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Ranking fuzzy sets and fuzzy random variables by means of stochastic orders

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Abstract

This paper establishes a theory of decision making under uncertainty with fuzzy utilities. The extension of expected utility and stochastic dominance to the comparison of sets of random variables plays a crucial role. Their properties as fuzzy rankings are studied, and their definitions are further generalized to the comparison of fuzzy random variables. Also, a connection between expected utility for fuzzy random variables and the comparison of the lower/upper probabilities they induce is proven.

Keywords: Fuzzy rankings, fuzzy random variables, stochastic orders, imprecise probabilities, possibility measures, p-boxes.

1. Introduction

Since they were introduced in [23], fuzzy sets have been widely used as a mathematical model in a context of incomplete information. Their use was boosted further by the introduction by Puri and Ralescu of the notion of *fuzzy random variable* [17], a generalization of random variables to situations where the images are fuzzy sets.

The widespread use of fuzzy sets has lead naturally to the consideration of decision making problems with fuzzy information. In order to solve them, it is necessary to use methods that allow to rank fuzzy sets and fuzzy random variables. The first of these two cases has been widely investigated (see for instance [2, 20, 22]) but, to the best of our knowledge no methods have been proposed to rank fuzzy random variables.

In this paper, we extend the preliminary results we presented in [15] and we address the problem in two different ways: on the one hand, we define a second order possibility on the set of probabilities and we apply Walley's approach [19] to derive from it a lower and upper probability. This allows to turn the comparison of the fuzzy random variables to that of their lower/upper expectations. On the other hand, we extend a number of stochastic orders from random variables towards fuzzy random variables. With respect to this second approach, in earlier works [13, 14], we extended stochastic orders to the comparison of *sets* of random variables, as a first step when modeling imprecise information. Here we use some of the results in those papers by considering a number of imprecise probability models that are related to fuzzy random variables, such as probability boxes and possibility measures. As a side result, we study the axiomatic properties of these imprecise stochastic orders as fuzzy rankings.

After giving some preliminary concepts in Section 2, Section 3 introduces stochastic orders for the particular models we have mentioned. Then, in Section 4 we investigate the properties of imprecise stochastic orders as fuzzy rankings, and in Section 5 we study how these stochastic orders may be extended towards the comparison of fuzzy random variables. The paper concludes in Section 6 with some additional discussion.

2. Fuzzy sets and fuzzy random variables

A *fuzzy set* X tells us to which extent the elements of a possibility space Ω satisfy a property, often corresponding to linguistic information. It is determined by its membership function $\mu_X : \Omega \to [0,1]$. The set of elements with strictly positive membership value is called the *support* of X, and denoted by supp X. We shall denote by $\mathcal{F}(\Omega)$ the class of all fuzzy sets over a referential space Ω. The membership function can be extended to subsets of Ω, so that the acceptability in which the fuzzy concept is satisfied by set A is given by

$$\Pi(A) = \sup_{\omega \in A} \mu_X(\omega).$$

This function is a *possibility measure* [7, 24], and the membership function μ_X is its associated *possibility distribution.*

We will focus in this paper on one prominent family of fuzzy sets, that of fuzzy numbers. A fuzzy set is a *fuzzy number* if there exists a closed non-empty interval $[a,b]$ such that:

$$\mu(x) = \begin{cases} 1 & \text{for } x \in [a,b]; \\ l(x) & \text{for } x < a; \\ r(x) & \text{for } x > b, \end{cases}$$

where $l : (-\infty, a) \to [0,1]$ is a non-decreasing and right-continuous function such that $l(x) = 0$ for $x < \omega_1$, and $r : (b, \infty) \to [0,1]$ is a non-increasing and left-continuous function such that $r(x) = 0$ for $x > \omega_2$, for some $\omega_1, \omega_2 \in \mathbb{R}$. From this definition, it follows that the α-cuts of a fuzzy number are closed intervals.

In this paper we focus on fuzzy random variables, which extend the notion of random variable to the case where the images are fuzzy sets.

Definition 1 *[11] Let $(\Omega, \mathcal{A}, P)$ be a probability space. A fuzzy random variable is a map $\widetilde{X} : \Omega \to \mathcal{F}(\mathbb{R})$ such that the α-cuts $\widetilde{X}_\alpha : \Omega \to \mathcal{P}(\mathbb{R})$ given by*

$$\widetilde{X}_\alpha(\omega) = \{t \in \mathbb{R} : \widetilde{X}(\omega)(t) \geq \alpha\}$$

are random sets*, meaning that*

$$\{\omega \in \Omega : \widetilde{X}_\alpha(\omega) \cap A \neq \emptyset\} \in \mathcal{A}\ \forall A \in \beta_{\mathbb{R}}.$$

Fuzzy random variables were introduced by Féron in [9]. In this work we follow the *epistemic* interpretation developed by Kruse and Meyer [11], and regard a fuzzy random variable $\widetilde{X}$ as a model for the imprecise knowledge of a random variable U_0, in the sense that for any $\omega' \in \mathbb{R}$, $\widetilde{X}(\omega)(\omega')$ is interpreted as the acceptability degree of the proposition "$U_0(\omega) = \omega'$". Following these lines, we can define a fuzzy set on the class of measurable functions from Ω to $\mathbb{R}$, $\mu_{\widetilde{X}}$, that associates the value

$$\mu_{\widetilde{X}}(U) = \inf\{\widetilde{X}(\omega)(U(\omega)) : \omega \in \Omega\} \qquad (1)$$

with any measurable function $U : \Omega \to \mathbb{R}$. This value can then be understood as the acceptability degree of the proposition "$U = U_0$". Using this interpretation, Couso [1] defined the probabilistic envelope of a fuzzy random variable.

