A new framework for the statistical analysis of set-valued random elements

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Abstract

The space of nonempty convex and compact (fuzzy) subsets of $\mathbb{R}^p$, $\mathcal{K}_c(\mathbb{R}^p)$, has been traditionally used to handle imprecise data. Its elements can be characterized via the support function, which agrees with the usual Minkowski addition, and naturally embeds $\mathcal{K}_c(\mathbb{R}^p)$ into a cone of a separable Hilbert space. The support function embedding holds interesting properties, but it lacks of an intuitive interpretation for imprecise data. As a consequence, it is not easy to identify the elements of the image space that correspond to sets in $\mathcal{K}_c(\mathbb{R}^p)$. Moreover, although the Minkowski addition is very natural when $p = 1$, if $p > 1$ the shapes which are obtained when two sets are aggregated are apparently unrelated to the original sets, because it tends to convexify. An alternative and more intuitive functional representation will be introduced in order to circumvent these difficulties. The imprecise data will be modeled by using star-shaped sets on $\mathbb{R}^p$. These sets will be characterized through a center and the corresponding polar coordinates, which have a clear interpretation in terms of *location* and *imprecision*, and lead to a natural directionally extension of the Minkowski addition. The structures required for a meaningful statistical analysis from the so-called ontic perspective are introduced, and how to determine the representation in practice is discussed.

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1. Introduction

For the last decades the statistical analysis of imprecise-valued random variables has awakened a great interest from both the epistemic and the ontic viewpoints (see, for extensive comparative discussions, e.g., [4, 7]). These random variables associate outcomes of a random experiment (modelled through probability spaces) with elements in generalized spaces, such as the space of compact real intervals, the space of convex, and compact subsets of $\mathbb{R}^p$, the space of fuzzy numbers, or the space of convex and compact p-dimensional fuzzy sets. From the 'ontic' perspective, the one considered in this paper, (fuzzy) set-valued data are regarded as whole entities (see, e.g., [2, 4, 5, 6]), in contrast to the epistemic approach, which deals with (fuzzy) set-valued data as imprecise measurements of precise data (see, e.g., [7, 8, 11, 17]).

The elements of the above-mentioned spaces are generally parametrized by vectors/functions to derive more operational statistical techniques. For instance, any real compact interval can be characterized by its contour or endpoints (infimum/supremum), or by its midpoint (related to the location) and spread (related to the imprecision). For the space of fuzzy numbers, each level set is a real compact interval, so the corresponding level-wise characterization is usually considered. Zadeh's extension principle, which agrees with the Minkowski arithmetic, is fully meaningful and has been extensively used for statistical purposes (see, e.g., [9, 16, 19]). For the $p-$dimensional case, the contour of the compact and convex (fuzzy) sets is usually identified by the support function (see, e.g.,[10, 22]). By operating with it, the location/imprecision can also be characterized with the so-called generalized mid-spread representation [26] or alternative characterizations based on the Steiner point and shape deviations expressed in terms of the support function (see, e.g., [1, 23] and references therein). Nevertheless, neither the support function, nor the elements of the related representations, have a fully intuitive meaning, even if theoretically they provide a valid characterization. Moreover, the convexification property of the Minkowski arithmetic also makes it difficult to relate the meaning of two aggregated compact and convex sets with the original shapes when $p > 1$. These shortcomings are discussed for the set-valued case and illustrated through examples. An alternative functional representation based on the theory of star-shaped sets (see, e.g., [22]) is proposed, and the foundations to develop statistics in this new framework are set up.

The rest of the manuscript is structured as follows. In Section 2 the current paradigm based on the support function is formally introduced and some examples showing its lack of interpretability in some cases are discussed. In Section 3 a new parametrization is considered, based on a point related to the location, and a polar function, related to the imprecision. The main properties of this characterization are discussed. The new framework considers star-shaped sets as a natural setting, but it includes the usual convex and compact sets as particular cases. Examples to illustrate the new parametrization are provided. Section 4 introduces a pre-processing step to general star-shaped sets in order to establish their location in practice in a robust way. Finally, Section 5 summarizes some conclusions and future research.

2. Current framework

Let $\mathcal{K}_c(\mathbb{R}^p)$ be the space of nonempty compact and convex subsets of $\mathbb{R}^p$. This space is normally endowed with the Minkowski arithmetic, which generalizes the standard interval arithmetic as follows:

$$A + \gamma B = \{a + \gamma b \,|\, a \in A, b \in B\} \text{ for all } A, B \in \mathcal{K}_c(\mathbb{R}^p) \text{ and } \gamma \in \mathbb{R}.$$

Given $A \in \mathcal{K}_c(\mathbb{R}^p)$, the well-known support function characterizes the contour of A (see, e.g., [22]). It is defined as $s_A : \mathbb{S}^{p-1} \to \mathbb{R}$ such that

$$s_A(u) = \sup_{a \in A} \langle a, u \rangle \text{ for all } u \in \mathbb{S}^{p-1},$$

where $\mathbb{S}^{p-1}$ denotes the unit sphere in $\mathbb{R}^p$ and $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^p$ with associated norm $\|\cdot\|$. The support function s_A is continuous and square-integrable on $\mathbb{S}^{p-1}$.

From now on, given $a \in \mathbb{R}^p$ and $\epsilon > 0$, $B(a,\epsilon)$ will denote the open ball centered at a and with radius ϵ, that is

$$B(a,\epsilon) = \{x \in \mathbb{R}^p \mid \|a - x\| < \epsilon\},$$

and $\overline{B}(a,\epsilon)$ will denote the corresponding closed ball.

The space $\mathcal{K}_c(\mathbb{R}^p)$ can be embedded into the separable Hilbert space of the square integrable functions $\mathcal{H} = \mathcal{L}^2(\mathbb{S}^{p-1},\zeta)$, with ζ the normalized Lebesgue surface measure on $\mathbb{S}^{p-1}$, by means of the mapping $s : \mathcal{K}_c(\mathbb{R}^p) \to \mathcal{L}^2(\mathbb{S}^{p-1})$ defined by $s(A) = s_A$. The support function is not linear, but semi-linear, i.e.

$$s_{A+\gamma B}(u) = s_A(u) + |\gamma| s_B(\mathrm{sign}(\gamma)u),$$

for all $A, B \in \mathcal{K}_c(\mathbb{R}^p)$, $\gamma \in \mathbb{R}$ and $u \in \mathbb{S}^{p-1}$. Thus, s preserves the Minkowski addition and product by non-negative scalars, and it makes $\mathcal{K}_c(\mathbb{R}^p)$ to be isomorph to a cone of $\mathcal{L}^2(\mathbb{S}^{p-1},\zeta)$ by using the induced L_2 distance on $\mathcal{K}_c(\mathbb{R}^p)$. The embedding s allows the development of statistical analyses in $\mathcal{K}_c(\mathbb{R}^p)$ by applying powerful techniques for Hilbert spaces (see, for instance, [15]). It should be underlined that once the support functions are used to characterize compact convex sets, the different developments refer to equivalence classes, since the elements in $\mathcal{L}^2(\mathbb{S}^{p-1},\zeta)$ represent classes of functions that are equal almost surely.

Unfortunately, the support function of a convex set is not easy to visualize. In general, a graphical representation of s_A seems to be unrelated with the shape of A, as it is illustrated in the following simple example.

Example 1: Figure 1 illustrates the computation of the support function of the rectangle drawn in green color, centered at (0,0) and with corners (±10,±1). The unit sphere is the solid black line circumference. For any fixed point u in the unit sphere, the rectangle is orthogonally projected on the line passing by the origin and with direction given by u (dotted black line), leading to an interval over this line. The value of the support function at u is the maximum value of this projection in the direction given by u (the length of the red segment in this particular case). Given any u in the unit sphere, the function drawn in blue represents the point where such a maximum is attained, and the distance to the origin corresponds to the value of the support function. This representation can only be done if the origin belongs to the set, otherwise there will be directions in the unit sphere for which the support function becomes negative. The graphic is complemented with the representation of the support function in the sphere parametrized by angles in $[0, 2\pi)$ (Figure 2).

Both the blue shapes generated by the projection of the sets over all the directions and the value of the support function itself are difficult to relate with the original shapes, even it they determine the contour uniquely.

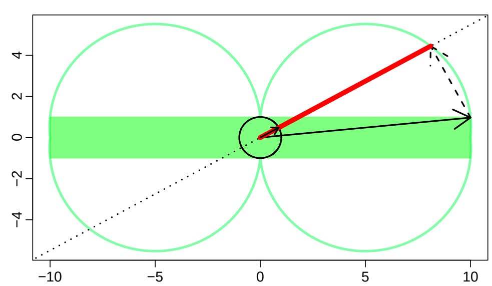

Figure 1: Graphical computation and representation of the support function of the green rectangle – $\mathcal{K}_c(\mathbb{R}^2)$

In an attempt to alleviate the lack of interpretability of the support function, the so-called mid-spread representation was introduced in [26] by decomposing it as

$$s_A = \text{mid}_A + \text{spr}_A$$

where mid_A, $\text{spr}_A \in \mathcal{H}$ are defined such that

$$\text{mid}_A(u) = (s_A(u) - s_A(-u))/2$$

$$\text{spr}_A(u) = (s_A(u) + s_A(-u))/2$$

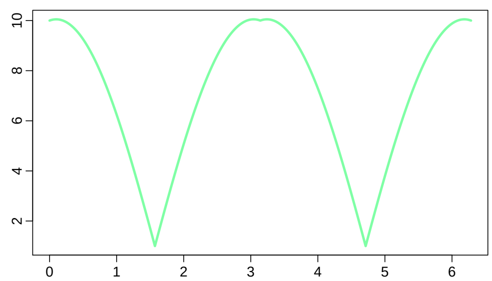

Figure 2: Support function of the green rectangle shown in Figure 1 being the unit sphere parametrized by angles in $[0, 2\pi)$

for all $u \in \mathbb{S}^{p-1}$. Note that the projection of A in the direction u leads to the interval $[-s_A(-u), s_A(u)]$ and, consequently, $\text{mid}_A(u)$ coincides with the midpoint of the projection of A over the direction $u \in \mathbb{S}^{p-1}$, and $\text{spr}_A(u)$ is the spread of this projection, which is always non-negative. In addition, mid_A is an odd function and spr_A is an even function. Thus, both functions are orthogonal in $\mathcal{H}$.

