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Commutativity Theorems in Rings

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Ahmed Fekri Serag Aboubakr

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Teoremas de Conmutatividad en Anillos

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Ahmed Fekri Serag Aboubakr

Tesis doctoral dirigida por:

Prof. D. **Santos González Jiménez**, Universidad de Oviedo

Prof. D^a. **Consuelo Martínez López**, Universidad de Oviedo

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RESUMEN (en español)

Una importante y activa línea de investigación en Algebra No-Conmutativa es la denominada como "Teoremas de Conmutatividad" cuyo objetivo final es encontrar condiciones que garanticen la conmutatividad de un anillo.

La presente tesis doctoral se enmarca en este campo de trabajo y, concretamente, en el estudio de condiciones de conmutatividad que involucran varias tipos de generalizaciones de derivaciones. Se obtienen resultados para anillos primos y semiprimos.

Se han estudiado varios tipos de generalizaciones de derivaciones y las consecuencias que tiene su existencia en relación con la conmutatividad del anillo en el que están definidas o bien en el hecho de que existan ideales centrales, cuando las condiciones impuestas sobre las derivaciones son más débiles.

Así, por ejemplo, se han introducido las nociones de derivación generalizada reversa a izquierda y a derecha y se ha probado que la existencia de una aplicación de este tipo en un anillo semiprimo implica la existencia de un ideal central no trivial. Además, si el anillo es libre de 2-torsion, entonces las nociones de derivación generalizada reversa a izquierda, a derecha, derivación generalizada a izquierda y a derecha coinciden.

Se han estudiado también aplicaciones sobre un anillo semiprimo que son derivaciones generalizadas a derecha sobre un ideal de Lie U. En este caso se ha probado, por ejemplo, que si (F, d) es una derivación generalizada a izquierda (resp. a derecha) y $F^2(U) = (0)$, entonces $d(U)=F(U)=(0)$ y $d(R), F(R) \subseteq C_R(U)$. Por otro lado, si (F, d) y (G, g) son derivaciones generalizadas a derecha e izquierda, respectivamente, y $F(u)v = uG(v)$ por todo u, v en U, entonces $d(U), g(U) \subseteq C_R(U)$.

Por otra parte se han estudiado distintos elementos de "tipo central" y se han definido varios "centros generalizados", probándose que todos ellos coinciden en el caso de un anillo primo.

También se ha considerado la noción de ortogonalidad, ligada a una derivación generalizada a izquierda y una derivación generalizada a derecha y se han encontrado varias condiciones equivalentes a la ortogonalidad. Se han estudiado las consecuencias de la existencia de derivaciones generalizadas ortogonales en la composición de las aplicaciones.

Hemos prestado atención a los anillos con una bi-derivación generalizada, viendo que si ésta satisface algunas condiciones algebraicas sobre elementos de un ideal de Jordan J se obtienen consecuencias en la estructura del anillo. En particular, si el anillo R es primo y la bi-derivación no es cero, concluimos que $J \subseteq Z(R)$.

Finalmente se han considerado derivaciones multiplicativas generalizadas de anillos semiprimas, estudiando de nuevo condiciones algebraicas que se traducen en propiedades de la multiplicación en el anillo.

RESUMEN (en Inglés)

An important and active research line in Non-Commutative Algebra is the line known as "Commutative Theorems". The final aim in it is to find conditions that guarantee the commutativity of a ring.

This thesis can be placed in this area of work and, in a concrete way, conditions studied are related to several types of generalizations of derivations. In this thesis, conditions have been applied to prime or semiprime rings.

Several types of generalizations of derivations have been studied, as well as the consequences that their existence have in relation to the commutativity of the ring or in the existence of central ideals, depending of how strong conditions are considered.

So the notions of l-generalized and r-generalized reverse derivations have been introduced. These notions extend the one of reverse derivation. In particular we have proved that the existence of such a map left generalized or right generalized reverse derivation) in a semiprime ring implies the existence of a non-zero central ideal. Furthermore, if the ring R is 2-torsion free, then the notions of left generalized, right generalized reverse derivations, left generalized and right generalized derivations coincide.

We have considered also maps on a semiprime ring that are left or right generalized derivations on a Lie ideal U. In particular we proved that if (F, d) is a left and right generalized derivation and $F^2(U) = (0)$ then $d(U)=F(U)=(0)$ and $d(R), F(R) \subseteq C_R(U)$. On the other side, if (F, d) and (G, g) are right and left generalized derivations, respectively, and $F(u)v=uG(v)$ for all u, v in U, then $d(U), g(U) \subseteq C_R(U)$.

On the other side we have studied different types of center-like elements. The corresponding generalized centers have been defined and they have been proved to be all equal when the ring R is prime.

The notion of orthogonality has been considered, linked to a left generalized and a right generalized derivation, finding necessary and sufficient conditions for their existence and some consequences of the existence of orthogonal generalized derivation on the composition of the maps have been obtained.

"The human mind has never invented a labor-saving machine equal to algebra"
Author Unknown

"Pure mathematics is, in its way, the poetry of logical ideas"
Albert Einstein

"Pure mathematics is the magician's real wand"
Novalis

"This thesis is dedicated to the memory of my father, FEKRI ABOUBAKR. *I miss him every day, but I am glad to know he saw this process through to its completion, offering the support to make it possible, as well as plenty of friendly encouragement"*

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Contents

Introducción

En teoría de anillos es natural estudiar condiciones que, en último término, implican la conmutatividad del anillo. Se puede considerar el bien conocido Teorema de Wedderburn (un anillo de división finito es un cuerpo) como el primer precedente.

Jacobson extendió este teorema y probó que si el anillo R satisface que para todo $a \in R$, existe un $n(a) > 1$ tal que $a^{n(a)} = a$, entonces el anillo R es conmutativo.

Estos anillos (usualmente llamados anillos periódicos) han sido objeto de un extenso estudio y este tipo de problemas son conocidos usualmente bajo el nombre de "Teoremas de Conmutatividad".

Una línea de investigación conectada a la anterior, pero diferente, trata de encontrar las condiciones bajo las cuales una aplicación, usualmente una derivación, definida sobre el anillo fuerza su conmutatividad. La presente tesis puede enmarcarse en este contexto.

No se pueden esperar resultados impactantes y revolucionarios en el tema de "Anillos con Derivaciones", sin embargo han sido objeto de estudio de muchos autores en los últimos 60 años, especialmente en lo que respecta a las relaciones entre derivaciones y la estructura de los anillos en los que están definidas.

Una de las cuestiones que aparecen a menudo en álgebra y análisis es determinar cuándo una aplicación puede estar definida por su propiedades locales. Por ejemplo, la cuestión de saber si una aplicación que actúa como una derivación sobre el producto de Lie de alguna subálgebra importante de un anillo primo viene inducida por una derivación ordinaria es un problema bien conocido planteado por Herstein en [74]. El primer resultado obtenido en esta dirección aparece en un trabajo sin publicar de Kaplansky (ver [74], pg 529) quién consideró álgebras de matrices sobre un cuerpo. En presencia de idempotentes, esta cuestión ha sido examinada por Martindale [82] para anillos primitivos. El problema de Herstein fue resuelto, de modo general gracias a las potentes técnicas de las identidades funcionales (ver, por ejemplo, [19], [20], [21], [41]).

La noción "anillos con derivación" juega un papel significativo en la interacción de análisis, geometría algebraica y álgebra. En los 40 se encontró que la teoría de Galois de ecuaciones algebraicas se puede transferir a la teoría de ecuaciones diferenciales lineales ordinarias (la teoría de Picard-Vessiot, incluyendo la teoría para ecuaciones diferenciales y para ecuaciones en diferencias). Usualmente, la teoría de Picar Vessiot significa una teoría de Galois para ecuaciones diferenciales lineales ordinarias (ver Van der Put y Singer [55] para los detalles).

Una aplicación $d : R \longrightarrow R$ es una derivación de un anillo R si d es aditiva y satisface la regla de Leibnitz: $d(ab) = d(a)b + ad(b)$ para todo $a, b \in R$. Un ejemplo simple es la derivación usual en anillos de polinomios. Un ejemplo básico en anillos no conmutativos viene dado por $d : R \longrightarrow R$, $d(x) = [x, a]$ para todo $x \in R$ y siendo a un elemento fijo de R. La función d así definida es una derivación.

El estudio de derivaciones en anillos, aunque iniciado hace tiempo, consiguió su ímpetu sólo después de que Posner [114] en 1957 estableciera dos resultados sorprendentes sobre derivaciones en anillos primos.

Las afirmaciones específicas de los teoremas de Posner, a los cuales haremos referencia frecuente en lo sucesivo, son:

Primer Teorema de Posner. Si R es un anillo primo de característica distinta de 2 y d_1, d_2 son derivaciones de R tales que la composición d_1d_2 es una derivación, entonces al menos una de ellas es cero.

El primer teorema de Posner nos dice que la composición de dos derivaciones no nulas de un anillo primo, de característica distinta de 2, no puede nunca ser una derivación. Este teorema ha sido generalizado de muchos modos por diversos autores (ver por ejemplo Bergen [32], Chebotar [47], Chuang [49], [50], Hirano et al. [75], Hvala [79], Jensen [84], Krempa [95], Lanski [99], Martindale [83] y [129])).

La invariancia de ciertos ideales bajo derivaciones es otro tema investigado por numerosos autores. Se sabe que derivaciones acotadas sobre álgebras de Banach dejan invariantes los ideales primitivos [122]. Creedon demostró que si P es un ideal primo de un anillo R, siendo car $R/P \neq 2$, y el producto de dos derivaciones deja invariante P, entonces una de las derivaciones debe dejar P invariante. También probó que si d es una derivación sobre un anillo R, P es un ideal semiprimo de R, y para un entero positivo fijo k se tiene $d^k(P) \subseteq P$, entonces $d(P) \subseteq P$. Para mas resultados relacionados ver, por ejemplo, Bell [29], Bell y Argac [30], Hirano et al. [73], Jensen [84], Krempa [96], Lanski [99] y Wang [127].

Segundo Teorema de Posner: Sea R un anillo primo. Si existe una derivación centralizadora de R entonces R es conmutativo.

Este teorema asegura la conmutatividad de un anillo primo que posee una derivación centralizadora. No es clara la motivación de Posner para probar este teorema, ni las razones que le llevaron a conjeturar el resultado luego probado. En todo caso, es un hecho que el teorema ha ejercido una notable influencia y que, al menos indirectamente, ha iniciado el estudio de las derivaciones conmutadoras.

Recordemos que d es una aplicación centralizadora si $[d(x), x)] \in Z(R)$ para todo $x \in R$. Bajo hipótesis suaves, una aplicación centralizadora es necesariamente conmutadora, es decir, $[d(x), x] = 0$ para todo $x \in R$ (ver, por ejemplo [36, Prop.3.1]). El segundo teorema de Posner no se puede extender a anillos arbitrarios.

Durante las últimas décadas ha surgido gran cantidad de trabajo en relación con las derivadas generalizadas, especialmente en un contexto de álgebras normadas (el lector puede consultar [79]). Por ejemplo, una aplicación de la forma $x \to ax + xb$, siendo a, b elementos fijos en el álgebra A es una derivación generalizada. Tales aplicaciones se llaman, usualmente, derivaciones internas generalizadas. Dentro de la teoría de álgebras de operadores se consideran como una clase importante de los llamados operadores elementales, $x \to \sum_{i=1}^{n} a_i x b_i$.

Si F es una derivación interna generalizada sobre un anillo R dada por $F(x) = ax + xb$, notemos que $F(xy) = F(x)y + xI_b(y)$, dónde $I_b(y) = yb - by$ es la derivación interna definida por el elemento b de R. Motivado por esta observación, en 1991 Bresar introdujo el concepto de derivación generalizada en [37] como sigue: Sea S un subconjunto no vacio de R. Una aplicación aditiva $F : R \to R$ se dice derivación generalizada sobre S si existe una derivación $d : R \to R$ tal que $F(xy) = F(x)y + xd(y)$ para todo $x, y \in S$.

Recientemente, Hvala [79] inició el estudio algebraico de las derivaciones generalizadas, extendiendo a ellas algunos resultados de derivaciones. De hecho, el concepto de derivación generalizada incluye el de derivación y el de derivación interna generalizada. Además, si $d = 0$, la derivación generalizada es un multiplicador a izquierda, es decir, una aplicación aditiva f satisfaciendo $f(xy) = f(x)y$

$\forall x, y \in R$. Han sido muy estudiados en análisis funcional y se han obtenido varios resultados interesantes sobre ellos (ver, por ejemplo Are y Mathiew [9], Sinclair [122] y Wendel [127]).

En [79], Havla encontró condiciones necesarias y suficientes para que el producto de dos derivaciones generalizadas sea una derivación generalizada. Consideró derivaciones generalizadas f_1, f_2 de R satisfaciendo la relación $[f_1(x), f_2(x)] = 0$ $\forall x \in R$.

En 1999 Nakajima [111] dió algunas propiedades elementales de las derivaciones generalizadas y determinó relaciones functoriales entre $gDer_k(S, M)$ y $Der_k(S, M)$ dónde $gDer_k(S, M)$ denota el conjunto de derivaciones generalizadas y $Der_k(S, M)$ el conjunto de derivaciones de S a M. Otros resultados relacionados se pueden ver en Komatsu y Nakajima [94], Nakajima [112] o Nakajima y Sapanci [113]).

En 1999 Lee in [102] extendió la definición de derivación generalizada como sigue: una derivación generalizada es una aplicación aditiva $F : \varrho \to U$ tal que $F(xy) = F(x)y + xd(y)$ $\forall x, y \in \varrho$ dónde ϱ es un ideal denso de R y d es una derivación de ϱ in U, el anillo cociente de Utumi. Lee probó que toda derivación generalizada se puede extender de modo único a una derivación generalizada de U. De hecho, existe un elemento $a \in U$ y una derivación d de U tal que $F(x) = ax + d(x)$ para todo $x \in U$ ([102, teorema 3]). Luego podemos suponer, sin pérdida de generalidad, que una derivación generalizada de R, en este contexto, es una aplicación de $U \to U$.

En [93], Kharchenko describió identidades con derivaciones y sus resultados son una potente herramienta para reducir una identidad diferencial a una identidad polinomial generalizada. Así, para estudiar identidades con derivaciones generalizadas parece razonable encontrar un teorema correspondiente para identidades con derivaciones generalizadas.

Durante la última década se ha experimentado un creciente interés por las relaciones entre la conmutatividad de un anillo y la existencia de ciertos tipos de derivaciones de R. Recientemente, varios autores [30, 27, 36] y [76] han obtenido la conmutatividad de anillos primos y semiprimos teniendo derivaciones que satisfacen ciertas restricciones polinomiales. En 2001, Ashraf y Nadeem [15] establecieron que un anillo primo R con un ideal I que admite una derivación d satisfaciendo $d(xy) + xy \in Z(R)$ o $d(xy) - xy \in Z(R)$ $\forall x, y \in I$ es conmutativo. Motivado por estas observaciones, Ali [7] explora la conmutatividad de un anillo R satisfaciendo una de las siguientes propiedades (i) $F(xy) - xy \in Z(R)$, (ii) $F(xy) + xy \in Z(R)$, (iii) $F(xy) - yx \in Z(R)$, (iv) $F(xy) + yx \in Z(R)$, (v) $F(x)F(y) - xy \in Z(R)$ y (vi) $F(x)F(y) + xy \in Z(R)$, $\forall x, y \in I$. Para mas resultados relacionados ver, por ejemplo, [13, 14, 37, 79, 86, 103, 112, 115, 121].

Por otra parte, derivaciones y derivaciones generalizadas son también temas importantes en álgebras y superálgebras de Lie. En el estudio de factores de Levi en álgebras derivación de álgebras de Lie nilpotentes derivaciones generalizadas, cuasiderivaciones, centroides y cuasicentroides juegan papeles esenciales (ver [31]), Melville se ocupa en particular, de los centroides de álgebras de Lie nilpotentes (ver [110]). La investigación mas importante y sistemática sobre derivaciones generalizadas de un álgebra de Lie y sus subálgebras de Lie se debe a Leger y Luks [104]. En [31] se obtienen algunas propiedades acerca de derivaciones generalizadas, cuasiderivaciones y cohomología de álgebras de Lie. En particular, investigan las estructura del álgebra derivación generalizada y caracterizan álgebras de Lie que satisfacen ciertas condiciones. Apuntan también algunas conexiones entre cuasiderivaciones y cohomología de álgebras de Lie. Para mas resultados relacionados ver, por ejemplo, [58, 78, 85, 87, 88, 110, 124, 133] y [97].

En esta tesis se estudia si la existencia de ciertas aplicaciones en el anillo garantizan su conmutatividad. El objetivo esencial es la búsqueda de dichas condiciones. En algunos casos se consideran condiciones mas débiles, que no permiten obtener la conmutatividad, sino alguna información sobre el centro.

El estudio de derivaciones y extensiones naturales de este concepto es, en última instancia, el motivo guía de la tesis. La investigación realizada se centra en encontrar relaciones entre la conmutatividad del anillo R y la existencia de aplicaciones (distintos tipos de generalizaciones de derivaciones) definidas sobre R.

La tesis está estructurada en 7 capítulos más la bibliografía.

El capítulo 1 recuerda algunas definiciones básicas, conceptos y comentarios que serán usados en los capítulos sucesivos. El material se distribuye en una introducción y otras dos secciones, en las que se fija terminología y se dan las definiciones de distintos tipos de aplicaciones y algunos ejemplos.

Nuevos resultados empiezan a aparecer en el capítulo 2 "Derivaciones generalizadas reversas sobre anillos semiprimos", en el que se introduce la noción de derivación generalizada reversa, que generaliza la de derivación reversa, es decir, una aplicación aditiva satisfaciendo $d(xy) = d(y)x + yd(x)$ para todo $x, y \in R$.

Definimos derivaciones reversas generalizadas a izquierda y a derecha como aplicaciones aditivas $F : R \to R$ que satisfacen $F(xy) = F(y)x + yd(x)$ ($F(xy) = d(y)x + yF(x)$) para todo $x, y \in R$, dónde d es una derivación reversa de R.

Estudiaremos relaciones entre derivaciones reversas generalizadas definidas sobre un ideal de un anillo semiprimo y las consecuencias que se deducen para dicho anillo de la existencia de una derivación reversa generalizada a izquierda (resp. a derecha).

El material de este capítulo ha aparecido en el artículo:
A. Aboubakr and S. González: Generalized reverse derivations on semiprime rings, *Siberian Mathematical Journal*, 56, no. 2, 199-205, 2015.

El capítulo 3 se ocupa del estudio de derivaciones generalizadas sobre ideales de Lie en anillos semiprimos. Herstein [69] probó que dado un anillo semiprimo libre de 2-torsión R, si una derivación interna d_t satisface $d_t^2(U) = 0$ para un ideal de Lie U de R, entonces $d_t(U) = 0$. Carini [45] extendió este resultado para una derivación arbitraria d probando que $d^2(U) = 0$ implica que $d(U) \subseteq Z(R)$. El objetivo de este capítulo es extender los resultados mencionados para derivaciones generalizadas a derecha y a izquierda.

Los resultados de este capítulo se recogen en el artículo:

Ahmed Aboubakr and Santos González: Generalized Derivations on Lie Ideals in Semiprime Rings, *Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry*, volume 57, number 4, pages 841-850, 2016.

En el capítulo 4 prestamos atención al estudio de subconjuntos de tipo central en anillos primos con una derivación generalizada. Bell y Daif en [25] definen subconjuntos de tipo central$Z^*(R, f)$, $Z^{**}(R, f)$ y $Z_1(R, f)$, dónde R es un anillo y f una aplicación de R en R. Prueban que si f es una derivación y R es primo, entonces estos conjuntos coinciden con el centro de R. El objetivo de este capítulo será extender estos resultados a derivaciones generalizadas. Además, se buscarán contraejemplos para demostrar que las restricciones impuestas en las hipótesis de nuestros resultados no son superfluas.

Los resultados de este capítulo han sido sometidos a publicación en el artículo:

Ahmed Aboubakr and Santos González, Center-like Subsets in Prime Rings with Generalized Derivations, Sometido.

En el capítulo 5 se presentarán algunos resultados relativos a la ortogonalidad de una derivación generalizada a izquierda y una a derecha sobre ideales de anillos semiprimos. Estudiaremos tanto las consecuencias de la existencia de tales derivaciones generalizadas ortogonales como condiciones necesarias y suficientes para que existan en un anillo semiprimo R. También estudiaremos las conexiones entre la ortogonalidad de derivaciones generalizadas y propiedades de su composición. Los resultados de esta sección están relacionados con algunos resultados de M. Breŝar y J. Vukman en [43] que extienden el mencionado teorema 1 de Posner

[114]. También se mostrarán ejemplos que justifiquen las restricciones impuestas.

Los resultados de este capítulo han sido sometidos a publicación en el artículo:

Ahmed Aboubakr and Santos González, Orthogonality of Two Left and Right Generalized Derivations on Ideals in Semiprime Rings, Sometido.

El capítulo 6 se dedica al estudio de la conmutatividad de un anillo que tiene una biderivación generalizada, que satisface ciertas condiciones algebraicas. La noción de biderivación generalizada fue introducida por Breŝar en [40]. Una aplicación biaditiva $G : R \times R \to R$ es una biderivación generalizada ligada a una biderivación $B : R \times R \to R$ si para todo $x \in R$ las aplicaciones $y \to G(x, y)$ e $y \to G(y, x)$ son derivaciones generalizadas de R ligadas a las derivaciones $B(x, .)$ y $B(., x)$ respectivamente.

La aplicación $g : R \to R$ (resp. $f : R \to R$) definida por $g(x) = G(x, x)$ (resp. $f(x) = B(x, x)$) es la traza de G (res. de B). En este capítulo estudiaremos la conmutatividad de un anillo primo que admite una biderivación generalizada G ligada a una biderivación B que satisface distintas condiciones algebraicas, por ejemplo:

i. $B(g(u), u) = 0$, $G(f(u), u) = 0$ y $G(g(u), u) = 0$ para todo $u \in J$.

ii. $B(g(u), g(v)) = 0$, $(g(u))^2 = 0$ y $g^2(u) - f^2(u) = fg(u) - gf(u)$ para todo $u, v \in J$.

Aquí J es un ideal de Jordan no cero de R.

Finalmente, en el Capítulo 7 discutimos la conmutatividad de anillos primos y semiprimos que involucran derivaciones generalizadas multiplicativas.

Una aplicación $F : R \to R$ (no necesariamente aditiva) se dice que es una derivación multiplicativa generalizada si $F(xy) = F(x)y + xg(y)$ para todo $x, y \in R$, dónde g es una aplicación (no necesariamente derivación). En este capítulo se impondrán condiciones algebraicas del tipo:

i. $[F(x), F(y)] = \pm[x, y]$, $[F(x), y] = \pm[x, G(y)]$ y $[g(x), F(y)] = \pm[x, y]$ para todo $x, y \in L$.

ii. $F(x)y = \pm xG(y)$, $F(x)y \pm xG(y) \in Z(R)$ y $F(xy) = \pm F(yx)$ para todo $x, y \in L$.

iii. $F([x, y]) = \pm[x, y]$ y $F(x \circ y) = \pm(x \circ y)$ para todo $x, y \in L$.

Aquí G denota otra derivación generalizada multiplicativa y L es un ideal bilátero de R. También se incluirán algunos ejemplos.

Introduction

In Ring Theory it is natural to study conditions that finally imply commutativity of the ring. The well known Wedderburn Theorem (a finite division ring is a field) can be considered the first precedent.

This result was extended by Jacobson who proved that if the ring R satisfies that $\forall a \in R$, $\exists n(a) > 1$ such that $a = a^{n(a)}$, then the ring R is commutative.

These rings (usually called periodic) have been widely studied and this frame of problems is usually referred to as "Commutative Theorems".

A connected, but different, line of research tries to find conditions on a map, usually a derivation, defined on the ring that forces the commutativity of the ring. The present thesis can be set up in this context.

Rings with derivations are not the kind of subject that undergoes tremendous revolutions. However, this has been studied by many authors in the last 60 years, specially the relationships between derivations and the structure of rings.

One of the questions which often appeared in algebra and analysis is whether a map can be defined by its local properties. For example, the question whether a map, which acts like a derivation on the Lie product of some important Lie subalgebra of prime rings, is induced by an ordinary derivation was a well-known problem posed by Herstein [74]. The first result in this direction was obtained in an unpublished work of Kaplansky (cf. Herstein [74], p. 529), who considered matrix algebras over a field. In the presence of idempotents, this question has been examined by Martindale [82] for primitive rings. Herstein's problem was solved in full generality only after the powerful techniques of functional identities was developed (see for example; [19], [20], [21], [41]).

The notion "ring with derivation" is quite old and plays a significant role in the integration of analysis, algebraic geometry and algebra. In the 1940's it was found that the Galois theory of algebraic equations can be transferred to the theory of ordinary linear differential equations (Picard-Vessiot theory, including Picard-Vessiot theories for differential equations and for difference equations). In the usual sense, "Picard-Vessiot theory" means a Galois theory for linear ordinary differential equations (cf. Van der Put and Singer [55] for details).

A map $d : R \longrightarrow R$ is a derivation of a ring R if d is additive and satisfies the Leibnitz's rule; $d(ab) = d(a)b + ad(b)$, for all $a, b \in R$. A simple example is of course the usual derivative in polynomial rings. Basic examples in non-commutative rings are quite different. For a fixed $a \in R$, define $d : R \longrightarrow R$ by $d(x) = [x, a]$ for all $x \in R$. The function d so defined can be easily checked to be additive and $d(xy) = [xy, a] = x[y, a] + [x, a]y = xd(y) + d(x)y$, for all $x, y \in R$.

The study of derivations in rings though initiated long ago, got impetus only after Posner [114] who in 1957 established two very striking results on derivations in prime rings.

The specific statements of Posner's theorems, to which we shall referer frequently, are the following:

Posner's First Theorem. If R is a prime ring of characteristic not 2 and d_1, d_2 are derivations on R such that the composition $d_1 d_2$ is also a derivation, then at least one of d_1, d_2 is zero.

Posner's First Theorem tells us that the composition of two nonzero derivations of a prime ring R can not be a derivation if the characteristic of R is different from 2. Thereafter, a number of authors have generalized this theorem in several ways (see for example Bergen [32], Chebotar [47], Chuang [49], [50], Hirano et al. [75], Hvala [79], Jensen [84], Krempa [95], Lanski [99], Martindale [83] and Ye et al. [129]).

Many authors have investigated the invariance of certain ideals under derivations. It is known that bounded derivations on Banach algebras leave primitive ideals invariant [122]. Creedon showed that if P is a prime ideal of a ring R, where the characteristic of R/P is not two, and the product of two derivations leaves P invariant, then one of the derivations must leave P invariant. He also proved that, if d is a derivation on a ring R and P is a semiprime ideal of R, for a fixed positive integer k and $d^k(P) \subseteq P$, then $d(P) \subseteq P$. For more related results see e.g.; Bell [29], Bell and Argac [30], Hirano et al. [73], Jensen [84], Krempa [96], Lanski [99] and Wang [127].

Posner's Second Theorem. Let R be a prime ring. If there is a nonzero centralizing derivation of R, then R is commutative.

This theorem says that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative.

Considering this theorem from some distance it is not entirely clear what was Posner's motivation for proving it and which reasons were underling to conjecture that the theorem is true. In any case, it is a fact that the theorem has been extremely influential and, at least indirectly, it has initiated the study of commuting derivations.

Let's remember that d is a centralizing map of R if $[d(x), x] \in Z(R)\ \forall x \in R$. It has been proved that under rather wild assumptions a centralizing map is necessarily commuting, that is, $[d(x), x] = 0\ \forall x \in R$. (see, for instance [36, Prop.3.1]). Posner's Second Theorem can not be extended to arbitrary rings.

During the last few decades there has been a great deal of work concerning generalized derivation, specially in a context of normed algebras (reader may see [79]). For instance a map of the form $x \to ax + xb$, where a and b are fixed elements in the algebra A is generalized derivation. Such maps are usually called generalized inner derivations since they can be seen as a generalization of the concept of inner derivations (i.e., the map of the form $x \to xa - ax$). In the theory of operator algebras, they are considered as an important class of the so called elementary operators, that is, operators where $x \to \sum_{i=1}^{n} a_i x b_i$.

Now let's consider a ring R. If F is a generalized inner derivation on R given by $F(x) = ax+xb$, let us notice that $F(xy) = F(x)y+xI_b(y)$ where $I_b(y) = yb-by$ is the inner derivation defined by $b \in R$. In 1991, motivated by this observation, Breŝar [37] introduced the concept of generalized derivation in rings as follows: Let S be a non-empty subset of R. An additive map $F : R \to R$ is said to be a generalized derivation on S if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in S$.

Recently, Hvala [79] initiated the algebraic study of generalized derivations, extending to them some results concerning to derivations. In fact, the concept of generalized derivation covers both, the concept of derivation and the concept of generalized inner derivation. Moreover, generalized derivations with $d = 0$ cover the concept of left multipliers, that is, additive maps f satisfying $f(xy) = f(x)y$, for all $x, y \in R$. This has widely been studied in functional analysis and several interesting results have been obtained (see, for example; Ara and Mathieu [9], Sinclair [122] and Wendel [127]).

In [79], Hvala found necessary and sufficient conditions for the product of two generalized derivations to be a generalized derivation. He also considered generalized derivations f_1, f_2 of R satisfying the relation $[f_1(x), f_2(x)] = 0$ for all $x \in R$.

In 1999, Nakajima [111] gave some elementary properties of generalized derivations, and determined functorial relations between $gDer_k(S,M)$ and $Der_k(S,M)$ where $gDer_k(S,M)$ denotes the set of generalized derivations and $Der_k(S,M)$ the set of derivations from S to M. Other related results can be found in Komatsu and Nakajima [94], Nakajima [112] or Nakajima and Sapanci [113]).

In 1999, Lee in [102] extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive map $F : \varrho \to U$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in \varrho$, where ϱ is a dense right ideal of R and d is a derivation from ϱ into U, the right Utumi quotient ring. He proved that every generalized derivation can be uniquely extended to a generalized derivation of U. In fact, there exists $a \in U$ and a derivation d of U such that $F(x) = ax + d(x)$ for all $x \in U$ ([102, Theorem 3]). Therefore we may assume, without loss of generality, that a generalized derivation of R, in this setting, is a map $U \to U$.

In [93], Kharchenko described identities with derivations and his results are a powerful tool for reducing a differential identity to a generalized polynomial identity. Thus, to study identities with generalized derivations, it seems reasonable to find a corresponding theorem for identities with generalized derivations.

During the last decade, an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain types of derivations of R has emerged. Recently, many authors [30, 27, 36] and [76] have obtained commutativity of prime and semiprime rings having derivations that satisfy certain polynomial constraints. In 2001, Ashraf and Nadeem [15] established that a prime ring R with a nonzero ideal I that admits a derivation d satisfying $d(xy) + xy \in Z(R)$ or $d(xy) - xy \in Z(R)$ for all $x, y \in I$, is commutative. Motivated by these observations, Ali in [7] explores the commutativity of a ring R satisfying one of the following properties: (i) $F(xy) - xy \in Z(R)$, (ii) $F(xy) + xy \in Z(R)$, (iii) $F(xy) - yx \in Z(R)$, (iv) $F(xy) + yx \in Z(R)$, (v) $F(x)F(y) - xy \in Z(R)$ and (vi) $F(x)F(y) + xy \in Z(R)$, for all $x, y \in I$. For more related results see for example ([13, 14, 37, 79, 86, 103, 112, 115, 121]).

On the other hand, derivation and generalized derivation algebras are also an important subject in Lie algebras and superalgebras. In the study of Levi factors in derivation algebras of nilpotent Lie algebras, generalized derivations, quasiderivations, centroids, and quasicentroids play key roles. Melville dealt particularly with centroids of nilpotent Lie algebras (see [110]). The most important and systematic research on generalized derivations of a Lie algebra and their Lie subalgebras was due to Leger and Luks [104]. In [31], some nice properties about generalized derivations, quasiderivations and centroids have been obtained. In particular, they investigated the structure of the generalized derivation algebra and characterized Lie algebras satisfying certain conditions. They also pointed some connections between quasiderivations and cohomology of Lie algebras. For details about the generalized derivation algebra of a general non-associative algebra, readers are referred to [58, 78, 85, 87, 88, 110, 124, 133] and [97].

The subject of this thesis moves around the existence of maps satisfying some properties guaranties the commutativity of the ring. The essential aim is the search of conditions that give the commutativity of the ring, or the existence of maps that satisfy some specific conditions. In some other cases, weaker conditions are considered and then, instead of commutativity, some information about the center is obtained.

Outline

The overall theme of this thesis is the study of derivations and its various generalizations in the setting of rings. We investigate the relationship between the commutativity of the ring R and the existence of maps on R of some particular types.

The structure of the thesis is at the follows.

Chapter 1 reminds some basic definitions, concepts and remarks that will be used in the sequel. The material in this chapter is organized into two main sections. In the second section we will fix some required terminology on rings. The third section contains definitions of some types of maps together with some examples.

Now results appear first in Chapter 2, "Generalized Reverse Derivations on Semiprime Rings", in which we introduce the notion of generalized reverse derivations that generalizes the one of reverse derivation that is, a map satisfying $d(xy) = d(y)x + yd(x)$ $\forall x, y \in R$. We define *reverse l-generalized derivation (reverse r-generalized derivation)* as an additive map $F : R \to R$, satisfying $F(xy) = F(y)x + yd(x)$ $(F(xy) = d(y)x + yF(x))$ for all $x, y \in R$, where d is a reverse derivation of R. We study the relationship between generalized reverse derivations and generalized derivations on an ideal in a semiprime ring and the implications for a prime ring R of the existence of a l-generalized (or r-generalized) reverse derivation.