Definition 2 *[1, Definition 5.1.1] Let $\widetilde{X} : \Omega \to \mathcal{F}(\mathbb{R})$ be a fuzzy random variable. The* probabilistic envelope *of $\widetilde{X}$ is the map $P_{\widetilde{X}} : \mathcal{A} \to \mathcal{F}([0,1])$ such that the membership function $P_{\widetilde{X}}(A)(p)$ is given by:*

$$\sup\{\mu_{\widetilde{X}}(U) \text{ such that } U : \Omega \to \mathbb{R} \text{ r.v.}, P_U(A) = p\}$$

for any $A \in \mathcal{A}$ and $p \in [0,1]$.

Thus, $P_{\widetilde{X}}(A)(p)$ can be interpreted as the acceptability degree of the proposition "$P_{U_0}(A) = p$". With this idea, we can also possible to define the envelope of the cumulative distribution function of $\widetilde{X}$ as the map $F_{\widetilde{X}} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ such that

$$F_{\widetilde{X}}(t)(p) = \sup\{\mu_{\widetilde{X}}(U) : F_U(t) = p\}. \qquad (2)$$

Then, $F_{\widetilde{X}}(t) = P_{\widetilde{X}}((-\infty, t])$ for any $t \in \mathbb{R}$, and $F_{\widetilde{X}}(t)(p)$ can be interpreted as the acceptability degree of the proposition "$F_{U_0}(t) = p$".

Following [1], the fuzzy version of any parameter can be defined for fuzzy random variables. Formally, if the parameter belongs to the parametric space Θ, its fuzzy version is defined by:

$$\theta_{\widetilde{X}} \in \mathcal{F}(\Theta),\ \theta_{\widetilde{X}}(\theta') = \sup\{\mu_{\widetilde{X}}(U) : \theta(P_U) = \theta'\}.$$

$\theta_{\widetilde{X}}(\theta')$ represents the acceptability degree of the proposition "$\theta(P_{U_0}) = \theta'$". In particular, the expectation of a fuzzy random variable is given by:

$$E(\widetilde{X})(t) = \sup\{\mu_{\widetilde{X}}(U) : E(U) = t\}, \qquad (3)$$

and $E(\widetilde{X})(t)$ can be interpreted as the acceptability degree of the proposition "$E(U_0) = t$".

3. Imprecise stochastic orders

Stochastic orders are methods that compare random variables by means of their probabilistic information [16]. The most important one is *expected utility*: given two random variables X, Y defined on a probability space $(\Omega, \mathcal{A}, P)$, we define

$$X \succeq_{\mathrm{E}} Y \Leftrightarrow E(X) \geq E(Y).$$

In this paper, we also consider stochastic dominance.

Definition 3 *Let X and Y be two random variables and let F_{X} and F_{Y} be their respective cumulative distribution functions. X is said to* stochastically dominate *Y, and we denote it $X \succeq_{\mathrm{SD}} Y$, if*

$$F_{\mathrm{X}}(t) \leq F_{\mathrm{Y}}(t) \text{ for any } t \in \mathbb{R}.$$

It is well known that $X \succeq_{\mathrm{SD}} Y$ implies $X \succeq_{\mathrm{E}} Y$.

These two stochastic orders were extended in [13, 14] towards the comparison of *sets* of random variables. The following definitions were considered:

Definition 4 *[13, Def. 5] Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets of random variables, and let $\succeq$ be a stochastic order. It is said that:*

1. *$\mathcal{X} \succeq_1 \mathcal{Y}$ if and only if for any $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ it holds that $X \succeq Y$.*
2. *$\mathcal{X} \succeq_2 \mathcal{Y}$ if and only if there is some $X \in \mathcal{X}$ such that $X \succeq Y$ for any $Y \in \mathcal{Y}$.*
3. *$\mathcal{X} \succeq_3 \mathcal{Y}$ if and only if for any $Y \in \mathcal{Y}$ there is some $X \in \mathcal{X}$ such that $X \succeq Y$.*
4. *$\mathcal{X} \succeq_4 \mathcal{Y}$ if and only if there are $X \in \mathcal{X}, Y \in \mathcal{Y}$ such that $X \succeq Y$.*
5. *$\mathcal{X} \succeq_5 \mathcal{Y}$ if and only if there is some $Y \in \mathcal{Y}$ such that $X \succeq Y$ for any $X \in \mathcal{X}$.*
6. *$\mathcal{X} \succeq_6 \mathcal{Y}$ if and only if for any $X \in \mathcal{X}$ there is $Y \in \mathcal{Y}$ such that $X \succeq Y$.*

It follows from this definition that $\mathcal{X} \succeq_1 \mathcal{Y} \Rightarrow \mathcal{X} \succeq_2 \mathcal{Y} \Rightarrow \mathcal{X} \succeq_3 \mathcal{Y} \Rightarrow \mathcal{X} \succeq_4 \mathcal{Y}$ and that $\mathcal{X} \succeq_1 \mathcal{Y} \Rightarrow \mathcal{X} \succeq_5 \mathcal{Y} \Rightarrow \mathcal{X} \succeq_6 \mathcal{Y} \Rightarrow \mathcal{X} \succeq_4 \mathcal{Y}$; no additional implication holds in general.

When $\succeq$ is given by expected utility or stochastic dominance, we shall refer to the extensions $\succeq_i$ for $i = 1, \ldots, 6$ as *imprecise expected utility* or *imprecise stochastic dominance*, and we shall denote them by $\succeq_{\mathrm{E_i}}$ or $\succeq_{\mathrm{SD_i}}$, respectively.

Next, we are going to show how the definition above can be applied in two particular cases: the comparison of possibility measures and sets of distribution functions. Since these two models are related to fuzzy random variables, our results in this section shall be useful when comparing fuzzy random variables in Section 5.