These functions generalize the concepts of midpoint and spread of the intervals in $\mathcal{K}_c(\mathbb{R})$ to $\mathcal{K}_c(\mathbb{R}^p)$, by considering the interval-valued projections of A in all the directions $u \in \mathbb{S}^{p-1}$. This representation shows some advantages from an interpretation perspective, which is exploited in order to define a generalized $\mathcal{L}^2$-type metric in $\mathcal{K}_c(\mathbb{R}^p)$ weighting the importance of the location against the imprecision (see [26]) as

$$d_{\theta,2}(A, B) = \sqrt{||\text{mid}_A - \text{mid}_B||^2 + \theta||\text{spr}_A - \text{spr}_B||^2}\ , \tag{1}$$

where $\theta \in (0, 1]$ and $||\cdot||$ is the usual $\mathcal{L}^2$-type norm for functions defined on $\mathcal{S}^{p-1}$ with respect to ζ, i.e. $||f||^2 = \int_{\mathcal{S}^{p-1}} f^2(u) d\zeta(u)$, $f \in \mathcal{H}$.

The generalized mid-spread representation inherits, nevertheless, some drawbacks from the support function from an operational view. Namely, a non-negative function $f \in \mathcal{L}^2(\mathbb{S}^{p-1}, \zeta)$ can be identified with the spread function of a given convex set, but it is not easy to check whether a general function $f \in \mathcal{L}^2(\mathbb{S}^{p-1}, \zeta)$ characterizes the midpoint function of an element in $\mathcal{K}_c(\mathbb{R}^p)$, as it happens with s_A. In other words, given an element $f \in \mathcal{L}^2(\mathbb{S}^{p-1}, \zeta)$, it can remain unknown whether f is the support function (or the generalized midpoint) of any set in $\mathcal{K}_c(\mathbb{R}^p)$. Any sub-linear function $f : \mathbb{R}^p \to \mathbb{R}$ determines a convex and compact set [22], however verifying the condition of sub-linearity is not obvious, so such a characterization is of very limited use in practice in this sense. This fact is a challenge when, after applying a statistical technique in the Hilbert space by using any of these representations, one tries to go back and get statistical conclusions on the original elements in $\mathcal{K}_c(\mathbb{R}^p)$. The same issues apply for the alternative decompositions based on the support function (e.g., [1, 23]).

On the other hand, it is well-known that the embedding via the support function is coherent with the Minkowski arithmetic in the sense that

$$s(A + \gamma B) = s(A) + \gamma s(B) \text{ for all } A, B \in \mathcal{K}_c(\mathbb{R}^p) \text{ and } \gamma \geq 0.$$

That is, the arithmetic in $\mathcal{K}_c(\mathbb{R}^p)$ agrees with the usual arithmetic for functions in $\mathcal{L}^2(\mathbb{S}^{p-1}, \zeta)$ via the isometry. The Minkowski addition is very natural when dealing with intervals, nevertheless, it is not always the best choice in $\mathcal{K}_c(\mathbb{R}^p)$ for $p > 1$, since it tends to convexify shapes. Thus, the Minkowski sum of two or more convex bodies may change completely the shape of the resulting set, and it does not preserve the area, as illustrated in Example 2.

Example 2: The Minkowski sum, also known as dilation, makes the area of the sum to increase enormously when the imprecision follows (almost) orthogonal directions. For instance, in Figure 3, the set obtained by computing the Minkowski sum (gray rectangle) of the horizontal green rectangle and the vertical red one is shown. Here, the sum of two elements in $\mathcal{K}_c(\mathbb{R}^2)$ with areas of 40 and 60 respectively results in a convex set with area of 506 and with a quite different shape (dilated by the combination the two orthogonal directions of imprecision). The contrast is even higher if the green and red rectangles are replaced by segments with null area with a resulting sum of arbitrary area, nevertheless these null sets are of little interest in $\mathcal{K}_c(\mathbb{R}^2)$.

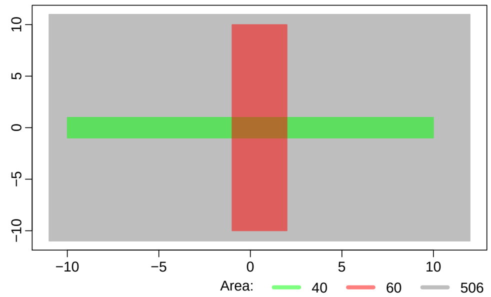

Figure 3: Minkowski sum of green and red rectangles – $\mathcal{K}_c(\mathbb{R}^2)$

3. New framework: Basic concepts

The new proposal is based on an alternative directional extension of the Minkowski addition that is connected with a natural center-radial characterization of the sets. In the interval case, given $A \in \mathcal{K}_c(\mathbb{R})$ consider $c_A \in A$, $A^* = A - c_A$ and the function $\rho_{A^*} : \{-1, 1\} \to [0, \infty)$ with $\rho_{A^*}(-1) = \inf A^*$, $\rho_{A^*}(1) = \sup A^*$. This leads to a center-radial or polar characterization of A by means of the pair (c_A, ρ_{A^*}) being usually c_A chosen as the mid point of the interval. The function ρ_{A^*} simply identifies the boundaries of the interval A^*. It can be equivalently rewritten as $\rho_{A^*}(u) = \sup\{\alpha \,|\, \alpha u \in A^*\}$ and agrees with the support function of A^* for the interval case. Thus, for any $A, B \in \mathcal{K}_c(\mathbb{R})$ and $\gamma \in \mathbb{R}$ the Minkowski addition and product by scalar can be expressed in the following terms:

- Minkowski addition: $(c_A, \rho_{A^*}) + (c_B, \rho_{B^*}) = (c_A + c_B, \rho_{A^*} + \rho_{B^*})$
- Product by scalar: $\gamma(c_A, \rho_{A^*}(\cdot)) = (\gamma c_A, |\gamma| \rho_{A^*}(\mathrm{sign}(\gamma) \cdot))$

In higher dimensions the polar representation of a generalization of the convex bodies, namely, the star-shaped sets (see, e.g., [22]) will be considered. A subset of $\mathbb{R}^p$, $A \subset \mathbb{R}^p$, is called a *star-shaped set* iff there exists $c \in A$ so that

$$\gamma c + (1 - \gamma) a \in A \text{ for all } a \in A \text{ and all } \gamma \in [0, 1] \tag{2}$$

In such a case, A is also called *star-shaped set* with respect to c, emphasizing with this terminology that Equation 2 is verified for this particular point $c \in A$.

If the set A is closed, then all the segments joining any point in the boundary of the set and the "central" point c belong to the set, which means that the set can be characterized by a center, with serves a location of the set, and a polar or radial function.

It should be noted that a set $A \subset \mathbb{R}^p$ could be a star-shaped set w.r.t. different points $c \in A$. The set of points in A verifying Equation (2) is called *kernel* of A and is denoted as $\ker(A)$, that is

$$\ker(A) = \{c \in A \,|\, \gamma c + (1-\gamma)a \in A \text{ for all } a \in A \text{ and all } \gamma \in [0,1]\}.$$

The kernel of a star-shaped set is the intersection of all its maximal convex subsets (see [25]). Thus, it is convex, and if the star-shaped set is closed, then its kernel is closed too. In addition, if A is a star-shaped set w.r.t. $0 \in A$, then for any $c \in \mathbb{R}^p$ the set $c + A$ is a star-shaped set w.r.t. c. So, initially, we will focus on the star-shaped sets w.r.t. 0, and the selection of an appropriate center to represent the location of any star-shaped set will be discussed later on.

3.1. Characterization of a class of star-shaped sets w.r.t. 0

The space of star-shaped sets of $\mathbb{R}^p$ w.r.t. 0 will be denoted by $\mathcal{X}_0(\mathbb{R}^p)$, that is,

$$\mathcal{X}_0(\mathbb{R}^p) = \big\{A \subset \mathbb{R}^p \,|\, \gamma a \in A \text{ for all } a \in A \text{ and all } \gamma \in [0,1]\big\}.$$

Obviously, $\mathcal{X}_0(\mathbb{R}^p)$ contains all the elements $A \in \mathcal{K}_c(\mathbb{R}^p)$ for which $0 \in A$, but there are star-shaped sets w.r.t. 0 that are not convex sets (see, e.g., Figure 5).

In order to characterize the star-shaped sets w.r.t. 0, the polar or radial function will be used. As usual, $\overline{\mathbb{R}}$ will denote the extended real line.

Definition 3.1. *Given $A \subset \mathbb{R}^p$ so that $0 \in A$, the polar or radial function $\rho_A : \mathbb{S}^{p-1} \to \overline{\mathbb{R}}$ is defined as*

$$\rho_A(u) = \sup\{\gamma \in [0,\infty) \,|\, \gamma u \in A\} \textit{ for all } u \in \mathbb{S}^{p-1}.$$

Conversely, every $f : \mathbb{S}^{p-1} \to \mathbb{R}$ has an associated star-shaped set w.r.t. 0 given by

$$K_f = \{\gamma u \,|\, \gamma \in [0,\infty),\, \gamma \leq |f(u)|,\, u \in \mathbb{S}^{p-1}\} \in \mathcal{X}_0(\mathbb{R}^p).$$

In the previous definition the polar function has been defined generically for any $A \subset \mathbb{R}^p$ containing the origin of coordinates. Nevertheless, its main utility is when dealing with star-shaped sets w.r.t. 0.

Given $A \subset \mathbb{R}^p$ the following two disjoint subsets of $\mathbb{R}^p$ are considered:

$$A_E = \left\{ u\rho_A(u) \,|\, u \in \mathbb{S}^{p-1},\, \rho_A(u) < \infty \right\} \subset \mathbb{R}^p,$$

$$A_I = \left\{ u\gamma \,|\, u \in \mathbb{S}^{p-1},\, \gamma \in [0, \rho_A(u)) \right\} \subset \mathbb{R}^p.$$

By construction, $K_{\rho_A} = A_E \cup A_I$, A_E is contained in the boundary of A and, obviously, $A \subset K_{\rho_A}$ and $A_I = \emptyset$ if, and only if, $A = \{0\}$. In addition, if $A \in \mathcal{X}_0(\mathbb{R}^p)$ then $A_I \subset A$.

Let $A \subset \mathbb{R}^p$ so that $A_E \subset A$ (which happens, for instance, if A is a closed set). By using the previous decomposition of K_{ρ_A}, it is easy to check that $A \in \mathcal{X}_0(\mathbb{R}^p)$ if, and only if, $A = K_{\rho_A}$. In particular, the radial function ρ_A characterizes any closed set $A \in \mathcal{X}_0(\mathbb{R}^p)$ by a directional identification of its boundary (see Figure 4).

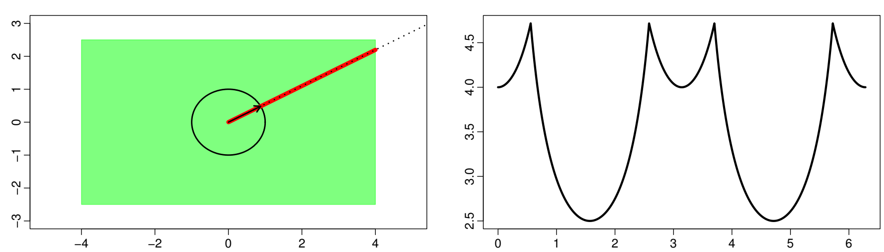

Figure 4: Computation of the polar function ρ of a set in $\mathbb{X}_0(\mathbb{R}^2)$ (green rectangle). The polar function corresponds to the distance to 0 of the boundary point in the direction $u \in \mathbb{S}^1$.