The material of this chapter has appeared in the following publication:
A. Aboubakr and S. González: Generalized reverse derivations on semiprime rings, *Siberian Mathematical Journal*, volume 56, number 2, pages 199-205, 2015.

Chapter 3 deals with the study of generalized derivations on Lie ideals in semiprime rings. Herstein [69] proved that given a semiprime 2-torsion free ring R and an inner derivation d_t, if $d_t^2(U) = 0$ for a Lie ideal U of R then $d_t(U) = 0$. Carini [45] extended this result for an arbitrary derivation d, proving that $d^2(U) = 0$ implies $d(U) \subseteq Z(R)$. The aim of this chapter is to extend the results mentioned above for right (resp. left) generalized derivations.

The results of this chapter have been published in:
Ahmed Aboubakr and Santos González: Generalized Derivations on Lie Ideals in Semiprime Rings, *Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry*, volume 57, number 4, pages 841-850, 2016.

In Chapter 4 we turn out our attention to the study of center-like subsets in prime ring with a generalized derivation. Bell and Daif in [25] defined the center-like subsets $Z^*(R, f)$, $Z^{**}(R, f)$ and $Z_1(R, f)$, where R is a ring and f is a map from R to R. They proved that if f is a derivation and R is prime then these sets coincide with the center of R. In this chapter we extend these results to generalized derivations. Further, some counter examples have also been given to demonstrate that the restrictions imposed on the hypotheses of the various results are not superfluous.

The results of this chapter are collected in the article:
Ahmed Aboubakr and Santos González, Center-like Subsets in Prime Rings with Generalized Derivations, Submitted.

In Chapter 5, we present some results concerning orthogonality of l-generalized derivations and r-generalized derivations on ideals in semiprime rings. In Section 5.2, we consider the situation in which we have one l-generalized derivation and a r-generalized derivation and find necessary and sufficient conditions for them to be orthogonal on a nonzero ideal of a semiprime ring R. In Section 5.3, we also study the connections between orthogonality and some properties of the composition of a l-generalized derivation and a r-generalized derivation. These results are related to some results of M. Brešar and J. Vukman in [43], that extend above mentioned Theorem 1 by E. Posner [114] about products of derivations on prime rings. Finally we provide several examples to justify that the different restrictions imposed in the hypotheses of our theorems are not superfluous.

The results of this chapter are collected in the article:

Ahmed Aboubakr and Santos González, Orthogonality of Two Left and Right Generalized Derivations on Ideals in Semiprime Rings, Submitted.

Chapter 6 is based on the study of commutativity of a ring having a generalized biderivation that satisfy certain algebraic conditions. The notion of generalized biderivation was introduced by Breŝar in [40]. A biadditive map $G : R \times R \to R$ is a generalized biderivation linked to a biderivation $B : R \times R \to R$ if for every $x \in R$, the maps $y \to G(x, y)$ and $y \to G(y, x)$ are generalized derivations of R linked to the derivations $B(x, .)$ and $B(., x)$, respectively. The map $g : R \to R$ (resp. $f : R \to R$) defined by $g(x) = G(x, x)$ (resp. $f(x) = B(x, x)$) is the trace of G (resp. B). In this chapter we study the commutativity of a prime ring which admits a generalized biderivation G linked to a biderivation B satisfying the following algebraic conditions for instance:

i. $B(g(u), u) = 0$, $G(f(u), u) = 0$ and $G(g(u), u) = 0$ for all $u \in J$.

ii. $B(g(u), g(v)) = 0$, $(g(u))^2 = 0$ and $g^2(u) - f^2(u) = fg(u) - gf(u)$ for all $u, v \in J$.

Here J is a nonzero Jordan ideal of R.

Finally, in Chapter 7, we discuss the commutativity of prime and semiprime rings involving multiplicative (generalized)-derivations. A map $F : R \to R$ (not necessarily additive) is called a multiplicative (generalized)-derivation if $F(xy) = F(x)y+xg(y)$ is fulfilled for all $x, y \in R$, where $g : R \to R$ is a map (not necessarily a derivation). Algebraic conditions imposed in this chapter are of the type:

i. $[F(x), F(y)] = \pm[x, y]$, $[F(x), y] = \pm[x, G(y)]$ and $[g(x), F(y)] = \pm[x, y]$ for all $x, y \in L$.

ii. $F(x)y = \pm xG(y)$, $F(x)y \pm xG(y) \in Z(R)$ and $F(xy) = \pm F(yx)$ for all $x, y \in L$.

iii. $F([x, y]) = \pm[x, y]$ and $F(x \circ y) = \pm(x \circ y)$ for all $x, y \in L$.

Here G is another multiplicative (generalized)-derivation and L is a left-sided ideal of R. Some examples are also given.

Chapter 1

Ring Terminology and Some Types of Maps

1.1 Introduction

In the present chapter, we give a first overview on the subject stating more frequently used definitions, preliminary notions, more exciting examples and some elementary results required for the development of the subject in subsequent chapters of the present thesis. Of course, knowledge of some algebraic concepts such as groups, rings, ideals, fields, homomorphisms, etc. have been pre-assumed and no attempt will be made to discuss them. Some key results and well-known theorems related to our subject have been also incorporated for reader's convenience. With the same aim, the exposition has been made self-contained so as possible. Most of the material included in this chapter appears in standard literature namely, Beidar et. al. [18], Burton [44], Herstein [68, 71], Kharchenko [92], Lambek [98], McCoy [109] and Rowen [116].

1.2 Some elementary concepts

In the present section, we will give a brief exposition of some important terminology in ring theory. Throughout the thesis, unless otherwise mentioned, R will denote an associative ring (with or without unit) having at least two elements. For any pair of elements $x, y \in R$, the symbols $[x, y]$ and $(x \circ y)$ stand for the commutator $xy - yx$ and symmetrized product $xy + yx$. The symbols $Z(R)$ and $C_R(R)$ denote the center and centralizer of R respectively. We start our discussion with the following definitions:

Definition 1.2.1. (**Prime ideal**). An ideal P in a ring R is said to be a *prime ideal* if $P \neq R$ and for any two ideals A and B of R, $AB \subseteq P$ implies $A \subseteq P$ or

$B \subseteq P$.

Remark 1.2.1. If P is an ideal in a ring R with identity, then the following conditions are equivalent:

i. P is a prime ideal of R.

ii. If $a, b \in P$ such that $aRb \subseteq P$, then $a \in P$ or $b \in P$.

iii. If (a) and (b) are principal ideals in R such that $(a)(b) \subseteq P$, then $a \in P$ or $b \in P$.

iv. If U and V are right (or left) ideal in R such that $UV \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.

Definition 1.2.2. (**Prime ring**). A ring R is said to be a *prime* if zero ideal of R is a prime ideal in R.

Remark 1.2.2. In a ring R, the following conditions are equivalent:

i. R is prime.

ii. If A and B are ideals in R such that $AB = 0$, then $A = 0$ or $B = 0$.

iii. If $a, b \in R$, $aRb = 0$, then $a = 0$ or $b = 0$.

iv. If (a) and (b) are principal ideals in R such that $(a)(b) = (0)$, then $a = 0$ or $b = 0$.

Definition 1.2.3. (**Semiprime ideal**). An ideal S in a ring R is said to be a *semiprime ideal* in R if for every ideal A of R, $A^2 \subseteq S$ implies $A \subseteq S$.

Remark 1.2.3. If S is an ideal in a ring R, then the following conditions are equivalent:

i. S is a semiprime ideal of R.

ii. If $a \in R$ such that $aRa \subseteq S$, then $a \in S$.

iii. If (a) is principal ideal in R such that $(a)^2 \subseteq S$, then $a \in S$.

iv. If U is right (or left) ideal in R such that $U^2 \subseteq S$, then $U \subseteq S$.

Remark 1.2.4. i. The intersection of any set of semiprime ideals is semiprime.

ii. Any prime ideal is a semiprime ideal.

iii. The converse of (ii) is not true in general. Indeed, considering the ring $\mathbb{Z}$ of integers, the ideal $(6) = (2) \cap (3)$ is semiprime by (i), but it is not prime.

Definition 1.2.4. (**Idempotent element**). An element e in a ring R is called *idempotent* if $e^2 = e$. It is obvious that zero is an idempotent element of every ring. Moreover, if R contains unity 1, then 1 is also idempotent.

Remark 1.2.5. For any idempotent e in a ring R, $e + ex - exe$ and $e + xe - exe$ for all $x \in R$ are also idempotents.

Definition 1.2.5. (**Nilpotent element**). An element x in a ring R is said to be a *nilpotent* element if there exists a positive integer n such that $x^n = 0$. If such a positive integer exists, then the smallest of all integers $n \geq 1$ satisfying the condition is called the *nilpotency index* of x.

Remark 1.2.6. For an idempotent $e \in R$ and any $x \in R$, the elements $ex - exe$ and $xe - exe$ are nilpotent in R.

Remark 1.2.7. If R is a prime ring without non-zero nilpotent elements then R does not zero divisor. In fact, if $ab = 0$, then $(ba)^2 = (ba)(ba) = b(ab)a = 0$. By hypothesis $ba = 0$. Furthermore, if $ab = 0$, then $(ab)x = 0$. As we have mentioned above, this implies that $a(bx) = 0$ for all $x \in R$, i.e., $(bx)a = 0$ for all $x \in R$ and hence $bRa = 0$. Since R is prime, either $a = 0$ or $b = 0$ i.e., R does not have zero divisors.

Remark 1.2.8. It is trivial that the zero element of a ring is nilpotent, that is, the nilpotency index of an element $x \in R$ is 1 if and only if $x = 0$. Moreover, every nilpotent element is a zero divisor. Indeed, if $a \neq 0$, and n is the smallest positive integer such that $a^n = 0$, then $n > 1$ and $a(a^{n-1}) = 0$ with $a^{n-1} \neq 0$.

Definition 1.2.6. (**Nilpotent ideal**). An ideal A of R is called a *nilpotent ideal* if $A^n = 0$ for some positive integer n. If every element of A is nilpotent, then A is said to be a *nil ideal.*

Example 1.2.1. Let M be the ring of all 2×2 upper triangular matrices over integers. Then the ideal generated by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent.

Remark 1.2.9. Every nilpotent ideal is necessarily a nil ideal, but the converse is not true in general.

Definition 1.2.7. (**Semiprime ring**). A ring R is said to be *semiprime* if it has no nonzero nilpotent ideal.

Remark 1.2.10. In a ring R, the following conditions are equivalent:

i. R is semiprime.

ii. 0 is a semiprime ideal.

iii. If A is an ideal of R such that $A^2 = 0$, then $A = 0$.

iv. If $a \in R$, $aRa = 0$, then $a = 0$.

Remark 1.2.11. A prime ring is necessarily a semiprime ring but the converse need not be true in general.

Example 1.2.2. i. The ring $\mathbb{Z}_6$ of residue classes modulo 6 is a semiprime ring but not a prime ring.

ii. Let R_1, R_2 are prime rings, then $R_1 \oplus R_2$ is semiprime ring but not prime.

Definition 1.2.8. (**Characteristic of a ring**). Let R be a ring. If there exists a positive integer n such that $nx = 0$ for all $x \in R$, then the smallest positive integer with this property is called the *characteristic* of the ring R and is denoted as $\text{char}(R) = n$. If no such positive integer exists, then R is said to be of *characteristic zero*.

Example 1.2.3. Consider the ring of integers modulo n i.e., $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \bar{2}, ..., n \bar{-} 1\}$, then characteristic of $\mathbb{Z}_n$ will be n.

Example 1.2.4. The ring of integers, rational numbers and real numbers are standard examples of rings having characteristic zero. On the other hand let $P(X)$ be the set of all subsets of a given non-empty set X. If $A, B \in P(X)$, we define AB to be $A \cap B$ and $A + B$ is defined to be the symmetric difference of A and B i.e., $A \Delta B = (A \backslash B) \cup (B \backslash A)$, then with respect to these addition and multiplication $P(X)$ is a commutative ring with identity. Moreover, since $2A = A \Delta A = \emptyset$ for every subset $A \subseteq X, P(X)$ is a ring of characteristic 2.

Remark 1.2.12. The characteristic of an integral domain is either zero or a prime.

Definition 1.2.9. (**Torsion element**). An element $x \in R$ is called *torsion element* if there is n a positive integer such that $nx = 0$ (it has finite order in the additive subgroup of R).

Remark 1.2.13. Let R be a prime ring. If there exist a nonzero torsion element in R, then R has finite characteristic and it is prime.

Proof. Assume that $0 \neq a \in R$ is torsion, then $pa = 0$ for some integer $p \geqslant 1$, therefore for all $r \in R$, we have $paRr = 0 = aRpr$. Primeness of R gives $pr = 0$ for all $r \in R$. Then R has positive characteristic. $\square$

Notice that 1.2.13 does not hold if R is semiprime as shown by the following example.

Example 1.2.5. Define $R = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It is easy to check that R is semiprime ring, we have $2.(\bar{0}, \bar{1}) = (\bar{0}, \bar{0})$, then $(\bar{0}, \bar{1})$ is 2-torsion element in R. However characteristic of R is zero.

Definition 1.2.10. (**Torsion free ring**). A ring R is *torsion free* if 0 is its only torsion element.

Definition 1.2.11. (**Simple ring**). A ring $R \neq 0$ is said to be *simple* if the only ideals of R are 0 and R.

Definition 1.2.12. (**Center of a ring**). The *center* of a ring R is the set of elements of R which commute with every element of R. It is denoted by $Z(R)$ i.e., $Z(R) = \{x \in R \mid xr = rx \text{ for all } r \in R\}$.

Remark 1.2.14. i. The center of a prime ring does not contain zero divisors.

ii. A ring R is commutative if and only if $Z(R) = R$.

Remark 1.2.15. The center of a semiprime ring does not contain nonzero nilpotent element.

Proof. Let x be a non-zero nilpotent element of R such that $x \in Z(R)$. Suppose that index of nilpotency is n. If $n = 2$, then $x^2r = 0$ for all $r \in R$ i.e., $x(xr) = 0$ gives $xrx = 0$. This implies that $x = 0$. If $n > 2$ then $2n - 2 > n$ and we have $(x^{n-1})^2 = 0$ i.e. $(x^{n-1})^2r = 0$ for all $r \in R$. This implies $x^{n-1}rx^{n-1} = 0$. Since R is semiprime $x^{n-1} = 0$ a contradiction. □

Definition 1.2.13. (**Centralizer of a set**). Let S be a nonempty subset of R. Then the *centralizer* of S in R is defined by $C_R(S) = \{x \in R \,|\, sx = xs \text{ for all } s \in S\}$.

Lemma 1.2.6. *([48, Lemma 1.1.5]). Let R be a semiprime ring and let I be a right ideal of R, then $Z(I) \subseteq Z(R)$.*

Proof. If $a \in Z(I)$ and $x \in R$ then, since $ax \in I$, $a(ax) = (ax)a$, that is $a(ax - xa) = 0$. If $r \in R$, then $a(a(xr) - (xr)a) = 0$. Moreover, $a(xr) - (xr)a = (ax - xa)r + x(ar - ra)$ consequently $ax(ar - ra) = 0$ for all $x, r \in R$. But this gives $(ar - ra)R(ar - ra) = 0$. Since R is semiprime, we conclude that $ar - ra = 0$ for all $r \in R$, hence $a \in Z(R)$. □

Definition 1.2.14. (**Annihilator**). If M is a subset of a ring R, then the *right annihilator* of M denoted as $A_r(M)$ is the totality of all $r \in R$, such that

$$A_r(M) = \{r \in R \mid \ mr = 0 \text{ for all } m \in M\}.$$

Similarly, the *left annihilator* of M denoted as $A_l(M)$ is the set of all $r \in R$ such that

$$A_l(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}.$$

The intersection $Ann_R(M) = A_r(M) \cap A_l(M)$ is called an *annihilator* of M in R.

Remark 1.2.16. The right annihilator of a nonzero right ideal of a prime ring is zero.

Remark 1.2.17. If I is an ideal of a semiprime ring R. Then

i. $Ann_R(I) = r_R(I) = l_R(I)$.

ii. $I \cap A_r(I) = 0$.

Definition 1.2.15. (**Martindale ring of quotients**). Let R be a prime ring and $\Im$ be the set of all pairs (U, f) where $U \neq 0$ is an ideal of R and $f : U \to R$ is a homomorphism of right R-modules from U to R. Define a relation $\sim$ on $\Im$ by $(U, f) \sim (V, g)$ if $f = g$ on some ideal $W \neq 0$ of R where $W \subset U \cap V$. The primeness of R trivially implies that $\sim$ is an equivalence relation on $\Im$. Let Q be the set of equivalence classes of $\Im$. Denote the equivalence class determined by (U, f) as $\tilde{f}$. For $\tilde{f} = cl(U, f)$, $\tilde{g} = cl(V, g) \in Q$, we can define an addition $\tilde{f} + \tilde{g} = cl(U \cap V, f + g)$ and a product $\tilde{f}.\tilde{g} = cl(UV, fg)$ it is easy to verify that two operations are will defined. Thus, Q forms an associative ring with unit element $\tilde{1} = cl(R, id)$ will respect to above defined operations known as *Martindale (or simply) ring of quotients*.

Remark 1.2.18. Q satisfies the following properties:

- R can be isomorphically embedded in Q.
- If $0 \neq q \in Q$ then there exists an ideal $U \neq \{0\}$ of R such that $\{0\} \neq qU \subset R$.
- Primeness of R gives Q is also prime.

Definition 1.2.16. (**Extended centroid**). The center C of Q is known as the *extended centroid* of R.

Remark 1.2.19. The extended centroid is a field.

Definition 1.2.17. (**Central closure**). Let R be a prime ring and C be its extended centroid, then subring generated by R and C is called *central closure* of R in Q and denoted by RC or R_C.

Definition 1.2.18. (**Lie and Jordan products**). Given any associative ring R, one can consider two new operations in R as follows:

- for all $x, y \in R$, the *Lie product* $[x, y] = xy - yx$.
- for all $x, y \in R$, the *Jordan product* $x \circ y = xy + yx$.

Definition 1.2.19. (**Lie and Jordan subrings**). An additive subgroup A of R is said to be a *Lie* (resp. *a* Jordan) *subring* of R if $[a, b] \in A$ (resp. $a \circ b \in A$) for all $a, b \in A$.

Definition 1.2.20. (**Lie and Jordan ideals**). An additive subgroup U of R is said to be a *Lie* (resp. a *Jordan*) ideal of R if $[U, R] \subseteq U$ (resp. $U \circ R \subseteq U$).

Example 1.2.7. Let $R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_2\}$. Then it can be easily seen that $U = \{\begin{pmatrix} a & b \\ c & a \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2\}$ is a Lie ideal of R and $J = \{\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2\}$ is a Jordan ideal of R.

Definition 1.2.21. (**Square closed Lie ideal**). A Lie ideal U of R is said to be a *square closed Lie ideal* of R if $u^2 \in U$ for all $u \in U$.

Throughout this thesis, we will make extensive use of the basic commutator identities:

Remark 1.2.20. For any $x, y, z \in R$, the following identities are obvious:

$$[x, yz] = [x, y]z + y[x, z]; [xy, z] = x[y, z] + [x, z]y. \tag{1.1}$$

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z. \tag{1.2}$$

$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]. \tag{1.3}$$

$$[[x, y], z]] + [[y, z], x]] + [[z, x], y]] = 0 \quad (\text{Jacobi's identity}) \tag{1.4}$$

1.3 Derivations and generalizations

Definition 1.3.1. (**Derivation**). An additive map $d : R \to R$ is said to be a *derivation* of the ring R if it satisfies $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

Example 1.3.1. The most natural example of a non-trivial derivation is the usual derivation of polynomials in the ring $F[x]$ over a field F.

Definition 1.3.2. (**Inner derivation**). Let a be a fixed element of R. If we define the map $I_a : R \to R$ by $I_a(x) = [x, a]$, for all $x \in R$, then it can be easily proved that I_a is a derivation of R. This map is called *Inner derivation* of R determined by a.

Remark 1.3.1. It is obvious that every inner derivation on a ring R is a derivation. But the converse is not true in general.

Example 1.3.2. Let $R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}\}$ be the ring of 2×2 matrices over $\mathbb{Z}$ the ring of integers. Define a map $d : R \to R$ as follows:

$$d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$$

Then it can be verified that d is a derivation but not an inner derivation on R.

Remark 1.3.2. The set of all derivations of the ring R is denoted by $DerR$. This set is closed with respect to the commutator operation i.e., if d_1, d_2 are derivations of R, then $[d_1, d_2]$ is also a derivation. Therefore, $DerR$ is a Lie subring of $End(R)$.

Remark 1.3.3. If d is a derivation on R and $r \in Z(R)$, then $d(r) \in Z(R)$.

Definition 1.3.3. (**Jordan derivation**). An additive map $d : R \to R$ is said to be a *Jordan derivation* of the ring R if it satisfies $d(x^2) = d(x)x + xd(x)$ for all $x \in R$.

Remark 1.3.4. Every derivation on a ring R is a Jordan derivation but the converse is not true in general as the following example shows.

Example 1.3.3. Let R be a ring and $a \in R$ such that $xax = 0$ for all $x \in R$ and $xay \neq 0$ for some $y \in R$, $(y \neq x)$. Define a map $d : R \to R$ by $d(x) = ax$. Then it can be verified that d is a Jordan derivation but not a derivation.

Definition 1.3.4. (**(σ,τ)-derivation**). Let $\sigma, \tau : R \to R$ be two maps. Then an additive map $d : R \to R$ is said to be a *(σ,τ)-derivation* on R if it satisfies $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$.

Example 1.3.4. Let $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}\}$. If we define the maps $d, \sigma, \tau : R \to R$ by $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $\tau\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Then d is a (σ,τ)-derivation of R.

Definition 1.3.5. (**Generalized inner derivation**). An additive map $F : R \to R$ is called a *generalized inner derivation* if $F(x) = ax + xb$ for some $a, b \in R$. It is straight forward to noting that if F is a generalized inner derivation, then for any $x, y \in R$, $F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y)$ where I_b is an inner derivation.

Definition 1.3.6. (**Generalized derivation**). An additive map $F : R \to R$ is said to be a *generalized derivation or left generalized derivation* of R if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

Remark 1.3.5. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$ for all $x \in R$ is a generalized derivation, where c is a fixed element of R and d is a derivation of R.

Definition 1.3.7. (**Right generalized derivation**). An additive map $F : R \to R$ is said to be a *right generalized derivation* of R if there exists a derivation $d : R \to R$ such that $F(xy) = xF(y) + d(x)y$ for all $x, y \in R$.

Example 1.3.5. Let S be any ring and $R = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S\}$. Define $F : R \to R$ such that $F(x) = 2e_{11}x - xe_{11}$. Then it can be easily seen that F is a generalized derivation with associated derivation $d(x) = e_{11}x - xe_{11}$.

Remark 1.3.6. It easy to see that every derivation is a generalized derivation but the converse is not true in general. The following example shows a generalized derivation that is not a derivation.

Example 1.3.6. Let $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2\}$. If we define the maps $F, d : R \to R$ such that $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation of R with associated derivation d but not a derivation of R.

Definition 1.3.8. (**Jordan generalized derivation**). An additive map $F : R \to R$ is said to be a *Jordan generalized derivation* if there exists a Jordan derivation d such that $F(x^2) = F(x)x + xd(x)$ for all $x \in R$.

Remark 1.3.7. Clearly, every generalized derivation is a Jordan generalized derivation but converse need not be true in general.

Example 1.3.7. Let S be a ring such that the square of each element in S is zero, but the product of some elements in S is nonzero. Next, let $R = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in S\}$. Define a map $F : R \to R$ such that $F\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. Then with derivation $d = 0$, it can be easy see that $F(r^2) = F(r)r$ for all $r \in R$, i.e. F is Jordan generalized derivation. Take $r_1 = \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $r_1 = \begin{pmatrix} x_2 & 0 \\ 0 & 0 \end{pmatrix}$ such that $x_1 \neq x_2$, then $F(r_1r_2) \neq F(r_1)r_2$ i.e, F is not generalized derivation associated to $d = 0$.

Definition 1.3.9. (**Centralizer map**). An additive map $f : R \to R$ is said to be a *left centralizer (right centralizer)* if $f(xy) = f(x)y$ ($f(xy) = xf(y)$) for all $x, y \in R$. It is called a *centralizer* if f is both left and right centralizer.

Definition 1.3.10. (**Centralizing and Commuting maps**). Let S be an additive subgroup of R. An additive map $f : S \to S$ is said to be *centralizing* on S if $[f(x), x] \in Z(R)$ for all $x \in S$. Furthermore, f is said to be *commuting* on S if $[f(x), x] = 0$ for all $x \in S$.

Example 1.3.8. Let us consider $R = \{\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}\}$. Now, define $T : R \to R$ such that $T\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, then T is a centralizer map.

Definition 1.3.11. (**Commutativity preserving map**). Let R and S be rings. A map $f : R \to S$ is said to be *commutativity preserving map* if for all $x, y \in R$, whenever $[x, y] = 0$ implies $[f(x), f(y)] = 0$.

Definition 1.3.12. (**Strong commutativity preserving (SCP) map**). Let S be a nonempty subset of a ring R. A map $f : R \to R$ is said to be *strong commutativity preserving (SCP)* on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$.

Definition 1.3.13. (**Symmetric and trace maps**). A map $B : R \times R \to R$ is said to be *symmetric* if $B(x, y) = B(y, x)$ for all $x, y \in R$. The map $f : R \to R$ defined by $f(x) = B(x, x)$ is called the *trace* of B.

Remark 1.3.8. If B is a symmetric map which is also biadditive (i.e., additive in both arguments), the trace f of B satisfies that $f(x+y) = f(x)+f(y)+2B(x, y)$, for all $x, y \in R$, hence f is not an additive map.

Definition 1.3.14. (**Symmetric biderivation**). A symmetric biadditive map $B : R \times R \to R$ is said to be a *symmetric biderivation* if $B(xy, z) = B(x, z)y + xB(y, z)$ is fulfilled for all $x, y, z \in R$.

Example 1.3.9. Typical examples are maps of the form $(x, y) \to \lambda[x, y]$ where $\lambda \in C$ where C is extended centroid, which called *inner biderivations*.

Definition 1.3.15. (**Symmetric generalized biderivation**). A symmetric biadditive map $G : R \times R \to R$ is said to be a *symmetric generalized biderivation* if there exists a symmetric biderivation $B : R \times R \to R$ such that $G(xy, z) = G(x, z)y + xB(y, z)$ for all $x, y, z \in R$.

Example 1.3.10. Let R be a ring. If B is any symmetric biderivation of R and $\tau : R \times R \to R$ is a symmetric biadditive map such that $\tau(x, yz) = \tau(x, y)z$ for all $x, y, z \in R$, then $B + \tau$ is a symmetric generalized biderivation of R associated to B.

Chapter 2

Generalized Reverse Derivations on Semiprime Rings

2.1 Introduction

The notion of reverse derivation arises in one early paper of Herstein in [67], when he studied Jordan derivations on prime associative rings. This notion is related to some generalizations of derivations. A reverse derivation is an additive map d from a ring R into itself satisfying $d(xy) = d(y)x + yd(x)$, for all $x, y \in R$. So, every reverse derivation is a Jordan derivation (but the converse is not true in general). In the anticommutative case, every reverse derivation is an antiderivation and every antiderivation is a reverse derivation. Reverse derivations in prime Lie and prime Malcev algebras were studied by N. Hopkins and V. Filippov. In those papers, some examples of non-zero reverse derivations for the 3-dimensional simple Lie algebra sl_2 were found, (see [78]), and prime Lie algebras admitting a non-zero reverse derivation were characterized by Filippov (see [59, 60]). In particular, Filippov proved that every prime Lie algebra, admitting a non-zero reverse derivation is a polynomial algebra. Filippov also described all reverse derivations of prime Malcev algebras(see [61]). The super case of reverse derivations (anti-superderivations) of simple Lie superalgebras was studied by I. Kaygorodov in [89] and [90]. He proved, that every reverse superderivation of simple finite-dimensional Lie superalgebra over an algebraically closed field of characteristic zero is a zero map. After that, he proved that every reverse r-generalized (or l-generalized) derivation of simple (non-Lie) Malcev algebra is a zero map (see [91]).

In his paper [67], Herstein showed that if R is a prime ring, and d a nonzero reverse derivation of R, then R is a commutative integral domain and d is a derivation. Later, in [118] Samman and Alyamani extended the result by Herstein to semiprime rings proving that, if R is a semiprime ring, then a reverse derivation is just a derivation from R to its center.

Generalized derivations were defined by Brešar [37] in 1991. An additive map $f : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. One may observe that the concept of generalized derivation includes both the concept of derivation, and the one of left multiplier (when $d = 0$). Gölbaşi and Kaya [64] distinguish between l-generalized derivation associated to a derivation d (Brešar generalized derivations) and r-generalized derivations associated to a derivation d, additive map $F : R \to R$ satisfying: $F(xy) = d(x)y + xF(y)$ for all $x, y \in R$.

In this chapter we generalize the notion of reverse derivation introducing generalized reverse derivations:

Definition 2.1.1. Let R be a ring and d a reverse derivation of R and $F : R \to R$ an additive map is said to be a *l-generalized reverse derivation* of R associated to d if

$$F(xy) = F(y)x + yd(x), \quad \text{for all}\, x, y \in R.$$

F is said to be a *r-generalized reverse derivation* associated to d if

$$F(xy) = d(y)x + yF(x), \quad \text{for all}\, x, y \in R.$$

The main purpose of this chapter is to extend the above mentioned results to generalized reverse derivations. If R is a semiprime ring, I is an ideal of R and $F : I \to R$ is a l-generalized reverse derivation (r-generalized reverse derivation), we will show that F is r-generalized derivation (l-generalized derivation) and applies I into $C_R(I)$. In particular, R contains a non zero central ideal.

Generalized Jordan derivations are considered by Wei and Xiao in [128]. A generalized Jordan derivation of a ring R is a map $f : R \to R$ that satisfies $f(x^2) = f(x)x + xd(x)$ for all $x \in R$, for some Jordan derivation d of R. In Theorem 2.7 of [128] authors proved that any generalized Jordan derivation of a 2-torsion free semiprime ring R is a generalized derivation. Clearly, the notion of r-generalized Jordan derivation can also be considered. A r-generalized Jordan derivation is a map $g : R \to R$ that satisfies $g(x^2) = d(x)x + xg(x)$ for d a Jordan derivation of R. The proof of the above mentioned Theorem 2.7 in [128] can be adapted to prove the same result for a r-generalized Jordan derivation, that is, any r-generalized Jordan derivation of a 2-torsion free semiprime ring R is a r-generalized derivation. Using this extended version of Theorem 2.7 we will prove in this chapter that, in case of 2-torsion free semiprime rings, the notions of r-generalized reverse, l-generalized reverse , r-generalized and l-generalized derivations coincide.

Results in this chapter have been published in [1].

2.2 Preliminaries and examples

The following lemmas will be widely used in our results.

Lemma 2.2.1. [26, Theorem 3]. *Let R be a semiprime ring and I a nonzero left ideal. If R admits a nonzero derivation d which is centralizing on I, then R contains a nonzero central ideal.*

Lemma 2.2.2. [128, Theorem 2.7]. *Let R be a 2-torsion free semiprime ring. Then any generalized Jordan derivation on R is a generalized derivation(left or right).*

Proposition 2.2.3. [26, Fact IV]. *In a prime ring, the centralizer of any nonzero one-sided ideal is equal to the center of R. In particular, if R has a nonzero central ideal, then R is commutative.*

The following examples explore possible relationships between l-generalized reverse derivations, r-generalized reverse derivations, l-generalized derivations and r-generalized derivations.

Example 2.2.4. Let S be a ring and consider the ring $R = \{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in S\}$. Let us define maps $F : R \to R$ and $d : R \to R$ as follows:

$F(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $d(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & a-c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

It is easy to check that d is both a reverse derivation and a derivation, F is l-generalized reverse derivation and r-generalized reverse derivation associated to d. F is also l-generalized derivation and r-generalized derivation associated to d. As a matter of fact, F is also a reverse derivation.

Remark 2.2.1. A map F can be (reverse) l-generalized (r-generalized) derivation with respect to two different reverse derivations. Indeed, in Example 2.2.4 above F is a (reverse) l-generalized (r-generalized) derivation with respect to F and d. But if the ring R is semiprime, then the reverse derivation associated to a (reverse) l-generalized (r-generalized) derivation is unique.

Example 2.2.5. Consider the ring R as in Example 2.2.4. Define maps $F : R \to R$ and $d : R \to R$ as follows:
$F(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $d(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & a & c-a \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$.

It is easy to check that d is a derivation and a reverse derivation and F is l-generalized derivation with respect to d. However F is not r-generalized derivation with respect to d. Furthermore, F is neither l-generalized reverse derivation not r-generalized reverse derivation with respect to d.

The next two examples will show that a l-generalized reverse derivation (r-generalized reverse derivation) with respect to a reverse derivation d that is also a derivation is not necessarily a r-generalized derivation (l-generalized derivation) with respect to d.

Example 2.2.6. Consider the ring $R = \{\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix} | a, b, c, d, e \in \mathbb{R}\}$, where

$\mathbb{R}$ is the set of all real numbers. Define maps, $F : R \to R$ and $d : R \to R$ as follows:
$F(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 & b+e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $d(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 & b-d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Then, d is a derivation and a reverse derivation, and F is a l-generalized reverse derivation associated to d, but not r-generalized derivation associated to d.

Example 2.2.7. Consider the ring R as in Example 2.2.6. Define maps $F : R \to R$ and $d : R \to R$ as follows:
$F(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $d(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 & b-d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

It is easy to verify that F is r-generalized reverse derivation associated to d, but not l-generalized derivation associated to d.

