: in this case we need to impose the projectors eq. (5.6) and, to cancel the first term we have to impose eq. (5.4) unless $P_{IJKL\,m}$ satisfies[21]

$$P_{IJKM\,m}\mathcal{J}^M{}_L - 3|M|^{-2}M^{MN}P_{MN[IJ|\,m}M_{K]L} = 0\,, \tag{5.8}$$

which implies the "no-hypers" condition eq. (4.56).

For $N = 5$ we also have to consider the KSE eq. (3.4): this equation is immediately solved by the condition eq. (4.8), or equivalently (4.9), which is a particular instance of eq. (5.8) implying once again the "no-hypers" condition (4.56). Therefore, in the $N = 5$ case we only need to impose the projection eq. (5.6).

Using the supersymmetry conditions that we have used to solve the previous KSEs plus $\mathfrak{D}\mathcal{J} = 0$, it is easy to see that the 0^{th} component of the KSE eq. (3.1) is satisfied, while the m^{th} component reduces to the equation in 3-dimensional transverse space

$$\mathfrak{D}_m\epsilon_I - |M|^{-2}\mathfrak{D}_m M_{IK}M^{JK}\epsilon_J = 0\,, \tag{5.9}$$

where

$$\mathfrak{D}_m\epsilon_I = \left(\partial_m + \frac{1}{4}\varpi_{mnp}\gamma^{np}\right)\epsilon_I - \Omega^J{}_I\epsilon_J = \partial_m\epsilon_I + \left[\pm\frac{i}{4}\varpi_{mnp}\varepsilon^{npq}(\sigma^q)^J{}_I - \Omega^J{}_I\right]\epsilon_J\,, \tag{5.10}$$

upon use of the condition eq. (5.4).[22]

From eqs. (4.44) and (4.49) we obtain

$$\pm\frac{i}{4}\varpi_{mnp}\varepsilon^{npq}\sigma^q = \mp\mathcal{J}\Omega\mathcal{J} \pm \frac{1}{2}\mathrm{Tr}\,[\mathcal{J}\Omega]\,, \tag{5.11}$$

and from $\mathfrak{D}\mathcal{J} = 0$ we get

$$\mathcal{J}\Omega\tilde{\mathcal{J}} = \mathcal{J}d\mathcal{J} = \frac{1}{4}(\mathcal{J}d\mathcal{J} + \sigma^m d\sigma^m)\,. \tag{5.12}$$

The second term in eq. (5.9) can be put in the form

$$\begin{aligned} |M|^{-2}\mathfrak{D}M_{IK}M^{JK}\epsilon_J &= \frac{1}{2}\left[\mathfrak{D}\mathcal{J}^J{}_I + |M|^{-2}\mathfrak{D}M_{MN}M^{MN}\mathcal{J}^J{}_I\right]\epsilon_J \\ &= \frac{1}{2}\left[2i\xi + \frac{1}{2}|M|^{-2}\partial|M|^2 - \mathrm{Tr}(\mathcal{J}\Omega)\right]\epsilon_J\,, \end{aligned} \tag{5.13}$$

where

$$\xi \equiv \frac{i}{4}|M|^{-2}(dM^{MN}M_{MN} - dM_{MN}M^{MN})\,. \tag{5.14}$$

[21] Here we have used eq. (4.36) to simplify the expression.

[22] Acting on this equation with the projector $\tilde{\mathcal{J}}^I{}_L$ we find the integrability condition $\mathfrak{D}\mathcal{J} = 0$.

Putting all this information together and choosing the upper sign so the terms $\mathrm{Tr}(\mathcal{J}\Omega_m)$ cancel, we can rewrite the reduced KSE using 3-dimensional differential forms as

$$d\hat{\epsilon} - \hat{\epsilon}\left[i\xi + \frac{1}{4}(\mathcal{J}d\mathcal{J} + \sigma^m d\sigma^m)\right] = 0\,, \tag{5.15}$$

where we have defined the U(N) row vector $\hat{\epsilon}_I \equiv |M|^{-1/2}\epsilon_I$. The integrability condition of this equation

$$\mathcal{J}\left[id\xi + \frac{1}{4}(d\mathcal{J}\wedge d\mathcal{J} + d\sigma^m \wedge d\sigma^m)\right] = 0\,, \tag{5.16}$$

is identically satisfied.[23]

This shows that the necessary conditions for supersymmetry enumerated at the beginning of this section are also sufficient. Furthermore, we have shown that the Killing spinors generically satisfy the condition eq. (5.5), which preserves $2/N$ supersymmetries; if the supergravity or matter vector field strengths are non-vanishing, then they also satisfy the condition eq. (5.6), which breaks a further 1/2 of the supersymmetries and, if one of the scalar Vielbein projections $P_{IJKL\,m}\mathcal{J}^I{}_M\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_Q$ or $P_{i\,IJ\,m}\mathcal{J}^I{}_M\tilde{\mathcal{J}}^J{}_N$ does not vanish, then the Killing spinor must satisfy one condition eq. (5.4) (with the upper sign only) for each value of m, each of which breaks the supersymmetry a further factor of 1/2 up to a maximum $1/(4N)$, which is the fraction of supersymmetry preserved by a generic configuration.

6 Equations of motion

The supersymmetric configurations found in the previous section do not necessarily satisfy all the equations of motion. In order to find supersymmetric solutions, we have seen in section 3 that it is enough to require that the supersymmetric configurations satisfy the 0^{th} components of the Maxwell equations and Bianchi identities because the rest of the equations of motion are then, according to the KSIs, automatically satisfied. In this section we are going to find the 0th component of the Maxwell equations and Bianchi identities and we will check that the KSIs are satisfied for the supersymmetric configurations that we have obtained. This will serve as a powerful cross-check of our results.

Let us start with the Maxwell equations and Bianchi identities: it is convenient to construct a symplectic vector of 2-forms $\mathcal{F}$ containing the field strengths F^Λ and their symplectic duals $\tilde{F}_\Lambda$, by $\mathcal{F}^T \equiv \left(F^\Lambda, \tilde{F}_\Lambda\right)$. The Bianchi identities and Maxwell equations can be written in the form $d\mathcal{F} = 0$.

The field strengths F^Λ can be easily deduced from the equations obtained in section (4.1) and read

$$F^\Lambda = F^{\Lambda+} + F^{\Lambda-} \equiv V^{-2}[\hat{V}\wedge E^\Lambda - \star(\hat{V}\wedge B^\Lambda)]\,, \tag{6.1}$$

where

$$\begin{aligned} E^\Lambda =& C^{\Lambda+} + C^{\Lambda+} = d(|M|^2\mathcal{R}^\Lambda)\,,\\ B^\Lambda =& -i(C^{\Lambda+} - C^{\Lambda+}) = -\frac{i}{2}\left\{M^{IJ}\mathfrak{D}f^\Lambda{}_{IJ} + f^{*\Lambda}{}_{IJ}\mathfrak{D}M^{IJ} - \text{c.c.}\right\}\,, \end{aligned} \tag{6.2}$$

[23] Here and in eq. (5.11) we have used $\mathcal{J}d\sigma^m\mathcal{J} = 0$.

Using the same results one can deduce

$$\tilde{F}_\Lambda = \mathcal{N}^*_{\Lambda\Sigma}F^{\Sigma+} + \mathcal{N}_{\Lambda\Sigma}F^{\Lambda-} \equiv V^{-2}[\hat{V}\wedge E_\Lambda - \star(\hat{V}\wedge B_\Lambda)]\,, \tag{6.3}$$

where

$$\begin{aligned} E_\Lambda &= \mathcal{N}^*_{\Lambda\Sigma}C^{\Sigma+} + \mathcal{N}_{\Lambda\Sigma}C^{\Lambda-} = d(|M|^2\mathcal{R}_\Lambda)\,, \\ B_\Lambda &= -i(\mathcal{N}^*_{\Lambda\Sigma}C^{\Sigma+} - \mathcal{N}_{\Lambda\Sigma}C^{\Lambda-}) = -\frac{i}{2}\left\{M^{IJ}\mathfrak{D}h_{\Lambda\,IJ} + h^*{}_{\Lambda\,IJ}\mathfrak{D}M^{IJ} - \text{c.c.}\right\}\,. \end{aligned} \tag{6.4}$$

Combining the two expressions one can see that the symplectic vector F is given by

$$\mathcal{F} \;=\; V^{-2}\left\{\hat{V}\wedge d(|M|^2\mathcal{R}) + \frac{i}{2}\left[\hat{V}\wedge\left(M^{IJ}\mathfrak{D}\mathcal{V}_{IJ} + \mathcal{V}^{*\,IJ}\mathfrak{D}M_{IJ} - \text{c.c.}\right)\right]\right\}\,. \tag{6.5}$$

Using the equation for ω (4.37) and $\mathfrak{D}\mathcal{J} = 0$, it can be rewritten in the form

$$\mathcal{F} \;=\; -\frac{1}{2}d(\mathcal{R}\hat{V}) - \frac{1}{2}\star(\hat{V}\wedge d\mathcal{I})\,, \tag{6.6}$$

The combined Maxwell equations and Bianchi identities (i.e. $d\mathcal{F} = 0$) then imply the equations

$$d\star(\hat{V}\wedge d\mathcal{I}) = 0\,, \tag{6.7}$$

which, can be rewritten in the form

$$\mathcal{E}^a = \frac{1}{\sqrt{2}}|M|\delta^a{}_0\nabla^2_{(3)}\mathcal{I} = 0\,, \tag{6.8}$$

in full agreement with the fact, derived from the KSIs, that the Maxwell and Bianchi equations only have nontrivial 0^{th} component.

