ERROR ESTIMATES FOR DIRICHLET CONTROL PROBLEMS IN POLYGONAL DOMAINS: QUASI-UNIFORM MESHES

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Abstract. The paper deals with finite element approximations of elliptic Dirichlet boundary control problems posed on two-dimensional polygonal domains. Error estimates are derived for the approximation of the control and the state variables. Special features of unconstrained and control constrained problems as well as general quasi-uniform meshes and superconvergent meshes are carefully elaborated. Compared to existing results, the convergence rates for the control variable are not only improved but also fully explain the observed orders of convergence in the literature. Moreover, for the first time, results in nonconvex domains are provided.

1. **Introduction.** In this paper we will study the finite element approximation of the control problem

$$(\text{P})\begin{cases} \min J(u) = \frac{1}{2}\|Su - y_\Omega\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \\ \text{subject to } (Su, u) \in H^{1/2}(\Omega) \times L^2(\Gamma), \\ u \in U_{ad} = \{u \in L^2(\Gamma): \ a \le u(x) \le b \ \text{ for a.a. } \ x \in \Gamma\}, \end{cases}$$

where Su is the very weak solution y of the state equation

$$-\Delta y = 0 \text{ in } \Omega, \ y = u \text{ on } \Gamma, \tag{1}$$

the domain $\Omega \subset \mathbb{R}^2$ is bounded and polygonal, Γ is its boundary, $a < b$ and $\nu > 0$ are real constants, and y_Ω is a function whose precise regularity will be stated when necessary. We assume that $0 \in [a, b]$ and comment on the opposite case in Remark 5.4. Abusing notation, we will allow the case $a = -\infty$ and $b = +\infty$ to denote the absence of one or both of the control constraints.

First order optimality conditions read as (see [1, Lemma 3.1])

Lemma 1.1. *Suppose* $y_\Omega \in L^2(\Omega)$. *Then problem (P) has a unique solution* $\bar{u} \in L^2(\Gamma)$ *with related state* $\bar{y} \in H^{1/2}(\Omega)$ *and adjoint state* $\bar{\varphi} \in H_0^1(\Omega)$. *The following optimality system is satisfied:*

$$(\nu \bar{u} - \partial_n \bar{\varphi}, u - \bar{u})_{L^2(\Gamma)} \geq 0 \ \forall u \in U_{ad}, \tag{2a}$$

$$-\Delta \bar{y} = 0 \ in \ \Omega, \ \bar{y} = \bar{u} \ on \ \Gamma, \ in \ the \ very \ weak \ sense, \tag{2b}$$

$$-\Delta \bar{\varphi} = \bar{y} - y_\Omega \ in \ \Omega, \ \bar{\varphi} = 0 \ on \ \Gamma, \ in \ the \ weak \ sense. \tag{2c}$$

The variational inequality (2a) is equivalent to

$$\bar{u}(x) = \mathrm{Proj}_{[a,b]} \left(\frac{1}{\nu} \partial_n \bar{\varphi}(x) \right) \ \text{for a.e.} \ x \in \Gamma, \tag{3}$$

where $\mathrm{Proj}_{[a,b]}$ denotes the pointwise projection on the interval $[a, b]$.

The aim of this paper is to investigate a finite element solution of the system (2a)–(2c), in particular to derive discretization error estimates. The precise description of the regularity of the solution of the first order optimality system is an important ingredient of such estimates. They were proven in our previous paper [1]; we recall these results in Section 2. There were two interesting observations which we may illustrate in the following example.

Example 1.2. Consider the L-shaped domain. The 270° angle leads in general to a singularity of type $r^{2/3}$ in the solution of the adjoint equation; the regularity can be characterized by $\bar{\varphi} \in H^s(\Omega)$ with $s < \frac{2}{3}$. Hence, the control has a $r^{-1/3}$-singularity in the unconstrained case, $\bar{u} \in H^s(\Gamma)$ for all $s < \frac{1}{6}$. In the constrained case, however, the control is in general constant in the vicinity of the singular corner since the normal derivative of the adjoint state has a pole there, we get $\bar{u} \in H^s(\Gamma)$ for all $s < \frac{3}{2}$. This regularity is determined by the largest convex angle and by the kinks due to the constraints.

Unfortunately, this is not the whole truth. In exceptional cases, e.g. when the data enjoy certain symmetry, the leading singularity of type $r^{2/3}$ may not appear in the adjoint state. Instead, the solution may have a $r^{4/3}$-singularity whose normal derivative has a $r^{1/3}$-singularity which is not flattened by the projection $\mathrm{Proj}_{[a,b]}$. The control is less regular. In this case $\bar{u} \in H^s(\Gamma)$ for all $s < \frac{5}{6}$; see [1, Example 3.6].

Hence, dealing with these exceptional cases is not fun but necessary. If in the unconstrained case a stress intensity factor vanishes , i.e., the leading singularity does not occur, then the convergence result is still true, one may only see a better convergence in numerical tests. See Figure 6, right hand side. However, in the constrained case, the situation is the opposite. The exceptional case leads to the worst-case estimate. To deal with the "worst-case" and the "generic-case" in an unified way, we introduce in (7) some numbers related to the singular exponents.

We distinguish two cases for the investigation of the discretization errors. After proving a general result in Section 3 we study the unconstrained case in Section 4
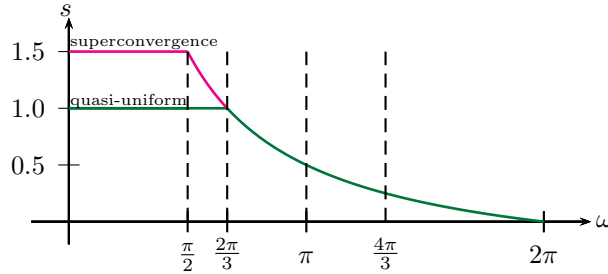
FIGURE 1. Convergence rates depending on the maximal interior angle in the unconstrained case

and the constrained case in Section 5. We focus on quasi-uniform meshes and distinguish general meshes and certain superconvergent meshes. In order not to overload the present paper, we postpone the study of graded meshes to a forthcoming paper. The numerical tests in Section 6 confirm the theoretical results.

The study of error estimates for Dirichlet control problems posed on polygonal domains can be traced back to [8], where a control constrained problem governed by a semilinear elliptic equation posed in a convex polygonal domain is studied. An order of convergence of h^s is proved for all $s < \min(1, \pi/(2\omega_1))$, where ω_1 is the largest interior angle, in both the control and the state variable. Later, in [18], it is proven that for unconstrained linear problems posed on convex domains, the state variable exhibits a better convergence property. The corresponding proof is based on a duality argument and estimates for the controls in weaker norms than $L^2(\Gamma)$. However, to the best of our knowledge, the argumentation is restricted to unconstrained problems. For the error of the controls in $L^2(\Gamma)$, the order shown in [8] is not improved.

Nevertheless, the regularity of the control and the existing numerical experiments, see [18, 17], suggested that for the control variable the order should be greater: h^s for all $s < \min(1, \pi/\omega_1 - 1/2)$ if one uses standard quasi-uniform meshes, and for all $s < \min(3/2, \pi/\omega_1 - 1/2)$ if one uses certain quasi-uniform meshes which allow for superconvergence effects, see Definition 4.5. Our main results, Theorems 4.1 and 5.3, fully explain the observed orders of convergence in the literature for the control variable, improve existing results for the state variable in constrained linear-quadratic problems posed in convex domains, and provide the first available results in nonconvex domains.

Let us summarize our results in detail. To be brief we introduce the maximal order $\mathcal{O} = 1$ for general meshes and $\mathcal{O} = 3/2$ for superconvergent meshes, $\lambda = \pi/\omega_1$ and $\Lambda = \min\{\lambda_j : \lambda_j > 1\}$. Note that we neglect terms with arbitrary small positive ε, and that Λ is defined slightly differently in (8). In the unconstrained case we prove in Theorem 4.1 convergence order $\min\{\mathcal{O}, \lambda - \frac{1}{2}\}$, see Figure 1 for a plot of the convergence order against the maximal angle of the domain. In the constrained case, we get the same result for convex domains, see Theorems 5.1 and the left part of Figure 2. For nonconvex domains we can prove order $\frac{1}{2}$ which is sharp when the maximal convex angle of the domain is close to π, see Theorem 5.5. A better result is obtained under the structural assumption 5.2 on the discrete control. In the worst case we get the order $\min\{\mathcal{O}, 2\lambda - \frac{1}{2}, \Lambda - \frac{1}{2}\}$ and in the generic case $\min\{\mathcal{O}, 2\lambda, \Lambda - \frac{1}{2}\}$, see Theorem 5.3 where a more involved definition of Λ is
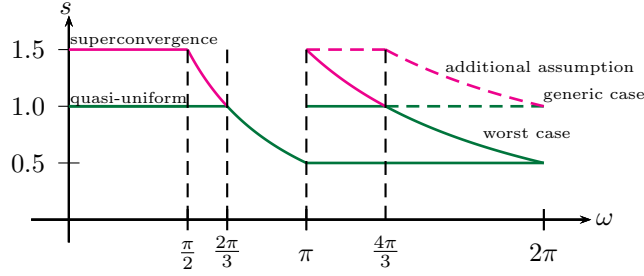
FIGURE 2. Convergence rates depending on the maximal interior angle in the constrained case

used. Figure 2 shows the order on the right hand side. To illustrate the dependence of the order on the nonconvex angle, the figure assumes that convex angles are small enough, such that these corners do not influence the convergence order.

2. **Notation and regularity results.** Let us denote by M the number of sides of Γ and $\{x_j\}_{j=1}^M$ its vertexes, ordered counterclockwise. For convenience denote also $x_0 = x_M$ and $x_{M+1} = x_1$. We will denote by Γ_j the side of Γ connecting x_j and x_{j+1}, and by $\omega_j \in (0, 2\pi)$ the angle interior to Ω at x_j, i.e., the angle defined by Γ_j and Γ_{j-1}, measured counterclockwise. Notice that $\Gamma_0 = \Gamma_M$. We will use (r_j, θ_j) as local polar coordinates at x_j, with $r_j = |x - x_j|$ and θ_j the angle defined by Γ_j and the segment $[x_j, x]$. In order to describe the regularity of the functions near the corners, we will introduce for every $j = 1, \ldots, M$ a positive number R_j and an infinitely differentiable cut-off function $\xi_j : \mathbb{R}^2 \to [0, 1]$ such that the sets

$$N_j = \{x \in \mathbb{R}^2 : 0 < r_j < 2R_j, \ 0 < \theta_j < \omega_j\},$$

satisfy $N_j \subset \Omega$ for all j and $N_i \cap N_j = \emptyset$ if $i \neq j$ and $\xi_j \equiv 1$ in the set $\{x \in \mathbb{R}^2 : r_j < R_j\}$, $\xi_j \equiv 0$ in the set $\{x \in \mathbb{R}^2 : r_j > 2R_j\}$.

For every $j = 1, \ldots, M$ we will call λ_j the *in general* leading singular exponent associated with the operator corresponding to the corner x_j. For the Laplace operator it is well known that $\lambda_j = \pi/\omega_j$. Since in general the regularity of the solution of a boundary value problem depends on the smallest singular exponent, it is customary to denote

$$\lambda = \min\{\lambda_j : \ j = 1, \ldots, M\} \text{ and } p_D = \frac{2}{1 - \min\{1, \lambda\}}. \quad (4)$$

Our main estimates are for data $y_\Omega \in W^{1, p^*}(\Omega)$ for some $p^* > 2$. To get these estimates it is key to use the sharp regularity results of the optimal control, state and adjoint state provided in [1]. For both the control and the state it is enough to know the Hilbert Sobolev-Slobodetskiĭ space they belong to, but for the adjoint state we will need to know with some more detail the development in terms of powers of the singular exponents. To write this development, we must proceed in two steps in order to be able to define the *effectively* leading singularity in each corner.