2.3 Characterization of generalized reverse derivations on ideals

Theorem 2.3.1. *Let R be a semiprime ring and I a nonzero ideal of R. If there exists $F : I \to R$ a l-generalized reverse derivation associated to a nonzero reverse derivation d on I, then $d(I), F(I) \subseteq C_R(I)$, d is a derivation on I and F is r-generalized derivation with respect to d on I. And conversely.*

Proof. Assume that F is a l-generalized reverse derivation on I. Then

$$F(u^2v) = F(v)u^2 + vd(u^2) = F(v)u^2 + v(d(u)u + ud(u)) \quad \text{for all}\, u, v \in I.$$

That is

$$F(u^2v) = F(v)u^2 + vd(u)u + vud(u) \quad \text{for all}\, u, v \in I. \tag{2.1}$$

Also,

$$F(u(uv)) = F(uv)u + uvd(u) = (F(v)u + vd(u))u + uvd(u) \quad \text{for all}\, u, v \in I.$$

Hence,

$$F(u(uv)) = F(v)u^2 + vd(u)u + uvd(u) \quad \text{for all}\, u, v \in I. \tag{2.2}$$

From (2.1) and (2.2), we get

$$[u, v]d(u) = 0 \quad \text{for all}\, u, v \in I. \tag{2.3}$$

Replacing v by rv in (2.3) and using (2.3) we have

$$[u, r]vd(u) = 0 \quad \text{for all}\, u, v \in I, r \in R. \tag{2.4}$$

Replacing v by $d(u)s[u, r]$ in (2.4) yields

$$[u, r]d(u)s[u, r]d(u) = 0 \quad \text{for all}\, u \in I, r, s \in R. \tag{2.5}$$

Since R is semiprime, by (2.5) we get

$$[u, r]d(u) = 0 \quad \text{for all}\, u \in I, r \in R. \tag{2.6}$$

A linearization of (2.6) leads to

$$[w, r]d(u) + [u, r]d(w) = 0 \quad \text{for all}\, u, w \in I, r \in R.$$

Thus,

$$[w, r]d(u) = -[u, r]d(w) \quad \text{for all}\, u, w \in I, r \in R. \tag{2.7}$$

Replacing v by $d(w)s[w, r]$ in (2.4) and using (2.7) we get

$$0 = [u, r]d(w)s[w, r]d(u) = -[u, r]d(w)s[u, r]d(w) \quad \text{for all}\, u, w \in I, r, s \in R.$$

That is,

$$[u, r]d(w)R[u, r]d(w) = (0) \quad \text{for all}\, u, w \in I, r \in R. \tag{2.8}$$

Since R is semiprime, then

$$[u, r]d(w) = 0 \quad \text{for all}\, u, w \in I, r \in R. \tag{2.9}$$

Replacing r by rt in (2.9) we have

$$[u,r]td(w) = 0 \quad \text{for all}\, u, w \in I, r, t \in R. \tag{2.10}$$

Put $r = d(w)$ and replace t by tu in (2.10) we get $[u, d(w)]tud(w) = 0$ for all $w, u \in I$, $t \in R$. Multiply from right (2.10) by u, we obtain $[u, d(w)]td(w)u = 0$. Subtracting the last two relations, we get $[u, d(w)]R[u, d(w)] = (0)$ for all $w, u \in I$. From the semiprimeness of R its follows that $[u, d(w)] = 0$ for all $w, u \in I$, that is $d(I) \subseteq C_R(I)$. Thus $d(xy) = d(y)x + yd(x) = d(x)y + xd(y)$ for all $x, y \in I$, what proves that d is a derivation on I. On the other hand, since F is l-generalized reverse derivation, we have

$$F(uv^2) = F(v^2)u + v^2d(u) = (F(v)v + vd(v))u + v^2d(u) \quad \text{for all}\, u, v \in I.$$

That is

$$F(uv^2) = F(v)vu + vd(v)u + v^2d(u) \quad \text{for all}\, u, v \in I. \tag{2.11}$$

Also,

$$F((uv)v) = F(v)uv + vd(uv) = F(v)uv + v(d(v)u + vd(u)) \quad \text{for all}\, u, v \in I.$$

$$F((uv)v) = F(v)uv + vd(v)u + v^2d(u) \quad \text{for all}\, u, v \in I. \tag{2.12}$$

Combining (2.11) and (2.12), we get

$$F(v)[u, v] = 0 \quad \text{for all}\, u, v \in I. \tag{2.13}$$

Using the same techniques that we have used above we get that $F(I) \subseteq C_R(I)$. Hence $F(xy) = F(y)x + yd(x) = xF(y) + d(x)y$ for all $x, y \in I$, and F is r-generalized derivation with respect to the derivation d. The converse is trivial. □

Theorem 2.3.2. *Let R be a semiprime ring and I a nonzero ideal of R. If there exists $F : I \to R$ a r-generalized reverse derivation associated to a nonzero reverse derivation d on I, then $d(I), F(I) \subseteq C_R(I)$, d is a derivation on I and F is l-generalized derivation with respect to d on I. And conversely.*

Proof. The proof follows the same lines than the one of Theorem 2.3.1. □

Corollary 2.3.3. *Let R be a semiprime ring. If there is $F : R \to R$ a l-generalized reverse (or r-generalized) derivation associated to a nonzero reverse derivation d on R, then R contains a non-zero central ideal.*

Proof. By Theorem 2.3.1 (or Theorem 2.3.2), we have $d(R) \subseteq Z(R)$, that is $[d(x), y] = 0$ for all $x, y \in R$. Then d is centralizing on R. By Lemma 2.2.1(with $I = R$), R contains a non-zero central ideal. □

The following corollary gives the relationship between l-generalized reverse derivations and l-generalized derivations.

Corollary 2.3.4. *Let R be 2-torsion free semiprime ring. If there exists $F : R \to R$ a l-generalized reverse (r-generalized) derivation associated to a nonzero reverse derivation d of R, then F is r-generalized reverse (l-generalized) derivation with respect to d. Furthermore, d is a derivation and F is l-generalized (r-generalized) derivation related to d.*

Proof. Let F be a l-generalized reverse derivation. Then by Theorem 2.3.1 d is a derivation, $F(I), d(I) \subseteq C_R(I)$. By hypotheses, $F(xy) = F(y)x + yd(x)$ for all $x, y \in R$. Doing $y = x$, we have $F(x^2) = F(x)x + xd(x)$ for all $x \in R$, then F is generalized Jordan derivation on R. By using Lemma 2.2.2, F is l-generalized derivation on R. Using the converse of theorem 2.3.2, we know that F is a reverse r-generalized derivation on R. The case of r-generalized reverse derivation follows the same lines. □

Corollary 2.3.5. *A map $d : I \to R$, where I is a two-sided ideal of a semiprime ring R, is a reverse derivation if and only if it is a derivation and it is centralizing on I, i.e. $d(I) \subseteq C_R(I)$.*

Definition 2.3.1. A map F is a **generalized reverse derivation**, if it is a l- and r-generalized reverse derivation.

From Theorems 2.3.1 and 2.3.2, it follows.

Corollary 2.3.6. *A map F on a semiprime ring R is a generalized reverse derivation if and only if it is a central generalized derivation.*

Corollary 2.3.7. *Let R be a prime ring and I a nonzero ideal of R. If there exists $F : I \to I$ a l-generalized reverse (r-generalized) derivation on I associated to a nonzero reverse derivation $d : I \to I$, then R is commutative.*

Proof. Using Corollary 2.3.3, we know that I (that clearly is also prime) contains a nonzero central ideal. By Proposition 2.2.3, I is a commutative ring. But Proposition 2.2.3 implies that $I \subseteq C_R(I) = Z(R)$, so I is a nonzero central ideal of R and R is commutative. □

The following example shows that the semiprimeness condition for the ring R is not superfluous.

Example 2.3.8. Consider the ring R in Example 2.2.6, and let

$$I = \{\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R}\} \text{ be an ideal of } R.$$

Define $F : R \to R$ and $d : R \to R$ as follows:

$$F(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 & -c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; d(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to see that d is a nonzero reverse derivation and a derivation on I, F is r-generalized reverse derivation on I, but not l-generalized derivation, and $d(I), F(I) \nsubseteq C_R(I)$.

Chapter 3

Generalized Derivations on Lie Ideals in Semiprime Rings

3.1 Introduction and preliminaries

In [114], Posner proved that if R is a prime ring with characteristic different from 2 and d_1d_2, the composition of two derivations d_1 and d_2, is a derivation, then at least one of them must be zero. Furthermore, he proved that if a prime ring R admits a nonzero derivation d such that $ad(a) - d(a)a \in Z(R)$ for all $a \in R$, then R is commutative. This result is no longer true for semiprime rings. That's why Breŝar and Vukman introduced the notion of orthogonal derivations, and proved that in a semiprime 2-torsion free ring R two derivations d_1 and d_2 are orthogonal if and only if d_1d_2 is a derivation. In particular, $d^2 = 0$ implies $d = 0$. The result of Breŝar and Vukman is still true assuming only orthogonality over a nonzero ideal I of R. Herstein [69] proved that given a semiprime 2-torsion free ring R and an inner derivation d_t, if $d_t^2(U) = 0$ for a Lie ideal U of R then $d_t(U) = 0$. Carini [45] extended this result for an arbitrary derivation d, proving that $d^2(U) = 0$ implies $d(U) \subseteq Z(R)$.

Coming back to prime rings R, char$R \neq 2$, it was proved in [101] that if $0 \neq d \in Der(R)$ satisfies $d(R) \subseteq Z(R)$, then R is commutative. If $d^2(U) \subseteq Z(R)$, then $U \subseteq Z(R)$, where U is a nonzero Lie ideal of R. The same result is still true if we consider a generalized derivation instead of a derivation, as it was proved by Dalgin in [54]. And Gölbaşi and Koç [65] proved that if (F, d), (G, g) are, respectively, left and right generalized derivations of R satisfying $F(u)v = uG(v)$ for all $u, v \in U$, where U is a Lie ideal of R, then $U \subseteq Z(R)$.

In Section 3.2 we extend the above mentioned results to semiprime rings and generalized derivations. We will prove for a semiprime 2-torsion free ring R, a non-central Lie ideal U of R and (F, d) a left generalized derivation that $F^2(U) = (0)$ implies $d^3(U) = (0)$, and $(d^2(U))^2 = 0$. Furthermore, if $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$, then $d(U) = F(U) = (0)$ and $d(R), F(R) \subseteq C_R(U)$.

In Section 3.3, we extend the results of Gölbaşi and Koç [65] to semiprime rings, and consequently it is shown that if $(F, d), (G, g)$ are, respectively, left and right generalized derivations that satisfying $G(u)v = uF(v)$ for all $u, v \in U$, then $d(U), g(U) \subseteq C_R(U)$.

These results extends and unify some previous results by Herstein [69], Carini [45], Lee and Lee [100], Gölbaşi and Koç [65], and Dalgin [54].

The results of this chapter have been published in [2].

Next, we will use the following lemmas in our results.

Lemma 3.1.1. [77, Corollary 2.1] *Let R be a semiprime 2-torsion free ring, U a Lie ideal of R, $U \not\subseteq Z(R)$, and $a, b \in U$. Then*
(1) If $aUa = (0)$, then $a = 0$.
(2) If $aU = (0)$ (or $Ua = (0)$), then $a = 0$.
(3) If U is square-closed and $aUb = 0$, then $ab = 0$ and $ba = 0$.

Lemma 3.1.2. [45, Lemma 1] *Let R be a semiprime 2-torsion free ring with a derivation d and U a Lie ideal of R. If $d^2(U) = 0$, then $d(U) \subseteq Z(R)$.*

Lemma 3.1.3. [34], Lemma 2] *Let R be a prime ring and U a Lie ideal of R. If $U \not\subseteq Z(R)$, then $C_R(U) = Z(R)$.*

Lemma 3.1.4. [5, Lemma 3] *Let R be a semiprime 2-torsion free ring and I a nonzero ideal of R. If d is a nonzero derivation of R such that $Id^2(I) = (0)$, then $I \subseteq Z(R)$.*

Lemma 3.1.5. [34, Lemmas 6 and 11] *Let R be a prime ring with $charR \neq 2$, d a nonzero derivation of R and U a Lie ideal of R.*
(i) If $d(U) \subseteq Z(R)$, then $U \subseteq Z(R)$.
(ii) If $d^3(U) = (0)$, then $d^3 = (0)$.

Lemma 3.1.6. [18, Theorem 2.3.2] *Let R be a semiprime ring, $Q = Q_{mr}(R)$, ${}_RU_R \subseteq {}_RQ_R$ a subbimodule of Q and $f : {}_RU_R \to {}_RQ_R$ a homomorphism of bimodules. Then there exists an element $\lambda \in C$ such that $f(u) = \lambda u$ for all $u \in U$.*

3.2 The case $F^2(U) = (0)$

In what follows R denotes a ring (associative but not necessarily commutative) and Q its Martindale quotient ring. The center C of Q is called the extended centroid of R (see [71] and [80] for details).

We will use the following lemma in our results.

Lemma 3.2.1. *Let R be a semiprime ring, U a Lie ideal of R, $U \nsubseteq Z(R)$. If $a \in R$ satisfies $a[x,r] = 0$ (resp. $[x,r]a = 0$) for every element $x \in U$, $r \in R$, then $a \in C_R(U)$.*

Proof. By assumption, for all $x \in U, r \in R$, we have

$$a[x,r] = 0. \tag{3.1}$$

If we substitute r by ra in (3.1) we get

$$0 = a[x,ra] = ar[x,a] + a[x,r]a = ar[x,a]. \tag{3.2}$$

If we substitute r by xr in (3.2) we get $axr[x,a] = 0$. Multiplying (3.2) by x to the left we get $xar[x,a] = 0$. Then $[x,a]R[x,a] = (0)$ for all $x \in U$. It follows from semiprimeness of R that $[x,a] = 0$, that is, $a \in C_R(U)$. $\square$

Now we can prove the main results of this section.

Theorem 3.2.2. *Let R be a semiprime 2-torsion free ring, U a noncentral square-closed Lie ideal of R and F a right generalized derivation associated to a derivation d. If $F^2(U) = (0)$ and $F(U), d(U) \subseteq U$, then $d^3(U) = (0)$ and $(d^2(U))^2 = (0)$. Moreover, if $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$ (that is, F is also a left generalized derivation on U), then $d(U) = 0$, $F(U) = 0$ and $d(R), F(R) \subseteq C_R(U)$.*

Proof. By assumption we have

$$F^2(u) = (0) \quad \text{for all}\, u \in U. \tag{3.3}$$

If we replace u by $2uv$ in (3.3), we get

$$2F(F(u)v+ud(v)) = 2(F^2(u)v+F(u)d(v)+F(u)d(v)+ud^2(v)) = 0 \quad \text{for all}\, u, v \in U. \tag{3.4}$$

By (3.3), and using that R is 2-torsion free it follows that

$$2F(u)d(v) + ud^2(v) = 0 \quad \text{for all}\, u, v \in U. \tag{3.5}$$

We can replace u by $F(u)$ in (3.5) and obtain

$$2F^2(u)d(v)+F(u)d^2(v)=0 \quad \text{for all}\, u,v\in U. \tag{3.6}$$

Now, using (3.3) we have

$$F(u)d^2(v)=0 \quad \text{for all}\, u,v\in U. \tag{3.7}$$

Replacing v by $d(v)$ in (3.5) we have

$$2F(u)d^2(v)+ud^3(v)=0 \quad \text{for all}\, u,v\in U. \tag{3.8}$$

From (3.7) and (3.8), it follows that $ud^3(v)=0$, for all $u,v\in U$, that is $Ud^3(v)=(0)$. Since $d^3(v)\in U$, Lemma 3.1.1, gives that $d^3(U)=(0)$.
On the other hand, if we replace u by $2ud^2(w)$ in (3.5) we obtain

$$2(2F(u)d^2(w)d(v)+2ud^3(w)d(v)+ud^2(w)d^2(v))=0 \quad \text{for all}\, u,v,w\in U. \tag{3.9}$$

Using (3.7), and the fact that $d^3(U)=(0)$ in (3.9), we get $ud^2(w)d^2(v)=0$ for all $u,v,w\in U$, that is $U(2d^2(w)d^2(v))=(0)$, and again Lemma 3.1.1, says that

$$d^2(w)d^2(v)=0 \quad \text{for all}\, v,w\in U. \tag{3.10}$$

In particular $(d^2(v))^2=0$ for all $v\in U$, that is $(d^2(U))^2=(0)$.
Now, let's assume that $F(uv)=uF(v)+d(u)v$ for all $u,v\in U$. We replace u by $2wu$ in (3.5) to get

$$2wF(u)d(v)+2d(w)ud(v)+wud^2(v)=0 \quad \text{for all}\, u,v,w\in U. \tag{3.11}$$

By the other side, multiplying (3.5) to the left by w we have

$$2wF(u)d(v)+wud^2(v)=0 \quad \text{for all}\, u,v,w\in U. \tag{3.12}$$

Thus,

$$d(w)ud(v)=0 \quad \text{for all}\, u,v,w\in U. \tag{3.13}$$

In particular $d(v)Ud(v)=(0)$ for all $v\in U$. Again Lemma 3.1.1, gives $d(U)=(0)$. Thus $d([u,r])=0$ for all $u\in U, r\in R$, that is $[d(u),r]+[u,d(r)]=0$. Thus $[u,d(r)]=0$ for all $u\in U, r\in R$. Hence $d(R)\subseteq C_R(U)$. On the other hand, $d(U)=(0)$, implies $F(uv)=F(u)v=uF(v)$ for all $u,v\in U$. Thus,

$$0=F(F(uv-vu))=F(F(u)v-vF(u))=F^2(u)v-F(v)F(u) \quad \text{for all}\, u,v\in U. \tag{3.14}$$

Since we are assuming that $F^2(U)=(0)$, then $F(u)F(v)=0$ for all $u,v\in U$. It suffices to replace u by $2vw$, to get $F(v)UF(v)=(0)$. Hence $F(U)=(0)$ again by

Lemma 3.1.1 . In particular $0 = F(ur - ru) = -F(r)u + ud(r) = (-F(r) + d(r))u$ for all $u \in U, r \in R$ since $d(r) \in C_R(U)$ as was proved above. Let us define $G(r) = F(r) - d(r)$. Thus

$$G(r)u = 0 \quad \text{for all}\, u \in U, r \in R. \tag{3.15}$$

If we multiply (3.15) to the left by v and substitute u by $2vu$, we get $vG(r)u = 0$, $G(r)vu = 0$. Hence $[G(r), v]u = 0$ for all $u, v \in U, r \in R$, and again by Lemma 3.1.1, we conclude that $[G(r), v] = 0$, that is $0 = [F(r) - d(r), v] = [F(r), v]$ for all $v \in U, r \in R$. Thus $F(R) \subseteq C_R(U)$. □

The following four results follow immediately from Theorem 3.2.2.

Corollary 3.2.3. *Let R be a semiprime 2-torsion free ring, U a noncentral square-closed reduced Lie ideal of R and F a right generalized derivation associated to a derivation d. If $F^2(U) = (0)$ and $F(U), d(U) \subseteq U$, then $d(U) \subseteq Z(R)$.*

Proof. By Theorem 3.2.2, we have $(d^2(U))^2 = (0)$. Since U is reduced (that is, $u^2 = 0$ implies $u = 0$), then $d^2(U) = (0)$. Lemma 3.1.2 gives that $d(U) \subseteq Z(R)$. □

Corollary 3.2.4. *Let R be a semiprime 2-torsion free ring, I a nonzero reduced ideal of R and F a right generalized derivation associated to a derivation d. If $F^2(I) = (0)$ and $F(I), d(I) \subseteq I$, then $d(I) = 0$ and $I \subseteq Z(R)$.*

Proof. By Theorem 3.2.2, we have $(d^2(I))^2 = (0)$. Since R is reduced, then $d^2(I) = (0)$. Lemma 3.1.4 gives that $I \subseteq Z(R)$. On the other hand, $0 = d^2(y^2) = d(d(y)y + yd(y)) = 2(d(y))^2$ for all $y \in I$. So we get $d(I) = 0$. □

Corollary 3.2.5. *Let R be a prime ring with char$R \neq 2$, U a noncentral square-closed Lie ideal of R and F a right generalized derivation associated to a nonzero derivation d. If $F^2(U) = (0)$, then $d^3 = 0$.*

Proof. It immediately follows from Theorem 3.2.2 and Lemma 3.1.5(ii). □

Corollary 3.2.6. *Let R be a prime ring with char$R \neq 2$, U a square-closed reduced Lie ideal of R and F a right generalized derivation associated to a non-zero derivation d. If $F^2(U) = (0)$ and $F(U), d(U) \subseteq U$, then $U \subseteq Z(R)$.*

Proof. Let's assume that $U \nsubseteq Z(R)$. Then we can apply Corollary 3.2.3 to conclude that $d(U) \subseteq Z(R)$. But Lemma 3.1.5 (i) now gives $U \subseteq Z(R)$, a contradiction. □

The following proposition describes the structure of a left and right generalized derivation associated to a derivation d on a semiprime ring.

Proposition 3.2.7. *Let R be a semiprime ring with an extended centroid C. If R admits a left and right generalized derivation F associated to a derivation d, then there exists an element $\lambda \in C$ such that $F(x) = d(x) + \lambda x$ for all $x \in R$.*

Proof. Take $T := F - d$. Since F is a left and right generalized derivation, then

$$T(xy) = T(x)y = xT(y) \quad \text{for all } x, y \in R. \tag{3.16}$$

In particular we can used Lemma 3.1.6, we get $T(x) = \lambda x$ for all $x \in R$. That is $F(x) = d(x) + \lambda x$ for all $x \in R$. $\square$

The following example shows a map F on a semiprime ring R, that is not a derivation, but (F, d) is a right generalized derivation. Furthermore, it is also a left generalized derivation over a Lie ideal U of R.

Example 3.2.8. Let $\mathbb{Z}$ be the set of integers, and $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z} \right\}$. Then R is a semiprime ring. Now, take $U = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in \mathbb{Z} \right\}$. It can be easily checked that U is a Lie ideal of R. Since, $u^2 = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix} \in U$, U is an square-closed Lie ideal. We define the maps $F : R \to R$ and $d : R \to R$ as follows:

$$F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}.$$

Then it is easy to check that d is the inner derivation given by $d(x) = [x, 2e_{11} + e_{22}]$, F is a right generalized derivation associated to d, and $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$, that is, F is left generalized derivation on U. However, F is not a derivation on R.

Now we remove the assumptions $F(U), d(U) \subseteq U$ in Theorem 3.2.2.

Theorem 3.2.9. *Let R be a semiprime 2-torsion free ring, U a noncentral square-closed Lie ideal of R and F a right generalized derivation associated to a derivation d. If $F^2(U) = (0)$ and $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$, then $d(U) \subseteq C_R(U)$.*

Proof. Following the same lines that were used in the proof of Theorem 3.2.2, we can reach the equation (3.13), that is $d(w)ud(v) = 0$ for all $u, v, w \in U$. Multiplying to the left by w_1 we have $w_1 d(w)ud(v) = 0$. If we replace in (3.13) u by $2w_1 u$ we obtain $d(w)w_1 ud(v) = 0$, and subtracting these two relations we have $[d(w), w_1]ud(v) = 0$. In the same way, we get $[d(w), w_1]u[d(v), v_1] = 0$ for all $u, v, v_1, w, w_1 \in U$. Doing $w = v, w_1 = v_1$, we obtain $[d(v), v_1]U[d(v), v_1] = (0)$ for all $u, v_1 \in U$. Lemma 3.1.1, gives $[d(v), v_1] = 0$ for all $u, v_1 \in U$, that is, $d(U) \subseteq C_R(U)$. □

The following example shows that the semiprimeness condition in Theorem 3.2.2 is not superfluous.

Example 3.2.10. Let $\mathbb{Z}$ be the set of integers, and $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}$. For any $0 \neq b \in \mathbb{Z}$, $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = (0)$, then R is not a semiprime ring. Now, take $U = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, a, b \in \mathbb{Z} \right\}$. It can be easily checked that U is a Lie ideal of R. Since, $u^2 = \begin{pmatrix} a^2 & ab \\ 0 & a^2 \end{pmatrix} \in U$, U is an square-closed Lie ideal. We define maps $F : R \to R$ and $d : R \to R$ as follows:

$$F\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to check that d is the inner derivation given by $d(x) = [x, e_{12}]$, F is a right generalized derivation associated to d, $d(U), F(U) \subset U$, $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$ and $F^2(U) = (0)$. However, $F(U) \neq (0)$.

3.3 The case $F(u)v = uG(v)$

Now, we will extend Theorem 3.4 in [65] to semiprime rings.

Theorem 3.3.1. *Let R be a semiprime ring, U a noncentral Lie ideal of R,and F, G maps satisfying $G(u)v = uF(v)$ for all $u, v \in U$. If F is a right generalized derivation associated to a derivation d and G is a left generalized derivation associated to a derivation g, then $d(U), g(U) \subseteq C_R(U)$.*

Proof. Let's start by considering that

$$G(u)v = uF(v) \quad \text{for all}\, u, v \in U. \tag{3.17}$$

If we replace in (3.17) the element v by $[v, r]v$, $r \in R$, (and by $[v, r]$), we get

$$G(u)[v, r]v = uF([v, r])v + u[v, r]d(v) \quad \text{for all}\, u, v \in U, r \in R. \tag{3.18}$$

And

$$G(u)[v,r] = uF([v,r]) \quad \text{for all}\, u,v \in U, r \in R. \tag{3.19}$$

Consequently,

$$u[v,r]d(v) = 0 \quad \text{for all}\, u,v \in U, r \in R. \tag{3.20}$$

Now we can replace u by $[u,s]$, $s \in R$, in (3.20), and get $0 = [u,s][v,r]d(v) = us[v,r]d(v) - su[v,r]d(v) = us[v,r]d(v)$ for all $u,v \in U, r,s \in R$. Again we replace u by $[v,r]$ and s by $d(v)s$, to have $[v,r]d(v)s[v,r]d(v) = 0$ for all $u,v \in U, r,s \in R$. Thus $[v,r]d(v)R[v,r]d(v) = (0)$. Since R is semiprime, we conclude that

$$[v,r]d(v) = 0 \quad \text{for all}\, v \in U, r \in R. \tag{3.21}$$

Linearizing (3.21), we get

$$[v,r]d(u) = -[u,r]d(v) \quad \text{for all}\, u,v \in U, r \in R. \tag{3.22}$$

Now, if we replace r by rs in (3.21), and use again (3.21), we get

$$[v,r]sd(v) = 0 \quad \text{for all}\, u,v \in U, r,s \in R. \tag{3.23}$$

In particular, $0 = [v,r]d(u)s[u,r]d(v) = -[v,r]d(u)s[v,r]d(u)$, that is $[v,r]d(u)R[v,r]d(u) = (0)$. Since R is semiprime we have $[v,r]d(u) = 0$, for all $u,v \in U, r \in R$. From Lemma 3.2.1 it follows that $d(U) \subseteq C_R(U)$.
On the other hand, we replace in (3.17) u by $u[u,r]$ (and by $[u,r]$), we get

$$uG([u,r])v + g(u)[u,r]v = u[u,r]F(v) \quad \text{for all}\, u,v \in U, r \in R. \tag{3.24}$$

And,

$$G([u,r])v = [u,r]F(v) \quad \text{for all}\, u,v \in U, r \in R. \tag{3.25}$$

Consequently, $g(u)[u,r]v = 0$. So, we can follow the same lines as above and conclude that $g(U) \subseteq C_R(U)$. □

An immediate consequence of Theorem 3.3.1 is the following corollary.

Corollary 3.3.2. *Let R be a prime ring with $charR \neq 2$, U a Lie ideal of R and F, G maps satisfying $G(u)v = uF(v)$ for all $u,v \in U$. If F is a right generalized derivation associated to a nonzero derivation d and G is a left generalized derivation associated to a nonzero derivation g, then $U \subseteq Z(R)$.*

Proof. Suppose that $U \nsubseteq Z(R)$. Theorem 3.3.1, gives $d(U) \subseteq C_R(U)$ and $C_R(U) = Z(R)$ by Lemma 3.1.3, hence $d(U) \subseteq Z(R)$. Lemma 3.1.5 (i) gives $U \subseteq Z(R)$, the contradiction. □

The following example shows that the semiprimeness condition in Theorem 3.3.1 is not superfluous.

Example 3.3.3. Consider the ring R as in Example 3.2.10.

$$\text{Take} \quad U = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, a, b \in \mathbb{Z} \right\}.$$

It can be easily checked that U is a Lie ideal of R. Define maps $G, g, F, d : R \to R$ as follows:

$$G\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & b+c \\ 0 & c \end{pmatrix}, g\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$

$$F\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}, d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to check that g and d are the inner derivations given by $g(x) = [x, e_{11} + 2e_{22}]$ and $d(x) = [x, e_{12}]$, F is a right generalized derivation associated to d, G is a left generalized derivation associated to g and $G(u)v = uF(v)$ for all $u, v \in U$. However, $d(U) \nsubseteq C_R(U)$.

Chapter 4

Center-like Subsets in Prime Rings with Generalized Derivations

4.1 Introduction

Several results in the literature assert that certain subsets of a ring R, defined by some sort of commutativity conditions, coincide with $Z(R)$. We call such subsets center-like subsets. A classical result of Herstein [70] states that the hypercenter $S(R)$, defined as $\{a \in R \mid \forall x \in R \quad \exists n = n(x,a) \quad \text{s.t.} \quad ax^n = x^n a\}$, coincides with $Z(R)$ if R has no nonzero nil ideals. Following Herstein, Chacron [46] introduced the cohypercenter $G(R)$ as follows: $a \in G(R)$ if and only if for each $x \in R$ there exists a polynomial $p(X) \in \mathbb{Z}[X]$, depending on a and x, such that $[a, x - x^2 p(x)] = 0$; and he established equality of $Z(R)$ and $G(R)$ for semiprime R. Similar results can be found in [22], [28], [33], [48], [62] and [63].

Our purpose is to study center-like subsets, whose definition involves a map $f : R \to R$. Apparently the first example of such center-like subset was $H(R, d) = \{a \in R \mid ad(x) = d(x)a \quad \forall x \in R\}$, where d is a derivation. Herstein introduced this set, in [72], and proved that if R is prime with $\text{char}(R) \neq 2$ and d is a nonzero derivation, then $H(R, d) = Z(R)$. We will extend this result considering $H(R, F) = \{a \in R \mid aF(x) = F(x)a \quad \forall x \in R\}$ where F is a left and a right generalized derivation and proving that if R is prime with $\text{char}(R) \neq 2$, then $H(R, F) = Z(R)$.

In [23] it is proved that a semiprime ring with a derivation d on R such that $[x, y] = [d(x), d(y)]$ for all $x, y \in R$ is necessarily commutative; and in [24] it is shown that a prime ring is commutative if for some nonzero derivation d,

$[d(x), d(y)] = [d(x), y] + [x, d(y)]$ for all $x, y \in R$.

Motivated by these results, Bell and Daif considered in [25] a ring R equipped with a map $f : R \to R$ and defined the following subsets:

- $Z^*(R, f) = \{y \in R \mid [x, y] = [f(y), f(x)] \,\text{for all}\, x \in R\}$;
- $Z^{**}(R, f) = \{y \in R \mid [x, y] = [f(x), f(y)] \,\text{for all}\, x \in R\}$;
- $Z_1(R, f) = \{y \in R \mid [f(x), f(y)] = [f(y), x] + [y, f(x)] \,\text{for all}\, x \in R\}$.

They proved that if R is a semiprime ring and d a derivation, then $Z^*(R, d) = Z(R) = Z^{**}(R, d)$. Moreover if R is prime, $\text{char}R \neq 2$, and d is a non zero derivation, then $Z_1(R, d) = Z(R)$.

In the present chapter, we shall extend the results of Bell and Daif [25] to generalized derivations. Further, we also will give some examples to show that the restrictions imposed on the hypotheses of the various results are not superfluous.

The following definitions seem quite natural:

- $Z^\star 1(R, f) = \{y \in R \mid [f(x), f(y)] = [f(x), y] + [x, f(y)] \,\text{for all}\, x \in R\}$.
- $Z_2(R, f) = \{y \in R \mid [y, x] = [f(y), x] + [y, f(x)] \,\text{for all}\, x \in R\}$.
- $Z_2^\star(R, f) = \{y \in R \mid [y, x] = [f(x), y] + [x, f(y)] \,\text{for all}\, x \in R\}$.
- Note that: $Z^*(R, f) \cap Z_1(R, f) \subseteq Z_2(R, f)$ and $Z^{**}(R, f) \cap Z_1(R, f) \subseteq Z_2^\star(R, f)$.

First of all let's notice that there exist maps that are both right and left generalized derivations, related to the same derivation d, but that are not derivations, as the following example shows.

Example 4.1.1. Let R be an arbitrary ring with a nonzero central element c. We define the map F on R by $F(x) = cx + d(x)$ for all $x \in R$, where d is a derivation of R. Then it is easy to check that F is both, right and left generalized derivation associated to d. However, F is not a derivation on R.

4.2 Preliminaries

Let's start by reminding a lemma that will be used in our results.

Lemma 4.2.1. [[35], Lemma 4] *Let R be a 2-torsion free semiprime ring and let $a, b \in R$. If $axb + bxa = 0$ holds for all $x \in R$, then $axb = bxa = 0$ for all $x \in R$.*

Now, we prove two lemmas which will be essential to get our main results.