To calculate $\mathcal{E}_{00}$ we need to use eq. (4.41) to express the second derivatives of $|M|$ in terms of symplectic sections. Then

$$-\nabla^2\langle\,\mathcal{R}\mid\mathcal{I}\,\rangle = 2\langle\,\nabla^2\mathcal{I}\mid\mathcal{R}\,\rangle + 2\langle\,\nabla_m\mathcal{I}\mid\nabla_m\mathcal{R}\,\rangle\,. \tag{6.9}$$

Using in the second term eq. (A.24) we find that

$$\begin{aligned} \mathcal{E}_{00} =& G_{00} + \frac{1}{24}\alpha_1 P^{*\,IJKL}{}_m P_{IJKLm} + \frac{1}{2}\alpha_2 P^{*\,iIJ}{}_m P_{iIJ\,m} - 8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_{0m}F^{\Sigma\,-}{}_{0m} \\ =& -2|M|^4\langle\,\nabla^2_{(3)}\mathcal{I}\mid\mathcal{R}\,\rangle + \frac{1}{2}|M|^2\Big[R(\gamma) + 6|M|^{-2}\Pi^{IJ}{}_{KL}\mathfrak{D}_m M_{IJ}\mathfrak{D}_m M^{KL} \\ & + \frac{1}{12}\alpha_1\left(\delta^{IJ}{}_{KL} - 6\alpha_1^{-1}|M|^{-2}M^{IJ}M_{KL}\right)P_{IJMN\,m}P^{*\,KLMN}{}_m \\ & + \alpha_2\left(\delta^{IJ}{}_{KL} - \alpha_2^{-1}|M|^{-2}M^{IJ}M_{KL}\right)P_{iIJ\,m}P^{*\,iKL}{}_m\Big]\,. \end{aligned} \tag{6.10}$$

It is straightforward to show that $\mathcal{E}_{0m} = 0$ identically, and, for simplicity, we compute

$$\begin{aligned} |M|^{-2}\left[\mathcal{E}_{mn} + \frac{1}{2}\delta_{mn}\mathcal{E}_\mu{}^\mu\right] =& -\frac{\sqrt{2}}{|M|^3}\langle\,\mathcal{E}^0\mid\mathcal{R}\,\rangle + R(\gamma)_{mn} - 2|M|^{-2}\Pi^{IJ}{}_{KL}\mathfrak{D}_{(m|}M_{IJ}\mathfrak{D}_{|n)}M^{KL} \\ & + \frac{1}{12}\alpha_1\left(\delta^{IJ}{}_{KL} - 6\alpha_1^{-1}|M|^{-2}M^{IJ}M_{KL}\right)P_{IJMN\;(m}P^{*\;KLMN}{}_{n)} \\ & + \alpha_2\left(\delta^{IJ}{}_{KL} - \alpha_2^{-1}|M|^{-2}M^{IJ}M_{KL}\right)P_{iIJ\;(m}P^{*\;iKL}{}_{n)}\,. \end{aligned} \tag{6.11}$$

Finally, from eqs. (4.14) and (4.16) we find that the scalar equations of motion are given by:

$N = 2$::

$$
\begin{aligned}
-|M|^{-2}\mathcal{E}^{iIJ} =& \mathfrak{D}_m P^{*\,iIJ}{}_m - 2|M|^{-2}\mathfrak{D}_m M^{IJ} M_{KL} P^{*\,iKL}{}_m \\
& - \frac{1}{2}|M|^{-2} P^{*\,iIJ\,A} P^{*\,jk}{}_A M^{KL} M^{MN} P_{jKL\,m} P_{kMN\,m} \,.
\end{aligned}
\tag{6.12}
$$

$N = 3$::

$$
-|M|^{-2}\mathcal{E}^{iIJ} = \mathfrak{D}_m P^{*\,iIJ}{}_m - 2|M|^{-2}\mathfrak{D}_m M^{IJ} M_{KL} P^{*\,iKL}{}_m \,, \tag{6.13}
$$

or, in terms of the dual variables

$$
-|M|^{-2}\tilde{\mathcal{E}}^i{}_I = \mathfrak{D}_m \tilde{P}^i{}_{I\,m} - 2|\tilde{M}|^{-2}\mathfrak{D}_m \tilde{M}_I \tilde{M}^J \tilde{P}^i{}_{J\,m} \,. \tag{6.14}
$$

$N = 4$::

$$
\begin{aligned}
-|M|^{-2}\mathcal{E}^{IJKL} =& \mathfrak{D}_m P^{*\,IJKL}{}_m - 12|M|^{-2} M_{MN} P^{*\,MN[IJ|}{}_m \mathfrak{D}_m M^{|KL]} \\
& - \frac{1}{2}|M|^{-2} P^{*\,IJKL\,A} P^{*\,ij}{}_A M^{MN} M^{PQ} P_{i\,MN\,m} P_{i\,PQ\,m} \,,
\end{aligned}
\tag{6.15}
$$

or

$$
-|M|^{-2}\mathcal{E} = \mathfrak{D}_m P_m - 2|M|^{-2} M_{IJ} \mathfrak{D}_m M^{IJ} P_m - \frac{1}{2}|M|^{-2} M^{MN} M^{PQ} P_{iMN\,m} P_{jPQ\,m} \,, \tag{6.16}
$$

and

$$
\begin{aligned}
-|M|^{-2}\mathcal{E}^{iIJ} =& \mathfrak{D}_m P^{*\,iIJ}{}_m - 2|M|^{-2}\left[\mathfrak{D}_m M^{IJ} + \frac{1}{2} M_{MN} P^{*\,MNIJ}{}_m\right] M_{KL} P^{*\,iKL}{}_m \\
& - |M|^{-2}\varepsilon^{IJKL}\left[\mathfrak{D}_m M_{KL} + \frac{1}{2} M^{MN} P_{MNKL\,m}\right] M^{PQ} P_{iPQ\,m} \,.
\end{aligned}
\tag{6.17}
$$

$N = 5$::

$$
-|M|^{-2}\mathcal{E}^{IJKL} = \mathfrak{D}_m P^{*\,IJKL}{}_m - 12|M|^{-2} M_{MN} P^{*\,MN[IJ|}{}_m \mathfrak{D}_m M^{|KL]} \,, \tag{6.18}
$$

or

$$
-|M|^{-2}\tilde{\mathcal{E}}_I = \mathfrak{D}_m \tilde{P}_{I\,m} - 2|M|^{-2}\mathfrak{D}_m \tilde{M}_{IJK} \tilde{M}^{JKL} P_{L\,m} \,. \tag{6.19}
$$

