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Mathematical tools for hesitant sets. Applications

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RESUMEN DEL CONTENIDO DE TESIS DOCTORAL

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RESUMEN (en español)

La lógica difusa fue introducida por L.A. Zadeh en 1965 a partir del denominado principio de incompatibilidad: *Conforme la complejidad de un sistema aumenta, nuestra capacidad para ser precisos y construir instrucciones sobre su comportamiento disminuye hasta el umbral más allá del cual, la precisión y el significado son características excluyentes.* Esta lógica busca proporcionar un marco matemático que permita modelar la incertidumbre que aparece en los procesos cognitivos humanos.

En este medio siglo, la lógica difusa ha sido ampliamente estudiada por la comunidad científica, introduciendo extensiones de la definición original de conjunto difuso, tales como *interval-valued fuzzy sets*, *Atanassov's intuitionistic fuzzy sets* o *type-2 fuzzy sets*.

En los últimos años, aparece una interesante extensión de la lógica difusa, la llamada *hesitant fuzzy logic*. En este marco, se han definido distintos tipos de conjuntos, centrándonos en este trabajo en los llamados *interval-valued hesitant fuzzy sets*. Las propiedades de este tipo de conjuntos permiten generalizar las extensiones de los conjuntos difusos más importantes. Como consecuencia, todos los resultados introducidos para estos conjuntos pueden ser aplicados a otros tipos de conjuntos.

En este trabajo, se han estudiado diferentes conceptos para esta clase de conjuntos, comenzando con la definición de algunas relaciones de orden para conjuntos finitamente generados, los cuales son la base de los *interval-valued hesitant fuzzy sets*. Otras importantes definiciones dadas para estos conjuntos son las de norma y conorma triangulares (t-norma y t-conorma, respectivamente), las cuales se complementan con algunos ejemplos y casos particulares que serán utilizados a lo largo de la investigación.

A continuación, y desde un punto de vista axiomático, se proporciona una definición de cardinalidad, y se prueban varias de sus propiedades más importantes. Teniendo en cuenta sus características, se han estudiado algunos casos particulares que nos permiten obtener algunas de las definiciones clásicas de cardinalidad.

El siguiente concepto tratado ha sido el de entropía, a través de la cual se mide la incertidumbre asociada a un conjunto. La complejidad de estos conjuntos nos ha llevado a definir dicha entropía como una terna de tres aplicaciones (*fuzziness*, *lack of knowledge* y *hesitance*). Con el objetivo de facilitar su obtención, se han probado varios resultados y caracterizaciones.

Finalmente, se han generalizado algunos resultados sobre particionado difuso al caso de *interval-valued hesitant fuzzy sets*, incluyendo dos definiciones distintas para este tipo de particionado, así como algunas caracterizaciones. Particularizando los resultados alcanzados, ha sido posible obtener algunas definiciones clásicas de particionado para conjuntos difusos, tales como la definición de Ruspini.

La segunda parte de este trabajo es el desarrollo de dos aplicaciones de la lógica difusa generalizada a diferentes campos.

La primera de ellas se centra en la protección de la privacidad en microdatos. Este tipo de datos se puede encontrar en campos tales como el médico o el económico. El procedimiento habitual es la aplicación de particiones nítidas a los atributos no sensibles para así proteger los atributos sensibles. Entre las diferentes técnicas para medir el nivel de protección proporcionado por una partición, hemos seleccionado tres de las más destacadas (*k-anonymity*, *l-diversity* y *t-closeness*). Nuestra propuesta ha sido utilizar particiones difusas en lugar de particiones nítidas para dar una mejor protección a la tabla liberada. Además, hemos adaptado las técnicas de protección previamente elegidas a esta nueva situación, obteniendo tres nuevas técnicas que nos permiten medir el nivel de protección proporcionado por este nuevo tipo de particiones. Finalmente, hemos llevado a cabo una experimentación para comprobar la bondad de dicho método.

La segunda aplicación está relacionada con la detección de bordes en imágenes en escala de grises. El punto de partida ha sido un método de construcción de *interval-valued fuzzy relations* a partir de una relación difusa. Hemos estudiado la influencia de la variación de los parámetros involucrados en dicho método. Además, se ha introducido un nuevo método incluyendo pesos y un paso de suavizado, realizando además una comparación experimental con el original que nos permite afirmar que la nueva propuesta es una alternativa eficaz.

RESUMEN (en Inglés)

The fuzzy logic has been introduced by L.A. Zadeh in 1965 from the incompatibility principle: *As a system complexity increases, our ability to make absolute, precise and significant statements about the system's behavior diminishes until a threshold, fuzzily defined, is reached. Beyond that threshold precision and significance are mutually exclusive*. This logic tries to provide a mathematical framework to model the uncertainty in human cognitive processes.

During this half century, the fuzzy logic has been widely studied by the scientific community, introducing extensions of the original definition of fuzzy set, such as *interval-valued fuzzy sets*, *Atanassov's intuitionistic fuzzy sets* or *type-2 fuzzy sets*.

In the last years, an interesting extension of the fuzzy logic has been provided, the called *hesitant fuzzy logic*. In this framework, different types of sets have been defined, while in this work we focus on the *interval-valued hesitant fuzzy sets*. The properties of this type of sets make it possible to generalize the most important extensions of fuzzy sets. As a consequence, all the results introduced for these sets can be applied to other types.

In this work, different concepts for this type of sets have been studied, starting with the definition of some ordering relations for finitely generated sets, which are the basis of *interval-valued*

hesitant fuzzy sets. Another important definitions given for these sets are the ones of triangular norm and conorm (t-norm and t-conorm, respectively), which are complemented with some examples and particular cases used along the research.

From an axiomatic point of view, a cardinality definition is provided next, and several remarkable properties are proved. Taking into account its characteristics, some particular cases have been studied, as classical definitions of cardinality can be obtained from them.

The next concept that has been studied is the one of entropy, which measures the uncertainty associated to a set. The complexity of these sets leads us to define this entropy as a tuple of three mappings *(fuzziness*, *lack of knowledge* and *hesitance*). In order to ease their obtaining, several results and characterizations have been proved.

Lastly, some results about fuzzy partitioning have been generalized *to interval-valued hesitant fuzzy sets*, including two different definitions for this type of partitioning, as well as some characterizations. Particularizing the obtained results, it has been possible to obtain certain classical definitions of partitioning for fuzzy sets, such as Ruspini one.

The second part of this work is the development of two applications of the fuzzy logic generalized to different fields.

The first one is focused on the protection of privacy in microdata. This type of data can be found in fields such as the medical or economical ones. The classical procedure is to apply crisp partitions to the non-sensitive attributes in order to protect the sensitive ones. Among the different techniques to measure the level of protection provided by a partition, we have selected three of the most renowned ones (*k-anonymity*, *l-diversity* and *t-closeness*). Our proposal has been the use of fuzzy partitions instead of crisp ones in order to better protect the released data. In addition, the selected protection techniques have been adapted to this new situation, obtaining three new techniques that measure the level of protection provided by this new type of partitions. Finally, an experiment has been carried out in order to prove the goodness of our method.

The second application is related to the detection of edges in grey scale images. Our start point has been a construction method of *interval-valued fuzzy relations* from a fuzzy relation. We have studied the influence of the variation of the parameters involved on it. Furthermore, a new method including weights and a smoothing step has been proposed, and in addition, it has been compared experimentally to the original one in order to affirm that it is an efficient alternative.

Agradecimientos

El proceso hasta terminar esta tesis comenzó cuando en mi último año de carrera, decidí probar cómo sería la investigación científica realizando el trabajo académico dirigido, y ver si de cara al futuro, me gustaría continuar en esta línea. Así fue. Después de cuatro años, sigo investigando. Tras pasar por un máster al año siguiente de terminar la licenciatura, y continuar con la investigación en el trabajo fin de máster, me embarqué en empezar el doctorado, que aquí termina.

Esta investigación la he hecho junto a los tres mismos investigadores durante todo este tiempo, Pedro, Irene y Susana, por lo que debo agradecerles el haberme ayudado y dirigido durante estos años, sobre todo al principio, cuando no tenía ningún tipo de experiencia investigadora. Además, extendiendo estos agradecimientos al resto del grupo de investigación UNIMODE, al haber formado parte también de esta aventura investigadora.

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Abstract

The fuzzy logic has been introduced by L.A. Zadeh in 1965 from the incompatibility principle: *As a system complexity increases, our ability to make absolute, precise and significant statements about the system's behavior diminishes until a threshold, fuzzily defined, is reached. Beyond that threshold precision and significance are mutually exclusive.* This logic tries to provide a mathematical framework to model the uncertainty in human cognitive processes.

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In the last years, an interesting extension of the fuzzy logic has been provided, the called hesitant fuzzy logic. In this framework, different types of sets have been defined, while in this work we focus on the interval-valued hesitant fuzzy sets. The properties of this type of sets make it possible to generalize the most important extensions of fuzzy sets. As a consequence, all the results introduced for these sets can be applied to other types.

In this work, different concepts for this type of sets have been studied, starting with the definition of some ordering relations for finitely generated sets, which are the basis of interval-valued hesitant fuzzy sets. Another

important definitions given for these sets are the ones of triangular norm and conorm (t-norm and t-conorm, respectively), which are complemented with some examples and particular cases used along the research.

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Resumen

La lógica difusa fue introducida por L.A. Zadeh en 1965 a partir del denominado principio de incompatibilidad: *Conforme la complejidad de un sistema aumenta, nuestra capacidad para ser precisos y construir instrucciones sobre su comportamiento disminuye hasta el umbral más allá del cual, la precisión y el significado son caracterásticas excluyentes.* Esta lógica busca proporcionar un marco matemático que permita modelar la incertidumbre que aparece en los procesos cognitivos humanos.

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para conjuntos finitamente generados, los cuales son la base de los *interval-valued hesitant fuzzy sets*. Otras importantes definiciones dadas para estos conjuntos son las de norma y conorma triangulares (t-norma y t-conorma, respectivamente), las cuales se complementan con algunos ejemplos y casos particulares que serán utilizados a lo largo de la investigación.

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Foreword

Many frequently used terms are not completely precise. This imprecision is noticeable in expressions like *tall people* or *high temperature*. A classical point of view to model this imprecision is to give a threshold in order to discern between who is characterized by the term and who is not. However, it is not reasonable to say that a 1.80 meters person is tall, but a 1.78 meters person is not. It looks prudent not to interpret terms like these in this way.

Fuzzy logic is a great tool to tackle this type of uncertainty associated to certain terms. It has been defined for the first time by Zadeh in 1965 (see [82]). The basis of this logic is the membership degree assigned to each element with respect to a set, allowing the modeling of terms like the ones aforementioned. The usual range of values of such degree is the unit interval $[0, 1]$, where the greater the value, the stronger the membership. In this way, the possible uncertainty is modelled by the membership degree of a fuzzy set.

This logic was well received, and as a result, many researchers in the last 50 years focused their studies on the fuzzy logic and its applications. Consequently, along this half century, different modifications of the original fuzzy sets defined by Zadeh in 1965 were given. It was Sambuc who proposed the interval-valued fuzzy sets in 1975 (see [67]) to overcome the possible problems determining the membership functions using intervals instead of a single value as degrees. An equivalent generalization was defined by Atanassov in 1986 (see [3]), the Atanassov's intuitionistic fuzzy sets, where every element has a membership degree and a non-membership degree associated. A more complex generalization was given by the own Zadeh in 1975 (see [83]), the type-2 fuzzy sets.

This work is focused on a recent logic developed in the last years: the hesitant fuzzy logic, which was introduced by Torra in 2009 (see [75]). This logic has as the main advantage the fact of generalizing fuzzy sets, interval-valued fuzzy sets and Atanassov's intuitionistic fuzzy sets, and at the same time, providing certain properties that make it much more manageable than the type-2 fuzzy sets. The membership degrees assigned by a hesitant fuzzy set are any subset of the unit interval $[0, 1]$.

From the starting point given by Torra with hesitant fuzzy sets, different modifications have been defined. The main one is the typical hesitant fuzzy sets (see [7, 8]), where it is required that the membership degree must be finite. However, an important drawback is that this type of sets does not generalize interval-valued fuzzy sets nor Atanassov's intuitionistic fuzzy sets. Pérez *et al.* (see [56]) defined the finite interval-valued hesitant fuzzy sets (from here on out, interval-valued hesitant fuzzy sets) as a modification of another generalization provided by Chen (see [20]). The membership degrees of this type of sets are given by finitely generated sets of the unit interval. In other words, these sets are the finite union of closed subintervals of the unit interval. The goodness of these sets is the fact of generalizing interval-valued fuzzy sets and Atanassov's intuitionistic fuzzy sets as well. In addition, these membership functions are more controllable than type-2 fuzzy sets ones.

The previous reasoning lead us to focus on this type of sets: interval-valued hesitant fuzzy sets. Along this work, several developments have been carried out around this kind of sets, such as ordering relations, entropy measures or cardinality definitions. Furthermore, thanks to the goodness of this generalization, two different applications have been developed in two very different fields: privacy protection and edge image detection.

After this explanation of the background of this work, its structure is explained in the remainder of this foreword.

In Chapters 1 and 2 each necessary basic concept along this work is explained.

In Chapter 1, fuzzy logic is explained in depth. Firstly, an historical review of the different types of sets with their pros and cons, definitions and different notations is given. After this, the section is split into two sections to deal with two of these types of sets: fuzzy sets and interval-valued fuzzy sets. Several concepts have been explained there, focusing on the necessary concepts in the forthcoming chapters of this memory.

On the other hand, Chapter 2 is centered on the hesitant fuzzy logic. The different types of sets included in this logic are deeply explained. Interval-valued hesitant fuzzy sets are analyzed in further detail as they are the cornerstone of the next chapter.

The new material provided in this work is found in Chapters 3 and 4, whose content is the following.

The full work about interval-valued hesitant fuzzy sets is in Chapter 3, which is split into five sections.

In the first one, two ordering relations for finitely generated sets are given, as well as their generalization to interval-valued hesitant fuzzy sets. In addition, several properties are given in order to ease the other results given in this very chapter.

In the second part, the definitions of t-norm and t-conorm are adapted to this type of sets, followed by a pair of functions satisfying their definitions.

These two first sections serve as the basis for the developments given in the next three sections.

In the third section a cardinality definition is provided from an axiomatic point of view in order to avoid sticking to a single function. Furthermore, remarkable properties that these cardinalities satisfy and several results are also given.

The fourth section is centered on the definition of an entropy measure. Due to the shape of the membership functions of this type of sets, the entropy definition given is split into three different functions (fuzziness, lack of knowledge and hesitance) in order to detect different types of entropy. In addition, several results and characterizations are formulated, followed by a full-detailed example.

In the last part of this chapter, partitioning concepts are adapted to interval-valued hesitant fuzzy sets, along with some properties and results.

The two applications shown in this memory can be found in the two sections that Chapter 4 is split into.

Protection of privacy in microdata is tackled in the first one, replacing the classical approach with crisp partitions for another using fuzzy partitions. Three techniques to measure the level of privacy have been adapted to the fuzzy case. In addition, an experimental comparison has been carried out.

The second section deals with the detection of edges in grey scale images, through a construction method of interval-valued fuzzy relations from a fuzzy relation. The influence of certain parameters of the method has been studied in detail. Furthermore, a new method has been defined including weights and a smoothing step in the initial method. As well as in the previous application, we carried out an experimental comparison.

Finally, in Chapter 4.2.5 the main conclusions are presented, along with a brief summary of the obtained accomplishments along this work.

Contents

Chapter 1

Basic Concepts: Fuzzy Logic

The classical interpretation of set theory states that an element has two options with respect to its membership to a set: it belongs to it, or it does not. However, this reading is not useful when it comes to dealing with certain concepts that are usually employed. If a set represents the concept *"Temperature higher than 25^o C"*, it is obvious that the classical interpretation shapes this concept perfectly, as a temperature of 30^{o}C belongs to the set, while a 15^{o}C one does not. Nevertheless, if the set is *"High temperature"*, this approach is not the proper one to define this concept.

At this point, fuzzy logic comes into play. Sets like the one previously given are modeled by this logic using membership degrees of each element to the set. These degrees vary in the closed interval $[0, 1]$, where 0 represents that the element does not belong to the set at all, while 1 is interpreted as a total membership. Therefore, the concept *"High temperature"* can be modelled by a fuzzy set, where both temperatures 30^{o}C and 35^{o}C would be considered high, but the degree would be higher for the latter.

The main advantage that fuzzy logic provides, as it has been shown with the previous example, is that it can deal with the uncertainty associated to usual terminology, not only related to temperature, but also to other features such as people height, weight or age. For example, a typical situation related to the age of a person arises when the question *"Is this person young?"* is formulated. If this person is 16 years old, the answer is obvious, but what happens if the age is 35?. It depends on the interpretation of *"Young"* made in the context. This interpretation can be perfectly captured by a fuzzy set.

Fuzzy logic presents such good properties, that has been deeply studied and developed since the first definition given by Zadeh in 1965 (see [82]). Coherently, this is the first definition in this chapter.

Definition 1.1 *Let X be a non-empty set. A fuzzy set A on X is defined by its membership function*

$$\mu_A : X \to [0, 1],$$

where $\forall x \in X$, $\mu_A(x)$ represents the membership degree of the element x to the fuzzy set A.

As stated in the previous definition, a fuzzy set is given by a mapping, called membership function. For each element in X, it assigns a value in the closed interval $[0, 1]$ which represents the membership degree of such element to the fuzzy set.

Remark 1.2 *$FS(X)$ denotes the set of all fuzzy sets in X.*

It must be noted that any crisp set is a fuzzy set whose membership function only takes value 0 or 1. Given a crisp set A on X, then $A \in FS(X)$. A basic example of a fuzzy set is shown next.

Example 1.3 *In Figure 1.1 the graphical representation of the fuzzy set $A \in FS([0,4])$ is given, whose membership function is given by*

$$\mu_A(x) = \begin{cases} x-1, & \textit{if } x \in [1,2], \\ 3-x, & \textit{if } x \in (2,3], \\ 0, & \textit{in other case.} \end{cases}$$

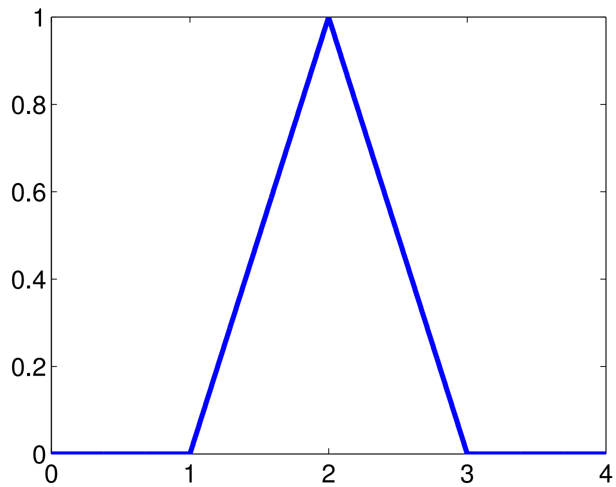


Figure 1.1: Graphical representation of the fuzzy set A.

The membership function must be determined by some experts. In some situations, this task is hard to be carried out accurately. In order to overcome this possible uncertainty, a new type of fuzzy sets has been developed: the interval-valued fuzzy sets. This type of sets represents a generalization of classical fuzzy sets, where the membership degree is an interval instead of just one value in the interval $[0,1]$. This concept was given in 1975 for the first time by Sambuc (see [67]).

Definition 1.4 *Let X be a non-empty set. An interval-valued fuzzy set A on X is defined by its membership function*

$$\mu_A : X \to L([0,1]),$$

where $L([0,1])$ denotes the family of all closed subintervals of $[0,1]$, and $\forall x \in X$, $\mu_A(x)$ represents the membership degree of the element x to the interval-valued fuzzy set A.

In addition, for each $x \in X$, the membership degree can be split into two values

$$\mu_A(x) = [\mu_A^L(x), \mu_A^U(x)].$$

The interval that shapes each membership degree makes it possible to capture the uncertainty that the experts may have when it comes to determining the membership function of a fuzzy set.

Remark 1.5 *$IVFS(X)$ denotes the set of all interval-valued fuzzy sets in X.*

It is also obvious that a fuzzy set is an interval-valued fuzzy set with $\mu_A^L(x) = \mu_A^U(x)$ for all $x \in X$, i.e., $FS(X) \subseteq IVFS(X)$. A simple example of an interval-valued fuzzy set is given next.

Example 1.6 *In Figure 1.2 the graphical representation of the fuzzy set $A \in FS([0,4])$ is shown, whose membership function is given by*

$$\mu_A^L(x) = \begin{cases} x-1, & \text{if } x \in [1,2], \\ 3-x, & \text{if } x \in (2,3], \\ 0, & \text{in other case,} \end{cases}$$

and

$$\mu_A^U(x) = \begin{cases} \frac{x}{2}, & \textit{if } x \in [0,2], \\ 2 - \frac{x}{2}, & \textit{if } x \in (2,4], \\ 0, & \textit{in other case.} \end{cases}$$

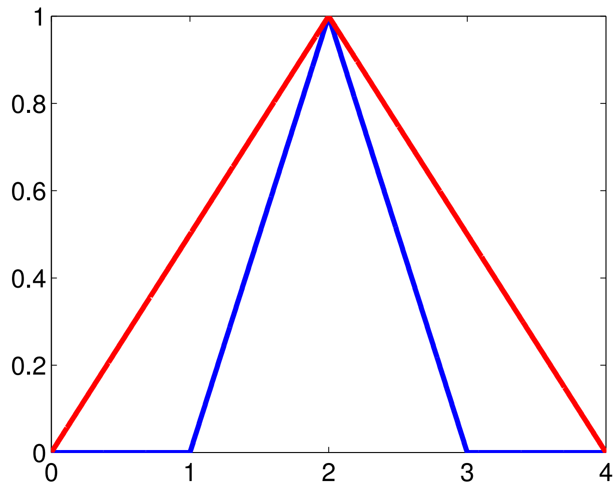


Figure 1.2: Graphical representation of the interval-valued fuzzy set A (μ_A^L in blue, μ_A^U in red).

Another generalization of fuzzy sets developed in order to overcome the uncertainty is the Atanassov's intuitionistic fuzzy sets, where the set is determined by a membership function and a non-membership function. These sets have been developed in 1986 by Atanassov (see [3]).

Definition 1.7 *Let X be a non-empty set. An Atanassov's intuitionistic fuzzy set A on X is defined by its membership and non-membership functions*

$$\mu_A, \upsilon_A : X \to [0,1],$$

such that

$$0 \leq \mu_A(x) + \upsilon_A(x) \leq 1, \ \forall x \in X,$$

where $\forall x \in X$, $\mu_A(x)$ *and* $\upsilon_A(x)$ *represent the membership and non-membership degree of the element* x *to the fuzzy set* A*, respectively.*

Remark 1.8 $AIFS(X)$ *denotes the set of all interval-valued fuzzy sets in* X*.*

However, interval-valued fuzzy sets and Atanassov's intuitionistic fuzzy sets are mathematically equivalent (see [73]). For this reason, we have decided to study just one of them, the interval-valued fuzzy sets.

Finally, type-2 fuzzy sets are a generalization of all the previously defined sets, which has been defined also by Zadeh in 1975 (see [83]).

Definition 1.9 *Let* X *be a non-empty set. A type-2 fuzzy set* A *on* X *is defined by its membership function*

$$\mu_A : X \to FS([0,1]),$$

where $\forall x \in X$, $\mu_A(x)$ *represents the membership degree of the element* x *to the fuzzy set* A*.*

Remark 1.10 $T2FS(X)$ *denotes the set of all type-2 fuzzy sets in* X*.*

As it has been denoted in the definition, this type of sets assigns another fuzzy set as a membership degree. As it is obvious, fuzzy sets, interval-valued fuzzy sets and Atanassov's intuitionistic fuzzy sets are particular cases of type-2 fuzzy sets.

Remark 1.11 *The types of sets defined in this chapter are related as follows*

$$FS(X) \subseteq IVFS(X) \equiv AIFS(X) \subseteq T2FS(X).$$

However, type-2 fuzzy sets have an important drawback: their membership functions are hard to handle, and as a consequence, they are barely used.

From here on out, we will focus in two of these type of sets: fuzzy sets and interval-valued fuzzy sets, developing important features and characteristics that are crucial in the research carried out in this work. It is split into two sections, where concepts for fuzzy sets and interval-valued fuzzy sets are given, respectively.

1.1 Fuzzy sets

As fuzzy sets are the basis of the fuzzy logic, it is the most studied and developed type of sets since its first definition in 1965 by Zadeh (see [82]). In this section, the different concepts needed about fuzzy sets are split into six subsections.

1.1.1 Basic operators

In this subsection, four important concepts are defined for fuzzy sets: negation, complement, α-cut and order for fuzzy sets. All of them are well known, and can be found in a wide range of sources, such as [5, 6, 43, 57].

A good summary of negation definitions for fuzzy sets can be found in [6].

Definition 1.12 *Let* $N : [0,1] \to [0,1]$ *be a function. Then,*

- N *is a fuzzy negation if it satisfies*

  1. $N(0) = 1$ *and* $N(1) = 0$,
  2. $a \leq b \Rightarrow N(b) \leq N(a)$, $\forall a, b \in [0,1]$.

- N *is a strict fuzzy negation if it is a fuzzy negation and satisfies*

  3. N *is continuous,*
  4. $a < b \Rightarrow N(b) < N(a)$, $\forall a, b \in [0,1]$.

- *N is a strong fuzzy negation if it is a fuzzy negation and involutive, i.e.,*

    5. $N(N(a)) = a, \ \forall a \in [0,1]$.

It must be noted that a strong fuzzy negation is a strict fuzzy negation, although the reverse is not satisfied.

Example 1.13 *The functions $N : [0,1] \to [0,1]$ given by $N(a) = 1 - a^{\alpha}$, are strict fuzzy negations, for every $\alpha > 0$. The graphical representations for $\alpha = 0.25, 0.5, 1, 2, 4$ are shown in Figure 1.3.*

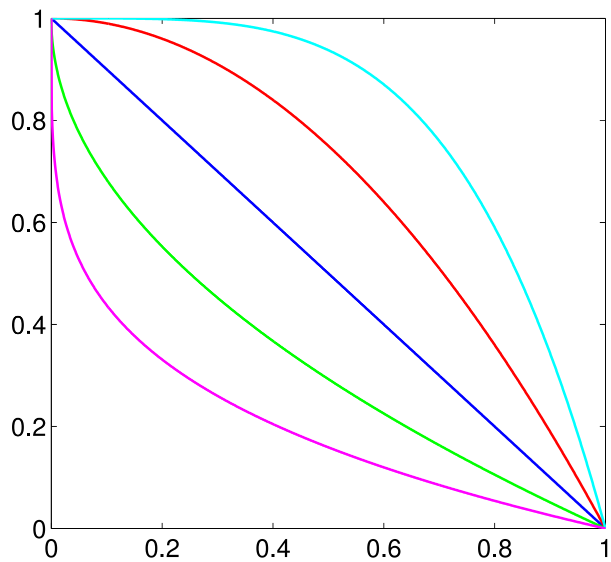


Figure 1.3: Graphical representation of the negation $N(a) = 1 - a^{\alpha}$ ($\alpha =$ 0.25 magenta, $\alpha = 0.25$ green, $\alpha = 1$ blue, $\alpha = 0.25$ red, $\alpha = 4$ cyan).

The special situation where $\alpha = 1$, i.e., $N(a) = 1 - a$, is known as the standard negation (blue function in Figure 1.3). Furthermore, it is the only *value for α such that N is involutive, and as a consequence, a strong fuzzy negation.*

The concept of negation in its different variations is closely related to another important definition, the one of complement, which is studied next. This concept is also easily found in fuzzy logic bibliography, for example, in [5].

Definition 1.14 *Let X be a non-empty set, $A \in FS(X)$ and N be a fuzzy negation. Then, the complement of A is denoted by $A^N \in FS(X)$ and is defined by $\mu_{A^N}(x) = N(\mu_A(x))$.*

Remark 1.15 *In this work, the complement obtained by the standard negation is used, unless otherwise noted. This complement is denoted by A^c.*

A particular fuzzy set is necessary in the forthcoming sections, and it is given in next definition (see [57]).

Definition 1.16 *The set $\xi \in FS(X)$ is called equilibrium set if $\mu_A(x) = 0.5,\ \forall x \in X$.*

The following concept given is the one of α-cut of a fuzzy set, which is well explained in [43]. An α-cut groups the elements of the universal set X based on the membership degree to the fuzzy set.

Definition 1.17 *Let X be a non-empty set, $A \in FS(X)$, and $\alpha \in [0,1]$. The α-cut and strong α-cut of A are the crisp sets A_α and $A_{\overline{\alpha}}$, respectively, given by*

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\},$$
$$A_{\overline{\alpha}} = \{x \in X | \mu_A(x) > \alpha\}.$$

Next example shows the meaning of α-cut.

Example 1.18 *Given the fuzzy set $A \in FS([0,4])$,*

$$\mu_A(x) = \begin{cases} x-1, & \text{if } x \in [1,2], \\ 3-x, & \text{if } x \in (2,3], \\ 0, & \text{in other case,} \end{cases}$$

the α-cut and strong α-cut of A for $\alpha = 0.4$ are given, respectively, as

$$A_{0.4} = \{x \in [0,4] | \mu_A(x) \geq 0.4\} = [1.4, 2.6],$$
$$A_{\overline{0.4}} = \{x \in [0,4] | \mu_A(x) > 0.4\} = (1.4, 2.6).$$

The graphical representation of the 0.4-cut is shown in Figure 1.4.

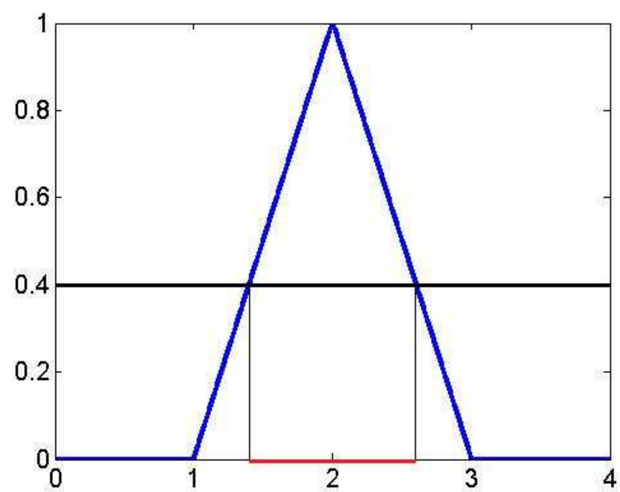


Figure 1.4: Graphical representation of the α-cut of A for $\alpha = 0.4$ (red interval).

In the next proposition, several properties satisfied by α-cuts and strong α-cuts are given. Most of the proofs are straightforward, although they can be found in [43].

Proposition 1.19 *Let X be a non-empty set, $A \in FS(X)$, and $\alpha, \beta \in [0,1]$. Then,*

(i) $A_0 = X$,

(ii) $A_{\overline{\alpha}} \subseteq A_\alpha$,

(iii) *if* $\alpha < \beta$, *then* $A_\beta \subseteq A_\alpha$ *and* $A_{\overline{\beta}} \subseteq A_{\overline{\alpha}}$,

(iv) $(A^c)_\alpha = (A_{\overline{1-\alpha}})^c$.

Finally, an ordering between fuzzy sets is also necessary, and the one selected in this work is given next, which can be found as the most generalized order for this type of sets, due to it simple and straightforward definition.

Definition 1.20 *Let $A, B \in FS(X)$, and μ_A and μ_B their membership functions, respectively. $\leq_F$ is a partial ordering relation for fuzzy sets, given by*

$$A \leq_F B \Leftrightarrow \mu_A(x) \leq \mu_B(x),\ \forall x \in X.$$

These four concepts are widely used, and part of the basis of the concepts explained in the remainder of this section.

1.1.2 t-norms and t-conorms

Two classical operations between sets are their intersection and union. In fuzzy logic, these concepts are defined as triangular norms and triangular conorms. The usual abbreviations are t-norm and t-conorm, respectively.

All the definitions and results given in this subsection are well known and has been widely used and studied. Therefore, they can be found in several books and papers related to fuzzy theory. A good source is [42].

First of all, let us start by giving the definition of triangular norm (t-norm from here on out). This concept is associated to the one of intersection.

Definition 1.21 *A function* $T : [0,1] \times [0,1] \to [0,1]$ *is a t-norm if it satisfies,* $\forall x, y, z \in [0,1]$,

T1 *Commutativity:* $T(x,y) = T(y,x)$,

T2 *Associativity:* $T(x, T(y,z)) = T(T(x,y), z)$,

T3 *Monotonicity:* $T(x,y) \leq T(x,z)$, *whenever* $y \leq z$,

T4 *Neutral element:* $T(x,1) = x$.

On the other hand, a triangular conorm (t-conorm from here on out) is associated to the concept of union.

Definition 1.22 *A function* $S : [0,1] \times [0,1] \to [0,1]$ *is a t-conorm if it satisfies,* $\forall x, y, z \in [0,1]$,

S1 *Commutativity:* $S(x,y) = S(y,x)$,

S2 *Associativity:* $S(x, S(y,z)) = S(S(x,y), z)$,

S3 *Monotonicity:* $S(x,y) \leq S(x,z)$, *whenever* $y \leq z$,

S4 *Neutral element:* $S(x,0) = x$.

Obviously, there exists a way to relate these two concepts through the duality.

Definition 1.23 *Let N be a strong negation, let T and S be a t-norm and t-conorm, respectively. Then,*

- *S is the dual t-conorm of T when*

$$S(x, y) = N(T(N(x), N(y))),$$

- *T is the dual t-norm of S when*

$$T(x, y) = N(S(N(x), N(y))).$$

Note that T and S are duals if one of the two expressions above holds, since both of them are equivalent.

Remark 1.24 *The previous definition of duality is usually applied with the standard negation $N(a) = 1 - a$. In this work, when talking about duality, the standard negation is considered, unless otherwise noted.*

The next two definitions provide us with a way to relate two t-norms (or t-conorms) with an order relation.

Definition 1.25 *Let T_1, T_2 be two t-norms (respectively t-conorms). T_1 is said to be lower than or equal to T_2, and it is denoted by $T_1 \leq T_2$, if and only if $\forall x, y \in [0, 1]$, $T_1(x, y) \leq T_2(x, y)$.*

Definition 1.26 *Let T_1, T_2 be two t-norms (respectively t-conorms). T_1 is said to be strictly lower than T_2, and it is denoted by $T_1 < T_2$, if and only if $T_1 \leq T_2$ and $\exists x_0, y_0 \in [0, 1]$ such that $T_1(x_0, y_0) < T_2(x_0, y_0)$.*

In the following example, four pairs of dual t-norms and t-conorms are given, which are the most usual ones in the literature (see [43]).

Example 1.27 *The following pairs of t-norm and t-conorm are duals:*

$$\textit{Maximum-minimum:} \begin{cases} T_M(x,y) = \min(x,y), \\ S_M(x,y) = \max(x,y). \end{cases}$$

$$\textit{Product:} \begin{cases} T_P(x,y) = xy, \\ S_P(x,y) = x + y - xy. \end{cases}$$

$$\textit{Lukasiewicz:} \begin{cases} T_L(x,y) = \max(x+y-1, 0), \\ S_L(x,y) = \min(x+y, 1). \end{cases}$$

$$\textit{Drastic product:} \begin{cases} T_D(x,y) = \begin{cases} \min(x,y), & \textit{if } x = 1 \textit{ or } y = 1, \\ 0, & \textit{in other case,} \end{cases} \\ S_D(x,y) = \begin{cases} \max(x,y), & \textit{if } x = 0 \textit{ or } y = 0, \\ 1, & \textit{in other case.} \end{cases} \end{cases}$$

The 3D graphical representation of these four t-norms and t-conorms are shown in Figures 1.5 and 1.6, respectively.

These t-norms and t-conorms are related by the order previously given.

Definition 1.28 *Given the dual pairs* (T_M, S_M), (T_P, S_P), (T_L, S_L) *and* (T_D, S_D), *then*

$$T_D < T_L < T_P < T_M,$$
$$S_M < S_P < S_L < S_D.$$

In addition, let T and S be any t-norm and t-conorm, respectively. Then,

$$T_D \leq T \leq T_M,$$
$$S_M \leq S \leq S_D.$$

Although t-norms and t-conorms are binary operations, the aforementioned examples have been generalized in order to obtain their n-ary extensions (see [42]).

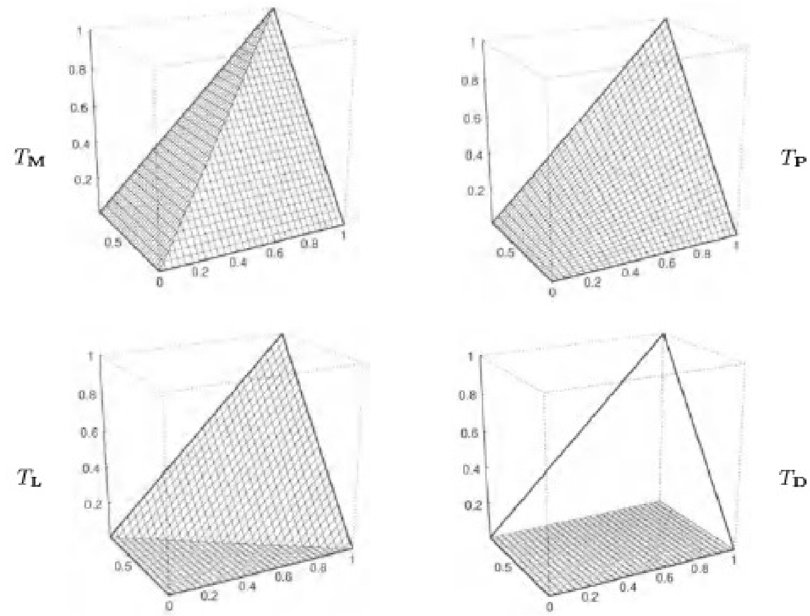


Figure 1.5: 3D graphical representation of t-norms T_M, T_P, T_L and T_D.

Example 1.29 *The n-ary extensions of the dual pairs (T_M, S_M), (T_P, S_P), (T_L, S_L) and (T_D, S_D) are given by:*

$$\textit{Maximum-minimum:} \begin{cases} T_M(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n), \\ S_M(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n), \end{cases}$$

$$\textit{Product:} \begin{cases} T_P(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i, \\ S_P(x_1, \ldots, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i), \end{cases}$$

$$\textit{Lukasiwicz:} \begin{cases} T_L(x_1, \ldots, x_n) = \max(\sum_{i=1}^{n} x_i - (n-1), 0), \\ S_L(x_1, \ldots, x_n) = \min(\sum_{i=1}^{n} x_i, 1), \end{cases}$$

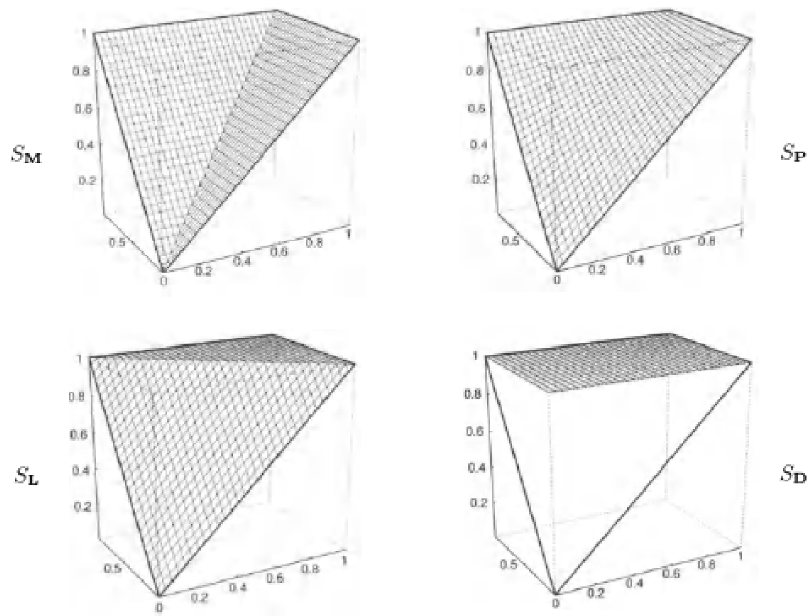


Figure 1.6: 3D graphical representation of t-conorms S_M, S_P, S_L and S_D.

$$\textit{Drastic product:} \left\{ \begin{array}{l} T_D(x_1,\ldots,x_n) = \left\{ \begin{array}{ll} x_i, & \textit{if } x_j = 1 \ \forall j \neq i, \\ 0, & \textit{in other case,} \end{array} \right. \\ S_D(x_1,\ldots,x_n) = \left\{ \begin{array}{ll} x_i, & \textit{if } x_j = 0 \ \forall j \neq i, \\ 1, & \textit{in other case.} \end{array} \right. \end{array} \right.$$

In addition to these four pairs of t-norms and t-conorms, there are families of t-norms and t-conorms. These families depend on parameters, and the previously defined pairs are usually obtainable. In the following definitions, Frank and Sugeno-Weber t-norm families are given, although others such as Yager t-norm family (see [81]) and Dombi t-norm family (see [29]) are also well-known ones.