By definition, ρ_A is a non-negative function. Given $A \in \mathcal{X}_0(\mathbb{R}^p)$, for each directional interval

$$I_u(A) = \{\gamma \in \mathbb{R} \,|\, \gamma u \in A\},$$

$\inf I_u(A) = -\rho_A(-u)$ and $\sup I_u(A) = \rho_A(u)$ for all $u \in \mathbb{S}^{p-1}$. There are elements $A \in \mathcal{X}_0(\mathbb{R}^p)$ which are not perfectly characterized by its polar function in the sense that $A \neq K_{\rho_A}$. In this respect, the next theorem states that if $A \in \mathcal{X}_0(\mathbb{R}^p)$ is a Lebesgue measurable subset of $\mathbb{R}^p$, then K_{ρ_A} provides a good approximation of A. To this aim let λ_B denote the Lebesgue measure on $B \subset \mathbb{R}^p$, whenever it can be defined.

Theorem 3.1. *Let $A \in \mathcal{X}_0(\mathbb{R}^p)$ be a Lebesgue measurable set. Then $\lambda_{\mathbb{R}^p}(A_E) = 0$ and, consequently, $K_{\rho_A} = A$ almost sure-$\lambda_{\mathbb{R}^p}$.*

Proof. Let $A \in \mathcal{X}_0(\mathbb{R}^p)$ be a Lebesgue measurable set and let $A_\epsilon = (1+\epsilon)A \setminus \frac{1}{1+\epsilon}A$. Obviously, for all $\epsilon > 0$, $A_E \subset A_\epsilon$.

If A is bounded, then

$$\lambda_{\mathbb{R}^p}(A_\epsilon) = \left((1+\epsilon)^p - \left(\frac{1}{1+\epsilon}\right)^p\right)\lambda_{\mathbb{R}^p}(A),$$

which tends to 0 as $\epsilon \to 0$. Consequently, A_E is Lebesgue measurable and $\lambda_{\mathbb{R}^p}(A_E) = 0$.

If A is not bounded, let $A^n = A \cap \overline{B}(0,n)$ for all $n \in \mathbb{N}$. Since A^n is bounded, $\lambda_{\mathbb{R}^p}(A^n_E) = 0$ for all $n \in \mathbb{N}$. It is easy to check that $A_E \subset \cup_{n\in\mathbb{N}} A^n_E$, whence $\lambda_{\mathbb{R}^p}(A_E) = 0$.

Finally, given that $A_I \subset A \subset A_E \cup A_I = K_{\rho_A}$,

$$K_{\rho_A} \setminus A \subset K_{\rho_A} \setminus A_I = A_E.$$

Thus, $K_{\rho_A} \setminus A$ is a null Lebesgue measurable set, i.e., $K_{\rho_A} = A$ a.s.-$\lambda_{\mathbb{R}^p}$. □

Note that if $A \neq \{0\}$, then given $B = A_I \neq \emptyset$, it is verified that $\rho_B = \rho_A$. Thus, $B_E = A_E$ and $B_I = A_I$, and by applying the previous theorem to $B = A_I$, the next result follows.

Corollary 3.1. *Let $A \in \mathcal{X}_0(\mathbb{R}^p)$. Then A is a Lebesgue measurable set if, and only if, A_I is a Lebesgue measurable set.*

Formally, two elements $A, B \in \mathcal{X}_0(\mathbb{R}^p)$ will be considered equivalent whenever $K_{\rho_A} = K_{\rho_B}$ (or equivalently, whenever $\rho_A = \rho_B$), and thus, a representative set of each of the equivalence classes is given by K_f for $f : \mathbb{S}^{p-1} \to \overline{\mathbb{R}}$. According to Theorem 3.1 and Corollary 3.1, given $A \in \mathcal{X}_0(\mathbb{R}^p)$ a Lebesgue measurable set, if $B \in \mathcal{X}_0(\mathbb{R}^p)$ is equivalent to A, then $B_I = A_I$, $B_E = A_E$ and $K_{\rho_B} = K_{\rho_A}$. As a results, B is also Lebesgue measurable and $B = A$ almost sure-$\lambda_{\mathbb{R}^p}$.

From now on, $\mathbb{X}_0(\mathbb{R}^p)$ will represent the quotient space, and $A \in \mathbb{X}_0(\mathbb{R}^p)$ will denote the unique element in the equivalence class $[A]$ so that $A = K_{\rho_A}$. Thus, the previous theorem states that any $[A] \in \mathbb{X}_0(\mathbb{R}^p)$ is characterized by its polar representation whenever A is Lebesgue measurable. The practical interest will focus on Lebesgue measurable star-shaped sets, thus, measurability conditions will be investigated later on (see Theorem 3.2).

Given $A, B \in \mathbb{X}_0(\mathbb{R}^p)$ and $\gamma \in \mathbb{R}$, the addition and product by scalar are defined as $A + B = K_{\rho_A+\rho_B} \in \mathbb{X}_0(\mathbb{R}^p)$ and $\gamma A = K_{|\gamma|\rho_A(\mathrm{sign}(\gamma)\cdot)} \in \mathbb{X}_0(\mathbb{R}^p)$. Consequently, $\rho_{A+B} = \rho_A + \rho_B$ and $\rho_{\gamma A}(\cdot) = |\gamma|\rho_A(\mathrm{sign}(\gamma)\cdot)$. This arithmetic is associated with a directional propagation of the "imprecision", and extends the interval Minkowski arithmetic as follows,

$$I_u(A + B) = I_u(A) + I_u(B) \text{ for all } u \in \mathbb{S}^{p-1}.$$

Obviously, both internal operations agree in the sense that $A + A = 2A$.

In general, $A + B \notin \mathcal{K}_c(\mathbb{R}^p)$, even if A and B are convex sets, which is the reason for the consideration of star-shaped sets (see Figure 5).

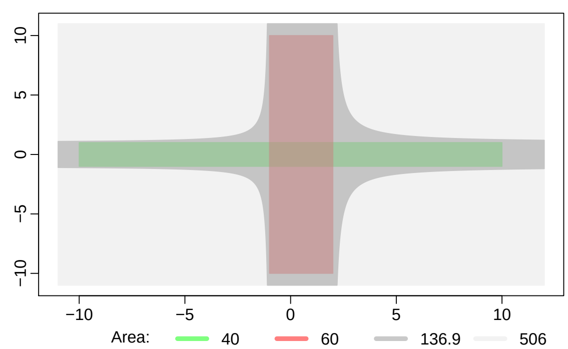

Figure 5: Addition (dark gray set) of two star-shaped sets (green and red rectangles). The Minkowski addition is depicted in light gray for comparative purposes.

The unnormalized Lebesgue surface measure on $\mathbb{S}^{p-1}$ defined by the cone volume will be considered from now on. The measure is given by,

$$\vartheta_p(A) = \lambda_{\mathbb{R}^p}(\{\alpha a \,|\, \alpha \in (0,1],\, a \in A\}) \text{ for all } A \in \beta_{\mathbb{S}^{p-1}},$$

being $\beta_{\mathbb{S}^{p-1}}$ the Borel σ-algebra of the subspace $\mathbb{S}^{p-1} \subset \mathbb{R}^p$. The complete measurable space $(\mathbb{S}^{p-1}, \overline{\beta}_{\mathbb{S}^{p-1}}, \vartheta_p)$ is obtained in the usual way. The so-called normalized Lebesgue (surface) measure on $\mathbb{S}^{p-1}$ reduces to

$$\zeta(A) = \vartheta_p(A)/\vartheta_p(\mathbb{S}^{p-1}).$$

Consider the mapping $\phi : \mathbb{R}^p \setminus \{0\} \to \mathbb{S}^{p-1} \times (0, \infty)$ given by $\phi(x) = (x/||x||, ||x||^p)$ for all $x \in \mathbb{R}^p \setminus \{0\}$ with inverse mapping given by $\phi^{-1}(u, \gamma) = \gamma u$ for all $u \in \mathbb{S}^{p-1}$ and all $\gamma \in (0, \infty)$. The mapping ϕ is a continuous bijection with continuous inverse (see, e.g. [13]) verifying that,

$$\lambda_{\mathbb{R}^p}\big(\phi^{-1}((a, b] \times E))\big) = \lambda((a, b])\vartheta_p(E),$$

for all $0 < a < b < \infty$ and all $E \in \beta_{\mathbb{S}^{p-1}}$. Consequently

$$\lambda_{\mathbb{R}^p}\big(\phi^{-1}(U)\big) = (\lambda \times \vartheta_p)(U) \text{ and } \lambda_{\mathbb{R}^p}\big(A\big) = (\lambda \times \vartheta_p)\big(\phi(A)\big)$$

for all $U \in \beta_{\mathbb{S}^{p-1}} \times \beta_{(0,\infty)}$ and all $A \in \beta_{\mathbb{R}^p \setminus \{0\}}$. Thus, ϕ is a measure-preserving homeomorphism. In addition, by applying the Theorem 2.49 in [13] to the function I_{K_f} (note that $\sigma = p\vartheta_p$ in that theorem), the following relationship for polar integration is fulfilled,

$$\int_{\mathbb{S}^{p-1}} |f(u)|^p \vartheta_p(du) = \lambda_{\mathbb{R}^p}(K_f) \text{ for all } f \in \mathcal{L}^p(\mathbb{S}^{p-1}, \vartheta_p). \tag{3}$$

According to Equation (3), the defined addition does not preserve the volume, that is, $\lambda_{\mathbb{R}^p}(A + B)$ does not coincide in general with $\lambda_{\mathbb{R}^p}(A) + \lambda_{\mathbb{R}^p}(B)$. Indeed, the volume can be computed as the integral of the corresponding function ρ^p w.r.t. ϑ_p so, for instance, if $p = 2$, ρ^2_{A+B} contains the extra factor $2\rho_A\rho_B$ and consequently area$(A + B) \geq$ area(A)+area(B). In some settings an arithmetic preserving the volume could be interesting, for instance, in problems where comparing average volumes is needed (e.g., [21]). Although that is not the aim of this work, it would be possible to consider such an arithmetic with a simple modification within this framework.