$N = 6$::

$$
\begin{aligned}
-|M|^{-2}\mathcal{E}^{IJKL} =& \mathfrak{D}_m P^{*\,IJKL}{}_m - 12|M|^{-2}\Big[M_{MN} P^{*\,MN[IJ|}{}_m \mathfrak{D}_m M^{|KL]} \\
& + \frac{1}{4} M_{MN} P^{*\,MN[IJ|}{}_m M_{OP} P^{*\,OP|KL]}{}_m \Big] \\
& - |M|^{-2}\varepsilon^{IJKLMN}\left[\mathfrak{D}_m M_{MN} + \frac{1}{2} M^{PQ} P_{PQMN\,m}\right] M^{RS} P_{RS\,m} \,,
\end{aligned}
\tag{6.20}
$$

or

$$
\begin{aligned}
-|M|^{-2}\mathcal{E}_{IJ} =& \mathfrak{D}_m P_{IJ\,m} - |M|^{-2}\left[\mathfrak{D}_m\tilde{M}_{IJKL} + \frac{1}{2}M_{IJ}P_{KL\,m}\right]\tilde{M}^{KLMN}P_{MN\,m} \\
&- 2|M|^{-2}\left[\mathfrak{D}_m M_{IJ} + \frac{1}{2}\tilde{M}_{IJKL}P^{*\,KL}{}_m\right]M^{RS}P_{RS\,m}\,,
\end{aligned}
\tag{6.21}
$$

and finally

$N=8$::

$$
\begin{aligned}
&-|M|^{-2}\mathcal{E}^{IJKL} = \mathfrak{D}_m P^{*\;IJKL}{}_m \\
&\quad - 12|M|^{-2}\left[M_{MN}P^{*\,MN[IJ|}{}_m\mathfrak{D}_m M^{|KL]} + \frac{1}{4}M_{MN}P^{*\,MN[IJ|}{}_m M_{OP}P^{*\,OP|KL]}{}_m\right] \\
&\quad - \frac{1}{2}|M|^{-2}\varepsilon^{IJKLMNPQ}\left[M^{RS}P_{RS[MN|\,m}\mathfrak{D}_m M_{|PQ]} + \frac{1}{4}M^{RS}P_{RS[MN|\,m}M^{TU}P_{TU|PQ]\,m}\right].
\end{aligned}
\tag{6.22}
$$

6.1 Checking the KSIs

Let us start by checking the KSI eq. (3.19). Substituting the above expression, we get

$$
\langle\, \nabla^2_{(3)}\mathcal{I} \mid \mathcal{I}\rangle = 0\,. \tag{6.23}
$$

The r.h.s. vanishes identically due to the integrability condition of the equation that defines the 1-form ω, eq. (4.42), whose existence is a necessary condition of supersymmetry.

To check the KSI eq. (3.16) we need to compute $\langle\, \mathcal{E}^0 \mid \mathcal{R}\,\rangle$:

$$
\langle\, \mathcal{E}^0 \mid \mathcal{R}\,\rangle = \frac{1}{\sqrt{2}}|M|^3\langle\, \nabla^2_{(3)}\mathcal{I} \mid \mathcal{R}\,\rangle\,. \tag{6.24}
$$

Comparing this with the expression for $\mathcal{E}_{00}$ given in eq. (6.10) we find that supersymmetry, requires the following relation between the curvature of the 3-dimensional space and the scalars

$$
\begin{aligned}
R(\gamma) =& -\frac{1}{12}\alpha_1\left(\delta^{IJ}{}_{KL} - 6\alpha_1^{-1}\mathcal{J}^I{}_K\mathcal{J}^I{}_L\right)P_{IJMN\,m}P^{*\,KLMN}{}_m \\
&- \alpha_2\left(\delta^{IJ}{}_{KL} - \alpha_2^{-1}\mathcal{J}^I{}_K\mathcal{J}^I{}_L\right)P_{iIJ\,m}P^{*\,iKL}{}_m\,,
\end{aligned}
\tag{6.25}
$$

a result we will comment upon shortly.

As for the KSI (3.15) we point out that, as we mentioned in the previous section, $\mathcal{E}_{0m}$ vanishes identically; from eq. (6.11) we see that $\mathcal{E}_{mn}$ vanishes if eq. (6.25) is satisfied and furthermore that

$$
\begin{aligned}
R(\gamma)_{mn} =& -\frac{1}{12}\alpha_1\left(\delta^{IJ}{}_{KL} - 6\alpha_1^{-1}\mathcal{J}^I{}_K\mathcal{J}^I{}_L\right)P_{IJMN\;(m}P^{*\;KLMN}{}_{n)} \\
&- \alpha_2\left(\delta^{IJ}{}_{KL} - \alpha_2^{-1}\mathcal{J}^I{}_K\mathcal{J}^I{}_L\right)P_{iIJ\;(m}P^{*\;iKL}{}_{n)}\,.
\end{aligned}
\tag{6.26}
$$

This is the only equation we really need to impose on the 3-dimensional metric as eq. (6.25) is nothing but its trace. One can show (case by case, for each N) that this expression is completely equivalent to eqs. (4.55), which are satisfied by the supersymmetric configurations.

We can then check those KSIs that relate the equations of motion of the scalars to the 0^{th} component of the Maxwell and Bianchi equations. It is convenient to first compute them for the result for a generic value of N, and then consider a specific value. For generic N one obtains

$$\begin{aligned}\langle\, \mathcal{E}^{0} \mid \mathcal{V}^{*\,i}\,\rangle =& \frac{1}{2\sqrt{2}}|M|\Big\{\mathfrak{D}_m P^{*\ iIJ}{}_m M_{IJ} - 2|M|^{-2} P^{*\ iIJ}{}_m M_{IJ} M_{KL} \mathfrak{D}_m M^{KL} \\ & - M^{IJ}\left[P_{jIJ\,m} P^{*\,ij}{}_m + \frac{1}{2} P_{IJKL\,m} P^{*\,iKL}{}_m\right]\Big\}\,.\end{aligned} \tag{6.27}$$

and

$$\begin{aligned}\langle\, \mathcal{E}^{0} \mid \mathcal{V}^{*\,IJ}\,\rangle =& \frac{1}{2\sqrt{2}}|M|\Big\{\mathfrak{D}_m P^{*\ IJKL}{}_m M_{KL} - 2|M|^{-2} \mathfrak{D}_m P^{*\ IJKL}{}_m M_{KL} M_{MN} \mathfrak{D}_m M^{MN} \\ & - \frac{1}{2} M^{MN}\left[P^{*\,IJKL}{}_m P_{KLMN\,m} + 2P^{*\,iIJ}{}_m P_{i\,MN\,m}\right]\Big\}\,.\end{aligned} \tag{6.28}$$

N = 2:: it is enough to check the KSI eq. (3.27) using the form of the equation of motion derived before eq. (6.12) being careful with the P^2 and P^4 terms. A detailed calculation shows that they cancel each other, in agreement with the results of ref. [17].

N = 3:: For the case $N = 3$ we have to check the KSI eq. (3.37) using the form of the equation of motion derived before eq. (6.13). Again, it is readily found to be satisfied by using the condition eq. (4.12) and the covariant constancy of $\mathcal{J}$.

N = 4:: For the case $N = 4$ we have to check the KSIs eqs. (3.38) and (3.39) using eqs. (6.16) and eq. (6.17) respectively. The first KSI is easily seen to be satisfied. The second KSI is satisfied up to a term of the form

$$\mathfrak{D}_m (P_{i\,MN\,m} \mathcal{J}^{M}{}_{[I} \tilde{\mathcal{J}}^{N}{}_{J]})\,, \tag{6.29}$$

which vanishes automatically after use of the constraint eqs. (4.54) and (4.46). This term can be seen as the equation of motion for the hypers of the associated $N = 2$ truncation and, as it happens in the $N = 2$ theory, it is automatically satisfied for the supersymmetric configurations independently of whether the Maxwell equations and Bianchi identities are satisfied or not.

N = 5:: For the case $N = 5$ we have to check the KSI eq. (3.40) using eq. (6.19). In this case the crucial property that makes it to be satisfied is eq. (4.9).

N = 6:: In the $N = 6$ case we find the the KSI eq. (3.41) is satisfied eq. (6.21) up to a term of the form eq. (6.29), which is also seen to vanish identically.

N = 8:: Finally, in the $N = 8$ case we find the the KSI eq. (3.42) is satisfied eq. (6.22) up to a term of the form

$$\mathfrak{D}_m (P_{IJKL\,m}\, \mathcal{J}^{I}{}_{[M} \tilde{\mathcal{J}}^{J}{}_{N} \tilde{\mathcal{J}}^{K}{}_{P} \tilde{\mathcal{J}}^{L}{}_{Q]})\,, \tag{6.30}$$

which vanishes upon use of eqs. (4.53) and (4.46).

In conclusion we see that the KSIs are always satisfied.