3.1. Sets of distribution functions

One model that we shall relate to fuzzy random variables in this paper are sets of distribution functions, or p-boxes.

Definition 5 *Let $\underline{F}, \overline{F} : \mathbb{R} \to [0,1]$ be two increasing functions satisfying $\underline{F} \leq \overline{F}$ and such that $\underline{F}(-\infty) = \overline{F}(-\infty) = 0$ and $\underline{F}(\infty) = \overline{F}(\infty) = 1$. Then the p-box $(\underline{F}, \overline{F})$ is the set of distribution functions bounded between $\underline{F}$ and $\overline{F}$.*

Note that $\underline{F}, \overline{F}$ may not be distribution functions, because we are not requiring them to be right-continuous; they are only the cumulative functions associated with a finitely additive probability. In this paper, we shall follow the work in [14] and regard *p*-boxes as sets of σ-additive distribution functions.

If we want to compare two *p*-boxes $(\underline{F}_{\rm X}, \overline{F}_{\rm X})$ and $(\underline{F}_{\rm Y}, \overline{F}_{\rm Y})$, we can use the extensions of expected utility and stochastic dominance. With respect to imprecise expected utility, it is easy to establish the following when the lower and upper distribution functions belong to the *p*-box:

Proposition 1 *Consider two p-boxes $(\underline{F}_{\rm X}, \overline{F}_{\rm X})$ and $(\underline{F}_{\rm Y}, \overline{F}_{\rm Y})$, with bounded support and including their respective lower and upper distribution functions.*

1. $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_1} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int id \ {\rm d}\overline{F}_{\rm X} \geq \int id \ {\rm d}\underline{F}_{\rm Y}$.
2. $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_2} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow (\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_3} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int id \ {\rm d}\underline{F}_{\rm X} \geq \int id \ {\rm d}\underline{F}_{\rm Y}$.
3. $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_4} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int id \ {\rm d}\underline{F}_{\rm X} \geq \int id \ {\rm d}\overline{F}_{\rm Y}$.
4. $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_5} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow (\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_6} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int id \ {\rm d}\overline{F}_{\rm X} \geq \int id \ {\rm d}\overline{F}_{\rm Y}$.

With respect to imprecise stochastic dominance, some results were already established in [13, Thm. 8] and [14, Prop. 3] for the comparison of arbitrary sets of distribution functions. Using [14, Cor. 1] it is not difficult to show that the converse of [14, Prop. 3] holds when the *p*-boxes include their lower and upper distribution functions:

Proposition 2 *Consider two p-boxes $(\underline{F}_{\rm X}, \overline{F}_{\rm X})$ and $(\underline{F}_{\rm Y}, \overline{F}_{\rm Y})$ with bounded support and including their respective lower and upper distribution functions. If we denote by $\mathcal{U}$ the set of increasing functions, it holds that:*

1. $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_1} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int u{\rm d}\overline{F}_{\rm X} \geq \int u{\rm d}\underline{F}_{\rm Y}$ *for any* $u \in \mathcal{U}$.
2. $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_2} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow (\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_3} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int u{\rm d}\underline{F}_{\rm X} \geq \int u{\rm d}\underline{F}_{\rm Y}$ *for any* $u \in \mathcal{U}$.
3. $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_4} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int u{\rm d}\underline{F}_{\rm X} \geq \int u{\rm d}\overline{F}_{\rm Y}$ *for any* $u \in \mathcal{U}$.
4. $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_5} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow (\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_6} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int u{\rm d}\overline{F}_{\rm X} \geq \int u{\rm d}\overline{F}_{\rm Y}$ *for any* $u \in \mathcal{U}$.

It follows from these two results that $\succeq_{\rm SD_i}$ implies $\succeq_{\rm E_i}$ for any $i = 1, \ldots, 6$ in this context.

3.2. Possibility measures

As we said, one imprecise probability model closely related to fuzzy set theory are possibility measures [24]: the membership function of a fuzzy set can be interpreted as a possibility distribution, and as a consequence its associated possibility measure extends the membership function towards subsets of the referential space.

Here we shall consider possibility measures on $\mathbb{R}$ induced by fuzzy numbers. For them, we can establish a simple characterization of expected utility and stochastic dominance.

Given a possibility measure Π, we can define a set of probabilities, named *credal set*, by:

$$\mathcal{M}(\Pi) = \{P \text{ prob} : P(A) \leq \Pi(A) \ \forall A \in \beta_{\mathbb{R}}\}.$$

Consider Π_1, Π_2 two possibility measures on the power set of $\mathbb{R}$ associated with fuzzy numbers, and let $\mathcal{M}(\Pi_1), \mathcal{M}(\Pi_2)$ be their respective credal sets. We shall also denote by $\mathcal{F}_1, \mathcal{F}_2$ the corresponding sets of cumulative distribution functions. Next lemma shows that these sets are indeed determined by the possibility measures and their conjugate necessity measures, where the *necessity measure* determined by a possibility measure Π is given by $N(A) = 1 - \Pi(A^c) \ \forall A \subseteq \Omega$. The key in the proof is to consider a random set with this possibility measure as upper probability, and such that the possibility measure is the maximum of the measurable selections.