Our first result concerns the regularity of the adjoint state and is a consequence of [1, Theorem 3.2 and Theorem 5.1]; see also [14]. For $m \in \mathbb{Z}$, $t \in \mathbb{R}$ and $1 < p \le +\infty$ we define

$$\mathbb{J}_{t,p}^m = \left\{ j \in \{1, \ldots, M\} \text{ such that } 0 < m\lambda_j < 2 + t - \frac{2}{p} \text{ and } m\lambda_j \notin \mathbb{Z} \right\}. \quad (5)$$

Lemma 2.1. *Suppose* $y_\Omega \in L^\infty(\Omega)$. *Let* $\bar{\varphi} \in H_0^1(\Omega)$ *be the optimal adjoint state, solution of* (2c). *Then, there exist a unique function* $\bar{\varphi}_r \in W^{2,p}(\Omega)$ *and unique real numbers* $(c_{j,m})_{j \in \mathbb{J}_{0,p}^m}$, *for all* $p < +\infty$ *for constrained problems and* $p < p_D$ *for unconstrained problems, such that*

$$\bar{\varphi} = \bar{\varphi}_r + \sum_{m=1}^{3} \sum_{j \in \mathbb{J}_{0,p}^m} c_{j,m} \xi_j r_j^{m\lambda_j} \sin(m\lambda_j \theta_j). \tag{6}$$

Note that $p_D = +\infty$ in convex domains such that we obtain for constrained as well as for unconstrained problems the same regularity of the optimal adjoint state. However, in nonconvex domains, the control and hence the state, as part of the right hand side of the adjoint equation, may be unbounded in the unconstrained case, which leads to the restriction $p < p_D$ for the regularity of $\bar{\varphi}_r$. Moreover, it may happen that the effectively leading singularity corresponding to a corner x_j is not the first one. This means that the associated coefficient $c_{j,1}$ in the asymptotic representation (6) is equal to zero. However, this will be of interest only for constrained problems in case of nonconvex corners x_j, i.e., $\lambda_j < 1$. To be able to cover this, we define the numbers

$$\Lambda_j = \begin{cases} \lambda_j & \text{if } \lambda_j > 1 \text{ or } c_{j,1} \neq 0 \\ 2\lambda_j & \text{if } \lambda_j < 1 \text{ and } c_{j,1} = 0 \end{cases} \tag{7}$$

for each corner. In addition, we introduce

$$\Lambda = \min\{\Lambda_j : \Lambda_j > 1, \ j = 1, \ldots, M\}. \tag{8}$$

In convex domains, $\lambda = \Lambda$ will determine the regularity of both the optimal control and state. This holds for unconstrained as well as for constrained problems. However, in nonconvex domains, different cases may appear. If we have no control constraints then the regularity of the optimal control and state will again be determined by λ. If the problem is constrained then in the vicinity of any corner x_j, where the coefficient of the corresponding first singularity $c_{j,1}$ is unequal to zero, the optimal control is flattened there due to the projection formula and consequently smooth. This is the usual case. If $c_{j,1} = 0$ then the optimal control in the neighborhood of such a corner is at least as regular as the normal derivative of the corresponding second singular function. In the control constrained case, Λ will determine the regularity of the optimal control, at least in a worst case sense. The regularity of the optimal state may depend on λ as well since singular terms may occur within its asymptotic representation independent of the adjoint state.

For unconstrained problems the following regularity result holds, see [1, Corollary 5.3, Corollary 4.2, Theorem 3.4].

Lemma 2.2 (unconstrained case). *Suppose* $-a = b = \infty$ *and* $y_\Omega \in H^t(\Omega) \cap L^2(\Omega)$ *for all* $t < \min\{1, \lambda - 1\}$. *Then*

$$\bar{u} \in H^s(\Gamma), \quad \bar{y} \in H^{s+\frac{1}{2}}(\Omega) \quad \forall s < \min\{\tfrac{3}{2}, \lambda - \tfrac{1}{2}\}. \tag{9}$$

Remark 2.3. In general Lipschitz domains, there is not a satisfactory definition of $H^s(\Gamma)$ for $s > 1$. But for polygonal domains and $s < 3/2$, there is no problem in naming $H^s(\Gamma) = \text{tr}_\Gamma H^{s+1/2}(\Omega)$, the space of traces of $H^{s+1/2}(\Omega)$.

Notice that for $1/2 < s < 3/2$, $\text{tr}_\Gamma H^{s+1/2}(\Omega) = \{u \in \Pi_{i=1}^m H^s(\Gamma_i) : \ u \in C(\Gamma)\}$; cf. [14, Th. 1.5.2.8].

For constrained problems, we can improve this result, see [1, Corollary 4.2, Theorem 3.4].

Lemma 2.4 (control constrained case). *Suppose* $-\infty < a < b < \infty$, $y_\Omega \in H^t(\Omega) \cap L^2(\Omega)$ *for all* $t < \min\{1, \lambda - 1\}$. *Assume that the optimal control has a finite number of kink points. Then*

$$\bar{u} \in H^s(\Gamma) \qquad\qquad \forall s < \min\{\tfrac{3}{2}, \Lambda - \tfrac{1}{2}\}, \tag{10}$$

$$\bar{y} \in H^{s+\frac{1}{2}}(\Omega) \qquad\qquad \forall s < \min\{\tfrac{3}{2}, \Lambda - \tfrac{1}{2}, \lambda + \tfrac{1}{2}\}. \tag{11}$$

We also have the following result from [1, Proof of Theorem 3.4].

Lemma 2.5. *Suppose* $-\infty < a < b < \infty$ *and* $y_\Omega \in L^2(\Omega)$. *If* $\lambda_j < 1$ *and* $c_{j,1} \neq 0$, *then one of the control constraints is active near the corner* x_j, *i.e., there exists* $\rho_j > 0$ *such that for* $x \in \Gamma$ *with* $|x - x_j| < \rho_j$ *either* $\bar{u} \equiv a$ *or* $\bar{u} \equiv b$.

Finally, we can write the representation of the adjoint state for regular enough data. For $m \in \mathbb{Z}$, $t \in \mathbb{R}$ and $1 < p \leq +\infty$ we will also need

$$\mathbb{L}_{t,p}^m = \left\{ j \in \{1, \ldots, M\} \text{ such that } 0 < m\lambda_j < 2 + t - \frac{2}{p} \text{ and } m\lambda_j \in \mathbb{Z} \right\}. \tag{12}$$

The following result is a consequence of [1, Corollary 4.4]; see also [14].

Lemma 2.6. *Suppose that* Ω *is convex or* $-\infty < a < b < +\infty$, *and that* $y_\Omega \in W^{1,p^*}(\Omega)$ *with* $p^* > 2$. *Then, for* $p > 2$ *such that*

$$\frac{3p - 2}{\lambda_j p} \notin \mathbb{Z} \ \ \text{for all } j \in \{1, \ldots, M\}$$

and

$$p \leq p^*, \ p < p_D, \ p < \frac{2}{2 - \min\{\lambda, 2\}}$$

there exist a unique function $\bar{\varphi}_\mathrm{r} \in W^{3,p}(\Omega)$ *and unique real numbers* $(c_{j,m})_{\mathbb{J}_{1,p}^m}$ *and* $(d_{j,m})_{\mathbb{L}_{1,p}^m}$, *such that*

$$\bar{\varphi} = \bar{\varphi}_\mathrm{r} + \sum_{m=1}^{5} \sum_{j \in \mathbb{J}_{1,p}^m} c_{j,m} \xi_j r_j^{m\lambda_j} \sin(m\lambda_j \theta_j)$$

$$+ \sum_{m=1,3} \sum_{j \in \mathbb{L}_{1,p}^m} d_{j,m} \xi_j r_j^2 \left(\log(r_j) \sin(2\theta_j) + \theta_j \cos(2\theta_j) \right).$$

Notice that the coefficients $c_{j,m}$ that appear in both expansions in Lemmata 2.1 and 2.6 coincide, due to the uniqueness of the expansion. In the expansion of Lemma 2.6 new terms appear that belong to $W^{2,p}(\Omega)$ for all $p < +\infty$ but not to $W^{3,p}(\Omega)$ for $p > 2$ satisfying the conditions in Lemma 2.6.

3. **A general discretization error estimate.** In this section we will present a general discretization error estimate in Theorem 3.2. The terms in this general estimate have to be estimated in particular cases. This work will be done in later sections.

For the discretization, consider a family of regular triangulations $\{\mathcal{T}_h\}$ depending on a mesh parameter h in the sense of Ciarlet [10]. Notice, that a triangulation $\mathcal{E}_h$ of the boundary is naturally induced by $\mathcal{T}_h$. We assume that the space Y_h is the space of conforming piecewise linear finite elements. The space U_h is the space of

piecewise linear functions generated by the trace of elements of Y_h on the boundary Γ. We denote the subspace of Y_h with vanishing boundary values by $Y_{0,h}$. In the following, we assume that the mesh parameter h fulfills $h < h_0$ with some $h_0 < 1$ small enough and we denote c a generic constant that may depend on the data of the problem but is independent of h.

We also introduce the discrete solution operator $S_h : L^2(\Gamma) \to Y_h$. For $u \in L^2(\Gamma)$ the function $S_h u \in Y_h$ is defined as the unique solution of

$$(\nabla S_h u, \nabla z_h)_{L^2(\Omega)} = 0 \quad \forall z_h \in Y_{0,h} \text{ and } (S_h u - u, v_h)_{L^2(\Gamma)} = 0 \ \forall v_h \in U_h. \quad (13)$$

We emphasize that on the boundary $S_h u$ coincides with the L^2-projection of u on U_h. Thus we get $S_h u_h = u_h$ on Γ for $u_h \in U_h$. Notice as well that (13) is not a conforming discretization of the very weak formulation of the state equation. However, according to [2, 7], its applicability is guaranteed.

In our discretized optimal control problem we aim to minimize the objective function

$$(P_h) \begin{cases} \min J_h(u_h) = \dfrac{1}{2}\|S_h u_h - y_\Omega\|_{L^2(\Omega)}^2 + \dfrac{\nu}{2}\|u_h\|_{L^2(\Gamma)}^2 \\ \text{subject to } u_h \in U_{ad}^h := \{u_h \in U_h : a \le u_h(x) \le b \quad \text{for all } x \in \Gamma\}. \end{cases}$$

The first order optimality conditions of this problem were derived in [8] and are stated in the next lemma.

Lemma 3.1. *Problem (P_h) has a unique solution $\bar{u}_h \in U_{ad}^h$, with related discrete state $\bar{y}_h = S_h \bar{u}_h \in Y_h$ and adjoint state $\bar{\varphi}_h \in Y_{0,h}$. The following discrete optimality system is satisfied*

$$(\nu \bar{u}_h - \partial_n^h \bar{\varphi}_h, u_h - \bar{u}_h)_{L^2(\Gamma)} \ge 0 \text{ for all } u_h \in U_{ad}^h, \tag{14a}$$

$$(\nabla \bar{y}_h, \nabla z_h)_{L^2(\Omega)} = 0 \text{ for all } z_h \in Y_{0,h} \text{ and } \bar{y}_h|_\Gamma = \bar{u}_h, \tag{14b}$$

$$(\nabla \bar{\varphi}_h, \nabla z_h)_{L^2(\Omega)} = (\bar{y}_h - y_\Omega, z_h)_{L^2(\Omega)} \text{ for all } z_h \in Y_{0,h}, \tag{14c}$$

where the discrete normal derivative $\partial_n^h \bar{\varphi}_h \in U_h$ is defined as the unique solution of

$$(\partial_n^h \bar{\varphi}_h, z_h)_{L^2(\Gamma)} = -(\bar{y}_h - y_\Omega, z_h)_{L^2(\Omega)} + (\nabla \bar{\varphi}_h, \nabla z_h)_{L^2(\Omega)} \text{ for all } z_h \in Y_h. \quad (15)$$

An important tool in the numerical analysis is the construction of a discrete control $u_h^* \in U_{ad}^h$ which interpolates $\bar{u}$ in a certain sense, see Lemma 4.2 and Lemma 5.6, and satisfies

$$(\nu \bar{u} - \partial_n \bar{\varphi}, u_h^* - \bar{u})_{L^2(\Gamma)} = 0. \tag{16}$$

If the optimal control $\bar{u} \in H^s(\Gamma)$ with $s < 1$ then we use a quasi-interpolant introduced by Casas and Raymond in [8]: Denote the boundary nodes of the mesh by x_Γ^j, $1 \le j \le N(h)$, and let e_j, $1 \le j \le N(h)$, be the nodal basis of U_h. We set

$$\bar{d}(x) = \nu \bar{u}(x) - \partial_n \bar{\varphi}(x),$$

$$I_j = \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar{d}(x) e_j(x)\, dx,$$

and define a control $u_h^* = \sum_{j=1}^{N(h)} u_{h,j}^* e_j$ by its coefficients

$$
u_{h,j}^* = \begin{cases} \dfrac{1}{I_j} \displaystyle\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar{d}(x)\bar{u}(x)e_j(x)\,d\sigma(x) & \text{if } I_j \neq 0, \\[4mm] \dfrac{1}{h_{j-1}+h_j} \displaystyle\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar{u}(x)\,d\sigma(x) & \text{if } I_j = 0. \end{cases} \tag{17}
$$

According to [8, Lemma 7.5] the function u_h^* belongs to U_{ad}^h. Moreover, it is constructed such that $u_h^* = \bar{u}$ on the active set, and it fulfills (16).

If $\bar{u} \in H^s(\Gamma)$ with $s \geq 1$, we use a modification of the standard Lagrange interpolant $I_h\bar{u}$ of $\bar{u}$, again denoted by $u_h^* \in U_{ad}^h$, which is defined by its coefficients as follows

$$
u_{h,j}^* = \begin{cases} a & \text{if } \min_{[x_\Gamma^{j-1},x_\Gamma^{j+1}]} \bar{u}(x) = a, \\ b & \text{if } \max_{[x_\Gamma^{j-1},x_\Gamma^{j+1}]} \bar{u}(x) = b, \\ \bar{u}(x_\Gamma^j) & \text{otherwise}, \end{cases} \tag{18}
$$

cf. [9, Section 2]. Of course, if we consider control problems without control constraints, that is $-a = b = \infty$, the interpolant u_h^* is just the Lagrange interpolant. In case of control bounds $a, b \in \mathbb{R}$, in order to get a unique definition of u_h^*, we need to assume that on each element only one control bound is active. However, due to the Hölder continuity of $\bar{u}$, which we have for $\bar{u} \in H^s(\Gamma)$ with $s \geq 1$, there exists a mesh size $h_0 > 0$ such that for all $h < h_0$ the above definition of the interpolant is unique. Obviously, this interpolant belongs to U_{ad}^h. Moreover, it satisfies (16) by construction. Indeed, whenever $\nu\bar{u}(x) - \partial_n\bar{\varphi}(x) \neq 0$, we have $u_h^*(x) - \bar{u}(x) = 0$.