7 Conclusions

The results presented in this paper are a first step towards a full characterization of all the four-dimensional supersymmetric solutions preserving at least one supercharge. It is clear that further work is needed in order to make the general solutions presented here more explicit for each N: first of all, convenient parametrizations of the matrices M^{IJ} satisfying all the required properties (in particular all the supersymmetry constraints involving the projector $\mathcal{J}$) and general ways to construct the generalized Pauli matrices σ^m have to be found, the stabilization equations have to be solved (this is in general hard, and might prove impossible); furthermore, the scalar fields need to be resolved; the would-be vector-scalars should be resolved in terms of the harmonic functions and the would-be hyperscalars should be found the hard way by solving the relevant equations (4.53), (4.54) and their consistent interplay with the connection on the 3-dimensional base space, eq. (4.44). Only then will we have explicit expressions for the supersymmetric solutions. The problem is similar to, but definitely more involved than, finding supersymmetric solutions in $d = 4$ $N = 2$ supergravities coupled to vector and hypermultiplets [18]. A further issue that needs to be investigated and which does not arise in the $N = 2$ $d = 4$ case is the classification of supersymmetric solutions preserving more than the minimal amount of supersymmetry.

The supersymmetric black hole solutions of the 4-dimensional supergravities are a very interesting subclass of the supersymmetric solutions identified here. They are "hyper-less" (i.e. they have a flat 3-dimensional base space) solutions and, therefore, simpler to construct. The black-hole solutions of $N = 8$ are particularly interesting due to the possible ultraviolet-finiteness of the theory, e.g. [88]. There are many partial results in the literature [76–79] including very large families of solutions obtained via $N = 2$ truncations of the theory [67] but the derivation of a manifestly $E_{7(7)}$-invariant family of solutions on which the conjectures concerning the $E_{7(7)}$-invariant entropy formula [80] could be explicitly checked is highly desirable. Our results provide a starting point for this derivation [81].

The attractor mechanism [82–85] (see also the more recent reference [86]) has been one of the main tools for the study of supersymmetric black-hole solutions. Our results establish a clear distinction between the scalars which are driven by the electric and magnetic charges of the vector fields (which would belong to the would-be vector multiplets of the associated $N = 2$ truncation) and, therefore, subject to the attractor mechanism, and those that are not (which would belong to the would-be hypermultiplets of the associated $N = 2$ truncation). A simple derivation of the attractor flow equations for the first kind of scalars based on the general form of the solutions found here can be readily given [87].

Another interesting class of timelike supersymmetric solutions which deserves to be studied in more detail is the class of domain walls associated to the supersymmetry projectors $\Pi^{m\pm I}{}_J$ and, therefore, to the would-be hyperscalars of the associated $N = 2$ truncation.

Finally, to complete the program of characterizing all supersymmetric solutions, the supersymmetric solutions in the null class need to be identified. In the null class the $\mathrm{U}(N)$ R-symmetry group is broken to $\mathrm{U}(1) \times \mathrm{U}(N-1)$ and there is an "$N = 1$ truncation" associated to the $\mathrm{U}(1)$ subgroup [89]. The solutions will then be analogous to the super-

N	3	4	5	6	8
n	n	n	0	1	0
$\bar{n}$	$n+3$	$n+6$	10	16	28

Table 1. This table details, for a given N, the number of vector supermultiplets, n, and the integer $\bar{n}$ needed for an embedding into the symplectic formulation.

symmetric solutions of the ungauged $N = 1$ theories with no superpotential, classified in refs. [20] and [21], and include waves, strings and domain walls.

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A Generic scalar manifolds

All the scalar manifolds can be described by a $\mathrm{Usp}(\bar{n},\bar{n})$ matrix U which is constructed in terms of the matrices[24]

$$f \equiv ({f^{\Lambda}}_{IJ}, {f^{\Lambda}}_{i}) \,, \qquad h \equiv (h_{\Lambda\, IJ}, h_{\Lambda\, i}) \,, \tag{A.1}$$

where $I, J = 1, \ldots N$ are the graviton-supermultiplet, or equivalently $\mathrm{U}(N)$, indices and $i(= 1, \ldots n)$ are indices labeling the vector multiplets, and the embedding then imposes that $\bar{n} = n + N(N-1)/2$; this information is detailed in table 1.[25]

Using the above matrices one can then embed the generic scalar manifolds as

$$U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} f + ih & f^* + ih^* \\ f - ih & f^* - ih^* \end{pmatrix} . \tag{A.2}$$

The condition that $U \in \mathrm{Usp}(\bar{n},\bar{n})$

$$\begin{aligned} U^{-1} = & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^{\dagger} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U^{T} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ = & \frac{1}{\sqrt{2}} \begin{pmatrix} f^{\dagger} - ih^{\dagger} & -(f^{\dagger} + ih^{\dagger}) \\ -(f - ih) & f + ih \end{pmatrix} , \end{aligned} \tag{A.3}$$

[24] When we multiply these matrices we must include a factor 1/2 for each contraction of pairs of anti-symmetric indices IJ.

[25] Observe that $N = 6$ has $n = 1$, even though there are no vector supermultiplets in this case. This will be explained in appendix (B).

leads to the following conditions for f and h:

$$i(f^\dagger h - h^\dagger f) = 1\,, \qquad f^T h - h^T f = 0\,. \tag{A.4}$$

In terms of the symplectic vectors

$$\mathcal{V}_{IJ} = \begin{pmatrix} f^\Lambda{}_{IJ} \\ h_{\Lambda IJ} \end{pmatrix}\,, \qquad \mathcal{V}_i = \begin{pmatrix} f^\Lambda{}_i \\ h_{\Lambda\, i} \end{pmatrix}\,, \tag{A.5}$$

these constraints take the form[26]

$$\begin{aligned} \langle \mathcal{V}_{IJ} \mid \mathcal{V}^{*\,KL} \rangle &= -2i\delta^{KL}{}_{IJ}\,, \\ \langle \mathcal{V}_i \mid \mathcal{V}^{*\,j} \rangle &= -i\delta_i^{\,j}\,, \end{aligned} \tag{A.7}$$

with the rest of the symplectic products vanishing.

The left-invariant Maurer-Cartan 1-form can be split into the Vielbeine P and the connection Ω as follows:

$$\Gamma \equiv U^{-1}dU = \begin{pmatrix} \Omega & P^* \\ P & \Omega^* \end{pmatrix}\,. \tag{A.8}$$

Thus, the different components of the connection are

$$\Omega = \begin{pmatrix} \Omega^{KL}{}_{IJ} & \Omega^j{}_{IJ} \\ \Omega^{KL}{}_i & \Omega^j{}_i \end{pmatrix} = \begin{pmatrix} i\langle d\mathcal{V}_{IJ} \mid \mathcal{V}^{*\,KL}\rangle & i\langle d\mathcal{V}_{IJ} \mid \mathcal{V}^{*\,j}\rangle \\ i\langle d\mathcal{V}_i \mid \mathcal{V}^{*\,KL}\rangle & i\langle d\mathcal{V}_i \mid \mathcal{V}^{*\,j}\rangle \end{pmatrix}\,, \tag{A.9}$$

and those of the Vielbeine are

$$P = \begin{pmatrix} P_{KLIJ} & P_{jIJ} \\ P_{KLi} & P_{ij} \end{pmatrix} = \begin{pmatrix} -i\langle d\mathcal{V}_{IJ} \mid \mathcal{V}_{KL}\rangle & -i\langle d\mathcal{V}_{IJ} \mid \mathcal{V}_j\rangle \\ -i\langle d\mathcal{V}_i \mid \mathcal{V}_{KL}\rangle & -i\langle d\mathcal{V}_i \mid \mathcal{V}_j\rangle \end{pmatrix}\,. \tag{A.10}$$

The period matrix $\mathcal{N}_{\Lambda\Sigma}$ is defined by

$$\mathcal{N} = hf^{-1} = \mathcal{N}^T\,, \tag{A.11}$$

which implies properties which should be familiar from the $N = 2$ case: for instance

$$\mathfrak{D}h_\Lambda = \mathcal{N}^*_{\Lambda\Sigma}\mathfrak{D}f^\Lambda\,, \qquad h_\Lambda = \mathcal{N}_{\Lambda\Sigma}f^\Sigma\,, \tag{A.12}$$

and

$$-\frac{1}{2}(\Im\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma} = \frac{1}{2}f^\Lambda{}_{IJ}f^{*\Sigma IJ} + f^\Lambda{}_i f^{*\Sigma\, i}\,, \tag{A.13}$$

which can be derived from the definition of $\mathcal{N}$ and eq. (A.4).