The hypograph of the non-negative function ρ_A, with $A \in \mathcal{X}_0(\mathbb{R}^p)$, can be defined as follows,

$$Hypo(\rho_A) = \{(u, \gamma) \in \mathbb{S}^{p-1} \times [0, \infty), \gamma \leq \rho_A(u)\}.$$

Note that the usual notion of hypograph of a general function (non necessarily non-negative) is, actually, $Hypo(\rho_A) \cup \big(\mathbb{S}^{p-1} \times (-\infty, 0)\big)$. Thus, the measurability of both hypographs is equivalent for non-negative functions. In addition,

$$Hypo(\rho_A^p) = \phi(A \setminus \{0\}) \cup \big(\mathbb{S}^{p-1} \times \{0\}\big).$$

The following result establishes an important relationship between the measurability of A and ρ_A with respect to the corresponding Lebesgue measures.

Theorem 3.2. *Let $A \in \mathcal{X}_0(\mathbb{R}^p)$, then the following properties are equivalent:*

(a) *A is a Lebesgue measurable set of $\mathbb{R}^p$.*

(b) *The hypograph of ρ_A^p is a Lebesgue measurable set of $\mathbb{S}^{p-1} \times [0, \infty)$.*

(c) *ρ_A is a Lebesgue measurable function.*

In addition, A has finite Lebesgue measure iff $\rho_A \in \mathcal{L}^{\max\{2,p\}}(\mathbb{S}^{p-1}, \vartheta_p)$.

Proof. The equivalence between (b) and (c) is direct for $\mathbb{R}$ valued functions (see [3]). Moreover, note that $\rho_A(u) = \sup_{n \in \mathbb{N}} \min(\rho_A(u), n)$ for all $u \in \mathbb{S}^{p-1}$ and consequently ρ_A is a Lebesgue measurable function if, and only if, the function $\min(\rho_A, n)$ is Lebesgue measurable for all $n \in \mathbb{N}$ (see [12]). The truncated function $\min(\rho_A, n)$ can be understood to take values on $\mathbb{R}$ and thus the equivalence between (b) and (c) is deduced straightforwardly. In addition, since ϕ is an homeomorphism, $A \subset \mathbb{R}^p$ is a Lebesgue measurable set if, and only if, $\phi(A \setminus \{0\}) \cup \big(\mathbb{S}^{p-1} \times \{0\}\big) = Hypo(\rho_A^p)$ is a Lebesgue measurable set, which proves the equivalence between (a) and (b).

Finally, note that according to Equation 3 and Theorem 3.1 $\lambda_{\mathbb{R}^p}(A) = \lambda_{\mathbb{R}^p}(K_{\rho_A}) = \int_{\mathbb{S}^{p-1}} \rho_A^p(u) \vartheta_p(du)$, where the measurability of the involved sets is guaranteed by the equivalence stated in the current Theorem. In the particular case of $p = 1$, the measure of the interval A is finite if, and only if, ρ_A is bounded, and thus, in particular, $\rho_A \in \mathcal{L}^2(\mathbb{S}^0)$. $\square$

The interest will focus on those elements $A \in \mathbb{X}_0(\mathbb{R}^p)$ so that ρ_A is integrable, that is

$$\mathbb{X}_0^1(\mathbb{R}^p) = \{A \in \mathbb{X}_0(\mathbb{R}^p) \,|\, \rho_A \in \mathcal{L}^1(\mathbb{S}^{p-1}, \vartheta_p)\}.$$

Actually, as it will be clarified in Section 3.2, $\mathcal{L}^2$ polar functions will be considered. This is a usual and non-restrictive condition when dealing with functions. As a consequence of Theorem 3.2, any $A \in \mathbb{X}_0^1(\mathbb{R}^p)$ is a Lebesgue measurable set that is characterized by its polar representation, as $\lambda_{\mathbb{R}^p}(A_E) = 0$. That is, $A = K_{\rho_A}$ a.s.-$\lambda_{\mathbb{R}^p}$. The space $\mathbb{X}_0^1(\mathbb{R}^p)$ is composed of equivalent classes, induced by those of $\mathcal{L}^1(\mathbb{S}^{p-1}, \vartheta_p)$. Thus, given $A, B \in \mathbb{X}_0(\mathbb{R}^p)$ so that ρ_A and ρ_B are integrable, then $A \sim B$ if, and only if, $\rho_A = \rho_B$ a.s-ϑ_p. The next results states an alternative characterization for $A \sim B$.

Theorem 3.3. *Let $A, B \in \mathbb{X}_0^1(\mathbb{R}^p)$, then*

$$A \sim B \textit{ if, and only if, } A = B \textit{ a.s.-}\lambda_{\mathbb{R}^p}$$

Proof. Firstly, $A \sim B$ iff $\rho_A = \rho_B$ a.s.$-\vartheta_p$ iff $\rho_A^p = \rho_B^p$ a.s.$-\vartheta_p$. Let assume that $\rho_A^p \leq \rho_B^p$, otherwise consider the minimum and maximum functions. Let $S \subset \mathbb{S}^p$ denote the set where ρ_A^p and ρ_B^p differ. By using the measure-preserving homeomorphism ϕ introduced in Section 3.1,

$$\lambda_{\mathbb{R}^p}(A \cup B \setminus A \cap B) = (\vartheta_p \times \lambda_{\mathbb{R}})(\{\phi(A \cup B \setminus A \cap B)\}) =$$

$$(\vartheta_p \times \lambda_{\mathbb{R}})\big(\{(u, \alpha) \mid \alpha \in [\rho_A^p(u), \rho_B^p(u)],\, u \in S\}\big) = \int_{\mathbb{S}^p} \big(\rho_B^p(u) - \rho_A^p(u)\big)\vartheta_p(du),$$

which implies the result. $\square$

As a consequence, $A \sim B$ if, and only if, $A = K_{\rho_A} = K_{\rho_B} = B$ a.s.-$\lambda_{\mathbb{R}^p}$.

Note that given a non-negative function $f \in \mathcal{L}^1(\mathbb{S}^{p-1}, \vartheta_p)$ then the set $f^{-1}(\{\infty\})$ has null Lebesgue measure and, consequently, the function $f^0 = f \cdot I_{f^{-1}([0,\infty))}$ is measurable and almost sure equal to f. In addition, it is easy to check that the star-shaped sets K_f and K_{f^0} are almost sure equal. As a result, $A \sim B$ if, and only if, $K_{\rho_A^0} = K_{\rho_B^0}$ (or equivalently, whenever $\rho_A^0 = \rho_B^0$), and thus, a representative set of each of the equivalence classes is given by K_f for $f : \mathbb{S}^{p-1} \to \mathbb{R}$ (note that $\overline{\mathbb{R}}$ does not need to be considered for the representatives of the classes). From now on $A \in \mathbb{X}_0(\mathbb{R}^p)$ will denote an element in the equivalence class $[A]$ so that $A = K_{\rho_A^0}$.

3.2. Star-shaped sets parametrization

General star-shaped sets will be parametrized by a location and the polar function of the "centered" set. To this aim, the space $(\mathbb{R}^p \times \mathbb{X}_0^1(\mathbb{R}^p), +, \cdot)$ is considered in order to represent location and imprecision. The arithmetic is naturally defined by:

- $(x, A) + (y, B) = (x + y, A + B)$,
- $\gamma(x + A) = (\gamma x, \gamma A)$,

for all $(x, A), (y, B) \in \mathbb{R}^p \times \mathbb{X}_0^1(\mathbb{R}^p)$ and all $\gamma \in \mathbb{R}$.

Consider now the vector space $(\mathbb{R}^p \times \{f : \mathbb{S}^{p-1} \to \mathbb{R}\}, +, \cdot)$, where for each $(x_1, f_1), (x_2, f_2) \in \mathbb{R}^p \times \{f : \mathbb{S}^{p-1} \to \mathbb{R}\}$ and $\gamma \in \mathbb{R}$, the addition and product by scalar are given by $(x_1, f_1) + (x_2, f_2) = (x_1 + x_2, f_1 + f_2)$ and $\gamma(x_1, f_1) = (\gamma x_1, \gamma f_1)$. If we define the mapping

$$\Gamma : \mathbb{R}^p \times \mathbb{X}_0^1(\mathbb{R}^p) \to \mathbb{R}^p \times \{f : \mathbb{S}^{p-1} \to \mathbb{R}\}$$

given by $\Gamma(x, A) = (x, \rho_A)$ for all $(x, A) \in \mathbb{R}^p \times \mathbb{X}_0^1(\mathbb{R}^p)$, it is direct to check that $\Gamma(\mathbb{R}^p \times \mathbb{X}_0^1(\mathbb{R}^p)) = \mathbb{R}^p \times \{f : \mathbb{S}^{p-1} \to \mathbb{R}^+\}$ is a convex cone of the vector space $(\mathbb{R}^p \times \{f : \mathbb{S}^{p-1} \to \mathbb{R}\}, +, \cdot)$ with inverse mapping $\Gamma^{-1}(x, f) = (x, K_f)$ for all $(x, f) \in \mathbb{R}^p \times \{f : \mathbb{S}^{p-1} \to \mathbb{R}^+\}$.

By considering an appropriate metric structure for the vector space, the mapping Γ will allow us to endow the location-imprecision star-shaped-based sets space with a useful metric. The aim is to take advantage of the structure of Hilbert spaces, which are metric vector spaces very useful in order to develop inferential statistics. They are widely used for handling functional data or fuzzy data among others (see, for instance, [14], [20], and [26]). In this sense, recall that $(\mathcal{L}^2(\mathbb{S}^{p-1}, \vartheta_p), +, \cdot, \langle \cdot, \cdot \rangle_{\mathcal{L}^2})$ is a separable Hilbert space with associated norm $||.||_{\mathcal{L}^2}$, being the inner product defined for all $f_1, f_2 \in \mathcal{L}^2(\mathbb{S}^{p-1}, \vartheta_p)$ as

$$\langle f_1, f_2 \rangle_{\mathcal{L}^2} = \int_{\mathbb{S}^{p-1}} f_1(u) f_2(u) \vartheta_p(du).$$

Consider the space

$$\mathcal{H} = \mathbb{R}^p \times \mathcal{L}^2(\mathbb{S}^{p-1}, \vartheta_p).$$

Fix $\tau \in (0,1)$, and define for any $(x_1, f_1), (x_2, f_2) \in \mathcal{H}$ and $\gamma \in \mathbb{R}$, $(x_1, f_1) + (x_2, f_2) = (x_1 + x_2, f_1 + f_2)$, $\gamma(x_1, f_1) = (\gamma x_1, \gamma f_1)$ and

$$\big\langle (x_1, f_1), (x_2, f_2) \big\rangle_\tau = (1-\tau)\big\langle x_1, x_2 \big\rangle + \tau \big\langle f_1, f_2 \big\rangle_{\mathcal{L}^2}.$$

Then, $\big\langle \cdot, \cdot \big\rangle_\tau$ is an inner product in $\mathcal{H}$ with norm $||\cdot||_\tau$ and $(\mathcal{H}, +, \cdot, \big\langle \cdot, \cdot \big\rangle_\tau)$ is a separable Hilbert space. The parameter $\tau \in (0,1)$ will allow us to weight the relative importance of "location" versus "imprecision" by using the mapping Γ. A similar approach to the one in [24] could be followed in order to chose an appropriate value for τ in practice.