Definition 1.30 ([34]) *Frank t-norms and t-conorms are defined,* $\forall \lambda \in [0, \infty]$*, as follows*

$$T_\lambda^F(x,y) = \begin{cases} T_M(x,y), & \text{if } \lambda = 0, \\ T_P(x,y), & \text{if } \lambda = 1, \\ T_L(x,y), & \text{if } \lambda = \infty, \\ log_\lambda(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}), & \text{in other case.} \end{cases}$$

$$S_\lambda^F(x,y) = \begin{cases} S_M(x,y), & \text{if } \lambda = 0, \\ S_P(x,y), & \text{if } \lambda = 1, \\ S_L(x,y), & \text{if } \lambda = \infty, \\ 1 - log_\lambda(1 + \frac{(\lambda^{1-x} - 1)(\lambda^{1-y} - 1)}{\lambda - 1}), & \text{in other case.} \end{cases}$$

Remark 1.31 $\forall \lambda_1 < \lambda_2$,

$$T_{\lambda_1}^F(x,y) > T_{\lambda_2}^F(x,y),$$
$$S_{\lambda_1}^F(x,y) < S_{\lambda_2}^F(x,y).$$

Definition 1.32 ([69, 77]) *Sugeno-Weber t-norms and t-conorms are defined,* $\forall \lambda \in [-1, \infty]$*, as follows*

$$T_\lambda^{SW}(x,y) = \begin{cases} T_D(x,y), & \text{if } \lambda = -1, \\ T_P(x,y), & \text{if } \lambda = \infty, \\ \max(\frac{x+y-1+\lambda xy}{1+\lambda}, 0), & \text{in other case.} \end{cases}$$

$$S_\lambda^{SW}(x,y) = \begin{cases} S_P(x,y), & \text{if } \lambda = -1, \\ S_D(x,y), & \text{if } \lambda = \infty, \\ \min(x+y+\lambda xy, 1), & \text{in other case.} \end{cases}$$

Remark 1.33 • $\forall \lambda_1 < \lambda_2$,

$$T^{SW}_{\lambda_1}(x,y) < T^{SW}_{\lambda_2}(x,y),$$
$$S^{SW}_{\lambda_1}(x,y) < S^{SW}_{\lambda_2}(x,y).$$

- T^{SW}_{λ} *and* S^{SW}_{μ} *are dual if* $\mu = -\frac{\lambda}{1+\lambda}$.
- *(*$T^{SW}_{-1}, S^{SW}_{\infty}$*) and (*$T^{SW}_{\infty}, S^{SW}_{-1}$*) are also duals.*

The definitions of t-norm and t-conorm are a cornerstone of this work, as they are generalized to the sets of study (hesitant logic) in order to model intersection and union for them.

1.1.3 Relations

The concept of relation is widely used in many areas of mathematics, where ordering and equivalence relations are, obviously, repeatedly used. Such relations are deeply studied in crisp theory due to their good properties and utility. As a consequence, the concept of relation has been generalized to the fuzzy logic as well. A good reference is [43] for detailed information about fuzzy relations.

Definition 1.34 *Let* $X_1, \ldots, X_n$ *be* n *non-empty sets. A fuzzy relation* R *on* $X_1, \ldots, X_n$ *is a fuzzy set in the cartesian product* $X_1 \times \cdots \times X_n$. $R(x_1, \ldots, x_n)$ *is the membership degree of the element* $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i$ *to* R*, which represents the strength of the relation between such elements.*

Remark 1.35 $FR(X_1, \ldots, X_n)$ *denotes the set of all fuzzy relations in* $X_1, \ldots, X_n$.

However, the most usual situation arises for two sets X and Y, i.e., binary relations.

Definition 1.36 *Let X and Y be two non-empty sets. A fuzzy relation R on X and Y is a fuzzy set in $X \times Y$. $R(x, y)$ is the membership degree of the element $(x, y) \in X \times Y$ to R, which represents the strength of the relation between x and y.*

Remark 1.37 *$FR(X, Y)$ denotes the set of all fuzzy relations in X and Y.*

Example 1.38 *Consider $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. The relation $R \in FR(X, Y)$ is defined by a matrix:*

$$R = \begin{bmatrix} 0.8 & 0.4 & 0.6 \\ 0.8 & 1 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix},$$

where $R(i, j)$ is the membership degree of the elements x_i and y_j to the relation, i.e., the strength of the relation between both elements.

Furthermore, when $X = Y$, a special case arises. The set of binary fuzzy relations where both sets match are denoted by $FR(X, X) = FR(X^2)$.

1.1.4 Entropy and dissimilarity measures

A whole chapter in this memory is devoted to the study of entropy and dissimilarity measures in the hesitant fuzzy logic. As a consequence, the basic definitions for fuzzy sets must be explained for a better understanding of the research.

The aim of an entropy is to quantify the uncertainty associated with either a fuzzy set or a generalization of it. In the next result, the definition of entropy for fuzzy sets is given. The definitions of entropy and dissimilarity measure in the classical fuzzy sets are well known and can be found in several sources, such as [31].

Definition 1.39 *Consider $A, B \in FS(X)$. A mapping $E : FS(X) \to [0,1]$ is an entropy measure if it satisfies the following properties:*

1. $E(A) = 0 \Leftrightarrow A$ *is crisp,*
2. $E(A) = 1 \Leftrightarrow A$ *is the equilibrium set,*
3. $E(A) = E(A^c)$,
4. $E(A) \leq E(B)$ *if* $|\mu_A(x) - \mu_\xi(x)| \geq |\mu_B(x) - \mu_\xi(x)|, \ \forall x \in X$.

Dissimilarity measures are being widely used in different fields. The usual definition in a fuzzy environment is given as follows.

Definition 1.40 *Consider $A, B, C \in FS(X)$. A mapping $D : FS(X) \times FS(X) \to [0,1]$ is a dissimilarity measure if it satisfies the following properties:*

1. $D(A, B) = D(B, A)$,
2. $D(A, A) = 0$,
3. *if* $A \leq B \leq C$, *then* $D(A, B) \leq D(A, C)$ *and* $D(B, C) \leq D(A, C)$.

Some authors replace condition (2) by

2. $D(A, B) = 0 \Leftrightarrow A = B$.

Some others consider a particular case of these dissimilarity measures, the ones obtained by considering the idea of restricted dissimilarity function given by Bustince et al. (see [17, 18]), that is, fulfilling the condition:

4. $D(A, A^c) = 1 \Leftrightarrow A$ is a crisp set.

We will work only with restricted dissimilarities from now on.

Another important concept related to dissimilarity and entropy measures is the similarity measure. However, from a dissimilarity measure, a similarity measure is easily obtainable by $S(A, B) = h(D(A, B))$, with h monotone decreasing such that $h(1) = 0$ and $h(0) = 1$ (that is, for any negation). For this reason, it is enough to study just one of the two measures.

1.1.5 Cardinality

The capability to count elements in a set is straightforward when working with crisp sets. However, this task is harder to carry out when we are dealing with another types of sets, such as fuzzy sets.

For this reason, several authors have developed different ways to measure the cardinality of this type of sets. Some proposals for fuzzy sets have been tried, such as σ-count, given by De Luca and Termini (see [25]).

Definition 1.41 *Consider $X = \{x_1, \ldots, x_N\}$ and $A \in FS(X)$. The σ-count cardinality of A is given by*

$$|A|_\sigma = \sum_{i=1}^{N} \mu_A(x_i).$$

However, this cardinality is hard to read, as most of the time the obtained value will be a non integer number.

One of the most important attempts to achieve the cardinality of fuzzy sets has been given by Ralescu (see [61]), where he proposes a non fuzzy cardinality for this type of sets.

Definition 1.42 *Consider* $X = \{x_1, \ldots, x_N\}$ *and* $A \in FS(X)$ *where* μ_A *is its membership function. The values* $\mu_A(x_1), \ldots, \mu_A(x_N)$ *are ordered decreasingly, where* $\mu_{(i)}$ *denotes the i-th largest value, such that*

$$1 = \mu_{(0)} \geq \mu_{(1)} \geq \cdots \geq \mu_{(N)} \geq \mu_{(N+1)} = 0.$$

Then, the non-fuzzy cardinality is defined by

$$|A|_R = \begin{cases} 0, & \text{if } A = \emptyset, \\ j, & \text{if } A \neq \emptyset \text{ and } \mu_{(j)} \geq 0.5, \\ j-1, & \text{if } A \neq \emptyset \text{ and } \mu_{(j)} < 0.5. \end{cases}$$

where

$$j = \max\{1 \leq t \leq N | \mu_{(t-1)} + \mu_{(t)} > 1\}.$$

In the same work, Ralescu proves the next result that will be useful in our paper.

Proposition 1.43 *Consider* $X = \{x_1, \ldots, x_N\}$ *and* $A, B \in FS(X)$*. Then,* $|A \cup B|_R = |A|_R + |B|_R - |A \cap B|_R$.

This cardinality has been applied in different works, such as in [60], where an approach to protect the privacy of microdata using fuzzy partitions is developed.

Another remarkable definition of cardinality for fuzzy sets has been given by Wygralak (see [78]), where instead of giving a fixed function as Ralescu did, he proposed an axiomatic definition of cardinality. But before the definition, it is necessary to define an special fuzzy set (see [27]).

Definition 1.44 *Let* X *be a non-empty set,* $a \in [0, 1]$ *and* $x \in X$*. The set* $a/x \in FS(X)$ *is given by the membership function* $\mu_{a/x}$ *where* $\mu_{a/x}(x) = a$ *and* $\mu_{a/x}(y) = 0,\ \forall y \neq x$.

Definition 1.45 *Consider* $X = \{x_1, \ldots, x_N\}$*. The mapping* $|\cdot| : FS(X) \to [0, \infty)$ *is a scalar cardinality measure for fuzzy sets if it satisfies the following properties, for all* $A, B \in FS(X)$ *,* $x, y \in X$ *and* $a, b \in [0, 1]$*:*

1. $|1/x| = 1$ *(coincidence),*
2. $a \leq b \Rightarrow |a/x| \leq |b/y|$ *(monotonicity),*
3. $|A \cup B| = |A| + |B|$ *if* $A \cap B = \emptyset$ *(additivity).*

In the next result we have proven that Ralescu non-fuzzy cardinality is a Wygralak scalar cardinality.

Proposition 1.46 *The non-fuzzy cardinality* $|\cdot|_R$ *is a scalar cardinality for fuzzy sets.*

Proof. The three axioms in Definition 1.45 must be proven:

1. Given $1/x \in FS(X)$, it is obvious that $1 = \mu_{(1)} > \mu_{(2)} = \cdots = \mu_{(N)} = 0$. Therefore, $j = max\{1 \leq t \leq N | \mu_{(t-1)} + \mu_{(t)} > 1\} = 1$, and as $\mu_{(1)} = 1 \geq 0.5$, $|1/x|_R = 1$.

2. Consider $a \leq b$ and the sets $a/x, b/y \in FS(X)$. Let $\mu_{a/x}(z)$ be the membership degrees associated to the set a/x, for every $z \in X$. By definition, these membership degrees are ordered decreasingly as follows
$$a > 0 = \cdots = 0.$$
In the same way, for the set b/y this order is given as
$$b > 0 = \cdots = 0.$$
However, by hypothesis $a \leq b$, and therefore, it is obvious that $|a/x|_R \leq |b/y|_R$.

3. Given $A, B \in FS(X)$, by Proposition 1.43, $|A \cup B|_R = |A|_R + |B|_R - |A \cap B|_R$. However, as $A \cap B = \emptyset$, by definition of non-fuzzy cardinality, $|A \cap B|_R = 0$. Hence, $|A \cup B|_R = |A|_R + |B|_R$.

The three axioms have been proven, and as a result, the proposition. ■

Another definitions of cardinality for fuzzy sets have been developed. However, the ones given in this subsection are the ones that will be important in the developments given in this work, as they fit the necessities of our research.

1.1.6 Partitions

In this subsection, a brief explanation of the basic definitions given by Montes *et al.* in [51] are provided, as they will be extended to the hesitant fuzzy logic in the forthcoming chapters of this memory.

First of all, the classical definition of crisp partition must be given.

Definition 1.47 *The family* $\Pi = \{A_i | i \in I\}$ *of crisp sets, where* I *is a finite subset of* $\mathbb{N}$*, is a partition of* A *if and only if*

1. $A_i \neq \emptyset, \ \forall i \in I,$
2. $\bigcup_{i \in I} A_i = A,$
3. $A_i \cap A_j = \emptyset, \ \forall i \neq j.$

The first definition of fuzzy partition was given by Ruspini in 1969 (see [65]), which will be the one used in the last chapter of applications in this memory, in the privacy of microdata section.

Definition 1.48 *Let X be a non-empty set and $A \in FS(X)$. The family $\Pi = \{A_i \in FS(X) | i \in I\}$ where I is a finite subset of $\mathbb{N}$, is a Ruspini partition if and only if*

$$\sum_{i \in I} A_i(x) = A(x),\ \forall x \in X.$$

This is the most remarkable definition of fuzzy partition, although different approaches has been developed by several authors. Bezdek *et al.* (see [10]), Dumitrescu (see [32]) or Ovchinnikov (see [53]) gave their definition of fuzzy partition using t-norms and t-conorms. Markechová (see [48]) relates it to soft fuzzy σ-algebras. De Baets and Mesiar (see [24]) give the concept of T-partition in order to establish a correspondence between fuzzy partitions and fuzzy equivalence relations, or an adaptation of their definition given by Chakraborty and Das (see [19]), among others.

However, the definition of fuzzy partition we have focused on is the one of δ-ϵ-partition given by Montes *et al.* in [51].

Definition 1.49 *Let X be a non-empty set and $A \in FS(X)$. The family $\Pi = \{A_i \in FS(X) | i \in I\}$ where I is a finite subset of $\mathbb{N}$, is a δ-ϵ-partition with $0 \leq \epsilon < \delta \leq 1$ if and only if*

1. $(\underset{i \in I}{S}(A_i))_\alpha = A_\alpha$,
2. $T(A_i, A_j)_\alpha = \emptyset,\ \forall i \neq j$,

where A_α is the α cut for all $\alpha \in (\epsilon, \delta)$, and T and S are a t-norm and a t-conorm respectively.

From this definition, an important characterization was given by the same authors in order to obtain a δ-ϵ-partition with other properties, that in some situations, could be easier to prove or to obtain.

Theorem 1.50 *Let X be a non-empty set and $A \in FS(X)$. The family $\Pi = \{A_i \in FS(X) | i \in I\}$ where I is a finite subset of $\mathbb{N}$, is a δ-ϵ-partition with $0 \le \epsilon < \delta \le 1$ if and only if, $\forall x \in X$:*

$$1'. \quad \begin{cases} \underset{i \in I}{S}(A_i)(x) \ge \delta, & \text{if } \mu_A(x) \ge \delta, \\ \underset{i \in I}{S}(A_i)(x) = \mu_A(x), & \text{if } \mu_A(x) > \epsilon \text{ and } \mu_A(x) < \delta, \\ \underset{i \in I}{S}(A_i)(x) \le \epsilon, & \text{if } \mu_A(x) \le \epsilon, \end{cases}$$

$$2'. \quad T(A_i, A_j)(x) \le \epsilon, \ \forall i \ne j,$$

where T and S are a t-norm and a t-conorm respectively.

Another definition of fuzzy partition, ϵ-ϵ-partition, closely related to δ-ϵ-partition is given as follows:

Definition 1.51 *Let X be a non-empty set and $A \in FS(X)$. The family $\Pi = \{A_i \in FS(X) | i \in I\}$ where I is a finite subset of $\mathbb{N}$, is a ϵ-ϵ-partition with $\epsilon \in [0,1]$ if and only if*

$$1. \quad \begin{cases} A_{\overline{\epsilon}} \subseteq (\underset{i \in I}{S}(A_i))_{\epsilon}, \\ (\underset{i \in I}{S}(A_i))_{\overline{\epsilon}} \subseteq A_{\epsilon}, \end{cases}$$

$$2. \quad T(A_i, A_j)_{\overline{\epsilon}} = \emptyset, \ \forall i \ne j,$$

where T and S are a t-norm and a t-conorm respectively.

Finally, a theorem including both definitions of δ-ϵ-partition and ϵ-ϵ-partition is given next.

Theorem 1.52 *Let X be a non-empty set, $A \in FS(X)$, and the family $\Pi = \{A_i \in FS(X) | i \in I\}$ where I is a finite subset of $\mathbb{N}$ a δ-ϵ-partition with $0 \le \epsilon \le \delta \le 1$. Then Π is a δ'-ϵ'-partition $\forall \epsilon', \delta' \in [0,1]$ such that $\epsilon \le \epsilon' \le \delta' \le \delta$.*

All the results given in this subsection are generalized to interval-valued hesitant fuzzy sets in the forthcoming chapters of this memory. Therefore, definitions of partitions for this type of sets will be provided as a generalization of the aforementioned ones.

In this previous section the basic concepts about fuzzy sets that will be necessary in the forthcoming chapters of this memory are provided. In an analogous structure, next section is devoted to explain these concepts for interval-valued fuzzy sets.

1.2 Interval-valued fuzzy sets

Besides fuzzy sets, interval-valued fuzzy sets have been widely used since its first definition by Sambuc in 1975 (see [67]). The capability to deal with uncertainty is the main characteristic that makes them so useful in certain fields, such as medicine (see [1]), decision making (see [22]) or image processing (see [5]). More concretely, this kind of construction methods are often applied to the detection of edges in grey scale images, which has its most important application in the medical field (see [64]) and other branches of science (see [15]).

As in the previous section, this one is split into several subsections in order to organize every necessary concept related to interval-valued fuzzy sets.

1.2.1 Basic operators

Negation, complement and α-cut are also defined for interval-valued fuzzy sets as well as orders for interval-valued fuzzy sets. These definitions can be found in sources such as [5, 6, 11, 57].

In this case, for the definition of interval-valued fuzzy negation, an ordering for this type of sets is necessary as well as an ordering of intervals in $L([0,1])$.

Definition 1.53 *Let $a, b \in L([0,1])$, such that $a = [a^L, a^U]$ and $b = [b^L, b^U]$. $\leq_I$ is a partial ordering relation for $L([0,1])$ defined by:*

$$a \leq_I b \Leftrightarrow a^L \leq b^L \text{ and } a^U \leq b^U.$$

It must be noted that in this work, another ordering relation for intervals in $L([0,1])$ is used in the forthcoming chapters, developed by Xu and Yager (see [80]), as a total ordering relation is necessary in certain situations.

Definition 1.54 *Let $a, b \in L([0,1])$, and the functions score and accuracy $Sc(a) = a^U - a^L$ and $H(a) = a^L + a^U$ respectively. Then, the total ordering relation $\leq_{XY}$ is defined as follows:*

$$a \leq_{XY} b \Longleftrightarrow \begin{cases} H(a) < H(b), \\ or \\ H(a) = H(b) \quad and \quad Sc(a) < Sc(b). \end{cases}$$

The selected ordering relation for interval-valued fuzzy sets is given next, which is well known and can be found in several sources such as [9, 55].

Definition 1.55 *Let $A, B \in IVFS(X)$, and μ_A and μ_B their membership functions, respectively. $\leq_{IV}$ is a partial ordering relation for interval-valued fuzzy sets defined by:*

$$A \leq_{IV} B \Leftrightarrow \mu_A(x) \leq_I \mu_B(x), \ \forall x \in X,$$

With this prior order for interval-valued fuzzy sets, the definition of interval-valued fuzzy negation is given next as in [5].

Definition 1.56 *Let $N_{IV} : L([0,1]) \rightarrow L([0,1])$ be a function. Then,*

- *N_{IV} is an interval-valued fuzzy negation if it satisfies*

  1. *$N_{IV}([0,0]) = [1,1]$ and $N([1,1]) = [0,0]$,*
  2. *$a \leq_I b \ \Rightarrow N_{IV}(b) \leq_I N_{IV}(a)$, $\forall a, b \in L([0,1])$.*

- *N_{IV} is a strict interval-valued fuzzy negation if it is an interval-valued fuzzy negation and satisfies*

  3. *N_{IV} is continuous,*
  4. *$a <_I b \Rightarrow N_{IV}(b) <_I N_{IV}(a)$, $\forall a, b \in L([0,1])$.*

- *N_{IV} is a strong interval-valued fuzzy negation if it is an interval-valued fuzzy negation and involutive, i.e.,*

  5. *$N_{IV}(N_{IV}(a)) = a$, $\forall a \in L([0,1])$.*

An important result relating fuzzy negations and interval-valued fuzzy negations is given by Deschrijver in [26].

Theorem 1.57 *Let $N_{IV} : L([0,1]) \to L([0,1])$. Then, N_{IV} is a strong interval-valued fuzzy negation if and only if there exists a strong fuzzy negation $N : [0,1] \to [0,1]$ such that*

$$N_{IV}([a^L, a^U]) = [N(a^U), N(a^L)].$$

Example 1.58 *Given the standard fuzzy negation $N(a) = 1 - a$, which is a strong fuzzy negation, applying the previous theorem, the mapping $N_{IV} : L([0,1]) \to L([0,1])$ given by*

$$N_{IV}([a^L, a^U]) = [N(a^U), N(a^L)] = [1 - a^U, 1 - a^L],$$

is a strong interval-valued fuzzy negation. This negation is the standard interval-valued fuzzy negation.

As it happens with fuzzy negations, the concept of interval-valued fuzzy negation is also connected to the definition of complement of interval-valued fuzzy sets (see [5]).

Definition 1.59 *Let X be a non-empty set, and $A \in IVFS(X)$, then given an interval-valued fuzzy negation N_{IV}, the complement of A is denoted by $A^{N_{IV}} \in IVFS(X)$ and it is defined by $\mu_{A^{N_{IV}}}(x) = N_{IV}(\mu_A(x))$.*

Remark 1.60 *In this work, the complement obtained by the standard interval-valued fuzzy negation is used, unless otherwise noted. This complement is denoted by A^c, and given $\forall x \in X$ by*

$$\mu_{A^c}(x) = [1 - \mu_A^U(x), 1 - \mu_A^L(x)],$$

where $\mu_A(x) = [\mu_A^L(x), \mu_A^U(x)]$.

The following particular interval-valued fuzzy set is important in the forthcoming research.

Definition 1.61 *The set $A \in IVFS(X)$ such that $\mu_A(x) = [0, 1]$, $\forall x \in X$ is called the pure interval-valued fuzzy set.*

Taking into account the mathematical duality between both concepts (see [73]), the concept of pure interval-valued fuzzy set is obtained directly from the one of pure Atanassov intuitionistic fuzzy set introduced in [54].

The notion of α-cut is also applicable to interval-valued fuzzy sets. A detailed explanation is given in [36].

Definition 1.62 *Let X be a non-empty set, $A \in IVFS(X)$, and $\alpha \in L([0, 1])$. The α-cut and strong α-cut of A are the crisp sets A_α and $A_{\overline{\alpha}}$, respectively, given by*

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\},$$
$$A_{\overline{\alpha}} = \{x \in X | \mu_A(x) > \alpha\},$$

where $\leq$ represents any ordering relation for $L([0, 1])$ (such as the one given in Definition 1.53).

Example 1.63 *Consider $A \in IVFS([0,4])$ such that*

$$\mu_A^L(x) = \begin{cases} x-1, & \textit{if } x \in [1,2], \\ 3-x, & \textit{if } x \in (2,3], \\ 0, & \textit{in other case,} \end{cases}$$

and

$$\mu_A^U(x) = \begin{cases} \frac{x}{2}, & \textit{if } x \in [0,2], \\ 2-\frac{x}{2}, & \textit{if } x \in (2,4], \\ 0, & \textit{in other case,} \end{cases}$$

selecting the ordering relation for $L([0,1])$ $\leq_I$ (Definition 1.53), the α-cut and strong α-cut of A for $\alpha = [0.4, 0.5]$ are given, respectively, as

$$\begin{aligned} A_{[0.4,0.5]} &= \{x \in [0,4] | \mu_A(x) \geq_I [0.4, 0.5]\} = \\ &= \{x \in [0,4] | \mu_A^L(x) \geq 0.4 \textit{ and } \mu_A^U(x) \geq 0.5\} = \\ &= [1.4, 2.6] \cap [1,3] = [1.4, 2.6], \\ A_{\overline{[0.4,0.5]}} &= \{x \in [0,4] | \mu_A(x) >_I [0.4, 0.5]\} = \\ &= \{x \in [0,4] | \mu_A^L(x) > 0.4 \textit{ and } \mu_A^U(x) > 0.5\} = \\ &= (1.4, 2.6) \cap (1,3) = (1.4, 2.6). \end{aligned}$$

The graphical representation of the $[0.4, 0.5]$-cut is shown in Figure 1.7.

Analogously to fuzzy α-cuts, some properties satisfied by α-cuts and strong α-cuts are also satisfied for interval-valued fuzzy sets, whose proofs are straightforward from the ones given by [43] for fuzzy sets.

Proposition 1.64 *Let X be a non-empty set, $A \in IVFS(X)$, and $\alpha, \beta \in L([0,1])$. Given the ordering relation $\leq_I$ for $L([0,1])$, then,*

(i) $A_0 = X$,

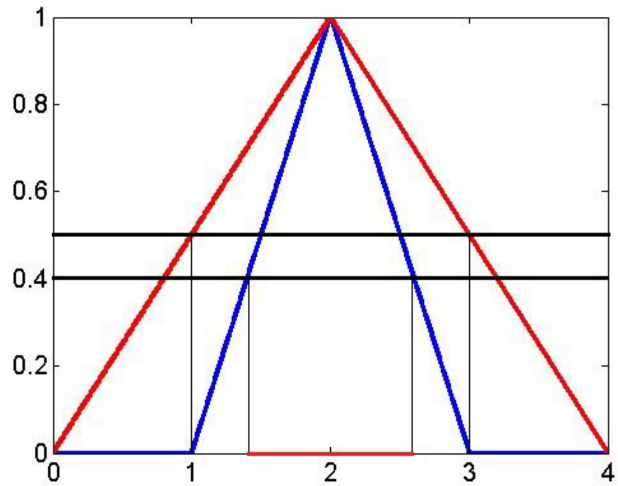


Figure 1.7: Graphical representation of the α-cut of A for $\alpha = [0.4, 0.5]$ (red interval).

(ii) $A_{\overline{\alpha}} \subseteq A_{\alpha}$,

(iii) *if* $\alpha <_I \beta$, *then* $A_{\beta} \subseteq A_{\alpha}$ *and* $A_{\overline{\beta}} \subseteq A_{\overline{\alpha}}$.

Next subsection is focused on the concepts of t-norm and t-conorm, which are of great importance in the study of interval-valued fuzzy sets, and are widely used in this work.

1.2.2 t-norms and t-conorms

The concepts of t-norm and t-conorm for interval-valued fuzzy sets are as valuable as they are in the fuzzy logic, as they make it possible to model intersection and union in this sets as well. These definitions can be found in papers such as [27].

Definition 1.65 *A function* $T : L([0,1]) \times L([0,1]) \to L([0,1])$ *is an interval-valued fuzzy t-norm if it satisfies,* $\forall x, y, z \in L([0,1])$*,*

T1 *Commutativity:* $T(x, y) = T(y, x)$*,*

T2 *Associativity:* $T(x, T(y, z)) = T(T(x, y), z)$*,*

T3 *Monotonicity:* $T(x, y) \leq T(x, z)$*, whenever* $y \leq z$ *(where* $\leq$ *represents any ordering relation for* $L([0,1])$*),*

T4 *Neutral element:* $T(x, [1,1]) = x$*.*

Definition 1.66 *A function* $S : L([0,1]) \times L([0,1]) \to L([0,1])$ *is an interval-valued fuzzy t-conorm if it satisfies,* $\forall x, y, z \in L([0,1])$*,*

S1 *Commutativity:* $S(x, y) = S(y, x)$*,*

S2 *Associativity:* $S(x, S(y, z)) = S(S(x, y), z)$*,*

S3 *Monotonicity:* $S(x, y) \leq S(x, z)$*, whenever* $y \leq z$ *(where* $\leq$ *represents any ordering relation for* $L([0,1])$*),*

S4 *Neutral element:* $S(x, [0,0]) = x$*.*

There also exists a way to relate these two concepts for interval-valued fuzzy sets through the duality.

Definition 1.67 *Let* N_{IV} *be a strong negation, let* T *and* S *be a t-norm and t-conorm for interval-valued fuzzy sets, respectively. Then,*

- S *is the dual t-conorm of* T *when*

$$S(x, y) = N_{IV}(T(N_{IV}(x), N_{IV}(y))),$$

- *T is the dual t-norm of S when*

$$T(x, y) = N_{IV}(S(N_{IV}(x), N_{IV}(y))).$$

It must be noted that T and S are duals if one of the two expressions above holds, since both of them are equivalent.

Remark 1.68 *The previous definition of duality is usually applied with the standard interval-valued fuzzy negation $N_{IV}([a^L, a^U]) = [1 - a^U, 1 - a^L]$. In this work, when talking about duality, the standard negation is considered, unless otherwise noted.*

In the next result, a way to obtain interval-valued fuzzy t-norms (and t-conorms) from fuzzy t-norms (and t-conorms) is defined.

Definition 1.69 *Let T_1 and T_2 be two t-norms, and let S_1 and S_2 be two t-norms for fuzzy sets, such that $T_1 \leq T_2$ and $S_1 \leq S_2$. Then, the mappings $T_{T_1,T_2}, S_{S_1,S_2} : L([0,1]) \times L([0,1]) \to L([0,1])$, defined as follows $\forall x, y \in L([0,1])$,*

$$T_{T_1,T_2}(x, y) = [T_1(x^L, y^L), T_2(x^U, y^U)],$$
$$S_{S_1,S_2}(x, y) = [S_1(x^L, y^L), S_2(x^U, y^U)],$$

are an interval-valued fuzzy t-norm and t-conorm, and are denoted as t-representable t-norm and t-conorm, respectively.

This definition enables us to obtain different t-norms and t-conorms for interval-valued fuzzy sets using the known ones for fuzzy sets given in the previous section.

Example 1.70 *Consider* $T_1 = T_P$, $T_2 = T_M$, $S_1 = S_M$ *and* $S_2 = S_P$. *They satisfy* $T_1 < T_2$ *and* $S_1 < S_2$. *Then, applying the previous definition,*

$$\begin{aligned} T_{T_1,T_2}(x,y) &= [T_1(x^L,y^L),T_2(x^U,y^U)] = [x^L y^L, \min(x^U,y^U)], \\ S_{S_1,S_2}(x,y) &= [S_1(x^L,y^L),S_2(x^U,y^U)] = [\max(x^L,y^L), x^U + y^U - x^U y^U], \end{aligned}$$

T_{T_1,T_2} *and* S_{S_1,S_2} *are the t-representable t-norm and t-conorm obtained with* T_P, T_M, S_M *and* S_P.

However, there are t-norms and t-conorms that are not t-representable, such as the ones given in the next example.

Example 1.71 *The next pairs of interval-valued fuzzy t-norms and t-conorms, are dual and non t-representable, given* $x, y \in L([0,1])$

$$\begin{cases} T(x,y) = [\max(0, x^L + y^L - (1-x^U)(1-y^U) - 1), \max(0, x^U + y^U - 1)], \\ S(x,y) = [\min(1, x^L + y^L), \min(1, x^U + y^U + x^L y^L)], \end{cases}$$

$$\begin{cases} T(x,y) = [\max(0, \min(x^L + y^U - 1, y^L + x^U - 1)), \max(0, x^U + y^U - 1)], \\ S(x,y) = [\min(1, x^L + y^L), \min(1, \max(y^U + x^L, x^U + y^L))], \end{cases}$$

$$\begin{cases} T(x,y) = [\max(0, x^L + y^L - 1), \\ \qquad\qquad \max(0, y^U - 2(1-x^L), x^U - 2(1-y^L), x^L + y^L - 1)], \\ S(x,y) = [\min(1, y^L + 2x^U, x^L + 2y^U, x^U + y^U), \min(1, x^U + y^U)]. \end{cases}$$

As it has been stated in the definitions for fuzzy sets, t-norms and t-conorms are also of great importance in the study of interval-valued fuzzy sets.

1.2.3 Relations

The extension to relations can be applied to interval-valued fuzzy sets in the same way as it has been done for fuzzy sets. These concepts are used and applied in different papers such as [3, 58].

Definition 1.72 *Let $X_1, \ldots, X_n$ be n non-empty sets. An interval-valued fuzzy relation R on $X_1, \ldots, X_n$ is an interval-valued fuzzy set in the cartesian product $X_1 \times \cdots \times X_n$.*

Remark 1.73 *$IVFR(X_1, \ldots, X_n)$ denotes the set of all interval-valued fuzzy relations in $X_1, \ldots, X_n$.*

As it has been stated in the fuzzy relations subsection, the most usual situation arises again for two sets X and Y, i.e., binary relations.

Definition 1.74 *Let X and Y be two non-empty sets. An interval-valued fuzzy relation R on X and Y is a fuzzy set in $X \times Y$.*

Remark 1.75 *$IVFR(X, Y)$ denotes the set of all interval-valued fuzzy relations in X and Y. If $X = Y$, it is denoted by $IVFR(X, X) = IVFR(X^2)$*

A brief illustrative example is shown next.

Example 1.76 *Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$. Given a relation $R \in IVFR(X, Y)$, it is defined by the a matrix:*

$$R = \begin{bmatrix} [0.8, 1] & [0.2, 0.4] & [0.5, 0.6] \\ [0, 0.3] & [0.5, 0.6] & [0.8, 0.8] \end{bmatrix},$$

where $R(i, j)$ is the membership degree of the elements x_i and y_j to the relation, i.e., the strength of the relation between both elements.

The extension from sets to relations is valuable when working in certain fields, such as image processing (see [58]), where data is given in a matrix structure.

1.2.4 Entropy and dissimilarity measures

The definitions of entropy and dissimilarity measures have been adapted to interval-valued fuzzy sets. In [54], the authors adapt the concept of entropy to Atanassov's intuitionistic fuzzy sets. This entropy is split into two functions, E_F and E_L, representing each one a different meaning of entropy. The former describes the fuzziness of the set, i.e., it measures how similar it is to a crisp set. The latter function outlines the lack of knowledge, which shows the similarity with a fuzzy set.

Next definition provides the same concepts for interval-valued fuzzy sets based on the fact that Atanassov's intuitionistic fuzzy sets and interval-valued fuzzy sets are mathematically equivalent (see [73]), so results given for one of them can be translated to the other.

Definition 1.77 *Let $E_F, E_L : IVFS(X) \to [0,1]$ be two mappings. The pair (E_F, E_L) is said to be a two-tuple entropy measure if E_F satisfies the following properties, where $A, B \in IVFS(X)$:*

1. $E_F(A) = 0 \Leftrightarrow A$ *is crisp or it is the pure interval-valued fuzzy set,*

2. $E_F(A) = 1 \Leftrightarrow A$ *is the equilibrium set,*

3. $E_F(A) = E_F(A^c)$*,*

4. $E_F(A) \leq E_F(B)$ *if* $\forall x \in X$

   $\mu_A(x) \leq_I \mu_B(x) \leq_I \mu_\xi(x)$ *for* $\mu_B^L(x) + \mu_B^U(x) \leq 1$ *or*

   $\mu_\xi(x) \leq_I \mu_B(x) \leq_I \mu_A(x)$ *for* $\mu_B^L(x) + \mu_B^U(x) \geq 1$*,*

and E_L satisfies the following properties, where $A, B \in IVFS(X)$:

1. $E_L(A) = 0 \Leftrightarrow A \in FS(X)$*,*

2. $E_L(A) = 1 \Leftrightarrow A$ *is the pure interval-valued fuzzy set,*

3. $E_L(A) = E_L(A^c)$,

4. $E_L(A) \leq E_L(B)$ *if* $\mu_A^U(x) - \mu_A^L(x) \leq \mu_B^U(x) - \mu_B^L(x)$, $\forall x \in X$.

The concept of dissimilarity can be also generalized to interval-valued fuzzy sets. This concept can be found in [50] for intuitionistic fuzzy sets, which is adapted to interval-valued fuzzy sets as follows.

Definition 1.78 *A mapping* $D : IVFS(X) \times IVFS(X) \to [0, 1]$ *is a (restricted) dissimilarity measure if it satisfies the following properties, where* $A, B, C \in IVFS(X)$*:*

1. $D(A, B) = D(B, A)$,

2. $D(A, A^c) = 1 \Leftrightarrow A$ *is crisp,*

3. $D(A, B) = 0 \Leftrightarrow A = B$,

4. *if* $A \leq B \leq C$, *then* $D(A, B) \leq D(A, C)$ *and* $D(B, C) \leq D(A, C)$.

The approach of the previous definition of entropy is adapted to the hesitant fuzzy logic in this work, where entropy is split into several mappings representing different features of a set.

1.2.5 Cardinality

When it comes to interval-valued fuzzy cardinality, few studies have been done around them. One of the most remarkable cardinalities is given as a generalization of Wygralak's scalar cardinality for fuzzy sets in [78], developed by Deschrijver and Král in [27]. As in the fuzzy case, an special interval-valued fuzzy set is necessary beforehand (see [27]).

Definition 1.79 *Let X be a non-empty set, $a \in L([0,1])$ and $x \in X$. The set $a/x \in IVFS(X)$ is given by the membership function $\mu_{a/x}$ where $\mu_{a/x}(x) = a$ and $\mu_{a/x}(y) = 0,\ \forall y \neq x$.*

Definition 1.80 *Consider $X = \{x_1, \ldots, x_N\}$. The mapping $|\cdot| : IVFS(X) \to L([0,\infty))$ is a scalar cardinality measure for interval-valued fuzzy sets if it satisfies the following properties, for all $A, B \in IVFS(X)$, $x, y \in X$ and $a, b \in L([0,1])$:*

1. $|[1,1]/x| = [1,1]$ *(coincidence),*
2. $a \leq_I b \Rightarrow |a/x| \leq_I |b/y|$ *(monotonicity),*
3. $|A \cup B| = |A| + |B|$ *if* $A \cap B = \emptyset$ *(additivity).*

In this work, we study the cardinality for hesitant fuzzy sets, and work with a extension of Wygralak definition as well, although we have followed a different path that the described by Deschrijver and Král in [27], as our cardinality assigns a value in $\mathbb{R}^+$ for every set.

In this chapter, we have developed all the basic concepts about the fuzzy logic necessary to carry out this work. It has been centered on fuzzy sets and interval-valued fuzzy sets, which are the ones analyzed in depth along this work.

Several concepts have been provided, from basic operators such as complement and negation, to more specialized ones, such as t-norms and t-conorms, entropies and dissimilarity measures.

From here on out, the memory is focused on the study of the hesitant fuzzy logic, which is well explained in the next section.

Chapter 2

Basic Concepts: Hesitant Fuzzy Logic

In the previous chapter, we have dealt with the basic concepts related to fuzzy logic. As it has been stated in that chapter, this logic appeared as a way to formalize certain concepts which the classical set theory can not properly model.