We also quote the completeness relation

$$\frac{1}{2} \mid \mathcal{V}_{IJ}\rangle\langle \mathcal{V}^{*\,IJ} \mid \; -\frac{1}{2} \mid \mathcal{V}^{*\,IJ}\rangle\langle \mathcal{V}_{IJ} \mid \; + \mid \mathcal{V}_i\rangle\langle \mathcal{V}^{*\,i} \mid \; - \mid \mathcal{V}^{*\,i}\rangle\langle \mathcal{V}_i \mid = i\,. \tag{A.14}$$

[26]We use the convention

$$\langle \mathcal{A} \mid \mathcal{B}\rangle \equiv \mathcal{B}^\Lambda\mathcal{A}_\Lambda - \mathcal{B}_\Lambda\mathcal{A}^\Lambda\,. \tag{A.6}$$

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Defining the $H_{\rm Aut}\times H_{\rm Matter}$ covariant derivative according to

$$\mathfrak{D}\mathcal{V}=d\mathcal{V}-\mathcal{V}\Omega\,, \tag{A.15}$$

and using eq. (A.12) we obtain from (A.9)

$$\Omega^{KL}{}_{i}=\Omega^{j}{}_{IJ}=0\,, \tag{A.16}$$

and from (A.10)

$$P_{IJKL} = -2f^{\Lambda}{}_{IJ}\Im{\rm m}\mathcal{N}_{\Lambda\Sigma}\,\mathfrak{D}f^{\Sigma}{}_{KL}\,, \tag{A.17}$$
$$P_{iIJ} = -2f^{\Lambda}{}_{i}\Im{\rm m}\mathcal{N}_{\Lambda\Sigma}\,\mathfrak{D}f^{\Sigma}{}_{IJ}\,, \tag{A.18}$$
$$P_{ij} = -2f^{\Lambda}{}_{i}\Im{\rm m}\mathcal{N}_{\Lambda\Sigma}\,\mathfrak{D}f^{\Sigma}{}_{j}\,. \tag{A.19}$$

The above equation can be inverted to give

$$\mathfrak{D}f^{\Lambda}{}_{IJ} = f^{*\,\Lambda i}P_{iIJ}+\frac{1}{2}f^{*\Lambda KL}P_{IJKL}\,, \tag{A.20}$$
$$\mathfrak{D}f^{\Lambda}{}_{i} = f^{*\,\Lambda j}P_{ij}+\frac{1}{2}f^{*\Lambda IJ}P_{iIJ}\,, \tag{A.21}$$

using eq. (A.13).

The definition of the covariant derivative leads to the identities

$$\langle\,\mathfrak{D}\mathcal{V}\mid\mathcal{V}^*\,\rangle=0\,,\qquad \langle\,\mathfrak{D}\mathcal{V}\mid\mathcal{V}\,\rangle=\langle\,d\mathcal{V}\mid\mathcal{V}\,\rangle=iP\,. \tag{A.22}$$

The inverse Vielbeine $P^{*\,IJKL}, P^{*\,iIJ}, P^{*\,ij}$, satisfy (here A labels the physical fields)

$$P^{*\,IJKL\,A}P_{MNOP\,A}=4!\delta^{IJKL}{}_{MNOP}\,,\qquad P^{*\,iIJ\,A}P_{jKL\,A}=2\delta^{i}{}_{j}\delta^{IJ}{}_{KL}\,. \tag{A.23}$$

Their crossed products vanish but their products with $P_{ij\,A}$ do not.

We find

$$\langle\,\mathfrak{D}_A\mathcal{V}_{IJ}\mid\mathfrak{D}_B\mathcal{V}^{*\;KL}\,\rangle = \frac{i}{2}P_{IJMNA}P^{*\;KLMN}{}_B+iP_{iIJA}P^{*\;iKL}{}_B\,, \tag{A.24}$$
$$\langle\,\mathfrak{D}_A\mathcal{V}_{IJ}\mid\mathfrak{D}_B\mathcal{V}^{*\;i}\,\rangle = \frac{i}{2}P_{IJKLA}P^{*\;iKL}{}_B+iP_{jIJA}P^{*\;ij}{}_B\,, \tag{A.25}$$
$$\langle\,\mathfrak{D}_A\mathcal{V}_{i}\mid\mathfrak{D}_B\mathcal{V}^{*\;j}\,\rangle = \frac{i}{2}P_{iIJA}P^{*\;iIJ}{}_B+iP_{ikA}P^{*\;jk}{}_B\,, \tag{A.26}$$

while $\langle\,\mathfrak{D}_A\mathcal{V}_{IJ}\mid\mathfrak{D}_B\mathcal{V}_{KL}\,\rangle=\langle\,\mathfrak{D}_A\mathcal{V}_{IJ}\mid\mathfrak{D}_B\mathcal{V}_{i}\,\rangle=\langle\,\mathfrak{D}_A\mathcal{V}_{i}\mid\mathfrak{D}_B\mathcal{V}_{j}\,\rangle=0$.

Using the definition of the period matrix eq. (A.11), equation (A.12) and the first of eqs. (A.4) we get

$$d\mathcal{N}=4i\Im{\rm m}\mathcal{N}\,\mathfrak{D}f\,f^{\dagger}\,\Im{\rm m}\mathcal{N}\,. \tag{A.27}$$

This expression can be expanded in terms of the Vielbeine, using eqs. (A.20) and (A.21)

$$d\mathcal{N}_{\Lambda_\Sigma}=i\Im{\rm m}\mathcal{N}_{\Gamma(\Lambda}\Im{\rm m}\mathcal{N}_{\Sigma)\Omega}\left[P_{IJKL}f^{*\,\Gamma IJ}f^{*\,\Omega KL}+4P_{iIJ}f^{*\,\Gamma i}f^{*\,\Omega IJ}+4P_{ij}f^{*\,\Gamma i}f^{*\,\Omega j}\right]\,. \tag{A.28}$$

$N=3$								
	${e^a}_\mu$	$\psi_{I\mu}$	${A^{IJ}}_\mu$	χ_{IJK}	${A^i}_\mu$	λ_{iI}	λ_{iIJK}	$P_{iIJ\mu}$
$\sharp$	1	3	3	1	n	$3n$	n	$(3+3)n$

Table 2. The field content of the $N=3$ supergravity multiplet, first 4 entries, and the n vector supermultiplets.

and, using eqs. (A.23) and taking into account that their contraction with P_{ij} does not necessarily vanish, implies

$$P^{*\,IJKL\,A}\frac{\partial}{\partial\phi^A}\mathcal{N}_{\Lambda\Sigma} = 4! i \Im\mathrm{m}\mathcal{N}_{\Omega(\Lambda}\Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta} f^{*\,\Omega[IJ|} f^{*\,\Delta|KL]}\,, \tag{A.29}$$

$$P^{*\,iIJ\,A}\frac{\partial}{\partial\phi^A}\mathcal{N}_{\Lambda\Sigma} = 8 i \Im\mathrm{m}\mathcal{N}_{\Omega(\Lambda}\Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta} f^{*\,\Omega i} f^{*\,\Delta IJ}\,. \tag{A.30}$$

$$P^{*\,IJKL\,A}\frac{\partial}{\partial\phi^A}\mathcal{N}^*_{\Lambda\Sigma} = -4 i \Im\mathrm{m}\mathcal{N}_{\Omega(\Lambda}\Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta} P^{*\,IJKL\,A} {P^{*\,ij}}_A {f^\Omega}_i {f^\Delta}_j\,, \tag{A.31}$$

$$P^{*\,iIJ\,A}\frac{\partial}{\partial\phi^A}\mathcal{N}^*_{\Lambda\Sigma} = -4 i \Im\mathrm{m}\mathcal{N}_{\Omega(\Lambda}\Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta} P^{*\,iIJ\,A} {P^{*\,jk}}_A {f^\Omega}_i {f^\Delta}_j\,. \tag{A.32}$$