Lemma 1 *Let Π be the possibility measure induced by a fuzzy number, and let $\mathcal{F}$ be the set of cumulative distribution functions associated with $\mathcal{M}(\Pi)$. Then, the lower and upper envelopes of $\mathcal{F}$ belong to $\mathcal{F}$ and they coincide with the ones determined by N and Π.*

Using this lemma, we can establish the following characterization of stochastic dominance for possibility measures:

Proposition 3 *Consider two possibility measures $\Pi_{\rm X}$ and $\Pi_{\rm Y}$ on $\mathbb{R}$ determined by fuzzy numbers, and denote by $\mathcal{F}_{\rm X}, \mathcal{F}_{\rm Y}$ their associated sets of distribution functions. The following statements hold:*

1. $\mathcal{F}_{\rm X} \succeq_{\rm SD_1} \mathcal{F}_{\rm Y} \Leftrightarrow \Pi_{\rm X}((-\infty, t]) \leq N_{\rm Y}((-\infty, t])$ *for any* t.
2. $\mathcal{F}_{\rm X} \succeq_{\rm SD_2} \mathcal{F}_{\rm Y} \Leftrightarrow \mathcal{F}_{\rm X} \succeq_{\rm SD_3} \mathcal{F}_{\rm Y} \Leftrightarrow N_{\rm X}((-\infty, t]) \leq N_{\rm Y}((-\infty, t])$ *for any* t.

3. $\mathcal{F}_X \succeq_{SD_4} \mathcal{F}_Y \Leftrightarrow N_X((-\infty,t]) \leq \Pi_Y((-\infty,t])$ *for any* t.
4. $\mathcal{F}_X \succeq_{SD_5} \mathcal{F}_Y \Leftrightarrow \mathcal{F}_X \succeq_{SD_6} \mathcal{F}_Y \Leftrightarrow \Pi_X((-\infty,t]) \leq \Pi_Y((-\infty,t])$ *for any* t.

Proposition 2 established a connection between the comparison of p-boxes by imprecise stochastic dominance and the comparison of integrals. Next we establish an analogous result for particular case of p-boxes induced by possibility measures.

Corollary 1 *Let* Π_X *and* Π_Y *be two possibility measures on* $\mathbb{R}$ *determined by fuzzy numbers, and let* $\mathcal{F}_X, \mathcal{F}_Y$ *denote their associated sets of distribution functions. The following statements hold:*

1. $\mathcal{F}_X \succeq_{SD_1} \mathcal{F}_Y \Leftrightarrow \int u \mathrm{d}F_{\Pi_X} \geq \int u \mathrm{d}F_{N_Y}$ *for every* $u \in \mathcal{U}$.
2. $\mathcal{F}_X \succeq_{SD_2} \mathcal{F}_Y \Leftrightarrow \mathcal{F}_X \succeq_{SD_3} \mathcal{F}_Y \Leftrightarrow \int u \mathrm{d}F_{N_X} \geq \int u \mathrm{d}F_{N_Y}$ *for every* $u \in \mathcal{U}$.
3. $\mathcal{F}_X \succeq_{SD_4} \mathcal{F}_Y \Leftrightarrow \int u \mathrm{d}F_{N_X} \geq \int u \mathrm{d}F_{\Pi_Y}$ *for every* $u \in \mathcal{U}$.
4. $\mathcal{F}_X \succeq_{SD_5} \mathcal{F}_Y \Leftrightarrow \mathcal{F}_X \succeq_{SD_6} \mathcal{F}_Y \Leftrightarrow \int u \mathrm{d}F_{\Pi_X} \geq \int u \mathrm{d}F_{\Pi_Y}$ *for every* $u \in \mathcal{U}$.

A similar result can be established with respect to imprecise expected utility.

Corollary 2 *Let* Π_X *and* Π_Y *be two possibility measures on* $\mathbb{R}$ *associated with fuzzy numbers, and let* $\mathcal{F}_X, \mathcal{F}_Y$ *denote their associated sets of distribution functions.*

1. $\mathcal{F}_X \succeq_{E_1} \mathcal{F}_Y \Leftrightarrow \int id \; \mathrm{d}F_{\Pi_X} \geq \int id \; \mathrm{d}F_{N_Y}$.
2. $\mathcal{F}_X \succeq_{E_2} \mathcal{F}_Y \Leftrightarrow \mathcal{F}_X \succeq_{E_3} \mathcal{F}_Y \Leftrightarrow \int id \; \mathrm{d}F_{N_X} \geq \int id \; \mathrm{d}F_{N_Y}$.
3. $\mathcal{F}_X \succeq_{E_4} \mathcal{F}_Y \Leftrightarrow \int id \; \mathrm{d}F_{N_X} \geq \int id \; \mathrm{d}F_{\Pi_Y}$.
4. $\mathcal{F}_X \succeq_{E_5} \mathcal{F}_Y \Leftrightarrow \mathcal{F}_X \succeq_{E_6} \mathcal{F}_Y \Leftrightarrow \int id \; \mathrm{d}F_{\Pi_X} \geq \int id \; \mathrm{d}F_{\Pi_Y}$.

Taking these results into account, whenever we compare two possibility measures Π_X, Π_Y, we shall consider only definitions $\succeq_1, \succeq_2, \succeq_4$ and $\succeq_5$ if $\succeq$ refers to stochastic dominance or expected utility, since in both cases $\succeq_2$ is equivalent to $\succeq_3$ and $\succeq_5$ is equivalent to $\succeq_6$. Moreover, we shall use the notation $\Pi_X \succeq_i \Pi_Y$ to refer to $\mathcal{F}_X \succeq_i \mathcal{F}_Y$ for simplicity.

In addition, it follows that imprecise stochastic dominance implies imprecise expected utility also when applied to the possibility measures that fuzzy numbers determine. This is consistent with the relation between stochastic dominance and expected utility explained at the beginning of the section.

4. Imprecise stochastic orders as fuzzy rankings

A *fuzzy ranking* is a method that allows to establish an order between fuzzy sets. Several fuzzy rankings have been proposed in the literature (see [20, 21, 22] for critical reviews and [5, 8, 26] for recent works).