Starting from the classical fuzzy sets developed by Zadeh in 1965 (see [82]), a big problem arises: the possible uncertainty in the definition of the membership functions. As a result, some extensions were provided by different authors, such as interval-valued fuzzy sets (see [67]), Atanassov's intuitionistic fuzzy sets (see [3]) or type-2 fuzzy sets (see [83]).

Type-2 fuzzy sets represent the most general type of fuzzy sets, as it was stated in Remark 1.11. However, the membership function of a type-2 fuzzy set assigns another fuzzy set to each element of the set, and as a consequence, the use of these sets is not that convenient.

In order to obtain a generalization of fuzzy sets, interval-valued fuzzy sets and Atanassov's intuitionistic fuzzy sets, with a more manageable fuzzy sets than type-2 fuzzy sets, Torra introduced hesitant fuzzy sets in 2009 (see [74, 75]) as an intermediate type of fuzzy sets. This type of sets assigns a subset of the interval $[0,1]$ to each element instead of a fuzzy set. In fact, Grattan-Guinness already introduced in 1976 this type of sets (see [38]) under the name of set-valued fuzzy sets. Nevertheless, unlike Grattan-Guinness, Torra provided functional definitions of union and intersection for them.

This type of sets presents good properties which make them suitable for researching (see [7, 79]), and specially, in decision making (see [22, 46, 76]). Several extensions of hesitant fuzzy sets have been defined lately (see [62]). In our work, the choice has been the so called finite interval-valued hesitant fuzzy sets, given for the first time by Pérez *et al.* in 2014 (see [56]), whose membership function assigns a union of a finite number of disjoint closed subintervals of $[0,1]$.

However, in this chapter, we will go through the hesitant fuzzy sets and their different extensions which have been developed in the previous years, although as it has been aforementioned, the one which we have selected to our studies has been the finite interval-valued hesitant fuzzy sets.

Let $\mathcal{P}([0,1])$ denote the family of subsets of the closed interval $[0,1]$. Firstly, the definition of a hesitant fuzzy set is given next (see [75]):

Definition 2.1 *Let X be a non-empty set. A hesitant fuzzy set is defined by its membership function*

$$\mu_A : X \to \mathcal{P}([0,1]),$$

where $\forall x \in X$, $\mu_A(x)$ represents the membership degree of the element x to the hesitant fuzzy set A.

Remark 2.2 *$HFS(X)$ denotes the set of all hesitant fuzzy sets in X.*

In a hesitant fuzzy set, the membership degree of an element to the set is given by a subset of the closed interval $[0,1]$. However, an obvious restriction is developed, under the name of typical hesitant fuzzy sets (see [7, 8]).

Definition 2.3 *Let X be a non-empty set and $\mathbb{H} \subset \mathcal{P}([0,1])$ the set of all finite non-empty subsets of the interval $[0,1]$. A typical hesitant fuzzy set is defined by its membership function*

$$\mu_A : X \to \mathbb{H},$$

where $\forall x \in X$, $\mu_A(x)$ represents the membership degree of the element x to the typical hesitant fuzzy set A.

Remark 2.4 *$THFS(X)$ denotes the set of all typical hesitant fuzzy sets in X.*

The membership function of a typical hesitant fuzzy sets assigns to each element of X a finite subset of the interval $[0,1]$. However, another type of sets has been recently developed by Pérez *et al.* (see [56]), the finite interval-valued hesitant fuzzy sets, as a modification of the interval-valued hesitant fuzzy sets given by Chen *et al.* (see [20]). Finite interval-valued hesitant fuzzy sets replace finite subsets by subsets which are generated by a union of a finite number of closed intervals.

Therefore, it is required to introduce the concept of finitely generated set.

Definition 2.5 *Let $n \in \mathbb{N}$. The class of n-finitely generated sets in $[0,1]$ is defined by:*

$$FG_n([0,1]) = \{I \subseteq [0,1] | I = \bigcup_{i=1}^{n} I_i \text{ with } I_i \cap I_j = \emptyset,\ I_i \in L([0,1]),\ \forall i \neq j\}.$$

The class of finitely generated sets in $[0,1]$ *is defined by:*

$$FG([0,1]) = \bigcup_{n=1}^{\infty} FG_n([0,1]).$$

Furthermore, a definition of complement for finitely generated sets is introduced next, where as stated in Section 1, the standard negation is used.

Definition 2.6 *Let* $I = I_1 \cup \cdots \cup I_n \in FG([0,1])$*. Its complement is defined as* $I^c = I_1^c \cup \cdots \cup I_n^c$*, with* $I_i^c = [1 - \max(I_i), 1 - \min(I_i)]$*, for* $i = 1, \ldots, n$*.*

After these prior concepts, the definition of a finite interval-valued hesitant fuzzy set is introduced as follows (see [56]).

Definition 2.7 *Let* X *be a non-empty set. A finite interval-valued hesitant fuzzy set is defined by its membership function*

$$\mu_A : X \to FG([0,1]),$$

where $\forall x \in X$*,* $\mu_A(x)$ *represents the membership degree of the element* x *to the finite interval-valued hesitant fuzzy set* A*.*

Remark 2.8 $IVHFS(X)$ *denotes the set of all finite interval-valued hesitant fuzzy sets in* X*.*

From here on out, we will refer to finite interval-valued hesitant fuzzy sets as just interval-valued hesitant fuzzy sets for simplicity.

For every interval-valued hesitant fuzzy set, and for each point $x \in X$, the membership function belongs to $FG_{n_x}([0,1])$ for some $n_x \in \mathbb{N}$, which represents the number of closed subintervals that generates the finitely generated set. Obviously, in a interval-valued hesitant fuzzy set some of the closed subintervals can be degenerated, i.e., singletons. If all the intervals are degenerated, then we recover typical hesitant fuzzy sets.

Example 2.9 *Figure 2.1 shows the graphical representation of the interval-valued hesitant fuzzy set* $A \in IVHFS([0,4])$*. Its membership function for each* $x \in [0,4]$ *is determined by the functions*

$$f^L(x) = \begin{cases} x-1, & \textit{if } x \in [1,2], \\ 3-x, & \textit{if } x \in (2,3], \\ 0, & \textit{in other case,} \end{cases}$$

and

$$f^U(x) = \begin{cases} \frac{x}{2}, & \textit{if } x \in [0,2], \\ 2-\frac{x}{2}, & \textit{if } x \in (2,4], \\ 0, & \textit{in other case.} \end{cases}$$

and the point calculated by the function

$$g(x) = \left| \sqrt{1 - \left(\frac{x-2}{2}\right)^2} \right|.$$

For example, for $x = 1$*, as* $f^L(1) = 0, f^U(1) = 0.5$ *and* $g(1) = \frac{\sqrt{3}}{2} = 0.866$*, the membership function is given by* $\mu_A(1) = [0, 0.5] \cup 0.866$*.*

As previously stated, one of the main interests of the interval-valued hesitant fuzzy sets lies on the fact that they generalize fuzzy sets and interval-valued fuzzy sets. The different relations are given in the next remark.

Remark 2.10 *The different types of sets previously introduced are related as follows:*

$$FS(X) \subseteq IVFS(X) \equiv AIFS(X) \subseteq IVHFS(X) \subseteq T2FS(X),$$
$$FS(X) \subseteq THFS(X) \subseteq IVHFS(X) \subseteq T2FS(X).$$

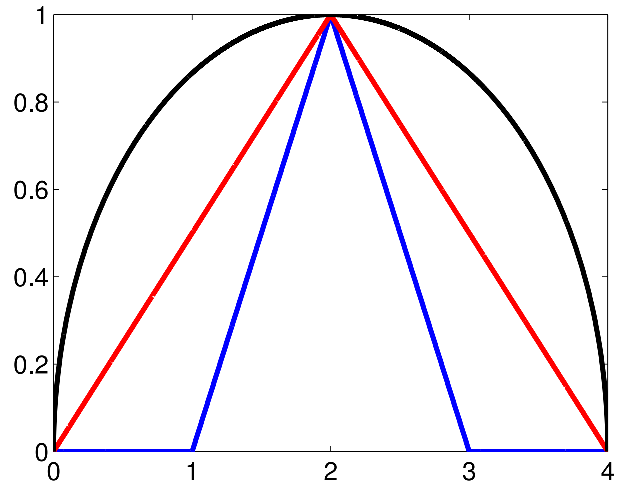


Figure 2.1: Graphical representation of the fuzzy set A (f^L in blue, f^U in red, g in black).

In the next chapter, we develop and prove several results around the hesitant fuzzy logic, which are the cornerstone of this work. The different applications shown in the subsequent chapters are based in these developments.

Chapter 3

Mathematical tools for interval-valued hesitant fuzzy sets

As it has been widely explained in Chapter 2, hesitant fuzzy logic is a recent theory which is still being developed. Since the introduction in 2009 of hesitant fuzzy sets by Torra in [74, 75], different extensions has been studied such as typical hesitant fuzzy sets (see [7, 8]) or interval-valued hesitant fuzzy sets (see [56]).

The developments that have been done related to interval-valued hesitant fuzzy sets are split into five sections. In the former, ordering relations for finitely generated sets and interval-valued hesitant fuzzy sets are developed. In the second part, the definition of t-norm and t-conorm and some examples are provided for this type of sets. In the third one, an axiomatic definition of cardinality is given. The fourth one is focused on the study of entropy. In the latter, some definitions of partitions are proposed.

3.1 Ordering relations

Once the prior concepts about both fuzzy logic (Chapter 1) and hesitant fuzzy logic (Chapter 2) have been introduced, the main goal of this memory is started in this section. Different results related to ordering relations for finitely generated sets are proposed. They will be the basis of the studies in the following sections, where further properties for interval-valued hesitant fuzzy sets are introduced and proven.

As it has been defined in Chapter 2, the membership function of an interval-valued hesitant fuzzy set assigns a finitely generated set to each element. As a result, it is of great importance to analyze this type of sets in order to carry out developments for interval-valued hesitant fuzzy sets. This chapter is focused on the study of ordering relations for the finitely generated sets used in this work.

In the literature, there exist some authors who have developed orders for finitely generated sets, such as Pérez *et al.* (see [56]). In this paper, they propose several orders related to a new concept, denoted by α^{sg}-points. However, these orders do not suit our needs in this work. As a consequence, we have developed two different ordering relations. Each one has certain properties that fit the best in the different studies we have carried out through this work.

The first order for finitely generated sets is based on Xu and Yager order for $L([0,1])$ structure given in Definition 1.54.

Proposition 3.1 *consider $I, J \in FG([0,1])$ such that:*

$$I = \bigcup_{i=1}^{n_I} I_i = \bigcup_{i=1}^{n_I} [I_i^L, I_i^U] \text{ and } J = \bigcup_{i=1}^{n_J} J_i = \bigcup_{i=1}^{n_J} [J_i^L, J_i^U],$$

where, for simplicity and without loss of generality, it is supposed that the sets are ordered increasingly, i.e., $I_i \leq_{XY} I_{i+1}$, $J_i \leq_{XY} J_{i+1}$, and $I_i \cap I_j = \emptyset$ and $J_i \cap J_j = \emptyset$, $\forall i \neq j$. Let Sc and H the following functions (score and accuracy, respectively):

$$Sc(I) = \sum_{i=1}^{n_I} (I_i^U - I_i^L),\ H(I) = \frac{1}{n_I} \sum_{i=1}^{n_I} \left[\frac{I_i^L + I_i^U}{2} \right].$$

Then,

$$I \leq_1 J \iff \begin{cases} (a)\ H(I) < H(J) \\ \ or \\ (b)\ H(I) = H(J) \text{ and } Sc(I) < Sc(J) \\ \ or \\ (c)\ H(I) = H(J),\ Sc(I) = Sc(J) \text{ and } n_I < n_J \\ \ or \\ (d)\ H(I) = H(J),\ Sc(I) = Sc(J),\ n_J = n_J,\ \text{and} \\ I_i^U \leq J_i^U \text{ and } I_i^L \leq J_i^L,\ \forall i = 1, \ldots, n_I, \end{cases}$$

is an ordering relation for finitely generated sets.

Proof. Let us prove the reflexivity, anti-symmetry and transitivity of the relation, given $I, J, K \in FG([0,1])$:

- **Reflexivity:** it is obvious by condition (d) with equalities that $I \leq_1 I$.
- **Anti-symmetry:** given $I \leq_1 J$ and $J \leq_1 I$, the only situation that does not lead to a contradiction is when both inequalities are given by condition (d) with equalities. As a result, it is obvious that $I = J$.

- **Transitivity:** given $I \leq_1 J$ and $J \leq_1 K$, let us analyze the possible situations:

  - if $I \leq_1 J$ is satisfied by condition (a), $H(I) < H(J)$.

    Furthermore, as $J \leq_1 K$, by any condition, $H(J) \leq H(K)$.

    Therefore, $H(I) < H(K)$ and $I \leq_1 K$,

  - if $I \leq_1 J$ is satisfied by condition (b), $H(I) = H(J)$ and $Sc(I) < Sc(J)$.

    If $J \leq_1 K$ is given by condition (a), $H(J) < H(K)$. In other case, $Sc(J) \leq Sc(K)$.

    Therefore, $Sc(I) < Sc(K)$ and $I \leq_1 K$.

  - if $I \leq_1 J$ is satisfied by condition (c), $H(I) = H(J)$, $Sc(I) = Sc(J)$ and $n_I < n_J$.

    If $J \leq_1 K$ is given by condition (a), $H(J) < H(K)$, so $H(I) < H(K)$. If $J \leq_1 K$ is given by condition (b), $Sc(J) < Sc(K)$, so $Sc(I) < Sc(K)$. In other case, $n_J \leq n_K$, so $n_I < n_K$.

    As a result, $I \leq_1 K$.

  - if $I \leq_1 J$ is satisfied by condition (d), $H(I) = H(J)$, $Sc(I) = Sc(J)$, $n_I = n_J$ and $I_i^U \leq J_i^U$ and $I_i^L \leq J_i^L$, $\forall i = 1, \ldots, n_I$.

    If $J \leq_1 K$ is satisfied by condition (a), (b) or (c), then $H(I) < H(K)$, $Sc(I) < Sc(K)$ or $n_I < n_K$ respectively, and as a result, $I \leq_1 K$. If it is given by condition (d) too, then $I_i^U \leq K_i^U$ and $I_i^L \leq K_i^L$, $\forall i = 1, \ldots, n_I$.

    Therefore, $I \leq_1 K$.

  As a consequence, $I \leq_1 K$ is satisfied in every situation.

The three axioms have been proved, so $\leq_1$ is an ordering relation. ■.

It must be noted that this order is a partial ordering relation for finitely generated sets. The incomparability of sets will be studied later in this chapter. Let us show now how this order runs with an example.

Example 3.2 *Let* $I, J \in FG([0,1])$ *defined as* $I = [0, 0.2] \cup \{0.8\}$ *and* $J = [0.2, 0.4] \cup [0.5, 0.7]$. *As* $H(I) = H(J) = 0.45$ *and* $Sc(I) = 0.2 < Sc(J) = 0.4$, *then* $I \leq_1 J$.

Remark 3.3 *It is straightforward to see that this ordering relation, when it is restricted to* $[0,1]$, *matches the usual order in* $\mathbb{R}$, *as* $H(\{a\}) = a$, $\forall a \in [0,1]$.

Along the next pages of this chapter, $\leq_1$ will be the object of the study, where several results and properties are developed.

Proposition 3.4 *Consider* $\leq_1$ *and* $I, J \in FG([0,1])$. *If* $I \leq_1 J$, $H(I) = H(J)$, $Sc(I) = Sc(J)$ *and* $n_I = n_J$, *then* $I = J$.

Proof. By hypothesis, $I \leq_1 J$ must be satisfied by condition (d), so $I_i^U \leq J_i^U$ and $I_i^L \leq J_i^L$, $\forall i = 1, \ldots, n_I$. Furthermore:

$$\begin{cases} H(I) = H(J), \\ Sc(I) = Sc(J), \\ n_I = n_J (= n). \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \frac{I_i^U + I_i^L}{2} = \frac{1}{n}\sum_{i=1}^{n} \frac{J_i^U + J_i^L}{2}, \\ \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} (I_i^U - I_i^L) = \frac{1}{n}\sum_{i=1}^{n} (J_i^U - J_i^L), \end{cases} \Leftrightarrow \begin{cases} \displaystyle\sum_{i=1}^{n} I_i^U = \sum_{i=1}^{n} J_i^U, \\ \displaystyle\sum_{i=1}^{n} I_i^L = \sum_{i=1}^{n} J_i^L, \end{cases} .$$

Let us suppose that $I \neq J$. Then, $\exists i' \in \{1, \ldots, n\}$ such that $I_{i'}^U < J_{i'}^U$

or $I^L_{i'} < J^L_{i'}$. Without loss of generality, $I^U_{i'} < J^U_{i'}$. Then, it is obvious that:

$$\sum_{i=1}^{n} I_i^U < \sum_{i=1}^{n} J_i^U,$$

which is a contradiction and as a result, $I = J$. ■

However, it is not a total ordering relation but a partial one. Therefore, it is possible for two finitely generated sets to be unrelated.

Definition 3.5 *Consider $I, J \in FG([0,1])$. I and J are incomparable with respect to the ordering relation $\leq_1$ if none of the possible relations $I <_1 J$, $J <_1 I$ or $I = J$ hold. This is denoted by $I||_1 J$.*

Some important results related to incomparability of $\leq_1$ ordering relation are introduced in the next propositions.

Proposition 3.6 *Consider $I, J \in FG([0,1])$ such that $I \neq J$, then:*

$$I||_1 J \Leftrightarrow H(I) = H(J),\ Sc(I) = Sc(J),\ n_I = n_J.$$

Proof. Let us suppose that $I||_1 J$. If $H(I) \neq H(J)$, they are comparable, so $H(I) = H(J)$. If $Sc(I) \neq Sc(J)$, they are comparable, so $Sc(I) = Sc(J)$. If $n_I \neq n_J$, they are comparable, so $n_I = n_J$.

The opposite implication is obvious by definition. With this hypothesis, the only possible condition of the order is (d). However, if $I_i^U \leq J_i^U$ and $I_i^L \leq J_i^L$, $\forall i = 1, \ldots, n_I$, and $\exists i$ such that $I_i^U < J_i^U$ or $I_i^L < J_i^L$, then $H(I) < H(J) = H(I)$, which is a contradiction. ■

The following example illustrates the application of the previous result.

Example 3.7 *Let $I, J \in FG([0,1])$ be defined as $I = [0, 0.2] \cup \{0.7\}$ and $J = \{0.3\} \cup [0.6, 0.8]$. As $H(I) = H(J) = 0.5$, $Sc(I) = Sc(J) = 0.2$, $n_I = n_J = 2$, by Proposition 3.6, $I||_1 J$.*

In the previous proposition, a characterization of incomparability for the ordering relation $\leq_1$ is presented. In the following result, a situation when two finitely generated sets are always comparable is shown.

Corollary 3.8 *Let $I \in FG([0,1])$ and $\delta \in [0,1]$. Then, I and $\{\delta\}$ are always comparable.*

Proof. If $I = \{\delta\}$, they are obviously comparable. If $I \neq \{\delta\}$, by Proposition 3.6:

$$I||_1\{\delta\} \Leftrightarrow H(I) = H(\{\delta\}) = \delta,\ Sc(I) = Sc(\{\delta\}) = 0,\ n_I = n_{\{\delta\}} = 1.$$

However, these conditions leads to the next equalities:

$$\frac{I^U + I^L}{2} = \delta,$$
$$I^U - I^L = 0,$$

which is obviously equivalent to $I^U = I^L = \delta$. Thus, $I = \{\delta\}$, and it contradicts our hypothesis, so they are comparable. ■

Another important feature of an order is the possibility of finding an element z such that $x < z < y$, given $x < y$. In the next two propositions, this is analyzed for finitely generated sets with the ordering relation $\leq_1$. The first proposition shows this is satisfied in a certain case.

Proposition 3.9 *Let $I \in FG([0,1])$ and $\delta \in [0,1]$ such that $I <_1 \{\delta\}$. Then, $\exists J \in FG([0,1])$ such that $I <_1 J <_1 \{\delta\}$.*

Proof. Let us distinguish the possible situations:

- if $I <_1 \{\delta\}$ is satisfied by condition (a), $H(I) < H(\{\delta\}) = \delta$. In addition, $\exists \mu \in [0,1]$ such that $H(I) < \mu < \delta$. Then, $J = \{\mu\} \in FG([0,1])$ satisfies that $I <_1 J <_1 \{\delta\}$.

- if $I <_1 \{\delta\}$ is satisfied by condition (b), $Sc(I) < Sc(\{\delta\}) = 0$, which leads to a contradiction, as the score function is non-negative.

- if $I <_1 \{\delta\}$ is satisfied by condition (c), $n_i < n_{\{\delta\}} = 1$, which leads to a contradiction, as at least one interval or point must shape the finitely generated set.

- if $I <_1 \{\delta\}$ is satisfied by condition (d), then $H(I) = H(\{\delta\}) = \delta$, $Sc(I) = Sc(\{\delta\}) = 0$ and $n_I = n_{\{\delta\}} = 1$. However, this only holds if $I = \{\delta\}$, which leads to a contradiction, as by hypothesis, $I <_1 \{\delta\}$.

Hence, the only possibility is that there exists a finitely generated set J such that $I <_1 J <_1 \{\delta\}$. ■

In the following proposition, a characterization of the no existence of such element z is given.

Proposition 3.10 *Let $I, J \in FG([0,1])$ such that $I <_1 J$, then:*

$$\left\{\begin{array}{l} \nexists K \in FG([0,1]) \\ \text{ such that } I <_1 K <_1 J \end{array}\right. \iff \left\{\begin{array}{l} H(I) = H(J),\ Sc(I) = Sc(J),\ \text{and} \\ n_I < n_J = n_I + 1. \end{array}\right.$$

Proof. Firsly, let us suppose that $\nexists K \in FG([0,1])$ such that $I <_1 K <_1 J$. Let us distinguish the possible situations:

- if $I <_1 J$ is satisfied by condition (a), $H(I) < H(J)$. In addition, $\exists \mu \in [0,1]$ such that $H(I) < \mu < H(J)$. Then, $K = \{\mu\} \in FG([0,1])$ satisfies that $I <_1 K <_1 J$.

- if $I <_1 J$ is satisfied by condition (b), $H(I) = H(J)$ and $Sc(I) < Sc(J)$. In addition, $\exists \mu \in [0,1]$ such that $Sc(I) < \mu < Sc(J)$. Then, $K \in FG([0,1])$ such that $H(K) = H(I)$ and $Sc(K) = \mu$ satisfies that $I <_1 K <_1 J$.

- if $I <_1 J$ is satisfied by condition (c), $H(I) = H(J)$, $Sc(I) = Sc(J)$ and $n_I < n_J$. If $n_J > n_I + 1$, then $K \in FG([0,1])$ such that $H(K) = H(I)$ and $Sc(K) = Sc(I)$ and $n_K = n_I + 1$ satisfies that $I <_1 K <_1 J$.

- if $I <_1 J$ is satisfied by condition (d), it is a contradiction, as the strict inequality can not be satisfied by condition (d).

So the only possible situation arises when $H(I) = H(J)$, $Sc(I) = Sc(J)$ and $n_I < n_J = n_I + 1$.

Conversely, let us suppose that $H(I) = H(J)$, $Sc(I) = Sc(J)$ and $n_I < n_J = n_I + 1$. It is straightforward that if there exists $K \in FG([0,1])$ such that $I <_1 K <_1 J$, $H(K) = H(I)$, $Sc(K) = Sc(I)$ and $n_I < n_K < n_J = n_I + 1$, which leads to a contradiction. ■

The second order applied in this work is related to classical membership and it is defined as follows.

Proposition 3.11 *Consider $I, J \in FG([0,1])$. Then,*

$$I \leq_2 J \Leftrightarrow I \subseteq J,$$

is an ordering relation.

Proof. Reflexivity, anti-symmetry and transitivity must be proven. In this case, it is obvious that this relation satisfy all of them. ■

The order $\leq_2$ is, obviously, a partial ordering relation.

Example 3.12 *Consider $I, J \in FG([0,1])$ defined as $I = [0, 0.2] \cup \{0.8\}$ and $J = [0, 0.4] \cup [0.7, 0.8]$. As $I \subseteq J$, then $I \leq_2 J$.*

However, this last ordering relation is so closely related to the classical set theory, that most necessary properties are already developed.

In order to finish off this chapter, the extensions of both ordering relations for finitely generated sets $\leq_1$ and $\leq_2$ to interval-valued hesitant fuzzy sets are studied.

Proposition 3.13 *Let X be a non-empty set with cardinality N, $A, B \in IVHFS(X)$ such that $\forall x \in X$,*

$$\mu_A(x) = \bigcup_{i=1}^{n_x^A} A_i^x = \bigcup_{i=1}^{n_x^A} [A_i^{x^L}, A_i^{x^U}] \text{ and } \mu_B(x) = \bigcup_{i=1}^{n_x^B} B_i^x = \bigcup_{i=1}^{n_x^B} [B_i^{x^L}, B_i^{x^U}]$$

where, for simplicity and without loss of generality, it is supposed that the sets are ordered increasingly $\forall x \in X$, i.e., $A_i^x \leq_{XY} A_{i+1}^x$, $B_i^x \leq_{XY} B_{i+1}^x$. Furthermore, $A_i^x \cap A_j^x = \emptyset$ and $B_i^x \cap B_j^x = \emptyset$ for $i \neq j$.

Then, given Sc and H the score and accuracy functions,

$A \leq_{1I} B$ if only if

(a) $H(\mu_A(x)) \leq H(\mu_B(x))$ $\forall x \in X$ *and* $\exists x'$ *s. t.* $H(\mu_A(x')) < H(\mu_B(x'))$ *or*

(b) $H(\mu_A(x)) = H(\mu_B(x))$ $\forall x \in X$ *and*

(b1) $Sc(\mu_A(x)) \leq Sc(\mu_B(x))$ $\forall x \in X$ *and* $\exists x'$ *s. t.* $Sc(\mu_A(x')) < Sc(\mu_B(x'))$ *or*

(b2) $Sc(\mu_A(x)) = Sc(\mu_B(x))$ $\forall x \in X$ *and*

(b2.1) $n_x^A \leq n_x^B$ $\forall x \in X$ *and* $\exists x'$ *s. t.* $n_{x'}^A < n_{x'}^B$ *or*

(b2.2) $n_x^A = n_x^B, A_i^{x^U} \leq B_i^{x^U}$ *and* $A_i^{x^L} \leq B_i^{x^L}$, $\forall x \in X$ *and* $\forall i = 1, \ldots, n_x^A$,

is and ordering relation.

Proof. In order to prove that this relation is an ordering relation, it must be proven that it is reflexive, anti-symmetric and transitive.

(i) **Reflexivity:** it is obvious, as all the conditions in the definition of the relation are fulfilled with equalities (Condition ($b2.2$)).

(ii) **Anti-symmetry:** given $A, B \in IVHFS(X)$ such that $A \leq_{1I} B$ and $B \leq_{1I} A$. Let us see that the only possible situation in $A = B$, distinguishing situations depending on the condition satisfied for each inequality.

- If $A \leq_{1I} B$ satisfies (a) and $B \leq_{1I} A$ satisfies (a), then

$$H(\mu_A(x)) \leq H(\mu_B(x)) \leq H(\mu_A(x)), \ \forall x \in X,$$

  but there exists $x' \in X$ such that $H(\mu_A(x')) < H(\mu_B(x')) \leq H(\mu_A(x'))$, which is a contradiction.

- If $A \leq_{1I} B$ satisfies (a) and $B \leq_{1I} A$ satisfies (b),

$$H(\mu_A(x)) \leq H(\mu_B(x)) = H(\mu_A(x)), \ \forall x \in X,$$

  but there exists $x' \in X$ such that $H(\mu_A(x')) < H(\mu_B(x')) = H(\mu_A(x'))$, which is a contradiction.

- Analogously, it is proven that it is a contradiction for every combination unless $A \leq_{1I} B$ satisfies ($b2.2$) and $B \leq_{1I} A$ satisfies ($b2.2$), where

$$A_i^{x^L} \leq B_i^{x^L} \leq A_i^{x^L} \text{ and } A_i^{x^U} \leq B_i^{x^U} \leq A_i^{x^U},$$

  $\forall x \in X$ and $\forall i = 1, \ldots, n_x$. Therefore, $A_i^{x^L} = B_i^{x^L}$ and $A_i^{x^U} = B_i^{x^U}$, and then $A = B$.

(iii) **Transitivity:** given $A, B, C \in IVHFS(X)$ such that $A \leq_{1I} B$ and $B \leq_{1I} C$, let us see that $A \leq_{1I} C$.

- If $A \leq_{1I} B$ satisfies (a) and $B \leq_{1I} C$ satisfies (a):

$$H(\mu_A(x)) \leq H(\mu_B(x)) \leq H(\mu_C(x)), \ \forall x \in X,$$

  but there exists $x' \in X$ such that $H(\mu_A(x')) < H(\mu_B(x')) \leq H(\mu_C(x'))$, so A and C satisfies (a) and hence, $A \leq_{1I} C$.

- If $A \leq_{1I} B$ satisfies (a) and $B \leq_{1I} C$ satisfies $(b1)$ or $(b2.1)$ or $(b2.2)$, then

$$H(\mu_A(x)) \leq H(\mu_B(x)) = H(\mu_C(x)), \ \forall x \in X,$$

  but there exists $x' \in X$ such that $H(\mu_A(x')) < H(\mu_B(x')) = H(\mu_C(x'))$, so A and C satisfies (a) and hence, $A \leq_{1I} C$.

- If $A \leq_{1I} B$ satisfies $(b1)$ and $B \leq_{1I} C$ satisfies $(b2.1)$ or $(b2.2)$, both $A \leq_{1I} B$ and $B \leq_{1I} C$ satisfy (b), then

$$H(\mu_A(x)) = H(\mu_B(x)) = H(\mu_C(x)), \ \forall x \in X.$$

  As $A \leq_{1I} B$ satisfies (b1) and $B \leq_{1I} C$ satisfies $(b2.1)$, then

$$Sc(\mu_A(x)) \leq Sc(\mu_B(x)) = Sc(\mu_C(x)) \ \forall x \in X.$$

  In addition, $\exists x'$ s. t. $Sc(\mu_A(x')) < Sc(\mu_B(x'))$. As $Sc(\mu_B(x)) = Sc(\mu_C(x)) \ \forall x \in X$, in particular, $Sc(\mu_B(x')) = Sc(\mu_C(x'))$. Therefore,

$$\exists x' \text{ s. t. } Sc(\mu_A(x')) < Sc(\mu_B(x')) = Sc(\mu_C(x')).$$

  Thus $A \leq_{1I} C$ by (b1).

- If $A \leq_{1I} B$ satisfies $(b2.1)$ and $B \leq_{1I} C$ satisfies $(b2.2)$, both $A \leq_{1I} B$ and $B \leq_{1I} C$ satisfy (b), then

$$H(\mu_A(x)) = H(\mu_B(x)) = H(\mu_C(x)),\ \forall x \in X.$$

As $A \leq_{1I} B$ satisfies (b2.1) and $B \leq_{1I} C$ satisfies $(b2.2)$, then

$$Sc(\mu_A(x)) = Sc(\mu_B(x)) = Sc(\mu_C(x))\ \forall x \in X.$$

In addition,

$$n_x^A \leq n_x^B\ \forall x \in X \text{ and } \exists x' \text{ s. t. } n_{x'}^A < n_{x'}^B,$$

and

$$n_x^B = n_x^C, B_{\sigma(i)}^{x^U} \leq C_{\sigma(i)}^{x^U} \text{ and } B_{\sigma(i)}^{x^L} \leq C_{\sigma(i)}^{x^L}\ \forall x \in X, \forall i = 1, \ldots, n_x^A.$$

Thus,

$$n_x^A \leq n_x^B = n_x^C\ \forall x \in X.$$

In addition, as $\exists x'$ s. t. $n_{x'}^A < n_{x'}^B$, and $n_x^B = n_x^C \forall x \in X$, then

$$n_{x'}^A < n_{x'}^B = n_{x'}^C.$$

That means $\exists x'$ s. t. $n_{x'}^A < n_{x'}^C$ and therefore $A \leq_{1I} C$ because it satisfies $(b2.1)$.

- If $A \leq_{1I} B$ satisfies $(b2.2)$ and $B \leq_{1I} C$ satisfies $(b2.2)$, both $A \leq_{1I} B$ and $B \leq_{1I} C$ satisfy (b), then

$$H(\mu_A(x)) = H(\mu_B(x)) = H(\mu_C(x)),\ \forall x \in X.$$

As both $A \leq_{1I} B$ and $B \leq_{1I} C$ satisfy $(b2.2)$, then

$$Sc(\mu_A(x)) = Sc(\mu_B(x)) = Sc(\mu_C(x))\ \forall x \in X.$$

In addition,

$n_x^A = n_x^B, A_{\sigma(i)}^{x^U} \leq B_{\sigma(i)}^{x^U}$ and $A_{\sigma(i)}^{x^L} \leq B_{\sigma(i)}^{x^L}$ $\forall x \in X, \forall i = 1, \ldots, n_x^A$,

$n_x^B = n_x^C, B_{\sigma(i)}^{x^U} \leq C_{\sigma(i)}^{x^U}$ and $B_{\sigma(i)}^{x^L} \leq C_{\sigma(i)}^{x^L}$ $\forall x \in X, \forall i = 1, \ldots, n_x^A$.

Thus,

$n_x^A = n_x^C, A_{\sigma(i)}^{x^U} \leq C_{\sigma(i)}^{x^U}$ and $A_{\sigma(i)}^{x^L} \leq C_{\sigma(i)}^{x^L}$ $\forall x \in X, \forall i = 1, \ldots, n_x^A$,

and therefore, $A \leq_{1I} C$, since $(b2.2)$ is fulfilled.

Thus, reflexivity, anti-symmetry and transitivity have been proven, and as a result, $\leq_{1I}$ is an ordering relation. ■

Proposition 3.14 *Let X be a non-empty set, and $A, B \in IVHFS(X)$. Then,*

$$A \leq_{2I} B \Leftrightarrow \mu_A(x) \leq_2 \mu_B(x),\ \forall x \in X.$$

Proof. As the relation given in Proposition 3.11 is an ordering relation for finitely generated sets, it is obvious that this one also satisfies reflexivity, anti-symmetry and transitivity, and the result is held. ■

These results are quite useful in the study of t-norms, t-conorms and the other concepts which are developed along the chapter.

3.2 t-norms and t-conorms

As for fuzzy logic (see Chapter 1), the definition of t-norm and t-conorm is of great importance for interval-valued hesitant fuzzy sets, as they make it possible to model the concept of intersection and union.

The specific definitions keep the same structure than for fuzzy sets and interval-valued fuzzy sets, although some attention must be paid to the ordering relation in the monotonicity axioms.

Definition 3.15 *A function* $T : FG([0,1]) \times FG([0,1]) \to FG([0,1])$ *is a t-norm if it satisfies,* $\forall I, J, K \in FG([0,1])$*,*

T1 *Commutativity:* $T(I,J) = T(J,I)$*,*

T2 *Associativity:* $T(I,T(J,K)) = T(T(I,J),K)$*,*

T3 *Monotonicity:* $T(I,J) \leq_{FG} T(I,K)$*, whenever* $J \leq_{FG} K$*, where* $\leq_{FG}$ *is an ordering relation for* $FG([0,1])$*,*

T4 *Neutral element:* $T(I,\{1\}) = I$*.*

Definition 3.16 *A function* $S : FG([0,1]) \times FG([0,1]) \to FG([0,1])$ *is a t-conorm if it satisfies,* $\forall I, J, K \in FG([0,1])$*,*

S1 *Commutativity:* $S(I,J) = S(J,I)$*,*

S2 *Associativity:* $S(I,S(J,K)) = S(S(I,J),K)$*,*

S3 *Monotonicity:* $S(I,J) \leq_{FG} S(I,K)$*, whenever* $J \leq_{FG} K$*, where* $\leq_{FG}$ *is an ordering relation for* $FG([0,1])$*,*

S4 *Neutral element:* $S(I, \{0\}) = I$.

The different orders that we have considered for the monotonicity axioms are the ones given in the previous section, $\leq_1$ and $\leq_2$.

Some operations between finitely generated sets were introduced by Chen *et al.* in [20]. Among them, the most important ones for the study of the cardinality are their union and intersection definitions, which are detailed in the next proposition. In addition, it is proven that they are a t-norm and t-conorm for interval-valued hesitant fuzzy sets, respectively.

Proposition 3.17 *Let* $I, J \in FG([0,1])$ *be such that* $I = \bigcup_{i=1}^{n_I} I_i$ *and* $J = \bigcup_{i=1}^{n_J} J_i$*, where* $I_i = [I_i^L, I_i^U]$ *and* $J_i = [J_i^L, J_i^U]$ *are pairwise disjoint closed subintervals of* $[0,1]$ *respectively. Then,*

- $I \wedge J = \dot{\bigcup}\{[\min(I_i^L, J_j^L), \min(I_i^U, J_j^U)] | \forall i = 1, \ldots, n_I, \ \forall j = 1, \ldots, n_J\}$,
- $I \vee J = \dot{\bigcup}\{[\max(I_i^L, J_j^L), \max(I_i^U, J_j^U)] | \forall i = 1, \ldots, n_I, \ \forall j = 1, \ldots, n_J\}$,

are respectively a t-norm and t-conorm for interval-valued hesitant fuzzy sets considering the partial ordering for finitely generated sets $\leq_2$*, and where* $\dot{\bigcup}$ *denotes a disjoint union.*

Proof. Let us start with $\wedge$ and see that it satisfies the axioms $T1 - T4$. Consider $I, J, K \in FG([0,1])$:

T1 Commutativity: it is obvious by definition.

T2 Associativity:

$$\begin{aligned}(I \wedge J) \wedge K &= \dot{\bigcup}\{[\min(\min(I_i^L, J_j^L), K_k^L), \min(\min(I_i^U, J_j^U), J_k^U)] \mid \\ &\quad \forall i = 1, \ldots, n_I,\ \forall j = 1, \ldots, n_J,\ \forall k = 1, \ldots, n_K\} = \\ &= \dot{\bigcup}\{[\min(I_i^L, J_j^L, K_k^L), \min(I_i^U, J_j^U, J_k^U)] \mid \\ &\quad \forall i = 1, \ldots, n_I,\ \forall j = 1, \ldots, n_J,\ \forall k = 1, \ldots, n_K\} = \\ &= \dot{\bigcup}\{[\min(I_i^L, \min(J_j^L, K_k^L)), \min(I_i^U, \min(J_j^U, K_k^U))] \mid \\ &\quad \forall i = 1, \ldots, n_I,\ \forall j = 1, \ldots, n_J,\ \forall k = 1, \ldots, n_K\} = \\ &= I \wedge (J \wedge K).\end{aligned}$$

T3 Monotonicity: by definition of $\leq_2$, if $J \leq_2 K$, then $J \subseteq K$. Then, $\forall j \in \{1, \ldots, n_J\}$, $\exists k \in \{1, \ldots, n_K\}$ such that $J_j \subseteq K_k$, i.e., $K_k^L \leq J_j^L \leq J_j^U \leq K_k^U$. Furthermore, $\forall i \in \{1, \ldots, n_I\}$, it is obvious that:

$$\min(I_i^L, K_k^L) \leq \min(I_i^L, J_j^L),$$
$$\min(I_i^U, J_j^U) \leq \min(I_i^U, K_k^U).$$

As a result,

$$[\min(I_i^L, J_j^L), \min(I_i^U, J_j^U)] \subseteq [\min(I_i^L, K_k^L), \min(I_i^U, K_k^U)].$$

As for every interval of the set J, there exists another interval of K such that this membership hold for every $i = 1, \ldots, n_I$, it is obvious that $I \wedge J \subseteq I \wedge K$, and therefore, $I \wedge J \leq_2 I \wedge K$.

T4 Neutral element: is obvious, as $[\min(I_i^L, 1), \min(I_i^U, 1)] = [I_i^L, I_i^U]$.