Lemma 4.2.2. *Let R be a ring, F a left and a right generalized derivation associated to a derivation d. Then $F(Z(R)) \subseteq Z(R)$.*

Proof. Suppose that $a \in Z(R)$, that is $ax = xa$ for all $x \in R$. Then $F(ax) = F(xa)$ that is (by using that F is left and right generalized derivation) $F(a)x + ad(x) = xF(a) + d(x)a$. This gives $[F(a), x] + [a, d(x)] = 0$. But $[a, d(x)] = 0$ because $a \in Z(R)$. So $[F(a), x] = 0$ for all $x \in R$, that is $F(a) \in Z(R)$. □

Lemma 4.2.3. *Let R be a prime ring, $charR \neq 2$ and F a left and a right generalized derivation associated to a nonzero derivation d. Then $H(R, F) = Z(R)$.*

Proof. Since $Z(R) \subseteq H(R, F)$ we only need to show that $H(R, F) \subseteq Z(R)$. Let $a \in H(R, F)$, then

$$[a, F(x)] = 0 \quad \text{for all } x \in R. \tag{4.1}$$

Let us replace x by xy in (4.1), to obtain

$$F(x)[a, y] + [a, F(x)]y + [a, x]d(y) + x[a, d(y)] = 0 \quad \text{for all } x \in R. \tag{4.2}$$

Using (4.1) in (4.2) we get

$$F(x)[a, y] + [a, x]d(y) + x[a, d(y)] = 0 \quad \text{for all } x, y \in R. \tag{4.3}$$

and substituting x by yx in (4.3), we have, for all $x, y \in R$

$$yF(x)[a, y] + d(y)x[a, y] + y[a, x]d(y) + [a, y]xd(y) + yx[a, d(y)] = 0. \tag{4.4}$$

Using (4.3) in (4.4), we obtain

$$d(y)x[a, y] + [a, y]xd(y) = 0 \quad \text{for all } x, y \in R. \tag{4.5}$$

Lemma 4.2.1 gives $d(y)x[a, y] = 0$ for all $y \in R$. Given a fixed $y \in R$, either $d(y) = 0$ or $[a, y] = 0$. But $R_1 = \{y \in R \mid d(y) = 0\}$ and $R_2 = \{y \in R \mid [a, y] = 0\}$ are both additive subgroups of R and $R = R_1 \cup R_2$. So either $R = R_1$ or $R = R_2$. But $R = R_1$ contradict our assumption $d \neq 0$. Then $R = R_2$ that is, $[a, y] = 0$ for all $y \in R$, so $H(R, F) \subseteq Z(R)$. □

4.3 On center-like subsets $Z^*(R,F)$, $Z^{**}(R,F)$ and $Z_1(R,F)$

Theorem 4.3.1. *Let R be a prime ring, $charR \neq 2$, and F a left and a right generalized derivation associated to a nonzero derivation d. Then $Z^*(R,F) = Z(R)$.*

Proof. Let $z \in Z(R)$. Since $F(Z(R)) \subseteq Z(R)$(by Lemma 4.2.2), and thus $[F(z), F(x)] = 0 = [x, z]$ for all $x \in R$, that is $Z(R) \subseteq Z^*(R,F)$. Now we only need to show that $Z^*(R,F) \subseteq Z(R)$. Let $t \in Z^*(R,F)$, i.e.,

$$[t, x] = [F(x), F(t)] \quad \text{for all}\, x \in R. \tag{4.6}$$

Let us replace x by xy in (4.6) to get

$$\begin{aligned} [t,x]y + x[t,y] = x[F(y), F(t)] + [x, F(t)]F(y) \\ +d(x)[y, F(t)] + [d(x), F(t)]y \end{aligned} \quad \text{for all}\, x, y \in R \tag{4.7}$$

Using (4.6) in (4.7), we have

$$[t,x]y = [x, F(t)]F(y) + d(x)[y, F(t)] + [d(x), F(t)]y \quad \text{for all}\, x, y \in R \tag{4.8}$$

Substituting y by yw in (4.8) and simplifying using (4.8), we get

$$\begin{aligned} [t,x]yw = [x, F(t)]F(y)w + [x, F(t)]yd(w) + d(x)[y, F(t)]w \\ +d(x)y[w, F(t)] + [d(x), F(t)]yw \end{aligned} \quad \text{for all}\, x, y, w \in R \tag{4.9}$$

Using (4.8) in (4.9), we obtain

$$[x, F(t)]yd(w) + d(x)y[w, F(t)] = 0 \quad \text{for all}\, x, y, w \in R. \tag{4.10}$$

Doing $w = x$ in (4.10), we get $[x, F(t)]yd(x) + d(x)y[x, F(t)] = 0$ for all $x, y \in R$, Lemma 4.2.1 gives that $[x, F(t)]yd(x) = 0$ for all $x, y \in R$. Given a fixed $x \in R$, either $d(x) = 0$ or $[x, F(t)] = 0$. But $R_1 = \{x \in R \mid d(x) = 0\}$ and $R_2 = \{x \in R \mid [x, F(t)] = 0\}$ are both additive subgroups of R and $R = R_1 \cup R_2$. So either $R = R_1$ or $R = R_2$. If $R = R_1$, that is a contradiction to our assumption $d \neq 0$. Then $R = R_2$ that is $[x, F(t)] = 0$ for all $x \in R$. So that $F(t) \in Z(R)$. By (4.6) we obtain that $t \in Z(R)$. □

Theorem 4.3.2. *Let R be a prime ring, $charR \neq 2$, and F is a left and right generalized derivation associated to a nonzero derivation d. Then $Z^{**}(R,F) = Z(R)$.*

Proof. The proof follows the same lines of the previous one. $\square$

Theorem 4.3.3. *Let R be a prime ring, char$R \neq 2$, and F a left and a right generalized derivation associated to a nonzero derivation d. Then $Z_1(R,F) = Z(R)$.*

Proof. We only need to prove that $Z_1(R,F) \subseteq Z(R)$, since the other inclusion is immediate from Lemma 4.2.2. Let $t \in Z_1(R,d)$, so that

$$[F(x),F(t)] = [F(t),x] + [t,F(x)] \quad \text{for all } x \in R. \tag{4.11}$$

Substituting x by xy in (4.11), we get

$$\begin{aligned} &[F(x),F(t)]y + F(x)[y,F(t)] + x[d(y),F(t)] + [x,F(t)]d(y) \\ &\quad = [F(t),x]y + x[F(t),y] + [t,F(x)]y \quad \text{for all } x,y \in R. \\ &\quad +F(x)[t,y] + [t,x]d(y) + x[t,d(y)] \end{aligned} \tag{4.12}$$

Using (4.11) in (4.12), we get

$$\begin{aligned} &F(x)[y,F(t)] + x[d(y),F(t)] + [x,F(t)]d(y) = \\ &x[F(t),y] + F(x)[t,y] + [t,x]d(y) + x[t,d(y)] \end{aligned} \quad \text{for all } x,y \in R. \tag{4.13}$$

We can replace x by zx in (4.13), we obtain

$$\begin{aligned} &zF(x)[y,F(t)] + d(z)x[y,F(t)] + zx[d(y),F(t)] + z[x,F(t)]d(y) \\ &\quad +[z,F(t)]xd(y) = zx[F(t),y] + zF(x)[t,y] + d(z)x[t,y] \quad \text{for all } x,y,z \in R. \\ &\quad +z[t,x]d(y) + [t,z]xd(y) + zx[t,d(y)] \end{aligned} \tag{4.14}$$

Using (4.13) in (4.14), we have

$$d(z)x[y,F(t)] + [z,F(t)]xd(y) = d(z)x[t,y] + [t,z]xd(y) \quad \text{for all } x,y,z \in R. \tag{4.15}$$

That is

$$d(z)x[y,F(t)+t] + [z,F(t)+t]xd(y) = 0 \quad \text{for all } x,y,z \in R. \tag{4.16}$$

Doing $z = y$ in (4.16), we obtain

$$d(y)x[y,F(t)+t] + [y,F(t)+t]xd(y) = 0 \quad \text{for all } x,y \in R. \tag{4.17}$$

Lemma 4.2.1 gives that $d(y)R[y,F(t)+t]=0$ for all $y\in R$. Then using the same arguments used in the proof of Theorem 4.3.1 (Brauer's trick), we get that $d(y)=0$ for all $y\in R$ or $[y,F(t)+t]=0$ for all $y\in R$. If $d=0$ it is a contradiction to our assumption, then $[y,F(t)+t]=0$ for all $y\in R$, so $F(t)+t\in Z(R)$; thus $[F(t)+t,F(x)]=[F(t),F(x)]+[t,F(x)]=0$, i.e $[F(t),F(x)]=[F(x),t]$. From (4.11) we conclude that $[F(t),x]=0$ for all $x\in R$, so that $F(t)\in Z(R)$. From (4.11) we obtain that $[t,F(R)]=0$, so $Z_1(R,F)\subseteq H(R,F)$, hence by Lemma 4.2.3, $Z_1(R,F)\subseteq Z(R)$.

□

Theorem 4.3.4. *Let R be a prime ring, $charR\neq 2$, and F a left and a right generalized derivation associated to a nonzero derivation d. Then $Z_1^\star(R,F)=Z(R)$.*

Proof. The proof follows the same lines of the previous one. □

If in Theorem 4.3.3 R is assumed to be semiprime instead of prime, we do not get any longer that $Z_1(R,F)\subseteq Z(R)$, as the following example shows.

Example 4.3.5. Let R_1 a commutative domain, d_1 a nonzero derivation on R_1, and R_2 is a noncommutative prime ring. If we define $R=R_1\oplus R_2$, then R is a semiprime ring. Let's define $d:R\to R$ and $F:R\to R$ as following:

$$d((r_1,r_2))=(d_1(r_1),0)\quad \text{for all}\, r_1\in R_1, r_2\in R_2,$$

and

$$F(r)=cr+d(r)\quad \text{for all}\, r\in R,\, \text{and}\, c=(c_1,0), c_1\in R_1.$$

It's easy to check that d is a nonzero derivation, and F is a left and a right generalized derivation associated to d. If $y=(0,r_2)$, we have $F(y)=0$ and $[y,F(r)]=0$ for all $r\in R$; hence $y\in Z_1(R,F)$. That is $S=\{(0,r_2)|r_2\in R_2\}\subseteq Z_1(R,F)$, but $S\nsubseteq Z(R)$

4.4 On center-like subsets $Z_2(R,F)$ and $Z_2^\star(R,F)$

In this section, we consider the sets $Z_2(R,F)$ and $Z_2^\star(R,F)$

Theorem 4.4.1. *Let R be a prime ring, $charR\neq 2$, and F a left and a right generalized derivation associated to a nonzero derivation d. Then $Z_2(R,F)=Z(R)$.*

Proof. We only need to prove that $Z_2(R,F) \subseteq Z(R)$, since the other inclusion is clear. Let $t \in Z_2(R,d)$, then

$$[t,x] = [F(t),x] + [t,F(x)] \quad \text{for all}\, x \in R. \tag{4.18}$$

Substituting x by xy in (4.18), we get

$$\begin{aligned} [t,x]y + x[t,y] = [F(t),x]y \\ +x[F(t),y] + [t,F(x)]y + F(x)[t,y] + [t,x]d(y) + x[t,d(y)] \end{aligned} \quad \text{for all}\, x,y \in R. \tag{4.19}$$

Using (4.18) in (4.19), we get

$$x[t,y] = x[F(t),y] + F(x)[t,y] + [t,x]d(y) + x[t,d(y)] \quad \text{for all}\, x,y \in R. \tag{4.20}$$

We can replace x by zx in (4.20), we obtain

$$\begin{aligned} zx[t,y] = zx[F(t),y] + zF(x)[t,y] + d(z)x[t,y] \\ +z[t,x]d(y) + [t,z]xd(y) + zx[t,d(y)] \end{aligned} \quad \text{for all}\, x,y,z \in R. \tag{4.21}$$

Using (4.20) in (4.21), we have

$$d(z)x[t,y] + [t,z]xd(y) = 0 \tag{4.22}$$

Doing $z = y$ in (4.22), we obtain

$$d(y)x[t,y] + [t,y]xd(y) = 0 \quad \text{for all}\, x,y \in R. \tag{4.23}$$

Lemma 4.2.1 gives that $d(y)R[t,y] = 0$ for all $y \in R$. Then using the same arguments used in the proof of Theorem 4.3.1 (Brauer's trick), we get that $d(y) = 0$ for all $y \in R$ or $[t,y] = 0$ for all $y \in R$. Since $d = 0$ contradiction our assumption, we get $[y,t] = 0$ for all $y \in R$. So $Z_2(R,F) \subseteq Z(R)$. $\square$

Theorem 4.4.2. *Let R be a prime ring, char$R \neq 2$, and F is a left and right generalized derivation associated to a nonzero derivation d. Then $Z_2^\star(R,F) = Z(R)$.*

Proof. The proof follows the same lines of the previous one. $\square$

The following example shows that the primeness condition in the previous Theorems is not superfluous.

Example 4.4.3. Let $\mathbb{Z}$ be the set of integers, and $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} | x,y,z \in \mathbb{Z} \right\}$.

i. Now, let's define the maps $F: R \to R$ and $d: R \to R$ as follows:

$$F\left(\left(\begin{array}{cc} x & y \\ 0 & z \end{array}\right)\right) = \left(\begin{array}{cc} x & x-z \\ 0 & z \end{array}\right), d\left(\left(\begin{array}{cc} x & y \\ 0 & z \end{array}\right)\right) = \left(\begin{array}{cc} 0 & x-y-z \\ 0 & 0 \end{array}\right).$$

It is easy to check that d is the inner derivation given by $d(x) = [x, e_{11}+e_{12}]$, F is left and right generalized derivation associated to d. For an arbitrary element $W = \left(\begin{array}{cc} w_1 & w_2 \\ 0 & w_3 \end{array}\right)$, we have $[W, X] = [F(W), X] + [X, F(W)]$ for all $X \in R$; hence $W \in Z_2(R, F)$. Thus $R = Z_1(R, F)$. However, R is not commutative.

ii. If we define the maps $F_1 : R \to R$ and $d_1 : R \to R$ as follows:

$$F_1\left(\left(\begin{array}{cc} x & y \\ 0 & z \end{array}\right)\right) = \left(\begin{array}{cc} x & x+y-z \\ 0 & z \end{array}\right), d_1\left(\left(\begin{array}{cc} x & y \\ 0 & z \end{array}\right)\right) = \left(\begin{array}{cc} 0 & x-z \\ 0 & 0 \end{array}\right).$$

Then it is easy to check that d_1 is the inner derivation given by $d_1(x) = [x, e_{12}]$, F is a left and a right generalized derivation associated to d. For $W_1 = \left(\begin{array}{cc} w_1 & w_2 \\ 0 & w_1 \end{array}\right)$, we have $[W_1, X] = [F(W_1), F(X)]$ for all $X \in R$; hence $W_1 \in Z^{**}(R, F)$. However, $W_1 \notin Z(R)$.

iii. If we define the maps $F_2 : R \to R$ and $d_2 : R \to R$ as follows:

$$F_2\left(\left(\begin{array}{cc} x & y \\ 0 & z \end{array}\right)\right) = \left(\begin{array}{cc} x & -y \\ 0 & z \end{array}\right), d_2\left(\left(\begin{array}{cc} x & y \\ 0 & z \end{array}\right)\right) = \left(\begin{array}{cc} 0 & -2y \\ 0 & 0 \end{array}\right).$$

Then it is easy to check that d_2 is the inner derivation given by $d_2(x) = [x, e_{11} - e_{22}]$, F is a left and a right generalized derivation associated to d. For $W_2 = \left(\begin{array}{cc} w_1 & w_2 \\ 0 & w_3 \end{array}\right)$, we have $[W_3, X] = [F(X), F(W_3)]$ for all $X \in R$; hence $W_2 \in Z^*(R, F)$. However, $W_2 \notin Z(R)$. R is not commutative.

What happens in Theorems 4.3.3 and 4.4.1 if F is only right or only left generalized derivation? The following example shows that the result in the above theorems can not be recovered.

Example 4.4.4. Let $\mathbb{Z}$ be the set of integers, and $R = \left\{\left(\begin{array}{cc} x & y \\ z & w \end{array}\right) | x, y, z, w \in \mathbb{Z}\right\}$. Then R is a prime ring. We define the maps $F : R \to R$ and $d : R \to R$ as follows:

$$F\left(\left(\begin{array}{cc} x & y \\ z & w \end{array}\right)\right) = \left(\begin{array}{cc} x & 0 \\ 0 & -w \end{array}\right), d\left(\left(\begin{array}{cc} x & y \\ z & w \end{array}\right)\right) = \left(\begin{array}{cc} 0 & -y \\ z & 0 \end{array}\right).$$

Then it is easy to check that d is the inner derivation given by $d(x) = [x, 2e_{11}+e_{22}]$, F is a right generalized derivation associated to d, but F is NOT a left generalized derivation on R associated to d.

i. For $Y = \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix}$, we have $[F(X), F(Y)] = [F(Y), X] + [Y, F(X)]$ for all $X \in R$; hence $Y \in Z_1(R, F)$. Thus $S = \{\begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix} | y \in \mathbb{Z}\} \subseteq Z_1(R, F)$. That is, $Z_1(R, F) \nsubseteq Z(R)$.

ii. If we take $C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$, then $C \notin Z_1(R, F)$, but $C \in Z(R)$. That is $Z(R) \nsubseteq Z_1(R, F)$.

iii. On the other hand, if we take $Y = \begin{pmatrix} -y & 0 \\ 0 & 0 \end{pmatrix}$, we have $[Y, X] = [F(Y), X]+ [Y, F(X)]$ for all $X \in R$, hence $Y \in Z_2(R, F)$. Thus $S = \{\begin{pmatrix} -y & 0 \\ 0 & 0 \end{pmatrix} | y \in \mathbb{Z}\} \subseteq Z_2(R, F)$. That is, $Z_2(R, F) \nsubseteq Z(R)$.

Chapter 5

Orthogonality of Generalized Derivations on Semiprime Rings

5.1 Introduction and preliminaries

Two maps $d, g : R \to R$ are called orthogonal if,

$$d(x)Rg(y) = (0) = g(y)Rd(x) \quad \text{for all}\, x, y \in R.$$

The concept of orthogonal derivation was introduced by M. Brešar and J. Vukman in [43]. The following example shows that there exist l-generalized derivations and r-generalized derivations which are orthogonal.

Example 5.1.1. Let $\mathbb{Z}$ be the ring of integers, and $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z}\}$. Define maps $F : R \to R$, $d : R \to R$, $G : R \to R$, and $g : R \to R$ as follows:

$$F(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} 0 & a+c \\ 0 & 0 \end{pmatrix}, d(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix},$$

$$G(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} 0 & c-2a \\ 0 & 0 \end{pmatrix}, \text{ and } g(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} 0 & c-a \\ 0 & 0 \end{pmatrix}.$$

Then its easy to prove that F is a l-generalized derivation associated to the derivation d, G is a r-generalized derivation associated to the derivation g and $F(R)RG(R) = (0) = G(R)RF(R)$, that is F and G are orthogonal.

Posner studied, for prime rings, the composition of two derivations d_1 and d_2. He proved in [114] that $d_1 d_2$ is a derivation only when $d_1 = 0$ or $d_2 = 0$. He also proved that if $ad(a) - d(a)a \in Z(R)$ for every $a \in R$ then either R is commutative or $d = 0$.

In [43], M. Breŝar and J. Vukman introduced the concept of orthogonal derivations in order to extend Posner result to 2-torsion free semiprime rings. Now if $d_1 d_2$ is a derivation it does not follow that $d_1 = 0$ or $d_2 = 0$. We can only say that d_1 and d_2 are orthogonal. In a concrete way the following Theorem is proved in [43]. Let R be a 2-torsion free semiprime ring, d and g two derivations of R. Then the following assertions are equivalent, (a) d and g are orthogonal; (b) $dg = 0$; (c) $dg + gd = 0$; (d) $d(x)g(x) = 0$ for all $x \in R$; (e) dg is a derivation; (f) There exist $a, b \in R$ such that $(dg)(x) = ax + xb$ for all $x \in R$. In [131], Yenigul and Argac proved that the results of Breŝar and Vukman still hold assuming only orthogonality over a nonzero ideal I of R.

In [11], Argac, Nakajima and Albas extended the previous idea of Posner and Breŝar and Vukman to l-generalized derivations and Albas extended the results of Yenigul and Argac in [3]. In a more concrete way, authors proved in [11] the following Theorem. Let $(D, d), (G, g)$ be two l-generalized derivations of a 2-torsion free semiprime ring, then the following conditions are equivalent: (i) (D, d) and $(G; g)$ are orthogonal; (ii) For all $x, y \in R$, the following relations hold: (a) $D(x)G(y) + G(x)D(y) = 0$ and (b) $d(x)G(y) + g(x)D(y) = 0$; (iii) $D(x)G(y) = d(x)G(y) = 0$ for all $x, y \in R$; (iv) $D(x)G(y) = 0$ for all $x, y \in R$ and $dG = dg = 0$; (v) (DG, dg) is a generalized derivation and $D(x)G(y) = 0$ for all $x, y \in R$; (vi) There exist ideals U and V of R such that: (a) $U \cap V = 0$ and $U \oplus V$ is an essential ideal of R (An ideal I of R is called an essential ideal if I has nonzero intersection with every other nonzero ideal of R), (b) $D(R), d(R) \subseteq U$ and $G(R), g(R) \subseteq V$, (c) $D(V) = d(V) = 0$ and $G(U) = g(U) = 0$.

In this chapter, we consider the situation in which we have one l-generalized derivation and one r-generalized derivation and find necessary and sufficient conditions for them to be orthogonal on a nonzero ideal of a semiprime ring R. We also study the connections between orthogonality and some properties of the composition of a l-generalized derivation and a r-generalized derivation of R.

To prove our result we will use the following lemmas.

Lemma 5.1.2. [131, Lemma 1] *Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R. Then for any $a, b \in R$ the following conditions are equivalent:*
(i) $axb = 0$ for all $x \in I$.
(ii) $bxa = 0$ for all $x \in I$.
(iii) $axb + bxa = 0$ for all $x \in I$.

Moreover, if one of the three conditions is fulfilled and the left annihilator of I is zero ($l(I) = (0)$), then $ab = ba = 0$.

Lemma 5.1.3. [119, Lemma 2.1]**.** *Let R be a semiprime ring, I a nonzero ideal of R and $a \in I$. If $axa = 0$ for all $x \in I$, then $a = 0$.*

Remark 5.1.1. If f, g are orthogonal maps of a 2-torsion free semiprime ring then Lemma 5.1.2 gives that $f(x)g(y) = 0$ for all $x, y \in R$. Conversely, this condition also implies orthogonality if either f or g is a derivation. This fact, when both f and g are derivations, is part of the following:

Lemma 5.1.4. [131, Main Theorem]**.** *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R such that $l(I) = 0$ and d, g two derivations of R. Then the following assertions are equivalent.*
(a) d and g are orthogonal.
(b) $d(x)g(x) = 0$ for all $x \in I$.
(c) $dg = 0$.
(d) dg is a derivation.

Lemma 5.1.5. *Let R be a semiprime ring, I a nonzero ideal of R such that $l(I) = (0)$. If $a \in R$ and $axa = 0$ for all $x \in I$, then $a = 0$.*

Proof. Since $axa = 0$ for all $x \in I$, in particular, $ayra = 0$ for all $y \in I, r \in R$. Right multiplication by y gives $ayray = 0$. Semiprimeness of R gives $ay = 0$ for all $y \in I$. Since $l(I) = (0)$, then $a = 0$. □

5.2 Orthogonality of a left and a right generalized derivations

We begin with the following lemmas which are essential for the development of our main results.

Lemma 5.2.1. *Let R be a semiprime ring, and I a nonzero ideal of R. Suppose that the relation $axb + bxc = 0$ holds for all $x \in I$ and some $a, b, c \in R$. Then $bxc = cxb$ for all $x \in I$ and, consequently, $(a + c)xb = 0$ for all $x \in I$. Similarly $axb = bxa$ for all $x \in I$ and so $bx(a + c) = 0$.*

Proof. According to our assumption,

$$axb + bxc = 0 \quad \text{for all}\, x \in I. \tag{5.1}$$

If we replace in (5.1) x by xby, $y \in I$, we get

$$axbyb + bxbyc = 0 \quad \text{for all}\, x, y \in I. \tag{5.2}$$

On the other hand, if we multiply (5.1) to the right by yb we get

$$axbyb + bxcyb = 0 \quad \text{for all}\, x, y \in I. \tag{5.3}$$

Subtracting (5.3) from (5.2), we obtain

$$bx(byc - cyb) = 0 \quad \text{for all}\, x, y \in I. \tag{5.4}$$

Now if we substitute x by ycx, and multiply (5.4) to the left by cy, we get respectively

$$bycx(byc - cyb) = 0 \quad \text{for all}\, x, y \in I, \tag{5.5}$$

and

$$cybx(byc - cyb) = 0 \quad \text{for all}\, x, y \in I. \tag{5.6}$$

Subtracting (5.6) from (5.5), we obtain

$$(byc - cyb)x(byc - cyb) = 0 \quad \text{for all}\, x, y \in I. \tag{5.7}$$

Lemma 5.1.3 gives $byc = cyb$ for all $y \in I$. The equality $axb = bxa$ for all $x \in I$ is obtained in the same way.

□

Lemma 5.2.2. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R such that $l(I) = (0)$, (D, d) a l-generalized derivation and (G, g) a r-generalized derivation. If $D(I)IG(I) = (0)$ (resp. $G(I)ID(I) = (0)$), then $D(R)RG(R) = (0)$ (resp. $G(R)RD(R) = (0)$), and d, g are orthogonal.*

Proof. Notice that $D(I)IG(I) = (0)$ implies $G(I)ID(I) = (0)$ by lemma 5.1.2. And conversely.
Assume that

$$D(x)zG(y) = 0 \quad \text{for all}\, x, y, z \in I. \tag{5.8}$$

Lemma 5.1.2 gives

$$G(y)D(x) = 0 = D(x)G(y) \quad \text{for all}\, x, y \in I. \tag{5.9}$$

If we replace x by xr in (5.9), we get $G(y)xd(r) = 0$ for all $x, y \in I, r \in R$. Lemma 5.1.2 gives

$$G(y)d(r) = 0 = d(r)G(y) \quad \text{for all}\, y \in I, r \in R. \tag{5.10}$$

Similarly, replacing y by ry in (5.9), we get

$$g(r)D(x) = 0 = D(x)g(r) \quad \text{for all}\, x \in I, r \in R. \tag{5.11}$$

Now, we replace x by xs in (5.11) and use Lemma 5.1.2 to obtain

$$g(r)d(s) = 0 = d(s)g(r) \quad \text{for all}\, r, s \in R. \tag{5.12}$$

That is d and g are orthogonal. If we replace y by ys in (5.10), we get

$$d(r)yG(s) = 0 \quad \text{for all}\, y \in I, r, s \in R. \tag{5.13}$$

Similarly, replacing x by sx in (5.11), we get

$$D(s)xg(r) = 0 \quad \text{for all}\, x \in I, r, s \in R. \tag{5.14}$$

Replacement of x by rx, and of y by ys in (5.9), and the use of (5.12), (5.13) and (5.14), gives

$$D(r)xyG(s) = 0 \quad \text{for all}\, x, y \in I, r, s \in R. \tag{5.15}$$

Lemma 5.1.2 gives $D(r)xG(s) = 0$, and changing x by xt, $t \in R$, we get that $D(r)tG(s) = 0$ for all $r, t, s \in R$. In the same way $G(R)RD(R) = (0)$ can be proved. □

Lemma 5.2.3. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R such that $l(I) = (0)$. If (D, d) a l-generalized derivation associated to a derivation d and (G, g) a r-generalized derivation associated to a derivation g. Then the following conditions are equivalent :*
(i) For any $x, y \in I$, the following relations holds.
(a) $D(x)g(y) + d(x)G(y) = 0$
(b) $G(x)d(y) + g(x)D(y) = 0$
(ii) $D(x)g(y) = G(x)d(y) = d(x)g(y) = 0$ for all $x, y \in I$.
Moreover, if the conditions of (i) are fulfilled and $D(x)G(y) + G(x)D(y) = 0$ for all $x, y \in I$, then $D(x)G(y) = 0$ (and $G(x)D(y) = 0$) for all $x, y \in I$.

Proof. To start, notice that $D(x)g(y) = 0$ is equivalent to $g(y)D(x) = 0$ by Lemma 5.1.2. Similarly $G(x)d(y) = 0 \Leftrightarrow d(y)G(x) = 0$ and $d(x)g(y) = 0 \Leftrightarrow g(y)d(x) = 0$. So (ii) $\Rightarrow$ (i) is obvious.
(i) $\Rightarrow$ (ii). We replace x by xw in (i. a) (respectively, y by wy), to obtain

$$D(x)wg(y) + xd(w)g(y) + d(x)wG(y) + xd(w)G(y) = 0 \quad \text{for all}\, x, y, w \in I, \tag{5.16}$$

and

$$D(x)g(w)y + D(x)wg(y) + d(x)wG(y) + d(x)g(w)y = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.17}$$

Subtracting them we get

$$D(x)g(w)y + d(x)g(w)y - xd(w)G(y) - xd(w)g(y) = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.18}$$

Using (a) in (5.18) we get

$$D(x)g(w)y + d(x)g(w)y + xD(w)g(y) - xd(w)g(y) = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.19}$$

Now, we replace y by yt in (5.19) to obtain

$$x(D(w)yg(t) - d(w)yg(t)) = 0 \quad \text{for all}\, x, y, w, t \in I. \tag{5.20}$$

Lemma 5.1.3, gives that $D(w)yg(t) - d(w)yg(t) = 0$, that is $(D(w) - d(w))Ig(t) = (0)$ for all $w, t \in I$. Lemma 5.1.2 gives that

$$D(w)g(t) = d(w)g(t) \quad \text{for all}\, w, t \in I. \tag{5.21}$$

On the other hand, we replace x by xw in (i. b) and we obtain

$$g(x)wd(y) + g(x)wD(y) = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.22}$$

If we multiply (5.22) to the left by $d(y)$ and to the right by $g(x)$ we have

$$d(y)g(x)wd(y)g(x) + d(y)g(x)wD(y)g(x) = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.23}$$

Lemma 5.2.1 gives that

$$(D(y)g(x) + d(y)g(x))wd(y)g(x) = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.24}$$

Using (5.21) and 2-torsion freeness, (5.24) gives $d(y)g(x)wd(y)g(x) = 0$ for all $x, y, w \in I$. Lemma 5.1.5 implies $d(y)g(x) = 0$ for all $x, y \in I$. $D(y)g(x) = 0$ for all $x, y \in I$ by (5.21). By hypothesis (i.a), we get $d(y)G(x) = 0$ for all $x, y \in I$. If we replace y by wy, and use Lemma 5.1.2, gives that $G(x)d(y) = 0$ for all $x, y \in I$. This finishes the equivalence of (i) and (ii).
Now assume that, in addition to (i), we have

$$D(x)G(y) + G(x)D(y) = 0 \quad \text{for all}\, x, y \in I. \tag{5.25}$$

We can replace x by xz (respectively, y by zy), to get (again using Lemma 5.1.2 and the equivalence between (i) and (ii))

$$D(x)zG(y) + xG(z)D(y) = 0 \quad \text{for all}\, x, y, z \in I, \tag{5.26}$$

and

$$D(x)zG(y) + G(x)D(z)y = 0 \quad \text{for all}\, x, y, z \in I. \tag{5.27}$$

Subtracting the last two identities, and using (5.25), we get

$$G(x)D(z)y + xD(z)G(y) = 0 \quad \text{for all}\, x, y, z \in I. \tag{5.28}$$

Now, we replace z by zt in (5.28) to obtain

$$G(x)D(z)ty + xD(z)tG(y) = 0 \quad \text{for all}\, x, y, z, t \in I. \tag{5.29}$$

Multiplying (5.29) to the right by $D(z)$, and doing $y = x$, we get

$$G(x)D(z)txD(z) + xD(z)tG(x)D(z) = 0 \quad \text{for all}\, x, z, t \in I. \tag{5.30}$$

Lemma 5.2.1 gives that

$$xD(z)tG(x)D(z) = 0 \quad \text{for all}\, x, z, t \in I. \tag{5.31}$$

We replace t by $tG(y)$ in (5.31) and we obtain

$$xD(z)tG(y)G(x)D(z) = 0 \quad \text{for all}\, x, y, z, t \in I. \tag{5.32}$$

Multiplying (5.29) to the right by $G(x)D(z)$, and using (5.32), we obtain

$$G(x)D(z)tyG(x)D(z) = 0 \quad \text{for all}\, x, y, z, t \in I. \tag{5.33}$$

Lemma 5.1.5 implies $G(x)D(z)t = 0$ and so $G(x)D(z) = 0$, since $l(I) = 0$. □

Now we can prove the main Theorem.

Theorem 5.2.4. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R such that $l(I) = (0)$. If (D, d) is a l-generalized derivation associated to a derivation d and (G, g) is a r-generalized derivation associated to a derivation g. Then the following conditions are equivalent:*
(i) D and G are orthogonal.
(ii) For any $x, y \in I$, the following relations holds.
(a) $D(x)g(y) + d(x)G(y) = 0$,
(b) $G(x)d(y) + g(x)D(y) = 0$,
(c) $D(x)G(y) + G(x)D(y) = 0$.
(iii) $D(x)G(y) = D(x)g(y) = d(x)G(y) = 0$ for all $x, y \in I$.
(iv) $D(x)G(y) = 0$ for all $x, y \in I$, and $d(G(x)) = d(g(x)) = 0$, for all $x \in I$.
(v) (Dg, dg) is a left generalized derivation on I, and $D(x)G(y) = 0$ for all $x, y \in I$

Proof. (i)⇒ (ii). By assumption we have $D(r)sG(t) = 0 = G(t)sD(r)$, for all $r, s, t \in R$. Lemma 5.1.2, gives $D(r)G(t) = 0 = G(t)D(r)$. In particular, we get

$$D(x)G(y) = 0 = G(y)D(x) \quad \text{for all}\, x, y \in I. \tag{5.34}$$

This implies (ii)(c). On the other hand, if we replace x by xw in (5.34) we obtain

$$D(x)wG(y) + xd(w)G(y) = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.35}$$

That is $xd(w)G(y) = 0$ for all $x, y, w \in I$. Similarly, replacing y by wy , we get

$$D(x)wG(y) + D(x)g(w)y = 0 \quad \text{for all}\, x, y, w \in I, \tag{5.36}$$

and so $D(x)g(w)y = 0$ for all $x, y, w \in I$. Lemma 5.1.5 gives, $d(w)G(y) = 0 = G(y)d(w)$, $D(x)g(w) = 0 = g(w)D(x)$ for all $x, y, w \in R$. This shows (ii).