As already announced, we conclude this section by stating a general error estimate for the control and state errors which will serve as a basis for the subsequent error analysis.

Theorem 3.2. *For the solution of the continuous and the discrete optimal control problem we have*

$$
\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}
$$
$$
\leq c\left(\|\bar{u} - u_h^*\|_{L^2(\Gamma)} + \|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} + \sup_{\psi_h \in U_h} \frac{\left|(\nabla\bar{\varphi}, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} \right). \tag{19}
$$

Proof. First, let us define the intermediate error $e_h := u_h^* - \bar{u}_h$. Then, we obtain

$$
\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq \|\bar{u} - u_h^*\|_{L^2(\Gamma)} + \|e_h\|_{L^2(\Gamma)}
$$
$$
+ \|\bar{y} - S_h u_h^*\|_{L^2(\Omega)} + \|S_h e_h\|_{L^2(\Omega)}. \tag{20}
$$

To deal with the third term, we take into account the continuity of S_h:

$$
\|\bar{y} - S_h u_h^*\|_{L^2(\Omega)} \leq \|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} + \|S_h(\bar{u} - u_h^*)\|_{L^2(\Omega)}
$$
$$
\leq \|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} + c\|\bar{u} - u_h^*\|_{L^2(\Gamma)}, \tag{21}
$$

cf. [2, Lemma 2.3 and Corollary 3.3]. Accordingly, we only need estimates for the second and fourth term in (20). We begin with estimating the second one, but as we will see this also yields an estimate for the fourth term. There holds

$$
\nu\|e_h\|_{L^2(\Gamma)}^2 = \nu(u_h^* - \bar{u}, e_h)_{L^2(\Gamma)} + \nu(\bar{u} - \bar{u}_h, e_h)_{L^2(\Gamma)}. \tag{22}
$$

Next, we consider the second term of (22) in detail. By adding the continuous and discrete variational inequalities (2a) and (14a) with $u = \bar{u}_h \in U_{ad}$ and $u_h = u_h^* \in U_{ad}^h$, respectively, we deduce

$$(\nu(\bar{u}_h - \bar{u}) + \partial_n \bar{\varphi} - \partial_n^h \bar{\varphi}_h, e_h)_{L^2(\Gamma)} + (\nu\bar{u} - \partial_n\bar{\varphi}, u_h^* - \bar{u})_{L^2(\Gamma)} \geq 0.$$

Rearranging terms and using (16) leads to

$$\nu(\bar{u} - \bar{u}_h, e_h)_{L^2(\Gamma)} \leq (\partial_n\bar{\varphi} - \partial_n^h\bar{\varphi}_h, e_h)_{L^2(\Gamma)}.$$

Integration by parts (cf. [11, Lemma 3.4]) using $e_h = S_h e_h$ on Γ, (15), (2c) and (13) yield

$$
\begin{aligned}
\nu(\bar{u} - \bar{u}_h, e_h)_{L^2(\Gamma)} &\leq (\Delta\bar{\varphi} + (\bar{y}_h - y_\Omega), S_h e_h)_{L^2(\Omega)} + (\nabla(\bar{\varphi} - \bar{\varphi}_h), \nabla S_h e_h)_{L^2(\Omega)} \\
&= (\bar{y}_h - \bar{y}, S_h e_h)_{L^2(\Omega)} + (\nabla\bar{\varphi}, \nabla S_h e_h)_{L^2(\Omega)} \\
&= (S_h u_h^* - \bar{y}, S_h e_h)_{L^2(\Omega)} - \|S_h e_h\|_{L^2(\Omega)}^2 + (\nabla\bar{\varphi}, \nabla S_h e_h)_{L^2(\Omega)}.
\end{aligned}
\tag{23}
$$

By collecting the estimates (22) and (23) we obtain

$$
\begin{aligned}
\nu\|e_h\|_{L^2(\Gamma)}^2 &+ \|S_h e_h\|_{L^2(\Omega)}^2 \\
&\leq \nu(u_h^* - \bar{u}, e_h)_{L^2(\Gamma)} + (S_h u_h^* - \bar{y}, S_h e_h)_{L^2(\Omega)} + (\nabla\bar{\varphi}, \nabla S_h e_h)_{L^2(\Omega)} \\
&\leq \nu\|u_h^* - \bar{u}\|_{L^2(\Gamma)}\|e_h\|_{L^2(\Gamma)} + \|S_h u_h^* - \bar{y}\|_{L^2(\Omega)}\|S_h e_h\|_{L^2(\Omega)} \\
&\quad + \sup_{\psi_h \in U_h} \frac{\left|(\nabla\bar{\varphi}, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} \|e_h\|_{L^2(\Gamma)}.
\end{aligned}
\tag{24}
$$

From the Young inequality we can deduce

$$
\begin{aligned}
\|e_h\|_{L^2(\Gamma)} &+ \|S_h e_h\|_{L^2(\Omega)} \\
&\leq c\left(\|u_h^* - \bar{u}\|_{L^2(\Gamma)} + \|S_h u_h^* - \bar{y}\|_{L^2(\Omega)} + \sup_{\psi_h \in U_h} \frac{\left|(\nabla\bar{\varphi}, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}}\right).
\end{aligned}
\tag{25}
$$

Finally, the assertion is a consequence from (20), (21) and (25). □

4. **Problems without control constraints.** In the rest of the paper, we will always assume that $\{\mathcal{T}_h\}$ is a quasi-uniform family of meshes. However, if the underlying mesh has a certain structure then it is possible to improve the error estimates. These special quasi-uniform meshes are called superconvergent meshes or $O(h^2)$-irregular meshes; for the precise definition we refer to Definition 4.5. The main result of this section is the following one.

Theorem 4.1. *Suppose that either $\lambda < 1$ and $y_\Omega \in L^2(\Omega)$, or $y_\Omega \in W^{1,p^*}(\Omega)$ for some $p^* > 2$. Then it holds*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^s|\log h|^r$$
$$\forall s \in \mathbb{R} \text{ such that } s < \lambda - \tfrac{1}{2} \text{ and } s \leq 1, \tag{26}$$

where r is equal to one for $\lambda - \frac{1}{2} \in (1, \frac{3}{2}]$ and equal to zero otherwise. If, further, $\{\mathcal{T}_h\}$ is $O(h^2)$-irregular according to Definition 4.5, then

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^s \quad \forall s < \min\{\tfrac{3}{2}, \lambda - \tfrac{1}{2}\}. \tag{27}$$

For the proof, we are going to estimate the three terms that appear in the general estimate of Theorem 3.2. Whereas the first two terms in (19) can be estimated by standard techniques, the third one needs special care. Analogously to the derivation of (23), this term can formally be rewritten as

$$\sup_{\psi_h \in U_h} \frac{\left|(\nabla\bar\varphi, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} = \sup_{\psi_h \in U_h} \frac{\left|(\nabla(\bar\varphi - R_h\bar\varphi), \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}}$$
$$= \sup_{\psi_h \in U_h} \frac{\left|(\partial_n\bar\varphi - \partial_n^h R_h\bar\varphi, \psi_h)_{L^2(\Gamma)}\right|}{\|\psi_h\|_{L^2(\Gamma)}},$$

where $\partial_n^h R_h\bar\varphi$ is defined as in (15) just by replacing $\bar y_h$ with $\bar y$ and $\bar\varphi_h$ with the Ritz-projection $R_h\bar\varphi$ of $\bar\varphi$ on $Y_{0,h}$. Thus, we are interested in the error between the normal derivative of the adjoint state and the corresponding discrete normal derivative of its Ritz-projection. In order to estimate the above term, we will pursue two different strategies. The first one relies on local and global $W^{1,\infty}$-discretization error estimates. In case of general quasi-uniform meshes, this will result in a convergence order of $O(h^s|\log h|^r)$ for all $s \in \mathbb{R}$ such that $s < \lambda - \frac{1}{2}$ and $s \leq 1$, where r is equal to one for $\lambda - \frac{1}{2} \in (1, \frac{3}{2}]$ and equal to zero otherwise. The second strategy will rely on special superconvergent meshes as introduced in [5]. The idea to use such meshes in the context of Dirichlet boundary control problems originally stems from [12]. In contrast to the setting in that reference, we are not concerned with smoothly bounded domains but with polygonal domains. For that reason we need to extend the corresponding estimates to that case, that is, we have to deal with less regular functions due to the appearance of corner singularities. This will yield an approximation rate of $O(h^s)$ with $s < \min\{\frac{3}{2}, \lambda - \frac{1}{2}\}$, which results in an improvement in comparison with meshes not satisfying the superconvergence property, for domains with interior angles less than $2\pi/3$.

Lemma 4.2. *Suppose $y_\Omega \in H^t(\Omega) \cap L^2(\Omega)$ for all $t < \min\{1, \lambda - 1\}$. Then we have*

$$\|\bar u - u_h^*\|_{L^2(\Gamma)} + \|\bar y - S_h\bar u\|_{L^2(\Omega)} \leq ch^s \quad \forall s < \min\{\tfrac{3}{2}, \lambda - \tfrac{1}{2}\}.$$

Proof. We know from Lemma 2.2 that the control satisfies $\bar u \in H^s(\Gamma)$ for all $s < \min\{\frac{3}{2}, \lambda - \frac{1}{2}\}$. If $s < 1$, we choose u_h^* as defined in (17), and the estimate for the control follows from [8, Eq. (7.10)] by setting $s = 1 - \frac{1}{p}$ with $p \in (1, \infty)$. If $1 \leq s < \frac{3}{2}$, we have $\bar u \in H^s(\Gamma) \hookrightarrow C^{0,s-\frac{1}{2}}(\Gamma)$ due to the Sobolev embedding theorem. Thus, the modified Lagrange interpolant u_h^* from (18) is well-defined. Actually, in the present case, u_h^* is just the Lagrange interpolant. As a consequence, the error estimate for the control is given by a standard estimate for the Lagrange interpolant.

Again from Lemma 2.2, the optimal state satisfies $\bar y \in H^{s+\frac{1}{2}}(\Omega)$, for all $s < \min\{\frac{3}{2}, \lambda - \frac{1}{2}\}$. Thus, $\|\bar y - S_h\bar u\|_{H^1(\Omega)} \leq ch^{s-\frac{1}{2}}$ for all $s < \min\{\frac{3}{2}, \lambda - \frac{1}{2}\}$ if $\lambda \geq 1$. By the Aubin–Nitsche method we obtain

$$\|\bar y - S_h\bar u\|_{L^2(\Omega)} \leq ch^{s-\frac{1}{2}+\min\{1, s+\frac{1}{2}\}} \quad \forall s < \min\{\tfrac{3}{2}, \lambda - \tfrac{1}{2}\}, \tag{28}$$

cf. for instance [6]. Since $s + \frac{1}{2}$ can be chosen greater than $\frac{1}{2}$, we have the desired result in case that $\lambda \geq 1$. For $\lambda < 1$ we do not have $\bar y \in H^1(\Omega)$ such that standard techniques for estimating finite element errors fail. However, in this case we can directly refer to Remark 5.4 of [2]. □

Lemma 4.3. *Suppose that either $\lambda < 1$ and $y_\Omega \in L^2(\Omega)$, or $y_\Omega \in W^{1,p^*}(\Omega)$ for some $p^* > 2$. Then there is the estimate*

$$\sup_{\psi_h \in U_h} \frac{\left|(\nabla\bar\varphi, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} \le ch^s |\log h|^r \quad \forall s \in \mathbb{R} \text{ such that } s < \lambda - \tfrac{1}{2} \text{ and } s \le 1,$$

where r is equal to one for $\lambda - \tfrac{1}{2} \in (1, \tfrac{3}{2}]$ and equal to zero otherwise.

Proof. As above, we denote by R_h the operator that maps a function of $H_0^1(\Omega)$ to its Ritz-projection in $Y_{0,h}$. In addition, we introduce the extension operator $\tilde{S}_h$ which extends a function belonging to U_h to one in Y_h by zero. Using the norm equivalence in finite dimensional spaces on a reference domain we easily infer for any $\psi_h \in U_h$ and $q \in [1, \infty]$

$$\|\tilde{S}_h\psi_h\|_{L^q(\Omega)} + h\|\nabla\tilde{S}_h\psi_h\|_{L^q(\Omega)} \le ch^{1/q}\|\psi_h\|_{L^q(\Gamma)}. \tag{29}$$

Since $S_h\psi_h$ is discrete harmonic, we obtain together with the orthogonality properties of the Ritz-projection the identity

$$(\nabla\bar\varphi, \nabla S_h\psi_h)_{L^2(\Omega)} = (\nabla(\bar\varphi - R_h\bar\varphi), \nabla S_h\psi_h)_{L^2(\Omega)} = (\nabla(\bar\varphi - R_h\bar\varphi), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega)}, \tag{30}$$

where we employed that $(S_h - \tilde{S}_h)\psi_h$ belongs to $Y_{0,h}$.

Now, we distinguish the three cases $\omega_i < \pi/2$, $\omega_i < \pi$ and $\omega_i < 2\pi$ for $i = 1, \dots, M$.