Using the Maurer-Cartan equations $d\Gamma+\Gamma\wedge\Gamma=0$ and direct calculations we find that the curvatures of ${\Omega^{KL}}_{IJ}$ and ${\Omega^j}_i$ are

$$\begin{aligned}
{R^{KL}}_{IJ} &= d{\Omega^{KL}}_{IJ} + \tfrac{1}{2}{\Omega^{KL}}_{MN}\wedge{\Omega^{MN}}_{IJ} \\
&= -\tfrac{1}{2}P^{*KLMN}\wedge P_{MNIJ} - P^{*iKL}\wedge P_{iIJ} &\text{(A.33)}\\
&= -i\langle\,\mathfrak{D}\mathcal{V}_{IJ}\mid\mathfrak{D}\mathcal{V}^{*\,KL}\,\rangle\,, &\text{(A.34)}\\
{R^j}_i &= d{\Omega^j}_i + {\Omega^j}_k\wedge{\Omega^k}_i = -\tfrac{1}{2}P^{*\,jIJ}\wedge P_{iIJ} - P^{*ik}\wedge P_{ik} &\text{(A.35)}\\
&= -i\langle\,\mathfrak{D}\mathcal{V}_i\mid\mathfrak{D}\mathcal{V}^{*\,j}\,\rangle\,. &\text{(A.36)}
\end{aligned}$$

The vanishing of the curvature of ${\Omega^i}_{IJ}$ leads to

$$\frac{1}{2}P_{IJKL}\wedge P^{*\,iKL} + P_{jIJ}\wedge P^{*\,ij} = -i\langle\,\mathfrak{D}\mathcal{V}_{IJ}\mid\mathfrak{D}\mathcal{V}^{*\,i}\,\rangle = 0\,. \tag{A.37}$$

B Generic $N\geq 2$, $d=4$ multiplets

In this section we will spill out the field content of the relevant graviton- and vector-supermultiplet[27] by specifying said field content in tables 2–6 and discussing briefly the possible constraints that apply for each individual case.

In order to recover the $N=4$ field content we have to impose

$$N=4 :: \qquad P_{i\,IJ} = \frac{1}{2}\varepsilon_{IJKL}P^{*\,i\,KL}\,, \tag{B.1}$$

$$N=4 :: \qquad \lambda_{i\,I} = \frac{1}{3!}\varepsilon_{IJKL}\lambda^{i\,JKL}\,. \tag{B.2}$$

$N=4$									
	${e^a}_\mu$	$\psi_{I\mu}$	${A^{IJ}}_\mu$	χ_{IJK}	$P_{IJKL\mu}$	${A^i}_\mu$	λ_{iI}	λ_{iIJK}	$P_{iIJ\mu}$
$\sharp$	1	4	6	4	1+1	n	$4n$	$4n$	$(6+6)n$

Table 3. The field content of the $N = 4$ supergravity multiplet, first 5 entries, and the n vector supermultiplets.

$N=5$						
	${e^a}_\mu$	$\psi_{I\mu}$	${A^{IJ}}_\mu$	χ_{IJK}	χ^{IJKLM}	$P_{IJKL\mu}$
$\sharp$	1	5	10	10	1	$5+5$

Table 4. The field content of the $N = 5$ supergravity multiplet.

$N=6$										
	${e^a}_\mu$	$\psi_{I\mu}$	${A^{IJ}}_\mu$	χ_{IJK}	χ^{IJKLM}	$P_{IJKL\mu}$	A	λ_I	λ_{IJK}	P_{IJ}
$\sharp$	1	6	15	20	6	$15+15$	1	6	20	$15+15$

Table 5. The field content of the $N = 6$ supergravity multiplet, first 5 entries, and the auxiliar vector supermultiplet.

The situation for the $N = 6$ case is a little bit more involved. In spite of the fact that for $N = 6$ there are no vector multiplets, the graviton multiplet is obtained from the "general case" eq. (2.1) coupling an extra "vector multiplet". This is because the decomposition of $\mathrm{SO}^*(12)$ with respect to SU(6) produces a singlet (this is the "practical reason" why eq. (2.1) is not enough). The presence of the singlet comes together with the fact that $\mathrm{SO}^*(12)/\mathrm{U}(6)$ has a Special Geometry structure.

In order to recover the $N = 6$ field content we have to impose

$$N=6 :: \qquad \lambda_I = \frac{1}{5!}\varepsilon_{IJKLMN}\chi^{JKLMN}\,, \tag{B.3}$$

$$N=6 :: \qquad \chi_{IJK} = \frac{1}{3!}\varepsilon_{IJKLMN}\lambda^{LMN}\,, \tag{B.4}$$

$$N=6 :: \qquad P_{IJKL} = \frac{1}{2}\varepsilon_{IJKLMN}P^{*\,MN}\,. \tag{B.5}$$

In order to recover the $N = 8$ field content we have to impose

$$N=8 :: \qquad P_{IJKL} = \frac{1}{4!}\varepsilon_{IJKLMNOP}P^{*\,MNOP}\,, \tag{B.6}$$

$$N=8 :: \qquad \chi_{IJK} = \frac{1}{5!}\varepsilon_{IJKLMNOP}\chi^{LMNOP}\,. \tag{B.7}$$

[27]The information in this appendix taken from ref. [66], but adapted to the notations of ref. [16].

$N=8$						
	${e^a}_\mu$	$\psi_{I\mu}$	${A^{IJ}}_\mu$	χ_{IJK}	χ^{IJKLM}	$P_{IJKL\mu}$
$\sharp$	1	8	28	56	56	$70+70$

Table 6. The field content of the $N=8$ supergravity multiplet.

C Gamma matrices and spinors

We work with a purely imaginary representation

$$\gamma^{a\,*} = -\gamma^a\,, \tag{C.1}$$

and our convention for their anti-commutator is

$$\{\gamma^a, \gamma^b\} = +2\eta^{ab}\,. \tag{C.2}$$

Thus,

$$\gamma^0\gamma^a\gamma^0 = \gamma^{a\,\dagger} = (\gamma^a)^{-1} = \gamma_a\,. \tag{C.3}$$

The chirality matrix is defined by

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d\,, \tag{C.4}$$

and satisfies

$${\gamma_5}^\dagger = -{\gamma_5}^* = \gamma_5\,, \qquad (\gamma_5)^2 = 1\,. \tag{C.5}$$

With this chirality matrix, we have the identity

$$\gamma^{a_1\cdots a_n} = \frac{(-1)^{[n/2]}i}{(4-n)!}\epsilon^{a_1\cdots a_n b_1\cdots b_{4-n}}\gamma_{b_1\cdots b_{4-n}}\gamma_5\,. \tag{C.6}$$

Our convention for Dirac conjugation is

$$\bar{\psi} = i\psi^\dagger\gamma_0\,. \tag{C.7}$$

Using the identity eq. (C.6) the general $d=4$ Fierz identity for *commuting* spinors takes the form

$$\begin{aligned}(\bar{\lambda}M\chi)(\bar{\psi}N\varphi) =& \frac{1}{4}(\bar{\lambda}MN\varphi)(\bar{\psi}\chi) + \frac{1}{4}(\bar{\lambda}M\gamma^a N\varphi)(\bar{\psi}\gamma_a\chi) - \frac{1}{8}(\bar{\lambda}M\gamma^{ab}N\varphi)(\bar{\psi}\gamma_{ab}\chi) \\ & - \frac{1}{4}(\bar{\lambda}M\gamma^a\gamma_5 N\varphi)(\bar{\psi}\gamma_a\gamma_5\chi) + \frac{1}{4}(\bar{\lambda}M\gamma_5 N\varphi)(\bar{\psi}\gamma_5\chi)\,.\end{aligned} \tag{C.8}$$

We use 4-component chiral spinors whose chirality is related to the position of the SU(4)-index:

$$\gamma_5\chi_I = +\chi_I\,, \qquad \gamma_5\psi_{\mu\,I} = -\psi_{\mu\,I}\,, \qquad \gamma_5\epsilon_I = -\epsilon_I\,. \tag{C.9}$$

Both chirality and position of the SU(4)-index are reversed under complex conjugation, e.g.

$$\gamma_5\chi_I^* \equiv \gamma_5\chi^I = -\chi^I\,, \qquad \gamma_5\psi_{\mu\,I}^* \equiv \gamma_5{\psi_\mu}^I = +{\psi_\mu}^I\,, \qquad \gamma_5\epsilon_I^* \equiv \gamma_5\epsilon^I = +\epsilon^I\,. \tag{C.10}$$

We take this fact into account when Dirac-conjugating chiral spinors:

$$\bar{\chi}^I \equiv i(\chi_I)^\dagger\gamma_0\,, \quad \bar{\chi}^I\gamma_5 = -\bar{\chi}^I\,, \quad \text{etc.} \tag{C.11}$$

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D Fierz identities for bilinears

Here we are going to work with an arbitrary number N of chiral spinors. Whenever there are special results for particular values of N, we will explicitly say so. We should bear in mind that the maximal number of independent chiral spinors is 2 and $N(>2)$ spinors cannot be linearly independent at a given point. This trivial fact has important consequences.