(ii)$\Rightarrow$ (iii). It is clear by Lemma 5.2.3.
Note that Lemma 5.2.3 says that (ii)$\Rightarrow$ (iii) and $d(w)g(y) = 0$. But we will show that (iii) implies also $d(w)g(y) = 0$.
(iii)$\Rightarrow$ (iv). By assumption we have $D(x)g(y) = 0$ for all $x, y \in I$. If we replace y by wy (resp. x by xw) we get $D(x)wg(y) = 0$ and $D(x)wg(y) + xd(w)g(y) = 0$ for all $x, y, w \in I$, this gives $xd(w)g(y) = 0$ and so $d(w)g(y) = 0$ for all $y, w \in I$. Now from Lemma 5.1.4 it follows that $dg = 0$.
On the other side, using that $d(x)G(y) = 0$ and replacing x by xw, we get $d(x)wG(y) = 0$ for all $x, y, w \in I$. Lemma 5.2.2 gives $d(r)sG(t) = 0$ for all $r, s, t \in R$. Thus $0 = d(d(r)sG(t)) = d(d(r))sG(t) + d(r)d(s)G(t) + d(r)sd(G(t))$. That is $d(r)sd(G(t)) = 0$ for all $r, s, t \in R$. In particular $d(G(t))sd(G(t)) = 0$. By semiprimeness we get $dG = 0$.

(iv) $\Rightarrow$ (v). By assumption, $d(G(x)) = 0$. For all $x \in I$. So $0 = d(G(xy)) = d(x)G(y) + d(g(x))y + g(x)d(y)$. Using the hypothesis $dg = 0$ and Lemma 5.1.4, we get

$$d(x)G(y) = 0 \quad \text{for all}\, x, y \in I. \tag{5.37}$$

On the other hand, since $D(x)G(y) = 0$, in particular $0 = D(xw)G(y) = D(x)wG(y)$ by (5.37) for arbitrary $x, y, w \in I$. Now Lemma 5.1.2 gives $G(y)D(x) = 0$, for all $x, y \in I$. Replacing y by yw, to get $g(y)wD(x) = 0$, again Lemma 5.1.2 gives that

$$D(x)g(y) = 0 \quad \text{for all}\, x, y \in I. \tag{5.38}$$

By the other side. Since $Dg(xy) = Dg(x)y + g(x)d(y) + D(x)g(y) + xdg(y)$. Using our hypothesis, and (5.38), to get $Dg(xy) = Dg(x)y$ for all $x, y \in I$. Therefore (Dg, dg) is a l-generalized derivation from I to R.

(v) $\Rightarrow$ (i). By assumption (Dg, dg) is a l-generalized derivation from I to R, then $Dg(xy) = Dg(x)y + xdg(y)$. But $D(g(xy)) = Dg(x)y + g(x)d(y) + D(x)g(y) + xdg(y)$. So

$$g(x)d(y) + D(x)g(y) = 0 \quad \text{for all}\, x, y \in I. \tag{5.39}$$

Since dg is a derivation from I to R. So

$$g(x)d(y) + d(x)g(y) = 0 \quad \text{for all}\, x, y \in I. \tag{5.40}$$

Subtracting (5.40) from (5.39), we obtain

$$d(x)g(y) - D(x)g(y) = 0 \quad \text{for all}\, x, y \in I. \tag{5.41}$$

Replacing x by xw in (5.41), to get

$$d(x)wg(y) - D(x)wg(y) = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.42}$$

Now, we can replace w by $G(z)w$ and using $D(x)G(y) = 0$, to get

$$d(x)G(z)wg(y) = 0 \quad \text{for all}\, x, y, z, w \in I. \tag{5.43}$$

On the other hand. By assumption $D(x)G(y) = 0$. If we replace x by xw (resp. y by wy), we get

$$D(x)wG(y) + xd(w)G(y) = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.44}$$

And

$$D(x)wG(y) + D(x)g(w)y = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.45}$$

Subtracting the last two identities, we obtain

$$D(x)g(w)y + xd(w)G(y) = 0 \quad \text{for all}\, x, y, w \in I. \tag{5.46}$$

Multiply (5.43) to the left by t and using (5.46), we get $D(t)g(x)zwg(y) = 0$. Lemma 5.1.2 gives that $D(t)g(x)zg(y) = 0$. We can replace z by $zD(t)$, to obtain $D(t)g(x)zD(t)g(x) = 0$ for all $x, z, t \in I$. Lemma 5.1.5 gives that

$$D(t)g(x) = 0 \quad \text{for all}\, x, t \in I. \tag{5.47}$$

Using (5.47) in (5.45), we obtain $D(x)wG(y) = 0$ for all $x, y, w \in I$. Lemma 5.2.2 gives $D(R)RG(R) = (0) = G(R)RD(R)$. Thus D, G are orthogonal. $\square$

Now we give an example showing that if D and G are l-generalized and r-generalized derivation of a semiprime ring R respectively and $G(R)D(R) = 0$, then it does not imply that D and G are orthogonal.

Example 5.2.5. Let a be non-zero element of R. Let $D(x) = ax$ and $G(x) = xa$ be left and right multiplications by a, respectively. Let's assume that $a^2 = 0$. Then $(D, 0)$ and $(G, 0)$ are non-zero left and right generalized derivations, respectively, and $G(R)D(R) = 0$. If $(D, 0)$ and $(G, 0)$ are orthogonal, then $0 = G(R)RD(R) = RaRaR$. That is $aRaRaRa = 0$, If R is a semiprime ring, then $a = 0$ a contradiction.

Corollary 5.2.6. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R such that $l(I) = (0)$. If D is a l-generalized derivation associated to a derivation d and G a r-generalized derivation associated to a derivation g that are orthogonal on I, then D, g are orthogonal, G, d are orthogonal and g, d are orthogonal.*

Proof. Since by hypothesis, D and G are orthogonal on I, Lemma 5.2.2 gives that D and G are orthogonal on R. The result follows from Theorem 5.2.4. □

The following Example shows that the converse of Corollary 5.2.6 is not true.

Example 5.2.7. Consider R a semiprime ring and (D, g) and (G, d) as in Example 5.2.5. It is clear that D and g, G and d, d and g are orthogonal. However D and G are not orthogonal.

Remark 5.2.1. Corollary 5.2.6 shows that D and G do not play a symmetric role in Theorem 5.2.4. Indeed, $D(x)G(y) = D(x)g(y) = d(x)G(y) = 0 \Longrightarrow G(x)D(y) = 0$. However $G(x)D(y) = G(x)d(y) = g(x)D(y) = 0$ does not imply $D(x)G(y) = 0$. So the fact that D and G orthogonal is not equivalent to $G(x)D(y) = 0$ for every $x, y \in I$ and $g(D(x)) = g(d(x)) = 0$.
By the other side if D and G are orthogonal imply that (Dg, dg) and (gD, gd) are l-generalized derivations and (Gd, gd) and (dG, dg) are r-generalized derivations.

Corollary 5.2.8. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R such that $l(I) = (0)$, D a l-generalized derivation associated to a derivation d and G a r-generalized derivation associated to a derivation g. If $G(x)D(y) = 0$ for all $x, y \in I$, then D, g are orthogonal, G, d are orthogonal and g, d are orthogonal.*

Proof. Our hypothesises, $G(x)D(y) = 0$ for all $x, y \in I$. We replace x by xz (resp. y by zy), we obtain $g(x)zD(y) = 0$ (resp. $G(x)zd(y) = 0$) for all $x, y, z \in I$. Now the proof follow using the same lines in the proof of Corollary 5.2.6. □

5.3 Relationship between orthogonality and the composition of l- and r- generalized derivations

In this section, we study the connections between orthogonality and some properties of the composition of l-generalized derivation and a r-generalized derivation of a semiprime ring.

Theorem 5.3.1. *Let R be a 2-torsion free semiprime ring. If D is a l-generalized derivation associated to a derivation d and G is a r-generalized derivation associated to a derivation g, then D and G are orthogonal if and only if $DG = 0$ and $D(R)G(R) = (0)$.*

Proof. Assume that D and G are orthogonal, then $G(x)yD(z) = 0$ for all $x, y, z \in R$. Therefore

$$0 = D(G(x)yD(z)) = D(G(x))yD(z) + G(x)d(y)D(z) + G(x)yd(D(z)) \quad \text{for all } x, y, z \in R. \tag{5.48}$$

Since D and G are orthogonal, it follows from Corollary 5.2.6 that $D(G(x))yD(z) = 0$ for all $x, y, z \in R$. We replace z by $G(x)$ and use semiprimeness of R, then $D(G(x)) = 0$ for all $x \in R$.
On the other hand, we assume that $D(G(z)) = 0$ and $D(x)G(y) = 0$ for all $x, y, z \in R$. If we replace x by xw, we get

$$D(x)wG(y) + xd(w)G(y) = 0 \quad \text{for all } x, y, w \in R. \tag{5.49}$$

Now, we replace x by $G(x)$ in (5.49), and use $D(G(x)) = 0$, we obtain

$$G(x)d(w)G(y) = 0 \quad \text{for all } x, y, w \in R. \tag{5.50}$$

We replace x by xt in (5.50), we get

$$g(x)td(w)G(y) = 0 \quad \text{for all } x, y, w, t \in R. \tag{5.51}$$

Lemma 5.1.2 gives that $d(w)G(y)g(x) = 0$. Now we can replace w by wt, and use again Lemma 5.1.2, to get $G(y)g(x)d(w) = 0$. Replacing y by ys, which gives $g(y)sg(x)d(w) = 0$ for all $x, y, w, s \in R$, and using semiprimeness we get $g(x)d(w) = 0$ for all $x, w \in R$. By the other side, if we replace y by ty in (5.50), and use the fact $d(z)g(t) = 0$ for all $z, t \in R$, we obtain $G(x)d(w)tG(y) = 0$. Now it suffices to multiply to the right by $d(w)$, and use semiprimeness of R, to get $G(x)d(w) = 0$, thus $d(w)G(x) = 0$ for all $x, w \in R$. Substitution in (5.49), we obtain $D(x)wG(y) = 0$. Lemma 5.1.2 gives that $G(y)wD(x) = 0$ for all $x, w, y \in R$. This proves that D and G are orthogonal. □

The following example shows that semiprimeness condition in Theorems 5.2.4 and 5.3.1 is not superfluous.

Example 5.3.2. Let $\mathbb{Z}$ be the ring of integers, and

$$R = \{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{Z} \}.$$

Let's define maps $D : R \to R$ and $d : R \to R$ as follows:

$$D(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & a & a+b+c \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, d(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & -a & a-b-c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similarly $G : R \to R$, and $g : R \to R$ are defined as follows:

$$G(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & a & c-b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, g(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & -a & a-b+c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to prove that D is a l-generalized derivation associated to the derivation d, G is a r-generalized derivation associated to the derivation g and D, G are orthogonal. However $g(R)D(R) \neq 0$, $G(R)D(R) \neq 0$, $d(G(R)) \neq 0$, $d(g(R)) \neq 0$ and $D(G(R)) \neq 0$.

Chapter 6

Symmetric Generalized Biderivations on Jordan Ideals in Prime Rings

6.1 Introduction

If $B : R \times R \to R$ is a symmetric map ($B(x,y) = B(y,x)$ for all $x, y \in R$) the map $f : R \to R$ defined by $f(x) = B(x,x)$ is the trace of B. If B is also biadditive (i.e., additive in both arguments), its trace f satisfies $f(x+y) = f(x)+f(y)+2B(x,y)$, for all $x, y \in R$. A symmetric biadditive map $B : R \times R \to R$ is a symmetric biderivation if $B(xy, z) = B(x,z)y + xB(y,z)$ for all $x, y, z \in R$. The concept of symmetric biderivation was introduced by G. Maksa in [106]. A symmetric biadditive map $\tau : R \times R \to R$ is a symmetric left bicentralizer if $\tau(xy, z) = \tau(x,z)y$ (and consequently $\tau(x, yz) = \tau(x,y)z$) for all $x, y, z \in R$.

Symmetric biderivations were proved to be related to the general solution of some functional equations (see [107]). The maps $(x,y) \to \lambda[x,y]$, $\lambda \in C$, are typical examples of biderivations and they were called inner biderivations. Here C is the extended centroid of R, that is, the center of the two-sided Martindale quotient ring Q (we refer the reader to [18] for more details).
Breŝar, Martindale, and Miers in [42], shown that every biderivation of a noncommutative prime ring R is inner. In [39], Breŝar extended this result to semiprime rings. In [125], Vukman proved that if B is a nonzero symmetric biderivation, where R is a prime ring of characteristic not two, with the property:

$$B(x,x)x = xB(x,x), x \in R. \tag{6.1}$$

then R is commutative. He also proved that if B_1, B_2 are nonzero biderivations on R, D is a symmetric biadditive map and $f_1(f_2(x)) = d(x)$ holds for all $x \in R$, where f_1, f_2, and d are the traces of B_1, B_2, and D, respectively, then either $B_1 = 0$ or $B_2 = 0$. Let's mention two results proved in [126]. The first one states that if B_1 and B_2 are symmetric biderivations on a prime ring R, char$R \neq 2, 3$, such that $B_1(x,x)B_2(x,x) = 0$ holds for all $x \in R$, then either $B_1 = 0$ or $B_2 = 0$. The second result says that if $[[B(x,x), x], x] \in Z(R)$ for all $x \in R$, then R is commutative. Yenigul and Argac in [130, 12] extended the results of Vukmun in [125] assuming condition (6.1) over a nonzero ideal and a nonzero Lie ideal of a prime ring respectively.

The notion of generalized biderivation was introduced by Brešar in [40]. A bi-additive map $G : R \times R \to R$ is a generalized biderivation linked to a biderivation $B : R \times R \to R$ if for every $x \in R$, the maps $y \to G(x, y)$ and $y \to G(y, x)$ are generalized derivations of R linked to $B(x, .)$ and $B(., x)$. That is, $G(xy, z) = G(x, z)y + xB(y, z)$ and $G(x, yz) = G(x, y)z + yB(x, z)$ hold for all $x, y, z \in R$. Brešar shown that every generalized biderivation G of an ideal I ($G : I \times I \to R$) of a prime ring R with char$R \neq 2$, is of the form $G(x, y) = xay + ybx$ for some $a, b \in Q$, where Q is Martindale quotient ring of R (see [71] and [80] for details). In [4] Ali, Filippis and Shujat extended some results of Vukman contained in [125, 126] to generalized biderivations on prime and semiprime rings. Recently, in [17], symmetric generalized (σ, τ)-biderivations of a prime ring R with char$R \neq 2$ have been considered.

Remark 6.1.1. A symmetric left bicentralizer is a symmetric generalized biderivation linked to the biderivation $B = 0$

Example 6.1.1. Let R be a ring. If B is a biderivation of R and $\tau : R \times R \to R$ is a biadditive map such that $\tau(x, yz) = \tau(x, y)z$ and $\tau(xy, z) = \tau(x, z)y$ for all $x, y, z \in R$, then $B + \tau$ is a generalized biderivation of R linked to B.

In [57] some results concerning generalized derivations were proved. Here similar results are obtained for symmetric generalized biderivations. Precisely, we will consider the commutativity of a prime ring which admits a generalized biderivation G linked to a biderivation B satisfying one of the following algebraic conditions:

i. In section 6.3: $f(u) \in Z(R)$ and $g_1(u)u = ug_2(u)$ for all $u \in J$.

ii. In section 6.4: $B(g(u), u) = 0$, $G(f(u), u) = 0$ and $G(g(u), u) = 0$ for all $u \in J$.

iii. In section 6.5: $B(g(u), g(v)) = 0$, $[g_1(u), g_2(v)] = 0$, $(g(u))^2 = 0$ and $g^2(u) - f^2(u) = fg(u) - gf(u)$ for all $u, v \in J$.

Where J be a nonzero Jordan ideal of R. Here g, g_1, g_2 and f denote the traces of G, G_1, G_2 and B, respectively.

6.2 Preliminaries

We will use the following lemmas in our results.

Lemma 6.2.1. [108, Lemma 3]. *If the prime ring R contains a commutative nonzero right ideal I, then R is commutative.*

Lemma 6.2.2. [132, Lemma 2.6]. *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal of R. If $a, b \in R$ and $aJb = (0)$, then either $a = 0$ or $b = 0$.*

Lemma 6.2.3. [132, Lemma 2.7]. *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal of R. If J is commutative, then $J \subseteq Z(R)$.*

Lemma 6.2.4. [125, Theorem 4]. *Let R be a 2-torsion free semiprime ring. If there exists a symmetric biderivation $B(.,.) : R \times R \to R$ such that $B(f(x), x) = 0$ for all $x \in R$, where f denotes the trace of B, then $B = 0$.*

Remark 6.2.1. If J is a Jordan ideal of R, and $u \in J$, we have $u \circ u \in J$, therefore $2u^2 \in J$, for all $u \in J$.

Lemma 6.2.5. *Let R be a 2-torsion free ring, and $B : R \times R \to R$ be a symmetric biadditive map with trace f. If $f(r) = 0$ for all $r \in R$, then $B = 0$.*

Proof. Since $f(r) = 0$ for all $r \in R$, $f(r + s) = 0$, for all $r, s \in R$. This implies that $f(r) + f(s) + 2B(r, s) = 0$. Then $B(r, s) = 0$ for all $r, s \in R$. □

Lemma 6.2.6. *Let R be a ring. If $G : R \times R \to R$ is a symmetric generalized biderivation linked to a symmetric biderivation $B : R \times R \to R$, then the map $G - B : R \times R \to R$ is a symmetric left bicentralizer of R.*

Proof. The map $T = G - B$, is clearly biadditive. For all $x, y, z \in R$, we have

$$\begin{aligned} T(xy, z) &= (G - B)(xy, z) = G(xy, z) - B(xy, z) \\ &= G(x, z)y - B(x, z)y = T(x, z)y \end{aligned} \tag{6.2}$$

Therefore, T is a symmetric left bicentralizer of R. □

Note that Lemma 6.2.6 says that every symmetric generalized biderivation is obtained as in Example 6.1.1.

Lemma 6.2.7. *Let R be a prime ring, char$R \neq 2$, J a nonzero Jordan ideal of R. If $T : R \times R \to R$ is a symmetric left bicentralizer such that $T(u, u) = 0$ for all $u \in J$, then $T = 0$.*

Proof. Suppose that

$$T(u,u) = 0 \quad \text{for all } u \in J. \tag{6.3}$$

Then

$$2T(u,v) = 0 \quad \text{for all } u, v \in J. \tag{6.4}$$

Using the fact that characteristic of R is not 2, we get

$$T(u,v) = 0 \quad \text{for all } u, v \in J. \tag{6.5}$$

Let us replace u by $ur + ru$ in (6.5), to obtain

$$T(r,v)u = 0 \quad \text{for all } u, v \in J, r \in R. \tag{6.6}$$

That is, $T(r,v)JT(r,v) = 0$ for all $v, r \in J$. By Lemma 6.2.2, $T(r,v) = 0$ for all $v \in J, r \in R$. Now, we can replace v by $vs+sv$, for all $s \in R$. We have $T(r,s)v = 0$ for all $v \in J, r, s \in R$, that is $T(r,s)J = (0)$ for all $r, s \in R$. Lemma 6.2.2 gives $T(r,s) = 0$ for all $r, s \in R$. □

6.3 The cases $f(J) \subseteq Z(R)$ and $G_1(u,u)u = uG_2(u,u)$

We begin with the following lemma which is essential for the development of our main results.

Lemma 6.3.1. *Let R be a prime ring, char$R \neq 2$, J a nonzero Jordan ideal of R, and $B : R \times R \to R$ a symmetric biderivation with trace f. If $f(u) = 0$ for all $u \in J$, then either $J \subseteq Z(R)$ or $B = 0$.*

Proof. Suppose that

$$f(u) = 0 \quad \text{for all } u \in J. \tag{6.7}$$

Then

$$B(u,w) = 0 \quad \text{for all } u, w \in J. \tag{6.8}$$

If we replace u by $ur + ru \in J$ in (6.8), $r \in R$, we get

$$uB(r,w) + B(r,w)u = u \circ B(r,w) = 0 \quad \text{for all } u, w \in J, r \in R. \tag{6.9}$$

Replacing r by rv, where $v \in J$, and using (6.8), we get

$$0 = u \circ (B(r,w)v) + u \circ (rB(v,w)) = u \circ (B(r,w)v) \quad \text{for all } u, w, v \in J, r \in R, \tag{6.10}$$

By using the identity 1.2, we obtain

$$0 = u\circ(B(r,w)v) = (u\circ B(r,w))v - B(r,w)[u,v] \quad \text{for all}\, u,w,v \in J, r \in R. \tag{6.11}$$

From (6.9), (6.11) it follows that

$$B(r,w)[u,v] = 0 \quad \text{for all}\, u,w,v \in J, r \in R. \tag{6.12}$$

If we replace r by rs, $s \in R$, we get $B(r,w)s[u,v] = 0$ for all $u,w,v \in J$ and $r,s \in R$. Hence $B(r,w)R[u,v] = (0)$. By primeness of R it follows that either $[u,v] = 0$ for all $u,v \in J$ or $B(r,w) = 0$ for all $w \in J, r \in R$. If $[u,v] = 0$ for all $u,v \in J$, then $J \subseteq Z(R)$ by Lemma 6.2.3. In other case, $B(r,w) = 0$ for all $w \in J, r \in R$. Replacing w by $ws + sw$, we get

$$wB(r,s) + B(r,s)w = 0 \quad \text{for all}\, w \in J, r,s \in R, \tag{6.13}$$

Replacing s by su, where $u \in J$, we get

$$wB(r,s)u + B(r,s)uw = 0 \quad \text{for all}\, u,w \in J, r,s \in R, \tag{6.14}$$

If we multiply (6.13) by u to the right, and then subtract (6.14), we get $B(r,s)[u,w] = 0$ for all $u,w \in J, r,s \in R$. Replacing r by tr, $t \in R$, we get $B(t,s)R[u,w] = 0$. So by primeness of R either $B = 0$ or J is commutative. Now Lemma 6.2.3 gives the result. □

In [12], the following theorem was proved: Let R be a prime ring, $\text{char}R \neq 2$, and U be a nonzero Lie ideal of R. Let $B : R\times R \to R$ be a symmetric biderivation and f its trace. Then the following assertions are true (i) $f(U) = 0$ implies $U \subseteq Z(R)$ or $f = 0$. (ii) $f(U) \subseteq Z(R)$ and U square closed imply $U \subseteq Z(R)$ or $f = 0$.

Here we will show that this result can also be obtained for a nonzero Jordan ideal of R.

Theorem 6.3.2. *Let R be a prime ring, $\text{char}R \neq 2$, J a nonzero Jordan ideal of R, and $B : R\times R \to R$ a symmetric biderivation with trace f. If $f(u) \in Z(R)$ for all $u \in J$, then either $J \subseteq Z(R)$ or $B = 0$.*

Proof. By assumption we have

$$f(u) \in Z(R) \quad \text{for all}\, u \in J. \tag{6.15}$$

Then

$$2B(u,w) \in Z(R) \quad \text{for all}\, u,w \in J. \tag{6.16}$$

Let us replace u by $2u^2$ in (6.16). Then we have

$$4uB(u,w) + 4B(u,w)u = 8uB(u,w) \in Z(R) \quad \text{for all}\, u, w \in J. \tag{6.17}$$

Therefore, in particular $8uf(u) \in Z(R)$ for all $u \in J$. Then we have

$$0 = [uf(u), r] = [u,r]f(u) + u[f(u), r] = [u,r]f(u) \quad \text{for all}\, u \in J, r \in R. \tag{6.18}$$

So for every $r, s \in R$, $[u, rs]f(u) = 0 = r[u,s]f(u) + [u,r]sf(u) = [u,r]sf(u)$ for all $u \in J, r, s \in R$. By primeness, given a arbitrary element $u \in J$ we have either

$$f(u) = 0\, \text{or}\, u \in Z(R) \quad \text{for all}\, u \in J. \tag{6.19}$$

If $Z(R) \cap J = 0$, then $f(u) = 0\ \forall u \in J$.
Assume $Z(R) \cap J \neq 0$. If $J \not\subseteq Z(R)$, then $\exists v \in J \setminus Z(R)$. Then $\forall u \in Z(R) \cap J$, the elements $u + v, u - v \in J \setminus Z(R)$. Hence $f(u+v) = 0$ and $f(u-v) = 0$. Adding both equations $2f(u) = 0$ that is, $f(u) = 0$ using that R is 2-torsion free. In conclusion, we have proved that $f(u) = 0\ \forall u \in Z(R) \cap J$ and we already know by (6.19) that $f(u) = 0\ \forall u \in J \setminus Z(R)$. That is $f(u) = 0$ for all $u \in J$. By Lemma 6.3.1, either $J \subseteq Z(R)$ or $B = 0$. □

Corollary 6.3.3. *Let R be a prime ring, char$R \neq 2$, I a nonzero ideal of R, and $B : R \times R \to R$ a symmetric biderivation with a nonzero trace f. If $f(x) \in Z(R)$ for all $x \in I$, then R is commutative.*

Proof. It immediately follows from Theorem 6.3.2 and Lemma 6.2.1. □

Theorem 6.3.4. *Let R be a prime ring, char$R \neq 2$, J a nonzero Jordan ideal that is also a subring of R, and $G_1, G_2 : R \times R \to R$ two symmetric generalized biderivations linked to symmetric biderivations $B_1, B_2 : R \times R \to R$, respectively. If $G_1(u,u)u = uG_2(u,u)$ for all $u \in J$ and $B_2 \neq 0$, then $J \subseteq Z(R)$.*

Proof. Suppose that g_1, g_2, f_1, f_2 are traces of G_1, G_2, B_1, B_2, respectively. By hypothesis we have

$$g_1(u)u = ug_2(u) \quad \text{for all}\, u \in J. \tag{6.20}$$

Then

$$\begin{aligned} &g_1(u)v + g_1(v)u + 2G_1(u,v)u + 2G_1(u,v)v \\ &= ug_2(v) + vg_2(u) + 2uG_2(u,v) + 2vG_2(u,v) \end{aligned} \quad \text{for all}\, u, v \in J. \tag{6.21}$$

Substituting u by $-u$ in (6.21), we get

$$\begin{aligned} &g_1(u)v - g_1(v)u + 2G_1(u,v)u - 2G_1(u,v)v \\ &= -ug_2(v) + vg_2(u) + 2uG_2(u,v) - 2vG_2(u,v) \end{aligned} \quad \text{for all}\, u, v \in J. \tag{6.22}$$

Adding up (6.21) and (6.22), and using the fact that characteristic of R is not 2, we obtain

$$g_1(u)v + 2G_1(u,v)u = vg_2(u) + 2uG_2(u,v) \quad \text{for all } u, v \in J. \tag{6.23}$$

Substituting v by vu in (6.23), we have

$$\begin{aligned}(g_1(u)v + 2G_1(u,v)u - 2uG_2(u,v))u + 2vf_1(u)u \\ = vug_2(u) + 2uvf_2(u)\end{aligned} \quad \text{for all } u, v \in J. \tag{6.24}$$

Using (6.23) in (6.24), we get

$$vg_2(u)u + 2vf_1(u)u = vug_2(u) + 2uvf_2(u) \quad \text{for all } u, v \in J. \tag{6.25}$$

Replacing v by wv in (6.25), and subtracting the new identity from (6.25) multiplied by w to the left, we get

$$[w,u]vf_2(u) = 0 \quad \text{for all } u, v, w \in J. \tag{6.26}$$

This implies that $[w,u]Jf_2(u) = 0$ for all $u, w \in J$. Lemma 6.2.2, gives that for an arbitrary element $u \in J$ either $u \in Z(J)$ or $f_2(u) = 0$. If $Z(J) = 0$, then $f_2(J) = 0$. If $J = Z(J)$, then $J \subseteq Z(R)$ by Lemma 6.2.3. Let us assume that $J \neq Z(J) \neq 0$. Then $\exists u \in J \setminus Z(J)$. So $f_2(u) = 0$ since $f_2(v) = 0$ for all $v \in J \setminus Z(J)$. Take $0 \neq w \in Z(J)$. Then $u + w, u - w \in J \setminus Z(J)$ and so $B_2(u+w, u+w) = 0 = B_2(u-w, u-w)$, that is, $B_2(u,u) + 2B_2(u,w) + B_2(w,w) = 0$ and $B_2(u,u) - 2B_2(u,w) + B_2(w,w) = 0$. Adding the above two relations, we get $2f_2(w) = 0$. Since R is 2-torsion free, we have $f_2(w) = 0$ for all $w \in J$. Using Lemma 6.3.1 we get $J \subseteq Z(R)$. □

Theorem 6.3.5. *Let R be a prime ring, char$R \neq 2$, J a nonzero subring and Jordan ideal of R, and $G_1, G_2 : R \times R \to R$ two symmetric generalized biderivations linked to nonzero symmetric biderivations $B_1, B_2 : R \times R \to R$, respectively. If $G_1(u,u)u + uG_2(u,u) = 0$ for all $u \in J$ and $B_2 \neq 0$, then $J \subseteq Z(R)$.*

Proof. The proof follows the same lines of the previous one. □

Corollary 6.3.6. *Let R be a prime ring, char$R \neq 2$, I a nonzero ideal of R, $G_1, G_2 : R \times R \to R$ two symmetric generalized biderivations linked to nonzero symmetric biderivations $B_1, B_2 : R \times R \to R$, respectively. If $G_1(x,x)x = \pm xG_2(x,x)$ for all $x \in I$, then R is commutative.*

Proof. It immediately follows from Theorems 6.3.4, 6.3.5 and Lemma 6.2.1. □

Corollary 6.3.7. *Let R be a prime ring, char$R \neq 2$, J a nonzero subring and Jordan ideal of R, and $G : R \times R \to R$ a symmetric generalized biderivation associated to a symmetric biderivation $B : R \times R \to R$. If $[G(u,u),u] = 0$ (or $G(u,u) \circ u = 0$) for all $u \in J$, then $J \subseteq Z(R)$ or $B = (0)$.*

Corollary 6.3.8. *Let R be a prime ring, char$R \neq 2$, I a nonzero ideal of R, and $G : R \times R \to R$ a symmetric generalized biderivation associated to a symmetric biderivation $B : R \times R \to R$. If $G(x,x)x = \pm xB(x,x)$ for all $x \in I$, then R is commutative or G is a left bicentralizer.*

Proof. It immediately follows from Theorem 6.3.4 and Lemma 6.2.1. □

6.4 The cases $B(g(u),u) = 0$, $G(f(u),u) = 0$ and $G(g(u),u) = 0$

Theorem 6.4.1. *Let R be a prime ring, char$R \neq 2$, J a nonzero subring and Jordan ideal of R and $G : R \times R \to R$ a symmetric generalized biderivation linked to a symmetric biderivation $B : R \times R \to R$. Let g and f be the traces of G and B respectively. If $B(g(u),u) = 0$ for all $u \in J$, then either $B = 0$ or $J \subseteq Z(R)$.*

Proof. By assumption we have

$$B(g(u),v) + B(g(v),u) + 2B(G(u,v),u) + 2B(G(u,v),v) = 0 \quad \text{for all } u,v \in J. \tag{6.27}$$

Substituting u by $-u$ in (6.27), we get

$$B(g(u),v) - B(g(v),u) + 2B(G(u,v),u) - 2B(G(u,v),v) = 0 \quad \text{for all } u,v \in J. \tag{6.28}$$

Adding up (6.27) and (6.28), and using the fact that characteristic of R is not 2, we obtain

$$B(g(u),v) + 2B(G(u,v),u) = 0 \quad \text{for all } u,v \in J. \tag{6.29}$$

Substituting v by vz in (6.29), and using (6.29), we have

$$vB(g(u),z) + 2G(u,v)B(z,u) + 2B(v,u)B(u,z) + 2vB(B(u,z),u) = 0 \quad \text{for all } u,v,z \in J. \tag{6.30}$$

Replacing v by wv in (6.30), subtracting the new identity from (6.29) multiplied by w to the left, and characteristic of R is not 2, we get

$$\begin{aligned} G(u,w)vB(z,u) - wG(u,v)B(z,u) \\ +wB(u,v)B(z,u) + B(w,u)vB(u,z) = 0 \end{aligned} \quad \text{for all } u,v,z,w \in J. \tag{6.31}$$

Taking $w=v$ in (6.31), we obtain

$$\begin{aligned} G(u,v)vB(z,u)-vG(u,v)B(z,u)\\ +vB(u,v)B(z,u)+B(v,u)vB(u,z)=0 \end{aligned} \quad \text{for all } u,v,z,w\in J. \qquad (6.32)$$

This implies that

$$([G(u,v),v]+B(u,v^2))B(u,z)=0 \quad \text{for all } u,v,z\in J. \qquad (6.33)$$

Now, we can replace z by tz, $t\in J$ in (6.33) and we obtain

$$([G(u,v),v]+B(u,v^2))JB(u,z)=0 \quad \text{for all } u,v,z\in J.$$

Given a fixed $u\in J$, either $B(u,z)=0$ for all $z\in J$ or $[G(u,v),v]+B(u,v^2)=0$ for all $v\in J$. But $A_1=\{u\in J \mid [G(u,v),v]+B(u,v^2)=0 \quad \forall v\in J\}$ and $A_2=\{u\in J \mid B(u,z)=0 \quad \forall v\in J\}$ are both additive subgroups of J and $J=A_1\cup A_2$. So either $J=A_1$ or $J=A_2$. If $J=A_2$, then $B(u,z)=0$ for all $u,z\in J$. Theorem 6.3.2 gives that $J\subseteq Z(R)$ or $B=0$.
On the other hand, if $J=A_1$, then $[G(u,v),v]+B(u,v^2)=0$ for all $u,v\in J$. Replacing u by uw, where $w\in J$, we get

$$G(u,v)[w,v]+u[B(w,v),v]+[u,v]B(w,v)+uB(w,v^2)=0 \quad \text{for all } u,v,w\in J. \qquad (6.34)$$