In the first one, we know from Lemma 2.6 that the optimal adjoint state belongs to $W^{3,q}(\Omega)$ (for some $q > 2$), which is continuously embedded in $W^{2,\infty}(\Omega)$. Consequently, a global $W^{1,\infty}$-discretization error estimate from, e.g., [23, 13], and (29) yield

$$(\nabla(\bar\varphi - R_h\bar\varphi), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega)} \le \|\nabla(\bar\varphi - R_h\bar\varphi)\|_{L^\infty(\Omega)}\|\nabla\tilde{S}_h\psi_h\|_{L^1(\Omega)} \le ch\|\psi_h\|_{L^1(\Gamma)}, \tag{31}$$

which represents, together with (30) and the embedding $L^2(\Gamma) \hookrightarrow L^1(\Gamma)$, the desired result for $\omega_i < \pi/2$, $i = 1, \dots, M$.

Next, we consider the case $\omega_i < \pi$ for $i = 1, \dots, M$. For simplicity, we assume that the domain has only one corner with an interior angle greater or equal to $\pi/2$. However, the proof extends to the general case in a natural way. In the following, that corner is located at the origin. Furthermore, we denote its interior angle by ω_1, the distance to that corner by r_1, and the corresponding leading singular exponent by $\lambda_1 = \pi/\omega_1$. According to Lemma 2.6, the optimal adjoint state admits the splitting

$$\bar\varphi = \bar\varphi_{\mathrm{r}} + \bar\varphi_{\mathrm{s}}, \tag{32}$$

where $\bar\varphi_{\mathrm{r}}$ belongs to $W^{3,q}(\Omega)$ with some $q > 2$. Combining (30) and (32) yields the identity

$$(\nabla(\bar\varphi - R_h\bar\varphi), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega)} = (\nabla(\bar\varphi_{\mathrm{s}} - R_h\bar\varphi_{\mathrm{s}}), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega)}$$
$$+ (\nabla(\bar\varphi_{\mathrm{r}} - R_h\bar\varphi_{\mathrm{r}}), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega)}. \tag{33}$$

For the latter term, we can argue as in (31) to show first order convergence, i.e.,

$$(\nabla(\bar\varphi_{\mathrm{r}} - R_h\bar\varphi_{\mathrm{r}}), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega)} \le ch\|\psi_h\|_{L^1(\Gamma)} \le ch\|\psi_h\|_{L^2(\Gamma)}. \tag{34}$$

In order to estimate the singular term, we decompose the neighborhood of the critical corner in subdomains Ω_J which are defined by

$$\Omega_I := \{x : |x| \le d_I\} \quad \text{and} \quad \Omega_J := \{x : d_{J+1} \le |x| \le d_J\} \quad \text{for } J = I - 1, \dots, 1.$$

We set the radii d_J equal to 2^{-J} and choose the index I in such a way that $d_I = 2^{-I} = c_I h$ with a constant c_I. Below, this constant is chosen large enough such that on the one hand local $W^{1,\infty}$-finite element error estimates from [13, Corollary 1] are applicable on the strips Ω_J, see (37), and on the other hand the validity of the weighted error estimate (40) is guaranteed. Moreover, we set

$$\Omega_0 := \Omega\backslash\Omega_R \quad \text{with} \quad \Omega_R := \bigcup_{J=1}^{I}\Omega_J$$

and

$$\Omega'_J := \Omega_{J-1} \cup \Omega_J \cup \Omega_{J+1}$$

with the obvious modifications for $J = 0$ and $J = I$. Using this kind of covering, we obtain

$$(\nabla(\bar\varphi_{\rm s} - R_h\bar\varphi_{\rm s}),\nabla\tilde S_h\psi_h)_{L^2(\Omega)} = \sum_{J=0}^{I}(\nabla(\bar\varphi_{\rm s} - R_h\bar\varphi_{\rm s}),\nabla\tilde S_h\psi_h)_{L^2(\Omega_J)}$$

$$\leq \sum_{J=0}^{I}\|\nabla(\bar\varphi_{\rm s} - R_h\bar\varphi_{\rm s})\|_{L^\infty(\Omega_J)}\|\nabla\tilde S_h\psi_h\|_{L^1(\Omega_J)}. \tag{35}$$

Arguing as in (29), we get

$$\|\nabla\tilde S_h\psi_h\|_{L^1(\Omega_J)} \leq c\|\psi_h\|_{L^1(\partial\Omega'_J\cap\Gamma)}. \tag{36}$$

Having chosen the constant c_I large enough, local $W^{1,\infty}$-error estimates from [13, Corollary 1] yield

$$\|\nabla(\bar\varphi_{\rm s} - R_h\bar\varphi_{\rm s})\|_{L^\infty(\Omega_J)}$$
$$\leq c\left(\|\nabla(\bar\varphi_{\rm s} - I_h\bar\varphi_{\rm s})\|_{L^\infty(\Omega'_J)} + d_J^{-2}\|\bar\varphi_{\rm s} - R_h\bar\varphi_{\rm s}\|_{L^2(\Omega'_J)}\right), \tag{37}$$

where $I_h\bar\varphi$ denotes the Lagrange interpolant of $\bar\varphi$. Notice, according to [13, Remark 2], this inequality is only valid for any $J = 0,\ldots,I-2$ if the domain Ω is nonconvex, i.e., $\omega_1 > \pi$. Now, let $\sigma := r_1 + d_I$, which possesses the properties $\sigma \sim d_J$ for $x \in \bar\Omega'_J$ and $\min_{x\in\Omega}\sigma \sim h$. By combining (35)–(37), we infer

$$(\nabla(\bar\varphi_{\rm s} - R_h\bar\varphi_{\rm s}),\nabla\tilde S_h\psi_h)_{L^2(\Omega)}$$

$$\leq c\sum_{J=0}^{I}\left(\|\nabla(\bar\varphi_{\rm s} - I_h\bar\varphi_{\rm s})\|_{L^\infty(\Omega'_J)} + d_J^{-2}\|\bar\varphi_{\rm s} - R_h\bar\varphi_{\rm s}\|_{L^2(\Omega'_J)}\right)\|\psi_h\|_{L^1(\partial\Omega'_J\cap\Gamma)}$$

$$\leq c\left(\|\sigma^{1/2}\nabla(\bar\varphi_{\rm s} - I_h\bar\varphi_{\rm s})\|_{L^\infty(\Omega)} + h^{-1}\|\sigma^{-1/2}(\bar\varphi_{\rm s} - R_h\bar\varphi_{\rm s})\|_{L^2(\Omega)}\right)\|\sigma^{-1/2}\psi_h\|_{L^1(\Gamma)}. \tag{38}$$

The second derivatives of the singular part $\bar\varphi_{\rm s}$ behave like $r^{\lambda-2}$ for $\lambda \neq 2$ and like $\log r$ if $\lambda = 2$, cf. Lemma 2.6. Thus, by using standard interpolation error estimates (on the strips Ω_J), we get for $\lambda_1 > 1$, hence for $\omega_1 < \pi$,

$$\|\sigma^{1/2}\nabla(\bar\varphi_{\rm s} - I_h\bar\varphi_{\rm s})\|_{L^\infty(\Omega)} \leq ch^{\min\{1,\lambda-1/2\}}. \tag{39}$$

From [22, Corollary 3.62] (setting $\tau = \frac{1}{2}$ and $\gamma = 2 - \lambda$ there) we know that for c_I large enough there holds

$$\|\sigma^{-1/2}(\bar\varphi_{\rm s} - R_h\bar\varphi_{\rm s})\|_{L^2(\Omega)} \leq ch^{\min\{2,\lambda+1/2\}}|\log h|^{1/2}. \tag{40}$$

Notice that in that reference problems with Neumann boundary conditions are considered. However, the proof for the present problem is just a word by word repetition. Next, the Cauchy-Schwarz inequality and basic integration yield

$$\|\sigma^{-1/2}\psi_h\|_{L^1(\Gamma)} \le \|\sigma^{-1/2}\|_{L^2(\Gamma)}\|\psi_h\|_{L^2(\Gamma)} \le c|\log h|^{1/2}\|\psi_h\|_{L^2(\Gamma)}. \qquad (41)$$

By collecting the results from (38)–(41), we obtain

$$(\nabla(\bar\varphi_{\mathrm{s}} - R_h\bar\varphi_{\mathrm{s}}), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega)} \le ch^{\min\{1,\lambda-1/2\}}|\log h|\,\|\psi_h\|_{L^2(\Gamma)}, \qquad (42)$$

which yields together with (34), (33) and (30) the assertion in the second case.

Finally, we consider the case $\omega_i < 2\pi$ for $i = 1, \ldots, M$. Similar to the foregoing considerations, we assume that only the angle ω_1 is greater or equal to π and hence $1/2 < \lambda_1 \le 1$. According to (30), the Cauchy-Schwarz inequality, (29), and a standard finite element error estimate, we obtain

$$\begin{aligned}
(\nabla\bar\varphi, \nabla S_h\psi_h)_{L^2(\Omega)} &= (\nabla(\bar\varphi - R_h\bar\varphi), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega)} \\
&\le \|\nabla(\bar\varphi - R_h\bar\varphi)\|_{L^2(\Omega)}\|\nabla\tilde{S}_h\psi_h\|_{L^2(\Omega)} \\
&\le ch^s\|\psi_h\|_{L^2(\Gamma)}
\end{aligned}$$

for all $s < \lambda_1 - 1/2$. This ends the proof. $\qquad\square$

Remark 4.4. Related results to those of Lemma 4.3, which are established by using similar techniques, can be found in [3, 15, 19, 22].

According to the previous lemma, the critical term in the general estimate (19) converges with an order close to one provided that the interior angles are less $2\pi/3$. However, it is possible to improve the convergence rate if we assume a certain structure of the underlying mesh. The following definition for superconvergent meshes can be found in [5]. Those have been used in [12] in the context of Dirichlet boundary control problems in the case of smoothly bounded domains.

Definition 4.5. The triangulation $\mathcal{T}_h$ is called to be $O(h^{2\sigma})$-irregular if the following conditions hold:

1. The set of interior edges $\mathcal{E}$ of the triangulation $\mathcal{T}_h$ is decomposed into two disjoint sets $\mathcal{E}_1$ and $\mathcal{E}_2$ which fulfill the following properties:
   - For each $e \in \mathcal{E}_1$, let T and T' denote the two elements of the triangulation $\mathcal{T}_h$ that share this edge e. Then the lengths of any two opposite edges of the quadrilateral $T \cup T'$ differ only by $O(h^2)$.
   - $\sum_{e \in \mathcal{E}_2}(|T| + |T'|) = O(h^{2\sigma})$.
2. The set of the boundary vertexes $\mathcal{P}$ is decomposed into two disjoint sets $\mathcal{P}_1$ and $\mathcal{P}_2$ which satisfy the following properties:
   - For each vertex $x \in \mathcal{P}_1$, let e and e' be the two boundary edges sharing this vertex as an endpoint. Denote by T and T' the elements having e and e', respectively, as edges and let t and t' be the corresponding unit tangents. Furthermore, take e and e' as one pair of corresponding edges, and make a clockwise traversal of ∂T and $\partial T'$ to define two additional corresponding edge pairs. Then $|t - t'| = O(h)$ and the lengths of any two corresponding edges only differ by $O(h^2)$.
   - $|\mathcal{P}_2| = c$ with a constant c independent of h.

Next, let us recall a result from [12, Lemma 5.2], which leads us to Lemma 4.7.

Lemma 4.6. *Let Ω_h be any polygonal domain with boundary Γ_h. Suppose that the triangulation $\mathcal{T}_h$ of Ω_h is $O(h^{2\sigma})$ irregular and let $f \in W^{3,q}(\Omega_h)$ for some $q > 2$. Then for any $\phi_h \in Y_h$ there holds*

$$\left| \int_{\Omega_h} \nabla(f - I_h f) \cdot \nabla \phi_h \right| \leq c\|f\|_{W^{3,q}(\Omega_h)} \left(h^{1+\min\{1,\sigma\}} \|\phi_h\|_{H^1(\Omega_h)} + h^{3/2} \|\phi_h\|_{L^2(\Gamma_h)} \right),$$

where $I_h f \in Y_h$ denotes the piecewise linear Lagrange interpolant.