In order to use $\mathcal{H}$ as the image vector space, the class of star-shaped sets $\mathbb{X}_0^1(\mathbb{R}^p)$ needs to be constrained to those sets with $\mathcal{L}^2$ polar functions. As already mentioned, this is a usual and non-restrictive condition when dealing with functions. Thus, we will consider the space

$$\mathbb{X}_0^2(\mathbb{R}^p) = \{A \in \mathbb{X}_0^1(\mathbb{R}^p) \,|\, \rho_A \in \mathcal{L}^2(\mathbb{S}^{p-1}, \vartheta_p)\}.$$

As a consequence of Theorem 3.2, when $p \leq 2$, $A \in \mathbb{X}_0^2(\mathbb{R}^p)$ if, and only if, A has finite Lebesgue measure. Furthermore, if $p > 2$, there are sets in $A \in \mathbb{X}_0^2(\mathbb{R}^p)$ with no finite Lebesgue measure.

The space $\mathbb{R}^p \times \mathbb{X}_0^2(\mathbb{R}^p)$ can be endowed with a metric structure by means of the mapping Γ introduced previously. To this aim, given $\tau \in (0,1)$, for any $(x, A), (y, B) \in \mathbb{R}^p \times \mathbb{X}_0^2(\mathbb{R}^p)$, define

$$\begin{aligned} d_\tau((x, A), (y, B)) &= ||\Gamma(x, A) - \Gamma(y, B)||_\tau \\ &= \sqrt{(1-\tau)\|(x-y)\|^2 + \tau \big\|\rho_A - \rho_B\big\|_{\mathcal{L}^2}}, \end{aligned} \tag{4}$$

and consider the space of equivalence classes in $\mathbb{R}^p \times \mathbb{X}_0^2(\mathbb{R}^p)$ (that will be denoted in the same way) induced by the following equivalence relation: $(x, A) \sim (y, B)$ iff $x = y$ and $A \sim B$, for all $x, y \in \mathbb{R}^p$ and all $A, B \in \mathbb{X}_0^2(\mathbb{R}^p)$. Let $\mathcal{L}^2(\mathbb{S}^{p-1}, \vartheta_p)^+ = \{f \in \mathcal{L}^2(\mathbb{S}^{p-1}, \vartheta_p) \,|\, f \geq 0\}$ (i.e. the equivalence classes containing at least one positive function). Then,

Theorem 3.4. *The following properties hold,*

(a) $\Gamma(\mathbb{R}^p \times \mathbb{X}_0^2) = \mathbb{R}^p \times \mathcal{L}^2(\mathbb{S}^{p-1}, \vartheta_p)^+$ *is a closed and convex cone.*

(b) $(\mathbb{R}^p \times \mathbb{X}_0^2, d_\tau)$ *is a complete and separable metric space.*

(c) $\Gamma : \mathbb{R}^p \times \mathbb{X}_0^2 \to \mathbb{R}^p \times \mathcal{L}^2(\mathbb{S}^{p-1}, \vartheta_p)^+$ *is a isometric isomorphism.*

(d) d_τ *is invariant by isometries in the following sense: given* $T_1, T_2 : \mathbb{R}^p \to \mathbb{R}^p$ *two isometries with* $T_2(0) = 0$*, then*

$$d_\tau((T_1(x), T_2(A)), (T_1(y), T_2(B))) = d_\tau((x, A), (y, B))$$

for all $(x, A), (y, B) \in \mathbb{R}^p \times \mathbb{X}_0^2(\mathbb{R}^p)$*.*

(e) $d_\tau^2((0, A), (0, \{0\}))/\tau = \int_{\mathbb{S}^1} \rho_A^2(u)\vartheta_p(du) = area(A)$ *for all* $A \in \mathbb{X}_0^2(\mathbb{R}^2)$*.*

Proof. It is straightforward to check (a), (e) and to prove that d_τ is a metric (recall that $\mathbb{R}^p \times \mathbb{X}_0^2$ is composed of equivalence classes). By construction, (c) holds. In addition, $\mathbb{R}^p \times \mathcal{L}^2(\mathbb{S}^{p-1}, \vartheta_p)^+$ is a closed subspace of the complete and separable metric space $\mathbb{R}^p \times \mathcal{L}^2(\mathbb{S}^{p-1}, \vartheta_p)$, and thus, it is complete and separable. Consequently, as Γ is an isometric isomorphism, then $\mathbb{R}^p \times \mathbb{X}_0^2$ is also complete and separable, so (b) holds. Finally, (d) can be proven by using the same arguments as in the proof of Proposition 4.4 in [26]. □

3.3. Random L_2 Star-shaped sets

Let us consider a probabilistic setting in which the characteristic of interest associated with the outcome of a random experiment is expressed through star-shaped sets. This random mechanism can be suitably modeled by means of a random L_2 star-shaped set (R2S), defined as follows.

Definition 3.2. Let $(\Omega, \mathcal{A}, P)$ be a probability space. A mapping $X : \Omega \to \mathbb{R}^p \times \mathbb{X}_0^2$ will be called a *random L_2 star-shaped set* if it is a Borel measurable mapping with respect to $\mathcal{A}$ and the Borel σ-field generated by the topology induced by the metric d_τ on $\mathbb{R}^p \times \mathbb{X}_0^2$.

The statistical analysis of random star-shaped sets will extend the traditional setting, by considering the essential concepts and tools to develop meaningful statistics for this kind of variable. Furthermore, the embedding of the space of experimental data $\mathbb{R}^p \times \mathbb{X}_0^2$ into $\mathbb{R}^p \times \mathcal{L}^2(\mathbb{S}^{p-1})$ will allow us to take advantage of powerful statistical results for random elements in separable Hilbert spaces.

Given a mapping $X : \Omega \to \mathbb{R}^p \times \mathbb{X}_0^2$, consider the marginal mappings $c_X : \Omega \to \mathbb{R}^p$ and $A_X : \Omega \to \mathbb{X}_0^2$ so that $X = (c_X, A_X)$. Let $\rho_X : \Omega \to \mathcal{L}^2(\mathbb{S}^{p-1})$, given by $\rho_{X(\omega)} = \rho_{A_X(\omega)}$, be the associated polar mapping. Note that $\Gamma(X) = \Gamma(c_X, A_X) = (c_X, \rho_X)$. In this way, c_X represents the location of X and the ρ_X, the imprecision.

Then the concept of random L_2 star-shaped set can be equivalently formalized as follows.

Proposition 3.1. Given $(\Omega, \mathcal{A}, P)$ a probability space, a mapping $X : \Omega \to \mathbb{R}^p \times \mathbb{X}_0^2$ is an R2S if, and only if, c_X and A_X are random elements, i.e. they both are Borel measurable mappings w.r.t. the corresponding σ-fields generated by the topologies induced by the euclidean metric and the L_2 norm, respectively. Moreover, X is an R2S if, and only if, $\Gamma \circ X$ is an $\mathcal{H}-$valued random element, and this happens if, and only if, c_X and ρ_X are random elements w.r.t. the corresponding σ-fields.

According to Proposition 3.1, if X is an R2S, then it is measurable w.r.t. β_{d_τ} for any value $\tau \in (0, 1)$. The Borel measurability condition for R2Ss allows us to properly consider traditional statistical notions such as the distribution induced by an R2S or the stochastic independence of R2Ss, among others.

The most used characterizing summary measures (moments) of the distribution of a random variable are the expected value, describing its central tendency values, and the variance, to measure the variability or dispersion of the values of the variable w.r.t. that central element. The probabilistic formalization of R2Ss together with the consideration of a metric space of star-shaped sets allow us to introduce these summary measures based on Fréchet concepts in general metric spaces. Namely, the mean value can be defined as the element in the space minimizing the mean squared distance to the values of the variables, and the variance is defined as this minimum. This approach agrees with the classical definition of expected value in Hilbert spaces based on the Bochner integral.

Formally,

Definition 3.3. Given $(\Omega, \mathcal{A}, P)$ a probability space and $X : \Omega \to \mathbb{R}^p \times \mathbb{X}_0^2$ an R2S associated with it so that $E(\|c_X\|) < \infty$ and $E(\|\rho_X\|_{\mathcal{L}^2}) < \infty$, then the expected value of X is defined as the element $E(X) \in \mathbb{R}^p \times \mathbb{X}_0^2$ so that $\Gamma(E(X)) = E(\Gamma(X))$, where the expectations in $\mathcal{H}$ are considered in terms of Bochner integrals.

Remark 3.1. Under the considered conditions on X, both Bochner and Pettis expectations agree and, particularly, the expected value verifies that

$$\left\langle E(\Gamma(X)), u\right\rangle_\tau = E\left(\left\langle\Gamma(X), u\right\rangle_\tau\right) \text{ for all } u \in \mathcal{H} \text{ and any } \tau \in (0,1).$$

Moreover, if $E(\|c_X\|^2) < \infty$ and $E(\|\rho_X\|^2_{\mathcal{L}^2}) < \infty$ (equivalently $E(d^2_\tau(X,0) < \infty$ irrespectively of the value of $\tau \in (0,1)$ chosen), then $E(X)$ is the unique element in $\mathbb{R}^p \times \mathbb{X}_0^2$ so that

$$E(d^2_\tau(X, E(X))) = \min_{(c,A)\in\mathbb{R}^p\times\mathbb{X}_0^2} E(d^2_\tau(X,(c,A))). \tag{5}$$

In this situation the variance can be defined as the value of this minimum.

Definition 3.4. Given $(\Omega, \mathcal{A}, P)$ a probability space and $X : \Omega \to \mathbb{R}^p \times \mathbb{X}_0^2$ an R2S associated with it so that $E(d^2_\tau(X,0) < \infty$, the variance of X, σ^2_X or $Var(X)$, is defined as

$$Var(X) = E\left(d^2_\tau(X, E(X))\right) = E(||\Gamma(X) - E(\Gamma(X))||^2_\tau) = Var(\Gamma(X)) < \infty.$$

To complement these concepts, the covariance, usually employed in regression contexts, is considered. In this sense,

Definition 3.5. Given $(\Omega, \mathcal{A}, P)$ a probability space and $X, Y : \Omega \to \mathbb{R}^p \times \mathbb{X}_0^2$ two R2Ss associated with it so that $E(d^2_\tau(X,0)) < \infty$ and $E(d^2_\tau(Y,0)) < \infty$, the covariance of X and Y, $\sigma_{X,Y}$ or $Cov(X,Y)$, is defined as

$$Cov(X,Y) = E\left(\left\langle\Gamma(X) - E(\Gamma(X)), \Gamma(Y) - E(\Gamma(Y))\right\rangle_\tau\right) \in \mathbb{R}$$

The covariance between X and Y cannot be formulated in terms of the original variables, but based on the transformed ones, $\Gamma(X)$ and $\Gamma(Y)$, due to the lack of linearity of the original space.