Taking $w=v$ in (6.34), we get

$$u[f(v),v]+[u,v]f(v)+uB(v,v^2)=0 \quad \text{for all } u,v\in J. \qquad (6.35)$$

Substituting u by tu in (6.35), gives

$$[t,v]uf(v)=0 \quad \text{for all } u,v,t\in J. \qquad (6.36)$$

Then using the same arguments used in the proof of Theorem 6.3.4, we get that $J\subseteq Z(R)$ or $B=0$. □

Theorem 6.4.2. *Let R be a prime ring, char$R\neq 2$, J a nonzero subring and Jordan ideal of R and $G: R\times R\to R$ a symmetric generalized biderivation linked to a symmetric biderivation $B: R\times R\to R$. Let f be the trace of B. If $G(f(u),u)=0$ for all $u\in J$, then either $B=0$ or $J\subseteq Z(R)$.*

Proof. By assumption we have

$$\begin{aligned} G(f(u),v)+G(f(v),u)+2G(B(u,v),u)\\ +2G(B(u,v),v)=0 \end{aligned} \quad \text{for all } u,v\in J. \qquad (6.37)$$

Substituting u by $-u$ in (6.37), we get

$$\begin{aligned} G(f(u),v)-G(f(v),u)+2G(B(u,v),u) \\ -2G(B(u,v),v)=0 \end{aligned} \quad \text{for all } u,v\in J. \tag{6.38}$$

Adding up (6.37) and (6.38), and using the fact that characteristic of R is not 2, we obtain

$$G(f(u),v)+2G(B(u,v),u)=0 \quad \text{for all } u,v\in J. \tag{6.39}$$

Substituting v by vz in (6.39), and using (6.39), we have

$$\begin{aligned} vB(f(u),z)+2B(u,v)B(z,u)+2G(v,u)B(u,z) \\ +2vB(B(u,z),u)=0 \end{aligned} \quad \text{for all } u,v,z\in J. \tag{6.40}$$

Replacing v by wv in (6.40), subtracting the new identity from (6.39) multiplied by w to the left, and characteristic of R is not 2, we get

$$\begin{aligned} G(u,w)vB(u,z)-wG(v,u)B(u,z) \\ +wB(v,u)B(u,z)+B(u,w)vB(u,z)=0 \end{aligned} \quad \text{for all } u,v,z,w\in J. \tag{6.41}$$

Taking $w=v$ in (6.41), we obtain

$$\begin{aligned} G(u,v)vB(z,u)-vG(u,v)B(z,u) \\ +vB(u,v)B(z,u)+B(v,u)vB(u,z)=0 \end{aligned} \quad \text{for all } u,v,z,w\in J. \tag{6.42}$$

This implies that

$$([G(u,v),v]+B(u,v^2))B(u,z)=0 \quad \text{for all } u,v,z\in J. \tag{6.43}$$

Then using the same arguments that were used to prove Theorem 6.4.1, we get that $J\subseteq Z(R)$ or $B=0$. □

Theorem 6.4.3. *Let R be a prime ring, $charR\neq 2$, J a nonzero subring and Jordan ideal of R and $G: R\times R\to R$ a symmetric generalized biderivation associated to a symmetric biderivation $B: R\times R\to R$. Let g be the trace of G. If $G(g(u),u)=0$ for all $u\in J$, then either $B=0$ or $J\subseteq Z(R)$.*

Proof. By assumption we have

$$\begin{aligned} G(g(u),v)+G(g(v),u)+2G(G(u,v),u) \\ +2G(G(u,v),v)=0 \end{aligned} \quad \text{for all } u,v\in J. \tag{6.44}$$

Substituting u by $-u$ in (6.44), we get

$$\begin{aligned} G(g(u),v)-G(g(v),u)+2G(G(u,v),u) \\ -2G(G(u,v),v)=0 \end{aligned} \quad \text{for all } u,v\in J. \tag{6.45}$$

Adding up (6.44) and (6.45), and using the fact that characteristic of R is not 2, we obtain

$$G(g(u),v)+2G(G(u,v),u)=0 \quad \text{for all } u,v\in J. \tag{6.46}$$

Substituting v by vz in (6.46), and using (6.46), we have

$$\begin{aligned} vB(g(u),z)+4G(u,v)B(z,u) \\ +2vB(B(u,z),u)=0 \end{aligned} \quad \text{for all } u,v,z\in J. \tag{6.47}$$

Replacing v by wv in (6.47), subtracting the new identity from (6.46) multiplied by w to the left, and characteristic of R is not 2, we get

$$\begin{aligned} G(u,w)vB(z,u)-wG(u,v)B(z,u) \\ +wB(u,v)B(u,z)=0 \end{aligned} \quad \text{for all } u,v,z,w\in J. \tag{6.48}$$

Taking $w=v$ in (6.48), we obtain

$$\begin{aligned} G(u,v)vB(z,u)-vG(u,v)B(z,u) \\ +vB(u,v)B(z,u)=0 \end{aligned} \quad \text{for all } u,v,z,w\in J. \tag{6.49}$$

This implies that

$$(G(u,v^2)-vG(u,v))B(z,u)=0 \quad \text{for all } u,v,z\in J. \tag{6.50}$$

Now, we can replace z by tz, $t\in J$ in (6.50) and we obtain

$$(G(u,v^2)-vG(u,v))JB(z,u)=0 \quad \text{for all } u,v,z\in J. \tag{6.51}$$

Arguing as in the proof of Theorem 6.4.1, either $B(u,z)=0$ for all $u,z\in J$ or $G(u,v^2)-vG(u,v)=0$ for all $u,v\in J$. In the first case, Theorem 6.3.2 gives that $J\subseteq Z(R)$ or $B=0$. On the other hand, if $G(u,v^2)-vG(u,v)=0$ for all $u,v\in J$. Replacing u by uw, where $w\in J$, and using the previous relation, we get

$$uB(w,v^2)-vuB(w,v)=0 \quad \text{for all } u,v,w\in J. \tag{6.52}$$

Substituting u by tu in (6.52), and using (6.52) gives

$$[t,v]uB(w,v)=0 \quad \text{for all } u,v,w,t\in J. \tag{6.53}$$

Then using the same arguments that were used in the proof of Theorem 6.3.4, we get that $J\subseteq Z(R)$ or $B=0$. □

6.5 The cases $B(g(u), g(v)) = 0$, $[g_1(u), g_2(v)] = 0$ and $(g(u))^2 = 0$

Theorem 6.5.1. *Let R be a prime ring, char$R \neq 2$, J a nonzero subring and Jordan ideal of R and $G : R \times R \to R$ a symmetric generalized biderivation linked to a symmetric biderivation $B : R \times R \to R$. Let g be the trace of G. If $g(J) \subseteq J$, and $B(g(u), g(v)) = 0$ for all $u, v \in J$, then either $B = 0$ or $J \subseteq Z(R)$.*

Proof. By linearization, our assumptions imply

$$B(G(u, w), g(v)) = 0 \quad \text{for all}\, u, v, w \in J. \tag{6.54}$$

Substituting w by wz in (6.54), and using (6.54) we get

$$\begin{aligned} G(u, w)B(z, g(v)) + B(w, g(v))B(u, z) \\ +wB(B(u, z), g(v)) = 0 \end{aligned} \quad \text{for all}\, u, v, w, z \in J. \tag{6.55}$$

Substituting z by $g(z)$ in (6.55), and using our assumption, we have

$$B(w, g(v))B(u, g(z)) + wB(B(u, g(z)), g(v)) = 0 \quad \text{for all}\, u, v, w, z \in J. \tag{6.56}$$

Substituting w by sw in (6.56), and using (6.56), we obtain

$$B(w, g(v))sB(u, g(z)) = 0 \quad \text{for all}\, u, v, w, s, z \in J. \tag{6.57}$$

In particular, $B(u, g(u))JB(u, g(u)) = 0$ for all $u \in J$. Lemma 6.2.2, gives $B(u, g(u)) = 0$ for all $u \in J$. Theorem 6.4.1 gives that $J \subseteq Z(R)$ or $B = 0$. □

Theorem 6.5.2. *Let R be a prime ring with char$R \neq 2, 3$, J a nonzero subring and Jordan ideal of R and $G_1, G_2 : R \times R \to R$ two symmetric generalized biderivations linked to nonzero symmetric biderivations $B_1, B_2 : R \times R \to R$, respectively. Let g_1, g_2 be the traces of G_1, G_2 respectively. If $g_2(J) \subseteq J$ and $[g_1(u), g_2(v)] = 0$ for all $u, v \in J$, then $J \subseteq Z(R)$.*

Proof. By linearization,

$$[g_1(u), G_2(v, w)] = 0 \quad \text{for all}\, u, v, w \in J. \tag{6.58}$$

Replace w by wz in (6.58) and using (6.58), we get

$$\begin{aligned} G_2(v, w)[g_1(u), z] + [g_1(u), w]B_2(v, z) \\ +w[g_1(u), B_2(v, z)] = 0 \end{aligned} \quad \text{for all}\, u, v, w, z \in J. \tag{6.59}$$

Replace z by $g_2(z)$ in (6.59) and using our assumption, we get

$$[g_1(u), w]B_2(v, g_2(z)) + w[g_1(u), B_2(v, g_2(z))] = 0 \quad \text{for all}\, u, v, w, z \in J. \tag{6.60}$$

Substituting w by tw in (6.60), and using (6.60), we get

$$[g_1(u), t]wB_2(v, g_2(z)) = 0 \quad \text{for all}\, u, v, w, z, t \in J. \tag{6.61}$$

This implies that $[g_1(u), t]JB_2(v, g_2(z)) = 0$ for all $u, v, z, t \in J$. Lemma 6.2.2 guarantees that either $[g_1(u), t] = 0$ for all $u, t \in J$ or $B_2(v, g_2(z)) = 0$ for all $v, z \in J$. If $[g_1(u), t] = 0$, then Corollary 6.3.7 gives that $J \subseteq Z(R)$. On the other hand, if $B_2(v, g_2(z)) = 0$ then $J \subseteq Z(R)$ follows from Theorem 6.4.1. □

Theorem 6.5.3. *Let R be a prime ring with char$R \neq 2$, J a nonzero subring and Jordan ideal of R and $G : R \times R \to R$ a symmetric generalized biderivations linked to a symmetric biderivation $B : R \times R \to R$. Let g be the trace of G. If $g(J) \subseteq J$ and $(g(u))^2 = 0$ for all $u \in J$, then either $B = 0$ or $J \subseteq Z(R)$.*

Proof. By linearization (we refer readers to [134]), our assumptions imply

$$g(u)G(u, w) + G(u, w)g(u) = 0 \quad \text{for all}\, u, w \in J. \tag{6.62}$$

Replacing w by wz in (6.62) and using (6.62), we get

$$G(u, w)[z, g(u)] + g(u)wB(u, z) + wB(u, z)g(u) = 0 \quad \text{for all}\, u, w, z \in J. \tag{6.63}$$

Doing $z = g(u)$ in (6.63), we obtain

$$g(u)wB(u, g(u)) + wB(u, g(u))g(u) = 0 \quad \text{for all}\, u, w \in J. \tag{6.64}$$

Substituting z by $tg(u)$ in (6.63), and using (6.63), we get

$$g(u)wtB(u, g(u)) + wtB(u, g(u))g(u) = 0 \quad \text{for all}\, u, w, t \in J. \tag{6.65}$$

Replacing w by $g(u)$ in (6.65) and using our assumption, we obtain

$$g(u)tB(u, g(u))g(u) = 0 \quad \text{for all}\, u, t \in J. \tag{6.66}$$

Right multiplication of (6.66) by $B(u, g(u))$ gives

$$B(u, g(u))g(u)tB(u, g(u))g(u) = 0 \quad \text{for all}\, u, t \in J. \tag{6.67}$$

Now Lemma 6.2.2, gives

$$B(u,g(u))g(u)=0 \quad \text{for all } u\in J. \tag{6.68}$$

Using 6.68 in 6.64 we have

$$g(u)wB(u,g(u))=0 \quad \text{for all } u,w\in J. \tag{6.69}$$

This implies that $g(u)JB(u,g(u))=0$ for all $u\in J$. Let's fix some element u_1. Lemma 6.2.2 gives that $g(u_1)=0$ or $B(u_1,g(u_1))=0$. But $g(u_1)=0$ implies $B(u_1,g(u_1))=0$. Then we have $B(u,g(u))=0$ for all $u\in J$. Theorem 6.4.1 gives that $B=0$ or $J\subseteq Z(R)$. □

Theorem 6.5.4. *Let R be a prime ring with char$R\neq 2,3$, J a nonzero subring and Jordan ideal of R, and $G:R\times R\to R$ a symmetric generalized biderivation associated to a symmetric biderivation $B:R\times R\to R$. If $G(G(u,u),G(u,u))-B(B(u,u),B(u,u))=B(G(u,u),G(u,u))-G(B(u,u),B(u,u))$ for all $u\in J$,then one of the following holds:*
(i) $B=0$ or (ii) $G=B$. or (iii) $J\subseteq Z(R)$.

Proof. Let g,f be the traces of G,B, respectively. By hypothesis we have

$$g^2(u)-f^2(u)=fg(u)-gf(u) \quad \text{for all } u\in J. \tag{6.70}$$

The substitution of u by $u+v$ in (6.70), gives

$$\begin{aligned} &4g(G(u,v))+2G(g(u),g(v))+4G(g(u),G(u,v))\\ &+4G(g(v),G(u,v))-4f(B(u,v))-2B(f(u),f(v))\\ &-4B(f(u),B(u,v))-4B(f(v),B(u,v))\\ &=4f(G(u,v))+2B(g(u),g(v))+4B(g(u),G(u,v))\\ &+4B(g(v),G(u,v))-4g(B(u,v))-2G(f(u),f(v))\\ &-4G(f(u),B(u,v))-4G(f(v),B(u,v)) \end{aligned} \quad \text{for all } u,v\in J. \tag{6.71}$$

Substituting v by $-v$ in (6.71), we obtain

$$\begin{aligned} &4g(G(u,v))+2G(g(u),g(v))-4G(g(u),G(u,v))\\ &-4G(g(v),G(u,v))-4f(B(u,v))-2B(f(u),f(v))\\ &+4B(f(u),B(u,v))+4B(f(v),B(u,v))\\ &=4f(G(u,v))+2B(g(u),g(v))-4B(g(u),G(u,v))\\ &-4B(g(v),G(u,v))-4g(B(u,v))-2G(f(u),f(v))\\ &+4G(f(u),B(u,v))+4G(f(v),B(u,v)) \end{aligned} \quad \text{for all } u,v\in J. \tag{6.72}$$

Adding (6.71), (6.72), and using char$R\neq 2$, we get

$$\begin{aligned} 2g(G(u,v))+G(g(u),g(v))-2f(B(u,v))& \\ -B(f(u),f(v))=2f(G(u,v))+B(g(u),g(v))& \quad \text{for all } u,v\in J. \\ -2g(B(u,v))-G(f(u),f(v))& \end{aligned} \tag{6.73}$$

The substitution v by $v+w$ in (6.73), gives

$$\begin{aligned} 2G(G(u,v),G(u,w))+G(g(u),G(v,w))& \\ -2B(B(u,v),B(u,w))-B(f(u),B(v,w))& \quad \text{for all } u,v,w\in J. \\ =2B(G(u,v),G(u,w))+B(g(u),G(v,w))& \\ -2G(B(u,v),B(u,w))-G(f(u),B(v,w))& \end{aligned} \tag{6.74}$$

Let us take $T=G-B$, and denote $k=g+f$ the trace of $K=G+B$. By Lemma 6.2.6, T is a symmetric left bicentralizer of R. Then (6.74), reduces to

$$\begin{aligned} 2T(G(u,v),G(u,w))+T(g(u),G(v,w))& \quad \text{for all } u,v,w\in J. \\ +2T(B(u,v),B(u,w))+T(f(u),B(v,w))=0& \end{aligned} \tag{6.75}$$

Replacing w by wx in (6.75), we get

$$\begin{aligned} 2T(G(u,v),w)B(u,x)+T(g(u),w)B(v,x)& \quad \text{for all } u,v,w,x\in J. \\ +2T(B(u,v),w)B(u,x)+T(f(u),w)B(v,x)=0& \end{aligned} \tag{6.76}$$

That is

$$2T(K(u,v),w)B(u,x)+T(k(u),w)B(v,x)=0 \quad \text{for all } u,v,w,x\in J. \tag{6.77}$$

Doing $v=u$ in (6.77), and using char$R\neq 3$, we get

$$T(k(u),w)B(u,x)=0 \quad \text{for all } u,x,w\in J. \tag{6.78}$$

Doing $x=u$ in (6.77), we obtain

$$2T(K(u,v),w)f(u)+T(k(u),w)B(v,u)=0 \quad \text{for all } u,v,w\in J. \tag{6.79}$$

Using (6.78) in (6.79), and char$R\neq 2$, we have

$$T(K(u,v),w)f(u)=0 \quad \text{for all } u,v,w\in J. \tag{6.80}$$

Substituting v by vt in (6.80), gives

$$T(K(u,v)t+2vB(u,t),w)f(u)=0 \quad \text{for all } u,v,w,t\in J. \tag{6.81}$$

That is,

$$T(K(u,v),w)tf(u) + 2T(v,w)B(u,t)f(u) = 0 \quad \text{for all}\, u,v,w,t \in J. \tag{6.82}$$

By replacing w by wt in (6.80) and using the new equation in (6.82), we obtain

$$T(v,w)B(u,t)f(u) = 0 \quad \text{for all}\, u,v,w,t \in J. \tag{6.83}$$

Now, we can replace w by ww_1 in (6.83) and we obtain $T(v,w)JB(u,t)f(u) = 0\ \forall u,v,w,t \in J$. Lemma 6.2.2, gives either $T(v,w) = 0$ for all $v,w \in J$ or $B(u,t)f(u) = 0$ for all $u,t \in J$. In the first case, Lemma 6.2.7 proves that $T = 0$. Therefore $G = B$. On the other hand if $B(u,t)f(u) = 0$ for all $u,t \in J$, then replacing t by ut, we have $f(u)Jf(u) = (0)$ for all $u \in J$. By Lemma 6.2.2, $f(u) = 0$ for all $u \in J$. Theorem 6.3.2 gives that $J \subseteq Z(R)$ or $B = 0$. □

Corollary 6.5.5. *Let R be a prime ring, char$R \neq 2$, I a nonzero ideal of R, and $G : R \times R \to R$ a symmetric generalized biderivation associated to a symmetric biderivation $B : R \times R \to R$. If $G(G(x,x),G(x,x)) - B(B(x,x),B(x,x)) = B(G(x,x),G(x,x)) - G(B(x,x),B(x,x))$ for all $x \in I$, then R is commutative or $G = B$ or G is a left bicentralizer.*

Proof. It immediately follows from Theorem 6.5.4 and Lemma 6.2.1. □

The following example shows that the primeness condition in Theorems 6.3.4, 6.3.5, 6.4.1, 6.4.2, and 6.5.3 is not superfluous.

Example 6.5.6. Let $\mathbb{Z}$ be the set of integers, and $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} | a,b \in \mathbb{Z} \right\}$. Now, let's take $J = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, b \in \mathbb{Z} \right\}$. It can be easily checked that J is an ideal (so a Jordan ideal and a subring) of R. We define maps $G : R \times R \to R$ and $B : R \times R \to R$ as follows:

$$B\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ ac & 0 \end{pmatrix},\ G\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ bd & 0 \end{pmatrix}.$$

Then it is easy to check that B is a biderivation, G is a generalized biderivation associated to B, $[G(u,u),u] = 0$, $G(u,u) \circ u = 0$, $B(G(u,u),u) = 0$, and $(G(u,u))^2 = 0$ for all $u \in J$. However, $B \neq 0$, $B \neq G$ and $J \nsubseteq Z(R)$.

Chapter 7

Multiplicative (Generalized)-derivations on Semiprime Rings

7.1 Introduction

The idea of multiplicative derivation was introduced in 1991 by Daif [51] as follows: A map $D : R \to R$ (not necessarily additive) is called a *multiplicative derivation* of R if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. Daif's work [51] was motivated by the work of Martindale [81]. Further, the complete description of those maps were given by Goldmann and Ŝemrl in [66].

The notion of multiplicative derivation was extended to multiplicative generalized derivation by Daif and Tammam in [53] as follows: A map $F : R \to R$ (not necessarily additive) is said to be a multiplicative generalized derivation if there exist a multiplicative derivation $D : R \to R$ such that $F(xy) = F(x)y + xD(y)$ for all $x, y \in R$. Further, Dhara and Ali [56] generalized this definition of multiplicative generalized derivation by considering D as any map on R. So, a map $F : R \to R$ (not necessarily additive) is called *multiplicative (generalized)-derivation* if $F(xy) = F(x)y + xg(y)$ holds for all $x, y \in R$, where $g : R \to R$ is any map (not necessarily a derivation nor additive). Hence, the concept of multiplicative (generalized)-derivation covers the concept of multiplicative derivation. Moreover, multiplicative (generalized)-derivation with $g = 0$ covers the notion of *multiplicative centralizer* (not necessarily additive).

It is obvious that every generalized derivation is multiplicative (generalized)-derivation on R. However, the converse does not need to be true in general, as the following examples show:

Example 7.1.1. Consider $R = C[0,1]$, the ring of all continuous real functions and define maps $D : R \to R$, as follows:

$$D(f)(x) = \{ \begin{smallmatrix} f(x)log|f(x)| & \text{for all } f(x)\neq 0 \\ 0, & \text{otherwise.} \end{smallmatrix}$$

And $F : R \to R$, as follows,

$$F(f)(x) = \{ \begin{smallmatrix} f(x)(1+log|f(x)|) & \text{for all } f(x)\neq 0 \\ 0, & \text{otherwise.} \end{smallmatrix}$$

It is easy to verify that D, F are not additive maps, D is a multiplicative derivation and F is a multiplicative (generalized)-derivation associated to D.

Example 7.1.2. Consider the ring $R = \{ \begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{R}\}$. Define maps $F : R \to R$ and $D : R \to R$ as follows:

$$F(\begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 & bd \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } D(\begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 & c^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that D, F are not additive maps, D is a multiplicative derivation and F is a multiplicative (generalized)-derivation associated to D.

Over the last three decades, several authors have proved commutativity theorems for prime or semiprime rings admitting automorphisms or derivations which are centralizing or commuting on some appropriate subsets of R (see [10], [14], [16], [37], [123] and [119], where further references can be found).

Let S be a nonempty subset of R. A map $F : R \to R$ is called commutativity preserving on a subset S of R if $[x, y] = 0$ implies $[F(x), F(y)] = 0$, for all $x, y \in S$. The map F is called strong commutativity preserving (simply, SCP) on S if $[x, y] = [F(x), F(y)]$ for all $x, y \in S$. There is also a growing literature on strong commutativity preserving (SCP) maps and derivations (for reference see [23], [38], [105], etc.)

In [6], A. Ali, M. Yasen and M. Anwar showed that if R is a semiprime ring and f is an endomorphism which is a strong commutativity preserving (simply, SCP) map on a nonzero ideal U of R, then f is commuting on U. In [117], M. S. Samman proved that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing. Derivations, as well as SCP maps, have been extensively studied by researchers in the context of operator algebras, prime rings and semiprime rings too.

In [52], Daif and Bell proved that if R is a semiprime ring, U is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = [x, y]$ for all $x, y \in U$ then

$U \subseteq Z$. This result was considered for generalized derivations by Quadri et al. in [115]. It was extended by Shang in [120].

In [10] Argac proved that if R is a semiprime ring and I is a nonzero ideal of R, then a derivation d of R is commuting on I if one of the following conditions holds: (i) $d([x,y]) = \pm[x,y]$. (ii) $d(x \circ y) = \pm x \circ y$. (iii) $[d(x), d(y)] = \pm d([x,y])$. Here $d([x,y]) = \pm[x,y]$ means that for every pair of elements $x, y \in R$ $d([x,y]) = [x,y]$ (resp. $d([x,y]) = \pm[x,y]$). Similarly in the other conditions. Inspired by this result, Ashraf et al. [14] have studied the situations when the derivation d is replaced with a generalized derivation F in the setting of prime ring R. In [16] Ashraf et al. generalized this result to semiprime case and proved the following result: If F, G are generalized derivations of R associated to derivations d, g respectively, and they satisfy one of the following algebraic conditions: (i) $F(x)x = xG(x)$, (ii) $[F(x), d(y)] = [x,y]$, (iii) $F([x,y]) = [d(x), F(y)]$, (iv) $[F(x), y] = [x, G(y)]$, (vii) $F([x,y]) = [F(x), y] + [d(y), x]$ for all x, y in a nonzero ideal in R, then R contains a nonzero central ideal.

The purpose of this chapter is to prove some results related to multiplicative (generalized)-derivations on semiprime rings which are of independent interest. In fact, our results extend some known results by replacing a two-sided ideal I by a left-sided ideal L and a generalized derivation by a multiplicative (generalized)-derivation in the setting of semiprime rings. Let F, G be multiplicative (generalized)-derivations associated to maps d, g respectively. We will consider the following algebraic conditions:

i. In section 7.2: $[F(x), F(y)] = \pm[x,y]$ (SCP map F), $[F(x), y] = \pm[x, G(y)]$, $[g(x), F(y)] = \pm[x,y]$ and $[g(x), y] = \pm[x, F(y)]$ for all $x, y \in L$.

ii. In section 7.3: $F(x)y = \pm xG(y)$, $F(x)y \pm xG(y) \in Z(R)$ and $F(xy) = \pm F(yx)$ for all $x, y \in L$.

iii. In section 7.4: $F([x,y]) = \pm[x,y]$ and $F(x \circ y) = \pm(x \circ y)$ for all $x, y \in L$.

In this line of research, Dhara and Ali in [56] proved the following result: If F a multiplicative (generalized)-derivation associated to a map g and they satisfy one of the following algebraic conditions: (i) $F(xy) \pm xy \in Z(R)$, (ii) $F(x)F(y) \pm xy \in Z(R)$ for all x, y in a left ideal L of a semiprime ring R, then $L[g(x), x] = (0)$ for all $x \in L$.

The following lemmas will be used in our results.

Lemma 7.1.3. [22]. *Let R be a 2-torsion free semiprime ring and L a left ideal of R. If $a, b \in R$ and $axb + bxa = 0$ for all $x \in L$, then $axb = 0$ and $bxa = 0$.*

Lemma 7.1.4. [119, Lemma 2.1]. *Let R be a semiprime ring, I a nonzero two-sided ideal of R. If $a \in I$ and $axa = 0$ for all $x \in I$, then $a = 0$.*

Lemma 7.1.5. [71, Lemma 1.1.8]. *If R is a semiprime ring, then the center of a nonzero one sided ideal of R is contained in the center of R.*

Lemma 7.1.6. *Let R be a semiprime ring and L a nonzero left ideal of R. If $a \in R$ and $ax = 0$ for all $x \in L$, then $a \in C_R(L)$.*

Proof. Our assumption is $ax = 0$ for all $x \in L$. Replacing x by rx, where $r \in R$, we get $arx = 0$, that is, $xarxa = 0$ for all $r \in R$. Semiprimeness of R gives $xa = 0$ for all $x \in L$. Therefore, $ax = xa$ for all $x \in L$ and hence $a \in C_R(L)$. □

7.2 SCP multiplicative (generalized)-derivation on left ideal

Theorem 7.2.1. *Let R be a 2-torsion free semiprime ring, L a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated to the map g. If $F(xy) = xF(y) + g(x)y$ $x, y \in L$ and F is SCP on L, then $L[g(x), x] = 0$ and $L[F(x), x] = 0$ for all $x \in L$.*

Proof. Since F is SCP on L, thus

$$[F(x), F(y)] = [x, y] \quad \text{for all}\, x, y \in L. \tag{7.1}$$

Replacing y by yx in (7.1), we get

$$[F(x), F(y)]x+F(y)[F(x), x]+[F(x), y]g(x)+y[F(x), g(x)] = [x, y]x \,\text{for all}\, x, y \in L. \tag{7.2}$$

Multiplying (7.1) to the right by x, we have

$$[F(x), F(y)]x = [x, y]x \quad \text{for all}\, x, y \in L. \tag{7.3}$$

Combining (7.2) and (7.3), we obtain

$$F(y)[F(x), x] + [F(x), y]g(x) + y[F(x), g(x)] = 0 \quad \text{for all}\, x, y \in L. \tag{7.4}$$

Now if we substitute y by zy in (7.4) and use (7.4), we get

$$g(z)y[F(x), x] + [F(x), z]yg(x) = 0 \quad \text{for all}\, x, y, z \in L. \tag{7.5}$$

Take $z = x$, we have $g(x)y[F(x), x] + [F(x), x]yg(x) = 0$ for all $x, y \in L$. Lemma 7.1.3, gives

$$[F(x), x]yg(x) = 0 \quad \text{for all}\, x, y \in L. \tag{7.6}$$

Since $F(x^2) = F(x)x + xg(x) = xF(x) + g(x)x$ for all $x \in L$, this gives

$$[F(x), x] = [g(x), x] \quad \text{for all}\, x \in L. \tag{7.7}$$

Using (7.7) in (7.6) we have $[g(x), x]yg(x) = 0$ for all $x \in L$, that is $L[g(x), x]RL[g(x), x] = 0$ for all $x \in L$. Use semiprimeness of R, to get $L[g(x), x] = 0$ for all $x \in L$. Substitution in (7.7), gives $L[F(x), x] = 0$ for all $x \in L$.
The case $[F(x), F(y)] = -[x, y]$ for all $x, y \in L$ is similar. □

Remark 7.2.1. In a similar way, the result can be proved for the case $[F(x), F(y)] = -[x, y]$ for all $x, y \in L$.

Corollary 7.2.2. *Let R be a 2-torsion free semiprime ring and $F : R \to R$ a multiplicative (generalized)-derivation associated to the map g. If $F(xy) = xF(y)+ g(x)y$ for all $x, y \in R$ and F is SCP on L or $[F(x), F(y)] = -[x, y]$ for all $x, y \in R$, then g and F are commuting on R.*

Theorem 7.2.3. *Let R be a semiprime ring, L a nonzero left ideal of R and F, G two multiplicative (generalized)-derivations associated to maps d, g respectively. If $[F(x), y] = [x, G(y)]$ or $[F(x), y] = -[x, G(y)]$ for all $x, y \in L$, then $L[d(x), x] = 0$ and $L[g(x), x] = 0$ for all $x \in L$.*

Proof. Assume

$$[F(x), y] = [x, G(y)] \quad \text{for all}\, x, y \in L. \tag{7.8}$$

If we replace in (7.8) x by xy, we get

$$[F(x), y]y + [x, y]d(y) + x[d(y), y] = [x, G(y)]y + x[y, G(y)] \quad \text{for all}\, x, y \in L. \tag{7.9}$$

On the other hand, if we multiply (7.8) to the right by y we get

$$[F(x), y]y = [x, G(y)]y \quad \text{for all}\, x, y \in L. \tag{7.10}$$

Subtracting (7.10) from (7.9), we obtain

$$[x, y]d(y) + x[d(y), y] = x[y, G(y)] \quad \text{for all}\, x, y \in L. \tag{7.11}$$

Now if we substitute x by rx in (7.11), we get

$$r[x, y]d(y) + [r, y]xd(y) + rx[d(y), y] = rx[y, G(y)] \quad \text{for all}\, x, y \in L, r \in R. \tag{7.12}$$

If we multiply (7.11) to the left by r, we obtain

$$r[x,y]d(y) + rx[d(y),y] = rx[y,G(y)] \quad \text{for all}\, x,y \in L, r \in R. \tag{7.13}$$

Form (7.12) and (7.13), we get $[r,y]xd(y) = 0$. If we replace r by $d(y)$, we have

$$[d(y),y]xd(y) = 0 \quad \text{for all}\, x,y \in L. \tag{7.14}$$

If we multiply (7.14) to the right by y, we get $[d(y),y]xd(y)y = 0$. Replacing x by xy in (7.14), to get $[d(y),y]xyd(y) = 0$. Subtracting the last two identities, we obtain $[d(y),y]x[d(y),y] = 0$. Since L is left ideal, its follows that $L[d(y),y]RL[d(y),y] = 0$ for all $y \in L$, use semiprimeness of R, then $L[d(y),y] = 0$ for all $y \in L$.

On the other hand, if we replace y by yx in (7.8) we have

$$[F(x),y]x+y[F(x),x] = [x,G(y)]x+y[x,g(x)]+[x,y]g(x) \quad \text{for all}\, x,y \in L. \tag{7.15}$$

Using (7.8) in (7.15) we get

$$y[F(x),x] = y[x,g(x)] + [x,y]g(x) \quad \text{for all}\, x,y \in L. \tag{7.16}$$

We can follow the same lines as above and conclude that $L[g(y),y] = 0$ for all $y \in L$.