Proving that $\vee$ satisfies $S1 - S4$ is analogous. Thus, $\wedge$ and $\vee$ are interval-valued hesitant fuzzy t-norm and t-conorm, respectively. ■

The t-norm $\wedge$ and t-conorm $\vee$ given in Proposition 3.17 are essential in the study of the cardinality provided in the following section. However, as their definitions are slightly complex, a brief example is given next.

Example 3.18 *Let* $I = \{0.8\}$ *and* $J = [0.4, 0.6] \cup [0.7, 0.9]$*, i.e.,* $I, J \in FG([0,1])$*. Then,*

$$\begin{aligned} I \wedge J &= \dot{\bigcup}\{[\min(I_i^L, J_j^L), \min(I_i^U, J_j^U)] | \forall i = 1,\ \forall j = 1,2\} = \\ &= [0.4, 0.6] \cup [0.7, 0.8], \\ I \vee J &= \dot{\bigcup}\{[\max(I_i^L, J_j^L), \max(I_i^U, J_j^U)] | \forall i = 1,\ \forall j = 1,2\} = \\ &= \{0.8\} \cup [0.8, 0.9] = [0.8, 0.9]. \end{aligned}$$

In the next results, we will focus on the t-norm and t-conorm introduced in Proposition 3.17. In the next proposition, an important property of this intersection is given, which is useful in the next sections.

Proposition 3.19 *Let* $I, J \in FG([0,1])$ *such that* $I \wedge J = \emptyset$*. Then,* $I = \emptyset$ *or* $J = \emptyset$*.*

Proof. Let $I = \bigcup_{i=1}^{n_I} I_i$ and $J = \bigcup_{i=1}^{n_J} J_i$, where $I_i = [I_i^L, I_i^U]$ and $J_i = [J_i^L, J_i^U]$ are pairwise disjoint closed subintervals of $[0,1]$ respectively. By Definition 3.17,

$$I \wedge J = \dot{\bigcup}\{[\min(I_i^L, J_j^L), \min(I_i^U, J_j^U)] | \forall i = 1, \ldots, n_I,\ \forall j = 1, \ldots, n_J\}.$$

Let us suppose that $I \neq \emptyset$ and there exists $i' \in \{1, \ldots, n_I\}$ such that $I_{i'} \neq \{0\}$. Then, as $I \wedge J = \emptyset$, by the definition of intersection:

$$[\min(I_{i'}^L, J_j^L), \min(I_{i'}^U, J_j^U)] = \{0\},\ \forall j = 1, \ldots, n_J.$$

Besides, as $I_{i'} \neq \{0\}$, then $\min(I_{i'}^U, J_j^U) = J_j^U = 0$, and as a consequence, $J_j = \{0\}$ for every $j = 1, \ldots, n_J$, i.e., $J = \emptyset$. ■

The extension of the union and intersection explained in Definition 3.17 for interval-valued hesitant fuzzy sets is detailed in the next result.

Definition 3.20 *Let X be a non-empty set, and $A, B \in IVHFS(X)$. Then,*

- $A \vee B \in IVHFS(X)$ *such that* $\mu_{A \vee B}(x) = \mu_A(x) \vee \mu_B(x)$, $\forall x \in X$,
- $A \wedge B \in IVHFS(X)$ *such that* $\mu_{A \wedge B}(x) = \mu_A(x) \wedge \mu_B(x)$, $\forall x \in X$.

All these results are necessary in the forthcoming sections. As it has been repeatedly stated, the definitions of t-norm and t-conorm are essential in the fuzzy logic, and as a consequence, in the hesitant fuzzy logic.

3.3 Cardinality

The concept of cardinality has always been present in every type of sets. For crisp sets, it is an intuitive concept which is easy to define mathematically. However, when working with fuzzy sets, this definition is not straightforward, and different options have been given by several authors. De Luca and Termini in 1972 (see [25]) defined the σ-count cardinality, which is defined by the sum of all the membership degrees, although this one is hard to read. Ralescu proposed in 1995 (see [61]) the concepts of fuzzy and non-fuzzy cardinality, applied in other papers, such as [60], where it is used for the protection of privacy in microdata using fuzzy partitions. From an axiomatic point of view, Wygralak gives a definition of scalar cardinality for fuzzy sets in 2003 (see [78]), which includes the non-fuzzy cardinality given by Ralescu as it has been proven in Chapter 1. The axiomatic definition given by Wygralak has been extended to interval-valued fuzzy sets by Deschrijver and Král (see [27]), among results around this new definition.

As it has been mentioned before, the hesitant fuzzy logic is a fresh theory. Furthermore, the extension which is taken into account in this work, the interval-valued hesitant fuzzy sets, is even more recent. The aim of this subsection is to study the cardinality for this type of sets. In particular, it is provided an axiomatic definition of cardinality in order to get different cardinalities, without the restrictions of a fixed one. A particular case is also studied, which presents good properties when it is restricted to fuzzy sets, as it matches the non-fuzzy cardinality given by Ralescu. Other results related to the axiomatic definition of cardinality for interval-valued hesitant fuzzy sets are also studied, where some properties of this cardinality are specified.

First of all, the next interval-valued hesitant fuzzy set will be necessary in the axiomatic definitions of cardinality.

Definition 3.21 *Let X be a non-empty set, $a \in FG([0,1])$ and $x \in X$. The set $a/x \in IVHFS(X)$ is characterized by $\mu_{a/x}(x) = a$ and $\mu_{a/x}(y) = 0,\ \forall y \neq x$.*

It is obvious that the cases for fuzzy sets and interval-valued fuzzy sets are particularizations of the previous definition.

Remark 3.22 *Let $a/x \in IVHFS(X)$ as in the previous definition,*

- *if $a \in [0,1]$, then $a/x \in FS(X)$,*
- *if $a \in L([0,1])$, then $a/x \in IVFS(X)$.*

In the following result, the axiomatic definition of cardinality for interval-valued hesitant fuzzy sets is introduced, with the same structure of the one given by Wygralak in Definition 1.45, and which will be the centre of the study of this section.

Definition 3.23 *Consider $X = \{x_1, \ldots, x_N\}$. The mapping $|\cdot| : IVHFS(X) \to [0,\infty)$ is a scalar cardinality measure for interval-valued hesitant fuzzy sets if it satisfies the following properties, for all $A, B \in IVHFS(X)$, $x, y \in X$ and $a, b \in FG([0,1])$:*

1. *$|1/x| = 1$ (coincidence),*
2. *$a \leq_{FG} b \Rightarrow |a/x| \leq |b/y|$, where $a \leq_{FG} b$ represents an ordering relation for finitely generated sets (monotonicity),*
3. *$|A \vee B| = |A| + |B|$ if $A \wedge B = \emptyset$ (additivity).*

Some important properties that scalar cardinalities for interval-valued hesitant fuzzy sets satisfy are shown and proven in the next result.

Proposition 3.24 *Consider* $X = \{x_1, \dots, x_N\}$, $|\cdot| : IVHFS(X) \to [0, \infty)$ *a scalar cardinality measure for interval-valued hesitant fuzzy sets, and* $\leq_{FG}$ *any ordering relation for finitely generated sets. Then,* $|\cdot|$ *satisfies the following properties:*

(i) Given $A_1, \dots, A_n \in IVHFS(X)$ *such that* $A_i \wedge A_j = \emptyset$, $\forall i \neq j$. *Then,*

$$\left| \bigvee_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|.$$

(ii) If A *is a crisp set, then* $|A|$ *is the number of elements in* A.

(iii) Given $x, y \in X$ *and* $a \in FG([0,1])$, *then* $|a/x| = |a/y|$.

(iv) Given $A, B \in IVHFS(X)$ *and a bijection* $\sigma : \{1, \dots, N\} \to \{1, \dots, N\}$ *such that* $\mu_A(x_i) = \mu_B(x_{\sigma(i)})$, $\forall i \in \{1, \dots, N\}$, *then* $|A| = |B|$.

(v) $|\emptyset| = 0$ *and* $|X| = N$.

Proof.

(i) Let $A_1, \dots, A_n \in IVHFS(X)$ such that $A_i \cap A_j = \emptyset$, $\forall i \neq j$. Then:

$$\bigvee_{i=1}^{n} A_i = \left(\bigvee_{i=1}^{n-1} A_i \right) \bigvee A_n \quad \text{and} \quad \left(\bigvee_{i=1}^{n-1} A_i \right) \bigwedge A_n = \emptyset,$$

so by the third axiom of scalar cardinality,

$$\left| \bigvee_{i=1}^{n} A_i \right| = \left| \left(\bigvee_{i=1}^{n-1} A_i \right) \bigvee A_n \right| = \left| \bigvee_{i=1}^{n-1} A_i \right| + |A_n|.$$

Taking the $n-1$ first sets,

$$\bigvee_{i=1}^{n-1} A_i = \left(\bigvee_{i=1}^{n-2} A_i\right) \bigvee A_{n-1} \quad \text{and} \quad \left(\bigvee_{i=1}^{n-2} A_i\right) \bigwedge A_{n-1} = \emptyset,$$

and therefore

$$\left|\bigvee_{i=1}^{n-1} A_i\right| = \left|\left(\bigvee_{i=1}^{n-2} A_i\right) \bigvee A_{n-1}\right| = \left|\bigvee_{i=1}^{n-2} A_i\right| + |A_{n-1}|.$$

Repeating this process for every set,

$$\left|\bigvee_{i=1}^{n} A_i\right| = |A_1| + \cdots + |A_n| = \sum_{i=1}^{n} |A_i|.$$

(ii) If A is a crisp set, then:

$$A = \bigvee_{i \in Supp(A)} 1/x_i,$$

where $Supp(A) = \{x \in X | \mu_A(x) \neq \{0\}\} \subseteq \{1, \ldots, N\}$, which represents the support of the set. Applying the property (i) and the first axiom of scalar cardinality:

$$|A| = \left|\bigvee_{i \in Supp(A)} 1/x_i\right| = \sum_{i \in Supp(A)} |1/x_i| = |Supp(A)| \in \mathbb{N}.$$

However, as A is a crisp set, $Supp(A) = \{x \in X | \mu_A(x) = \{1\}\}$, and as a result, $|A|$ matches the number of elements in A.

(iii) Given $x, y \in X$ and $a \in FG([0,1])$, it is enough to apply the second axiom of scalar cardinality,

$$a \leq_{FG} a \Rightarrow |a/x| \leq |a/y|,$$
$$a \geq_{FG} a \Rightarrow |a/x| \geq |a/y|,$$

and therefore, $|a/x| = |a/y|$.

(iv) Let $A, B \in IVHFS(X)$ and a bijection $\sigma : \{1, \ldots, N\} \rightarrow \{1, \ldots, N\}$ such that $\mu_A(x_i) = \mu_B(x_{\sigma(i)}),\ \forall i \in \{1, \ldots, N\}$. Then,

$$A = \bigvee_{i=1}^{N} \mu_A(x_i)/x_i = \bigvee_{i=1}^{N} \mu_B(x_{\sigma(i)})/x_i \quad \text{and} \quad B = \bigvee_{i=1}^{N} \mu_B(x_i)/x_i.$$

For every $i \in \{1, \ldots, N\}$, $\exists j \in \{1, \ldots, N\}$ such that $\mu_B(x_j) = \mu_B(x_{\sigma(i)})$. As σ is a bijection, and by property (iii):

$$|\mu_B(x_j)/x_i| = |\mu_B(x_{\sigma(i)})/x_i|,\ \forall i \in \{1, \ldots, N\}.$$

Thus, applying property (i),

$$|A| = \left|\bigvee_{i=1}^{N} \mu_A(x_i)/x_i\right| = \left|\bigvee_{i=1}^{N} \mu_B(x_{\sigma(i)})/x_i\right| = \sum_{i=1}^{N} |\mu_B(x_{\sigma(i)})/x_i| = |B|.$$

(v) Given $A, \emptyset \in IVHFS(X)$, it is obvious that $A \wedge \emptyset = \emptyset$. Applying the third axiom of scalar cardinality:

$$|A| = |A \vee \emptyset| = |A| + |\emptyset|,$$

and therefore, $|\emptyset| = 0$.

In order to obtain the cardinality of X, this set can be decomposed as follows

$$X = \bigvee_{i=1}^{N} 1/x_i.$$

Applying the property (i) and the first axiom of scalar cardinality,

$$|X| = |\bigvee_{i=1}^{N} 1/x_i| = \sum_{i=1}^{N} |1/x_i| = N. \quad \blacksquare$$

Usually some definitions require certain conditions that are hard to satisfy. A way to overcome such problem is through results that simplify them. In the next theorem, a simplification is proposed in order to obtain scalar cardinalities in an easier way than the original definition.

Theorem 3.25 *Let $X = \{x_1, \ldots, x_N\}$, and $\leq_{FG}$ any ordering relation for finitely generated sets. Then, the mapping $|\cdot| : IVHFS(X) \to [0, \infty)$ is a scalar cardinality for interval-valued hesitant fuzzy sets if and only if, there exists a mapping $f : FG([0,1]) \to [0,1]$ that satisfies the following properties:*

1. $f(\{0\}) = 0$ *and* $f(\{1\}) = 1$,

2. *if* $a, b \in FG([0,1])$ *such that* $a \leq_{FG} b$, *then* $f(a) \leq f(b)$,

such that for every $A \in IVHFS(X)$:

$$|A| = \sum_{i=1}^{N} f(\mu_A(x_i)).$$

Proof. Firstly, let us suppose that $|\cdot|$ is a scalar cardinality. Let us define the mapping $f : FG([0,1]) \to [0,1]$, where $f(a) = |a/x|$, with $a \in FG([0,1])$ and whichever $x \in X$ (as it does not matter this choice by property (iii) in Proposition 3.24).

As any set $A \in IVHFS(X)$ can be decomposed as:

$$A = \bigvee_{i=1}^{N} \mu_A(x_i)/x_i,$$

by property (i) of Proposition 3.24,

$$|A| = \left|\bigvee_{i=1}^{N} \mu_A(x_i)/x_i\right| = \sum_{i=1}^{N} |\mu_A(x_i)/x_i| = \sum_{i=1}^{N} |\mu_A(x_i)/x| = \sum_{i=1}^{N} f(\mu_A(x_i)).$$

Therefore, it is enough to see that f satisfies the two properties of the theorem:

1. Applying property (v) of Proposition 3.24 and the first axiom of scalar cardinality respectively:

$$\begin{aligned} f(\{0\}) &= |0/x| = |\emptyset| = 0, \\ f(\{1\}) &= |1/x| = 1. \end{aligned}$$

2. Given $a, b \in FG([0,1])$ such that $a \leq_{FG} b$, then by the second axiom of scalar cardinality, $|a/x| \leq |b/x|$, and by definition of f, $f(a) \leq f(b)$.

In the second part of the proof it is supposed that there exists a mapping f as defined in the theorem. To see that $|\cdot|$ is a scalar cardinality, the three axioms of Definition 3.23 must be proven.

1. Given $x \in X$:

$$|1/x| = \sum_{i=1}^{N} f(\mu_{1/x}(x_i)) = f(\mu_{1/x}(x)) = f(\{1\}) = 1.$$

2. Given $x, y \in X$, and $a, b \in FG([0,1])$ such that $a \leq_{FG} b$, by hypothesis, $f(a) \leq f(b)$. By definition of the sets $|a/x|$ and $|b/y|$,

$$\begin{aligned} |a/x| &= \sum_{i=1}^{N} f(\mu_{a/x}(x_i)) = f(\mu_{a/x}(x)) = f(a) \leq f(b) = \\ &= f(\mu_{b/y}(y)) = \sum_{i=1}^{N} f(\mu_{b/y}(x_i)) = |b/y|. \end{aligned}$$

3. Let $A, B \in IVHFS(X)$ such that $A \wedge B = \emptyset$. By Proposition 3.19, it is known that $\mu_A(x) = \emptyset$ or $\mu_B(x) = \emptyset$ for every $x \in X$, which means that the intersection of the support is null. In our case:

$$Supp(A) \cap Supp(B) = \emptyset \text{ and } Supp(A \vee B) = Supp(A) \cup Supp(B).$$

Thus,

$$\begin{aligned}
|A \vee B| &= \sum_{i=1}^{N} f(\mu_{A\vee B}(x_i)) = \sum_{Supp(A\vee B)} f(\mu_{A\vee B}(x_i)) = \\
&= \sum_{Supp(A)} f(\mu_{A\vee B}(x_i)) + \sum_{Supp(B)} f(\mu_{A\vee B}(x_i)) = \\
&= \sum_{Supp(A)} f(\mu_{A}(x_i)) + \sum_{Supp(B)} f(\mu_{B}(x_i)) = \\
&= \sum_{i=1}^{N} f(\mu_{A}(x_i)) + \sum_{i=1}^{N} f(\mu_{B}(x_i)) = |A| + |B|,
\end{aligned}$$

as for every $x \notin Supp(A)$, $\mu_A(x) = \{0\}$ and as a result $f(\mu_A(x)) = 0$. Respectively for B and $A \vee B$.

Therefore, the three axioms have been proven, and the result is demonstrated. ■

In the following result, some added properties satisfied for scalar cardinalities are given for fixed orders.

Proposition 3.26 *Consider* $X = \{x_1, \dots, x_N\}$, $|\cdot| : IVHFS(X) \to [0, \infty)$ *a scalar cardinality measure for interval-valued hesitant fuzzy sets, and the ordering relation* $\leq_{2I}$. *Then,* $|\cdot|$ *satisfies the following properties:*

(i) Given $A, B \in IVHFS(X)$ *such that* $A \leq_{2I} B$, *then* $|A| \leq |B|$.

(ii) $0 \leq |A| \leq N$, $\forall A \in IVHFS(X)$.

Proof.

(i) Let $A, B \in IVHFS(X)$ such that $A \leq_{2I} B$. By definition:

$$A = \bigvee_{i=1}^{N} \mu_A(x_i)/x_i \quad \text{and} \quad B = \bigvee_{i=1}^{N} \mu_B(x_i)/x_i.$$

Furthermore,

$$\mu_A(x_i)/x_i \bigwedge \mu_A(x_j)/x_j = \emptyset \text{ and } \mu_B(x_i)/x_i \bigwedge \mu_B(x_j)/x_j = \emptyset, \ \forall i \neq j.$$

By Proposition 3.14,

$$A \leq_{2I} B \Leftrightarrow \mu_A(x_i) \leq_2 \mu_B(x_i), \ \forall x_i \in X.$$

and by the second axiom of a scalar cardinality,

$$|\mu_A(x_i)/x_i| \leq |\mu_B(x_i)/x_i|, \ \forall x_i \in X.$$

As a result, and applying the previous property,

$$\begin{aligned} |A| &= |\bigvee_{i=1}^{N} \mu_A(x_i)/x_i| = \sum_{i=1}^{N} |\mu_A(x_i)/x_i| \leq \\ &\leq \sum_{i=1}^{N} |\mu_B(x_i)/x_i| = |\bigvee_{i=1}^{N} \mu_B(x_i)/x_i| = |B|. \end{aligned}$$

(ii) Given $A \in IVHFS(X)$, it is obvious (see Proposition 3.14) that $\emptyset \leq_{2I} A \leq_{2I} X$. Applying property (i) to both inequalities and the property (v) from Proposition 3.24:

$$0 = |\emptyset| \leq A \leq |X| = N. \quad \blacksquare$$

Remark 3.27 *Previous properties are also satisfied for $\leq_{1I}$ (see Proposition 3.13), with a slight modification in the proof.*

Once the theoretical results has been stated, a subsection with particular cases and examples is given next.

3.3.1 Examples

This axiomatic definition classifies a wide range of functions as cardinalities for interval-valued hesitant fuzzy sets, avoiding the restriction to a fixed one. The first example studied, which is obtained using a previous result, has a resemblance to a well known cardinality for fuzzy sets: the σ-count cardinality.

Example 3.28 *Consider* $X = \{x_1, \ldots, x_N\}$, *and* $\leq_1$ *the ordering relation for finitely generated sets. Then,* $|\cdot| : IVHFS(X) \to [0, \infty)$ *defined as*

$$|A| = \sum_{i=1}^{N} H(\mu_A(x_i)), A \in IVHFS(X),$$

with H *the accuracy function, is a scalar cardinality for interval-valued hesitant fuzzy sets.*

In order to prove it, it is enough to see that the accuracy function H *satisfies both axioms of Theorem 3.25:*

1. $H(\{0\}) = 0$ *and* $H(\{1\}) = 1$ *by definition.*

2. *Given* $a, b \in FG([0,1])$ *such that* $a \leq_1 b$, *if this order satisfies condition* (a) *of Proposition 3.1, then* $H(a) < H(b)$. *If it satisfies condition* (b), *then* $H(a) = H(b)$. *Thus,* $H(a) \leq H(b)$.

Remark 3.29 *The scalar cardinality defined in the previous example matches the* σ*-count cardinality when it is restricted to fuzzy sets.*

Another particular case is highlighted in this section due to the special properties that it presents when it is restricted to fuzzy sets.

Firstly, a way to obtain a fuzzy set from an interval-valued hesitant fuzzy set using the accuracy function introduced in Proposition 3.1 is proposed. Furthermore, a result related to this definition is stated right after.

Definition 3.30 *Consider* $X = \{x_1, \ldots, x_N\}$*,* $A \in IVHFS(X)$ *and* H *the accuracy function.* $A' = \{(x, \mu_{A'}(x)) | x \in X\}$ *obtained from* A *as*

$$\mu_{A'}(x) = H(\mu_A(x)), \ \forall x \in X,$$

is a fuzzy set.

Proposition 3.31 *Let* $A, B \in IVHFS(X)$ *such that* $A \wedge B = \emptyset$*. Then,*

$$(A \vee B)' = A' \cup B',$$

where $\cup$ *represents the fuzzy t-norm of maximum.*

Proof. By hypothesis, $A \wedge B = \emptyset$, so $\mu_{A \wedge B}(x) = \emptyset$, and by Proposition 3.19, $\mu_A(x) = \emptyset$ or $\mu_B(x) = \emptyset$, $\forall x \in X$.

Fixed an element $x \in X$, let us suppose without loss of generality that $\mu_A(x) \neq \emptyset$ and $\mu_B(x) = \emptyset$, where $\mu_A(x)$ is given as follows:

$$\mu_A(x) = \bigcup_{i=1}^{n_x^A} A_i^x = \bigcup_{i=1}^{n_x^A} [A_i^{x^L}, A_i^{x^U}].$$

By Definition 3.17,

$$\mu_{A \vee B}(x) = \dot{\bigcup} \{[max(A_i^{x^L}, 0), max(A_i^{x^U}, 0)] | \forall i = 1, \ldots, n_x^A\} = \mu_A(x),$$

and therefore, $\mu_{(A \vee B)'}(x) = H(\mu_A(x))$.

On the other hand:

$$\mu_{A' \cup B'}(x) = max(H(\mu_A(x)), H(\mu_B(x))) = H(\mu_A(x)),$$

as $\mu_B(x) = \emptyset$ and as a result $H(\mu_B(x)) = 0$.

Thus, $\mu_{(A \vee B)'}(x) = \mu_{A' \cup B'}(x)$, $\forall x \in X$, and as a consequence,

$$(A \vee B)' = A' \cup B'. \quad \blacksquare$$

In the following example, the aforementioned particular case is shown, where it is structured in the same way as Ralescu non fuzzy cardinality detailed in Definition 1.42.

Example 3.32 *Let X be the set with elements $\{x_1, \ldots, x_N\}$, and the set $A \in IVHFS(X)$ where*

$$\mu_A(x_i) = \bigcup_{j=1}^{n_{x_i}} A_j^{x_i} = \bigcup_{j=1}^{n_{x_i}} [A_j^{x_i^L}, A_j^{x_i^U}],\ \forall i = 1, \ldots, N,$$

and $A' \in FS(X)$ obtained from the set A. The finitely generated sets $\mu_{A'}(x_i)$ $\forall i = 1, \ldots, N$, are ordered decreasingly, where $\mu_{(i)}$ denotes the i-th largest value such that:

$$1 = \mu_{(0)} \geq \mu_{(1)} \geq \cdots \geq \mu_{(N)} \geq \mu_{(N+1)} = 0.$$

Then, the function $|\cdot|_{RH} : IVHFS(X) \to [0, \infty)$ is defined by:

$$|A|_{RH} = \begin{cases} 0, & \textit{if } A = \emptyset, \\ j, & \textit{if } A \neq \emptyset \textit{ and } \mu_{(j)} \geq 0.5, \\ j-1, & \textit{if } A \neq \emptyset \textit{ and } \mu_{(j)} < 0.5. \end{cases}$$

where

$$j = max\{1 \leq t \leq N | \mu_{(t-1)} + \mu_{(t)} > 1\}.$$

Remark 3.33 *It is immediate to see that the function $|\cdot|_{RH}$ restricted to fuzzy sets matches Ralescu's cardinality for fuzzy sets (Definition 1.42), as if $A \in FS(X)$, $H(\mu_A(x_i)) = \mu_A(x)$ $\forall x \in X$, so $A' = A$, and the rest of the process is the same. It also must be noted that $|A|_{RH} = |A'|_R$.*

In the next result, it is proven that the function previously defined is a scalar cardinality.

Proposition 3.34 *Let $\leq_1$ be the selected ordering relation for finitely generated sets, and $\cap$ and $\cup$ be the minimum-maximum dual pair of t-norm and t-conorm for fuzzy sets. Then, the function $|\cdot|_{RH}$ is a scalar cardinality measure for interval-valued hesitant fuzzy sets with respect to $\leq_1$, $\cap$ and $\cup$.*

Proof. In order to see that it is a scalar cardinality, the three axioms of Definition 3.23 must be proven.

1. Given $1/x \in IVHFS(X)$, then $\mu_{1/x}(x) = 1$ and $\mu_{1/x}(y) = 0,\ \forall y \neq x$. Furthermore, $H(\mu_{1/x}(x)) = 1$ and $H(\mu_{1/x}(y)) = 0,\ \forall y \neq x$, i.e., $1/x' = 1/x$. Thus, the ordered membership degrees are the following:

$$1 = \mu_{(1)} > \mu_{(2)} = \cdots = \mu_{(N)} = 0.$$

   Hence, $j = 1$, and as $\mu_{(1)} = 1$, $|1/x|_{RH} = 1$.

2. Given $a, b \in FG([0,1])$ such that $a \leq_1 b$, and $x, y \in X$, then $|a/x|_{RH}$ and $|b/y|_{RH}$ are defined as follows:

$$\mu_{a/x}(x) = a, \quad \mu_{a/x}(z) = 0,\ \forall z \neq x,$$
$$\mu_{b/y}(y) = b, \quad \mu_{b/y}(z) = 0,\ \forall z \neq y.$$

   In order to obtain the sets a/x' and b/y' as in Definition 3.30, the accuracy function is applied to every membership degree,

$$\mu_{a/x'}(x) = H(\mu_{a/x}(x)) = H(a), \quad \mu_{a/x'}(z) = H(\mu_{a/x}(z)) = 0,\ \forall z \neq x,$$
$$\mu_{b/y'}(y) = H(\mu_{b/y}(y)) = H(b), \quad \mu_{b/y'}(z) = H(\mu_{b/y}(z)) = 0,\ \forall z \neq y.$$

   Thus, the ordered membership degrees for a/x' is given as follows,

$$H(a) \geq 0 = \cdots = 0,$$

   and for the set b/y',

$$H(b) \geq 0 = \cdots = 0.$$

And as a consequence, $j^{a/x} = j^{b/y} = 1$. Furthermore, as $a \leq_1 b$ by Proposition 3.1, if it is satisfied by condition (1), then $H(a) < H(b)$, and if it is satisfied by condition (2), then $H(a) = H(b)$, i.e., $H(a) \leq H(b)$. Now, let us distinguish the three possible situations:

- if $H(a) \leq H(b) < 0.5$, then $|a/x|_{RH} = |b/y|_{RH} = 0$,
- if $H(a) < 0.5 \leq H(b)$, then $|a/x|_{RH} = 0 < |b/y|_{RH} = 1$,
- $0.5 \leq H(a) \leq H(b)$, then $|a/x|_{RH} = |b/y|_{RH} = 1$,

and therefore, $|a/x|_{RH} \leq |b/y|_{RH}$.

3. Let $A, B \in IVHFS(X)$ such that $A \wedge B = \emptyset$, and $A', B' \in FS(X)$ the sets according to Definition 3.30.

   By hypothesis, $A \wedge B = \emptyset$. In addition, by Proposition 3.19,

$$\mu_A(x) = \emptyset \text{ or } \mu_B(x) = \emptyset,\ \forall x \in X.$$

   As a consequence, $\forall x \in X$:

$$\mu_{A'}(x) = H(\mu_A(x)) = H(0) = 0 \quad \text{or} \quad \mu_{B'}(x) = H(\mu_B(x)) = H(0) = 0,$$

   so $\min(\mu_{A'}(x), \mu_{B'}(x)) = 0,\ \forall x \in X$, and therefore, $A' \cap B' = \emptyset$. Furthermore, by Proposition 1.43 applied to $A', B' \in FS(X)$, as $|A' \cap B'|_R = 0$:

$$|A' \cup B'|_R = |A'|_R + |B'|_R,$$

   However, by Proposition 3.31, $(A \vee B)' = A' \cup B'$, and obviously, $(A \vee B)', A' \cup B' \in FS(X)$, so $|(A \vee B)'|_R = |A' \cup B'|_R$. Hence,

$$|(A \vee B)'|_R = |A'|_R + |B'|_R,$$

but as it has been stated in Remark 3.33, $|A|_{RH} = |A'|_R$, and as a result:

$$|(A \vee B)'|_R = |A'|_R + |B'|_R \Rightarrow |A \vee B|_{RH} = |A|_{RH} + |B|_{RH}.$$

As a consequence, the three axioms have been demonstrated and it is proven that $|\cdot|_{RH}$ is a scalar cardinality. ■

This previous result shows that the function given in Example 3.32 is a scalar cardinality for interval-valued hesitant fuzzy sets given by Definition 3.23. Furthermore, when restricted to fuzzy sets it matches Ralescu's cardinality, which is also a scalar cardinality for fuzzy sets given by Definition 1.45.

Obviously, the definition of scalar cardinality for interval-valued hesitant fuzzy sets, when restricted to fuzzy sets, matches the definition for fuzzy sets.

In conclusion, this section has been focused in an axiomatic definition of cardinality, along with several results that make it possible to prove certain properties, as well as various examples which, when restricted to fuzzy sets, match well known cardinalities for such sets.

3.4 Entropy

The study of entropy measures in the fuzzy set theory became an important part of this research, firstly defined by De Luca and Termini in 1972 (see [25]), whose aim was to quantify the uncertainty associated to a fuzzy set. This concept has been adapted to other types of fuzzy sets, such as Atanassov's intuitionistic fuzzy sets (see [54]), interval-valued fuzzy sets (see [16]) or even interval-valued hesitant fuzzy sets (see [33]).

Nevertheless, the existing definition of entropy for interval-valued hesitant fuzzy sets in [33] only reflects one type of uncertainty, associated to how distant a set is from a union of crisp sets. Our proposal along this section is to define a new entropy measure for interval-valued hesitant fuzzy sets, where three types of uncertainty are reflected through three mappings, instead of the classical concept of just one function for one type of uncertainty associated. In addition, several results have been developed in order to obtain such mappings with ease, and as a result, the entropy measure can be obtained with simpler conditions.

Remark 3.35 *Along this section, the ordering relations considered for finitely generated sets and interval-valued hesitant fuzzy sets are* $\leq_1$ *and* $\leq_{1I}$*, respectively.*

As it has been aforementioned, Farhadinia made in [33] a study of entropy for this type of sets. Firstly, his definition of dissimilarity for interval-valued hesitant fuzzy sets is given next, which we will use along this section.

Definition 3.36 *A mapping $D : IVHFS(X) \times IVHFS(X) \to [0,1]$ is a hesitant dissimilarity measure if it satisfies the following properties, where $A, B, C \in IVHFS(X)$:*

1. $D(A,B) = D(B,A)$,
2. $D(A, A^c) = 1 \Leftrightarrow A$ *is crisp,*
3. $D(A,B) = 0 \Leftrightarrow A = B$,
4. *If $A \leq_{1I} B \leq_{1I} C$, then $D(A,B) \leq D(A,C)$ and $D(B,C) \leq D(A,C)$.*

Some examples of hesitant dissimilarity measures were proposed by Xu and Xia (see [79]). In the next example, the Hamming distance is introduced as dissimilarity measure. This distance will be used further in this section.

Example 3.37 *Let X be a finite set with cardinality N, the Hamming distance is defined as*

$$D_H(A,B) = \frac{1}{N}\sum_{x\in X}\left[\frac{1}{2n_x}\sum_{i=1}^{n_x}(|A_i^{x^L} - B_i^{x^L}| + |A_i^{x^U} - B_i^{x^U}|)\right],$$

for all $A, B \in IVHFS(X)$ where $\mu_A(x) = \bigcup_{i=1}^{n_x} A_i^x = \bigcup_{i=1}^{n_x}[A_i^{x^L}, A_i^{x^U}]$ with $A_i^x \leq_{XY} A_{i+1}^x$ (Definition 1.54) for every $x \in X$ and $i \in \{1, \ldots, n_x - 1\}$, and analogously for the set B.

The entropy definition for interval-valued hesitant fuzzy sets provided by Farhadinia is detailed next. It is associated to a hesitant dissimilarity measure.

Definition 3.38 *Let $E : IVHFS(X) \to [0,1]$ be a mapping, and D a hesitant dissimilarity measure. E is said to be a hesitant entropy measure associated to D if it satisfies, where $A, B \in IVHFS(X)$:*

1. $E(A) = 0 \Leftrightarrow \mu_A(x) \subseteq \{0, 1\}\ \forall x \in X$,

2. $E(A) = 1 \Leftrightarrow A$ *is the equilibrium set,*

3. $E(A) = E(A^c)$,

4. $E(A) \leq E(B)$, *if* $D(A, \xi) \geq D(B, \xi)$.

However, this definition only takes into account the distance to the equilibrium set, which may not be enough to quantify the uncertainty associated to an interval-valued hesitant fuzzy set.

In order to overcome this, a different definition of entropy is studied. It is characterized by three mappings instead of just one, as Pal *et. al* (see [54]) did for Atanassov's intuitionistic fuzzy sets with two different mappings.

Hence, the new entropy proposed for interval-valued hesitant fuzzy sets is split into three functions: E_F (fuzziness), E_L (lack of knowledge) and E_H (hesitance). They are studied separately in the next three subsections, representing each one a different type of uncertainty associated to an interval-valued hesitant fuzzy set. This makes it possible to provide a more detailed entropy measure.

3.4.1 Fuzziness entropy measure

The first function of the interval-valued hesitant fuzzy entropy, is the one that represents the fuzziness of the set. The goal of this function is to measure how distant the set is from the union of a finite number of crisp sets. This mapping is similar to the one given by Definition 3.38, but with a modification in the first and last axioms, which makes it more efficient in order to represent this part of the entropy.

Definition 3.39 *Let $E_F : IVHFS(X) \to [0,1]$ be a mapping, $A, B \in IVHFS(X)$. E_F is said to be a fuzziness entropy measure associated to a hesitant dissimilarity measure D if it satisfies the following properties:*

1. $E_F(A) = 0 \Leftrightarrow \mu_A(x) \in \{0, 1, \{0,1\}, [0,1]\},\ \forall x \in X$,

2. $E_F(A) = 1 \Leftrightarrow A$ *is the equilibrium set,*

3. $E_F(A) = E_F(A^c)$,

4. $E_F(A) \leq E_F(B)$, *if* $D(A_x, \xi) \geq D(B_x, \xi)$ $\forall x \in X$, *where* $A_x, B_x \in IVHFS(X)$ *are given such that* $\mu_{A_x}(y) = \mu_A(x)$ *and* $\mu_{B_x}(y) = \mu_B(x)$, $\forall y \in X$.

The first axiom states that the fuzziness is null if the membership function is the union of crisp sets or the pure interval-valued fuzzy set. In the second axiom, the maximum fuzziness happens when the set is the equilibrium. The third one requires a set and its complement to take the same entropy. The fourth axiom states that two interval-valued hesitant fuzzy sets are compared with respect to E_F using the associated hesitant dissimilarity measure. In fact, the definition of fuzziness entropy is related to the dissimilarity, but it is not detailed in all the cases, since there is not ambiguity.

Furthermore, local property is obtained for this entropy measure in the case of finite referential sets. To develop it, some notation is necessary.

Definition 3.40 *Let $X = \{x_1, \ldots, x_N\}$ be a finite set with cardinality N. Given $A \in IVHFS(X)$ and $M \subseteq \{1, \ldots, N\}$, $A^{(M)}$ is an interval-valued* *hesitant fuzzy set whose membership function is defined* $\forall x_i \in X$ *as*

$$\mu_{A^{(M)}}(x_i) = \begin{cases} \mu_A(x_i) & \text{if } i \notin M, \\ \{0\} & \text{if } i \in M \text{ and } A_{x_i} \leq_{1I} \xi, \\ \{1\} & \text{if } i \in M \text{ and } A_{x_i} >_{1I} \xi. \end{cases}$$

Remark 3.41 *Note that as both sets* A_{x_i} *and* ξ *have constant membership functions (in* $\mu_A(x_i)$ *and* $\{0.5\}$ *respectively),* $A_{x_i} \leq_{1I} \xi$ *or* $A_{x_i} >_{1I} \xi$ *must hold, as this order is equivalent to the one given between the finitely generated sets* $\mu_A(x_i)$ *and* $\{0.5\}$ *with respect to the ordering relation* $\leq_1$ *(Corollary 3.8).*

It is clear that $A^{(M)}$ is only different to A in $x_i \in X$ with $i \in M$. In the particular case $M = \{j\}$, the notation is simplified to $A^{(j)}$. In next definition, the local property is given for this entropy.

Definition 3.42 *Let* X *be a finite set with cardinality* N *and the mapping* $E_F : IVHFS(X) \to [0,1]$ *a fuzziness entropy measure.* E_F *is said to be a local fuzziness entropy measure if there exists a function* $f : FG([0,1]) \to [0,1]$ *such that for every* $x_j \in X$*, given* $A \in IVHFS(X)$*,*

$$E_F(A) - E_F(A^{(j)}) = f(\mu_A(x_j)),$$

or equivalently, it only depends on the term $\mu_A(x_j)$*.*

Remark 3.43 *Note that* $E_F(A) - E_F(A^{(j)}) \in [0,1]$ *for all* $j = 1, \ldots, N$*. It is enough to see that* $D(A_{x_j}, \xi) \leq D(A^{(j)}_{x_j}, \xi)$*, as for the other* $x \in X$*, the equality is immediate.*

- *If* $A_{x_j} \leq_{1I} \xi$*, then* $A^{(j)}_{x_j} = \{(x,0) | x \in X\} = \emptyset$*. Hence,* $\emptyset = A^{(j)}_{x_j} \leq_{1I} A_{x_j} \leq_{1I} \xi$*. By the last property in Definition 3.36,* $D(A_{x_j}, \xi) \leq D(\emptyset, \xi) = D(A^{(j)}_{x_j}, \xi)$*, and by the last condition of a fuzziness entropy,* $E_F(A^{(j)}) \leq E_F(A)$*.*

- *If* $A_{x_j} >_{1I} \xi$*, then* $A^{(j)}_{x_j} = \{(x,1)|x \in X\} = X$*. Hence,* $\xi \leq_{1I} A_{x_j} \leq_{1I} A^{(j)}_{x_j} = X$*. By the last property in Definition 3.36,* $D(A_{x_j}, \xi) \leq D(X, \xi) = D(A^{(j)}_{x_j}, \xi)$*, and by the last condition of a fuzziness entropy,* $E_F(A^{(j)}) \leq E_F(A)$*.*

Henceforth, two results have been presented in order to ease the obtaining of local fuzziness entropy measures with functions whose properties are more manageable than the original ones in the definition of such entropy. Initially, the local fuzziness entropies are characterized by means of the following result.