Since a fuzzy set is formally equivalent to a possibility measure, we can use the ideas from Section 3.2 and regard the methods we have considered in the previous section as fuzzy rankings. There is, however, one fundamental difference with the majority of the fuzzy rankings: we are allowing for incomparability between the fuzzy sets, which in our view is natural under the epistemic interpretation we are giving to fuzziness in this paper. In this sense, our proposal aligns with the one by Dubois and Prade in [6]: they propose several indices for the comparison between two fuzzy sets but in case of contradiction leave the final choice to the decision maker, under the light of the information provided. A somewhat related idea (considering a ranking method whose output is a fuzzy set) was proposed in [26].

The use of stochastic orders as fuzzy rankings has already been investigated in [2]. One essential difference with our approach is that they assume some knowledge about the dependence between the fuzzy sets, according to which they express some stochastic orders in terms of the comparison of lower/upper expectations with respect to adequate functions. In contrast, our approach does not make any assumptions about the dependence between the fuzzy sets.

Next, we shall study the properties of imprecise stochastic dominance and imprecise expected utility as fuzzy rankings. We shall focus on the application of these orders to *trapezoidal fuzzy numbers*, for which the orders shall take a simple expression. Recall that a trapezoidal fuzzy number is a fuzzy set determined by four parameters (t_1, t_2, t_3, t_4):

$$\mu(x) = \begin{cases} 0 & \text{if } x < t_1 \text{ or } x > t_4. \\ \frac{x-t_1}{t_2-t_1} & \text{if } t_1 \leq x < t_2. \\ 1 & \text{if } t_2 \leq x \leq t_3. \\ \frac{t_4-x}{t_4-t_3} & \text{if } t_3 < x \leq t_4. \end{cases}$$

4.1. Imprecise expected utility as a fuzzy ranking

Given a possibility measure induced by a trapezoidal fuzzy number, we can establish a simple characterization of imprecise expected utility. To see how this comes about, note that if Π is determined by the membership function of a trapezoidal fuzzy number (t_1, t_2, t_3, t_4), then

$$\int id \; dF_N = \frac{t_3+t_4}{2} \text{ and } \int id \; dF_\Pi = \frac{t_1+t_2}{2}.$$

Applying Corollary 2, we deduce the following:

Proposition 4 *Let (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) be two trapezoidal fuzzy numbers, and denote by Π_X and Π_Y the possibility measures they determine.*

1. $\Pi_X \succeq_{E_1} \Pi_Y \Leftrightarrow \frac{x_1+x_2}{2} \geq \frac{y_3+y_4}{2}$.
2. $\Pi_X \succeq_{E_2} \Pi_Y \Leftrightarrow \frac{x_3+x_4}{2} \geq \frac{y_3+y_4}{2}$.
3. $\Pi_X \succeq_{E_4} \Pi_Y \Leftrightarrow \frac{x_3+x_4}{2} \geq \frac{y_1+y_2}{2}$.
4. $\Pi_X \succeq_{E_5} \Pi_Y \Leftrightarrow \frac{x_1+x_2}{2} \geq \frac{y_1+y_2}{2}$.

In [20, 21], Wang and Kerre discuss a number of desirable properties for fuzzy rankings $\succeq$. Here we shall consider the following:

(A0) For any pair of fuzzy numbers A, B, either $A \succeq B$ or $B \succeq A$. [Completeness]
(A1) $A \succeq A$ for any fuzzy number A. [Reflexivity]
(A2) $A \succeq B, B \succeq A \Rightarrow A = B$. [Antisymmetry]
(A3) $A \succeq B$ and $B \succeq C \Rightarrow A \succeq C$. [Transitivity]
(A4) $\inf \operatorname{supp}(A) > \sup \operatorname{supp}(B) \Rightarrow A \succ B$.
(A5) $A \succeq B \Rightarrow A+C \succeq B+C$ for any fuzzy number C.
(A6) $\operatorname{supp}(C) \subseteq [0, +\infty), A \succeq B \Rightarrow AC \succeq BC$.

Using Proposition 4, we establish the following:

Proposition 5 *Let $\succeq_{E_i}$ denote the extension of expected utility to the imprecise case by means of Definition 4. They satisfy the following properties as fuzzy rankings of trapezoidal fuzzy numbers:*

	(A0)	(A1)	(A2)	(A3)	(A4)	(A5)	(A6)
$\succeq_{E_1}$			•	•	•		
$\succeq_{E_2}$	•	•		•	•	•	
$\succeq_{E_4}$	•	•			•	•	•
$\succeq_{E_5}$	•	•		•	•	•	

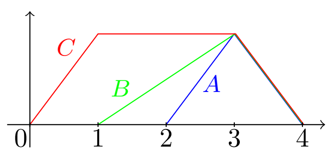

Figure 1: Graphical representation of the trapezoidal fuzzy sets.

It is also interesting to discuss the behavior of these orders in the controversial case discussed in [20, Section 1.2], that we depict in Figure 1: we consider the trapezoidal fuzzy sets $A = (2, 3, 3, 4), B = (1, 3, 3, 4)$ and $C = (0, 1, 3, 4)$. We deduce from Proposition 4 that

$$A \succ_{E_5} B \succ_{E_5} C, \quad A \equiv_{E_2} B \equiv_{E_2} C,$$
$$A \equiv_{E_4} B \equiv_{E_4} C,$$

and that they are incomparable with respect to $\succeq_{E_1}$. This is because the order $\succeq_{E_5}$ is looking at the lower limits of the fuzzy sets, for which we can establish a strict order, while $\succeq_{E_2}$ is looking at the upper limits, where the three fuzzy sets coincide.