Lemma 4.7. *Suppose that either $\lambda < 1$ and $y_\Omega \in L^2(\Omega)$, or $y_\Omega \in W^{1,p^*}(\Omega)$ for some $p^* > 2$. Suppose further that $\{\mathcal{T}_h\}$ is a family of $O(h^2)$-irregular meshes. Then it holds*

$$\sup_{\psi_h \in U_h} \frac{\left| (\nabla \bar\varphi, \nabla S_h \psi_h)_{L^2(\Omega)} \right|}{\|\psi_h\|_{L^2(\Gamma)}} \leq ch^s \quad \forall s \in \mathbb{R} \text{ such that } s < \lambda - \frac{1}{2} \text{ and } s \leq \frac{3}{2}.$$

Proof. First we observe that

$$\left| (\nabla \bar\varphi, \nabla S_h \psi_h)_{L^2(\Omega)} \right| = \left| (\nabla(\bar\varphi - I_h \bar\varphi), \nabla S_h \psi_h)_{L^2(\Omega)} \right|$$

since S_h represents the discrete harmonic extension operator and $\bar\varphi$ has zero boundary conditions. If at least one interior angle ω_i is greater or equal to $2\pi/3$, we have $\lambda \leq 3/2$ and therefore $\lambda - 1/2 \leq 1$. Consequently, there is no advantage in taking a superconvergent mesh and we can apply the result for quasi-uniform meshes. If $\omega_i < \pi/2$ for $i = 1, \ldots, M$, and hence $\lambda > 2$, we can directly apply the results of Lemma 4.6 since $\bar\varphi \in W^{3,q}(\Omega)$ for some $q > 2$ according to Lemma 2.6. For these reasons, we focus in the following only on the case $3/2 < \lambda \leq 2$. We are in this case if the largest interior angle, denoted by ω_1 in the following, fulfills $\pi/2 \leq \omega_1 < 2\pi/3$. For simplicity, we assume as in the proof of Lemma 4.3 that the remaining angles are less than $\pi/2$. However, the proof again extends to the general case in a natural way. According to Lemma 2.6 we have that

$$\bar\varphi = \bar\varphi_{\mathrm{s}} + \bar\varphi_{\mathrm{r}},$$

where $\bar\varphi_{\mathrm{r}}$ belongs to $W^{3,q}(\Omega)$ with some $q > 2$. For the regular part we can again employ Lemma 4.6 to obtain the order $3/2$. The singular part behaves at worst like r_1^λ or like $r_1^2 |\log r_1|$, respectively, if $\lambda = 2$. As before, we would like to use Lemma 4.6 to get the corresponding estimate. For that purpose, we decompose the domain into two disjoint subsets $\Omega_{h,1}$ and $\Omega_{h,2}$. The set $\Omega_{h,1}$ consists of the elements of the triangulation which have contact to the corner x_1, while $\Omega_{h,2} := \Omega \backslash \Omega_{h,1}$. Since the triangulation of Ω is $O(h^2)$ irregular, the triangulation of $\Omega_{h,2}$ is $O(h^2)$ irregular too. Applying Lemma 4.6 yields for any $q > 2$

$$\left| \int_{\Omega_{h,2}} \nabla(\bar\varphi_{\mathrm{s}} - I_h \bar\varphi_{\mathrm{s}}) \cdot \nabla S_h \psi_h \right| \leq c\|\bar\varphi_{\mathrm{s}}\|_{W^{3,q}(\Omega_{h,2})} \left(h^2 \|S_h \psi_h\|_{H^1(\Omega_{h,2})} \right.$$

$$\left. + h^{3/2} \left(\|S_h \psi_h\|_{L^2(\partial\Omega_{h,2} \cap \Omega)} + \|\psi_h\|_{L^2(\partial\Omega_{h,2} \cap \Gamma)} \right) \right). \tag{43}$$

Since the number of elements in $\Omega_{h,1}$ is bounded independently of h and $\partial\Omega_{h,1} \cap \Omega = \partial\Omega_{h,2} \cap \Omega$, we have that $|\partial\Omega_{h,2} \cap \Omega| \sim h$. Using this fact, the Hölder inequality, and a discrete Sobolev inequality, we obtain

$$\|S_h \psi_h\|_{L^2(\partial\Omega_{h,2} \cap \Omega)} \leq ch^{1/2} \|S_h \psi_h\|_{L^\infty(\Omega)} \leq ch^{1/2} |\log h|^{1/2} \|S_h \psi_h\|_{H^1(\Omega)}. \tag{44}$$

Define $\tilde{S}_h$ as the zero extension operator as in the proof of Lemma 4.3. Since $S_h \psi_h$ denotes the discrete harmonic extension of ψ_h, we infer

$$\|\nabla(S_h - \tilde{S}_h)\psi_h\|_{L^2(\Omega)} \leq \|\nabla \tilde{S}_h \psi_h\|_{L^2(\Omega)}.$$

Using this in combination with the Poincaré inequality yields

$$
\begin{aligned}
\|S_h \psi_h\|_{H^1(\Omega)} &\leq \|(S_h - \tilde{S}_h)\psi_h\|_{H^1(\Omega)} + \|\tilde{S}_h \psi_h\|_{L^2(\Omega)} + \|\nabla \tilde{S}_h \psi_h\|_{L^2(\Omega)} \\
&\leq c\|\nabla(S_h - \tilde{S}_h)\psi_h\|_{L^2(\Omega)} + \|\tilde{S}_h \psi_h\|_{L^2(\Omega)} + \|\nabla \tilde{S}_h \psi_h\|_{L^2(\Omega)} \\
&\leq \|\tilde{S}_h \psi_h\|_{L^2(\Omega)} + c\|\nabla \tilde{S}_h \psi_h\|_{L^2(\Omega)} \\
&\leq c h^{-1/2}\|\psi_h\|_{L^2(\Gamma)},
\end{aligned}
\tag{45}
$$

where we used (29) in the last step.

Next, we observe that the third derivatives of $\bar{\varphi}_s$ behave like $r_1^{\lambda-3}$ such that we can conclude for some arbitrary $\varepsilon > 0$ (depending on q)

$$
\|\bar{\varphi}_s\|_{W^{3,q}(\Omega_{h,2})} \leq c\|r_1^{\lambda-3}\|_{L^q(\Omega_{h,2})} \leq c h^{\lambda-2-\varepsilon}
\tag{46}
$$

since $\min_{x \in \Omega_{h,2}} r_1(x) \sim h$. Collecting (43)–(46) yields

$$
\left| \int_{\Omega_{h,2}} \nabla(\bar{\varphi}_s - I_h \bar{\varphi}_s) \cdot \nabla S_h \psi_h \right| \leq c h^{\lambda-1/2-\epsilon}\|\psi_h\|_{L^2(\Gamma)},
$$

which represents the desired result for the subdomain $\Omega_{h,2}$. Finally, for the subdomain $\Omega_{h,1}$, we conclude by inserting a standard interpolation error estimate and the a priori estimate for the operator S_h as before that

$$
\left| \int_{\Omega_{h,1}} \nabla(\bar{\varphi}_s - I_h \bar{\varphi}_s) \cdot \nabla S_h \psi_h \right| \leq c\|\nabla(\bar{\varphi}_s - I_h \bar{\varphi}_s)\|_{L^2(\Omega_{h,1})}\|\nabla S_h \psi_h\|_{L^2(\Omega_{h,1})}
$$

$$
\leq c h^{1/2}|\bar{\varphi}_s|_{H^2(\Omega_{h,1})}\|\psi_h\|_{L^2(\Gamma)}.
$$

After observing that the second derivatives of $\bar{\varphi}_s$ behave like $r_1^{\lambda-2}$ or $\log r_1$, respectively, if $\lambda = 2$, and that $\max_{x \in \Omega_{h,1}} r_1(x) \sim h$, we get the desired result for the subdomain $\Omega_{h,1}$. $\qquad\square$

Proof of Theorem 4.1. The result is obtained from the general error estimate in Theorem 3.2 using the estimates in Lemmata 4.2, 4.3 and 4.7. $\qquad\square$

Remark 4.8. It is also possible to achieve estimate (27) without the assumption of having a $O(h^2)$-irregular mesh using elements of higher order, e.g. Lagrange quadratic elements. In this case, the third term of the general error estimate can be treated as in the proof of Lemma 4.3, where one sees that the order comes from Ritz projections which are of higher order in case of quadratic elements.

5. **The control constrained case.** This section is devoted to the numerical analysis of control constrained Dirichlet control problems. As we will see, the convergence rates in convex domains coincide with those for the unconstrained problems. More precisely, we will prove the following theorem.

Theorem 5.1. *Suppose that either $\lambda < 1$ and $y_\Omega \in L^2(\Omega)$, or $y_\Omega \in W^{1,p^*}(\Omega)$ for some $p^* > 2$. Moreover, assume that the optimal control has a finite number of kink points. Then it holds*

$$
\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq c h^s |\log h|^r
$$

$$
\forall s \in \mathbb{R} \text{ such that } s < \lambda - \tfrac{1}{2} \text{ and } s \leq 1,
$$

where r is equal to one for $\lambda - \frac{1}{2} \in (1, \frac{3}{2}]$ and equal to zero otherwise. If, further, $\{\mathcal{T}_h\}$ is $O(h^2)$-irregular according to Definition 4.5, then

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^s \quad \forall s < \min\{\tfrac{3}{2}, \lambda - \tfrac{1}{2}\}.$$

The proof of this theorem is postponed to Section 5.1. As already observed, this is exactly the result which we have proven in the unconstrained case. However, if the underlying domain is nonconvex, the approximation rates in the control constrained case can be improved. In this regard, one of our results relies on a structural assumption on the discrete optimal control which we formulate next. Through this section we will shortly write

$$\mathbb{H} = \{j : \lambda_j < 1 \text{ and } c_{j,1} \neq 0\}.$$

Assumption 5.2. There exists some $h_0 > 0$ such that for every $j \in \mathbb{H}$, there exists $\tilde{\rho}_j > 0$ independent of h such that $\bar{u}_h(x) = \bar{u}(x)$ for all $h < h_0$ if $|x - x_j| < \tilde{\rho}_j$.

Let us comment on Assumption 5.2. In Lemma 2.5 it was established that in the neighbourhood of a nonconvex corner, the optimal control will normally be constant and either equal to the lower or the upper bound. Assumption 5.2 says that this property is inherited by the discrete optimal control.

One of our main results in the constrained case is now given as follows.

Theorem 5.3. Suppose $y_\Omega \in W^{1,p^*}(\Omega)$ for some $p^* > 2$. Moreover, let either $\lambda > 1$ or Assumption 5.2 be satisfied, and assume that the optimal control has a finite number of kink points. Then there is the estimate

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^s |\log h|^r$$
$$\forall s \in \mathbb{R} \text{ such that } s < \Lambda - \tfrac{1}{2} \text{ and } s \leq 1, \qquad (47)$$

where r is equal to one for $\Lambda - \frac{1}{2} \in (1, \frac{3}{2}]$ and equal to zero otherwise. If further $\{\mathcal{T}_h\}$ is $O(h^2)$ irregular, then

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^s \; \forall s \in \mathbb{R} \text{ such that } s < \min\{\tfrac{3}{2}, \Lambda - \tfrac{1}{2}, 2\lambda\}. \quad (48)$$

Remark 5.4. We only consider the case $a < 0 < b$. This is because it is known that for those corners such that $\Lambda_j > 1$ we have that $\partial_n \bar{\varphi}(x_j) = 0$. In the case $a < 0 < b$, the projection formula (3) implies that in a neighbourhood of x_j, the optimal control will satisfy $\bar{u}(x) = -\partial_n \bar{\varphi}(x)$, and hence its regularity will be determined by that of the adjoint state. If $0 \notin [a, b]$, then the same projection formula implies that in a neighborhood of x_j, $\bar{u}(x)$ will be equal to some of the control bounds. If we suppose, as in Assumption 5.2 that this property is inherited by the solutions of the discrete approximations, we have that the conclusions of Theorem 5.3 remain valid.

The proof of Theorem 5.3 is postponed to Section 5.1. Since $\Lambda > 1$ and $\lambda > 1/2$, we always have a convergence rate greater than $1/2$. This is a real improvement compared the unconstrained case since in the latter it may happen that the convergence rates tend to zero as the largest interior angle tend to 2π. However, one may ask for a justification of Assumption 5.2. In Lemma 5.10 we will see that there exist constants $\tilde{\rho}_{1,j}$ and $\tilde{\rho}_{1,2}$ greater than zero for all $j \in \mathbb{H}$, and a constant $h_0 > 0$ such that

$$\bar{u}_h(x_{h,i}) = \bar{u}(x_{h,i}) \quad \text{for all nodes } x_{h,i} \text{ with } |x_{h,i} - x_j| \in [\tilde{\rho}_{1,j} h |\log h|^{1/2}, \tilde{\rho}_{2,j}]. \quad (49)$$

Thus, we could relax Assumption 5.2 to an h-dependent neighborhood of those corners x_j with $j \in \mathbb{H}$. Moreover, due to (49), it is even possible to show the

following improved result in nonconvex domains without any structural assumption on the discrete optimal control, i.e., we can always expect a convergence rate close to $1/2$ in nonconvex domains.

Theorem 5.5. *Suppose $y_\Omega \in W^{1,p^*}(\Omega)$ for some $p^* > 2$, and assume that the optimal control has a finite number of kink points. Then it holds*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^{1/2}|\log h|^{1/4}. \tag{50}$$

The proof of Theorem 5.5 is given in Section 5.2.

5.1. **Proof of Theorems 5.1 and 5.3.** The results of Theorem 5.1, and Theorem 5.3 for $\lambda > 1$ directly follow from the general error estimate given in Theorem 3.2, the estimates for the adjoint state provided in Section 4 in Lemmata 4.3 and 4.7 and the error estimates for the control and the state established below in Lemma 5.6.