Theorem 3.44 *Let* X *be a finite set with cardinality* N*,* E_F *be the mapping* $E_F : IVHFS(X) \to [0,1]$ *and* D *a hesitant dissimilarity measure. Then,* E_F *is a local fuzziness entropy measure associated to* D *if and only if there exists a mapping* $h : FG([0,1]) \to [0,1]$ *such that*

$$E_F(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)),$$

which also satisfies the following four axioms, given $I, J \in FG([0,1])$*:*

1. $h(I) = 0 \Leftrightarrow I \in \{0, 1, \{0,1\}, [0,1]\}$,
2. $h(I) = 1 \Leftrightarrow I = \mu_\xi(x)$,
3. $h(I) = h(I^c)$,
4. $h(I) \leq h(J)$ *if* $D(X_I, \xi) \geq D(X_J, \xi)$*, where* $X_I = \{(x, I)|x \in X\}$ *and* $X_J = \{(x, J)|x \in X\}$*.*

Proof. First, let us suppose that E_F is a local fuzziness entropy. Then, by Definition 3.42, there exists a function $f : FG([0,1]) \to [0,1]$ such that:

$$E_F(A) - E_F(A^{(j)}) = f(\mu_A(x_j)),\ \forall j \in \{1, \ldots, N\}.$$

Given $A \in IVHFS(X)$, applying recursively the definition of local:

$$\begin{aligned} E_F(A) &= E_F(A^{(N)}) + f(\mu_A(x_N)) = \\ &= E_F((A^{(N)})^{(N-1)}) + f(\mu_{A^{(N)}}(x_{N-1}) + f(\mu_A(x_N)) = \\ &= E_F(A^{(N-1,N)}) + f(\mu_A(x_{N-1}) + f(\mu_A(x_N)) = \cdots = \\ &= E_F(A^{(1,\ldots,N)}) + \sum_{x \in X} f(\mu_A(x)). \end{aligned}$$

However, $\mu_{A^{(1,\ldots,N)}}(x) \in \{0,1\}$, $\forall x \in X$, i.e., $A^{(1,\ldots,N)}$ is a crisp set, and therefore, by the first axiom of fuzziness entropy, $E_F(A^{(1,\ldots,N)}) = 0$. Hence,

$$E_F(A) = \sum_{x \in X} f(\mu_A(x)).$$

In addition, it is known that $E_F(A) \in [0,1]$ for every $A \in IVHFS(X)$. Then, for all $x_i \in X$, applying the mapping E_F to the set $X_{\mu_A(x_i)}$:

$$E_F(X_{\mu_A(x_i)}) = \sum_{x \in X} f(\mu_A(x_i)) = Nf(\mu_A(x_i)) \in [0,1] \Rightarrow f(\mu_A(x_i)) \in [0, \frac{1}{N}].$$

Consequently, taking $h : FG([0,1]) \to [0,1]$ such that $h(I) = Nf(I)$, it is immediate that:

$$E_F(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)).$$

Now, let us see that h satisfies the four conditions of the theorem.

1. Given $I \in FG([0,1])$ and $X_I \in IVHFS(X)$, then:

   $$E_F(X_I) = \frac{1}{N} \sum_{x \in X} h(I) = h(I),$$

   so by the first axiom of Definition 3.39, it is known that:

   $$E_F(X_I) = h(I) = 0 \quad \Leftrightarrow \quad \mu_A(x) = I \in \{0, 1, \{0,1\}, [0,1]\}.$$

2. Given $I \in FG([0,1])$ and $X_I \in IVHFS(X)$ such that $E_F(X_I) = h(I)$. From the second axiom of E_F, it is obvious that:

$$E_F(X_I) = h(I) = 1 \quad \Leftrightarrow \quad \mu_A(x) = I = \{0.5\} = \mu_\xi(x).$$

3. Given $I \in FG([0,1])$, and $X_I \in IVHFS(X)$, it is obtained that $E_F(X_I) = h(I)$ and $E_F(X_{I^c}) = h(I^c)$, and as E_F satisfies the third axiom of Definition 3.39, $E_F(X_I) = E_F(X_{I^c})$ and hence,

$$h(I) = h(I^c).$$

4. Given $I, J \in FG([0,1])$, and $X_I, X_J \in IVHFS(X)$, it is supposed that $D(X_I, \xi) \geq D(X_J, \xi)$, where by construction, $E_F(X_I) = h(I)$ and $E_F(X_J) = h(J)$. Due to E_F being a fuzziness entropy, the fourth axiom is satisfied and:

$$E_F(X_I) \leq E_F(X_J) \Leftrightarrow h(I) \leq h(J).$$

Now, in order to proceed with the second part of the proof, it is supposed that h satisfies the four conditions of the theorem, so it is necessary to prove that E_F is a local fuzziness entropy. First, let us see that it satisfies the four axioms of Definition 3.39:

1. Given $A \in IVHFS(X)$:

$$0 = E_F(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) \Leftrightarrow h(\mu_A(x)) = 0, \forall x \in X,$$

and as h satisfies the first item of the theorem, this only happens when:

$$\mu_A(x) \in \{0, 1, \{0,1\}, [0,1]\}, \; \forall x \in X.$$

2. Given $A \in IVHFS(X)$:

$$1 = E_F(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) \Leftrightarrow h(\mu_A(x)) = 1, \forall x \in X,$$

which is the same as $\mu_A(x) = 0.5, \forall x \in X$, as h fulfills the second axiom of the theorem.

3. Given $A \in IVHFS(X)$ and A^c its complement, as h satisfies the third item of the theorem, it is known that $h(J) = h(J^c)$ for every finitely generated set, therefore:

$$\begin{aligned} E_F(A) &= \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)^c) = \\ &= \frac{1}{N} \sum_{x \in X} h(\mu_{A^c}(x)) = E_F(A^c). \end{aligned}$$

4. Let $A, B \in IVHFS(X)$ such that $D(A_x, \xi) \geq D(B_x, \xi)$ $\forall x \in X$. By the fourth axiom of the theorem for $I = \mu_A(x)$ and $J = \mu_B(x)$, $h(\mu_A(x)) \leq h(\mu_B(x))$ $\forall x \in X$, hence by construction of the mapping E_F:

$$E_F(A) \leq E_F(B).$$

In order to close the proof, let us see that it is also a local fuzziness entropy measure (Definition 3.42):

(L) Given $A \in IVHFS(X)$, for every $x_j \in X$:

$$\begin{aligned} & E_F(A) - E_F(A^{(j)}) = \\ &= \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) - \frac{1}{N} \left(\sum_{x \in X \setminus \{x_j\}} h(\mu_A(x)) + h(\mu_{A^{(j)}}(x_j)) \right) = \\ &= \frac{1}{N} (h(\mu_A(x_j)) - h(\mu_{A^{(j)}}(x_j)) = \frac{1}{N} h(\mu_A(x_j)) = f(\mu_A(x_j)), \end{aligned}$$

i.e., it only depends on the term $\mu_A(x_j)$ for every j as $\mu_{A^{(j)}}(x_j) \in \{0,1\}$ and by hypothesis, $h(\mu_{A^{(j)}}(x_j)) = 0$. Therefore, it is local. ■

After the simplification provided by the previous theorem, the next result goes another step forward and ease even more the obtaining of a local fuzziness entropy.

Corollary 3.45 *Let X be a finite set with cardinality N, let E_F be the mapping $E_F : IVHFS(X) \to [0,1]$ and let D be a hesitant dissimilarity measure where $D(A,\xi)$ is defined in function of the terms $|A_i^{x^U} - 0.5|$ and $|A_i^{x^L} - 0.5|$, and where $D(A,\xi) = 0.5$ if and only if $A_i^{x^L}, A_i^{x^U} \in \{0,1\}$ for every $x \in X$ and $i \in \{1,\dots,n_x\}$.*

Then, E_F is a local fuzziness entropy associated to D if and only if there exists a mapping $g : [0,1] \to [0,1]$ such that

$$E_F(A) = \frac{1}{N}\sum_{x\in X} g(2D(A_x,\xi)),$$

which also satisfies the following properties:

1. *$g(a) = 0 \Leftrightarrow a = 1$,*
2. *$g(a) = 1 \Leftrightarrow a = 0$,*
3. *g is monotone decreasing.*

Proof. It is enough to see that the function $h(I) = g(2D(X_I,\xi))$ satisfies the four axioms in Theorem 3.44, and the result will be proven.

1. Let $I \in FG([0,1])$ such that $h(I) = g(2D(X_I,\xi)) = 0$, and by the second axiom that g satisfies:

$$h(I) = g(2D(X_I,\xi)) = 0 \Leftrightarrow 2D(X_I,\xi) = 1 \Leftrightarrow D(X_I,\xi) = 0.5.$$

Given $I = I_1 \cup \cdots \cup I_{n_I} \in FG_{n_I}([0,1])$ with $I_i = [I_i^L, I_i^U]$ $\forall i$, by the hypothesis about D, this only happens when $I_i^L, I_i^U \in \{0,1\}$ $\forall i$, or what is the same, $I_i \in \{0, 1, [0,1]\}$ $\forall i$. Equivalently, $I \in \{0, 1, \{0,1\}, [0,1]\}$.

2. Given $I \in FG([0,1])$ such that $h(I) = g(2D(X_I, \xi)) = 1$. This only holds when $D(X_I, \xi) = 0$ for the first axiom that g satisfies, and by the definition of hesitant distance (third axiom of Definition 3.36) $I = \{0,5\} = \mu_\xi(x)$.

3. Given $I \in FG([0,1])$, as 0.5 is the center of the interval $[0,1]$, by symmetry and the hypothesis about how D is defined, $D(X_I, \xi) = D(X_{I^c}, \xi)$, and it is immediate that $g(2D(X_I, \xi)) = g(2D(X_{I^c}, \xi))$.

4. Given $I, J \in FG([0,1])$ such that $D(X_I, \xi) \geq D(X_J, \xi)$, as g is monotone decreasing by the third axiom:

$$h(I) = g(2D(X_I, \xi)) \leq g(2D(X_J, \xi)) = h(J).$$

So the four axioms have been proven. ■

These two last results simplify the obtaining of fuzziness entropies given by Definition 3.39, where it is only needed a hesitant distance and a function h satisfying the three conditions from Corollary 3.45, which are much more manageable than the original ones.

In order to illustrate this first part of the entropy, an example is presented next, where a particular dissimilarity and function g are selected as in Corollary 3.45.

Example 3.46 *Let X be a finite set with cardinality N, and the mapping $E_F : IVHFS(X) \to [0,1]$ given by:*

$$E_F(A) = \frac{1}{N} \sum_{x \in X} \left[1 - 2D_H(\mu_A(x), \{0.5\})\right],$$

and where D_H is the hesitant normalized Hamming dissimilarity, which was first developed by [79] for hesitant fuzzy sets, and adapted to interval-valued hesitant fuzzy sets by [33]. The dissimilarity for finite interval-valued hesitant fuzzy sets has the expression given in Example 3.37.

Then, E_F is a local fuzziness entropy measure, as it is a particular situation of the Corollary 3.45, where $g(a) = 1 - a$ and $D = D_H$, both satisfying the required properties.

3.4.2 Lack of knowledge entropy measure

The second part of the entropy definition is obtained by a function which represents the lack of knowledge. With this function the distance of the set to the union of a finite number of classical fuzzy sets is measured. Thus, a different kind of uncertainty is considered.

Using the same notation as in the previous subsection, this function is defined as follows:

Definition 3.47 *Let $E_L : IVHFS(X) \to [0,1]$ be a mapping, $A, B \in IVHFS(X)$ with $\mu_A(x) = \bigcup_{i=1}^{n_x} A_i^x = \bigcup_{i=1}^{n_x} [A_i^{x^L}, A_i^{x^U}] \in FG_{n_x}([0,1])$ $\forall x \in X$, and respectively for B. E_L is said to be a lack of knowledge entropy measure if it satisfies the following properties:*

1. $E_L(A) = 0 \Leftrightarrow Sc(A_i^x) = 0$, $\forall i = 1, \ldots, n_x$, $\forall x \in X$,
2. $E_L(A) = 1 \Leftrightarrow A$ *is the pure interval-valued fuzzy set,*

3. *$E_L(A) = E_L(A^c)$,*

4. *$E_L(A) \leq E_L(B)$ if $\forall x \in X$ $Sc(\mu_A(x)) \leq Sc(\mu_B(x))$, where Sc denotes the score function given (Proposition 3.1).*

The first axiom states that the entropy is null when all the sets A_i^x are singletons, i.e., the set A is a classical fuzzy set. The maximum entropy is found when A is the pure interval-valued fuzzy set. In the third point, the entropy of a set and its complement must match. The last axiom shows how to compare two interval-valued hesitant fuzzy sets with respect to the lack of knowledge entropy measure, where it is taken into account the upper ($A_i^{x^U}$) and lower ($A_i^{x^L}$) bounds of each A_i^x for every $i = 1, \dots, n_x$ and $x \in X$.

As it has been done for the fuzziness entropy in the previous subsection, the concept of local lack of knowledge is also important.

Definition 3.48 *Let X be a finite set with cardinality N and the mapping $E_L : IVHFS(X) \to [0,1]$ a lack of knowledge entropy measure. E_L is said to be a local lack of knowledge entropy measure if there exists a function $f : FG([0,1]) \to [0,1]$ such that for every $x_j \in X$, given $A \in IVHFS(X)$:*

$$E_L(A) - E_L(A^{(j)}) = f(\mu_A(x_j)),$$

or equivalently, it only depends on the term $\mu_A(x_j)$.

Remark 3.49 *Note that $E_L(A) - E_L(A^{(j)}) \in [0,1]$ for all $j = 1, \dots, n$. By construction, $S(\mu_A(x)) = S(\mu_{A^{(j)}}(x))$, $\forall x \neq x_j$. Furthermore, $S(\mu_{A^{(j)}}(x_j)) = 0$, so it is obvious that $S(\mu_A(x_j)) \geq S(\mu_{A^{(j)}}(x_j))$, and by the last axiom of a lack of knowledge entropy, $E_L(A) \geq E_L(A^{(j)})$.*

From here on out, the following two results ease the obtaining of local lack of knowledge entropy measures, with functions whose properties are more manageable than the ones of the original definition.

Theorem 3.50 *Let X be a finite set with cardinality N and E_L be the mapping $E_L : IVHFS(X) \to [0,1]$. Then, E_L is a local lack of knowledge entropy measure if and only if there exists a mapping $h : FG([0,1]) \to [0,1]$ such that*

$$E_L(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)).$$

which also satisfies the following four axioms, given $I, J \in FG([0,1])$:

1. $h(I) = 0 \Leftrightarrow Sc(I_i) = 0, \ \forall i = 1, \dots, n_I$,

2. $h(I) = 1 \Leftrightarrow I = [0,1]$,

3. $h(I) = h(I^c)$,

4. $h(I) \leq h(J)$ *if* $Sc(I) \leq Sc(J)$.

Proof. First, let us suppose that E_L is a local lack of knowledge entropy. By definition, it exists a function $f : FG([0,1]) \to [0,1]$ such that:

$$E_L(A) - E_L(A^{(j)}) = f(\mu_A(x_j)), \ \forall j \in \{1, \dots, N\}.$$

Given $A \in IVHFS(X)$, applying recursively the definition of local:

$$E_L(A) = \dots = E_L(A^{(1,\dots,N)}) + \sum_{x \in X} f(\mu_A(x)).$$

However, $\mu_A(A^{(1,\dots,N)})(x) \in \{0,1\}$, $\forall x \in X$, i.e., the score function applied to each of them is equal to 0, and by the first axiom of lack of knowledge entropy, $E_L(A^{(1,\dots,N)}) = 0$. Hence,

$$E_L(A) = \sum_{x \in X} f(\mu_A(x)).$$

In addition, it is known that $E_L(A) \in [0,1]$ for every $A \in IVHFS(X)$. Then, for all $x_i \in X$, applying the mapping E_L to the set $X_{\mu_A(x_i)}$:

$$E_L(X_{\mu_A(x_i)}) = \sum_{x \in X} f(\mu_A(x_i)) = Nf(\mu_A(x_i)) \in [0,1] \Rightarrow f(\mu_A(x_i)) \in [0, \frac{1}{N}].$$

Consequently, taking $h : FG([0,1]) \to [0,1]$ such that $h(I) = Nf(I)$, it is immediate that:

$$E_L(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)).$$

Now, let us see that h satisfies the four conditions of the theorem.

1. Given $I \in FG([0,1])$ such that $I = I_1 \cup \cdots \cup I_{n_I}$, let us take $X_I \in IVHFS(X)$ such that $\mu_{X_I}(x) = I$ for all $x \in X$. Then:

   $$E_L(X_I) = \frac{1}{N} \sum_{x \in X} h(\mu_{X_I}(x)) = \frac{1}{N} \sum_{i=1}^{N} h(I) = h(I),$$

   and therefore $h(I) = 0 \Leftrightarrow E_L(X_I) = 0$. E_L satisfies the first axiom of lack of knowledge entropy, so $h(I) = 0 \Leftrightarrow Sc(I_i) = 0$, for all $i = 1, \dots, n_I$ and the first axiom is proved.

2. Given $I \in FG([0,1])$, and $X_I \in IVHFS(X)$, it is direct that

   $$h(I) = 1 \Leftrightarrow E_L(X_I) = 1,$$

   and as E_L satisfies the second axiom of Definition 3.47, $I = [0,1]$.

3. Given $I \in FG([0,1])$, and $X_I \in IVHFS(X)$, it is obtained that $E_L(X_I) = h(I)$ and $E_L(X_{I^c}) = h(I^c)$, and as E_L satisfies the third axiom of a lack of knowledge entropy, $E_L(X_I) = E_L(X_I^c) = E_L(X_{I^c})$ and hence,

   $$h(I) = h(I^c).$$

4. Let $I, J \in FG([0,1])$ such that $Sc(I) \leq Sc(J)$. Given $X_I, X_J \in IVHFS(X)$, as $Sc(I) \leq Sc(J)$ and E_L satisfies the fourth axiom of the lack of knowledge entropy, $E_L(X_I) \leq E_L(X_J)$. However, $E_L(X_I) = h(I)$ and $E_L(X_J) = h(J)$, so

$$h(I) \leq h(J).$$

Now, in order to proceed with the second part of the proof, it is supposed that h satisfies the four conditions of the theorem, so it is needed to prove that E_L is a local lack of knowledge entropy. First, let us prove the four conditions of Definition 3.47:

1. Given $A \in IVHFS(X)$,

$$0 = E_L(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) \Leftrightarrow h(\mu_A(x)) = 0, \ \forall x \in X,$$

and as h satisfies (1), then $Sc(\mu_A(x)) = 0, \ \forall x \in X$, and hence, it is a finite union of singletons.

2. Given $A \in IVHFS(X)$,

$$1 = E_L(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) \Leftrightarrow \sum_{x \in X} h(\mu_A(x)) = N,$$

and it is known by definition that $h(I) \in [0,1]$ for every finitely generated set, so the only possible situation is that

$$h(\mu_A(x)) = 1 \Leftrightarrow \mu_A(x) = [0,1], \ \forall x \in X.$$

3. Given $A \in IVHFS(X)$ and A^c its complement, as h satisfies the third item of the theorem, it is known that $h(J) = h(J^c)$ for every finitely

generated set, therefore:

$$\begin{aligned} E_L(A) &= \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)^c) = \\ &= \frac{1}{N}\sum_{x\in X} h(\mu_{A^c}(x)) = E_L(A^c). \end{aligned}$$

4. Given $A, B \in IVHFS(X)$ such that $\forall x \in X$:

$$Sc(\mu_A(x)) \leq Sc(\mu_B(x)).$$

From the last inequality, as h satisfies the fourth axiom of the theorem, $h(\mu_A(x)) \leq h(\mu_B(x))$. Therefore:

$$E_L(A) = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) \leq \frac{1}{N}\sum_{x\in X} h(\mu_B(x)) = E_L(B).$$

Finally, it must be proven that E_L is also local:

(L) Given $A \in IVHFS(X)$, for every $x_j \in X$:

$$\begin{aligned} &E_L(A) - E_L(A^{(j)}) = \\ &= \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) - \frac{1}{N}\left(\sum_{x\in X\setminus\{x_j\}} h(\mu_A(x)) + h(\mu_{A^{(j)}}(x_j))\right) = \\ &= \frac{1}{N}(h(\mu_A(x_j)) - h(\mu_{A^{(j)}}(x_j)) = \frac{1}{N}h(\mu_A(x_j)) = f(\mu_A(x_j)), \end{aligned}$$

i.e., it only depends on the term $\mu_A(x_j)$ for every j as $\mu_{A^{(j)}}(x_j) \in \{0, 1\}$ and by hypothesis, $h(\mu_{A^{(j)}}(x_j)) = 0$. Therefore, it is local. ■

With the support of the previous result, the next corollary provides a way to get local lack of knowledge entropies by a mapping with simpler achievable conditions.

Corollary 3.51 *Let X be a finite set with cardinality N and let E_L be the mapping $E_L : IVHFS(X) \to [0,1]$. Then, E_L is a local lack of knowledge entropy if and only if there exists a mapping $g : [0,1] \to [0,1]$ such that*

$$E_L(A) = \frac{1}{N} \sum_{x \in X} g(Sc(\mu_A(x))),$$

which also satisfies the following properties:

1. $g(a) = 0 \Leftrightarrow a = 0$,

2. $g(a) = 1 \Leftrightarrow a = 1$,

3. *g is monotone increasing.*

Proof. It is enough to see that the function $h(I) = g(Sc(I))$ fulfills the four conditions of Theorem 3.50, and the result would be proven.

1. Given $I \in FG([0,1])$,

$$h(I) = 0 = g(Sc(I)),$$

   but for the first property that g satisfies, $g(a) = 0 \Leftrightarrow a = 0$, and then $S(I) = 0$.

2. Given $I \in FG([0,1])$

$$h(I) = 1 = g(Sc(I)),$$

   and for the second property of g, $Sc(I) = 1$, which only happens when $I = [0,1]$.

3. Given $I \in FG([0,1])$ such that $I = I_1 \cup \cdots \cup I_n$, and the complement $I^c = I_1^c \cup \cdots \cup I_n^c$. Given any i:

$$I_i^c = [1 - I_i^U, 1 - I_i^L],$$

so

$$Sc(I_i^c) = I_i^U - I_i^L = Sc(I_i).$$

Bearing this in mind, $h(I) = h(I^c)$.

4. Let $I, J \in FG([0,1])$ such that $Sc(I) \leq Sc(J)$. The third property states that g is monotone increasing, hence:

$$g(Sc(I)) \leq g(Sc(J)) \Leftrightarrow h(I) \leq h(J).$$

Thus, the four axioms have been demonstrated. ■

With this two last results, it has been found a way to obtain local lack of knowledge entropy measures just with a mapping g satisfying the properties of Corollary 3.51, which are less complicated to obtain than the ones in the original definition of this entropy measure.

As it has been done with the first type of entropy, an example is given next, starting from the last corollary.

Example 3.52 *Let X be a finite set with cardinality N, and the mapping $E_L : IVHFS(X) \rightarrow [0,1]$ given by:*

$$E_L(A) = \frac{1}{N} \sum_{x \in X} \sum_{i=1}^{n_x} S(A_i^x),$$

where $\mu_A(x) = A_1^x \cup \cdots \cup A_{n_x}^x \in FG_{n_x}([0,1])$, $\forall x \in X$, with $A_i^x = [A_i^{x^L}, A_i^{x^U}]$, $\forall i$.

This is obviously a local lack of knowledge entropy, as it is the particular case of Corollary 3.51 with $g(a) = a$.

3.4.3 Hesitance entropy measure

The last part of the definition of entropy in an interval-valued hesitant environment is given by a function which measures the distance of a set to a single interval-valued fuzzy set. It has been called hesitance, and it is defined as follows:

Definition 3.53 *Let $E_H : IVHFS(X) \to [0,1]$ be a mapping, $A, B \in IVHFS(X)$. E_H is said to be a hesitance entropy measure if it satisfies the following properties:*

1. $E_H(A) = 0 \Leftrightarrow A \in IVFS(X)$,
2. $\lim_{n_x^A \to \infty} E_H(A) = 1 \; \forall x \in X$,
3. $E_H(A) = E_H(A^c)$,
4. $E_H(A) \leq E_H(B)$ *if* $\forall x \in X$:

$$n_x^A \leq n_x^B,$$

where

$$\mu_A(x) = \bigcup_{i=1}^{n_x^A} A_i^x \quad and \quad \mu_B(x) = \bigcup_{i=1}^{n_x^B} B_i^x \; \forall x \in X,$$

i.e., n_x^A and n_x^B represent the number of disjoint intervals that shapes the set $\mu_A(x)$ and $\mu_B(x)$ respectively.

As it has been already said, a null entropy happens when the set is an interval-valued fuzzy one. The second axiom remarks that the entropy tends to its maximum when the number of sets defining $\mu_A(x)$ for each point tends to infinite. In this axiom there is an abuse of notation: since A is fixed, also n_x^A is; but with this expression we would like to say that, for any

x, if we consider the infimum of the values of the entropies of the sets with n disjoint intervalar components, the limit when n tends to infinity is equal to 1. The third one, states that a set and its complement must have the same entropy value. In the latter property, a set is greater than another with respect to this entropy when for every point, the number of intervals defining the set is also greater.

Next, an extension of this definition is given, adding the property of local to hesitance entropy measures.

Definition 3.54 *Let X be a finite set with cardinality N and the mapping $E_H : IVHFS(X) \to [0,1]$ a hesitance entropy measure. E_H is said to be a local hesitance entropy measure if there exists a function $f : FG([0,1]) \to [0,1]$ such that for every $x_j \in X$, given $A \in IVHFS(X)$:*

$$E_H(A) - E_H(A^{(j)}) = f(\mu_A(x_j)),$$

or equivalently, it only depends on the term $\mu_A(x_j)$.

Remark 3.55 *Note that $E_H(A) - E_H(A^{(j)}) \in [0,1]$ for all $j = 1, \ldots, n$. By construction, $n_x^A = n_x^{A^{(j)}}$, $\forall x \neq x_j$. Furthermore, $n_{x_j}^{A^{(j)}} = 1$, so it is obvious that $n_{x_j}^A \geq n_{x_j}^{A^{(j)}}$, and by the last axiom of a hesitance entropy, $E_H(A) \geq E_H(A^{(j)})$.*

The next two results that are about to be developed, provide a way to obtain local hesitance entropies with simpler conditions, avoiding the more complex ones in the original definition previously given.

Theorem 3.56 *Let X be a finite set with cardinality N and E_H be the mapping $E_H : IVHFS(X) \to [0,1]$. Then, E_H is a local hesitance entropy measure if and only if there exists a mapping $h : FG([0,1]) \to [0,1]$ such* *that*

$$E_H(A) = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)).$$

which also satisfies the following four axioms, given $I, J \in FG([0,1])$ *such that* $I \in FG_{n_I}([0,1])$ *and* $J \in FG_{n_J}([0,1])$*:*

1. $h(I) = 0 \Leftrightarrow n_I = 1$,
2. $\lim_{n_I\to\infty} h(I) = 1$,
3. $h(I) = h(I^c)$,
4. $h(I) \leq h(J)$ *if* $n_I \leq n_J$.

Proof. First, let us suppose that E_H is a local hesitance entropy, and by the definition of local for hesitance entropy, it is known that there exists a function $f : FG([0,1]) \to [0,1]$ such that:

$$E_H(A) - E_H(A^{(j)}) = f(\mu_A(x_j)),\ \forall j \in \{1,\dots,N\}.$$

Given $A \in IVHFS(X)$, applying recursively the definition of local:

$$E_H(A) = \cdots = E_H(A^{(1,\dots,N)}) + \sum_{x\in X} f(\mu_A(x)).$$

However, $\mu_A(A^{(1,\dots,N)})(x) \in \{0,1\}$, $\forall x \in X$, i.e., $n_x^A = 1\ \forall x \in X$, and by the first axiom of hesitance entropy, $E_H(A^{(1,\dots,N)}) = 0$. Hence,

$$E_H(A) = \sum_{x\in X} f(\mu_A(x)).$$

In addition, it is known that $E_H(A) \in [0,1]$ for every $A \in IVHFS(X)$. Then, for all $x_i \in X$, applying the mapping E_H to the set $X_{\mu_A(x_i)}$:

$$E_H(X_{\mu_A(x_i)}) = \sum_{x\in X} f(\mu_A(x_i)) = Nf(\mu_A(x_i)) \in [0,1] \Rightarrow f(\mu_A(x_i)) \in [0, \frac{1}{N}].$$

Consequently, taking $h : FG([0,1]) \to [0,1]$ such that $h(I) = Nf(I)$, it is immediate that:

$$E_H(A) = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)).$$

Now, let us see that h satisfies the four conditions of the theorem.

1. Given $I \in FG_{n_I}([0,1]) \subseteq FG([0,1])$ and $X_I \in IVHFS(X)$ such that $\mu_{X_I}(x) = I$ for all $x \in X$. Then:

$$E_H(X_I) = \frac{1}{N}\sum_{x\in X} h(\mu_{X_I}(x)) = \frac{1}{N}\sum_{x\in X} h(I) = h(I),$$

   and therefore $h(I) = 0 \Leftrightarrow E_H(X_I) = 0$. E_H satisfies the first axiom of a hesitance entropy, so $h(I) = 0 \Leftrightarrow n_I = 1$, and the first axiom is proved.

2. Given $I \in FG_{n_I}([0,1]) \subseteq FG([0,1])$, and $X_I \in IVHFS(X)$, it is direct that

$$h(I) = 1 \Leftrightarrow E_H(X_I) = 1,$$

   and as E_H satisfies the second axiom of a hesitance entropy measure, $\lim_{n_x\to\infty} E_H(X_I) = 1$, $x \in X$, and by definition, $n_I = n_x$, so $\lim_{n_I\to\infty} h(I) = 1$.

3. Given $I \in FG([0,1])$, and $X_I \in IVHFS(X)$, it is obtained that $E_H(X_I) = h(I)$ and $E_H(X_{I^c}) = h(I^c)$, and as E_H satisfies the third axiom of a hesitance entropy, $E_H(X_I) = E_H(X_I^c) = E_H(X_{I^c})$ and hence,

$$h(I) = h(I^c).$$

4. Let $I, J \in FG([0,1])$ such that $I \in FG_{n_I}([0,1])$ and $J \in FG_{n_J}([0,1])$ and $n_I \leq n_J$, and $X_I, X_J \in IVHFS(X)$. By the last axiom of a hesitance entropy and the hypothesis $n_I \leq n_J$, $E_H(X_I) \leq E_H(X_J)$, and by definition of both interval-valued hesitant fuzzy sets, $h(I) \leq h(J)$.

To prove the converse, it is supposed that h satisfies the four conditions of the theorem, so it is needed to prove that E_H is a local hesitance entropy. On one hand, the properties of Definition 3.53 must be proven:

1. Given $A \in IVHFS(X)$,

$$0 = E_H(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) \Leftrightarrow h(\mu_A(x)) = 0, \ \forall x \in X,$$

   and as h satisfies (1), then $n_x = 1, \ \forall x \in X$ where $\mu_A(x) = \bigcup_{i=1}^{n_x} A_i^x$, or equivalently, $A \in IVFS(X)$.

2. Given $A \in IVHFS(X)$, and by the second condition of the theorem, $\lim_{n_I \to \infty} h(I) = 1$, for every finitely generated set. Therefore, $\forall x \in X$:

$$\lim_{n_x \to \infty} E_H(A) = \lim_{n_x \to \infty} \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) = \frac{1}{N} \sum_{x \in X} \lim_{n_x \to \infty} h(\mu_A(x)) = 1.$$

3. Given $A \in IVHFS(X)$ and A^c its complement. As h satisfies the third item of the theorem, it is known that $h(J) = h(J^c)$ for every

finitely generated set, therefore:

$$\begin{aligned} E_H(A) &= \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)^c) = \\ &= \frac{1}{N}\sum_{x\in X} h(\mu_{A^c}(x)) = E_H(A^c). \end{aligned}$$

4. Given $A, B \in IVHFS(X)$ where $\mu_A(x) = \bigcup_{i=1}^{n_x^A} A_i^x$ and $\mu_B(x) = \bigcup_{i=1}^{n_x^B} B_i^x$, $\forall x \in X$. Let us suppose that $n_x^A \leq n_x^B$ $\forall x \in X$, and by the fourth axiom that h satisfies:

$$h(\mu_A(x)) \leq h(\mu_B(x)),\ \forall x \in X,$$

and by construction of the mapping E_H:

$$E_H(A) = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) \leq \frac{1}{N}\sum_{x\in X} h(\mu_B(x)) = E_H(B).$$

On the other hand, let us prove that E_H is also local:

(L) Given $A \in IVHFS(X)$, for every $x_j \in X$:

$$\begin{aligned} &E_H(A) - E_H(A^{(j)}) = \\ &= \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) - \frac{1}{N}\left(\sum_{x\in X\setminus\{x_j\}} h(\mu_A(x)) + h(\mu_{A^{(j)}}(x_j))\right) = \\ &= \frac{1}{N}(h(\mu_A(x_j)) - h(\mu_{A^{(j)}}(x_j)) = \frac{1}{N}h(\mu_A(x_j)) = f(\mu_A(x_j)), \end{aligned}$$

i.e., it only depends on the term $\mu_A(x_j)$ for every j as $\mu_{A^{(j)}}(x_j) \in \{0, 1\}$ and by hypothesis, $h(\mu_{A^{(j)}}(x_j)) = 0$. Therefore, it is local. ■

The next result provides another step forward to simplify the conditions required to obtain a local hesitance entropy measure, where a new mapping is used to get it. Before the corollary, some notation is needed.

The mapping $NInt : FG([0,1]) \to \mathbb{N}$ provides the number of closed disjoint subintervals that shape the finitely generated set. Given $I = \bigcup_{i=1}^{n_I} I_i \in FG([0,1])$, $NInt(I) = n_I$.

Corollary 3.57 *Let X be a finite set with cardinality N and let E_H be the mapping $E_H : IVHFS(X) \to [0,1]$. Then, E_H is a local hesitance entropy if and only if there exists a mapping $g : \mathbb{N} \to [0,1]$ such that*

$$E_H(A) = \frac{1}{N} \sum_{x \in X} g(NInt(\mu_A(x))),$$

which also satisfies the following properties:

1. $g(a) = 0 \Leftrightarrow a = 1$,
2. $\lim_{a \to \infty} g(a) = 1$,
3. *g is monotone increasing.*

Proof. To prove the result, it is enough to see that the function $h(I) = g(NInt(I))$ satisfies the four axioms in Theorem 3.56.

1. Given $I \in FG_{n_I}([0,1]) \subseteq FG([0,1])$:

   $$h(I) = g(NInt(I)) = 0,$$

   but for the first property that g satisfies, $g(a) = 0 \Leftrightarrow a = 1$, and then $NInt(I) = n_I = 1$.

2. Given $I \in FG_{n_I}([0,1]) \subseteq FG([0,1])$:

$$\lim_{n_I \to \infty} h(I) = \lim_{n_I \to \infty} g(NInt(I)) = \lim_{n_I \to \infty} g(n_I),$$

but for the second property of g, $\lim_{a \to \infty} g(a) = 1$, and then $\lim_{n_I \to \infty} h(I) = 1$.

3. Given $I, I^c \in FG_{n_I}([0,1]) \subseteq FG([0,1])$ a finitely generated set I and its complement I^c, as both are generated by the same number of closed disjoint intervals:

$$h(I) = g(NInt(I)) = g(NInt(I^c)) = h(I^c).$$

4. Given $I \in FG_{n_I}([0,1]) \subseteq FG([0,1])$ and $J \in FG_{n_J}([0,1]) \subseteq FG([0,1])$ such that $n_I \leq n_J$, it is known by the increasing monotony of g that:

$$h(I) = g(NInt(I)) = g(n_I) \leq g(n_J) = g(NInt(J)) = h(J).$$

Thus, all the axioms have been proved. ■

As it has been done with the previous two mappings of this new definition of hesitant entropy, these last two results get rid of the difficulties associated to the third part of the entropy with functions which are easier to obtain than the one in the original definition.

Again, a brief example is shown next to illustrate an obtainable particular case of local hesitance entropy by these last two results.

Example 3.58 *Let X be a finite set with cardinality N. Let the mapping $E_H : IVHFS(X) \to [0,1]$ be given by:*

$$E_H(A) = \frac{1}{N} \sum_{x \in X} (1 - \frac{1}{n_x}),$$

where $\mu_A(x) = \bigcup_{i=1}^{n_x} A_i^x, \ \forall x \in X$.

Then, E_H *is a local hesitance entropy, as it is the particular case of Corollary 3.57 with* $g(a) = 1 - \dfrac{1}{a}$.

Once that the three mappings have been defined and studied separately in the last three subsections, the joint definition of entropy is given and analyzed in the next one.

3.4.4 Joint hesitant entropy measure

The definition of the hesitant entropy proposed in this work is given next, where the three mappings E_F, E_L and E_H are put together in order to measure different types of uncertainties associated to a hesitant fuzzy set.

Definition 3.59 *Let* $E_F, E_L, E_H : IVHFS(X) \to [0,1]$ *be three mappings. The triplet* (E_F, E_L, E_H) *is said to be a joint entropy measure in an interval-valued hesitant fuzzy environment if* E_F, E_L *and* E_H *satisfy the axioms of Definitions 3.39, 3.47 and 3.53 and the local properties of Definitions 3.42, 3.48 and 3.54, respectively.*

In order to illustrate the way that this entropy works and how it varies depending on the type of interval-valued hesitant fuzzy sets used, the next example has been carried out.

Example 3.60 *Let us obtain a joint entropy measure* (E_F, E_L, E_H) *through the different results developed in the previous sections, specifically, Corollaries 3.45, 3.51 and 3.57 for* E_F, E_L *and* E_H *respectively.*

The set that we are working with has four elements: $X = \{x_1, x_2, x_3, x_4\}$. *Given* $A \in IVHFS(X)$ *defined by* $\mu_A(x_i) = \bigcup_{j=1}^{n_{x_i}} A_j^{x_i} = \bigcup_{j=1}^{n_{x_i}} [A_j^{x_i^L}, A_j^{x_i^U}], \ \forall x_i \in$

X:

- *For* E_F*, the dissimilarity measure selected is the hesitant normalized Hamming dissimilarity, defined previously in Example 3.37, as well as the function* $g(a) = 1 - a^2$*, which satisfies the properties of Corollary 3.45. The local fuzziness entropy obtained is given as:*

$$E_F(A) = \frac{1}{4}\sum_{i=1}^{4}\left[1 - \left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}(|A_j^{x_i^U} - 0.5| + |A_j^{x_i^L} - 0.5|)\right)^2\right].$$

- *For* E_L*, the function* $g(a) = a^2$ *is selected, which satisfies the properties of Corollary 3.51. The local lack of knowledge entropy obtained is given as:*

$$E_L(A) = \frac{1}{4}\sum_{i=1}^{4}\left(\sum_{j=1}^{n_{x_i}} S(A_j^{x_i})\right)^2.$$

- *For* E_H*, the function* $g(a) = 1 - \frac{1}{a^2}$ *is selected, which satisfies the properties of Corollary 3.57. The local hesitance entropy obtained is given as:*

$$E_H(A) = \frac{1}{4}\sum_{i=1}^{4}\left(1 - \frac{1}{n_{x_i}^2}\right).$$

Once that the three mappings are defined, the value of each one has been obtained for different interval-valued hesitant fuzzy sets, as it is shown in the Table 3.1.

Let us analyze each situation separately:

- A_1*: which is a crisp set, as the only values that it takes are* 0 *and* 1*. As a result, all the entropies are null, i.e.,* $(E_F, E_L, E_H) = (0, 0, 0)$*, because the only values are* 0 *and* 1 *(*$E_F(A_1) = 0$*), they are singletons (*$E_L(A_1) = 0$*) and there is a single interval (point) for each* x_i

IVHFS(X)	A_1	A_2	A_3	A_4
x_1	{1}	{[0,0.4],[0.41,0.8],[0.81,1]}	{0.5}	{0}
x_2	{0}	{[0,0.4],[0.41,0.7],[0.71,1]}	{[0.45,0.5]}	{[0,0.004], [0.005,1]}
x_3	{0}	{[0,0.5],[0.51,0.7],[0.71,1]}	{[0.5,0.55], [0.56, 0.6]}	{[0.99,0.994], [0.995,1]}
x_4	{1}	{[0,0.5],[0.51,1]}	{[0.4,0.6]}	{1}
E_F	0	0.9032	0.9995	0.0197
E_L	0	0.9653	0.0126	0
E_H	0	0.8542	0.1875	0.375

Table 3.1: Different entropy values for four interval-valued hesitant fuzzy sets.