The case $[F(x),y] = -[x,G(y)]$ for all $x,y \in L$ is similar. □

Corollary 7.2.4. *Let R be a semiprime ring, I a nonzero two sided ideal and F,G two multiplicative (generalized)-derivations associated to maps d,g respectively. If $[F(x),y] = [x,G(y)]$ or $[F(x),y] = -[x,G(y)]$ for all $x,y \in I$, then d, g are commuting on I.*

Proof. By Theorem 7.2.3, we have $I[d(y),y] = 0$ for all $y \in I$. Multiplying to the left by $[d(y),y]$ we have $[d(y),y]I[d(y),y] = 0$. Lemma 7.1.4, gives $[d(y),y] = 0$ for all $y \in I$, that is, d is commuting on I. In the same way, we get g is commuting on I. □

Theorem 7.2.5. *Let R be a 2-torsion free semiprime ring, L a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated to the map g. If $F(xy) = xF(y) + g(x)y$ and $[g(x),F(y)] = [x,y]$ or $[g(x),F(y)] = -[x,y]$ for all $x,y \in L$, then $L[g(x),x] = 0$ and $L[F(x),x] = 0$ for all $x \in L$.*

Proof. Assume that

$$[g(x),F(y)] = [x,y] \quad \text{for all}\, x,y \in L. \tag{7.17}$$

Replacing y by yx in (7.17), to get

$$[g(x), F(y)]x + F(y)[g(x), x] + [g(x), y]g(x) = [x, y]x \quad \text{for all}\, x, y \in L. \tag{7.18}$$

Multiplying (7.17) to the right by x, we obtain

$$[g(x), F(y)]x = [x, y]x \quad \text{for all}\, x, y \in L. \tag{7.19}$$

Combining (7.18) and (7.19) we obtain

$$F(y)[g(x), x] + [g(x), y]g(x) = 0 \quad \text{for all}\, x, y \in L. \tag{7.20}$$

If we replace y by zy in (7.58), we get

$$zF(y)[g(x), x]+g(z)y[g(x), x]+z[g(x), y]g(x)+[g(x), z]yg(x) = 0 \quad \text{for all}\, x, y, z \in L. \tag{7.21}$$

Using (7.58) in (7.21) and take $z = x$, we get

$$g(x)y[g(x), x] + [g(x), x]yg(x) = 0 \quad \text{for all}\, x, y \in L. \tag{7.22}$$

Lemma 7.1.3, gives $[g(x), x]yg(x) = 0$ for all $x, y \in L$, this gives

$$L[g(x), x]RL[g(x), x] = 0 \quad \text{for all} \quad x \in L.$$

Since R is semiprime, we conclude that $L[g(x), x] = 0$ for all $x \in L$. Since $F(x^2) = F(x)x + xg(x) = xF(x) + g(x)x$ for all $x \in L$, this gives $[F(x), x] = [g(x), x]$, thus $L[F(x), x] = L[g(x), x] = 0$ for all $x \in L$.

The case $[g(x), F(y)] = -[x, y]$ for all $x, y \in L$ is similar. □

Corollary 7.2.6. *Let R be a 2-torsion free semiprime ring and $F : R \to R$ a multiplicative (generalized)-derivation associated to the map g. If $F(xy) = xF(y)+ g(x)y$ and $[g(x), F(y)] = [x, y]$ or $[g(x), F(y)] = -[x, y]$ for all $x, y \in R$, then g and F are commuting on R.*

Theorem 7.2.7. *Let R be a semiprime ring, L a nonzero left ideal of R and F a multiplicative (generalized)-derivation associated to a map g. If $[g(x), y] = [x, F(y)]$ or $[g(x), y] = -[x, F(y)]$ for all $x, y \in L$, then $L[d(x), x] = 0$ and $L[F(x), x] = 0$ for all $x \in L$.*

Proof. Assume that

$$[g(x), y] = [x, F(y)] \quad \text{for all}\, x, y \in L. \tag{7.23}$$

Replacing y by yx in (7.23), we get

$$[g(x), y]x + y[g(x), x] = [x, F(y)]x + y[x, g(x)] + [x, y]g(x) \quad \text{for all}\, x, y \in L. \tag{7.24}$$

Multiplying (7.23) to the right by x, we obtain

$$[g(x), y]x = [x, F(y)]x \quad \text{for all}\, x, y \in L. \tag{7.25}$$

Combining (7.24) and (7.25), we have

$$2y[g(x), x] = [x, y]g(x) \quad \text{for all}\, x, y \in L. \tag{7.26}$$

Now, we can replace y by zy in (7.26) and using (7.26), we get

$$[x, z]yg(x) = 0 \quad \text{for all}\, x, y, z \in L. \tag{7.27}$$

We replace z by rz in (7.27) we get

$$[x, r]zyg(x) = 0 \quad \text{for all}\, x, y, z \in Lr \in R. \tag{7.28}$$

Replacing z by $g(x)z$ in (7.28), to get $0 = [x, r]g(x)zyg(x)$, that is

$$[x, r]g(x)Rzyg(x) = (0) \quad \text{for all} \quad x, y, z \in L,\, r \in R.$$

Interchanging z and y and then subtracting one from the other, we have

$$[x, r]g(x)R[z, y]g(x) = (0) \quad \text{for all} \quad x, y, z \in L.$$

In particular (take $r = z$ and $y = x$), $[x, z]g(x)R[x, z]g(x) = (0)$ for all $x, z \in L$. Since R is semiprime, thus

$$[x, z]g(x) = 0 \quad \text{for all}\, x, z \in L. \tag{7.29}$$

multiplying (7.29) to the right by x, we get $[x, z]g(x)x = 0$ for all $x, z \in L$. Replacing z by zx , we get $[x, z]xg(x) = 0$ for all $x, z \in L$. Subtracting the last two identities, we get $[x, z][g(x), x] = 0$ for all $x, z \in L$. Replacing z by $g(x)z$ in the last identity, we obtain $[x, g(x)]z[x, g(x)] = 0$ for all $x, z \in L$, that is $L[g(x), x]RL[g(x), x] = (0)$ for all $x \in L$. Semiprimeness of R, gives $L[g(x), x] = (0)$ for all $x \in L$. By our assumption we have $[x, F(x)] = [g(x), x]$, that is $L[x, F(x)] = L[g(x), x] = 0$ for all $x \in L$.

The case $[g(x), y] = -[x, F(y)]$ for all $x, y \in L$ is similar. □

Corollary 7.2.8. *Let R be a semiprime ring, and F a multiplicative (generalized)-derivation associated to a map g. If $[g(x), y] = [x, F(y)]$ or $[g(x), y] = -[x, F(y)]$ for all $x, y \in R$, then F and g are commuting maps on R.*

7.3 The case $F(x)y \pm xG(y) \in Z(R)$

Theorem 7.3.1. *Let R be a semiprime ring, L a nonzero left ideal of R and F, G two multiplicative (generalized)-derivations associated to maps d, g respectively. If $F(x)y = xG(y)$ or $F(x)y = -xG(y)$ for all $x, y \in L$, then $Lg(L) = 0$ and $G(xy) = G(x)y$ for all $x, y \in L$. Moreover, if $F(xy) = xF(y) + d(x)y$ for all $x, y \in L$, then $d(R) \subseteq C_R(L)$.*

Proof. Assume that

$$F(x)y = xG(y) \quad \text{for all}\, x, y \in L. \tag{7.30}$$

We can replace y by yz in (7.30) and obtain

$$F(x)yz = xG(y)z + xyg(z) \quad \text{for all}\, x, y, z \in L. \tag{7.31}$$

Multiplying (7.30) to the right by z, we get

$$F(x)yz = xG(y)z \quad \text{for all}\, x, y \in L. \tag{7.32}$$

Combining (7.31) and (7.32), we obtain

$$xyg(z) = 0 \quad \text{for all}\, x, y, z \in L, \tag{7.33}$$

Replacing y by $g(z)rx$ where $r \in R$ in (7.33), we have $xg(z)rxg(z) = 0$ for all $x, z \in L, r \in R$. From the semiprimeness of R its follows that $xg(z) = 0$ for all $x, z \in L$, that is $Lg(L) = 0$. For all $x, y \in L$, we obtain $G(xy) = G(x)y + xg(y) = G(x)y$.

On the other hand, replacing x by rx in (7.30) yields

$$rF(x)y + d(r)xy = rxG(y) \quad \text{for all}\, x, y \in L, r \in R. \tag{7.34}$$

Multiplying (7.30) to the left by r we get

$$rF(x)y = rxG(y) \quad \text{for all}\, x, y \in L, r \in R. \tag{7.35}$$

Combining (7.34) and (7.35), we obtain

$$d(r)xy = 0 \quad \text{for all}\, x, y \in L, r \in R. \tag{7.36}$$

Replacing y by $sd(r)x$ where $r \in R$ in (7.36), we get $d(r)xsd(r)x = 0$ for all $x \in L, r, s \in R$. By semiprimeness we get $d(r)x = 0$ for all $x \in L, r \in R$. Lemma 7.1.6, gives $d(R) \subseteq C_R(L)$.

The case $F(x)y = -xG(y)$ for all $x, y \in L$ is similar. □

Corollary 7.3.2. *Let R be a semiprime ring and F and G two multiplicative (generalized)-derivations associated to maps d, g respectively. If $F(x)y = xG(y)$ or $F(x)y = -xG(y)$ for all $x, y \in R$, then $g = 0$ and $G(xy) = G(x)y$ for all $x, y \in R$. Moreover, if $F(xy) = xF(y) + d(x)y$ for all $x, y \in R$, then $d = 0$ and $F(xy) = F(x)y$ for all $x, y \in R$.*

Proof. Theorem 7.3.1, gives $Rg(R) = 0$. By semiprimeness of R, we get $g = 0$ that is $G(xy) = G(x)y$ for all $x, y \in R$. Similarly, $d = 0$ that is $F(xy) = F(x)y$ for all $x, y \in R$. □

Theorem 7.3.3. *Let R be a semiprime ring, L a nonzero left ideal of R and F, G two multiplicative (generalized)-derivations associated to maps d, g respectively. If $F(x)y+xG(y) \in Z(R)$ or $F(x)y-xG(y) \in Z(R)$ for all $x, y \in L$, then $L[g(x), x] = 0$ for all $x \in L$. Moreover if $F(xy) = xF(y) + d(x)y$ for all $x, y \in L$, then $L[d(x), x] = 0$ for all $x \in L$.*

Proof. Assume

$$F(x)y + xG(y) \in Z(R) \quad \text{for all}\, x, y \in L. \tag{7.37}$$

If we substitute y by yz in (7.37), we get

$$\begin{aligned} &F(x)yz + xG(y)z + xyg(z) \\ &= (F(x)yz + xG(y))z + xyg(z) \in Z(R) \end{aligned} \quad \text{for all}\, x, y, z \in L. \tag{7.38}$$

Using (7.37) in (7.38), we obtain

$$[xyg(z), z] = 0 \quad \text{for all}\, x, y, z \in L. \tag{7.39}$$

Now, we replace x by rx in (7.39), where $r \in R$, to obtain

$$\begin{aligned} 0 = [rxyg(z), z] &= r[xyg(z), z] + [r, z]xyg(z) \\ &= [r, z]xyg(z) \end{aligned} \quad \text{for all}\, x, y, z \in L, r \in R. \tag{7.40}$$

Following the same lines that were used in the proof of Theorem 7.2.7, eq. (7.28), we can prove that $L[g(z), z] = 0$ for all $z \in L$.

On the other hand, assume that $F(xy) = xF(y) + d(x)y$ for all $x, y \in L$. By using our hypothesis and replacing x by zx in (7.37), we get

$$\begin{aligned} zF(x)y + d(z)xy + zxG(y) &= z(F(x)y + xG(y)) \\ &+d(z)xy \in Z(R) \end{aligned} \quad \text{for all}\, x, y, z \in L. \tag{7.41}$$

Using (7.37) in (7.41) we obtain

$$[d(z)xy, z] = 0 \quad \text{for all}\, x, y, z \in L. \tag{7.42}$$

That is
$$d(z)x[y,z] + [d(z),z]xy = 0 \quad \text{for all } x,y,z \in L. \tag{7.43}$$

Replacing y by yt where $t \in L$ in (7.43) and using (7.43) we get $d(z)xy[t,z]$ for all $x,y,z,t \in L$, and hence $d(z)LRL[L,z] = (0)$ for all $z \in L$. Since R is semiprime and $R/P_\imath$ is prime. It must contain a family $\mathbf{P} = \{P_\imath | \imath \in \wedge\}$ of ideals such that $\cap P_\imath = (0)$ (see [8] for details). If P is any member of $\mathbf{P}$ and $z \in L$, it follows that
$$d(z)L \subseteq P \quad \text{or} \quad L[L,z] \subseteq P \quad \text{for all } z \in L. \tag{7.44}$$

These two conditions together imply that $d(z)L[L,z] \subseteq P$ for any $P \in \mathbf{P}$. Therefore, $d(z)L[L,z] \subseteq \cap_{\imath \in \wedge} P_\imath = (0)$ for all $z \in L$, that is $d(z)y[x,z] = 0$ for all $x,y,z \in L$. Using the last identity in (7.43), we have
$$[d(z),z]xy = 0 \quad \text{for all } x,y,z \in L. \tag{7.45}$$

Multiplying (7.45) to the right by $d(z)$ we get $[d(z),z]xyd(z) = 0$. Replacing y by $d(z)y$ in (7.45) we have $[d(z),z]xd(z)y = 0$ for all $x,y,z \in L$. From the last two identities we conclude that $[d(z),z]x[d(z),y] = 0$ for all $x,y \in L$. In particular, $L[d(z),z]RL[d(z),z] = 0$ for all $z \in L$. By semiprimeness we get $L[d(z),z] = 0$ for all $z \in L$.

The case $F(x)y - xG(y) \in Z(R)$ for all $x,y \in L$ is similar. □

Corollary 7.3.4. *Let R be a semiprime ring, and $F : R \to R$ a multiplicative (generalized)-derivation associated to the map g. If $F(x)y + xF(y) \in Z(R)$ or $F(x)y - xF(y) \in Z(R)$ for all $x,y \in R$, then g is commuting on R.*

Corollary 7.3.5. *Let R be a semiprime ring, I a nonzero ideal of R and F,G two multiplicative (generalized)-derivations associated to maps d,g respectively. If $F(x)y + xG(y) \in Z(R)$ or $F(x)y - xG(y) \in Z(R)$ for all $x,y \in I$, then g is commuting on I. Moreover, if $F(xy) = xF(y) + d(x)y$ for all $x,y \in I$, then d is commuting on I.*

Proof. Theorem 7.3.3, gives $I[d(y),y] = 0$ for all $y \in I$, that is $[d(y),y]I[d(y),y] = 0$. Lemma 7.1.4, gives d is commuting on I. Similarly, g is commuting on I. □

Theorem 7.3.6. *Let R be a semiprime ring, L a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated to a map g. If $F(xy) = F(yx)$ or $F(xy) = -F(yx)$ for all $x,y \in L$, then $L[g(x),x] = 0$ for all $x \in L$.*

Proof. Assume that,
$$F(xy) = F(yx) \quad \text{for all } x,y \in L. \tag{7.46}$$

If we replace x by xy in (7.46), we get

$$F(xy)y + xyg(y) = F(yx)y + yxg(y) \quad \text{for all } x, y \in L. \tag{7.47}$$

Multiplying (7.46) to the right by y, we obtain

$$F(xy)y = F(yx)y \quad \text{for all } x, y \in L. \tag{7.48}$$

Combining (7.47) and (7.48) we obtain

$$[x, y]g(y) = 0 \quad \text{for all } x, y \in L. \tag{7.49}$$

Replacing x by rx where $r \in R$, to get $[r, y]xg(y) = 0$ for all $x, y \in L, r \in R$. It suffices to replace r by $g(y)$, to get $L[g(y), y]RL[g(y), y] = 0$ for all $y \in L$. Use semiprimeness of R, then $L[g(y), y] = 0$ for all $y \in L$.

The case $F(xy) + F(yx) = 0$ for all $y \in L$ is similar. □

Corollary 7.3.7. *Let R be a semiprime ring, and $F : R \to R$ a multiplicative (generalized)-derivation associated to a map g. If $F(xy) = F(yx)$ or $F(xy) = -F(yx)$ for all $x, y \in R$, then g is commuting on R.*

7.4 The case $F([x, y]) = \pm[x, y]$

Theorem 7.4.1. *Let R be a semiprime ring, L a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated to a map g. If $F([x, y]) = [x, y]$ or $F([x, y]) = -[x, y]$ for all $x, y \in L$, then $L[g(x), x] = 0$ for all $x \in L$.*

Proof. Assume that

$$F([x, y]) = [x, y] \quad \text{for all } x, y \in L. \tag{7.50}$$

Replacing y by yx in (7.50), to get

$$F([x, y])x + [x, y]g(x) = [x, y]x \quad \text{for all } x, y \in L. \tag{7.51}$$

Multiplying (7.50) to the right by x, we obtain

$$F([x, y])x = [x, y]x \quad \text{for all } x, y \in L. \tag{7.52}$$

Combining (7.51) and (7.52), we have

$$[y, x]g(x) = 0 \quad \text{for all } x, y \in L. \tag{7.53}$$

Replacing y by ry where $r \in R$ in (7.53), we get $[r, x]yg(x) = 0$ for all $x, y \in L, r \in R$. It suffices to replace r by $g(x)$, to get $L[g(x), x]RL[g(x), x] = 0$ for all $x \in L$. Use semiprimeness of R, then $L[g(x), x] = 0$ for all $x \in L$.
The case $F([x, y]) = -[x, y]$ for all $x, y \in L$ is similar. □

Corollary 7.4.2. *Let R be a semiprime ring and $F : R \to R$ a multiplicative (generalized)-derivation associated to the map g. If $F([x,y]) = [x,y]$ or $F([x,y]) = -[x,y]$ for all $x, y \in R$, then g is commuting on R.*

Theorem 7.4.3. *Let R be a semiprime ring, L a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated to the map g. If $F(x \circ y) = (x \circ y)$ or $F(x \circ y) = -(x \circ y)$ for all $x, y \in L$, then $L[g(x), x] = 0$ for all $x \in L$.*

Proof. Assume that

$$F(x \circ y) = (x \circ y) \quad \text{for all}\, x, y \in L. \tag{7.54}$$

Replacing y by yx in (7.54), to get

$$F(x \circ y)x + (x \circ y)g(x) = (x \circ y)x \quad \text{for all}\, x, y \in L. \tag{7.55}$$

Multiplying (7.54) to the right by x, we obtain

$$F(x \circ y)x = (x \circ y)x \quad \text{for all}\, x, y \in L. \tag{7.56}$$

Combining (7.55) and (7.56), we get

$$(x \circ y)g(x) = 0 \quad \text{for all}\, x, y \in L. \tag{7.57}$$

Replacing y by yz in (7.57), we have

$$[x, y]zg(x) = 0 \quad \text{for all}\, x, y, z \in L. \tag{7.58}$$

Replacing y by ry where $r \in R$ in (7.58), we get $[r, x]yzg(x) = 0$ for all $x, y, z \in L, r \in R$. Following the same lines that were used in the proof of Theorem 7.3.3, we can reach $L[g(x), x] = 0$ for all $x \in L$.

The case $F(x \circ y) = -(x \circ y)$ for all $x, y \in L$. □

Corollary 7.4.4. *Let R be a semiprime ring, and $F : R \to R$ a multiplicative (generalized)-derivation associated to the map g. If $F(x \circ y) = (x \circ y)$ or $F(x \circ y) = -(x \circ y)$ for all $x, y \in R$, then g is commuting on R.*

7.5 Some Examples

The following example shows that the semiprimeness condition in the previous Theorems is not superfluous.

Example 7.5.1. Let $\mathbb{Z}$ be the set of integers, and, $R = \{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{Z}\}$.

For any $0 \neq b \in \mathbb{Z}$, $\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$, then R is not a semiprime ring. We define maps $F : R \to R$ and $g : R \to R$ as follows:

$$F(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } g(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & a^2 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Then its easy to check that F is a multiplicative (generalized)-derivation associated to the map g. It is straightforward to verify that: $F(x)y = yF(x) = 0$ for all $x, y \in R$. So, (i)$[F(x), y] = \pm[x, F(y)]$, (ii) $F(x)y = xF(y)$, (iii)$F(x)y + xF(y) \in Z(R)$, (iv) $F([x, y]) = [x, y]$, (v) $F(x \circ y) = (x \circ y)$ for all $x, y \in R$. However, $g \neq 0$, g is not commuting on R, and $F(xy) \neq F(x)y$.

Conclussions and future work

An important and active research line in Non-commutative Algebra is the line known as "Commutative Theorems". The final aim in it is to find conditions that guarantee the commutativity of a ring.

This thesis can be placed in this area of work and conditions studied are related to several types of generalizations of derivations. In a concrete way, conditions have been applied to prime or semiprime rings.

So in Chapter 2 the notions of l-generalized and r-generalized reverse derivations have been introduced. These notions extend the one of reverse derivation. In particular we have proved that the existence of such a map (l-generalized or r-generalized reverse derivation) in a semiprime ring implies the existence of a non-zero central ideal. Furthermore, if the ring is 2-torsion free, then the notions of l-generalized, r-generalized reverse derivations, l-generalized and r-generalized derivations coincide.

In Chapter 3 we have considered maps on a semiprime ring that are left or right generalized derivations on a Lie ideal U. In particular we proved that if (F, d) is a l- and r-generalized derivation and $F^2(U) = (0)$ then $d(U) = F(U) = (0)$ and $d(R), F(R) \subseteq C_R(U)$. On the other side, if (F, d) and (G, g) are right and left generalized derivations, respectively, and $F(u)v = uG(v)$ for all $u, v \in U$, then $d(U), g(U) \subseteq C_R(U)$.

In Chapter 4 center-like elements have been studied. Different generalized centers have been defined and they have been proved to be all equal to the center of the ring R when R is prime.

In Chapter 5 the notion of orthogonality has been considered. We have studied, in particular, the existence of a l-generalized and a r-generalized derivation that are orthogonal, finding necessary and sufficient conditions for their existence. We have got some consequence of orthogonality on the composition of the maps.

In Chapter 6 we have studied rings having a symmetric generalized biderivation that satisfies some algebraic conditions on elements of a Jordan ideal J. When the considered ring R is prime and the biderivation is non-zero then we could conclude that $J \subseteq Z(R)$.

Finally, in Chapter 7 multiplicative (generalized)-derivations of semiprime rings have been considered. We have studied some algebraic conditions to derive consequences on the multiplication in the ring.

We believe that some of the studied problems may have sense in a super-rings context. For instance we can consider the right notion of generalized derivation and study if their existence may imply supercommutativity. We consider that this is an interesting line of research that deserves to be explored and that we intend to consider in a near future.

Posner proved two important theorems in this context:

Th.1 If R is prime, ch$R \neq 2$ and d_1, d_2 derivations of R. Then $d_1 d_2$ is not a derivation.

Th.2 If R is prime and there is a centralizing derivation of R then R is commutative.

M. Mathieu proved that Th.1 implies Th.2 and studied properties of the composition of two derivations in a $\mathbb{C}^*$-algebra. He proved that $d_1 d_2$ is a derivation only if $d_1 d_2 = 0$ and that the operator $d_1 d_2$ is bounded if and only if both d_1 and d_2 are bounded.

Breŝar and Vukman introduced the notion of orthogonality in rings inspired by Posner's Th.1. They studied orthogonality of derivations in 2-torsion free semiprime rings deriving some equivalent conditions. In particular Posner Th.1 is a consequence of those results.

What consequences follow from the orthogonality in a $\mathbb{C}^*$-algebra or in another normed algebra?

We intend to explore those problems as part of our future work.

Conclusiones y trabajo futuro

Una importante y activa línea de investigación en Algebra No-commutativa es la denominada como "Teoremas de Conmutatividad" cuyo objetivo final es encontrar condiciones que garanticen la conmutatividad de un anillo.

La presente tesis se enmarca en este campo de trabajo y, concretamente, se estudian condiciones de conmutatividad que involucran varios tipos de generalizaciones de derivaciones. Se obtienen resultados para anillos primos y semiprimos.

Así en el capítulo 2 se han introducido las nociones de derivación generalizada reversa a izquierda y a derecha. En particular hemos probado que la existencia de una aplicacion de este tipo (a izquierda o a derecha) en un anillo semiprimo implica la existencia de un ideal central no trivial. Además, si el anillo es libre de 2-torsion, entonces las nociones de derivación generalizada reversa a izquierda, a derecha, derivación generalizada a izquierda y a derecha coinciden.

En el capítulo 3 hemos considerado aplicaciones sobre un anillo semiprimo que son derivaciones generalizadas a derecha sobre un ideal de Lie U. Se ha probado, en particular, que si (F, d) es una derivación generalizada a izquierda (resp. a derecha) y $F^2(U) = (0)$, entonces $d(U) = F(U) = (0)$ y $d(R), F(R) \subseteq C_R(U)$. Por otro lado, si (F, d) y (G, g) son derivaciones generalizadas a derecha e izqda, respectivamente, y $F(u)v = uG(v)\ \ \forall u, v \in U$, entonces $d(U), g(U) \subseteq C_R(U)$.

En el capítulo 4 se han estudiado elementos de "tipo central". Se han definido varios "centros generalizados" y se ha probado que todos ellos coinciden con el centro en el caso de un anillo primo.

En el capítulo 5 hemos considerado la noción de ortogonalidad ligada, en nuestro caso, a una derivación generalizada a izquierda y una derivación generalizada a derecha. Se han encontrado varias condiciones equivalentes a la ortogonalidad y se han obtenido consecuencias en la composición de las aplicaciones.

En el capítulo 6 se han estudiado anillos con una bi-derivación generalizada que satisface algunas condiciones algebraicas sobre elementos de un ideal de Jordan J. Cuando el anillo R es primo y la biderivación no es cero, concluimos que $J \subseteq Z(R)$.

Finalmente, en el capítulo 7 se han considerado las derivaciones multiplicativas generalizadas de anillos semiprimos. Se estudian condiciones algebraicas que implican consecuencias en la multiplicación en el anillo.

Creemos que algunos de los problemas estudiados pueden tener sentido en el contexto de superanillos. Asi, por ejemplo, parece natural buscar la noción adecuada de (super) derivación generalizada y estudiar en que situaciones su existencia puede implicar (super) conmutatividad. Consideramos que ésta es una línea de investigación que merece ser explorada a corto plazo.

Posner probó dos importantes resultado en este contexto:

Teorema 1. Si R es primo, $\mathrm{ch}R \neq 2$ y d_1,d_2 son derivaciones no nulas de R, entonces $d_1 d_2$ no es derivación.

Teorema 2. Si R es primo y existe una derivación de R que es "centralizadora", entonces R es conmutativo.

M. Mathieu probó que el teorema 1 implica el teorema 2 y estudió propiedades de la composición de dos derivaciones en una $\mathbb{C}^*$-álgebra. Probó que $d_1 d_2$ es una derivación sólo si $d_1 d_2 = 0$ y que el operador $d_1 d_2$ es acotado si y sólo si tanto d_1 como d_2 son acotadas. Es decir, el teorema 1 de Posner sigue siendo válido en una $\mathbb{C}^*$-álgebra.

Breŝar y Vukman introdujeron la noción de ortogonalidad de derivaciones en anillos inspirados en el teorema 1 de Posner. Estudiaron dicha ortogonalidad en anillos semiprimos libres de 2-torsión encontrando varias condiciones equivalentes. En particular el teorema 1 de Posner es una consecuencia de estos resultados.

¿Qué consecuencias se siguen de la ortogonalidad de derivaciones generalizadas en una $\mathbb{C}^*$-álgebra o en otra álgebra normada?

Intentamos explorar estos problemas como parte de nuestro trabajo futuro.

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Alphabetical index

Appendix A

Publications derived from the thesis

Title: GENERALIZED REVERSE DERIVATIONS ON SEMIPRIME RINGS.
Author: A. Aboubakr and S. Gonźalez.
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GENERALIZED REVERSE DERIVATIONS ON SEMIPRIME RINGS

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Abstract: We generalize the notion of reverse derivation by introducing generalized reverse derivations. We define an *l-generalized reverse derivation* (*r-generalized reverse derivation*) as an additive mapping $F : R \to R$, satisfying $F(xy) = F(y)x + yd(x)$ $(F(xy) = d(y)x + yF(x))$ for all $x, y \in R$, where d is a reverse derivation of R. We study the relationship between generalized reverse derivations and generalized derivations on an ideal in a semiprime ring. We prove that if F is an l-generalized reverse (or r-generalized) derivation on a semiprime ring R, then R has a nonzero central ideal.

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Keywords: semiprime ring, ideal, derivation, reverse derivation, l-generalized derivation, r-generalized derivation, l-generalized reverse derivation, r-generalized reverse derivation

1. Introduction

Throughout this paper R denotes an associative ring with center $Z(R)$. If I is a subset of R, then $C_R(I)$ denotes the centralizer of I which is defined by

$$C_R(I) = \{x \in R \mid xa = ax \quad \text{for all} \quad a \in I\}.$$

Recall that R is *prime* if $aRb = (0)$ implies that $a = 0$ or $b = 0$. The ring R is *semiprime* if $aRa = 0$ implies $a = 0$ (obviously, every prime ring is semiprime). As usual, $[x, y]$ denotes the commutator $xy - yx$. We will make extensive use of the basic commutator identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$. An additive mapping d from R into itself is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Given $a \in R$, the additive mapping $d : R \to R$ defined by $d(x) = [x, a]$ for all $x \in R$ is a derivation called the *inner derivation* of R determined by a.

The notion of reverse derivation arose in one early paper of Herstein [1], when he studied Jordan derivations on prime associative rings. The notion of reverse derivation has relations with some generalizations of derivations. A reverse derivation is an additive mapping d from a ring R into itself satisfying $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$. So, each reverse derivation is a Jordan derivation (but the converse is not true in general). In the anticommutative case each reverse derivation is an antiderivation, and each antiderivation is a reverse derivation. The reverse derivations in the case of prime Lie and prime Malcev algebras were studied by Hopkins and Filippov. Those papers provided some examples of nonzero reverse derivations for the simple 3-dimensional Lie algebra sl_2 (see [2]) and characterized the prime Lie algebras admitting a nonzero reverse derivation (see [3, 4]). In particular, Filippov proved that each prime Lie algebra, admitting nonzero reverse derivation is a PI-algebra. Filippov also described all reverse derivations of prime Malcev algebras [5]. The supercase of reverse derivations (antisuperderivations) of simple Lie superalgebras was studied by Kaygorodov in [6] and [7]. He proved that every reverse superderivation of a simple finite-dimensional Lie superalgebra over an algebraically closed field of characteristic zero is the zero mapping. After that, Kaygorodov proved that every r-generalized reverse (or l-generalized) derivation of a simple (non-Lie) Malcev algebra is the zero mapping (see [8]).

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In [1], Herstein showed that if R is a prime ring, and d is a nonzero reverse derivation of R, then R is a commutative integral domain and d is a derivation. Later Samman and Alyamani extended the result by Herstein to semiprime rings in [9], proving that if R is a semiprime ring then a reverse derivation is just a derivation from R to its center.

The generalized derivations were defined by Brešar [10] in 1991. An additive mapping $f : R \to R$ is called a *generalized derivation* if there exists a derivation $d : R \to R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. The concept of generalized derivation includes both the concept of derivation and that of left multiplier (when $d = 0$). Gölbasi and Kaya [11] distinguish between the l-generalized derivation associated to a derivation d (Brešar generalized derivations) and the r-generalized derivations associated to d, the additive mappings $F : R \to R$ satisfying $F(xy) = d(x)y + xF(y)$ for all $x, y \in R$.

In this paper we extend the notion of reverse derivation to that of generalized reverse derivation. Let R be a ring and let d be a reverse derivation of R. An additive mapping $F : R \to R$ is said to be an *l-generalized reverse derivation* of R associated with d if

$$F(xy) = F(y)x + yd(x) \quad \text{for all } x, y \in R.$$

F is said to be an *r-generalized reverse derivation* associated with d if

$$F(xy) = d(y)x + yF(x) \quad \text{for all } x, y \in R.$$

The main purpose of this paper is to extend the above results to generalized reverse derivations. If R is a semiprime ring, I is an ideal of R, and $F : I \to R$ is an l-generalized reverse derivation (r-generalized reverse derivation), then we will show that F is an r-generalized derivation (l-generalized derivation) and applies I into $C_R(I)$. In particular, R has a nonzero central ideal.

Generalized Jordan derivations are considered in [12]. A generalized Jordan derivation of a ring R is a mapping $f : R \to R$ that satisfies $f(x^2) = f(x)x + xd(x)$ for all $x \in R$ for some Jordan derivation d of R. In Theorem 2.7 the authors of [12] proved that each generalized Jordan derivation of a 2-torsion-free semiprime ring R is a generalized derivation. Clearly, the notion of r-generalized Jordan derivation can also be considered. An r-generalized Jordan derivation is a mapping $g : R \to R$ that satisfies $g(x^2) = d(x)x + xg(x)$ with d a Jordan derivation of R. The proof of Theorem 2.7 in [12] can be adapted to prove the same result for an r-generalized Jordan derivation; i.e., each r-generalized Jordan derivation of a 2-torsion-free semiprime ring R is an r-generalized derivation. Using this extended version of Theorem 2.7 we will prove in the paper that, in the case of 2-torsion-free semiprime rings, the notions of r-generalized reverse, l-generalized reverse, r-generalized, and l-generalized derivations coincide.

2. Preliminaries and Examples

The following lemmas will be widely used in our results.

Lemma 2.1 [13, Theorem 3]. *Let R be a semiprime ring and let I be a nonzero left ideal. If R admits a nonzero derivation d centralizing on I, then R has a nonzero central ideal.*

Lemma 2.2 [12, Theorem 2.7]. *Let R be a 2-torsion-free semiprime ring. Then each generalized Jordan derivation on R is a generalized derivation* (*left or right*).

The following fact is well known (see [13, Fact IV]).

Proposition 2.3. *In a prime ring, the centralizer of each nonzero one-sided ideal is equal to the center of R. In particular, if R has a nonzero central ideal, then R is commutative.*

The examples explore the possible relationships between l-generalized reverse derivations, r-generalized reverse derivations, l-generalized derivations, and r-generalized derivations.

Example 1. Let S be a ring and

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in S \right\}.$$

Hence R is a ring. Let us define the mappings $F : R \to R$ and $d : R \to R$ as follows:

$$F\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad d\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & a-c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that d is both a reverse derivation and a derivation, F is an l-generalized reverse derivation and an r-generalized reverse derivation associated with d. F is also an l-generalized derivation and an r-generalized derivation associated with d.