3.4. Estimation of moments and the Central Limit Theorem

Let X be a random L_2 star-shaped set, and consider $(X_1, X_2, \ldots)$ an i.i.d. sequence of R2Ss defined on the same probability space and with the same distribution as X. Recall that from the characterization of R2Ss shown in Proposition 3.1, we can consider for each $i \in \mathbb{N}$ the random elements $c_{X_i} : \Omega \to \mathbb{R}^p$, $A_{X_i} : \Omega \to \mathbb{X}_0^2$ and $\rho_{X_i} : \Omega \to \mathcal{L}^2(\mathbb{S}^{p-1})$ so that $X_i = (c_{X_i}, A_{X_i})$ and $\Gamma(X_i) = (c_{X_i}, \rho_{X_i})$. Thus, $\{c_{X_i}\}_{i\in\mathbb{N}}$ are i.i.d. random vectors in $\mathbb{R}^p$, $\{A_{X_i}\}_{i\in\mathbb{N}}$ are i.i.d. random elements taking values in $\mathbb{X}_0^2$ and $\{\rho_{X_i}\}_{i\in\mathbb{N}}$ are i.i.d. random elements taking values in the separable Hilbert space $\mathcal{L}^2(\mathbb{S}^{p-1})$ with common distribution c_X, A_X and ρ_X respectively. For any fixed $n \in \mathbb{N}$, $\{X_i\}_{i=1}^n$, $\{c_{X_i}\}_{i=1}^n$, $\{A_{X_i}\}_{i=1}^n$ and $\{\rho_{X_i}\}_{i=1}^n$ will denote the corresponding simple random samples.

The sample expectation or mean of X is defined in terms of the arithmetic on the space $\mathbb{R}^p \times \mathbb{X}_0^2$ as

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i. \tag{6}$$

The arithmetic with positive scalars is directly related with the arithmetic in $\mathbb{R}^p \times \mathcal{L}^2(\mathbb{S}^{p-1})$ through the embedding Γ. It is straightforward to show that $\Gamma(\overline{X}) = (c_{\overline{X}}, \rho_{\overline{X}})$ coincides with $(\overline{c_X}, \overline{\rho_X}) = \overline{\Gamma(X)}$.

The sample variance is defined as the sample counterpart of $Var(X)$ in (3.4), i.e.

$$\widehat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n} d_\tau^2(X_i, \overline{X}) \ . \tag{7}$$

Expressing $d_\tau^2(X_i, \overline{X})$ in terms of the corresponding distances on $\mathbb{R}^p$ and $\mathcal{L}^2(\mathbb{S}^{p-1})$ it is shown that $\widehat{\sigma}_X^2$ can be expressed as

$$\widehat{\sigma}_X^2 = (1-\tau)\widehat{\sigma}_{c_X}^2 + \tau\widehat{\sigma}_{\rho_X}^2, \tag{8}$$

which highlights the role of the parameter τ as a balance of the weight of the variance of locations and imprecisions in the overall variance.

As a consequence of the well-known Strong Law of Large Numbers (SLLN) in separable Banach spaces, the following result holds.

Theorem 3.5. *Let $(X_1, X_2, \ldots)$ be an i.i.d. sequence of R2Ss defined on the same probability space $(\Omega, \mathcal{A}, P)$.*

- *If* $E(\|c_{X_1}\|) < \infty$ *and* $E(\|\rho_{X_1}\|_{\mathcal{L}^2}) < \infty$*, then* $\overline{X} \xrightarrow{a.s.-P} E(X)$*. Indeed* $\overline{c_X} \xrightarrow{a.s.-P} E(c_X)$*,* $\overline{A_X} \xrightarrow{a.s.-P} E(A_X)$ *and* $\overline{\rho_X} \xrightarrow{a.s.-P} E(\rho_X)$*.*

- *If* $E(d_\tau^2(X_1, 0)) < \infty$*, then* $\widehat{\sigma}_X^2 \xrightarrow{a.s.-P} Var(X)$*. Indeed* $\widehat{\sigma}_{c_X}^2 \xrightarrow{a.s.-P} Var(c_X)$*,* $\widehat{\sigma}_{A_X}^2 \xrightarrow{a.s.-P} Var(A_X)$ *and* $\widehat{\sigma}_{\rho_X}^2 \xrightarrow{a.s.-P} Var(\rho_X)$*.*

The so-called covariance operator plays a fundamental role in the establishment of the Central Limit Theorem (CLT). In this sense, let X be an R2S so that $E(d_\tau^2(X,0)) < \infty$. The covariance operator of $\Gamma(X)$ is the linear mapping $C_{\Gamma(X)} : \mathcal{H} \to \mathcal{H}$ verifying

$$Cov\big(\langle\Gamma(X), u\rangle_\tau, \langle\Gamma(X), v\rangle_\tau\big) = \langle C_{\Gamma(X)}(u), v\rangle_\tau \text{ for all } u, v \in \mathcal{H},$$

that is,

$$C_{\Gamma(X)}(u) = E\big(\langle(\Gamma(X) - E(\Gamma(X))), u\rangle_\tau(\Gamma(X) - E(\Gamma(X)))\big) \text{ for all } u \in \mathcal{H}.$$

The condition $E(d_\tau^2(X,0)) < \infty$, implies that $C_{\Gamma(X)}$ is a self-adjoint, positive and nuclear operator (see [27]). Thus, according to the Spectral Theorem, there exists an orthonormal basis of $\mathcal{H}$, $\{v_n\}_{n\in\mathbb{N}}$, consisting of eigenvectors of $C_{\Gamma(X)}$ with corresponding non-negative eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ decreasing towards 0 so that the covariance operator can be decomposed as

$$C_{\Gamma(X)}(u) = \sum_{n=1}^{\infty} \lambda_n \langle u, v_n\rangle v_n, \quad \text{for all } u \in \mathcal{H}.$$

Consequently,

$$\Gamma(X) = E(\Gamma(X)) + \sum_{n=1}^{\infty} Z_n v_n$$

where $\{Z_n\}_{n\in\mathbb{N}}$ is the collection of centered and uncorrelated real random variables with corresponding variances $\{\lambda_n\}_{n\in\mathbb{N}}$ given by $Z_n = \langle\Gamma(X) - E(\Gamma(X)), v_n\rangle$ for all $n \in \mathbb{N}$. Note that the orthonormal basis $\{v_n\}_{n\in\mathbb{N}}$ entails a decomposition of the $\mathcal{H}-$valued random element $\Gamma(X)$ such that $Var(X) = Var(\Gamma(X)) = \sum_{n=1}^{\infty} \lambda_n$.

Theorem 3.6. Let $(X_1, X_2, \ldots)$ be an i.i.d. sequence of R2Ss defined on the same probability space $(\Omega, \mathcal{A}, P)$ so that $E(d_\tau^2(X_1, 0)) < \infty$, then

i) $n^{1/2}\left(\Gamma\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - E(\Gamma(X_1))\right) \to Z_{\Gamma(X_1)}$ weakly in $\mathcal{H}$,

ii) $n\, d_\tau^2\left(\frac{1}{n}\sum_{i=1}^{n} X_i, E(X_1)\right)_\tau \to \|Z_{\Gamma(X_1)}\|_\tau^2$ weakly in $\mathbb{R}$,

where $Z_{\Gamma(X_1)}$ is a centered Gaussian $\mathcal{H}-$valued random element with $C_Z = C_{\Gamma(X_1)}$.

Note that $i)$ is just the Central Limit Theorem in separable Hilbert spaces (see [18]) applied to the i.i.d. sequence $(\Gamma(X_1), \Gamma(X_2)\ldots)$. The result cannot be stated directly for the sequence of R2Ss $(X_1, X_2, \ldots)$ due to the lack of linearity of the space $\mathbb{R}^p \times \mathbb{X}_0^2$. Once the norm is computed, in view of its relation with the distance d_τ, the weak limit for the distance (or the squared distance) between the sample mean and the expectation can be deduced by applying the Continuous Mapping Theorem (item ii)).

4. Location of an arbitrary star-shaped set

Once a star-shaped set is observed, the first step in order to use the star-shaped sets parametrization introduced previously is to decompose it as a center plus a star-shaped set with respect to 0. A reasonable initial proposal is to choose the center of gravity of the kernel as location parameter.

The main drawback of this direct approach is associated with the lack of robustness of the kernel against slight perturbations of a star-shaped set. In order to illustrate this fact, consider the two closed star-shaped sets represented in Figure 6.

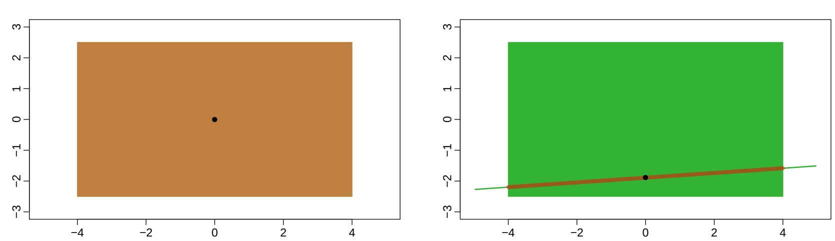

Figure 6: Kernel (brown) of two similar closed star-shaped sets and corresponding centers of gravity

The first one is a simple rectangle and, thus, its kernel, represented in brown color, is the whole set. The second one, which will be denoted by A, has been obtained as the union of the initial rectangle with a line segment. Its kernel reduces to the brown segment represented in the picture. Although both star-shaped sets are almost ‘identical’, their kernels are drastically different, and thus, the center of gravity of the kernels, represented by the black dots, are different.

Moreover, if c_A denotes the center of gravity of $ker(A)$, then it can be checked that the polar function of $A - c_A$ belongs to the same equivalence class as the polar function of $B - c_A$ (see Figure 7). That is, $K_{\rho_{A-c_A}}$ and $K_{\rho_{B-c_A}}$ are identical sets according to the distance d_τ, although their kernels are absolutely different.

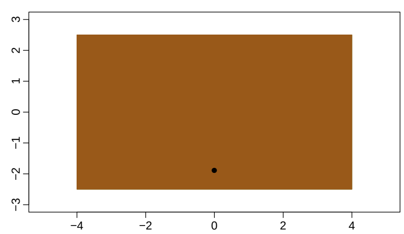

Figure 7: Star-shaped set B and an ’arbitrary’ center c_B (black dot)

This illustration highlights the need for a robust definition of center against simple perturbations. The main problem is associated with the possible existence of negligible subsets in the boundary of any star-shaped set A that would restrict the kernel of A to a lower dimensional subspace. These subsets should be removed, and this pre-processing step should lead to a new star-shaped set equivalent to the original one (and, thus, with the same measure).