($E_H(A_1) = 0$). This shows that it is possible to obtain a low value in all of them with the same set.

- *A_2: whose values are all close to or include the point 0.5 (high value of $E_F(A_2)$), the membership functions are close to the interval $[0, 1]$ (high value of $E_L(A_2)$) and for each point there are several intervals defining the membership function (high value of $E_H(A_2)$). Hence, the values of all the entropies are high, showing that this is possible in the same set.*
- *A_3: the memberships include and are all close to the point 0.5 (high value of $E_F(A_3)$), the total lengths of the memberships are small (low value of $E_L(A_3)$), and the number of intervals are one in three out of the four elements (low value of $E_H(A_3)$).*
- *A_4: the memberships are all close to the extremes 0 and 1 (low value of $E_F(A_4)$), the total lengths of the memberships are very small (low value of $E_L(A_4)$), and the number of intervals are one in two out of the four elements and two in two out of the four elements (low-medium value of $E_H(A_4)$).*

The last two sets show that the three mappings do not usually take similar values as it happened in the sets A_1 (low values) and A_2 (high values). In A_3, $E_F(A_3)$ is much higher than the other two, while in A_4, it is $E_H(A_4)$ which takes a greater value.

Taking into account the results given in this example, it is straightforward to see that the variety of uncertainties that the new definition of entropy quantify is reflected clearly in each shown example. Obviously, the uncertainty associated to the definition of entropy given by Farhadinia in [33], is included in our new proposal, as the first mapping of our definition (E_F) represents a similar concept.

In summary, this new approach makes it possible to obtain the classical concept of entropy for other types of sets, which is the distance to a crisp set, as well as another two uncertainties, related to the distance to a fuzzy set and to an interval-valued fuzzy set, being up to the researcher the importance given to each one in the studied situation.

3.5 Partitions

Partitioning a set of elements is a task which has been of great importance in statistics. The capability to classify these elements in groups with similar features is a key point in the study of surveys in order to identify groups of opinion. When the information is one-dimensional, the existence of an order makes the task of partitioning much easier. However, it gets tougher when data is multi-dimensional. Different approaches exist such as the classical k-means to deal with these situations.

However, when dealing with fuzzy information, partitioning becomes even harder to be carried out. Some fuzzy partitioning methods such as *Gustafson-Kessel* and *fuzzy c-means* are experimentally used in the next applications chapter.

In this section, we are generalizing the concepts about partitions given by Montes *et al.* (see [51]) for fuzzy sets in Chapter 1. These results include two different definitions of partitions for interval-valued hesitant fuzzy sets, as well as several results and characterizations of these definitions.

First of all, it is necessary to extend the concept of α-cut and strong α-cut for interval-valued hesitant fuzzy sets.

Definition 3.61 *Let X be a non-empty set, $A \in IVHFS(X)$ and $\alpha \in FG([0,1])$. Then:*

$$A_\alpha = \{x \in X | \mu_A(x) \geq_1 \alpha\},$$
$$A_{\overline{\alpha}} = \{x \in X | \mu_A(x) >_1 \alpha\},$$

represent the α-cut and strong α-cut of A respectively.

Remark 3.62 *As well as with the definition of t-norm and t-conorm, when the previous* α*-cut definition given for interval-valued hesitant fuzzy sets is restricted to fuzzy sets (*$A \in FS(X)$ *and* $\alpha \in [0,1]$*), it matches the classical definition of* α*-cut for such sets.*

Once this previous concept is adapted, δ-ϵ-partition definition for this type of sets is given as follows.

Definition 3.63 *Let* X *be a non-empty set and* $A \in IVHFS(X)$*. The family* $\Pi = \{A_i \in IVHFS(X) | i \in I\}$*, where* I *is a finite subset of* $\mathbb{N}$*, is a* δ-ϵ*-partition with* $0 \leq \epsilon < \delta \leq 1$ *if and only if*

1. $(\underset{i \in I}{S}(A_i))_\alpha = A_\alpha$,
2. $T(A_i, A_j)_\alpha = \emptyset, \ \forall i \neq j$,

for all $\alpha \in FG([0,1])$ *such that* $\{\epsilon\} <_1 \alpha <_1 \{\delta\}$*, where* T *and* S *are a t-norm and a t-conorm respectively.*

Remark 3.64 *As it has been stated previously along this chapter, t-norms (and t-conorms) and* α*-cuts defined in a hesitant environment, when they are restricted to a classical fuzzy situation, their definitions match the usual ones in this type of sets.*

As a result, it is immediate to see that Definition 3.63 of δ-ϵ*-partition, when restricted to fuzzy sets, matches the one given by Montes et. al in [51] for such sets.*

This definition is the basis of this section. As a consequence, the forthcoming results will be developed around it. Furthermore, an interesting characteristic of this definition is given in the following Remark.

Remark 3.65 *If Definition 3.63 is restricted to fuzzy sets, it matches the definition given originally for such sets. As a consequence, the classical definition of partition given by Ruspini (Definition 1.48) can be obtained by Lukasiewicz t-norm and t-conorm.*

The next theorem, is a characterization of this definition of δ-ϵ-partition for interval-valued hesitant fuzzy sets.

Theorem 3.66 *Let X be a non-empty set and $A \in IVHFS(X)$. The family $\Pi = \{A_i \in IVHFS(X) | i \in I\}$, where I is a finite subset of $\mathbb{N}$, is a δ-ϵ-partition with $0 \leq \epsilon < \delta \leq 1$ if and only if, $\forall x \in X$:*

$$(1') \quad \begin{cases} \underset{i\in I}{S}(A_i)(x) \geq_1 \{\delta\}, & \text{if } \mu_A(x) \geq_1 \{\delta\}, \\ \underset{i\in I}{S}(A_i)(x) = \mu_A(x), & \text{if } \mu_A(x) >_1 \{\epsilon\} \text{ and } \mu_A(x) <_1 \{\delta\}, \\ \underset{i\in I}{S}(A_i)(x) \leq_1 \{\epsilon\}, & \text{if } \mu_A(x) \leq_1 \{\epsilon\}, \end{cases}$$

$$(2') \quad T(A_i, A_j)(x) \leq_1 \{\epsilon\}, \ \forall i \neq j,$$

where T and S are a t-norm and a t-conorm respectively.

Proof. Assume first that Π is a δ-ϵ-partition, and prove that it satisfies both axioms from the theorem. In order to prove $(1')$, all possible situations are studied. Given $x \in X$:

- $\mu_A(x) \geq_1 \{\delta\}$: let us suppose that $\underset{i\in I}{S}(A_i)(x) <_1 \{\delta\}$. Then, by Proposition 3.9, $\exists \alpha \in FG([0,1])$ such that $\underset{i\in I}{S}(A_i)(x) <_1 \alpha <_1 \{\delta\}$, and we can suppose that also $\{\epsilon\} <_1 \alpha <_1 \{\delta\}$ (if not, applying Proposition 3.9 to α and $\{\delta\}$ repeatedly if necessary, an element $\beta \in FG([0,1])$ with such properties would be obtained). As a result, $x \in A_\alpha$ and $x \notin (\underset{i\in I}{S}(A_i))_\alpha$, which contradicts (1).
- $\mu_A(x) \leq_1 \{\epsilon\}$: let us suppose that $\underset{i\in I}{S}(A_i)(x) >_1 \{\epsilon\}$. Let us distinguish two situations:

- $\exists \alpha \in FG([0,1])$ such that $\{\epsilon\} <_1 \alpha <_1 \underset{i\in I}{S}(A_i)(x)$: as in the previous point, we can suppose that $\{\epsilon\} <_1 \alpha <_1 \{\delta\}$. However, $x \notin A_\alpha$ and $x \in (\underset{i\in I}{S}(A_i))_\alpha$, which contradicts (1).
- $\nexists \alpha \in FG([0,1])$ such that $\{\epsilon\} <_1 \alpha <_1 \underset{i\in I}{S}(A_i)(x)$: by Proposition 3.10, $H(\underset{i\in I}{S}(A_i)(x)) = H(\{\epsilon\}) = \epsilon$, $Sc(\underset{i\in I}{S}(A_i)(x)) = Sc(\{\epsilon\}) = 0$ and $n_{\{\epsilon\}} = 1 < n_{\underset{i\in I}{S}(A_i)(x)} = 2$.
  However, $\epsilon <_1 \delta$, so $H(\underset{i\in I}{S}(A_i)(x)) = H(\{\epsilon\}) < H(\{\delta\})$, and as a result $\underset{i\in I}{S}(A_i)(x) <_1 \{\delta\}$. Then, given $\alpha = \underset{i\in I}{S}(A_i)(x)$, $\{\epsilon\} <_1 \alpha$ and $\{\epsilon\} <_1 \alpha <_1 \{\delta\}$. As a result, $x \notin A_\alpha$ and $x \in (\underset{i\in I}{S}(A_i))_\alpha$, which contradicts (1).

- $\mu_A(x) <_1 \{\delta\}$ and $\mu_A(x) >_1 \{\epsilon\}$: let us suppose that $\underset{i\in I}{S}(A_i)(x) \neq \mu_A(x)$. Let us distinguish three situations:
  - $\underset{i\in I}{S}(A_i)(x) || \mu_A(x)$: given $\alpha = \mu_A(x)$, $\{\epsilon\} <_1 \alpha <_1 \{\delta\}$, so $x \in A_\alpha$ and $x \notin (\underset{i\in I}{S}(A_i))_\alpha$, which contradicts (1).
  - $\underset{i\in I}{S}(A_i)(x) <_1 \mu_A(x)$: given $\alpha = \mu_A(x)$, $\{\epsilon\} <_1 \alpha <_1 \{\delta\}$, so $x \in A_\alpha$ and $x \notin (\underset{i\in I}{S}(A_i))_\alpha$, which contradicts (1).
  - $\underset{i\in I}{S}(A_i)(x) >_1 \mu_A(x)$: it is necessary to study two different situations:
    - $\exists \alpha \in FG([0,1])$ such that $\mu_A(x) <_1 \alpha <_1 \underset{i\in I}{S}(A_i)(x)$: it is obvious that $x \notin A_\alpha$ and $x \in (\underset{i\in I}{S}(A_i))_\alpha$, which contradicts (1).
    - $\nexists \alpha \in FG([0,1])$ such that $\mu_A(x) <_1 \alpha <_1 \underset{i\in I}{S}(A_i)(x)$: by Proposition 3.10, $H(\underset{i\in I}{S}(A_i)(x)) = H(\mu_A(x))$, $Sc(\underset{i\in I}{S}(A_i)(x)) = Sc(\mu_A(x))$ and $n_{\mu_A(x)} < n_{\underset{i\in I}{S}(A_i)(x)} = n_{\mu_A(x)} + 1$.
      On one hand, $\mu_A(x) <_1 \{\delta\}$, and as it has been shown in the proof of Proposition 3.10, then $H(\mu_A(x)) < \delta$, and as

a consequence, $\underset{i\in I}{S}(A_i)(x) <_1 \{\delta\}$. On the other hand, by hypothesis $\{\epsilon\} <_1 \mu_A(x) <_1 \underset{i\in I}{S}(A_i)(x)$, so $\{\epsilon\} <_1 \underset{i\in I}{S}(A_i)(x)$. Then, given $\alpha = \underset{i\in I}{S}(A_i)(x)$, it is obvious that $x \notin A_\alpha$ and $x \in (\underset{i\in I}{S}(A_i))_\alpha$, which contradicts (1).

Once that (1′) has been proven, let us continue with the second axiom. Given i, j such that $i \neq j$, let us suppose that $T(A_i, A_j)(x) >_1 \{\epsilon\}$, and analyze the two different situations:

- $\exists \alpha \in FG([0,1])$ such that $\{\epsilon\} <_1 \alpha <_1 T(A_i, A_J)(x)$: as it has been done before in this proof, we can suppose that $\{\epsilon\} <_1 \alpha <_1 \{\delta\}$. However, it is obvious that $x \in (T(A_i, A_j))_\alpha$, which contradicts (2).

- $\nexists \alpha \in FG([0,1])$ such that $\{\epsilon\} <_1 \alpha <_1 T(A_i, A_J)(x)$: by Proposition 3.10, $H(T(A_i, A_J)(x)) = H(\{\epsilon\}) = \epsilon$, $Sc(T(A_i, A_J)(x)) = Sc(\{\epsilon\}) = 0$ and $n_{\{\epsilon\}} = 1 < n_{T(A_i,A_J)(x)} = n_{\{\epsilon\}} + 1 = 2$. In addition, $H(T(A_i, A_J)(x)) = H(\{\epsilon\}) < H(\{\delta\})$, so $\{\epsilon\} <_1 T(A_i, A_J)(x) <_1 \{\delta\}$. Given $\alpha = T(A_i, A_J)(x)$, $x \in (T(A_i, A_j))_\alpha$, which contradicts (2).

So the first implication has been demonstrated.

To prove the converse, let us start proving (1). Given $\alpha \in FG([0,1])$ such that $\{\epsilon\} <_1 \alpha <_1 \{\delta\}$, the equality $A_\alpha = (\underset{i\in I}{S}(A_i))_\alpha$ must be proven.
($\subseteq$) Given $x \in A_\alpha$, by definition, $\mu_A(x) \geq_1 \alpha$. Two different situations must be studied:

- $\{\epsilon\} <_1 \mu_A(x) <_1 \{\delta\}$: by property (1′), $\underset{i\in I}{S}(A_i)(x) = \mu_A(x)$, so $x \in (\underset{i\in I}{S}(A_i))_\alpha$.

- $\mu_A(x) \geq_1 \{\delta\}$: by property (1′), $\underset{i\in I}{S}(A_i)(x) \geq_1 \mu_A(x)$, so $x \in (\underset{i\in I}{S}(A_i))_\alpha$.

($\supseteq$) Given $x \in (\underset{i\in I}{S}(A_i))_\alpha$, by definition $\underset{i\in I}{S}(A_i)(x) \geq_1 \alpha$. Let us suppose that $x \notin A_\alpha$:

- $\mu_A(x) <_1 \alpha$: if $\{\epsilon\} <_1 \mu_A(x)$, then by property $(1')$, $\underset{i\in I}{S}(A_i)(x) = \mu_A(x)$, which is a contradiction. If $\mu_A(x) \leq_1 \{\epsilon\}$, then by property $(1')$, $\underset{i\in I}{S}(A_i)(x) \leq_1 \{\epsilon\} <_1 \alpha$, which is a contradiction.

- $\mu_A(x)||\alpha$: if $\mu_A(x) \leq_1 \{\epsilon\}$ or $\{\delta\} \leq_1 \mu_A(x)$, it would lead to a contradiction with the incomparability. If $\{\epsilon\} <_1 \mu_A(x) <_1 \{\delta\}$, then by property $(1')$, $\underset{i\in I}{S}(A_i)(x) = \mu_A(x)||\alpha$, which is also a contradiction.

In order to prove (2), by property $(2')$, it is known that $T(A_i, A_j)(x) \leq_1 \{\epsilon\}$, $\forall i \neq j, \forall x \in X$. However, given $\{\epsilon\} <_1 \alpha <_1 \{\delta\}$, $T(A_i, A_j)(x) \leq_1 \{\epsilon\} <_1 \alpha$, and $T(A_i, A_j)_\alpha = \emptyset$, $\forall i \neq j$.

So the second implication has been proven, and as a consequence, as well the whole result. ■

After this characterization, another definition of partition is given, ϵ-ϵ-partition for interval-valued hesitant fuzzy sets.

Definition 3.67 *Let X be a non-empty set and $A \in IVHFS(X)$. The family $\Pi = \{A_i \in IVHFS(X)|i \in I\}$, where I is a finite subset of $\mathbb{N}$, is a ϵ-ϵ-partition with $\epsilon \in [0,1]$ if and only if*

1. $\begin{cases} A_{\overline{\{\epsilon\}}} \subseteq (\underset{i\in I}{S}(A_i))_{\{\epsilon\}}, \\ (\underset{i\in I}{S}(A_i))_{\overline{\{\epsilon\}}} \subseteq A_{\{\epsilon\}}, \end{cases}$

2. $T(A_i, A_j)_{\overline{\{\epsilon\}}} = \emptyset,\ \forall i \neq j,$

where T and S are a t-norm and a t-conorm respectively.

The next result analyzes the properties of these two last definitions (δ-ϵ-partition and ϵ-ϵ-partition), studying how the relation between the parameters affect the preservation of the properties.

Theorem 3.68 *Let X be a non-empty set, $A \in IVHFS(X)$, and the family $\Pi = \{A_i \in IVHFS(X) | i \in I\}$, where I is a finite subset of $\mathbb{N}$ a δ-ϵ-partition with $0 \leq \epsilon \leq \delta \leq 1$. Then Π is a δ'-ϵ'-partition $\forall \epsilon', \delta' \in [0,1]$ such that $\epsilon \leq \epsilon' \leq \delta' \leq \delta$.*

Proof. If $\epsilon' < \delta'$, it is obvious by definition of δ-ϵ-partition. Let us suppose that $\epsilon' = \delta'$. Both axioms of Definition 3.67 must be proven:p

1. Firstly, given $x \in A_{\overline{\{\epsilon'\}}}$, by definition, $\mu_A(x) >_1 \{\epsilon'\}$. Let us distinguish two situations:

   - $\mu_A(x) <_1 \{\delta\}$: by Theorem 3.66, $\underset{i \in I}{S}(A_i)(x) = \mu_A(x) >_1 \{\epsilon'\}$, and therefore, $x \in (\underset{i \in I}{S}(A_i))_{\overline{\{\epsilon'\}}} \subseteq (\underset{i \in I}{S}(A_i))_{\{\epsilon'\}}$.
   - $\mu_A(x) \geq_1 \{\delta\}$: by Theorem 3.66, $\underset{i \in I}{S}(A_i)(x) \geq \{\delta\} \geq_1 \{\epsilon'\}$, and therefore, $x \in (\underset{i \in I}{S}(A_i))_{\{\epsilon'\}}$.

   Given $x \in (\underset{i \in I}{S}(A_i))_{\overline{\{\epsilon'\}}}$, by definition, $\underset{i \in I}{S}(A_i)(x) >_1 \{\epsilon'\}$. Let us study two different situations:

   - $\epsilon' > \epsilon$:
   $$x \in (\underset{i \in I}{S}(A_i))_{\overline{\{\epsilon'\}}} \subseteq (\underset{i \in I}{S}(A_i))_{\{\epsilon'\}} = A_{\{\epsilon'\}},$$
   by definition of δ-ϵ-partition.
   - $\epsilon' = \epsilon$: then $\exists \eta \in (\epsilon, \delta]$, such that $\underset{i \in I}{S}(A_i)(x) >_1 \{\epsilon'\} = \{\epsilon\} > \{\eta\}$. Therefore:
   $$x \in (\underset{i \in I}{S}(A_i))_{\overline{\{\eta\}}} \subseteq (\underset{i \in I}{S}(A_i))_{\{\eta\}} = A_{\{\eta\}} \subseteq A_{\{\epsilon\}} = A_{\{\epsilon'\}},$$
   by definition of δ-ϵ-partition and $\epsilon < \eta$.

2. As in the previous axiom, let us study two different situations, given $i \neq j$:

- $\epsilon' > \epsilon$:
$$T(A_i, A_j)_{\overline{\{\epsilon'\}}} \subseteq T(A_i, A_j)_{\{\epsilon'\}} = \emptyset,$$
- $\epsilon' = \epsilon$: let us suppose that $\exists x \in T(A_i, A_j)_{\overline{\{\epsilon'\}}}$, i.e., $T(A_i, A_j)(x) >_1 \{\epsilon'\}$. Then, $\exists \eta \in (\epsilon, \delta]$, such that $T(A_i, A_j)(x) >_1 \{\epsilon'\} = \{\epsilon\} >_1 \{\eta\}$, and therefore, $x \in T(A_i, A_j)_{\{\eta\}} = \emptyset$ by definition of δ-ϵ-partition, which is a contradiction. ■

Finally, from these definitions, equivalence relations are easily obtained, associated to ϵ and δ, as stated in the next definition.

Definition 3.69 *Let X be a non-empty set, $A, B \in IVHFS(X)$ and $0 \leq \epsilon \leq \delta \leq 1$. Then:*

$$A =_{(\epsilon,\delta)} B \iff A_\alpha = B_\alpha, \ \forall \alpha \in FG([0,1]) \text{ such that } \{\epsilon\} <_1 \alpha <_1 \{\delta\}.$$

Theorem 3.70 *Given $0 \leq \epsilon \leq \delta \leq 1$, the relation $=_{(\epsilon,\delta)}$ is an equivalence relation.*

Proof. Reflexivity, transitivity and symmetry must be proven. All three are shown just by applying the definition straightforwardly. ■

Additionally, the original definition of δ-ϵ-partition can be also rewritten in terms of this equivalence relation.

Proposition 3.71 *Let X be a non-empty set and $A \in IVHFS(X)$. The family $\Pi = \{A_i \in IVHFS(X) | i \in I\}$ where I is a finite subset of $\mathbb{N}$, is a δ-ϵ-partition with $0 \leq \epsilon < \delta \leq 1$ if and only if:*

(1) $\underset{i\in I}{S}(A_i)) =_{(\epsilon,\delta)} A$,

(2) $T(A_i, A_j) =_{(\epsilon,\delta)} \emptyset, \ \forall i \neq j$,

where T and S are a t-norm and a t-conorm respectively.

Proof. It is immediate to prove it, as by definition of $=_{(\epsilon,\delta)}$, and bearing in mind that $\emptyset_\alpha = \emptyset$, both pairs of axioms are equivalent. ■

Along this section, we have adapted some definitions of partition given by Montes *et al.* in [51], where some of the characterizations and results given for fuzzy sets, have been generalized to interval-valued hesitant fuzzy sets.

These new definitions, when they are restricted to fuzzy sets, match the original definitions, obtaining classical definitions of fuzzy partitions, such as Ruspini's by selecting Lukasiewicz t-norm and t-conorm.

Chapter 4

Applications

In this chapter we have carried out two applications: protection of privacy in microdata and detection of edges in grey scale images.

The first application is devoted to the study of the protection of privacy in microdata, and a new approach involving the use of fuzzy partitions instead of the classical crisp point of view in this type of problem. In addition, an experimentation has served as a basis to prove the goodness of our proposal.

The second one is focused on the development of a new method to detect edges in grey scale images, applying methods to obtain interval-valued fuzzy relations from a fuzzy relation, which represents the original image. As well as in the previous application, an experimentation has been carried out in order to compare the results with other existing methods.

This chapter is structured in two sections covering the protection of privacy in microdata and the detection of edge images, respectively.

4.1 Protection of privacy in microdata

The sharing and diffusion of information is maybe the most common activity in the networked society. Therefore, the microdata (i.e., those data not summarized by some statistics but related to individuals) mining is becoming basic in an open range of fields such as Medicine or Business. Microdata are often presented by tables containing information about individuals. With the purpose of avoiding individuals to be uniquely identified, a common practice for organizations is to remove explicit identifiers such as name, phone number or social security number. However, although sometimes the published table looks anonymous, the privacy of the released data is involuntary compromised. For example, joining the data available in a released table with some publicly available database (like census database) or other attributes (for example race or ZipCode) can be used to identify individuals. The problem of protecting private information is actually legislated in several countries. The most representative laws regulating this task are the United States Healthcare Information Portability and Accountability Act and European Union directive 95/46/EC.

Therefore, as some public administrations are required to make public certain information, there is a need to find the balance between the right to privacy and the data dissemination. In order to avoid the identification of individuals, some techniques have been developed. Most of them are based on grouping individuals into equivalence classes, as the well known k-

anonymity. Samarati in [66] and Sweeney *et al.* in [70] define this technique as the property that makes every individual in the data indistinguishable from al least other $k-1$ individuals. There are many techniques to efficiently achieve k-anonymous tables, as the ones proposed by [66] and [70], or some more elaborated techniques, as the one given by Maimon *et al.* (see [47]).

However, k-anonymity does not represent a good protection if sensitive values in an equivalence class lack diversity (homogeneity attack). A new technique was developed by Machanavajjhala *et al.* (see [35]) called l-diversity, that requires the sensitive attributes in each group of k indistinguishable individuals to have at least l well represented different values.

The l-diversity presents also some drawbacks, mainly based on bias and similarity attacks. For instance, if the sensitive attribute is numeric, the l-diversity does not take into account that some values can be very similar. To solve this similarity attacks, a new technique was recently developed by Li *et al.* in [45], known as t-closeness, which establishes that the distribution of the sensitive attribute in each equivalence class has to be similar to the one in the whole table. In this approach, the similarity is measured by means of the Earth Mover's Distance (see [45]).

There are other approaches to preserve privacy in data contexts. For example, in [21] it is introduced local suppression to achieve a tailored privacy model for trajectory data anonymization. Zhong in [84] studies how to maintain privacy in distributed mining of frequent itemsets without revealing each party's portion of the data to the other. Other highlighted works about this issue can be seen in [14, 30, 63, 68].

Each technique has a weak point. The proposal made in this section is based on the use of fuzzy sets properties to improve these techniques and to get a better protection. Therefore, each attribute will be masked by fuzzy partitions instead of the crisp ones. This is not a simple generalization, as it requires a different cardinality (see [28]). The goal of this section is to protect released data using fuzzy set theory and to check the performance of this approach.

4.1.1 Basic concepts

This subsection is devoted to describe the methods and tools related to privacy upon which this work is based. For a more detailed description see [23, 35, 44].

Microdata are represented by tables (denoted by T), where the rows represent the individuals, and the columns the attributes defining these individuals. In a privacy context, two types of attributes are defined, the sensitive ones (the ones to be protected, denoted by S), and the non sensitive ones (the others, denoted by Q). A quasi-identifier (denoted by QI) is a subset of the non sensitive attributes.

Example 4.1 *Table 4.1 shows the information of 12 individuals, with respect to three attributes, where the sensitive one is the Illness, as this information must not be publicly associated to the individual. On the other hand, the non sensitive attributes are the ZIP code and the Age. Formally,* $S = \{Illness\}$ *and* $Q = \{ZIP, Age\}$. Q, $Q1 = \{Illness\}$ *and* $Q2 = \{Age\}$ *are the possible quasi-identifiers in this case.*

However, it is easily detected the poor privacy protection provided by a table like the one given in the previous example in Table 4.1. A direct relation between the non sensitive attribute and the sensitive one makes it possible to obtain sensitive information with a small amount of knowledge (in Table 4.1, knowing an individual who is in the data and is 22 years old leads to discovering the illness of such person: gastritis).

Individual	ZIP	Age	Illness
1	47677	29	gastric ulcer
2	47602	22	gastritis
3	47678	27	stomach cancer
4	47905	43	gastritis
5	47979	52	flu
6	47906	47	broncuitis
7	47973	36	pneumonia
8	47607	32	stomach cancer
9	47906	55	heart attack
10	47925	56	heart attack
11	47923	61	angina
12	47923	67	pneumonia

Table 4.1: Example of microdata.

In order to overcome this problem, the classical procedure is to apply crisp partitions to the quasi-identifier, so this direct identification is not as straightforward as in the original data. The definition of crisp partition is the usual one, where the sets are pairwise disjoint and the union of all of them gets the whole set.

Example 4.2 *Given Table 4.1, crisp partitions are applied to non sensitive attributes. ZIP code can be split into two subsets, {476**, 479**}. Meanwhile, Age into four:* $\{[0, 30], (30, 40], (40, 50], (50, \infty]\}$*. Then, we obtain Table 4.2 as a result of applying these two partitions.*

In Table 4.2, the individuals have been grouped in function of the value of the quasi-identifier, where in each block, they are indistinguishable with respect to them.

Individuo	ZIP	Edad	Enfermedad
1	476**	[0,30]	gastric ulcer
2	476**	[0,30]	gastritis
3	476**	[0,30]	stomach cancer
7	479**	(30,40]	pneumonia
8	476**	(30,40]	stomach cancer
4	479**	(40,50]	gastritis
6	479**	(40,50]	broncuitis
5	479**	(50,∞]	flue
9	479**	(50,∞]	heart attack
10	479**	(50,∞]	heart attack
11	479**	(50,∞]	angina
12	479**	(50,∞]	pneumonia

Table 4.2: Example with crisp partitions applied.

Formally, a q^*-block of a table T is given by the individuals which are indistinguishable with respect to the quasi-identifier QI and which value for such attributes is q^*.

It is obvious that the simple use of partitions is not enough, as some blocks can be represented by a single individual, and as a result, the sensitive information is compromised.

Partitions are used to mask microdata in order to obtain a new table, which is meant to be released minimizing the risk of revealing private information. However, some additional properties are mandatory in order to obtain a proper protection. The ones studied here, are the most well-known techniques, and will be deeply analyzed in the next subsection.

4.1.2 Protection techniques

The three techniques studied in this subsection are k-anonymity, l-diversity and t-closeness, and as it will be explained, each one protects the information from different types of attacks.

k-anonymity

This first technique is of great importance in order to protect the data from attacks like the one shown in Example 4.2, where a block is shaped by a single individual. Samarati and Sweeney (see [66, 70]) define a k-anonymous table as the one that makes every individual in the data indistinguishable from al least other $k-1$ individuals.

Definition 4.3 *A table T satisfies k-anonymity if for all tuple $t \in T$, there exists other $k-1$ tuples indistinguishable with respect to the quasi-identifier, i.e., it exists $t_{i_1}, \ldots, t_{i_{k-1}} \in T$ such that $t[QI] = t_{i_1}[QI] = \cdots = t_{i_{k-1}}[QI]$, where $t[QI]$ denotes the values taken by the tuple t for the quasi-identifier QI.*

Example 4.4 *Given Table 4.1, and applying the partitions {476**, 479**} for the ZIP code and $\{[0,32],(32,52],(52,\infty]\}$ for Age, Table 4.3 is the resulting table.*

This table is 4-anonymous, as each block has at least 4 individuals, which are indistinguishable with respect to the quasi-identifier {Zip, Age}.

This technique also has some drawbacks. The most important one lies in the homogeneity of the sensitive attribute. If all the sensitive values of a block are the same, the privacy is again compromised. To prevent this type of attack a new metric is developed: l-diversity.

Individuo	ZIP	Age	Illness
1	476**	$[0, 32]$	gastric ulcer
2	476**	$[0, 32]$	gastritis
3	476**	$[0, 32]$	stomach cancer
8	476**	$[0, 32]$	stomach cancer
4	479**	$(32, 52]$	gastritis
5	479**	$(32, 52]$	flu
6	479**	$(32, 52]$	broncuitis
7	479**	$(32, 52]$	pneumonia
9	479**	$(52, \infty]$	heart attack
10	479**	$(52, \infty]$	heart attack
11	479**	$(52, \infty]$	angina
12	479**	$(52, \infty]$	pneumonia

Table 4.3: 4-anonymous example.

l-diversity

The main goal of this technique is to measure the diversity of the sensitive attribute in each block, thus dealing with and, as a result, avoiding homogeneity attacks (see [35]).

Definition 4.5 *A q^*-block is l-diverse if it contains at least l well-represented values for the sensitive attribute S. A table is l-diverse if every q^*-block is l-diverse.*

Associated to the l-diversity technique can be found the concepts of prior and posterior belief. The second one is obtained as stated in the next result through a generalization (from the original table T to the new one T^*), where the sensitive value is s given that the non sensitive value is q. It is computed as follows (see [35]).

Theorem 4.6 *Let q be a value of the non sensitive attributes Q in the table T; q^* be the generalized values of q in the private table T^*; s be a possible value of the sensitive attribute; $n(q^*, s')$ be the number of tuples $t^* \in T^*$ where $t^*[Q] = q^*$ and $t^*[S] = s'$; and $f(s'|q^*)$ be the conditional probability of the sensitive attribute conditioned on the fact that the non sensitive attributes Q can be generalized to q^*. Then the posterior belief is defined by:*

$$\beta_{(q,s,T^*)} = \frac{n_{(q^*,s)} \dfrac{f(s|q)}{f(s|q^*)}}{\displaystyle\sum_{s' \in S} n_{(q^*,s')} \dfrac{f(s'|q)}{f(s'|q^*)}}.$$

Example 4.7 *The protected Table 4.3, in addition to be 4-anonymous, it is also 3-diverse, as in each block, at least 3 different sensitive values are possible.*

Even if a table satisfies l-diversity, it could be attacked using similarity of values. If all the possible sensitive values in a block are similar or closely related, the attacker can get some undesirable extra information. In order to deal with such drawback, the last technique helps to deal with it: t-closeness.

t-closeness

t-closeness (see [45]) protects the data from the similarity attacks previously mentioned as a weak point of l-diversity. The procedure of this new technique is developed as follows.

Suppose an attacker has a prior belief about the sensitive attribute of an individual (denoted by B_0). He gets the information about the whole published table (denoted by W). Then his belief changes to B_1. After identifying the values of the quasi-identifier, the attacker identifies the block the individual belongs to, obtaining the distribution of the sensitive attribute in that block (denoted by P). Then, the attacker's belief changes to B_2.

$$\boxed{B_0 \xrightarrow{W} B_1 \xrightarrow{P} B_2}$$

Assuming that the information given by W is public, if the goal is to minimize $B_2 - B_0$, $B_2 - B_1$ should be minimized. To measure that distance the Earth Mover's Distance (see [45]) was used. The next definition is devoted to the Earth Mover's Distance for both quantitative and qualitative attributes.

Definition 4.8 *Let* $P = (p_1, \dots, p_m)$ *be the distribution of the sensitive attribute in the block,* $W = (w_1, \dots, w_m)$ *the distribution of the sensitive attribute in the whole table and* $\{v_1, \dots, v_m\}$ *the values assumed by the sensitive attribute. The Earth Mover's Distance is split into two cases, given* $r_i = p_i - w_i$, $\forall i = 1, \dots, m$*:*

- **Quantitative sensitive attribute:** $\{v_1, \dots, v_m\}$ *are numerical values ordered increasingly, then:*

$$D[P, W] = \frac{1}{m-1} \sum_{i=1}^{m} \left| \sum_{j=1}^{i} r_j \right|.$$

- **Qualitative sensitive attribute:**

$$D[P, W] = \frac{1}{2} \sum_{i=1}^{m} |r_i|.$$

Based on this distance, the definition of t-closeness is given as follows.

Definition 4.9 *A block is said to satisfy t-closeness if the distance between the distribution of a sensitive attribute in this class P and the distribution of the attribute in the whole table W is no more than a threshold t. A table is said to have t-closeness if all blocks have t-closeness.*

Example 4.10 *Let Table 4.4 be a protected one by a partition of the non sensitive attributes (ZIP code and Age), where the sensitive one is Salary.*

Individuo	ZIP	Age	Salary(k)
1	476**	$[0, 30)$	3
2	476**	$[0, 30)$	4
3	476**	$[0, 30)$	5
4	479**	$[40, \infty)$	6
5	479**	$[40, \infty)$	11
6	479**	$[40, \infty)$	8
7	476**	$[30, 40)$	7
8	476**	$[30, 40)$	9
9	476**	$[30, 40)$	10

Table 4.4: 3-anonymity, 3-diversity, 0.1667-closeness example.

There are 3 blocks in the table, with 3 individuals each one, hence, it is a 3-anonymous table. In each block, there are 3 different values of the sensitive attribute, so the table is 3-diverse.

Regarding the study of the t-closeness, given $\{3, \ldots, 11\}$ the set of values that the sensitive attribute take, the distributions in the whole table (W) and *in each block* (P_1, P_2, P_3) *are given by*

$$\begin{aligned}
W &= (\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}), \\
P_1 &= (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, 0, 0), \\
P_2 &= (0, 0, 0, \frac{1}{3}, 0, \frac{1}{3}, 0, 0, \frac{1}{3}), \\
P_3 &= (0, 0, 0, 0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, 0).
\end{aligned}$$

Applying the Earth Mover's Distance for each q^**-block, the obtained values are* $D[P_1, W] = 0.375$, $D[P_2, W] = 0.1667$ *and* $D[P_3, W] = 0.2361$. *Hence,* $t = \min(0.375, 0.1667, 0.2361) = 0.1667$.

In order to conclude this subsection, a diagram of the different types of attacks and the developed techniques is given in Figure 4.1.

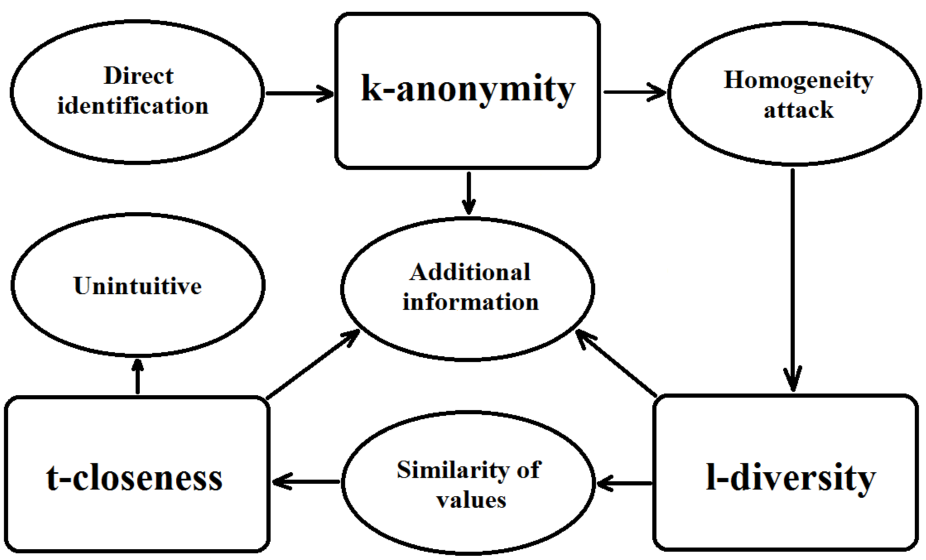


Figure 4.1: Relationships between privacy metrics and different attacks.

As it can be read in Figure 4.1, the additional information attack is common to all the techniques. Obviously, if an attacker has at his disposal a great amount of information to cross with the released data, he can discard possible values of the sensitive attribute, reducing the privacy of such data. However, this additional information can not be dealt with, as it is impossible to know the attacker's additional background knowledge.

4.1.3 Fuzzy approach

In this subsection, the new approach using fuzzy partitions instead of the usual crisp partitions is given. This new point of view uses the good properties of fuzzy sets to protect the privacy of data by the uncertainty associated to them.

In a k-anonymous table the equivalence classes are constructed so that each individual record is indistinguishable from at least $k-1$ records within the same class (with regard to the quasi-identifier). This condition might not be enough, as sensitive information could be homogeneously associated to individuals within the same class. This drawback could be overcome by introducing a fuzzy model. A priori, the fuzzyfied released table provides a first level of privacy as some uncertainty is introduced to protect the data against an attacker.

However, the introduction of fuzziness requires a redefinition in terms of fuzzy sets of the aforementioned metrics. Two concepts are needed to carry it out. The definitions of the selected fuzzy cardinality and fuzzy partition.

The cardinality selected for this application has been the non-fuzzy cardinality given by Ralescu in [61] (Definition 1.42). This cardinality, as it has been stated in Chapter 3, can be obtained as a scalar cardinality definition for interval-valued hesitant fuzzy sets (Definition 3.23) restricted to fuzzy sets, as given in Example 3.32. Furthermore, we associate a possibility measure to this cardinality, given by

$$Poss(|A|_R \geq k) = \begin{cases} \mu_{(k)}, & \text{if } k \geq j, \\ (1-\mu_{(j)}) \vee \mu_{(j)}, & \text{if } k < j, \end{cases}$$

where

$$j = \begin{cases} \max\{1 \leq s \leq n \mid \mu_{(s-1)} + \mu_{(s)} > 1\}, & \text{if } A \neq \emptyset, \\ 0, & \text{if } A = \emptyset, \end{cases},$$

and $\mu_{(1)}, \ldots, \mu_{(n)}$ represent the membership degrees of A ordered decreasingly.