4.2. Imprecise stochastic dominance as a fuzzy ranking

When the possibility measures to be compared are induced by trapezoidal fuzzy numbers, imprecise stochastic dominance also takes a simple expression:

Proposition 6 *Let (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) be two trapezoidal fuzzy numbers, and let Π_X and Π_Y be the possibility measures they determine.*

1. $\Pi_X \succeq_{SD_1} \Pi_Y \Leftrightarrow y_3 \leq x_1$ *and* $y_4 \leq x_2$.
2. $\Pi_X \succeq_{SD_2} \Pi_Y \Leftrightarrow y_3 \leq x_3$ *and* $y_4 \leq x_4$.
3. $\Pi_X \succeq_{SD_4} \Pi_Y \Leftrightarrow y_1 \leq x_3$ *and* $y_2 \leq x_4$.
4. $\Pi_X \succeq_{SD_5} \Pi_Y \Leftrightarrow y_1 \leq x_1$ *and* $y_2 \leq x_2$.

We can use this result to determine the properties of imprecise stochastic dominance as a fuzzy ranking.

Proposition 7 *Let $\succeq_{SD_i}$ denote the extension of stochastic dominance to the imprecise case by means of Definition 4. They satisfy the following properties as fuzzy rankings of trapezoidal fuzzy numbers:*

	(A0)	(A1)	(A2)	(A3)	(A4)	(A5)	(A6)
$\succeq_{SD_1}$			•	•	•	•	
$\succeq_{SD_2}$		•		•	•	•	•
$\succeq_{SD_4}$	•	•			•	•	•
$\succeq_{SD_5}$		•		•	•	•	•

It is also interesting to discuss the behavior of these orders in the case discussed in Figure 1: from Proposition 6, we immediately see that

$$A \succ_{SD_5} B \succ_{SD_5} C, \quad A \equiv_{SD_2} B \equiv_{SD_2} C,$$
$$A \equiv_{SD_4} B \equiv_{SD_4} C,$$

and that they are incomparable with respect to $\succeq_{SD_1}$.

5. Comparison of fuzzy random variables

Next, we shall apply the previous results to the comparison of fuzzy random variables. Throughout this section we shall assume that the images of the fuzzy random variables are fuzzy numbers, and also that they are *uniformly bounded*, meaning that for each fuzzy random variable $\widetilde{X}$ there is a compact interval $[a, b]$ such that the support $\widetilde{X}(\omega)$ is included in $[a, b]$ for every ω.

5.1. An imprecise probabilistic approach

The fuzzy set $\mu_{\widetilde{X}}(U)$ defined in Eq. (1) measures how compatible is the random variable U with the unknown random variable that $\widetilde{X}$ is modeling. In a similar manner we can define a fuzzy set on the set of probabilities:

$$\mu'_{\widetilde{X}}(P) = \sup\{\mu_{\widetilde{X}}(U) : P_U = P\}.$$

Following the same interpretation, $\mu'_{\widetilde{X}}(P)$ measures how compatible is P with the probability induced by the unknown random variable that $\widetilde{X}$ is modeling; as such, $\mu'_{\widetilde{X}}$ is a second order possibility over a set of probabilities. In [19], Walley introduced a method to reduce $\mu'_{\widetilde{X}}$ to a first order model. Next, we show how the resulting model can be used for the comparison of fuzzy random variables.

Walley defines the functions $\underline{P}_\alpha$ and $\overline{P}_\alpha$ by:

$$\underline{P}_\alpha(A) = \inf\{P(A) : \mu'_{\widetilde{X}}(P) \geq \alpha\} \text{ and}$$
$$\overline{P}_\alpha(A) = \sup\{P(A) : \mu'_{\widetilde{X}}(P) \geq \alpha\};$$

by representing these functions as conditional information and using the notion of *natural extension*, which models the implications of a number of assessments within the behavioural theory of imprecise probabilities, he then derives the (first order) lower and upper probabilities by:

$$\underline{P}(A) = \int_0^1 \underline{P}_\alpha(A)\mathrm{d}\alpha \text{ and } \overline{P}(A) = \int_0^1 \overline{P}_\alpha(A)\mathrm{d}\alpha$$

for every $A \subseteq \Omega$. When we consider fuzzy random variables whose images are fuzzy numbers, it holds that $\mu'_{\widetilde{X}}(P) \geq \alpha$ if and only if there exists a random variable U such that $P_U = P$ and $\mu_{\widetilde{X}}(U) \geq \alpha$. On the other hand,

$$\mu'_{\widetilde{X}}(P_U) \geq \alpha \Leftrightarrow U \in S(\widetilde{X}_\alpha).$$

Thus, $\{P \mid \mu'_{\widetilde{X}}(P) \geq \alpha\} = \{P_U \mid U \in S(\widetilde{X}_\alpha)\}$. In other words, $\underline{P}_\alpha$ and $\overline{P}_\alpha$ are the lower and upper expectations of the α-cuts. If we use the notation $\widetilde{X}_\alpha(\omega) = [l_\alpha(\omega), r_\alpha(\omega)]$, we can define the lower and upper expectation associated with $\mu'_{\widetilde{X}}$ by:

$$\underline{E}^W = \int_0^1 E(l_\alpha)\ \mathrm{d}\alpha \quad \text{and} \quad \overline{E}^W = \int_0^1 E(r_\alpha)\ \mathrm{d}\alpha. \tag{4}$$

Using the notions of interval dominance [25], minimax [18], maximax [10] and E-admissibility [12] of Imprecise Probability Theory, we can compare fuzzy random variables in the following manner:

Definition 6 *Let $\widetilde{X}$ and $\widetilde{Y}$ be two fuzzy random variables whose images are fuzzy numbers. Denote by $\underline{E}^W_{\widetilde{X}}, \overline{E}^W_{\widetilde{X}}$ and $\underline{E}^W_{\widetilde{Y}}, \overline{E}^W_{\widetilde{Y}}$ the lower and upper expectations obtained by Eq.* (4). *We say that $\widetilde{X}$ is preferred to $\widetilde{Y}$ with respect to:*

Interval dominance *when $\underline{E}^W_{\widetilde{X}} \geq \overline{E}^W_{\widetilde{Y}}$;*