Lemma 5.6. *Suppose $y_\Omega \in H^t(\Omega) \cap L^2(\Omega)$ for all $t < \min\{1, \lambda - 1\}$ and assume that the optimal control has a finite number of kink points. Then*

$$\|\bar{u} - u_h^*\|_{L^2(\Gamma)} \leq ch^s \quad \forall s < \min\{\tfrac{3}{2}, \Lambda - \tfrac{1}{2}\},$$
$$\|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} \leq ch^s \quad \forall s < \min\{\tfrac{3}{2}, \Lambda - \tfrac{1}{2}, 2\lambda\}.$$

Proof. The proof starts exactly following the lines of the proof of Lemma 4.2, using the regularity stated in Lemma 2.4. In this way, if $s < 1$ we again obtain the desired estimate for u_h^*, as defined in (17), from [8, Eq. (7.10)]. If $s \in [1, \tfrac{3}{2})$, u_h^* is given by (18). Since control constraints are now present, we have to derive error estimates for the modified Lagrange interpolant. To this end, let us consider two adjacent boundary elements E_{j-1} and E_j belonging to $\mathcal{E}_h$ which are determined by the line segments $(x_\Gamma^{j-1}, x_\Gamma^j)$ and $(x_\Gamma^j, x_\Gamma^{j+1})$, respectively. Since we assume a finite number of kink points of $\bar{u}$ due to the projection formula (3), we have to deal with the following situations (at least for h small enough): First, no kink is contained in $E_{j-1} \cup E_j$, second, there is exactly one kink of $\bar{u}$ in $E_{j-1} \cup E_j$ due to the projection formula. In the first case, we have that u_h^* coincides with the Lagrange interpolant on $E_{j-1} \cup E_j$ such that the desired estimate on these elements is obtained by standard discretization error estimates for the Lagrange interpolant employing the regularity results from Lemma 2.4, i.e.,

$$\|\bar{u} - u_h^*\|_{L^2(E_{j-1} \cup E_j)} \leq ch^s|\bar{u}|_{H^s(E_{j-1} \cup E_j)} \tag{51}$$

with $s < \min\{3/2, \Lambda - 1/2\}$. In the second case, we can assume without loss of generality that $u_{h,j-1}^* = b = \bar{u}(x_\Gamma^{j-1})$, $u_{h,j}^* = b \neq \bar{u}(x_\Gamma^j)$ and $u_{h,j+1}^* = \bar{u}(x_\Gamma^{j+1}) \in (a, b)$. Thus, u_h^* is equal to $b = \bar{u}(x_\Gamma^{j-1})$ on E_{j-1}. Using the regularity of the optimal control $\bar{u} \in H^s(\Gamma) \hookrightarrow C^{0, s-1/2}(\Gamma)$ with $s < \min\{3/2, \Lambda - 1/2\}$ from Lemma 2.4, we now estimate the interpolation error on each of the elements E_{j-1} and E_j. For the error on E_{j-1} we obtain by means of the Hölder continuity of $\bar{u}$

$$\|\bar{u} - u_h^*\|_{L^2(E_{j-1})} = \|\bar{u} - \bar{u}(x_\Gamma^{j-1})\|_{L^2(E_{j-1})} \leq c|x_\Gamma^{j-1} - x_\Gamma^j|^{s-1/2}|E_{j-1}|^{1/2} \sim ch^s. \tag{52}$$

Next, recall that the nodal basis function associated with x_Γ^j is denoted by e_j. Then we deduce for the error on E_j

$$\|\bar{u} - u_h^*\|_{L^2(E_j)} = \|(e_j + e_{j+1})\bar{u} - \bar{u}(x_\Gamma^{j-1})e_j - \bar{u}(x_\Gamma^{j+1})e_{j+1}\|_{L^2(E_j)}$$
$$= \|(\bar{u} - \bar{u}(x_\Gamma^{j-1}))e_j + (\bar{u} - \bar{u}(x_\Gamma^{j+1}))e_{j+1}\|_{L^2(E_j)}$$

$$\leq \|\bar{u} - \bar{u}(x_\Gamma^{j-1})\|_{L^2(E_j)} + \|\bar{u} - \bar{u}(x_\Gamma^{j+1})\|_{L^2(E_j)}$$

$$\leq c \left(|x_\Gamma^{j-1} - x_\Gamma^{j+1}|^{s-1/2} |E_j|^{1/2} + |x_\Gamma^j - x_\Gamma^{j+1}|^{s-1/2} |E_j|^{1/2} \right)$$

$$\sim ch^s, \tag{53}$$

where we again used the Hölder continuity of $\bar{u}$. Since we assume a finite number of kink points, the desired interpolation error estimate for u_h^* on Γ in case that $s \in [1, \frac{3}{2})$ is just a combination of (51)–(53).

Since the optimal control $\bar{u}$ belongs at least to $H^{1/2}(\Gamma)$, the optimal state $\bar{y}$ is a weak solution such that we can rely on standard techniques for the derivation of the second estimate of the assertion. More precisely, by employing the regularity of $\bar{y} \in H^{t+1}(\Omega)$ with $t < \min\{1, \Lambda - 1, \lambda\}$ and $\bar{u} \in H^r(\Gamma)$ with $r < \min\{\frac{3}{2}, \Lambda - \frac{1}{2}\}$ from Lemma 2.4, an application of a duality argument, cf. for instance [6], yields

$$\|\bar{y} - S_h \bar{u}\|_{L^2(\Omega)} \leq ch^{t+\min\{1, r+\frac{1}{2}, \lambda\}} \leq ch^{t+\min\{1, \lambda\}} \leq ch^s,$$

where $s < \min\{\frac{3}{2}, \Lambda - \frac{1}{2}, 2\lambda\}$. For the last two steps notice that $\Lambda > 1$ and $\lambda > 1/2$. $\qquad\square$

Since $\Lambda \geq \lambda$, a straightforward application of Theorem 3.2, and Lemmata 5.6, 4.3 and 4.7 leads to an order of convergence identical to the one we have for unconstrained problems. Notice that Lemmata 4.3 and 4.7 can be used since bounds on the control do not play any role there. Thus, Theorem 5.1 and Theorem 5.3 for $\lambda > 1$ are proved.

For the results of Theorem 5.3, in case that $\lambda < 1$ and Assumption 5.2 is valid, we use the above error estimates for the control and the state, and we show in Lemmata 5.8 and 5.9 below how to improve the result for the adjoint state. Then an adaptation of the general error estimate, see Theorem 5.7, which we are going to prove next, can finally be used to combine these results. Let us define

$$\tilde{\Gamma} := \{x \in \Gamma : \ |x - x_j| < \tilde{\rho}_j \text{ if } j \in \mathbb{H}\}.$$

Moreover, let

$$V_h := \{u_h \in U_h : \ u_h \equiv 0 \text{ on } \tilde{\Gamma}\}.$$

Under the structural Assumption 5.2 it is clear that $e_h = u_h^* - \bar{u}_h \in V_h$, so we have the following modification of the general error estimate (19).

Theorem 5.7. *Suppose Assumption 5.2 holds. Then*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$$

$$\leq c \left(\|\bar{u} - u_h^*\|_{L^2(\Gamma)} + \|\bar{y} - S_h \bar{u}\|_{L^2(\Omega)} + \sup_{\psi_h \in V_h} \frac{\left|(\nabla \bar{\varphi}, \nabla S_h \psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} \right).$$

Proof. Since $e_h = u_h^* - \bar{u}_h \in V_h$ due to Assumption 5.2, the result can be obtained in the same way as in the proof of Theorem 3.2 just by replacing

$$(\nabla \bar{\varphi}, \nabla S_h e_h)_{L^2(\Omega)} \leq \sup_{\psi_h \in U_h} \frac{\left|(\nabla \bar{\varphi}, \nabla S_h \psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} \|e_h\|_{L^2(\Gamma)}$$

in (24) by

$$(\nabla \bar{\varphi}, \nabla S_h e_h)_{L^2(\Omega)} \leq \sup_{\psi_h \in V_h} \frac{\left|(\nabla \bar{\varphi}, \nabla S_h \psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} \|e_h\|_{L^2(\Gamma)}.$$

$\qquad\square$

Next, we are concerned with discretization error estimates for the critical term in the general estimate of Theorem 5.7. First, we deal with estimates for general quasi-uniform meshes. Afterwards we show improved estimates if we assume $O(h^2)$-irregular meshes.

Lemma 5.8. *Let* $y_\Omega \in W^{1,p^*}(\Omega)$ *for some* $p^* > 2$. *Then there is the estimate*

$$\sup_{\psi_h \in V_h} \frac{\left|(\nabla\bar\varphi, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} \leq ch^s|\log h|^r \quad \forall s \in \mathbb{R} \text{ such that } s < \Lambda - \frac{1}{2} \text{ and } s \leq 1,$$

where r *is equal to one for* $\Lambda - 1/2 \in (1, 3/2]$ *and equal to zero otherwise.*

Proof. To be able to localize the effects in the neighborhood of all corners x_j with $j \in \mathbb{H}$, we introduce a cut-off function η_1 which is equal to one in a fixed neighborhood of these corners and decays smoothly. In addition, we set $\eta_0 = 1 - \eta_1$. Then we infer for the quantity of interest

$$\sup_{\psi_h \in V_h} \frac{\left|(\nabla\bar\varphi, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}}$$
$$\leq \sup_{\psi_h \in V_h} \frac{\left|(\nabla(\eta_0\bar\varphi), \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} + \sup_{\psi_h \in V_h} \frac{\left|(\nabla(\eta_1\bar\varphi), \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}}. \quad (54)$$

For the first term on the right hand side of this inequality, we directly apply Lemma 4.3 to conclude

$$\sup_{\psi_h \in V_h} \frac{\left|(\nabla(\eta_0\bar\varphi), \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} \leq ch^s|\log h|^r \quad (55)$$

$$\forall s \in \mathbb{R} \text{ such that } s < \Lambda - \tfrac{1}{2} \text{ and } s \leq 1, \quad (56)$$

where r is equal to one for $\Lambda - 1/2 \in (1, 3/2]$ and equal to zero otherwise, having in mind the regularity results of Lemma 2.6 for the adjoint state and noting that the singular terms coming from the corners x_j with $j \in \mathbb{H}$ do not have any influence due to the cut-off function η_0. To deal with the second term in (54), let $\tilde S_h$ denote the extension operator which extends a piecewise linear function ψ_h on the boundary by zero to a function in Y_h. Thus, $\tilde S_h\psi_h$ is equal to zero in $\tilde\Omega := \{x \in \Omega : |x - x_j| < \tilde\rho_j/2 \text{ if } j \in \mathbb{H}\}$ for any $\psi_h \in V_h$. Moreover, let R_h be the operator that maps a function in $H^1_0(\Omega)$ to its Ritz-projection in $Y_{0,h}$. Due the properties of the discrete harmonic extension S_h and the Ritz-projection R_h, we obtain

$$(\nabla(\eta_1\bar\varphi), \nabla S_h\psi_h)_{L^2(\Omega)} = (\nabla(\eta_1\bar\varphi - R_h(\eta_1\bar\varphi)), \nabla S_h\psi_h)_{L^2(\Omega)}$$
$$= (\nabla(\eta_1\bar\varphi - R_h(\eta_1\bar\varphi)), \nabla \tilde S_h\psi_h)_{L^2(\Omega)}$$
$$= (\nabla(\eta_1\bar\varphi - R_h(\eta_1\bar\varphi)), \nabla \tilde S_h\psi_h)_{L^2(\Omega\setminus\tilde\Omega)}. \quad (57)$$

By applying the Hölder inequality, local $W^{1,\infty}$-discretization error estimates for the Ritz-projection from [13, Corollary 1], and (29), we obtain

$$(\nabla(\eta_1\bar\varphi - R_h(\eta_1\bar\varphi)), \nabla \tilde S_h\psi_h)_{L^2(\Omega\setminus\tilde\Omega)} \quad (58)$$
$$\leq \|\nabla(\eta_1\bar\varphi - R_h(\eta_1\bar\varphi))\|_{L^\infty(\Omega\setminus\tilde\Omega)}\|\nabla\tilde S_h\psi_h\|_{L^1(\Omega\setminus\tilde\Omega)}$$
$$\leq c\left(\|\nabla(\eta_1\bar\varphi - I_h(\eta_1\bar\varphi))\|_{L^\infty(\Omega\setminus\tilde\Omega')} + \|\eta_1\bar\varphi - R_h(\eta_1\bar\varphi)\|_{L^2(\Omega)}\right)\|\psi_h\|_{L^1(\Gamma)},$$
$$\quad (59)$$

where $\tilde{\Omega}' := \{x \in \Omega : |x - x_j| < \tilde{\rho}_j/4 \text{ if } j \in \mathbb{H}\}$. Regarding the regularity results for the optimal adjoint state from Lemma 2.6 and by using standard interpolation error estimates and a standard finite element error estimate, we deduce

$$\|\nabla(\eta_1\bar{\varphi} - I_h(\eta_1\bar{\varphi}))\|_{L^\infty(\Omega\setminus\tilde{\Omega}')} + \|\eta_1\bar{\varphi} - R_h(\eta_1\bar{\varphi})\|_{L^2(\Omega)} \le c(h + h^s) \qquad (60)$$

which is valid for all $s \in \mathbb{R}$ such that $s < 2\lambda$ and $s \le 2$. Combining (54)–(60) ends the proof. $\qquad\square$

Lemma 5.9. *Let $y_\Omega \in W^{1,p^*}(\Omega)$ for some $p^* > 2$ and suppose further that $\{\mathcal{T}_h\}$ is a family of $O(h^2)$-irregular meshes. Then it holds*

$$\sup_{\psi_h \in V_h} \frac{|(\nabla\bar{\varphi}, \nabla S_h\psi_h)_{L^2(\Omega)}|}{\|\psi_h\|_{L^2(\Gamma)}} \le ch^s$$

for all $s \in \mathbb{R}$ such that $s < \min\{\Lambda - \frac{1}{2}, 2\lambda\}$ and $s \le \frac{3}{2}$.