Given N chiral commuting spinors ϵ_I and their complex conjugates ϵ^I we can constructed the following bilinears that are not obviously related via eq. (C.6):

1. A complex matrix of scalars

$$M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J \,, \qquad M^{IJ} \equiv \bar{\epsilon}^I \epsilon^J = (M_{IJ})^* \,, \tag{D.1}$$

   which is antisymmetric $M_{IJ} = -M_{JI}$.

2. A complex matrix of vectors

$$V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J \,, \qquad V_I{}^J{}_a \equiv i\bar{\epsilon}_I \gamma_a \epsilon^J = (V^I{}_{J\,a})^* \,, \tag{D.2}$$

   which is Hermitean:

$$(V^I{}_{J\,a})^* = V_I{}^J{}_a = V^J{}_{I\,a} = (V^I{}_{J\,a})^T \,. \tag{D.3}$$

3. A complex matrix of 2-forms

$$\Phi_{IJ\,ab} \equiv \bar{\epsilon}_I \gamma_{ab} \epsilon_J \,, \qquad \Phi^{IJ}{}_{ab} \equiv \bar{\epsilon}^I \gamma_{ab} \epsilon^J = (\Phi_{IJ\,ab})^* \,, \tag{D.4}$$

   which is symmetric in the SU(N) indices $\Phi_{IJ\,ab} = \Phi_{JI\,ab}$ and furthermore is imaginary anti-selfdual, i.e.

$$^\star\Phi_{IJ\,ab} = -i\Phi_{IJ\,ab} \quad \Rightarrow \quad \Phi_{IJ\,ab} = \Phi_{IJ}{}^+{}_{ab} \,. \tag{D.5}$$

   As we are going to see, this matrix of 2-forms can be expressed entirely in terms of the scalar and vector bilinears.

It is straightforward to derive identities for the products of these bilinears using the Fierz identity eq. (C.8). First, the products of scalars:

$$M_{IJ} M_{KL} = \frac{1}{2} M_{IL} M_{KJ} - \frac{1}{8} \Phi_{IL} \cdot \Phi_{KJ} \,, \tag{D.6}$$

$$M_{IJ} M^{KL} = -\frac{1}{2} V^L{}_I \cdot V^K{}_J \,. \tag{D.7}$$

From eq. (D.6) immediately follows

$$M_{I[J} M_{KL]} = 0 \,, \tag{D.8}$$

which is a Plücker identity and implies that $\text{rank}(M_{IJ}) \leq 2$.

We can define the SU(N)-dual of M_{IJ}

$$\tilde{M}^{I_1 \cdots I_{N-2}} \equiv \frac{1}{2} \varepsilon^{I_1 \cdots I_{N-2} KL} M_{KL} \,, \qquad \varepsilon^{1 \cdots N} = \varepsilon_{1 \cdots N} = +1 \,, \tag{D.9}$$

in terms of which we can express eq. (D.8) as

$$\tilde{M}_{IJ_1\cdots J_{N-3}}M^{IK} = 0\,. \tag{D.10}$$

From eq. (D.7) and the antisymmetry of M immediately follows

$$V^I{}_L \cdot V^K{}_J = -V^I{}_J \cdot V^K{}_L = -V^K{}_L \cdot V^I{}_J\,, \tag{D.11}$$

which implies that all the vector bilinears $V^I{}_{J\,a}$ are null:

$$V^I{}_J \cdot V^I{}_J = 0 \qquad\qquad \text{(no sum!)}\,, \tag{D.12}$$

On the other hand, from eqs. (D.11) and (D.7) it follows that the real, SU(N)-invariant combination of vectors $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^I{}_J \cdot V^J{}_I = 2M^{IJ}M_{IJ} \geq 0\,. \tag{D.13}$$

The products of M with the other bilinears[28] give

$$M_{IJ}V^K{}_{L\,a} = \frac{1}{2}M_{IL}V^K{}_{J\,a} + \frac{1}{2}\Phi_{IL\,ba}V^K{}_J{}^b\,, \tag{D.14}$$

$$M_{IJ}\Phi^{KL}{}_{ab} = V^L{}_{I\,[a|}V^K{}_{J\,|b]} - \frac{i}{2}\epsilon_{ab}{}^{cd}V^L{}_{I\,c}V^K{}_{J\,d}\,. \tag{D.15}$$

Now, let us consider the product of two arbitrary vectors:[29]

$$V^I{}_{J\,a}V^K{}_{L\,b} = \frac{i}{2}\epsilon_{ab}{}^{cd}V^I{}_{L\,c}V^K{}_{J\,d} + V^I{}_{L\,(a|}V^K{}_{J\,|b)} - \frac{1}{2}g_{ab}V^I{}_L \cdot V^K{}_J\,. \tag{D.16}$$

For V^2 this identity allows us to write the metric in the form

$$g_{ab} = 2V^{-2}[V_aV_b - V^I{}_{J\,a}V^J{}_{I\,b}]\,. \tag{D.17}$$

Following Tod [8], for $V^2 \neq 0$ we introduce

$$\mathcal{J}^I{}_J \equiv \frac{2M^{IK}M_{JK}}{|M|^2} = \frac{2V\cdot V^I{}_J}{V^2}\,, \qquad |M|^2 \equiv M^{LM}M_{LM} = \frac{1}{2}V^2\,. \tag{D.18}$$

Using eq. (D.6) we can show that it is a Hermitean projector whose trace equals 2:

$$\mathcal{J}^I{}_J\mathcal{J}^J{}_K = \mathcal{J}^I{}_K\,, \qquad \mathcal{J}^I{}_I = +2\,. \tag{D.19}$$

Further, using the general Fierz identity we find

$$\mathcal{J}^I{}_J\epsilon^J = \epsilon^I\,, \qquad \epsilon_I\mathcal{J}^I{}_J = \epsilon_J\,, \tag{D.20}$$

which should be understood for $N > 2$ of the fact that the ϵ^I are not linearly independent.[30] As a consequence of the above identity, the contraction of $\mathcal{J}$ with any of the bilinears is the identity. Using this result and eq. (D.15), we find

$$\Phi^{KL}{}_{ab} = \frac{2M^{IK}M_{IJ}}{|M|^2}\Phi^{JL}{}_{ab} = \frac{2M^{IK}}{|M|^2}V^L{}_{I\,[a}V_{b]} - i\frac{M^{IK}}{|M|^2}\epsilon_{ab}{}^{cd}V^L{}_{I\,c}V_d\,. \tag{D.21}$$

[28] We omit the product $M_{IJ}\Phi_{KL\,ab}$ which will not be used.
[29] The product $V^I{}_{J\,a}V_L{}^K{}_b$ gives a different identity that will not be used.
[30] For $N = 2$ we automatically have $\mathcal{J}^I{}_J = \delta^I{}_J$.

Other useful identities are

$$\frac{M_{IJ}M^{KL}}{|M|^2} = \mathcal{J}^K{}_{[I}\mathcal{J}^L{}_{J]}\,, \tag{D.22}$$

and

$$\mathcal{J}^I{}_J = \delta^I{}_J - \tilde{\mathcal{J}}^I{}_J\,, \tag{D.23}$$

where

$$\tilde{\mathcal{J}}^I{}_J \equiv \frac{(N-2)\tilde{M}^{IK_1\cdots K_{N-3}}\tilde{M}_{JK_1\cdots K_{N-3}}}{|\tilde{M}|^2}\,, \qquad |\tilde{M}|^2 \equiv \tilde{M}^{I_1\cdots I_{N-2}}\tilde{M}_{I_1\cdots I_{N-2}} = \frac{(N-2)!}{2}|M|^2\,, \tag{D.24}$$

is the complementary projector.