On the other hand, Ruspini's fuzzy partition definition is the one used for this proposal, given in [65] (Definition 1.48). In addition, as it has been mentioned in Chapter 3, the new definition of δ-ϵ-partition (Definition 3.63), when it is restricted to fuzzy sets, matches the original definition (Definition 1.49), being able to obtain the classical Ruspini's fuzzy partition definition, by selecting Lukasiewicz t-norm and t-conorm.

Next, different generalizations of the previous techniques are given and explained in detail. First, it is introduced a new privacy metric based on assigning a fuzzy number to anonymity. Secondly, it is checked that the traditional privacy metrics l-diversity and t-closeness can be extended using fuzzy partitions, along with several results and examples in order to get a better understanding of these new techniques.

Q-anonymity

The first technique is an adaptation of the classical k-anonymity. It is obtained in a different way than the other studied techniques, where a possibility measure and an aggregation operator are used.

Definition 4.11 *Let T be a table with attributes $\{A_1, \ldots, A_n\}$, and let FQ be a fuzzy quasi-identifier associated with it. The Q-anonymity of T with respect to FQ is given by:*

$$Poss(|T|_R \geq Q) = \mathcal{T}(\beta_{f_1}, \ldots, \beta_{f_s}),$$

where

- $T[FQ]_1, \ldots, T[FQ]_s$ *are the different fuzzy classes that shape the fuzzy partition,*
- $Poss$ *is the aforementioned possibility measure,*
- β_{f_i} *is the possibility that* $T[FQ]_i$ *has at least* Q *elements and*
- $\mathcal{T}$ *is an aggregation operator, i.e., a mapping* $\mathcal{T} : \cup_{n \in \mathbb{N}}[0,1]^n \to [0,1]$ *fulfilling the boundary conditions (*$\mathcal{T}(0, \ldots, 0) = 0$ *and* $\mathcal{T}(1, \ldots, 1) = 1$*), being the identity when unary (*$\mathcal{T}(x) = x, \forall x \in [0,1]$*) and being increasing (*$\forall n \in \mathbb{N} : x_1 \leq y_1, \ldots, x_n \leq y_n \Rightarrow \mathcal{T}(x_1, \ldots, x_n) \leq \mathcal{T}(y_1, \ldots, y_n)$*) (see [37]).*

An example is given next so it is easier for the reader to understand how to obtain the value provided by this technique.

Example 4.12 *The original data with which we are working in this example are given in Table 4.5, where the sensitive attribute, as usual in the examples along this section, is Illness.*

The fuzzy partition carried out is given by the sets $Young = (-\infty; 30; 36)$*,* $Adult = (30; 36; 54; 60)$ *and* $Advanced\ Age = (54; 60; \infty)$*. This partition leads us to Table 4.6, whose Q-anonymity value is obtained through the results shown in Table 4.7 by selecting as aggregation operator the average.*

Individual	Age	Illness
1	21	gastritis
2	24	flu
3	32	stomach ulcer
4	36	gastritis
5	45	pneumonia
6	56	flu
7	58	heart attack
8	62	heart attack
9	65	pneumonia

Table 4.5: Original data for Q-anonymity example.

Age	Illness
Young	gastritis flu stomach ulcer
Adult	gastritis pneumonia flu
Advanced Age	heart attack pneumonia

Table 4.6: Data after applying a fuzzy partition to attribute Age.

In this case, the value obtained for the Q*-anonymity of Table 4.6 through Table 4.7 is* $Q = 3$*, as it is given by the greatest value of* Q *where* $Poss(|T|_R \geq Q) \geq 0.5$.

	Q=1	Q=2	Q=3	Q=4
Poss(Young)	0.67	0.67	0.67	0
Poss(Adult)	0.67	0.67	0.67	0
Poss(Advanced Age)	0.67	0.67	0.67	0.33
Poss($\|T\|_R \geq Q$)	0.67	0.67	0.67	0.11

Table 4.7: Values associated with Q-anonymity.

Although this technique provides a better protection than the classical one with respect to the homogeneity attack thanks to the labels assigned to the non-sensitive attribute, there is a necessity for another technique in order to give further security against this type of attacks. To do so, the generalization of l-diversity is analyzed in the next part of the subsection.

l-diversity

The focus of the beginning of this part is on the generalization of the obtaining of the posterior belief (Theorem 4.6), or what is the same, of the expression

$$\beta_{(q,s,T^*)} = \frac{n_{(q^*,s)}\dfrac{f(s|q)}{f(s|q^*)}}{\displaystyle\sum_{s' \in S} n_{(q^*,s')}\dfrac{f(s'|q)}{f(s'|q^*)}}.$$

In order to do it, we are adapting each term one by one. Firstly, as in the fuzzy case the element q^* does not exist, it is substituted by the membership functions associated to each partition. Secondly, the values $n(s,q,T^*)$ are defined as follows:

Definition 4.13 *Let $s \in S$ and $q \in Q$, where S and Q are the sensitive and non sensitive attributes, respectively. Given the fuzzy partition*

$\{Q_1, \ldots, Q_n\}$, the term $n(s, q, T^*)$ denotes the number of elements in the generalized table whose values are s and q for the sensitive and non sensitive attributes respectively. Then,

$$n(s, q, T^*) = \sum_{i=1}^{n} \mu_{Q_i}(q) \cdot |Q_i \cap s|_R.$$

The adaptation of the term $\dfrac{f(s|q)}{f(s|q^*)}$ is given next. $f(s|q)$ is obtained analogously to the classical situation, while for the term $f(s|q^*)$ we proceed in a similar way to how we did for $n(s, q, T^*)$.

Definition 4.14 *Let $s \in S$ and $q \in Q$, where S and Q are the sensitive and non sensitive attributes, respectively. Given the fuzzy partition $\{Q_1, \ldots, Q_n\}$, then*

$$f(s|q^*) = \sum_{i=1}^{r} \mu_{Q_i}(q) \cdot f(s|Q_i),$$

where

$$f(s|Q_i) = \frac{|s \cap Q_i|_R}{|Q_i|_R} \quad \forall i \in \{1, \ldots, n\}.$$

Finally, it is necessary to prove that $f(\cdot|Q_i)$ is a probability as previously defined.

Theorem 4.15 *$f(\cdot|Q_i)$ given as in Definition 4.14 is a probability, for any $i = 1, \ldots, n$.*

Proof. Let us prove that $f(\cdot|Q_i)$ satisfies the three axioms of Kolmogorov.

- $f(s|Q_i) \geq 0$: Obvious by construction,

- $f(S|Q_i) = 1$:

$$f(S|Q_i) = \frac{|S \cap Q_i|_R}{|Q_i|_R} = \frac{|Q_i|_R}{|Q_i|_R} = 1.$$

- Given $(A_n)_n \subset S$, pairwise disjoint,

$$\begin{aligned} f(\cup_n A_n|Q_i) &= \frac{\left|\bigcup_n A_n \cap Q_i\right|_R}{|Q_i|_R} = \frac{\left|\bigcup_n (A_n \cap Q_i)\right|_R}{|Q_i|_R} \\ &= \frac{\sum_n |A_n \cap Q_i|_R}{|Q_i|_R} = \sum_n \frac{|A_n \cap Q_i|_R}{|Q_i|_R} = \sum_n f(A_n|Q_i). \end{aligned}$$

So the three axioms have been demonstrated, and as a result, $f(\cdot|Q_i)$ is a probability. ■

Bearing all this results in mind, it is possible to define the posterior belief in the fuzzy case as follows.

Definition 4.16 *Let $s \in S$ and $q \in Q$, where S and Q are the sensitive and non sensitive attributes, respectively. Given the fuzzy partition $\{Q_1, \ldots, Q_n\}$, then,*

$$\beta_{(q,s,T^*)} = \frac{\left(\sum_{i=1}^{n} \mu_{Q_i}(q) \cdot |Q_i \cap s|_R\right) \dfrac{f(s|q)}{\sum_{i=1}^{n} \mu_{Q_i}(q) \cdot \dfrac{|Q_i \cap s|_R}{|Q_i|_R}}}{\sum_{s' \in S} \left(\sum_{i=1}^{n} \mu_{Q_i}(q) \cdot |Q_i \cap s'|_R\right) \dfrac{f(s'|q)}{\sum_{i=1}^{n} \mu_{Q_i}(q) \cdot \dfrac{|Q_i \cap s'|_R}{|Q_i|_R}}}.$$

Therefore, the definition of l-diversity is consistent when using fuzzy partitions, only adapting the corresponding posterior believes to the case of using fuzzy sets. As a consequence, the definition of l-diversity for fuzzy partitions is given as follows.

Definition 4.17 *Given a table T with a fuzzy partition $\{Q_1, \ldots, Q_n\}$, T is l-diverse if for every Q_i, there exists at least l well-represented values for the sensitive attribute.*

Example 4.18 *Given the Table 4.6 with a fuzzy partition in the non sensitive attribute Age, it is straightforward to see that it satisfies a l-diversity for $l = 2$.*

The use of labels is again a weapon that makes this approach more interesting than the classical one, and as a result, it is better protected against similarity attacks. However, an extension of t-closeness is given next so a better level of protection is provided.

t-closeness

The consistence of t-closeness is checked when fuzzy partitions are used. Remember that t-closeness minimizes the distance between the distribution of the sensitive attribute in the whole table and the distribution of the sensitive attribute associated to each block of the table in the crisp case.

However, in the fuzzy case, this comparison is carried out between the whole distribution and the distribution associated to each individual of the table instead of each block. It is necessary to prove that the expression associated to an individual given in the next result is also a probability in order to define correctly the technique.

Theorem 4.19 *Let $A_1, \ldots, A_n$ be the non sensitive attributes and S be the sensitive one, where $s_1, \ldots, s_m$ are the values that it can assume. Given*

$\{Q_1, \ldots, Q_r\}$ a fuzzy partition, $W = (w_1, \ldots, w_m)$ the distribution of the sensitive attribute in the whole table, $P_1, \ldots, P_r$ the distribution in $Q_1, \ldots, Q_r$ respectively, with $P_i = (p_1^i, \ldots, p_m^i), \quad \forall i = 1, \ldots, r$, and μ_{Q_i} the membership function of Q_i for each i. Then, for all x individual,

$$P_x = \sum_{i=1}^{r} \mu_{Q_i}(x) \cdot P_i$$

is a probability distribution.

Proof. Consider $P_x = (p_1, \ldots, p_m)$ associated to each x. To check that P_x is a probability distribution, it is proved that P_x is non negative ($p_i \geq 0$) and $\sum_{i=1}^{m} p_i = 1$.

- $p_i \geq 0, \quad \forall i = 1, \ldots, m$:

  As $\mu_{Q_i}(x) \geq 0$ and $p_j^i \geq 0$, $\forall i, j$. Then:

  $$p_j = \sum_{i=1}^{r} \mu_{Q_i}(x) \cdot p_j^i \geq 0, \quad \forall j = 1, \ldots, m.$$

- $\sum_{i=1}^{m} p_i = 1$:

  As P_j are probability distributions, $\sum_{i=1}^{m} p_i^j = 1$ and by the definition of fuzzy partition,

  $$\sum_{i=1}^{r} \mu_{Q_i}(x) = 1.$$

Then:

$$\begin{aligned}
\sum_{j=1}^{m} p_j &= \sum_{j=1}^{m}\sum_{i=1}^{r} \mu_{Q_i}(x) \cdot p_j^i = \\
&= \sum_{i=1}^{r}\sum_{j=1}^{m} \mu_{Q_i}(x) \cdot p_j^i = \\
&= \sum_{i=1}^{r} \mu_{Q_i}(x) \cdot (\sum_{j=1}^{m} p_j^i) = \\
&= \sum_{i=1}^{r} \mu_{Q_i}(x) = 1.
\end{aligned}$$

Therefore, P_x is a probability distribution for every individual x. ■

Once that this result is given, the definition of t-closeness in the fuzzy case is consistent.

Definition 4.20 *An individual x satisfies t-closeness if the distance between the distribution of the sensitive attribute associated to the individual, P_x (see Theorem 4.19), and the distribution of the sensitive attribute in the whole table W is no more than a threshold t. A table is said to have t-closeness if every individual satisfies that property.*

As it has been done in the crisp case, the distance between distributions is done again by the Earth Mover's Distance. A brief example is given next.

Example 4.21 *An example where the sensitive attribute is the Salary is shown in Table 4.8.*

Let us apply to the Age attribute the fuzzy partition Young $= (-\infty, 30, 36)$, *Adult* $= (30, 36, 54, 60)$ *and Advanced Age* $= (54, 60, \infty)$. *Table 4.9 is the resulting one.*

Individual	Age	Salary(k)
1	21	3
2	24	4
3	32	5
4	36	6
5	45	7
6	56	4
7	58	8
8	62	8
9	65	7

Table 4.8: Original data for t-closeness example.

In order to get the parameter for fuzzy t-closeness, the distributions of the sensitive attribute associated to each individual must be obtained, given by:

$$P_1 = P_2 = P_Y, \qquad P_3 = \frac{2}{3}P_Y + \frac{1}{3}P_A,$$
$$P_4 = P_5 = P_A, \qquad P_6 = \frac{2}{3}P_A + \frac{1}{3}P_{AA},$$
$$P_7 = \frac{1}{3}P_A + \frac{2}{3}P_{AA}, \qquad P_8 = P_9 = P_{AA},$$

where P_Y, P_A, P_{AA} are the distributions associated to each set of the fuzzy partition. Finally, it must be calculated the Earth Mover's Distance of each individual to the distribution in the whole table, and select the minimum one as the parameter.

$$D[P_1, W] = 0.3556, \qquad D[P_2, W] = 0.3556, \qquad D[P_3, W] = 0.2444,$$
$$D[P_4, W] = 0.1111, \qquad D[P_5, W] = 0.1111, \qquad D[P_6, W] = 0.1222,$$
$$D[P_7, W] = 0.2222, \qquad D[P_8, W] = 0.3444, \qquad D[P_9, W] = 0.3444,$$

Age	Salary (k)
Young	3
	4
	5
Adult	4
	6
	7
Advanced Age	7
	9

Table 4.9: Data after applying a fuzzy partition to attribute Age.

In this example, $t = 0.1111$.

As a result, this technique helps to deal with the similarity attacks that l-diversity is weak to. With these three new techniques (Q-anonymity, l-diversity and t-closeness), we have developed a way to measure the level of protection provided by the fuzzy partitions.

In order to compare both approaches, crisp and fuzzy ones, an experimental comparison has been carried out, and it is developed in the next part of this section.

4.1.4 Experimentation

In this part of the section the performance of the proposed approach in terms of privacy preservation is checked. The level of protection obtained when a database is coded using either a fuzzy partition or a crisp one is studied. Firstly, the analyzed methods are explained, both crisp and fuzzy ones. Secondly, information about the selected database and the obtained results are shown.

Analyzed methods

In order to get a crisp partition, the k-means (see [41]) method is used, as it is widely used in clustering when dealing with crisp data. The algorithm describing the method is summarized as:

Step 1. Randomly place k points representing initial group centroids.

Step 2. Assign each object to the group that has the closest centroid.

Step 3. Recalculate the positions of the k centroids after assigning all objects.

Step 4. Repeat Steps 2 and 3 until the centroids no longer move.

The *Matlab (version 7.11.0, R2010b)* implementation of this method was used in this experiment (see [41]).

On the other hand, the selected methods to obtain fuzzy partitions have been fuzzy c-means and Gustafson-Kessel method.

Fuzzy c-means (see [39]) minimize the functional:

$$J(X;U,V) = \sum_{i=1}^{c}\sum_{k=1}^{N}(\mu_{ik})^m ||x_k - v_i||_A^2,$$

where X is the data set, $U = [\mu_{ik}]$ the membership matrix of each individual to each set, $V = [v_1, \ldots, v_c]$ the set of centroids and m a parameter controlling the fuzziness. The used inner product norm is:

$$D_{ikA}^2 = ||x_k - v_i||_A^2 = (x_k - v_i)^T A(x_k - v_i),$$

the one induced by $A = I$. The implementation of the used algorithm was *FCMclust* of the package *Fuzzy Clustering and Data Analysis Toolbox*

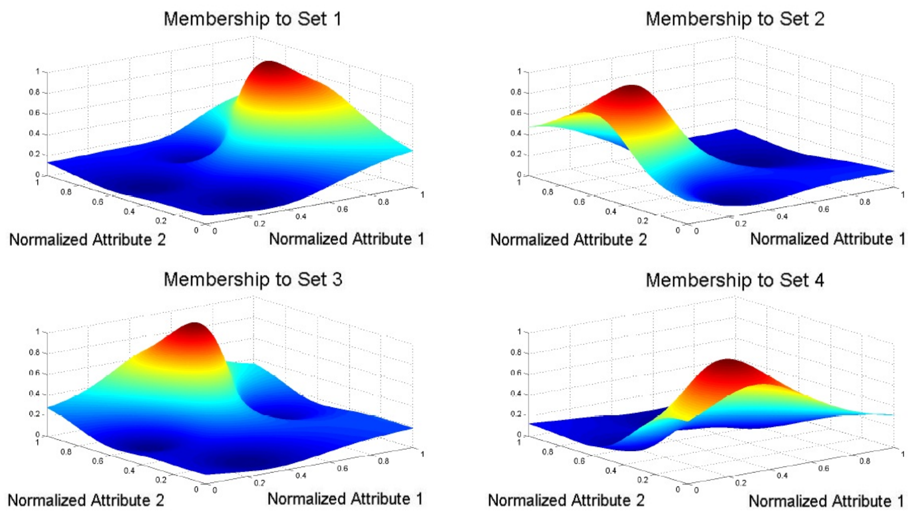


Figure 4.2: Fuzzy sets obtained by *fuzzy c-means* algorithm.

of *Matlab* (see [4]). In Figure 4.2, an example of a representation of this method is given for 4 sets.

Gustafson-Kessel (see [39]) method extends the standard fuzzy c-means algorithm by employing an adaptive distance norm, in order to detect clusters of different geometrical shapes in one data set. Each cluster has its own norm-inducing matrix A_i, which yields the inner-product norm $D^2_{ikA_i}$.

The matrices A_i represent optimization variables in the c-means functional, thus allowing each cluster to adapt the distance norm to the local topological structure of the data. The objective functional of the Gustafson-Kessel algorithm is defined by

$$J(X;U,V,A) = \sum_{i=1}^{c}\sum_{k=1}^{N}(\mu_{ik})^m D^2_{ikA_i}.$$

This algorithm is known as *GKclust* in the aforementioned *Matlab* package (see [4]).

As it has been previously done, in Figure 4.3, an example of a representation of Gustafson-Kessel method is given for 4 sets.

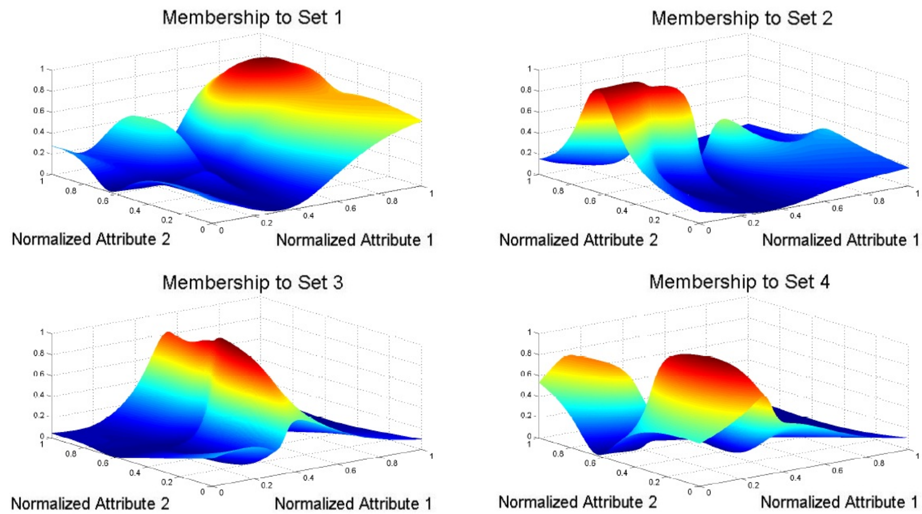


Figure 4.3: Fuzzy sets obtained by *Gustafson-Kessel* algorithm.

It is remarkable the difference between both fuzzy methods. While fuzzy c-means provides sets with soft curves and membership centered in a single area, Gustafson-Kessel's sets have irregular shapes and are not that centered in a single zone (see sets 3 and 4 from Figure 4.3). The reason behind these differences is the use of various matrices in the definition of Gustafson-Kessel instead of one, as in fuzzy c-means. These differences lead us to study both methods, as they provide solutions of different types.

Results

Results have been obtained for two different databases, CENSUS and EIA, both available at *sdcMicro R*-package (see [71]), in order to test the performance of the proposed approach.

CENSUS dataset was obtained on July 27, 2000 using the public Data Extraction System of the U.S. Bureau of the Census. It consists of 1080 examples characterized by 13 attributes (*afnlwgt, agi, emcontrb, ernval, fedtax, fica, intval, pearnval, pothval, ptotval, statetax, taxinc* and *walval*).

To test the performance of the approach the attributes *ptotval* (total person income) and *taxinc* (taxable income amount) have been selected as sensitive variable. Therefore two different experiments are considered, one with *ptotval* as sensitive value and other with *taxinc*.

In addition, the quasi-identifier is formed by any combination of two elements of the other eleven attributes, which are clustered according to the previously defined methods. According to the size of the dataset and in order to better compare the results, the quasi-identifier is coded using three sets (fuzzy or crisp).

On the other hand, EIA dataset was obtained from the U.S. Energy Information Authority. It consists of 4092 examples characterized by 15 attributes (*utilityid, utilname, state, year, month, resrevenue, ressales, comrevenue, comsales, indrevenue, indsales, othrevenue, othrsales, totrevenue* and *totsales*).

In order to study the performance of the attribute 13 (*othrsales*), the first five attributes have not been considered, as they contain administrative information. Again, the quasi-identifier is formed by any combination of two of the remaining 9 attributes.

After coding the quasi-identifier according to the three algorithms, k-anonymity, l-diversity, t-closeness and Q-anonymity are computed. Note that when fuzzy partitions are considered, the k-anonymity is computed taking as k the estimation obtained by Ralescu's non fuzzy cardinality (Definition 1.42).

Tables 4.10 and 4.11 show the number of experiments with the highest level of protection according to each metric and clustering method for CENSUS and EIA database respectively.

		k-means	*Gustafson-Kessel*	*fuzzy c-means*
k-anonymity		25%	18%	**57%**
Q-anonymity		-	14%	**86%**
Sensitive Attribute				
PTOTVAL	l-diversity	**53%**	16%	31%
	t-closeness	0%	**78%**	22%
TAXINC	l-diversity	**51%**	30%	48%
	t-closeness	0	**78%**	22%

Table 4.10: Summary of Results for CENSUS database.

		k-means	*Gustafson-Kessel*	*fuzzy c-means*
k-anonymity		13%	**63%**	24%
Q-anonymity		-	**60%**	40%
Sensitive Attribute				
OTHRSALES	l-diversity	31%	**44%**	25%
	t-closeness	0%	**69%**	31%

Table 4.11: Summary of Results for EIA database.

As it can be seen in Table 4.10, fuzzy c-means performs the best with regard to both k-anonymity and Q-anonymity. Anyhow, this behaviour does not remains when both l-diversity and t-closeness are studied. Focusing on l-diversity crisp methods seem to perform better. On the other side, Gustafson-Kessel algorithm performs the best with regard to t-closeness. Meanwhile in Table 4.11, it is Gustafson-Kessel the method which performs the best with regard to both k-anonymity and Q-anonymity as well as l-diversity and t-closeness.

As a complementary check, once the data are protected, the number of individuals with risk of re-identification higher than the rest is computed. This number is computed using the function *measure_risk* of package *sdcMicro* in R (see [71]). This measure of individuals in risk is computed as follows (see [40]):

- For each individual in the released table i^*, it is computed the probability of this individual to be related to another one in the original table (ρ_i).
- The individual risk of re-identification, r_i, that represents the same probability of ρ_i, but with the condition that the attacker tries to obtain the values of all the individuals of the released table.
- The output argument is the number of individuals of the table whose r_i is much bigger than the rest.

Tables 4.12 and 4.13 show the averages of the values of each parameter ($\overline{X}$) and the distance to the optimum value (D) (when the method is not the optimum) for CENSUS and EIA databases respectively.

It must be noted that the fuzzy methods perform better because their behaviour with regard to all the metrics is more stable, i.e., when the method is not the best, the distance to the optimum is low. The only exception appears in Table 4.13, where *measure_risk* is slightly better in the crisp method. Anyway, this fact is offset by the results obtained for k-anonymity, l-diversity and t-closeness, as they are much better in Gustafson-Kessel method than the crisp one k-means.

		k-means		*Gustafson-Kessel*		*fuzzy c-means*	
Criteria		$\overline{X}$	D	$\overline{X}$	D	$\overline{X}$	D
k-anonymity		213.62	100.27	200.78	107.05	**278.42**	**23.78**
measure_risk		164.42	189.36	236.78	257.83	**109.22**	**107.9**
Q-anonymity		-	-	184.36	99.3	**262.29**	**47.63**
PTOTVAL	l-diversity	97.73	41.88	89.55	32.54	**109.56**	**10.7**
	t-closeness	0.0867	0.0827	**0.0049**	**0.0042**	0.0129	0.0115
TAXINC	l-diversity	93.76	37.42	86.13	30.96	**104.6**	**10.19**
	t-closeness	0.0879	0.0841	**0.0047**	**0.004**	0.0122	0.0107

Table 4.12: Averages and distances to the optimal values for CENSUS database.

		k-means		*Gustafson-Kessel*		*fuzzy c-means*	
Criteria		$\overline{X}$	D	$\overline{X}$	D	$\overline{X}$	D
k-anonymity		127.11	80.42	**319.38**	**36.5**	134.69	81
measure_risk		**852.8**	**109.28**	944.44	294.42	960.8	195.17
Q-anonymity		-	-	**201.6**	**18.56**	119.69	85.5
OTHRSALES	l-diversity	125.11	68.72	**155.69**	**30.85**	121.42	68.56
	t-closeness	0.10833	0.105	**0.00417**	**0.0035**	0.00636	0.0047

Table 4.13: Averages and distances to the optimal values for EIA database.

In Tables 4.14 and 4.15 a brief summary of information about risk of re-identification for both databases are given.

Table 4.14 clearly shows that the risk of re-identification for the CENSUS experimentation when released data are encoded using a fuzzy partition is lower. First column of Table 4.14 shows the percentage of times each method obtained the lowest risk of re-identification. As it can be seen, fuzzy partitions prevent the risk of re-identification most of times. In addition,

	%	$\overline{X}$	D
k-means	34%	164.42	189.36
Gustafson-Kessel	24%	236.78	257.83
fuzzy c-means	42%	109.22	107.9

Table 4.14: Risk of re-identification for CENSUS database.

	%	$\overline{X}$	D
k-means	53%	852.8	109.28
Gustafson-Kessel	31%	944.44	294.42
fuzzy c-means	16%	960.8	195.17

Table 4.15: Risk of re-identification for EIA database.

the second column shows the number of re-identified elements (in average). Again, fuzzy partitions obtained by fuzzy c-means performs better. Finally, the third column of Table 4.14 shows the distance to the optimal, reinforcing the goodness of fuzzy c-means method. Table 4.15 shows the same information for the EIA experimentation, where as it has been stated above, the crisp method is slightly better with respect to this measure.

4.2 Detection of edges in grey scale images

This section studies the behaviour of a construction method for an interval-valued fuzzy relation built from a fuzzy relation. The behaviour of this construction method is analyzed depending on the used t-norms and t-conorms, showing that different combinations of them produce a big variation in the results. Furthermore, an hybrid construction method which considers weight functions and a smoothing procedure is also introduced. Among the different applications of this method, the detection of edges in images is one of the most challenging. Thus, the performance of the proposal in detecting image edges is tested, showing that the hybrid approach which combines weights and a smoothing procedure provides better results than the non-weighted methods.

Interval-valued fuzzy sets have been applied to many different domains such as medicine [1], decision making [22] or image processing [5]. More concretely, this kind of construction methods are often applied to the detection of edges in grey scale images, which has its most important application in the medical field (see [64]) and other branches of science (see [15]). The importance of image processing in several areas is proven by the huge amount of studies devoted to this topic, where different problems with different tools are considered (see, for instance, [2, 12, 52, 72]).

The aim of this section is twofold. First it is studied how the selection of different t-norms and t-conorms affects the construction of interval-valued fuzzy relations from fuzzy relations. The study shows how the relation between t-norms and t-conorms can be linked to the relation between the interval-valued fuzzy relations obtained from them. Secondly, a new construction method for interval-valued fuzzy relations is proposed. This method is based on adding weights to make the points closer to the one studied have a greater strength in the construction method than the ones that are not.

4.2.1 Construction method of interval-valued fuzzy relations

This construction method builds an interval-valued fuzzy relation, where the starting point is a fuzzy relation, as it has been previously stated. This process is carried out with two constructors (lower and upper constructors) in order to obtain both sides of each interval with the values of each new fuzzy relation. From this interval-valued fuzzy relation, another fuzzy relation is defined as the length of each interval. This is the relation used to apply the method to generate fuzzy edge images.

Definition 4.22 *Consider X and Y two finite universes of natural numbers $X = \{0, 1, \ldots, P-1\}$ and $Y = \{0, 1, \ldots, Q-1\}$, $R \in FR(X, Y)$ a fuzzy relation in $X \times Y$, two t-norms T_1, T_2, two t-conorms S_1, S_2, and $n, m \in \mathbb{N}$ such that $n \leq \frac{P-1}{2}$ and $m \leq \frac{Q-1}{2}$,*

- *the lower constructor associated to T_1, T_2, n and m is defined as follows:*

$$L^{n,m}_{T_1,T_2} : FR(X,Y) \to FR(X,Y), \quad \textit{where}$$
$$L^{n,m}_{T_1,T_2}[R](x,y) = \mathop{T_1}_{\substack{i=-n \\ j=-m}}^{\substack{m \\ n}} (T_2(R(x-i, y-j), R(x,y))),$$

- *the upper constructor associated to* S_1, S_2, n *and* m *is defined as follows:*

$$U^{n,m}_{S_1,S_2} : FR(X,Y) \to FR(X,Y), \quad \textit{where}$$
$$U^{n,m}_{S_1,S_2}[R](x,y) = \underset{\substack{i=-n\\ j=-m}}{\overset{\substack{m\\ n}}{S_1}} (S_2(R(x-i, y-j), R(x,y))),$$

$\forall (x,y) \in X \times Y$, *where* i, j *take values such that* $0 \leq x - i \leq P - 1$ *and* $0 \leq y - j \leq Q - 1$, n *and* m *indicate that the considered window is a matrix of dimension* $(2n+1) \times (2m+1)$ *and* $\overset{n}{\underset{i=1}{T}} x_i = T(x_1, \ldots, x_n)$.

The specific subsets of the natural numbers X and Y are considered in the definition above, since the main application taken into account here of this method is the edge image detection.

Definition 4.23 *Let* R *be a fuzzy relation in* $X \times Y$, $L^{n,m}_{T_1,T_2}[R]$ *a lower constructor and* $U^{n,m}_{S_1,S_2}[R]$ *an upper constructor, then* $R^{n,m}$ *defined by:*

$$R^{n,m}_{T_1,T_2,S_1,S_2}(x,y) = [L^{n,m}_{T_1,T_2}[R](x,y), U^{n,m}_{S_1,S_2}[R](x,y)],$$

for all $(x,y) \in X \times Y$ *is an interval-valued fuzzy relation in* $X \times Y$.

In the previous definition, when S_i is the dual t-conorm of T_i, the interval-valued fuzzy relation is just denoted by $R^{n,m}_{T_1,T_2}$.

After obtaining both lower and upper constructors from the initial fuzzy relation (Definition 4.22), and the interval-valued fuzzy relation generated by them (Definition 4.23), the last step of the construction method is to obtain another fuzzy relation from such interval-valued fuzzy relation. To do so, the next definition is given, where the length of each interval is used.

Definition 4.24 *Let* R *be a fuzzy relation in* $X \times Y$ *and let* $L^{n,m}_{T_1,T_2}[R]$ *and* $U^{n,m}_{S_1,S_2}[R]$ *be its lower and upper constructors, respectively, for two t-norms*

T_1 and T_2 and two t-conorms S_1 and S_2, the W-fuzzy relation associated to them is given by:

$$W[R^{n,m}_{T_1,T_2,S_1,S_2}](x,y) = U^{n,m}_{S_1,S_2}[R](x,y) - L^{n,m}_{T_1,T_2}[R](x,y).$$

Definitions 4.22, 4.23 and 4.24 (see [5]) establish the construction method procedure, as it is schematized in Algorithm 1.

Algorithm 1 Non-weighted construction method algorithm.

Input: $R \in FR(X,Y)$, $n,m \in \mathbb{N}$, T_1, T_2 t-norms, S_1, S_2 t-conorms

Output: W-fuzzy relation $W \in FR(X,Y)$

1: Obtain the lower constructor $L^{n,m}_{T_1,T_2}[R]$ associated to n, m, T_1 and T_2 (Def. 4.22)

2: Obtain the upper constructor $U^{n,m}_{S_1,S_2}[R]$ associated to n, m, S_1 and S_2 (Def. 4.22)

3: Construct the interval-valued fuzzy relation $R^{n,m}_{T_1,T_2,S_1,S_2}$ from $L^{n,m}_{T_1,T_2}$ and $U^{n,m}_{S_1,S_2}$ (Def. 4.23)

4: Obtain the W-fuzzy relation $W[R^{n,m}_{T_1,T_2,S_1,S_2}]$ from $R^{n,m}_{T_1,T_2,S_1,S_2}$ (Def. 4.24)

4.2.2 The problem of edges detection in grey scale images

The detection of edges in images has one of its most important applications in the medical field, where it can be used, for example, for brain tumor pattern recognition (see [64]).

In order to adapt the previous construction method, it is necessary to explain how to deal with grey scale images and their representation.

Definition 4.25 *A grey scale image R whose dimensions are $P \times Q$ pixels* *is a fuzzy relation where the finite sets used are* $X = \{0, 1, \ldots, P-1\}$ *and* $Y = \{0, 1, \ldots, Q-1\}$.

This means that the grey scale images are represented by fuzzy relations. With this premise, all the construction method can be applied, and the outputs are the following:

- **The lower constructor:** it represents a darker version of the original image. Depending on the t-norms chosen, this image can be more or less dark. In Figure 4.4, there is a representation of three lower constructors with different pairs of t-norms.

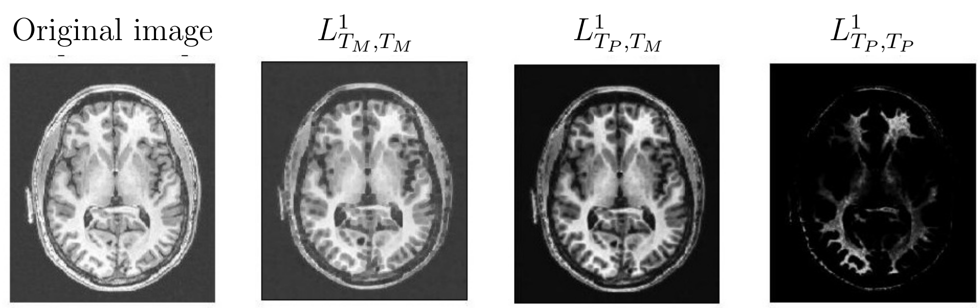


Figure 4.4: Comparative of lower constructors depending on the t-norms, where T_M and T_P are the minimum and product t-norms, respectively.

- **The upper constructor:** it represents a brighter version of the original image. Depending on the t-conorms chosen, this image can be more or less bright. In Figure 4.5, there is a representation of three upper constructors with different pairs of t-conorms.

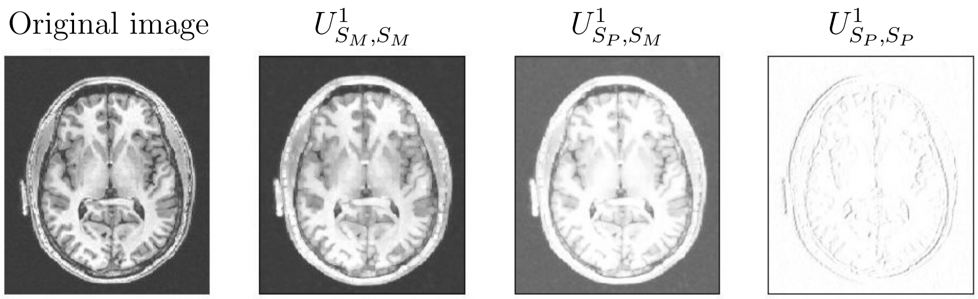


Figure 4.5: Comparative of upper constructors depending on the t-conorms, where S_M and S_P are the maximum and product t-conorms respectively.

- **The W-fuzzy edge image:** it represents the difference of contrast between both constructors. The edges can be identified in this image. In Figure 4.6, there is a representation of three W-fuzzy images with different pairs of t-norms and t-conorms.

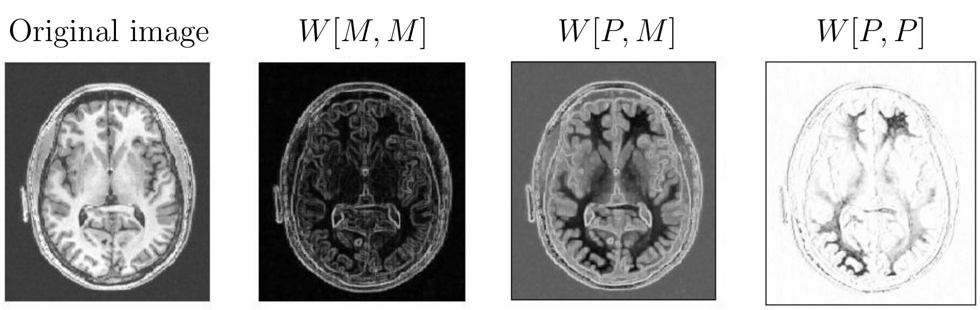


Figure 4.6: Comparative of W-fuzzy images depending on the pairs of t-norms and t-conorms, where $W[P, M] = U^{n,m}_{S_P,S_M} - L^{n,m}_{T_P,T_M}$, and analogously for the others.

Figures 4.4, 4.5 and 4.6 highlight the fact that different t-norms and t-conorms cause a variation in the resulting lower constructor, upper constructor and W-fuzzy image respectively. That is the reason to study in the next section certain properties that these relations keep from the selected t-norms and t-conorms.

4.2.3 Influence of the chosen t-norms and t-conorms

As it has been aforementioned, the construction method needs two t-norms and two t-conorms, and as the Figures 4.4, 4.5 and 4.6 show, the selection affects the resulting interval-valued fuzzy relation. It seems a natural step to study how the relation between the t-norms and t-conorms can be reflected in the constructors, and therefore, in the interval-valued fuzzy relation. In addition, some examples with the most usual t-norms (respectively t-conorms) are shown.

Proposition 4.26 *Let T_a, T_b, T_c, T_d be t-norms such that $T_a \leq T_b$ and $T_c \leq T_d$. Then, $L^{n,m}_{T_a,T_c} \leq L^{n,m}_{T_b,T_d}$.*

Proof: Let R be any fuzzy relation in $X \times Y$. Taking into account the t-norms monotony property:

$$\begin{aligned} L^{n,m}_{T_a,T_c}[R](x,y) &= \underset{\substack{i=-n\\j=-m}}{\overset{\substack{m\\n}}{T_a}} \left(T_c(R(x-i,y-j),R(x,y))\right) \leq \\ &\leq \underset{\substack{i=-n\\j=-m}}{\overset{\substack{m\\n}}{T_a}} \left(T_d(R(x-i,y-j),R(x,y))\right) \leq \\ &\leq \underset{\substack{i=-n\\j=-m}}{\overset{\substack{m\\n}}{T_b}} \left(T_d(R(x-i,y-j),R(x,y))\right) = L^{n,m}_{T_b,T_d}[R](x,y). \blacksquare \end{aligned}$$

The same relation is satisfied for t-conorms as the next proposition states.