Given any $B \subset \mathbb{R}^p$, consider the subspace determined by the linear span of all the finite collections of vectors determined by B. That is, given $b \in B$,

$$span(B) = \left\{ b + \sum_{i=1}^{k} \gamma_i (b_i - b) \,|\, k \in \mathbb{N}, b \neq b_i \in B,\, \gamma_i \in \mathbb{R} \right\}.$$

Let A be a star-shaped set and denote by clB, $intB$, ∂B and B^c the closure, interior, boundary and complementary of a subset B of $span(A)$ with respect to the topological subspace $span(A)$ respectively. Note that as $ker(A)$ is a convex set, then $\lambda_{\mathbb{R}^p}(ker(A)) > 0$ if, and only if, $span(ker(A)) = \mathbb{R}^p$. In this case, as the topological interior of $ker(A)$ is not the empty set, then the measurability of the set follows as it is stated in the next result. Considering the developments in $span(A)$, instead of in $\mathbb{R}^p$, will help to preserve as much information about the sets as possible, because, even if they are null sets in $\mathbb{R}^p$, their location could be relevant, and they should not be considered all equivalent to $\{0\}$.

Theorem 4.1. *Let A be a star-shaped set verifying that $\lambda_{\mathbb{R}^p}(ker(A)) > 0$ then, $\partial(A - c) = (A - c)_E$ for all $c \in int(ker(A))$. Consequently, A is a Lebesgue measurable set, being its boundary a null set.*

Proof. Without loss of generality it can be assumed that $0 = c \in int(ker(A))$. As $A_E \subset \partial A$, it remains to check that $\partial A \subset A_E$. Let $0 \neq a \in A$, then $a = ku$ for certain $u \in \mathbb{S}^{p-1}$ and $k > 0$. Assume that $a \notin A_E$, then there exists $0 < k < M < \infty$ so that $Mu \in A$. As $0 \in int(ker(A))$ there exists $\epsilon > 0$ so that $B_\epsilon \subset int(ker(A))$ being $B_\epsilon = \{x \in \mathbb{R}^p | \|x\|^2 < \epsilon\}$. In addition, since A is a star-shaped set, then

$$C = \{\alpha(Mu) + (1 - \alpha)v | \alpha \in (0, 1), \, v \in B_\epsilon\} \subset A.$$

Some algebra shows that $a = ku \in int(C)$, which implies that $a \in int(A)$.

Consequently, given $c \in int(ker(A))$, it follows that $(A - c)_I = int(A - c)$, which is a measurable set. Thus, according to Corollary 3.1 and Theorem 3.1, $A - c$ can be concluded to be a Lebesgue measurable set and its boundary is a null set, which finalizes the proof. $\square$

The common star-shaped sets in practice have null Lebesgue measurable boundary, so the pre-processing step will focus mainly on such sets. Indeed, to handle star-shaped sets in practice, the usual approach will consist in approximating them by means of p-polytopes parametrized by the directions of their vertexes and its norms, obtaining the rest of the p-polytopes by linear interpolation. The pre-processing for star-shaped sets with non-null Lebesgue measurable boundary is a challenging theoretical problem that is left partially open due to its lack of practical interest at this stage. Note that according to the previous theorem, star-shaped sets with non-null Lebesgue measurable boundary have its kernel restricted to a subspace of lower dimension than the span of the original set. In Section 4.1 an example of one of such sets contaminated by a null set that restricts, even more, its kernel is shown for illustrative purposes.

The aim is to establish a consistent pre-processing procedure in such a way that it will be based just on the set to be pre-processed, and not on the general space. Thus, given any star-shaped set A, the subspace spanned by A will be considered as working space. We can assume without loss of generality that $span(A) = \mathbb{R}^p$ (for the particular case in which A is a singleton the results are trivial, otherwise we can apply the corresponding isometry T so that $span(T(A)) = \mathbb{R}^d$ for certain $d < p$). Note that, once the span is considered, the previous theorem guarantees, in particular, the measurability of any convex subset of $\mathbb{R}^p$.

Definition 4.1. *Let $A \subset \mathbb{R}^p$ be a measurable star-shaped set verifying that $A = cl(int(A))$. Then, it is said that A is a prototype star-shaped set. The class of prototype star-shaped sets of $\mathbb{R}^p$ is denoted by $\mathcal{PS}(\mathbb{R}^p)$.*

Recall that the closure and interior is considered w.r.t. the topological subspace of the span of the set. Consequently, any non-empty closed and convex subset of $\mathbb{R}^p$ is a prototype star-shaped set, that is, $\mathcal{K}_c(\mathbb{R}^p) \subset \mathcal{PS}(\mathbb{R}^p)$. In addition, for any measurable star-shaped set A verifying that $\lambda_{\mathbb{R}^p}(A \cap \partial(A)) = 0$ (so $A = int(A)$ a.s.-$\lambda_{\mathbb{R}^p}$) and $\lambda_{\mathbb{R}^p}(\partial(int(A))) = 0$ (so $int(A) = cl(int(A))$ a.s.-$\lambda_{\mathbb{R}^p}$) it holds that $cl(int(A)) \in \mathcal{PS}(\mathbb{R}^p)$.

The following theorem provides with some properties that clarifies the usefulness of prototypes star-shaped sets.

Theorem 4.2. *Let $A, B \subset \mathbb{R}^p$ be Lebesgue measurable star-shaped sets with $span(A) = span(B) = \mathbb{R}^p$, then*

i) *$int(A) \cup ker(A)$ is a star-shaped set so that $ker(A) \subset ker(int(A) \cup ker(A))$*

ii) *$cl(A)$ is a star-shaped set so that $ker(A) \subset ker(cl(A))$*

iii) *If $int(A) \neq \emptyset$ then $ker(A) \subset ker(cl(int(A)))$*

iv) *If $A = cl(int(A))$ a.s.-$\lambda_{\mathbb{R}^p}$ and $B = cl(int(B))$ a.s.-$\lambda_{\mathbb{R}^p}$ then,*

$$A = B \text{ a.s.-}\lambda_{\mathbb{R}^p} \text{ if, and only if, } cl(int(A)) = cl(int(B))$$

v) *If $A \in \mathcal{PS}(\mathbb{R}^p)$ and $B = A$ a.s.-$\lambda_{\mathbb{R}^p}$ then $ker(B) \subset ker(A)$.*

Proof. Regarding item *i*), let $c \in ker(A)$. The convexity of $ker(A)$ guarantees that $\lambda c + (1-\lambda)a \in ker(A)$ for all $a \in ker(A)$ and all $\lambda \in [0,1]$. If $int(A) \neq \emptyset$, given $a \in int(A)$ with $a \neq c$ there exists $\epsilon > 0$ so that $B(a,\epsilon) \subset int(A) \subset A$. Thus, $\lambda c + (1-\lambda)x \in A$ for all $x \in B(a,\epsilon)$ and all $\lambda \in [0,1]$, which implies that $\lambda c + B((1-\lambda)a, (1-\lambda)\epsilon) \subset A$ for all $\lambda \in [0,1)$. Consequently, $\lambda c + (1-\lambda)a \in int(A) \cup ker(A)$ for all $\lambda \in [0,1]$, and the result is proven.

In order to prove item *ii*), let $c \in ker(A)$ (consider $c = 0$ without loss of generalization) and $a \in cl(A)$. Let us assume that there exists $\lambda \in (0,1)$ so that $\lambda c + (1-\lambda)a = (1-\lambda)a \notin cl(A)$. Then we can select $\epsilon > 0$ so that $B((1-\lambda)a,\epsilon) \cap A = \emptyset$. Thus, for any $x \in B((1-\lambda)a,\epsilon)$, it follows that $\delta x \notin A$ for any $\delta \geq 1$ and, consequently, $B(\delta(1-\lambda)a, \delta\epsilon) \cap A = \emptyset$ for all $\delta \geq 1$. By selecting $\delta = (1-\lambda)^{-1}$, we reach a contradiction, because $a \in cl(A)$. Thus, $\lambda c + (1-\lambda)a \in cl(A)$ for all $\lambda \in [0,1]$ and, as consequence, $cl(A)$ is an star-shaped set verifying that $ker(A) \subset ker(cl(A))$.

Concerning item *iii*), first of all it will be proved that if $int(A) \neq \emptyset$, then $ker(A) \subset cl(int(A))$. Following the proof of item *i*): given $a \in int(A) \neq \emptyset$ with $a \neq c$, then $\lambda c + (1-\lambda)a \in int(A)$ for all $\lambda \in [0,1)$. Thus, given $0 < \gamma < \|c-a\|$ for any $\lambda > 1 - \gamma/\|c-a\|$, it is verified that $\lambda c + (1-\lambda)a \in B(c,\gamma)$ and, consequently, $c \in cl(int(A))$.

If $int(A) \neq \emptyset$, by taking into account that $int(A) \cup ker(A) \subset A$, then $ker(A) \subset ker(int(A) \cup ker(A)) \subset ker(cl(int(A) \cup ker(A)))$, where the last content is guaranteed by item *ii*). Note that $int(A) \subset int(A) \cup ker(A) \subset int(A) \cup cl(int(A)) = cl(int(A))$, and thus, $cl(int(A) \cup ker(A)) = cl(int(A))$, which proves the assertion.

With respect to item iv), obviously, if $cl(int(A)) = cl(int(B))$, then $A = B$ a.s.-$\lambda_{\mathbb{R}^p}$. Conversely, let us assume that $A = B$ a.s.-$\lambda_{\mathbb{R}^p}$, then $cl(int(A)) = cl(int(B))$ a.s.-$\lambda_{\mathbb{R}^p}$, and thus, $cl(int(A)) \cup (cl(int(B)))^c$ is a null Lebesgue measurable set. The set $int(cl(int(A))) \cup (cl(int(B)))^c$ is contained in a null set, and as it is open, then it must be the empty set. Therefore, $int(cl(int(A))) \subset cl(int(B))$, and thus, $int(cl(int(A))) \subset int(cl(int(B)))$. In the same way, it can be checked that $int(cl(int(B))) \subset int(cl(int(A)))$, and thus, $int(cl(int(A))) = int(cl(int(B)))$, so the result follows.