Proof. As before, we introduce the circular sectors

$$\tilde{\Omega} := \{x \in \Omega : |x - x_j| < \tilde{\rho}_j/2 \text{ if } j \in \mathbb{H}\},$$

$$\tilde{\Omega}' := \{x \in \Omega : |x - x_j| < \tilde{\rho}_j/4 \text{ if } j \in \mathbb{H}\}.$$

For technical reasons we also need the circular sector

$$\tilde{\Omega}'' := \{x \in \Omega : |x - x_j| < \tilde{\rho}_j/8 \text{ if } j \in \mathbb{H}\}.$$

Let the operators $\tilde{S}_h$ and R_h be defined as in the proof of Lemma 5.8. Moreover, let η_1 be a smooth cut-off function which is equal to one in $\tilde{\Omega}''$ with $\text{supp}\,\eta_1 \subset \tilde{\Omega}'$. In addition, we choose η_1 such that $\text{supp}\,I_h\eta_1 \subset \tilde{\Omega}'$ which is possible without any restriction for h small enough. We set $\eta_0 := 1 - \eta_1$. Analogously to the foregoing proof, we infer

$$(\nabla\bar{\varphi}, \nabla S_h\psi_h)_{L^2(\Omega)} = (\nabla(\eta_1\bar{\varphi}), \nabla S_h\psi_h)_{L^2(\Omega)} + (\nabla(\eta_0\bar{\varphi}), \nabla S_h\psi_h)_{L^2(\Omega)}$$
$$= (\nabla(\eta_1\bar{\varphi} - R_h(\eta_1\bar{\varphi})), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega\setminus\tilde{\Omega})} + (\nabla(\eta_0\bar{\varphi} - I_h(\eta_0\bar{\varphi})), \nabla S_h\psi_h)_{L^2(\Omega)}. \qquad (61)$$

Observe that $\eta_0\bar{\varphi}$ is equal to zero in a fixed neighborhood of all corners x_j with $j \in \mathbb{H}$. Consequently, Lemma 4.6, applied as in the proof of Lemma 4.7, yields for the latter term in (61)

$$(\nabla(\eta_0\bar{\varphi} - I_h(\eta_0\bar{\varphi})), \nabla S_h\psi_h)_{L^2(\Omega)} \le ch^s\|\psi_h\|_{L^2(\Gamma)}$$

with $s \in \mathbb{R}$ such that $s < \Lambda - \frac{1}{2}$ and $s \le \frac{3}{2}$. By applying the Hölder inequality, local $W^{1,\infty}$-discretization error estimates for the Ritz-projection from [13, Corollary 1], and (29), we obtain for the first term in (61)

$$(\nabla(\eta_1\bar{\varphi} - R_h(\eta_1\bar{\varphi})), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega\setminus\tilde{\Omega})}$$
$$\le \|\nabla(\eta_1\bar{\varphi} - R_h(\eta_1\bar{\varphi}))\|_{L^\infty(\Omega\setminus\tilde{\Omega})}\|\nabla\tilde{S}_h\psi_h\|_{L^1(\Omega\setminus\tilde{\Omega})}$$
$$\le c\left(\|\nabla(\eta_1\bar{\varphi} - I_h(\eta_1\bar{\varphi}))\|_{L^\infty(\Omega\setminus\tilde{\Omega}')} + \|\eta_1\bar{\varphi} - R_h(\eta_1\bar{\varphi})\|_{L^2(\Omega\setminus\tilde{\Omega}')}\right)\|\psi_h\|_{L^1(\Gamma)}$$
$$\le c\|\eta_1\bar{\varphi} - R_h(\eta_1\bar{\varphi})\|_{L^2(\Omega\setminus\tilde{\Omega}')}\|\psi_h\|_{L^1(\Gamma)},$$

where we used that $\eta_1\bar{\varphi}$ and $I_h(\eta_1\bar{\varphi})$ are equal to zero in $\Omega\setminus\tilde{\Omega}'$. Usual error estimates for the Ritz-projection and a standard embedding yield

$$(\nabla(\eta_1\bar{\varphi} - R_h(\eta_1\bar{\varphi})), \nabla\tilde{S}_h\psi_h)_{L^2(\Omega\setminus\tilde{\Omega})} \le h^s\|\psi_h\|_{L^2(\Gamma)},$$

which is valid for all $s \in \mathbb{R}$ such that $s < 2\lambda$ and $s \le 2$. This ends the proof. $\qquad\square$

Finally, an application of Theorem 5.7, and Lemmata 5.6, 5.8 and 5.9, yield the results of Theorem 5.3 where $\lambda < 1$ and Assumption 5.2 is satisfied.

5.2. **Proof of Theorem 5.5.** In this subsection we show the results of Theorem 5.5. That is, we show a convergence rate close to $1/2$ for the optimal controls and states in the constrained case if the domain is nonconvex and even if the structural Assumption 5.2 does not hold. For that purpose, let us recall that $\{x_j\}$ denotes the corners of Γ, $\{x_\Gamma^i\}$ is the set of boundary nodes of the mesh and $\{e_i\}$ is the basis of U_h such that $e_i(x_\Gamma^k) = \delta_{ik}$. Thus, every function $u_h \in U_h$ can be written as

$$u_h = \sum_{i=1}^{N(h)} u_{h,i} e_i \quad \text{with } u_{h,i} = u_h(x_\Gamma^i).$$

By testing the discrete variational inequality appropriately, we deduce

$$\bar{u}_{h,i} = \begin{cases} a & \text{if } \int_\Gamma (\nu\bar{u}_h - \partial_n^h \bar{\varphi}_h)e_i > 0, \\ \\ b & \text{if } \int_\Gamma (\nu\bar{u}_h - \partial_n^h \bar{\varphi}_h)e_i < 0. \end{cases}$$

Lemma 5.10. *For each interior angle $\omega_j > \pi$, where $c_{j,1}$ from (6) is unequal to zero, there are two constants $\tilde{\rho}_{1,j}$ and $\tilde{\rho}_{2,j}$ greater than zero such that*

$$\bar{u}_h(x_\Gamma^i) = \begin{cases} a & \text{if } c_{j,1} > 0 \\ b & \text{if } c_{j,1} < 0 \end{cases}$$

for all nodes x_Γ^i with $|x_\Gamma^i - x_j| \in [\tilde{\rho}_{1,j} h |\log h|^{1/2}, \tilde{\rho}_{2,j}]$.

Proof. In the following we focus only on one nonconvex corner x_j. Without loss of generality let $c_{j,1}$ be greater than zero. Hence the normal derivative of $\bar{\varphi}$ is negative, and the lower bound of the control is active, and $\nu\bar{u} - \partial_n\bar{\varphi} > 0$ in the vicinity of this corner. We need to show that there are two constants $\tilde{\rho}_{1,j}$ and $\tilde{\rho}_{2,j}$ such that

$$\int_\Gamma (\nu\bar{u}_h - \partial_n^h \bar{\varphi}_h)e_{h,i} > 0$$

for all nodes $x_{h,i}$ with $|x_{h,i} - x_j| \in [\tilde{\rho}_{1,j} h |\log h|^{1/2}, \tilde{\rho}_{2,j}]$. According to [20, Theorem 3.4], we know that

$$c_{j,1} = (\bar{y} - y_d, \zeta_{j,1})_{L^2(\Omega)}$$

where the function $\zeta_{j,1}$ is of the form

$$\zeta_{j,1} = \pi^{-1/2}\xi_j r_j^{-\lambda_j} \sin(\lambda_j\theta_j) + z_{j,1},$$

where ξ_j denotes the cut-off function introduced at the beginning of Section 2 and the function $z_{j,1}$ denotes a function which solves

$$-\Delta z_{j,1} = [\Delta, \xi_j]\pi^{-1/2}r_j^{-\lambda_j} \sin(\lambda_j\theta_j) \text{ in } \Omega, \quad z_{j,1} = 0 \text{ on } \Gamma,$$

and $[a, b] = ab - ba$ denotes the commutator. According to Theorem 5.1, we deduce the existence of a constant $h_0 > 0$ such that for all $h < h_0$ there holds

$$\tilde{c}_{j,1} := (\bar{y}_h - y_d, \zeta_{j,1})_{L^2(\Omega)} = (\bar{y} - y_d, \zeta_{j,1})_{L^2(\Omega)} + (\bar{y}_h - \bar{y}, \zeta_{j,1})_{L^2(\Omega)}$$
$$\geq (\bar{y} - y_d, \zeta_{j,1})_{L^2(\Omega)} - c\|\bar{y}_h - \bar{y}\|_{L^2(\Omega)} > 0 \tag{62}$$

due to the assumption $c_{j,1} = (\bar{y} - y_d, \zeta_{j,1})_{L^2(\Omega)} > 0$. Using this result, we will show that the singular part of the function $\tilde{\varphi}$ which solves

$$-\Delta\tilde{\varphi} = \bar{y}_h - y_d \text{ in } \Omega, \quad \tilde{\varphi} = 0 \text{ on } \Gamma,$$

behaves like the singular part of $\bar{\varphi}$. Indeed, $\tilde{\varphi}$ admits the splitting

$$\tilde{\varphi} = \tilde{\varphi}_{\mathrm{r}} + \tilde{\varphi}_{\mathrm{s}} \tag{63}$$

according to Lemma 2.1. The regular part $\tilde{\varphi}_{\mathrm{r}}$ belongs to $W^{2,q}(\Omega)$, at least for some $q > 2$, since $\bar{y}_h - y_d$ belongs to $L^q(\Omega)$ due to the convergence result of Theorem 5.1. The singular part $\tilde{\varphi}_{\mathrm{s}}$ can be written as

$$\tilde{\varphi}_{\mathrm{s}} = \tilde{c}_{j,1}\xi_j r_j^{\lambda_j} \sin(\lambda_j\theta_j),$$

where the constant $\tilde{c}_{j,1}$ is greater than zero according to (62). Assuming that $|x_\Gamma^i - x_j|$ is already small enough such that $\xi_j \equiv 1$ on $\operatorname{supp} e_i$, we get by basic calculations

$$\int_\Gamma (\nu\bar{u}_h - \partial_n^h\bar{\varphi}_h)e_i = \int_\Gamma \nu\bar{u}_h e_i + \int_\Gamma (\partial_n\tilde{\varphi} - \partial_n^h\bar{\varphi}_h)e_i$$
$$- \int_\Gamma \partial_n\tilde{\varphi}_{\mathrm{r}} e_i - \int_\Gamma \partial_n\tilde{\varphi}_{\mathrm{s}} e_i$$
$$\geq -\int_\Gamma c e_i + \int_\Gamma (\partial_n\tilde{\varphi} - \partial_n^h\bar{\varphi}_h)e_i + \int_\Gamma c_{j,1} r_j^{\lambda_j-1} e_i \tag{64}$$

where we used that $\|\bar{u}_h\|_{L^\infty(\Gamma)} \leq \max\{|a|, |b|\}$ and that $\partial_n\tilde{\varphi}_{\mathrm{r}}$ is uniformly bounded in $L^\infty(\Gamma)$ due to the embedding $W^{2,q}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for $q > 2$. As before, let us denote by $\tilde{S}_h$ the operator which extends any function of U_h to one in Y_h by zero. Also observe that $\bar{\varphi}_h$ is the Ritz-projection $R_h\tilde{\varphi}$ of $\tilde{\varphi}$. Then integration by parts, the definition of $\partial_n^h\bar{\varphi}_h$ in (15) and (63) yield

$$\int_\Gamma (\partial_n\tilde{\varphi} - \partial_n^h\bar{\varphi}_h)e_i = \int_\Omega \nabla(\tilde{\varphi} - R_h\tilde{\varphi}) \cdot \nabla\tilde{S}_h e_i$$
$$= \int_\Omega \nabla(\tilde{\varphi}_{\mathrm{r}} - R_h\tilde{\varphi}_{\mathrm{r}}) \cdot \nabla\tilde{S}_h e_i + \int_\Omega \nabla(\tilde{\varphi}_{\mathrm{s}} - R_h\tilde{\varphi}_{\mathrm{s}}) \cdot \nabla\tilde{S}_h e_i$$
$$\leq \|\nabla(\tilde{\varphi}_{\mathrm{r}} - R_h\tilde{\varphi}_{\mathrm{r}})\|_{L^2(\Omega)}\|\nabla\tilde{S}_h e_i\|_{L^2(\Omega)} + \int_\Omega \nabla(\tilde{\varphi}_{\mathrm{s}} - R_h\tilde{\varphi}_{\mathrm{s}}) \cdot \nabla\tilde{S}_h e_i$$
$$\leq \int_\Gamma c e_i + \int_\Omega \nabla(\tilde{\varphi}_{\mathrm{s}} - R_h\tilde{\varphi}_{\mathrm{s}}) \cdot \nabla\tilde{S}_h e_i, \tag{65}$$

where we employed a standard discretization error estimate for the Ritz-projection, (29) and an inverse inequality in the last step. Now we proceed as in the proof of Lemma 4.3 between (38) and (42). Let σ and the subdomains Ω_J be defined as in that proof and let the index J be chosen such that $\operatorname{supp}\tilde{S}_h e_i \subset \Omega_J$. Assume that $|x_\Gamma^i - x_j| \geq c_I h$ with a constant c_I large enough such that local $W^{1,\infty}$-error estimates for the Ritz-projection are applicable. Then those estimates of [13, Corollary 1] and (29) yield

$$\int_\Omega \nabla(\tilde{\varphi}_{\mathrm{s}} - R_h\tilde{\varphi}_{\mathrm{s}}) \cdot \nabla\tilde{S}_h e_i \leq \|\nabla(\tilde{\varphi}_{\mathrm{s}} - R_h\tilde{\varphi}_{\mathrm{s}})\|_{L^\infty(\Omega)}\|\nabla\tilde{S}_h e_i\|_{L^1(\Omega)}$$
$$\leq c\left(\|\nabla(\tilde{\varphi}_{\mathrm{s}} - I_h\tilde{\varphi}_{\mathrm{s}})\|_{L^\infty(\Omega'_J)} + d_J^{-2}\|\tilde{\varphi}_{\mathrm{s}} - R_h\tilde{\varphi}_{\mathrm{s}}\|_{L^2(\Omega'_J)}\right)\|e_i\|_{L^1(\Gamma)}$$

$$\leq cd_J^{\lambda_j-2}\left(d_J^{2-\lambda_j}\|\nabla(\tilde{\varphi}_{\mathrm{s}}-I_h\tilde{\varphi}_{\mathrm{s}})\|_{L^\infty(\Omega'_J)}\right.$$

$$\left.+\|\sigma^{-\lambda_j}(\tilde{\varphi}_{\mathrm{s}}-R_h\tilde{\varphi}_{\mathrm{s}})\|_{L^2(\Omega'_J)}\right)\int_\Gamma e_i. \tag{66}$$