Proposition 4.27 *Let S_a, S_b, S_c, S_d be t-conorms such that $S_a \leq S_b$ and $S_c \leq S_d$. Then, $U^{n,m}_{S_a,S_c} \leq U^{n,m}_{S_b,S_d}$.*

After these results about both lower and upper constructors, the next step is to analize what happens with W-fuzzy relations.

Corollary 4.28 *Let (T_a, S_a), (T_b, S_b), (T_c, S_c) and (T_d, S_d) be dual pairs of t-norms and t-conorms such that $T_a \leq T_b$ and $T_c \leq T_d$. Then:*

$$W[R^{n,m}_{a,c}] \geq W[R^{n,m}_{b,d}].$$

Proof: Because of the duality, $S_a \geq S_b$ and $S_c \geq S_d$. Therefore, it is immediate, since

$$\begin{aligned} W[R^{n,m}_{a,c}](x,y) &= U^{n,m}_{S_a,S_c}[R](x,y) - L^{n,m}_{T_a,T_c}[R](x,y) \geq \\ &\geq U^{n,m}_{S_b,S_d}[R](x,y) - L^{n,m}_{T_b,T_d}[R](x,y) = W[R^{n,m}_{b,d}](x,y), \end{aligned}$$

for any $(x,y) \in X \times Y$. ■

The last result is proven straightforwardly from the previous results about t-norms and t-conorms. Keeping Corollary 4.28 in mind, it is possible to apply to some particular cases with well known t-norms and t-conorms, as it is shown in the next examples.

Example 4.29 *(Minimum-Maximum, Product and Lukasiewicz) Since $T_L < T_P < T_M$ and $S_M < S_P < S_L$, the W-fuzzy relations are related as given by Figure 4.7, where $W[M,P]$ denotes $W[R^{n,m}_{T_M,T_P}] = U^{n,m}_{S_M,S_P}[R] - L^{n,m}_{T_M,T_P}[R]$ for any $R \in FR(X,Y)$, and analogously for the others.*

Furthermore, in order to see the results in the edge image detection, in Figure 4.8, these nine combinations of W-fuzzy relations are shown for a grey scale image. It is easy to see that the results are reflected in this application. From this figure, it can be noted that the use of certain t-norms

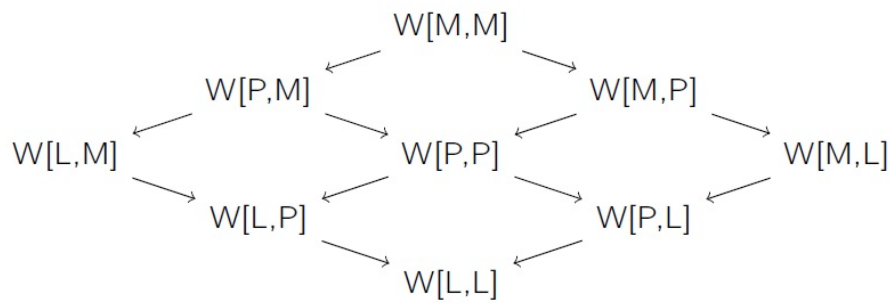


Figure 4.7: Relationships between the different W-fuzzy relations depending on t-norms and t-conorms.

and t-conorms builds an unclear W-fuzzy relation, like Lukasiewicz ones. This is the reason to skip them in the carried out experimentation.

Example 4.30 *(Frank t-norms) Taking into account the definition of this family, with $\lambda \in [0, \infty]$ (see [34]),*

$$T_\lambda^F(x,y) = \begin{cases} T_M(x,y), & \text{if } \lambda = 0, \\ T_P(x,y), & \text{if } \lambda = 1, \\ T_L(x,y), & \text{if } \lambda = \infty, \\ log_\lambda(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}), & \text{in other case,} \end{cases}$$

$$S_\lambda^F(x,y) = \begin{cases} S_M(x,y), & \text{if } \lambda = 0, \\ S_P(x,y), & \text{if } \lambda = 1, \\ S_L(x,y), & \text{if } \lambda = \infty, \\ 1 - log_\lambda(1 + \frac{(\lambda^{1-x} - 1)(\lambda^{1-y} - 1)}{\lambda - 1}), & \text{in other case,} \end{cases}$$

and given $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $\lambda_1 > \lambda_2$ and $\lambda_3 > \lambda_4$, then:

$$T_{\lambda_1}^F \le T_{\lambda_2}^F, \quad T_{\lambda_3}^F \le T_{\lambda_4}^F, \quad S_{\lambda_1}^F \ge S_{\lambda_2}^F, \quad S_{\lambda_3}^F \ge S_{\lambda_4}^F,$$

and therefore $L^{n,m}_{T^F_{\lambda_1}, T^F_{\lambda_3}} \le L^{n,m}_{T^F_{\lambda_2}, T^F_{\lambda_4}}$ and $U^{n,m}_{S^F_{\lambda_1}, S^F_{\lambda_3}} \le U^{n,m}_{S^F_{\lambda_2}, S^F_{\lambda_4}}$. As a result, $W[\lambda_2^F, \lambda_4^F] \le W[\lambda_1^F, \lambda_3^F]$.

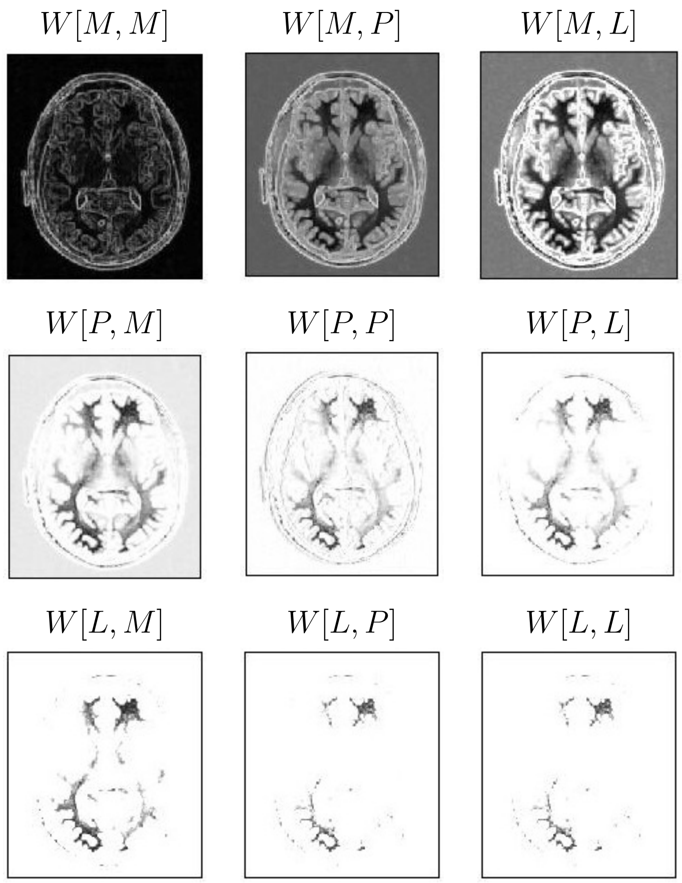


Figure 4.8: W-fuzzy edge images obtained by the combination of Minimum-Maximum, Product and Lukasiewicz t-norms and t-conorms.

These results can also be adapted to other families of t-norms and t-conorms like Yager, Dombi or Sugeno-Weber families.

All these results provided in this part are useful for choosing the right combination of t-norms and t-conorms depending on the purpose of the study, as the greater the value of the W-fuzzy image, the brighter the pixels in the image, and therefore, affecting the appearance of the edges.

4.2.4 Weighted construction method

In the method developed by [5], for each element of the relation, a centered window in that element is considered. Window dimension depends on natural numbers n and m, which are the parameters in both constructors.

This new approach tries to make that the closer the value to the center of the window, the greater the importance that it takes in the definition of the constructors. This goal is useful, as the detection of an edge must be more related to the pixels that are closer to the central one, and that is the main objective of this subsection: to develop a method to capture this reasoning.

In other words, our goal is to obtain the final lower and upper constructors which are obtained by weighting the original lower and upper constructor given in Definition 4.22 by means of weights in such a way that the smaller windows have more strength in the definition. Formally:

Definition 4.31 *Consider X and Y two finite universes of natural numbers $X = \{0, 1, \ldots, P-1\}$ and $Y = \{0, 1, \ldots, Q-1\}$, $R \in FR(X, Y)$ a fuzzy relation in $X \times Y$, two t-norms T_1, T_2, two t-conorms S_1, S_2, and $n, m \in \mathbb{N}$ such that $n \leq \frac{P-1}{2}$ and $m \leq \frac{Q-1}{2}$, for any $i = 1, 2 \ldots, \max(n, m)$ we consider the two fuzzy relations $L^i[R]$ and $U^i[R]$ defined by*

$$L^i[R](x, y) = L_{T_1,T_2}^{\min(i,n),\min(i,m)}[R](x, y)$$

and

$$U^i[R](x, y) = U_{S_1,S_2}^{\min(i,n),\min(i,m)}[R](x, y).$$

The next step is to weight these values in an appropriate way. Thus, we obtain the final lower and upper constructors associated to any fuzzy relation $R \in FR(X \times Y)$ as follows:

$$L[R](x,y) = \sum_{i=1}^{k} w_i L^i[R](x,y) \quad \text{and} \quad U[R](x,y) = \sum_{i=1}^{k} w_i U^i[R](x,y),$$

where k denotes the maximum of n and m.

Finally, it is necessary to determine the weights w_i such that they satisfy $w_i \geq w_{i+1}$, so the smaller windows have more strength. We have considered three cases:

- Average of the k windows:

$$\begin{cases} \displaystyle\sum_{i=1}^{k} w_i = 1, \\ w_1 = \cdots = w_k \in (0,1), \quad i = 1, \ldots, k, \end{cases}$$

  which leads us to the weights:

$$w_i = \frac{1}{k}, \quad i = 1, \ldots, k.$$

- An equidistant version of the weights with a constant increase given by the next restrictions:

$$\begin{cases} \displaystyle\sum_{i=1}^{k} w_i = 1, \\ w_i \in (0,1), & i = 1, \ldots, k, \\ w_i - w_{i+1} = C, & i = 1, \ldots, k-1, \\ w_k = C, \end{cases}$$

  where $C \in (0,1)$ is a constant. With some calculation, the expression of the weights is reached as follows:

$$w_{k-1} - w_k = C \Rightarrow w_{k-1} = 2C \Rightarrow \cdots \Rightarrow w_{k-i} = (i+1)C,$$

and applying it on the first condition:

$$1 = \sum_{i=1}^{k} w_i = \sum_{i=0}^{k-1} w_{k-i} = \sum_{i=0}^{k-1}(i+1)C = C\sum_{i=0}^{k-1}(i+1) = C\frac{k(k+1)}{2}.$$

Hence $C = \frac{2}{k(k+1)}$ and the weights are:

$$w_{k-i} = \frac{2(i+1)}{k(k+1)}, \quad i = 0, \ldots, k-1,$$

or equivalently,

$$w_i = \frac{2(k-i+1)}{k(k+1)}, \quad i = 1, \ldots, k.$$

- A constant relation between two consecutive weights, given $N \in \mathbb{N}\backslash\{1\}$:

$$\begin{cases} \sum_{i=1}^{k} w_i = 1, \\ w_i \in (0,1), & i = 1, \ldots, k, \\ w_i/w_{i+1} = N, & i = 1, \ldots, k-1, \\ w_k = C, \end{cases}$$

where $C \in (0,1)$ is a constant. From these conditions it follows than $w_i = N^{k-i}C$ for all i. Thus, after some calculation, the expression of the weights is reached as follows:

$$1 = \sum_{i=1}^{k} w_i = C\sum_{i=1}^{k} N^{k-i} = C\sum_{i=0}^{k-1} N^i = C\frac{N^k-1}{N-1} \Rightarrow$$
$$\Rightarrow C = \frac{N-1}{N^k-1} \Rightarrow$$
$$\Rightarrow w_i = N^{k-i}\frac{N-1}{N^k-1}, \; i = 1, \ldots, k.$$

Note that the bigger the value of N, the greater the importance on the central pixels. Moreover, note that the case $N = 1$ is not considered, because in that case the weights are selected according the average of the k windows method.

Once the values of the weights are calculated, and therefore, both lower and upper constructors, the remaining steps of the method in Algorithm 1 (Definitions 4.23 and 4.24) must be applied in order to get the new interval-valued fuzzy relation and the W-fuzzy relation.

It should be noted that the fact of using weights causes the appearance of some values very close to 0 or 1, but not the own value. The reason is that the use of the biggest windows can make a little influence in such value. To avoid this situation, a smoothing step is used such that the final W-fuzzy edge image W is modified with some cut point $\alpha \in (0, 0.5)$.

Given W a fuzzy relation, the smoothing step with cut point $\alpha \in (0, 0.5)$ is carried out as follows:

1. If $W(x, y) < \alpha$, then its value is modified such that $W^\alpha(x, y) = 0$.
2. If $W(x, y) > 1 - \alpha$, then its value is modified such that $W^\alpha(x, y) = 1$.
3. For the remaining values in the closed interval $[\alpha, 1 - \alpha]$, they are expanded to the closed interval $[0, 1]$ keeping the original proportion:

$$W(x, y) \to W^\alpha(x, y) = 0.5 + \frac{1}{1 - 2\alpha}(W(x, y) - 0.5).$$

The reason to take values in the interval $(0, 0.5)$ lies in the fact that when $\alpha \to 0$, the smoothing step leads us to the method without such step, as sets of points modified by parts 1 and 2 of it tends to the empty set. On the other hand, the remaining values get the modification:

$$W^\alpha(x, y) = 0.5 + \frac{1}{1 - 2\alpha}(W(x, y) - 0.5) \xrightarrow[\alpha \to 0]{} 0.5 + W(x, y) - 0.5 = W(x, y).$$

If the value α where greater or equal to 0.5, it would make no sense to apply the smoothing step, as there would be pixels whose value must be changed to 0 (step 1) and to 1 (step 2) at the same time.

The scheme of the weighted method with the smoothing step is shown in Algorithm 2. In order to obtain the weighted method without the smoothing step, point 8 of Algorithm 2 is skip, where $W = W_w$.

Algorithm 2 Weighted construction method algorithm.

Input: $R \in FR(X,Y)$, $n, m \in \mathbb{N}$, T_1, T_2 t-norms, S_1, S_2 t-conorms, $N \in \mathbb{N} \cup \{0\}$, $\alpha \in (0, 0.5)$

Output: W-fuzzy relation $W^\alpha \in FR(X,Y)$

1: Fix $k = \max(n, m)$

2: Obtain $L^i[R]$ associated to n, m, T_1 and T_2 $\forall i = 1, \ldots, k$ (Def. 4.31)

3: Obtain $U^i[R]$ associated to n, m, S_1 and S_2 $\forall i = 1, \ldots, k$ (Def. 4.31)

4: Obtain weights w_i for $i = 1, \ldots, k$, with the method assigned:
Average ($N = 0$): $w_i = \dfrac{1}{k}$,
Equidistant ($N = 1$): $w_i = \dfrac{2(k-i+1)}{k(k+1)}$,
Constant ($N \geq 2$): $w_i = N^{k-i}\dfrac{N-1}{N^k - 1}$.

5: Calculate the lower and upper constructors as:
$L[R](x,y) = \sum_{i=1}^{k} w_i L^i[R](x,y)$ and $U[R](x,y) = \sum_{i=1}^{k} w_i U^i[R](x,y)$

6: Construct the interval-valued fuzzy relation $R^{n,m}_{T_1,T_2,S_1,S_2}$ from L and U (Def. 4.23)

7: Obtain the W-fuzzy relation $W[R^{n,m}_{T_1,T_2,S_1,S_2}]$ from $R^{n,m}_{T_1,T_2,S_1,S_2}$ (Def. 4.24)

8: Calculate W^α from W with the smoothing step defined by the cut point α

In the experimentation carried out in the next subsection, the influence of α, along with the comparison of this new approach with the non-weighted one are analyzed with a grey scale database.

4.2.5 Experiments

In this part of the section, the weighted method is compared to the one introduced in [5]. To do that, a grey scale images database has been considered. These grey scale images were obtained from the *Berkeley Segmentation Dataset* (see [49]). This database contains original images and its corresponding edge images, which are used as the base to the comparison between all the methods of study.

The first 25 images from the test set were selected, whose dimensions are 481×321 (or 321×481) pixels. The t-norms and t-conorms selected for this study are the standard ones (Minimum-Maximum), as the goal of this experimentation is to check if the weighted method outperforms the non-weighted one under the same conditions.

The studied situations in this experimentation are:

- Non-weighted method with $n = m = 1$ (windows of size 3×3) (NW1).
- Non-weighted method with $n = m = 2$ (windows of size 5×5) (NW2).
- Weighted method with the combination of:
  - **Methods to obtain weights:** average (A), equidistant method (I) and constant relation method with $N = 2, 3, 4, 5$ (II, III, IV, V).
  - **Number of terms:** number of windows of different dimensions taken into account. $k = 2$, $k = 3$ or $k = 4$ terms (2,3,4).
  - **Smoothing step parameter:** $\alpha \in \{0, 0.05, 0.1, \ldots, 0.4, 0.45\}$.

Note that the equidistant method with 2 terms and the constant relation method with $N = 2$ with 2 terms too are the same. The non-weighted selected situations are the ones that obtain the best results (as it has been proven in [5]), and that is the reason to select these window sizes. In Figure 4.9 some of the W-fuzzy images obtained for each one of the test images are given.

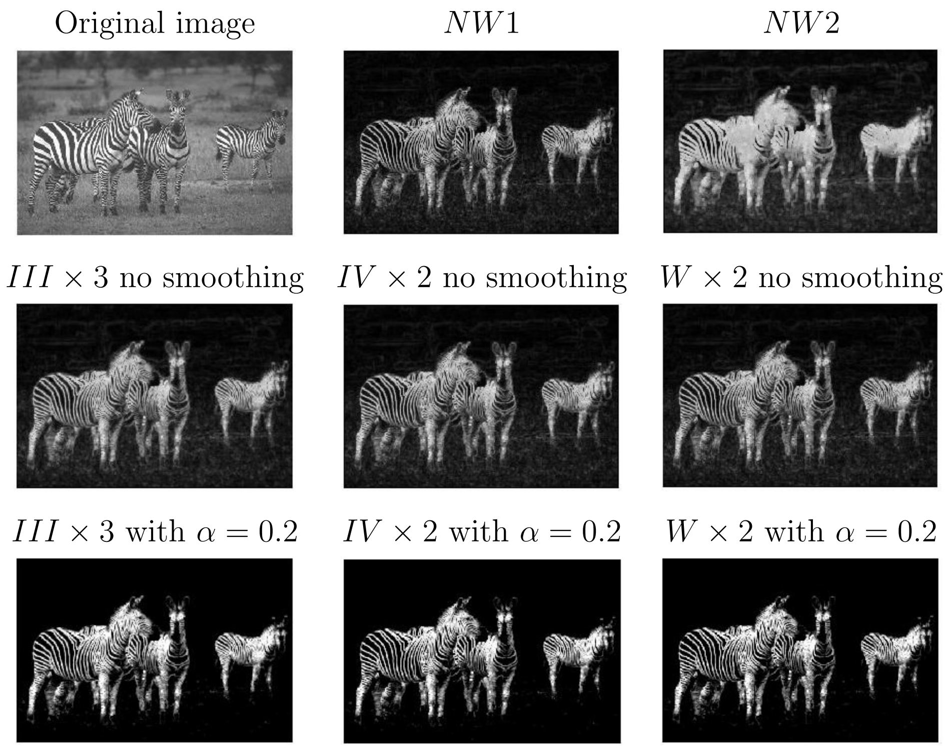


Figure 4.9: Comparative of W-fuzzy images depending on the experimental parameters, where two non-weighted methods ($NW1$, $NW2$), and $III \times 3$, $IV \times 2$ and $V \times 2$ without smoothing step and with $\alpha = 0.2$ are shown.

Each edge image is compared to the one given by the source. To make such comparison, for each image a value is assigned. Let S be the edge image given in [49] and let M be the one obtained by our method, the value assigned to M is given by the expression

$$v(M) = \frac{1}{\#(X \times Y)} \sum_{(x,y) \in X \times Y} |M(x, y) - S(x, y)|,$$

where $\#(X \times Y)$ represents the number of pixels of the image.

Notice that this value $v(M)$ is in fact the normalized Hamming distance between M and the edge image S (see, for instance, [43]), since fuzzy relations are just fuzzy sets of $X \times Y$.

The parameter α that defines the smoothing step is an important factor that must be taken into account when comparing the results. In Figure 4.10, five representations of an image with different values of α are presented.

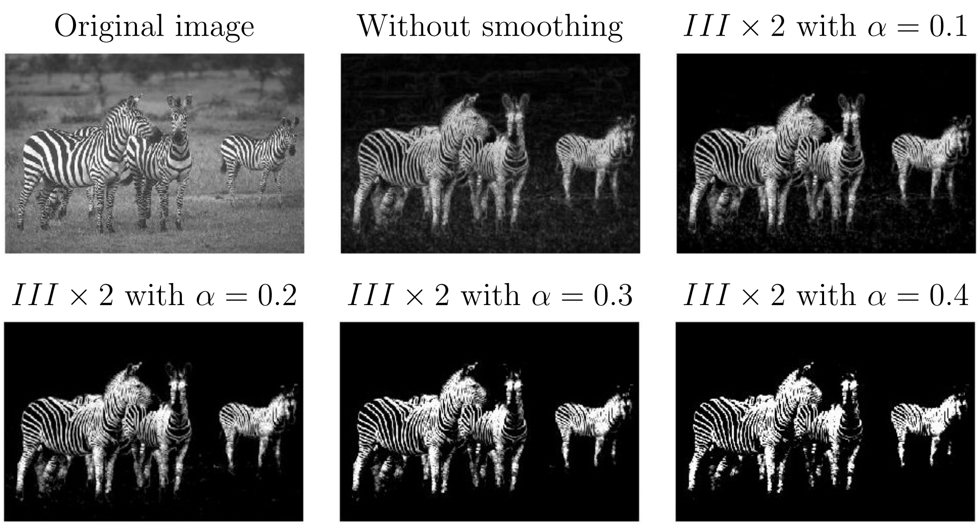
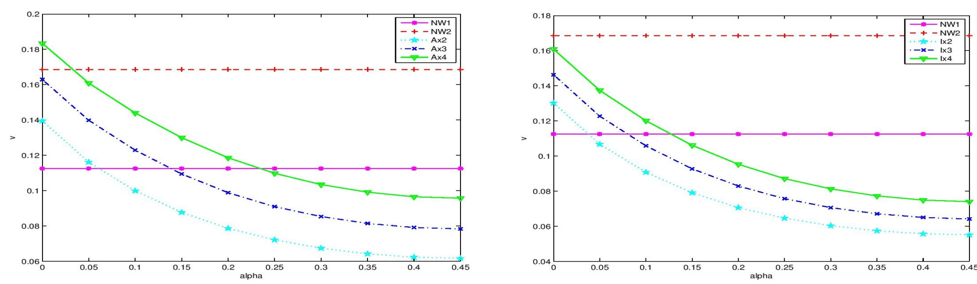


Figure 4.10: Comparative of W-fuzzy images depending on the parameter α, where it is used the constant relation method to obtain the weights with two terms ($III \times 2$).

To summarize the results obtained, the mean of the value $v(M)$ for all the selected 25 images is calculated for each method. Obviously, the smaller mean value provides the best result.

Figures 4.11, 4.12 and 4.13 provides the mean values of $v(M)$ obtained for each method, where X and Y axes represent the value α of the smoothing step and the mean value obtained for that method, respectively.

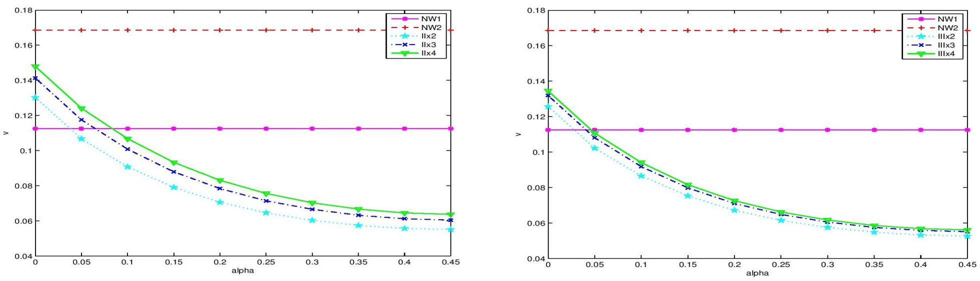


(a) (b)

Figure 4.11: (a) Non-weighted methods (NW1, NW2) and weighted methods with average weights (A). (b) Non-weighted methods (NW1, NW2) and weighted methods with equidistant weights (I).

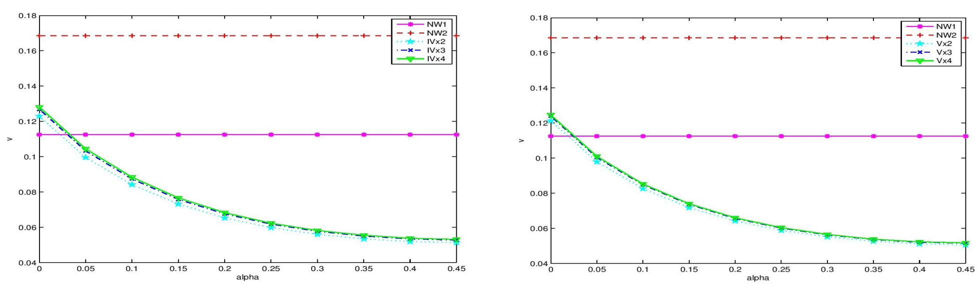
From the results showed by the Figures 4.11, 4.12 and 4.13, all the methods but one in the first graphic always obtain better results than the non-weighted method with $m = n = 2$ (NW2). Meanwhile, the other non-weighted method where $m = n = 1$ (NW1) is also improved by every combination of type of weights and number of terms, from an α value onwards. Depending on the method analyzed, this α value can be further or closer to 0, as it can be observed in the graphics.

These results prove that this new method where weights are added to the construction method, overcomes the one without its use, as long as the smoothing step is applied with a big enough α value.



(a) (b)

Figure 4.12: (a) Non-weighted methods (NW1, NW2) and weighted methods with constant relation weights for $N = 2$ (II). (b) Non-weighted methods (NW1, NW2) and weighted methods with constant relation weights for $N = 3$ (III).



(a) (b)

Figure 4.13: (a) Non-weighted methods (NW1, NW2) and weighted methods with constant relation weights for $N = 4$ (IV). (b) Non-weighted methods (NW1, NW2) and weighted methods with constant relation weights for $N = 5$ (V).

Conclusions

This research has been focused on the development of different tools for fuzzy sets, particularly, for interval-valued hesitant fuzzy sets. The membership function of this type of sets assigns to each element a finitely generated set, so it has been necessary to provide some results for this type of sets.

The first concept that has been treated is the one of ordering relation, for both finitely generated sets and interval-valued hesitant fuzzy sets. Firstly, two ordering relations have been defined for finitely generated sets, analyzing the incomparability of one of them by characterizing the possible situations in which it can occur. These two orders have been extended to interval-valued hesitant fuzzy sets. Both pairs of ordering relations have been essential in the remainder research.

Another remarkable concepts in the fuzzy logic, the ones of t-norm and t-conorm, have been necessary along this work, so we have defined both for interval-valued hesitant fuzzy sets, including a particular example of t-norm and t-conorm, necessary in the forthcoming results. The aforementioned ordering relations were necessary for these definitions, as the monotonicity requires it.

Bearing these results in mind, the next studied concept is the cardinality of interval-valued hesitant fuzzy sets, providing an axiomatic definition of it. Several properties that this type of cardinalities satisfy have been proved, as well as a characterization of the definition has been obtained. In addition, some examples have been presented, whose cardinalities, when they are restricted to fuzzy sets, match some well-known definitions (σ-count cardinality and Ralescu's cardinality).

The concept of entropy is a remarkable element of this memory. The aim of an entropy is to measure the amount of uncertainty associated to a set. In the case of interval-valued hesitant fuzzy sets, several types of uncertainty can be found, and as a consequence, the proposed definition is shaped by three different mappings (fuzziness, lack of knowledge and hesitance), where each one detects a different class of uncertainty associated to a set. Results and characterizations of each function have been proposed and proved. Finally, a global example is fully explained in order to show how the combination of these three mappings provide a good way to detect different types of uncertainty.

The last concept that has been analyzed about interval-valued hesitant fuzzy sets is the one of partitioning. Definitions of partitions (δ-ϵ-partition, ϵ-ϵ-partition) have been adapted to this new logic, as well as different results about them. Furthermore, these adaptations make it possible to obtain classical fuzzy definitions of partition such as Ruspini's one.

The second part of this memory is focused on the applications developed from the proposals of the present work: protection of privacy in microdata and edge detection in grey scale images.

The diffusion of information is one of the most important activities in the modern world, and as a result, the preservation of privacy is of great importance. We have focused on the protection of privacy in microdata. The classical procedure to protect such data is the use of crisp partitions to the non sensitive attributes in order to protect the sensitive ones. Our proposal has been based on the use of fuzzy partitions instead. In order to measure the level of protection of a particular case, the most usual techniques are k-anonymity, l-diversity and t-closeness. However, it was necessary to adapt them to the fuzzy case so it is possible to measure the protection in this situation.

It is also necessary to compare both procedures, so an experimental comparison has been carried out with two real databases. The obtained results, using two different databases (CENSUS and EIA), show that this new proposal is a good alternative to the classical one with respect to such techniques, thanks to the inclusion of the fuzzy logic.

The second application of this work is related to the edge detection in grey scale images. The starting point has been a construction method of interval-valued fuzzy relations from a fuzzy relation. The behaviour of the method has been analyzed taking into account, among others elements, the selected t-norms and t-conorms.

In addition, a new method have been developed from the initial one, including two new features. The first one is the inclusion of weights to the method, in order to provide a greater importance to certain pixels in the process. The second one complements the previous one, as it is a smoothing step that erases certain deviations created by small weights. Finally, an experimental comparison of both construction methods has been carried out, using a grey scale images database. The obtained results show that this new method, after adding weights and the smoothing step, is an efficient alternative to the initial one.

Accomplishments

During these last years, this research has been published and presented in several journals and conferences, respectively.

International conference presentations:

- *P. Quirós, P. Alonso, I. Díaz, V. Janis. An axiomatic definition of cardinality for finite interval-valued hesitant fuzzy sets. 16th World Congress of the International Fuzzy Systems Association (IFSA) and 9th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT). Gijón, Spain. June-july, 2015.*

- *A. Bouchet, P. Quirós, P. Alonso, S. Montes, I. Díaz. Medical Edge Detection combining Fuzzy Mathematical Morphology with Interval-Valued Relations. 10th International Conference on Soft Computing Models in Industrial and Environmental Applications. Burgos, Spain, June, 2015.*

- *P. Quirós, P. Alonso, I. Díaz, S. Montes. Introducing weight functions to construct interval valued fuzzy relations. 14th International Conference Computational and Mathematical Methods in Science and Engineering (CMMSE). Cádiz, Spain. July, 2014.*

- *P. Quirós, P. Alonso, I. Díaz, S. Montes. Protection of privacy in microdata. 13th International Conference Computational and Mathematical Methods in Science and Engineering (CMMSE). Almería, Spain. June, 2013.*

- *P. Quirós, P. Alonso, I. Díaz, S. Montes. Fuzzy sets as a tool for protecting sensitive information. 5th International Conference of the*

ERCIM WG on COMPUTING & STATISTICS (ERCIM). Oviedo, Spain. December, 2012.

JCR journal publications:

- *P. Quirós, P. Alonso, H. Bustince, I. Díaz, and S. Montes. An entropy measure definition for interval-valued hesitant fuzzy sets.* Knowledge-Based Systems*, 84:121–133, 2015*

- *P. Quirós, P. Alonso, I. Díaz, and S. Montes. Protecting data: a fuzzy approach.* International Journal of Computer Mathematics*, 92(9):1989–2000, 2015*

- *P. Quirós, P. Alonso, I. Díaz, A. Jurio, and S. Montes. An hybrid construction method based on weight functions to obtain interval-valued fuzzy relations.* Mathematical Methods in the Applied Sciences, DOI=10.1002/mma.3443*, DOI: 10.1002/mma.3443*

- *P. Quirós, P. Alonso, I. Díaz, and S. Montes. On the use of fuzzy partitions to protect data.* Integrated Computer-Aided Engineering*, 21(4):355–366, 2014*

- *A. Bouchet, P. Quirós, P. Alonso, V. Ballarin, I. Díaz, and S. Montes. Gray scale edge detection using interval-valued fuzzy relations.* (submitted)

Conclusiones

La investigación llevada a cabo se ha centrado en el desarrollo de distintas herramientas para conjuntos difusos, en particular para *interval-valued hesitant fuzzy sets*. La función de pertenencia de este tipo de conjuntos asigna a cada elemento un conjunto finitamente generado, por lo que ha sido necesario construir algunos resultados para este tipo de conjuntos.

El primer concepto que ha sido tratado es el de relación de orden, tanto para conjuntos finitamente generados como para *interval-valued hesitant fuzzy sets*. Primero, hemos definido dos órdenes para conjuntos finitamente generados, analizando la incomparabilidad de uno de ellos mediante la caracterización de las situaciones en las que puede suceder. Estos dos órdenes los hemos extendido a *interval-valued hesitant fuzzy sets*. Ambos pares de órdenes han sido indispensables para el resto de la investigación.

Otros conceptos importantes en la lógica difusa como son los de t-norma y t-conorma han sido necesarios a lo largo de este trabajo, por lo que hemos definido ambos conceptos para *interval-valued hesitant fuzzy sets*, incluyendo algunos ejemplos de t-norma y t-conorma, necesarios en desarrollos posteriores. Las relaciones de orden previamente mencionadas han sido necesarias en este punto de la investigación, ya que la condición de monoticidad así lo requiere.

Teniendo en cuenta estos resultados, el siguiente concepto que hemos estudiado ha sido el de cardinalidad para *interval-valued hesitant fuzzy sets*, dando una definición axiomática de la misma. Hemos probado algunas de las propiedades que dichas cardinalidades satisfacen, y se ha obtenido una caracterización de dicha definición. También se han presentado algunos ejemplos, cuya cardinalidad, cuando se restringe al caso de conjuntos difusos, coincide con algunas ya conocidas (σ-*count* y *Ralescu's cardinality*).

El concepto de entropía es un elemento a destacar en esta memoria. El objetivo de la entropía es medir la cantidad de incertidumbre asociada a un conjunto. En el caso de *interval-valued hesitant fuzzy sets*, se pueden encontrar varios tipos de incertidumbre, y como consecuencia, hemos propuesto una definición de entropía compuesta por tres aplicaciones diferentes (*fuzziness*, *lack of knowledge* y *hesitance*), donde cada una de ellas puede detectar diferentes tipos de incertidumbre asociada a un conjunto. Hemos propuesto y probado varios resultados y caracterizaciones de cada una de las funciones. Finalmente, y a través de un ejemplo, se ha podido observar cómo la combinación de dichas funciones proporciona un buen modo de detectar diferentes tipos de incertidumbre.

El último concepto tratado sobre *interval-valued hesitant fuzzy sets* es el de particionado. Hemos adaptado definiciones de partición (δ-ϵ-*partition*, ϵ-ϵ-*partition*) a esta nueva lógica, así como diferentes resultados relacionados. Además, estas adaptaciones nos han permitido obtener definiciones clásicas de partición tales como la dada por Ruspini.

La segunda parte de esta memoria está centrada en las aplicaciones desarrolladas a partir de los propuestas recogidas en el presente trabajo: la protección de privacidad en microdatos y la detección de bordes en imágenes en escala de grises.

La difusión de información es una de las actividades más importantes en el mundo actual, por lo que la preservación de la privacidad es de gran importancia. Nos hemos centrado en la protección de la privacidad en microdatos. El procedimiento habitual para proteger este tipo de datos es mediante el uso de particiones nítidas sobre los atributos no sensibles para así proteger los sensibles. Nuestra propuesta ha estado basada en el uso de particiones difusas en su lugar. Para poder medir el nivel de protección de un caso particular, las técnicas más usuales son *k-anonymity*, *l-diversity* y *t-closeness*. Sin embargo, ha sido necesario adaptarlas al caso difuso de modo que sea posible medir la protección en dicha situación.

También ha sido necesario comparar ambos enfoques, por lo que hemos llevado a cabo una comparación experimental con dos bases de datos reales. Los resultados obtenidos, utilizando dos bases de datos (CENSUS y EIA), muestran que esta nueva propuesta es una buena alternativa a la clásica con respecto a dichas técnicas, gracias a la inclusión de la lógica difusa.

La segunda aplicación de este trabajo está relacionada con la detección de bordes en imágenes en escala de grises. El punto de partida ha sido un método de construcción de *interval-valued fuzzy relations* a partir de una relación difusa. El comportamiento del método ha sido analizado teniendo en cuenta, entre otros elementos, las t-normas y t-conormas utilizadas.

Además, hemos desarrollado un nuevo método a partir del inicial, incluyendo dos nuevas características. La primera de ellas es la inclusión de pesos en el método, para poder así proporcionar una mayor importancia a ciertos píxeles en el proceso. El segundo complementa el anterior, y es un paso de suavizado que permite eliminar ciertas desviaciones creadas por pequeños pesos. Finalmente, se ha llevado a cabo una comparación experimental entre ambos métodos, utilizando para ello una base de datos de imágenes en escala de grises. Los resultados alcanzados muestran que este nuevo método, tras incluir los pesos y el paso de suavizado, es una alternativa eficiente a la inicial.

Logros

Durante estos últimos años, esta investigación ha sido publicada y presentada en varias revistas y congresos, respectivamente.

Presentaciones en congresos internacionales:

- *P. Quirós, P. Alonso, I. Díaz, V. Janis. An axiomatic definition of cardinality for finite interval-valued hesitant fuzzy sets. 16th World Congress of the International Fuzzy Systems Association (IFSA) and 9th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT). Gijón, Spain. June-july, 2015.*

- *A. Bouchet, P. Quirós, P. Alonso, S. Montes, I. Díaz. Medical Edge Detection combining Fuzzy Mathematical Morphology with Interval-Valued Relations. 10th International Conference on Soft Computing Models in Industrial and Environmental Applications. Burgos, Spain, June, 2015.*

- *P. Quirós, P. Alonso, I. Díaz, S. Montes. Introducing weight functions to construct interval valued fuzzy relations. 14th International Conference Computational and Mathematical Methods in Science and Engineering (CMMSE). Cádiz, Spain. July, 2014.*

- *P. Quirós, P. Alonso, I. Díaz, S. Montes. Protection of privacy in microdata. 13th International Conference Computational and Mathematical Methods in Science and Engineering (CMMSE). Almería, Spain. June, 2013.*

- *P. Quirós, P. Alonso, I. Díaz, S. Montes. Fuzzy sets as a tool for protecting sensitive information. 5th International Conference of the*

ERCIM WG on COMPUTING & STATISTICS (ERCIM). Oviedo, Spain. December, 2012.

Publicaciones en revistas del JCR:

- *P. Quirós, P. Alonso, H. Bustince, I. Díaz, and S. Montes. An entropy measure definition for interval-valued hesitant fuzzy sets.* Knowledge-Based Systems*, 84:121–133, 2015*
- *P. Quirós, P. Alonso, I. Díaz, and S. Montes. Protecting data: a fuzzy approach.* International Journal of Computer Mathematics*, 92(9):1989–2000, 2015*
- *P. Quirós, P. Alonso, I. Díaz, A. Jurio, and S. Montes. An hybrid construction method based on weight functions to obtain interval-valued fuzzy relations.* Mathematical Methods in the Applied Sciences, DOI=10.1002/mma.3443*, DOI: 10.1002/mma.3443*
- *P. Quirós, P. Alonso, I. Díaz, and S. Montes. On the use of fuzzy partitions to protect data.* Integrated Computer-Aided Engineering*, 21(4):355–366, 2014*
- *A. Bouchet, P. Quirós, P. Alonso, V. Ballarin, I. Díaz, and S. Montes. Gray scale edge detection using interval-valued fuzzy relations.* (submitted)

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