Remark. A mapping F can be an l-generalized (r-generalized) (reverse) derivation with respect to two different reverse derivations. Indeed, in Example 1 F is an l-generalized (r-generalized) (reverse) derivation with respect to F and d. But if the ring R is semiprime, then the reverse derivation associated to an l-generalized (r-generalized) (reverse) derivation is unique.

Example 2. Consider the ring R as in Example 1. Define the mappings $F : R \to R$ and $d : R \to R$ as follows:

$$F\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad d\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & a & c-a \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that d is a derivation and a reverse derivation and F is an l-generalized derivation with respect to d. But F is not an r-generalized derivation with respect to d. Furthermore, F is neither an l-generalized reverse derivation nor an r-generalized reverse derivation with respect to d.

The next two examples will show that an l-generalized reverse derivation (r-generalized reverse derivation) with respect to a reverse derivation d that is also a derivation is not necessarily an r-generalized derivation (l-generalized derivation) with respect to d.

Example 3. Consider the ring

$$R = \left\{ \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix} |a, b, c, d, e \in \mathbb{R} \right\},$$

where $\mathbb{R}$ is the set of all real numbers. Define the mappings $F : R \to R$ and $d : R \to R$ as follows:

$$F\left(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & b+e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$d\left(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & b-d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then d is a derivation and a reverse derivation, and F is an l-generalized reverse derivation associated with d, but not an r-generalized derivation associated to d.

Example 4. Consider the ring R as in Example 3. Define the mappings $F : R \to R$ and $d : R \to R$ as follows:

$$F\left(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$d\left(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & b-d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that F is an r-generalized reverse derivation associated with d, but not an l-generalized derivation associated to d.

3. Generalized Reverse Derivations on Ideals in Semiprime Rings

Theorem 3.1. *Let R be a semiprime ring and let I be a nonzero ideal of R. There exists $F : I \to R$, an l-generalized reverse derivation associated with a nonzero reverse derivation d on I, if and only if $d(I), F(I) \subseteq C_R(I)$, d is a derivation on I, and F is an r-generalized derivation with respect to d on I.*

Proof. Assume that F is an l-generalized reverse derivation on I. Then

$$F(u^2v) = F(v)u^2 + vd(u^2) = F(v)u^2 + v(d(u)u + ud(u)) \quad \text{for all } u, v \in I,$$

and so

$$F(u^2v) = F(v)u^2 + vd(u)u + vud(u) \quad \text{for all } u, v \in I. \tag{3.1}$$

Moreover,

$$F(u(uv)) = F(uv)u + uvd(u) = (F(v)u + vd(u))u + uvd(u) \quad \text{for all } u, v \in I.$$

Hence,

$$F(u(uv)) = F(v)u^2 + vd(u)u + uvd(u) \quad \text{for all } u, v \in I. \tag{3.2}$$

From (3.1) and (3.2) we get

$$[u, v]d(u) = 0 \quad \text{for all } u, v \in I. \tag{3.3}$$

Replacing v by rv in (3.3) and using (3.3) we have

$$[u, r]vd(u) = 0 \quad \text{for all } u, v \in I \text{ and } r \in R. \tag{3.4}$$

Replacing v by $d(u)s[u, r]$ in (3.4) yields

$$[u, r]d(u)s[u, r]d(u) = 0 \quad \text{for all } u \in I \text{ and } r, s \in R. \tag{3.5}$$

Since R is semiprime, by (3.5) we get

$$[u, r]d(u) = 0 \quad \text{for all } u \in I \text{ and } r \in R. \tag{3.6}$$

The linearization of (3.6) leads to

$$[w, r]d(u) + [u, r]d(w) = 0 \quad \text{for all } u, w \in I \text{ and } r \in R.$$

Thus,

$$[w, r]d(u) = -[u, r]d(w) \quad \text{for all } u, w \in I \text{ and } r \in R. \tag{3.7}$$

Replacing v by $d(w)s[w, r]$ in (3.4) and using (3.7) we get

$$0 = [u, r]d(w)s[w, r]d(u) = -[u, r]d(w)s[u, r]d(w) \quad \text{for all } u, w \in I \text{ and } r, s \in R;$$

i.e.,

$$[u, r]d(w)R[u, r]d(w) = (0) \quad \text{for all } u, w \in I \text{ and } r \in R. \tag{3.8}$$

Since R is semiprime; therefore,

$$[u,r]d(w)=0 \quad \text{for all } u,w\in I \text{ and } r\in R. \tag{3.9}$$

Replacing r by rt in (3.9) we have

$$[u,r]td(w)=0 \quad \text{for all } u,w\in I \text{ and } r,t\in R. \tag{3.10}$$

If we put $r=d(w)$ and replace t by tu in (3.10) then

$$[u,d(w)]tud(w)=0 \quad \text{for all } w,u\in I \text{ and } t\in R.$$

If we multiply to the right (3.10) by u, then $[u,d(w)]td(w)u=0$. Subtracting the last two relations, we get $[u,d(w)]R[u,d(w)]=(0)$ for all $w,u\in I$. From the semiprimeness of R it follows that $[u,d(w)]=0$ for all $w,u\in I$; i.e., $d(I)\subseteq C_R(I)$. Thus,

$$d(xy)=d(y)x+yd(x)=d(x)y+xd(y) \quad \text{for all } x,y\in I,$$

what proves that d is a derivation on I. On the other hand, since F is an l-generalized reverse derivation, we have

$$F(uv^2)=F(v^2)u+v^2d(u)=(F(v)v+vd(v))u+v^2d(u) \quad \text{for all } u,v\in I,$$

and so

$$F(uv^2)=F(v)vu+vd(v)u+v^2d(u) \quad \text{for all } u,v\in I. \tag{3.11}$$

Moreover,

$$F((uv)v)=F(v)uv+vd(uv)=F(v)uv+v(d(v)u+vd(u)) \quad \text{for all } u,v\in I,$$

$$F((uv)v)=F(v)uv+vd(v)u+v^2d(u) \quad \text{for all } u,v\in I. \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$F(v)[u,v]=0 \quad \text{for all } u,v\in I. \tag{3.13}$$

Using the same techniques as above we get that $F(I)\subseteq C_R(I)$. Hence,

$$F(xy)=F(y)x+yd(x)=xF(y)+d(x)y \quad \text{for all } x,y\in I,$$

and F is an r-generalized derivation with respect to d. The converse is trivial.

Theorem 3.2. *Let R be a semiprime ring and let I be a nonzero ideal of R. There exists $F: I\to R$, an r-generalized reverse derivation associated with a nonzero reverse derivation d on I, if and only if $d(I),F(I)\subseteq C_R(I)$, d is a derivation on I, and F is an l-generalized derivation with respect to d on I.*

Proof. This follows along the lines of Theorem 3.1.

Corollary 3.3. *Let R be a semiprime ring. If there is $F: R\to R$, an l-generalized (or r-generalized) reverse derivation associated with a nonzero reverse derivation d on R; then R has a nonzero central ideal.*

Proof. By Theorem 3.1 (or Theorem 3.2), we have $d(R)\subseteq Z(R)$, that is $[d(x),y]=0$ for all $x,y\in R$. Then d is centralizing on R. By Lemma 2.1 (with $I=R$), R has a nonzero central ideal.

The corollary gives the relationship between l-generalized reverse derivations and l-generalized derivations.

Corollary 3.4. *Let R be a 2-torsion-free semiprime ring. If there exists $F : R \to R$, an l-generalized (r-generalized) reverse derivation associated with a nonzero reverse derivation d of R; then F is an r-generalized (l-generalized) reverse derivation with respect to d. Furthermore, d is a derivation and F is an l-generalized (r-generalized) derivation related to d.*

Proof. Let F be an l-generalized reverse derivation. Then by Theorem 3.1 d is a derivation, $F(I), d(I) \subseteq C_R(I)$. By hypotheses, $F(xy) = F(y)x + yd(x)$ for all $x, y \in R$. Putting $y = x$, we have $F(x^2) = F(x)x + xd(x)$ for all $x \in R$, and so F is a generalized Jordan derivation on R. By using Lemma 2.2, F is an l-generalized derivation on R. Using the converse of Theorem 3.2, we know that F is an r-generalized reverse derivation on R. The case of an r-generalized reverse derivation follows the same lines.

Corollary 3.5. *A mapping $d : I \to R$ is a reverse derivation, where I is a two-sided ideal of a semiprime ring R, if and only if d is a derivation centralizing on I: $d(I) \subseteq C_R(I)$.*

Definition 3.6. A mapping F is a *generalized reverse derivation*, if it is an l- and r-generalized reverse derivation.

From Theorems 3.1 and 3.2 we have

Corollary 3.7. *A mapping F on a semiprime ring R is a generalized reverse derivation if and only if F is a central generalized derivation.*

Corollary 3.8. *Let R be a prime ring and let I be a nonzero ideal of R. If there exists $F : I \to I$, an l-generalized (r-generalized) reverse derivation on I associated with a nonzero reverse derivation $d : I \to I$; then R is commutative.*

Proof. Using Corollary 3.3, we know that I (which is clearly prime too) has a nonzero central ideal. By Proposition 2.3, I is a commutative ring. But Proposition 2.3 implies that $I \subseteq C_R(I) = Z(R)$, and so I is a nonzero central ideal of R and R is commutative.

The example shows that the semiprimeness condition for the ring R is not superfluous.

Example 5. Consider the ring R in Example 3, and let

$$I = \left\{ \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

be an ideal of R. Define $F : R \to R$ and $d : R \to R$ as follows:

$$F\left(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & -c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to see that d is a nonzero reverse derivation and a derivation on I, while F is an r-generalized reverse derivation on I, but not an l-generalized derivation, and $d(I), F(I) \nsubseteq C_R(I)$.

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A. Aboubakr
University of Fayoum, Fayoum, Egypt
University of Oviedo, Oviedo, Spain
E-mail address: `afs00@fayoum.edu.eg; uo242478@uniovi.es`

S. González
University of Oviedo, Oviedo, Spain
E-mail address: `santos@uniovi.es`

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ORIGINAL PAPER

Generalized derivations on Lie ideals in semiprime rings

Ahmed Aboubakr[1,2] · **Santos González**[2]

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Abstract Herstein (J Algebra 14:561–571, 1970) proved that given a semiprime 2-torsion free ring R and an inner derivation d_t, if $d_t^2(U) = 0$ for a Lie ideal U of R then $d_t(U) = 0$. Carini (Rend Circ Mat Palermo 34:122–126, 1985) extended this result for an arbitrary derivation d, proving that $d^2(U) = 0$ implies $d(U) \subseteq Z(R)$. The aim of this paper is to extend the results mentioned above for right (resp. left) generalized derivations. Precisely, we prove that if R admits a right generalized derivation F associated with a derivation d such that $F^2(U) = (0)$, then $d^3(U) = (0)$ and $(d^2(U))^2 = (0)$. Furthermore, if F is also a left generalized derivation on U, then $d(U) = F(U) = (0)$, and $d(R), F(R) \subseteq C_R(U)$. On the other hand, if (F, d), (G, g) are, respectively, right and left generalized derivations that satisfy $F(u)v = uG(v)$ for all $u, v \in U$, then $d(U), g(U) \subseteq C_R(U)$.

Keywords Semiprime ring · Lie ideal · Derivation · Generalized derivation

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✉ Ahmed Aboubakr
afs00@fayoum.edu.eg

Santos González
santos@uniovi.es

1 Department of Mathematics, University of Fayoum, Faiyum 63514, Egypt

2 Departamento de Matemáticas, Universidad de Oviedo, 33007 Oviedo, Spain

1 Introduction

Throughout this paper R denotes an associative ring (that is not assumed to be commutative) and $Z(R)$ its center. If A is a subset of R, $C_R(A)$ denotes the centralizer of A, that is defined by $C_R(A) = \{x \in R \mid xa = ax \text{ for all } a \in A\}$. Recall that R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$ and R is semiprime if $aRa = 0$ implies $a = 0$. A ring R (resp. a subset U of R) is said to be reduced if it has no nilpotent elements, or, equivalently, if $a^2 = 0$ implies that $a = 0$ [see, Lam (1991)]. An additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subseteq U$ where, as usual, $[x, y] = xy - yx$. A Lie ideal is said to be square-closed if $u^2 \in U$ for all $u \in U$, which implies that $2uv \in U$ for all $u, v \in R$. Indeed $(u+v)^2 = u^2 + v^2 + uv + vu \in U$ and $[u, v] = uv - vu \in U$ so $2uv \in U$, (but we do not know if given $u, v \in U$, $uv \in U$). Notice that if U is a Lie ideal of R, and $u \in U$, then $[u, r]u \in U$ and $u[u, r] \in U$ for all $r \in R$. An additive map $d : R \to R$ is a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. For a fixed $t \in R$, the map $d_t : R \to R$ given by $d_t(x) = [t, x]$ is a derivation which is called an inner derivation.

Following Bresar (1991), an additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. The concept of generalized derivations includes both the concept of derivation and the concept of left multiplier (i.e., an additive mapping $F : R \to R$ satisfying $F(xy) = F(x)y$ for all $x, y \in R$). In Golbasi and Kaya (2006), Gölbaşi and Kaya introduced the notion of a right generalized derivation and a left generalized derivation. Precisely, an additive mapping $F : R \to R$ is called a *right generalized derivation* associated with a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \to R$ is called a *left generalized derivation* associated with a derivation d if $F(xy) = d(x)y + xF(y)$ for all $x, y \in R$. Of course in case of commutative rings both concepts coincide. In Posner (1957), Posner proved that if R is a prime ring with characteristic different from 2 and $d_1 d_2$, the composition of two derivations d_1 and d_2, is a derivation, then at least one of them must be zero. Further, he proved that if a prime ring R admits a nonzero derivation d such that $ad(a) - d(a)a \in Z(R)$ for all $a \in R$, then R is commutative. This result is no longer true for semiprime rings. That's why Breŝar and Vukman introduced the notion of orthogonal derivations, and proved that in a semiprime 2-torsion free ring R two derivations d_1 and d_2 are orthogonal if and only if $d_1 d_2$ is a derivation. In particular, $d^2 = 0$ implies $d = 0$. The result of Breŝar and Vukman is still true assuming only orthogonality over a nonzero ideal I of R. In [9], Herstein considered semiprime 2-torsion free rings and proved that $d_t^2(U) = (0)$ for a Lie ideal U of R implies $d_t(U) = (0)$. Carina (1985) extended the above mentioned result to arbitrary derivations d and proved that $d^2(U) = (0)$ implies $d(U) \subseteq Z(R)$.

Coming back to prime rings R, char$R \neq 2$, it was proved in Lee and Lee (1983) that if $0 \neq d \in Der(R)$ satisfies $d(R) \subseteq Z(R)$, then R is commutative. If $d^2(U) \subseteq Z(R)$, then $U \subseteq Z(R)$, where U is a nonzero Lie ideal of R. The same result is still true if we consider a generalized derivation instead of a derivation, as it was proved by Dalgin (2010) and Golbasi and Koç (2011) proved that if (F, d), (G, g) are, respectively, left and right generalized derivations of R satisfying $F(u)v = uG(v)$ for all $u, v \in U$, where U is a Lie ideal of R, then $U \subseteq Z(R)$.

The aim of this paper is to extend the results mentioned above for semiprime rings and generalized derivations. We will prove for a semiprime 2-torsion free ring R, a noncentral Lie ideal U of R and (F, d) a left generalized derivation that $F^2(U) = (0)$ implies $d^3(U) = (0)$, and $(d^2(U))^2 = 0$. Furthermore, if $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$, then $d(U) = F(U) = (0)$, $d(R), F(R) \subseteq C_R(U)$. If (F, d), (G, g) are, respectively, left and right generalized derivations that satisfy $G(u)v = uF(v)$ for all $u, v \in U$, then $d(U), g(U) \subseteq C_R(U)$.

These results extends and unify some previous results by Herstein (1970), Carina (1985), Lee and Lee (1981), Golbasi and Koç (2011), and Dalgin (2010).

2 Preliminaries

In what follows R denotes a ring (associative but not necessarily commutative) and Q its Martindale quotient ring. The center C of Q is called the extended centroid of R (see Herstein (1976) and Martindale (1969) for details).

We will use the following lemmas in our results.

Lemma 2.1 [Motoshi (2011), Corollary 2.1] *Let R be a semiprime 2-torsion free ring, U a Lie ideal of R, $U \not\subseteq Z(R)$, and $a, b \in U$. Then*

(1) If $aUa = (0)$, then $a = 0$.
(2) If $aU = (0)$ (or $Ua = (0)$), then $a = 0$.
(3) If U is square-closed and $aUb = 0$, then $ab = 0$ and $ba = 0$.

Lemma 2.2 [Carina (1985), Lemma 1] *Let R be a semiprime 2-torsion free ring with a derivation d and U a Lie ideal of R. If $d^2(U) = 0$, then $d(U) \subseteq Z(R)$.*

Lemma 2.3 [Bergen et al. (1981), Lemma 2] *Let R be a prime ring and U a Lie ideal of R. If $U \not\subseteq Z(R)$, then $C_R(U) = Z(R)$.*

Lemma 2.4 [Ali and Shujat (2012), Lemma 3] *Let R be a semiprime 2-torsion free ring and I a nonzero ideal of R. If d is a nonzero derivation of R such that $Id^2(I) = (0)$, then $I \subseteq Z(R)$.*

Lemma 2.5 [Bergen et al. (1981), Lemmas 6 and 11] *Let R be a prime ring with $charR \neq 2$, d a nonzero derivation of R and U a Lie ideal of R.*

(i) If $d(U) \subseteq Z(R)$, then $U \subseteq Z(R)$.
(ii) If $d^3(U) = (0)$, then $d^3 = (0)$.

Lemma 2.6 [Beidar et al. (1996), Theorem 2.3.2] *Let R be a semiprime ring, $Q = Q_{mr}(R)$, ${}_RU_R \subseteq {}_RQ_R$ a subbimodule of Q and $f : {}_RU_R \to {}_RQ_R$ a homomorphism of bimodules. Then there exists an element $\lambda \in C$ such that $f(u) = \lambda u$ for all $u \in U$.*

Lemma 2.7 *Let R be a semiprime ring, U a Lie ideal of R, $U \not\subseteq Z(R)$. If $a \in R$ satisfies $a[x, r] = 0$ (resp. $[x, r]a = 0$) for every element $x \in U$, $r \in R$, then $a \in C_R(U)$.*

Proof By assumption, for all $x \in U, r \in R$, we have

$$a[x, r] = 0. \tag{2.1}$$

If we substitute r by ra in (2.1) we get

$$0 = a[x, ra] = ar[x, a] + a[x, r]a = ar[x, a]. \tag{2.2}$$

If we substitute r by xr in (2.2) we get $axr[x, a] = 0$. Multiplying (2.2) by x to the left we get $xar[x, a] = 0$. Then $[x, a]R[x, a] = (0)$ for all $x \in U$. It follows from semiprimeness of R that $[x, a] = 0$, that is, $a \in C_R(U)$. □

3 Main results

Now we can prove the main results of this paper.

Theorem 3.1 *Let R be a semiprime 2-torsion free ring, U a noncentral square-closed Lie ideal of R and F a right generalized derivation associated with a derivation d. If $F^2(U) = (0)$ and $F(U), d(U) \subseteq U$, then $d^3(U) = (0)$ and $(d^2(U))^2 = (0)$. Moreover, if $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$ (that is, F is also a left generalized derivation on U), then $d(U) = 0$, $F(U) = 0$, and $d(R), F(R) \subseteq C_R(U)$.*

Proof By assumption we have

$$F^2(u) = (0) \quad \text{for all } u \in U. \tag{3.1}$$

If we replace u by $2uv$ in (3.1), we get

$$2F(F(u)v + ud(v)) = 2(F^2(u)v + F(u)d(v) + F(u)d(v) + ud^2(v)) = 0 \quad \text{for all } u, v \in U. \tag{3.2}$$

By (3.1), and using that R is 2-torsion free it follows that

$$2F(u)d(v) + ud^2(v) = 0 \quad \text{for all } u, v \in U. \tag{3.3}$$

We can replace u by $F(u)$ in (3.3) and obtain

$$2F^2(u)d(v) + F(u)d^2(v) = 0 \quad \text{for all } u, v \in U. \tag{3.4}$$

Now, using (3.1) we have

$$F(u)d^2(v) = 0 \quad \text{for all } u, v \in U. \tag{3.5}$$

Replacing v by $d(v)$ in (3.3) we have

$$2F(u)d^2(v) + ud^3(v) = 0 \quad \text{for all } u, v \in U. \tag{3.6}$$

From (3.5) and (3.6), it follows that $ud^3(v) = 0$, for all $u, v \in U$, that is $Ud^3(v) = (0)$. Since $d^3(v) \in U$, Lemma 2.1, gives that $d^3(U) = (0)$.
On the other hand, if we replace u by $2ud^2(w)$ in (3.3) we obtain

$$2(2F(u)d^2(w)d(v) + 2ud^3(w)d(v) + ud^2(w)d^2(v)) = 0 \quad \text{for all } u, v, w \in U. \tag{3.7}$$

Using (3.5), and the fact that $d^3(U) = (0)$ in (3.7), we get $ud^2(w)d^2(v) = 0$ for all $u, v, w \in U$, that is $U(2d^2(w)d^2(v)) = (0)$, and again Lemma 2.1, says that

$$d^2(w)d^2(v) = 0 \quad \text{for all } v, w \in U. \tag{3.8}$$

In particular $(d^2(v))^2 = 0$ for all $v \in U$, that is $(d^2(U))^2 = (0)$.
Now, let's assume that $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$. We replace u by $2wu$ in (3.3) to get

$$2wF(u)d(v) + 2d(w)ud(v) + wud^2(v) = 0 \quad \text{for all } u, v, w \in U. \tag{3.9}$$

By the other side, multiplying (3.3) to the left by w we have

$$2wF(u)d(v) + wud^2(v) = 0 \quad \text{for all } u, v, w \in U. \tag{3.10}$$

Thus,

$$d(w)ud(v) = 0 \quad \text{for all } u, v, w \in U. \tag{3.11}$$

In particular $d(v)Ud(v) = (0)$ for all $v \in U$. Again Lemma 2.1, gives $d(U) = (0)$. Thus $d([u, r]) = 0$ for all $u \in U, r \in R$, that is $[d(u), r] + [u, d(r)] = 0$. Thus $[u, d(r)] = 0$ for all $u \in U, r \in R$. Hence $d(R) \subseteq C_R(U)$. On the other hand, $d(U) = (0)$, implies $F(uv) = F(u)v = uF(v)$ for all $u, v \in U$. Thus,

$$0 = F(F(uv - vu)) = F(F(u)v - vF(u)) = F^2(u)v - F(v)F(u) \quad \text{for all } u, v \in U. \tag{3.12}$$

Since we are assuming that $F^2(U) = (0)$, then $F(u)F(v) = 0$ for all $u, v \in U$. It suffices to replace u by $2vw$, to get $F(v)UF(v) = (0)$. Hence $F(U) = (0)$ again by Lemma 2.1 . In particular $0 = F(ur - ru) = -F(r)u + ud(r) = (-F(r) + d(r))u$ for all $u \in U, r \in R$ since $d(r) \in C_R(U)$ as was proved above. Let us define $G(r) = F(r) - d(r)$. Thus

$$G(r)u = 0 \quad \text{for all } u \in U, r \in R. \tag{3.13}$$

If we multiply (3.13) to the left by v and substitute u by $2vu$, we get $vG(r)u = 0$, $G(r)vu = 0$. Hence $[G(r), v]u = 0$ for all $u, v \in U, r \in R$, and again by Lemma 2.1, we conclude that $[G(r), v] = 0$, that is $0 = [F(r) - d(r), v] = [F(r), v]$ for all $v \in U, r \in R$. Thus $F(R) \subseteq C_R(U)$. □

The following four results follow immediately from Theorem 3.1.

Corollary 3.2 *Let R be a semiprime 2-torsion free ring, U a noncentral square-closed reduced Lie ideal of R and F a right generalized derivation associated with a derivation d. If $F^2(U) = (0)$ and $F(U), d(U) \subseteq U$, then $d(U) \subseteq Z(R)$.*

Proof By Theorem 3.1, we have $(d^2(U))^2 = (0)$. Since U is reduced (that is, $u^2 = 0$ implies $u = 0$), then $d^2(U) = (0)$. Lemma 2.2 gives that $d(U) \subseteq Z(R)$. □

Corollary 3.3 *Let R be a semiprime 2-torsion free ring, I a nonzero reduced ideal of R and F a right generalized derivation associated with a derivation d. If $F^2(I) = (0)$ and $F(I), d(I) \subseteq I$, then $d(I) = 0$ and $I \subseteq Z(R)$.*

Proof By Theorem 3.1, we have $(d^2(I))^2 = (0)$. Since R is reduced, then $d^2(I) = (0)$. Lemma 2.4 gives that $I \subseteq Z(R)$. On the other hand, $0 = d^2(y^2) = d(d(y)y + yd(y)) = 2(d(y))^2$ for all $y \in I$. So we get $d(I) = 0$. □

Corollary 3.4 *Let R be a prime ring with char$R \neq 2$, U a noncentral square-closed Lie ideal of R and F a right generalized derivation associated with a nonzero derivation d. If $F^2(U) = (0)$, then $d^3 = 0$.*

Proof It immediately follows from Theorem 3.1 and Lemma 2.5 (ii). □

Corollary 3.5 *Let R be a prime ring with char$R \neq 2$, U a square-closed reduced Lie ideal of R and F a right generalized derivation associated with a non-zero derivation d. If $F^2(U) = (0)$ and $F(U), d(U) \subseteq U$, then $U \subseteq Z(R)$.*

Proof Let's assume that $U \nsubseteq Z(R)$. Then we can apply Corollary 3.2 to conclude that $d(U) \subseteq Z(R)$. But Lemma 2.5 (i) now gives $U \subseteq Z(R)$, a contradiction. □

The following proposition describes the structure of a left and right generalized derivation associated with a derivation d on a semiprime ring.

Proposition 3.6 *Let R be a semiprime ring with an extended centroid C. If R admits a left and right generalized derivation F associated with a derivation d, then there exists an element $\lambda \in C$ such that $F(x) = d(x) + \lambda x$ for all $x \in R$.*

Proof Take $T := F - d$. Since F is a left and right generalized derivation, then

$$T(xy) = T(x)y = xT(y) \quad \text{for all } x, y \in R. \tag{3.14}$$

In particular we can used Lemma 2.6, we get $T(x) = \lambda x$ for all $x \in R$. That is $F(x) = d(x) + \lambda x$ for all $x \in R$. □

The following example shows a map F on a semiprime ring, that is not a derivation. However (F, d) is a right generalized derivation and, over a Lie ideal U of R, it is also a left generalized derivation.

Example 3.7 Let $\mathbb{Z}$ be the set of integers, and $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z} \right\}$. Then R is a semiprime ring. Now, take $U = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in \mathbb{Z} \right\}$. It can be easily checked that U is a Lie ideal of R. Since, $u^2 = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix} \in U$, U is an square-closed Lie ideal. We define the maps $F : R \to R$ and $d : R \to R$ as follows:

$$F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}.$$

Then it is easy to check that d is the inner derivation given by $d(x) = [x, 2e_{11} + e_{22}]$, F is a right generalized derivation associated with d, and $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$, that is F is left generalized derivation on U. However, F is not a derivation on R.

Now we remove the assumptions $F(U), d(U) \subseteq U$ in Theorem 3.1.

Theorem 3.8 *Let R be a semiprime 2-torsion free ring, U a noncentral square-closed Lie ideal of R and F a right generalized derivation associated with a derivation d. If $F^2(U) = (0)$ and $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$, then $d(U) \subseteq C_R(U)$.*

Proof Following the same lines that were used in the proof of Theorem 3.1, we can reach the equation (3.11), that is $d(w)ud(v) = 0$ for all $u, v, w \in U$. Multiplying to the left by w_1 we have $w_1 d(w)ud(v) = 0$. If we replace in (3.11) u by $2w_1 u$ we obtain $d(w)w_1 ud(v) = 0$, and subtracting these two relations we have $[d(w), w_1]ud(v) = 0$. In the same way, we get $[d(w), w_1]u[d(v), v_1] = 0$ for all $u, v, v_1, w, w_1 \in U$. Doing $w = v, w_1 = v_1$, we obtain $[d(v), v_1]U[d(v), v_1] = (0)$ for all $u, v_1 \in U$. Lemma 2.1, gives $[d(v), v_1] = (0)$ for all $u, v_1 \in U$, that is, $d(U) \subseteq C_R(U)$. □

The following example shows that the semiprimeness condition in Theorem 3.1 is not superfluous.

Example 3.9 Let $\mathbb{Z}$ be the set of integers, and $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}$. For any $0 \neq b \in \mathbb{Z}$, $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = (0)$, then R is not a semiprime ring. Now, take $U = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, a, b \in \mathbb{Z} \right\}$. It can be easily checked that U is a Lie ideal of R. Since, $u^2 = \begin{pmatrix} a^2 & ab \\ 0 & a^2 \end{pmatrix} \in U$, U is an square-closed Lie ideal. We define maps $F : R \to R$ and $d : R \to R$ as follows:

$$F\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a - c \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to check that d is the inner derivation given by $d(x) = [x, e_{12}]$, F is a right generalized derivation associated with d, $d(U), F(U) \subset U$, $F(uv) = uF(v) + d(u)v$ for all $u, v \in U$ and $F^2(U) = (0)$. However, $F(U) \neq (0)$.

Now, we will extend Theorem 3.4 in Golbasi and Koç (2011) to semiprime rings.

Theorem 3.10 *Let R be a semiprime ring, U a noncentral Lie ideal of R,and F, G maps satisfying $G(u)v = uF(v)$ for all $u, v \in U$. If F is a right generalized derivation associated with a derivation d and G is a left generalized derivation associated with a derivation g, then $d(U), g(U) \subseteq C_R(U)$.*

Proof Let's start by considering that

$$G(u)v = uF(v) \quad \text{for all } u, v \in U. \tag{3.15}$$

If we replace in (3.15) the element v by $[v, r]v$, $r \in R$, (and by $[v, r]$), we get

$$G(u)[v, r]v = uF([v, r])v + u[v, r]d(v) \quad \text{for all } u, v \in U, r \in R. \tag{3.16}$$

And

$$G(u)[v, r] = uF([v, r]) \quad \text{for all } u, v \in U, r \in R. \tag{3.17}$$

Consequently,

$$u[v, r]d(v) = 0 \quad \text{for all } u, v \in U, r \in R. \tag{3.18}$$

Now we can replace u by $[u, s]$, $s \in R$, in (3.18), and get $0 = [u, s][v, r]d(v) = us[v, r]d(v) - su[v, r]d(v) = us[v, r]d(v)$ for all $u, v \in U, r, s \in R$. Again we replace u by $[v, r]$ and s by $d(v)s$, to have $[v, r]d(v)s[v, r]d(v) = 0$ for all $u, v \in U, r, s \in R$. Thus $[v, r]d(v)R[v, r]d(v) = (0)$. Since R is semiprime, we conclude that

$$[v, r]d(v) = 0 \quad \text{for all } v \in U, r \in R. \tag{3.19}$$

Linearizing (3.19), we get

$$[v, r]d(u) = -[u, r]d(v) \quad \text{for all } u, v \in U, r \in R. \tag{3.20}$$

Now, if we replace r by rs in (3.19), and use again (3.19), we get

$$[v, r]sd(v) = 0 \quad \text{for all } u, v \in U, r, s \in R. \tag{3.21}$$

In particular, $0 = [v, r]d(u)s[u, r]d(v) = -[v, r]d(u)s[v, r]d(u)$, that is $[v, r]d(u) R[v, r]d(u) = (0)$. Since R is semiprime we have $[v, r]d(u) = 0$, for all $u, v \in U, r \in R$. From Lemma 2.7 it follows that $d(U) \subseteq C_R(U)$.

On the other hand, we replace in (3.15) u by $u[u, r]$ (and by $[u, r]$), we get

$$uG([u, r])v + g(u)[u, r]v = u[u, r]F(v) \quad \text{for all } u, v \in U, r \in R. \tag{3.22}$$

And,

$$G([u, r])v = [u, r]F(v) \quad \text{for all } u, v \in U, r \in R. \tag{3.23}$$

Consequently, $g(u)[u, r]v = 0$. So, we can follow the same lines as above and conclude that $g(U) \subseteq C_R(U)$. □

An immediate consequence of Theorem 3.10 is the following corollary.

Corollary 3.11 *Let R be a prime ring with char$R \neq 2$, U a Lie ideal of R and F, G maps satisfying $G(u)v = uF(v)$ for all $u, v \in U$. If F is a right generalized derivation associated with a nonzero derivation d and G is a left generalized derivation associated with a nonzero derivation g, then $U \subseteq Z(R)$.*

Proof Suppose that $U \nsubseteq Z(R)$. Theorem 3.10, gives $d(U) \subseteq C_R(U)$ and $C_R(U) = Z(R)$ by Lemma 2.3, hence $d(U) \subseteq Z(R)$. Lemma 2.5 (i) gives $U \subseteq Z(R)$, the contradiction. □

The following example shows that the semiprimeness condition in Theorem 3.10 is not superfluous.

Example 3.12 Consider the ring R as in Example 3.9. Take $U = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, a, b \in \mathbb{Z} \right\}$. It can be easily checked that U is a Lie ideal of R. Define maps $G, g, F, d : R \to R$ as follows:

$$G\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & b+c \\ 0 & c \end{pmatrix}, g\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$
$$F\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}, d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to check that g and d are the inner derivations given by $g(x) = [x, e_{11} + 2e_{22}]$ and $d(x) = [x, e_{12}]$, F is a right generalized derivation associated with d, G is a left generalized derivation associated with g and $G(u)v = uF(v)$ for all $u, v \in U$. However, $d(U) \nsubseteq C_R(U)$.

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