Maximin *when $\underline{E}^W_{\widetilde{X}} \geq \underline{E}^W_{\widetilde{Y}}$;*

Maximax *when $\overline{E}^W_{\widetilde{X}} \geq \overline{E}^W_{\widetilde{Y}}$;*

E-admissibility *when $\overline{E}^W_{\widetilde{X}} \geq \underline{E}^W_{\widetilde{Y}}$.*

5.2. Fuzzy expected utility

When we want to compare fuzzy random variables by means of expected utility, their expectations are given by fuzzy sets (see Eq. (3)), and as a consequence we must consider a fuzzy ranking on them. It is not difficult to show that

$$(E(\widetilde{X}))_\alpha = \left[\int \mathrm{id}\ dP_{*\widetilde{X}_\alpha}, \int \mathrm{id}\ dP^*_{\widetilde{X}_\alpha}\right],$$

and therefore that $E(\widetilde{X})$ is also a fuzzy number.

From the point of view of imprecise probabilities, we can also consider the possibility measures associated with these fuzzy sets, and one of imprecise stochastic orders we have discussed in Section 3.2. The following example illustrates both these possibilities.

Example 1 *We model two unknown random variables defined in $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$ by means of the fuzzy random variables $\widetilde{X}, \widetilde{Y} : [0,1] \to \mathcal{F}([0,1])$ given by:*

$$\widetilde{X}(\omega)(r) = \begin{cases} 2r & \text{if } r \leq \frac{1}{2}. \\ 2-2r & \text{if } r > \frac{1}{2}. \end{cases}$$
$$\widetilde{Y}(\omega)(r) = \begin{cases} 0 & \text{if } \omega \in [0, \frac{1}{2}] \text{ and } r \neq 0. \\ 1 & \text{if } \omega \in [0, \frac{1}{2}] \text{ and } r = 0. \\ 1 & \text{if } \omega \in (\frac{1}{2}, 1] \text{ and } r \geq \frac{1}{2}. \\ 0 & \text{if } \omega \in (\frac{1}{2}, 1] \text{ and } r < \frac{1}{2}. \end{cases}$$

Their expectations are given by:

$$E(\widetilde{X})(r) = \begin{cases} 2r & \text{if } r \in [0, \frac{1}{2}). \\ 2-2r & \text{if } r \in [\frac{1}{2}, 1]. \end{cases}$$
$$E(\widetilde{Y})(r) = \begin{cases} 1 & \text{if } r \in [\frac{1}{4}, \frac{1}{2}]. \\ 0 & \text{otherwise.} \end{cases}$$

Let us compare these two sets by means of the fuzzy ranking defined by de Campos and González Muñoz [4] with an optimism-pessimism index of 0.5, given by $A \succeq_{CM} B$ if and only if

$$CM(A) := \int_0^1 \frac{a^-_\alpha + a^+_\alpha}{2} d\alpha \geq \int_0^1 \frac{b^-_\alpha + b^+_\alpha}{2} d\alpha := CM(B), \tag{5}$$

where a^-_α, a^+_α (resp., b^-_α, b^+_α) denote the infimum and the supremum of the α-cut of A (resp., B). We obtain that

$$CM(E(\widetilde{X})) = \frac{1}{2} > \frac{3}{8} = CM(E(\widetilde{Y})),$$

and we conclude that $\widetilde{X}$ is preferred to $\widetilde{Y}$.

On the other hand, we can also interpret these expectations as possibility distributions, and we can thus compare them by means of imprecise expected utility. Denote by Π_{X} and N_{X} the possibility and necessity measures associated with $E(\widetilde{X})$ and by Π_{Y}

and N_Y the possibility and necessity measures associated with $E(\widetilde{Y})$. We have that

$$\int id \mathrm{d}F_{\Pi_X} = \frac{1}{4}, \quad \int id \mathrm{d}F_{N_X} = \frac{3}{4},$$
$$\int id \mathrm{d}F_{\Pi_Y} = \frac{1}{4}, \quad \int id \mathrm{d}F_{N_Y} = \frac{1}{2}.$$

Applying Corollary 2, we conclude that $\widetilde{X} \succ_{E_2} \widetilde{Y}$, $\widetilde{X} \equiv_{E_i} \widetilde{Y}$, for $i = 4, 5$ and that they are incomparable with respect to the first definition. ♦

Our next result establishes a connection between fuzzy expected utility, when the fuzzy ranking between the expectations is given by imprecise expected utility, and the comparison of the lower/upper expectations given by Walley's procedure. A somewhat similar result can be found in [3, Section 7]. Our proof is based on the properties of complete monotonicity that these functionals satisfy.

Theorem 1 *Let $\widetilde{X}$ be a fuzzy random variable whose images are fuzzy numbers, and let $\underline{E}^W, \overline{E}^W$ be given by Eq. (4). Let Π be the possibility measure associated with $E(\widetilde{X})$ and denote by $\underline{E}_\Pi, \overline{E}_\Pi$ the lower and upper expectations associated with Π. Then, $\underline{E}^W = \underline{E}_\Pi$ and $\overline{E}^W = \overline{E}_\Pi$.*

This allows us to show that the comparison of fuzzy random variables using Walley's approach is related to the comparison of the fuzzy expectations using imprecise expected utility. Using Proposition 4 and Theorem 1, we deduce that:

Theorem 2 *Let $\widetilde{X}$ and $\widetilde{Y}$ be two fuzzy random variables whose images are fuzzy numbers, and let $\underline{E}^W_{\widetilde{X}}, \overline{E}_{\widetilde{X}}, \underline{E}^W_{\widetilde{Y}}, \overline{E}^W_{\widetilde{Y}}$ be given by Eq. (4). Then:*

1. $E(\widetilde{X}) \succeq_{E_1} E(\widetilde{Y}) \Leftrightarrow \underline{E}^W_{\widetilde{X}} \geq \overline{E}^W_{\widetilde{Y}}$.
2. $E(\widetilde{X}) \succeq_{E_2} E(\widetilde{Y}) \Leftrightarrow \underline{E}^W_{\widetilde{X}} \geq \underline{E}^W_{\widetilde{Y}}$.
3. $E(\widetilde{X}) \succeq_{E_4} E(\widetilde{Y}) \Leftrightarrow \overline{E}^W_{\widetilde{X}} \geq \underline{E}^W_{\widetilde{Y}}$.
4. $E(\widetilde{X}) \succeq_{E_5} E(\widetilde{Y}) \Leftrightarrow \overline{E}^W_{\widetilde{X}} \geq \overline{E}^W_{\widetilde{Y}}$.