Finally, in order to prove item v), first of all it will be checked that $ker(B) \subset A$. In this way, let consider $c \in ker(B)$ and assume that $c \notin A$. Let $a \in int(A)$ so that $\delta = 2|c - a| > 0$. Note that $a \in B(c, \delta) \cap int(A)$, and thus, there exists $\epsilon > 0$ verifying that $B(a, \epsilon) \subset B(c, \delta) \cap int(A)$. Obviously, $\lambda_{\mathbb{R}^p}(B(a, \epsilon) \cap B^c) \leq \lambda_{\mathbb{R}^p}(A \cap B^c) = 0$ and, consequently,

$$\lambda_{\mathbb{R}^p}(B \cap B(c, \delta)) \geq \lambda_{\mathbb{R}^p}(B \cap B(a, \epsilon)) = \lambda_{\mathbb{R}^p}(B(a, \epsilon)) - \lambda_{\mathbb{R}^p}(B(a, \epsilon) \cap B^c) > 0.$$

Let us assume, without loss of generality, that $c = 0$. As $c \in A^c$, there exists $\eta > 0$ so that $B(0, \eta) \cap A = \emptyset$, with $\eta < \delta$. Note that $B \cap B(0, \delta)$ is a star-shaped set being $c = 0$ contained in its kernel, and thus,

$$(\eta/\delta)(B \cap B(0, \delta)) \subset (B \cap B(0, \eta)) \subset B \cap A^c.$$

However,

$$\lambda_{\mathbb{R}^p}((\eta/\delta)(B \cap B(0, \delta))) = (\eta/\ delta)^p \lambda_{\mathbb{R}^p}(B \cap B(0, \delta)) > 0,$$

which leads to a contradiction. Consequently, applying item ii) we have that $ker(B) = ker(B) \cap A \subset ker(cl(B)) \cap A \neq \emptyset$.

Given $a \in int(A)$ and $\epsilon > 0$ so that $B(a, \epsilon) \subset int(A)$, it is straightforward to check that $B(a, \epsilon) \cap B \neq \emptyset$, so it follows that $A \subset cl(A \cap B)$. Obviously, $cl(A \cap B) \subset cl(B) \cap A \subset A$, and thus, $A = cl(B) \cap A = cl(A \cap B)$. Consequently, $ker(cl(B)) \cap A \subset ker(A)$, so the result follows.

$\square$

According to Theorem 4.2, any two different prototype star-shaped sets (spanning the same space) are also almost surely different sets. Consequently, given $A \in \mathcal{PS}(\mathbb{R}^p)$, the family of measurable star-shaped sets of $\mathbb{R}^p$ that are almost sure equal to A in Lebesgue sense can be considered:

$$[A] = \big\{B \subset \mathbb{R}^p \text{ measurable star-shaped set so that}$$
$$span(A) = span(B) \text{ and } A = B \text{ a.s.} - \lambda_{span(A)}\big\}.$$

For any $A, B \in \mathcal{PS}(\mathbb{R}^p)$ with $A \neq B$ and $span(A) = span(B)$, it is verified that $[A] \cap [B] = \emptyset$. In addition, any $A \in \mathcal{PS}(\mathbb{R}^p)$ is the star-shaped set element of its class with "maximal" kernel (if $B \in [A]$ then $ker(B) \subset ker(A)$), so prototype star-shaped sets are appropriate to introduce the pursued definition of kernel robust against simple perturbations. Nevertheless, it has to be noted that there are measurable star-shaped sets not belonging to any prototype star-shaped class. These sets must have, in particular, non-null Lebesgue measurable boundary and its corresponding pre-processing is not addressed in this paper, as mentioned before. Irrespectively of this fact, it should be recalled that for any measurable star-shaped set A, once its location $c_A \in ker(A)$ is fixed, Theorem 3.1 guarantees that A is perfectly characterized by its center c_A and its polar function ρ_{A-c} (since $A - c = K_{\rho_{A-c}}$ a.s.-$\lambda_{\mathbb{R}^p}$), even if $\lambda_{\mathbb{R}^p}(\partial(A)) > 0$.

Let A be a star-shaped set and recall that it is assumed, without loss of generality, that $span(A) = \mathbb{R}^p$. The pre-processing will be as follows:

(1) If $\lambda_{\mathbb{R}^p}(A) = 0$, then replace A by $cl(ker(A))$

(2) If $\lambda_{\mathbb{R}^p}(A) > 0$ and there exists $B \in \mathcal{PS}(\mathbb{R}^p)$ verifying that $A \in [B]$, then replace A by B.

If $\lambda_{\mathbb{R}^p}(A) = 0$ then $ker(A) \subset A$ is a null Lebesgue measurable set. According to Theorem 4.1 also $\partial(ker(A))$ is a null set, and thus, A and its replacement (a closed convex set) are equal a.s-$\lambda_{\mathbb{R}^p}$. In this case, as any subset of A is a null-set, we consider that there is no need for a further identification of negligible subsets in its boundary that could restrict its kernel even more. Of course, more elaborated techniques are left to be developed in this respect. Consequently, in both cases the pre-processing replaces A by an equivalent (following the same idea as in Theorem 3.3) prototype star-shaped set. Note that in the Figure 6, the left star-shaped set is a prototype, and the right star-shaped set belongs to its class, so the pre-processing step transforms the contaminated star-shaped set into the "uncontaminated" one (an equivalent one with maximal kernel). In the particular case of A being a singleton, no replacement is carried out as $span(A) = A$. Obviously, there are also some star-shaped sets having non-null Lebesgue measurable boundary that are preserved with no modification (and thus, its kernel does not change with the pre-processing step). An example of one of such sets is provided in Section 4.1.

In practice, a class wide enough of prototype star-shaped sets is composed by those with bounded kernel. This class includes all kind of bounded prototype star-shaped sets and, more generally, all prototype star-shaped sets with finite Lebesgue measure w.r.t. the corresponding span. For this class of sets, the location is defined as follows:

Definition 4.2. *Let A be a prototype star-shaped set with bounded kernel. Then the location of A, c_A, is defined as the center of mass of $ker(A)$ w.r.t. the Lebesgue measure defined on $span(ker(A))$.*

Note that, in particular, if A is a singleton, i.e. $A = \{a\}$, then $c_A = a$. For measurable star-shaped sets not belonging to any prototype star-shaped class, the selection of an appropriate center remains an open problem for future theoretical research as it has been pointed out.

4.1. Star-shaped sets with non-null Lebesgue measurable boundary

The pre-processing of star-shaped sets with non-null Lebesgue measurable boundary is challenging. These are basically theoretical sets: it is not expectable to find one of those sets in practice, since it would be technically difficult to provide the information of the points composing one of those sets. Nevertheless, for illustrative purposes, a star-shaped set with non-null boundary will be shown. Some "noise" will be added to the set so that the corresponding kernel is restricted, and an appropriate pre-processing would be requited to remove it.

The so-called Smith-Volterra-Cantor set is considered as starting point. This set is constructed iteratively by removing some intervals from $[0, 1]$. If instead of the unit interval, the unit sphere in $\mathbb{R}^2$ is considered, proceed as follows: let Q_1 be the first quadrant of $\mathbb{S}^1$ (analogously with the other quadrants). In the first step, remove the "middle" 1/4 of the quadrant (those $u \in \mathbb{S}^1$ with $u \geq 0$ and $\langle u, (0, 1)\rangle \in (3/8, 5/8)$). The removed subset is open with mass $(1/4)\vartheta(Q_1)$. The remaining points split Q_1 in two subsets that are closed. In the n-th step, for each of the obtained 2^{n-1} closed subsets, remove its central open $1/2^{2n}$ part. So, a total mass of $(1/2^{n+1})\vartheta(S)$ is removed from the unit sphere at each step (when the procedure is carried out for all quadrants). As usual, the limit set C is a countable intersection of closed sets, so it is closed. Since its interior is empty, it is equal to its boundary. Moreover, as the remaining set has a mass of $\vartheta(S)\sum_{n=1}^{\infty}(1/2^{n+1}) = (1/2)\vartheta(S)$, then it has a mass of $(1/2)\vartheta(S)$ too.

Let $f(u) = 1$, if $u \in C$, and $f(u) = 0$ otherwise. Note that the hypograph of f is $C \times [0, 1]$, and $K_f = \phi^{-1}(C \times (0, 1]) \cup \{(0, 0)\}$ is a closed set of $\mathbb{R}^p$ (recall ϕ defined at the end of Section 3.1 and the fact that $C \times (0, 1]$ is closed in $\mathbb{S}^1 \times (0, \infty)$) with kernel $\{(0, 0)\}$). Indeed, its interior in $\mathbb{R}^p$ is empty, and its mass is $\lambda_{\mathbb{R}^p}(K_f) = \vartheta_p(C) = (1/2)\vartheta(S)$.

Consider now the set $A = [0, 1] \times K_f$, that is again closed, with empty interior, mass $(1/2)\vartheta(S)$ and kernel $[0, 1] \times \{0\} \times \{0\}$. Consider the closed and convex polygon B with vertexes $\{(0.5, 0, 0), (0, 0, 1), (0, 0, -1)\}$. The star-shaped set $A \cup B$ is closed, the interior is empty and the mass of its boundary is greater than 0, but its kernel is restricted to the set $[0, 0.5] \times \{0\} \times \{0\}$ due to the star-shaped set B. Note that B has mass 0, as it is restricted to a lower dimensional subspace. Thus, $A = A \cup B$ a.s.-$\lambda_{\mathbb{R}^p}$, and an appropriate pre-processing should be able to detect the existence of this null mass set and to remove it. Nevertheless, the task is not easy, and it could be non-computable in a finite time.

5. Conclusions and future research

Set-valued experimental data have been used to model imprecision in the characteristics observed in statistical experiments. The interval case is a remarkable example. When sets in higher dimensions are observed, the generalized setting of compact and convex sets is frequently considered. These are normally represented through the support function and, by extension of the interval case, the Minkowski arithmetic is considered. We have shown that this does not always have an intuitive meaning. The main aim has been to provide an alternative way to characterize sets, so that a meaningful statistical framework for set-valued experimental data can be established. For that, the general framework of star-shaped sets have been formalized, and the so-called center-radial parametrization has been analyzed in detail. This new representation is intuitive in terms of location/imprecision, and allows us to set up a statistical framework inherited by the powerful one available in Hilbert spaces. As happens with the representation based on the support function, the natural arithmetic endows the corresponding space with a conical structure, not linear, which is related to the propagation of the imprecision. Thus, this must be taken into account to formalize the specific statistics. Nevertheless, in contrast to the representation based on the support function, the identification of the cone in the corresponding Hilbert space is trivial, and analogous to the interval case (i.e., non-negativity constraints), which greatly simplifies the developments.

The main statistical objects (namely, the concept of random element, expectation, variance and covariance) have been introduced, and the basic probabilistic results (namely, the SLLN and the CLT) have been stated. On this basis, hypothesis testing and related inferences regarding the moments could be established. In order to show how to establish in practice the representation, a natural and robust way to determine the center has been discussed, but special attention should also be payed to the alignment of the sets, as rotations would affect artificially the results.

All the concepts and results in this paper can be extended level-wise to the case of fuzzy subsets of $\mathbb{R}^p$. The mathematical extension is simple, but the notation and practical implementation are cumbersome, and they are left as future research.

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