We can always use the 1-form $\hat{V} \equiv V_\mu dx^\mu$ to construct the 0^{th} component of a Vielbein basis $\{e^a\}$

$$e^0 \equiv \frac{1}{\sqrt{2}}|M|^{-1}\hat{V}\,. \tag{D.25}$$

Let us define the three 1-forms

$$\hat{V}^m \equiv |M|e^m\,, \qquad m = 1,2,3\,, \qquad V^{m\,\mu}V^n{}_\mu = -|M|^2\delta^{mn}\,, \tag{D.26}$$

and the spacetime-dependent Hermitean matrices

$$(\sigma^m)^I{}_J \equiv -\sqrt{2}\,V^{m\,\mu}V^I{}_{J\,\mu}\,, \tag{D.27}$$

so we can decompose the 1-forms $\hat{V}^I{}_J = V^I{}_{J\,\mu}dx^\mu$ as

$$\hat{V}^I{}_J = \frac{1}{2}\mathcal{J}^I{}_J\hat{V} + \frac{1}{\sqrt{2}}(\sigma^m)^I{}_J\hat{V}^m\,, \tag{D.28}$$

and

$$\hat{V}^I{}_{J\,a} = \frac{1}{\sqrt{2}}|M|\left[\delta_a{}^0\mathcal{J}^I{}_J + \delta_a{}^m(\sigma^m)^I{}_J\right]\,. \tag{D.29}$$

While this decomposition is unique, the matrices σ^m are defined only up to local SO(3) rotations of the $\hat{V}^m$.

The properties satisfied by the 1-forms $\hat{V}^I{}_J$ can be used to prove the following properties for the σ^x matrices:

$$\sigma^m\sigma^n = \delta^{mn}\mathcal{J} + i\varepsilon^{mnp}\sigma^p\,, \tag{D.30}$$

$$\mathcal{J}\sigma^m = \sigma^m\mathcal{J} = \sigma^m\,, \tag{D.31}$$

$$(\sigma^m)^I{}_I = 0\,, \tag{D.32}$$

$$\mathcal{J}^K{}_J\mathcal{J}^L{}_I = \frac{1}{2}\mathcal{J}^K{}_I\mathcal{J}^L{}_J + \frac{1}{2}(\sigma^m)^K{}_I(\sigma^m)^L{}_J\,, \tag{D.33}$$

$$M_{K[I}(\sigma^m)^K{}_{J]} = 0\,, \tag{D.34}$$

$$2|M|^{-2}M_{LI}(\sigma^m)^I{}_JM^{JK} = (\sigma^m)^K{}_L\,, \tag{D.35}$$

$$|M|^{-2}M^{IJ}M_{KL} = -\frac{1}{3}(\sigma^m)^{[I}{}_{[K}(\sigma^m)^{J]}{}_{L]}\,, \tag{D.36}$$

$$(\sigma^{[m|})^I{}_J(\sigma^{|n]})^K{}_L = -\frac{i}{2}\varepsilon^{mnp}[\mathcal{J}^I{}_L(\sigma^p)^K{}_J - (\sigma^p)^I{}_L\mathcal{J}^K{}_J]\,. \tag{D.37}$$

That is: they, together with $\mathcal{J}$, generate a $\mathfrak{u}(2)$ subalgebra of $\mathfrak{u}(N)$ in the eigenspace of $\mathcal{J}$ of eigenvalue $+1$ and provide a basis in the space of Hermitean matrices satisfying $\mathcal{J}A\mathcal{J} = A$: the last of the above properties is a completeness relation in that subspace since it implies that

$$A^L{}_J = \mathcal{J}^L{}_I A^I{}_K \mathcal{J}^K{}_J = \frac{1}{2}\mathrm{Tr}\,(A\mathcal{J})\,\mathcal{J}^L{}_J + \frac{1}{\sqrt{2}}\mathrm{Tr}\left[\frac{1}{\sqrt{2}}A\sigma^m\right](\sigma^m)^L{}_J\,. \tag{D.38}$$

Then, if A is an $N\times N$ Hermitean matrix such that $\mathrm{Tr}\,(A\mathcal{J}) = \mathrm{Tr}\,(A\sigma^x) = 0\,,\ \forall_{x=1,2,3}$, it satisfies $\mathcal{J}A\mathcal{J} = 0$ and it can be written in the form

$$A = (1-\mathcal{J})A\mathcal{J} + \mathcal{J}A(1-\mathcal{J}) + (1-\mathcal{J})A(1-\mathcal{J})\,. \tag{D.39}$$

It is not clear when a combination of global $\mathrm{U}(N)$ and local $\mathrm{SO}(3)$ transformations is enough to render the matrices σ^x constant; however, whenever it is possible, then the projector $\mathcal{J}$ will also be constant. Needless to say, in the $N=2$ case it is always possible.

E Connection and curvature of the conforma-stationary metric

A conforma-stationary metric has the general form

$$ds^2 = |M|^2(dt+\omega)^2 - |M|^{-2}\gamma_{\underline{mn}}dx^m dx^n\,, \qquad m,n = 1,2,3\,, \tag{E.1}$$

where all components of the metric are independent of the time coordinate t. Choosing the Vielbein basis

$$(e^a{}_\mu) = \begin{pmatrix} |M| & |M|\omega_{\underline{m}} \\ \\ 0 & |M|^{-1}v_{\underline{m}}{}^n \end{pmatrix}, \qquad (e^\mu{}_a) = \begin{pmatrix} |M|^{-1} & -|M|\omega_m \\ \\ 0 & |M|v_m{}^{\underline{n}} \end{pmatrix}, \tag{E.2}$$

where

$$\gamma_{\underline{mn}} = v_{\underline{m}}{}^p v_{\underline{n}}{}^q \delta_{pq}\,, \qquad v_m{}^{\underline{p}} v_{\underline{p}}{}^n v_n\,, \qquad \omega_m = v_m{}^{\underline{n}}\omega_{\underline{n}}\,, \tag{E.3}$$

we find that the spin connection components are

$$\begin{aligned} \omega_{00m} &= -\partial_m|M|\,, \quad \omega_{0mn} = \frac{1}{2}|M|^3 f_{mn}\,, \\ \omega_{m0n} &= \omega_{0mn}\,, \qquad \omega_{mnp} = -|M|\varpi_{mnp} - 2\delta_{m[n}\partial_{p]}|M|\,, \end{aligned} \tag{E.4}$$

where $\varpi_m{}^{np}$ is the 3-dimensional spin connection and

$$\partial_m \equiv v_m{}^{\underline{n}}\partial_{\underline{n}}\,, \qquad f_{mn} = v_m{}^{\underline{p}} v_n{}^{\underline{q}} f_{\underline{pq}}\,, \qquad f_{\underline{mn}} \equiv 2\partial_{[\underline{m}}\omega_{\underline{n}]}\,. \tag{E.5}$$

The components of the Riemann tensor are

$$\begin{aligned} R_{0m0n} =& \frac{1}{2}\nabla_m\partial_n|M|^2 + \partial_m|M|\partial_n|M| - \delta_{mn}(\partial|M|)^2 + \frac{1}{4}\nabla m|M|^6 f_{mp}f_{np}\,, \\ R_{0mnp} =& -\frac{1}{2}\nabla_m(|M|^4 f_{np}) + \frac{1}{2}f_{m[n}\partial_{p]}|M|^4 - \frac{1}{4}\delta_{m[n}f_{p]l}\partial_q|M|^4\,, \\ R_{mnpq} =& -|M|^2 R_{mnpq} + \frac{1}{2}|M|^6(f_{mn}f_{pq} - f_{p[m}f_{n]q}) \\ & -2\delta_{mn,pq}(\partial|M|)^2 + 4|M|\delta_{[m}{}^{[p}\nabla_{n]}\partial^{q]}|M|\,, \end{aligned} \tag{E.6}$$

where all the objects in the right-hand sides of the equations are referred to the 3-dimensional spatial metric γ and the 3-dimensional spin connection ϖ. The components of the Ricci tensor are

$$\begin{aligned}
R_{00} &= -\,|M|^2\nabla^2\log|M| - \frac{1}{4}|M|^6 f^2\,,\\
R_{0m} &= \frac{1}{2}\nabla_n(|M|^4 f_{nm})\,,\\
R_{mn} &= |M|^2\left\{R_{mn} + 2\partial_m\log|M|\partial_n\log|M| - \delta_{mn}\nabla^2\log|M| - \frac{1}{2}|M|^4 f_{mp}f_{np}\right\},
\end{aligned}\tag{E.7}$$

and the Ricci scalar is

$$R = -|M|^2\left\{R - \frac{1}{4}|M|^4 f^2 - 2\nabla^2\log|M| + 2(\partial\log|M|)^2\right\},\tag{E.8}$$

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