Example 2 *Consider again the fuzzy random variables $\widetilde{X}, \widetilde{Y}$ from Example 1, for which $E(\widetilde{X}) \succ_{E_2} E(\widetilde{Y})$ and $E(\widetilde{X}) \sim_{E_i} E(\widetilde{Y})$ for $i = 4, 5$, while they are incomparable for $\succeq_{E_1}$. Applying Theorem 2, $\widetilde{X}$ is preferred to $\widetilde{Y}$ with respect to the maximin criterion, while they are equivalent with respect to maximax and E-admissibility, and they are incomparable with respect to interval dominance.* ♦

5.3. Fuzzy stochastic dominance

Next we consider the extension of stochastic dominance. For fuzzy random variables $\widetilde{X}, \widetilde{Y}$, it follows from Eq. (2) that for any real number t, $F_{\widetilde{X}}(t), F_{\widetilde{Y}}(t)$ are fuzzy sets on $[0,1]$. Hence, we should compare them by means of a fuzzy ranking, and this gives rise to the following definition:

Definition 7 *Let $\succsim$ be a fuzzy ranking, and consider two fuzzy random variables $\widetilde{X}, \widetilde{Y}$. We say that $\widetilde{X}$ $\succsim$-stochastically dominates $\widetilde{Y}$ when $F_{\widetilde{Y}}(t) \succsim F_{\widetilde{X}}(t)$ for any real number t.*

Stochastic dominance is quite a strong relationship, and gives rise to many instances of incomparable random variables. This phenomenon is exacerbated when the fuzzy ranking $\succsim$ we consider does not produce a complete order, as is for instance the case with some versions of imprecise stochastic dominance or imprecise expected utility. Because of this, we think it makes more sense to use fuzzy stochastic dominance with respect to a complete fuzzy ranking. The following example illustrates the procedure:

Example 3 *Consider again the fuzzy random variables $\widetilde{X}, \widetilde{Y}$ from Example 1. Their (fuzzy) distribution functions are:*

$$F_{\widetilde{X}}(\omega)(r) = \begin{cases} 2\omega & \text{if } r \in (0,1], \omega \in [0,0.5). \\ 1 & \text{if } r = 0, \omega \in [0,0.5). \\ 2-2\omega & \text{if } r \in [0,1), \omega \in [0.5,1]. \\ 1 & \text{if } r = 1, \omega \in [0.5,1]. \end{cases}$$

while

$$F_{\widetilde{Y}}(\omega) = \begin{cases} I_{\{0.5\}} & \text{if } \omega \in [0,0.5). \\ I_{[0.5,1]} & \text{if } \omega \in [0.5,1). \\ I_{\{1\}} & \text{if } \omega = 1. \end{cases}$$

If we compare them by means of the fuzzy ranking in Eq. (5), $CM(F_{\widetilde{X}}(\omega)) = \omega$ for any $\omega \in [0,1]$ while

$$CM(F_{\widetilde{Y}}(\omega)) = \begin{cases} 0.5 & \text{if } \omega \in [0,0.5) \\ 0.75 & \text{if } \omega \in [0.5,1) \\ 1 & \text{if } \omega = 1. \end{cases}$$

As a consequence, $\widetilde{X}$ and $\widetilde{Y}$ are incomparable with respect to $\succsim_{CM}$-stochastic dominance. ♦

6. Conclusions

In this paper we have established a theory of fuzzy decision making under uncertainty by generalizing the notion of stochastic order. We have provided a number of extensions of two of the most prominent stochastic orders in the literature: expected utility and stochastic dominance. The choice of one particular extension over the others can be made according to a number of criteria: on the one hand, for those based on the imprecise stochastic orders in Definition 4, it should be remarked that the different notions take into account different underlying criteria, as discussed in [13]. On the other hand, some of the possibilities we have discussed require the use of an underlying fuzzy ranking. This second choice is a problem that has been widely analyzed in [20]. In this respect, one possibility is to make the choice by means of desirable axiomatic properties of the fuzzy ranking, such as the ones we have investigated in Section 4. We should also take into account the interpretation of the fuzzy information we have in the particular problem under consideration. Finally, it should also be considered that some of the new fuzzy rankings we have introduced in this paper do not produce a complete order.

In addition, we have also followed a different approach, by defining a second order possibility measure on the set of probabilities, which can be reduced to a lower and an upper probability model by means of Walley's approach. We have used this first order model to compare fuzzy random variables in a manner similar to standard techniques within Imprecise Probability Theory, and have proven that this procedure is related to the use of fuzzy expected utility where the expectations are compared by means of the imprecise expected utility.

In the future, we would like to deepen in the comparison between the different fuzzy stochastic orderings, by studying their behaviour in a number of real-life examples; from a more theoretical point of view, it would be interesting to generalize some of the results in this paper to fuzzy random variables whose images are not necessarily fuzzy numbers. Finally, it would be useful to obtain an axiomatic characterization of some of these orders, in the vein of some of the existing ones for the precise case.

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