The second derivatives of the singular part behave like $r_j^{\lambda_j-2}$. Thus, by means of standard interpolation error estimates and the results of [22, Corollary 3.62], we infer

$$d_J^{2-\lambda_j}\|\nabla(\tilde{\varphi}_{\mathrm{s}}-I_h\tilde{\varphi}_{\mathrm{s}})\|_{L^\infty(\Omega'_J)}+\|\sigma^{-\lambda_j}(\tilde{\varphi}_{\mathrm{s}}-R_h\tilde{\varphi}_{\mathrm{s}})\|_{L^2(\Omega'_J)}$$

$$\leq c\left(h+h|\log h|^{1/2}\right)\leq ch|\log h|^{1/2}. \tag{67}$$

Combining (64)–(67), we obtain

$$\int_\Gamma(\nu\bar{u}_h-\partial_n^h\bar{\varphi}_h)e_i\geq\int_\Gamma(c_{j,1}r_j^{\lambda_j-1}-c-ch|\log h|^{1/2}d_J^{\lambda_j-2})e_i$$

$$\geq(c_{j,1}d_J^{\lambda_j-1}-c-ch|\log h|^{1/2}d_J^{\lambda_j-2})\int_\Gamma e_i.$$

Finally, we observe that $|x_{h,i}-x_j|\sim d_J$. Thus, we are able to choose constants $\tilde{\rho}_{1,j}$ and $\tilde{\rho}_{2,j}$ such that

$$\frac{c_{j,1}}{2}d_J{}^{\lambda_j-1}-c>0$$

if $|x_\Gamma^i-x_j|\leq\tilde{\rho}_{2,j}$ and

$$\frac{c_{j,1}}{2}d_J{}^{\lambda_j-1}-ch|\log h|^{1/2}d_J^{\lambda_j-2}>0$$

if $|x_\Gamma^i-x_j|\geq\tilde{\rho}_{1,j}h|\log h|^{1/2}$. This proves the assertion. $\qquad\square$

Remark 5.11. By using the Cauchy-Schwarz inequality, estimates for the Ritz-projection from [4, Theorem 5.1], (29) and an inverse inequality, we infer

$$\int_\Omega\nabla(\tilde{\varphi}_{\mathrm{s}}-R_h\tilde{\varphi}_{\mathrm{s}})\cdot\nabla\tilde{S}_he_i\leq\|\nabla(\tilde{\varphi}_{\mathrm{s}}-R_h\tilde{\varphi}_{\mathrm{s}})\|_{L^2(\Omega)}\|\nabla\tilde{S}_he_i\|_{L^2(\Omega)}\leq ch^{\lambda_j}.$$

However, this is not enough to show that the discrete optimal control admits one of the control bounds in the direct vicinity of the corner x_j, since then

$$\int_\Gamma c_{j,1}r_j^{\lambda_j-1}e_i\geq ch^{\lambda_j}.$$

Now, we redefine the sets $\tilde{\Gamma}$ and $\tilde{\Omega}$ by

$$\tilde{\Gamma}:=\{x\in\Gamma:\ |x-x_j|<\tilde{\rho}_{2,j}\text{ if }j\in\mathbb{H}\}$$

and

$$\tilde{\Omega}:=\{x\in\Omega:\ |x-x_j|<\tilde{\rho}_{2,j}\text{ if }j\in\mathbb{H}\},$$

and we set again

$$V_h:=\{u_h\in U_h:\ u_h\equiv 0\text{ on }\tilde{\Gamma}\}.$$

Moreover, let $\Gamma^c:=\Gamma\backslash\tilde{\Gamma}$. We have the following modification for the general error estimate

Theorem 5.12. *For the solution of the continuous and the discrete optimal control problem we have*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma^c)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le c\Bigg(\|\bar{u} - \bar{u}_h\|_{L^2(\tilde{\Gamma})} + \|\bar{u} - u_h^*\|_{L^2(\Gamma^c)}$$

$$+ \|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} + \sup_{\psi_h \in V_h} \frac{\left|(\nabla\bar{\varphi}, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} \Bigg). \qquad (68)$$

Note that the first term on the left hand side of (68) is a norm with respect to Γ^c.

Proof. We proceed as in the proof of Theorem 3.2. In contrast, we will test the optimality conditions with different functions. For that purpose, let us introduce $\tilde{u} \in U_{ad}$ and $\tilde{u}_h \in U_{ad}^h$ by

$$\tilde{u} = \begin{cases} \bar{u} & \text{a.e. in } \tilde{\Gamma} \\ \bar{u}_h & \text{a.e. in } \Gamma^c \end{cases} \quad \text{and} \quad \tilde{u}_h = \begin{cases} \bar{u}_h & \text{a.e. in } \tilde{\Gamma} \\ u_h^* & \text{a.e. in } \Gamma^c \end{cases}.$$

Note that $\bar{u}$ and u_h^* are constant, even coincide, on $\tilde{\Gamma}$ and that $\bar{u}_h$ is equal to $\bar{u}$ at least for all $x_{h,i}$ with $|x_{h,i} - x_j| \in [\tilde{\rho}_{1,j}h|\log h|^{1/2}, \tilde{\rho}_{2,j}]$ for $j \in \mathbb{H}$ according to Lemma 5.10. Next, we define the intermediate error $e_h := \tilde{u}_h - \bar{u}_h$, which is equal to zero in $\tilde{\Gamma}$. Then, we obtain

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma^c)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le \|\bar{u} - u_h^*\|_{L^2(\Gamma^c)} + \|e_h\|_{L^2(\Gamma)}$$

$$+ \|\bar{y} - S_h\tilde{u}_h\|_{L^2(\Omega)} + \|S_h e_h\|_{L^2(\Omega)}. \qquad (69)$$

To deal with the third term, we take into account the continuity of S_h:

$$\|\bar{y} - S_h\tilde{u}_h\|_{L^2(\Omega)} \le \|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} + \|S_h(\bar{u} - \tilde{u}_h)\|_{L^2(\Omega)}$$

$$\le \|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} + c\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)}$$

$$\le \|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} + c(\|\bar{u} - \bar{u}_h\|_{L^2(\tilde{\Gamma})} + \|\bar{u} - u_h^*\|_{L^2(\Gamma^c)}). \qquad (70)$$

Accordingly, we only need estimates for the second and fourth terms in (20). We begin estimating the second one, but as we will see this also yields an estimate for the fourth term. There holds

$$\nu\|e_h\|_{L^2(\Gamma)}^2 = \nu(u_h^* - \bar{u}, e_h)_{L^2(\Gamma^c)} + \nu(\bar{u} - \bar{u}_h, e_h)_{L^2(\Gamma)}. \qquad (71)$$

Next, we consider the second term of (71) in detail. By adding the continuous and discrete variational inequality with $u = \tilde{u}$ and $u_h = \tilde{u}_h$, respectively, we deduce

$$(\nu(\bar{u}_h - \bar{u}) + \partial_n\bar{\varphi} - \partial_n^h\bar{\varphi}_h, e_h)_{L^2(\Gamma)} + (\nu\bar{u} - \partial_n\bar{\varphi}, u_h^* - \bar{u})_{L^2(\Gamma)} \ge 0.$$

Rearranging terms and using (16) leads to

$$\nu(\bar{u} - \bar{u}_h, e_h)_{L^2(\Gamma)} \le (\partial_n\bar{\varphi} - \partial_n^h\bar{\varphi}_h, e_h)_{L^2(\Gamma)}.$$

Integration by parts (cf. [21, Theorem 3.1.1]), (15), (2c) and (13) yield

$$\nu(\bar{u} - \bar{u}_h, e_h)_{L^2(\Gamma)} \le (\Delta\bar{\varphi} + (\bar{y}_h - y_\Omega), S_h e_h)_{L^2(\Omega)} + (\nabla(\bar{\varphi} - \bar{\varphi}_h), \nabla S_h e_h)_{L^2(\Omega)}$$

$$= (\bar{y}_h - \bar{y}, S_h e_h)_{L^2(\Omega)} + (\nabla\bar{\varphi}, \nabla S_h e_h)_{L^2(\Omega)}$$

$$= (S_h\tilde{u}_h - \bar{y}, S_h e_h)_{L^2(\Omega)} - \|S_h e_h\|_{L^2(\Omega)}^2 + (\nabla\bar{\varphi}, \nabla S_h e_h)_{L^2(\Omega)}. \qquad (72)$$

By collecting the estimates (71) and (72) we obtain

$$
\nu\|e_h\|_{L^2(\Gamma)}^2 + \|S_h e_h\|_{L^2(\Omega)}^2
$$
$$
\leq \nu(u_h^* - \bar{u}, e_h)_{L^2(\Gamma^c)} + (S_h \tilde{u}_h - \bar{y}, S_h e_h)_{L^2(\Omega)} + (\nabla\bar{\varphi}, \nabla S_h e_h)_{L^2(\Omega)}
$$
$$
\leq \nu\|u_h^* - \bar{u}\|_{L^2(\Gamma^c)}\|e_h\|_{L^2(\Gamma)} + \|S_h \tilde{u}_h - \bar{y}\|_{L^2(\Omega)}\|S_h e_h\|_{L^2(\Omega)}
$$
$$
+ \sup_{\psi_h \in V_h} \frac{\left|(\nabla\bar{\varphi}, \nabla S_h \psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}}\|e_h\|_{L^2(\Gamma)}.
$$

From the Young inequality we can deduce

$$
\|e_h\|_{L^2(\Gamma)} + \|S_h e_h\|_{L^2(\Omega)}
$$
$$
\leq c\left(\|u_h^* - \bar{u}\|_{L^2(\Gamma^c)} + \|S_h \tilde{u}_h - \bar{y}\|_{L^2(\Omega)} + \sup_{\psi_h \in V_h} \frac{\left|(\nabla\bar{\varphi}, \nabla S_h \psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}}\right).
$$
$$
(73)
$$

Finally, the assertion is a consequence from (69), (70) and (73). $\qquad\square$

Finally, by observing that

$$
\|\bar{u} - \bar{u}_h\|_{L^2(\tilde{\Gamma})} \leq ch^{1/2}|\log h|^{1/4}
$$

according to Lemma 5.10 and the uniform boundedness of $\bar{u}$ in $L^\infty(\Gamma)$, we deduce the desired result of Theorem 5.5 by combining Theorem 5.12 and Lemmata 5.6 and 5.8.

6. **Numerical experiments.** The experiments have been performed with Matlab R2015a on an Intel[(R)] Core[(TM)] i7 CPU 870 @2.93 GHz with 16GB RAM on Windows 7 64 bits. All the scripts and functions have been programmed by us.

To build an example with exactly known solution $\bar{u}$, we just define $\bar{\varphi} \in H_0^1(\Omega)$ and compute $\bar{u} = \text{Proj}_{[a,b]}\left(\frac{1}{\nu}\partial_n\bar{\varphi}\right)$, $\bar{y} \in H^1(\Omega)$ such that $-\Delta\bar{y} = 0$ in Ω, $\bar{y} = \bar{u}$ on Γ and $y_\Omega = \bar{y} + \Delta\bar{\varphi}$. In general, it is not possible to compute $\bar{y}$ exactly, so we will use its finite element approximation on the current mesh to compute an approximation of y_Ω.

Since the aim of the experiment is to measure the order of convergence of the $L^2(\Gamma)$ error in the control variable, we have solved the problems in two quasi-uniform families of J nested meshes obtained by diadic refinement from a rough initial mesh. One of them is built such that it does not have the superconvergence property (see Figure 3), while the other is obtained using regular refinement, which results in a $O(h^2)$-irregular family which has the superconvergence property (see Figure 4). The finest mesh has between 1 million and 3.15 million nodes, depending on the geometry of the domain. Notice that these fine meshes induce boundary meshes that only have between 4 thousand and 7 thousand nodes only. To solve the optimization problem, we have used a semismooth Newton method; see [16] for the details.

In the examples where the optimal control is continuous, we measure the error at the mesh at level $j = 1, \dots, J$ as

$$
e_j = \|\bar{u}_{h_j} - I_{h_j}\bar{u}\|_{L^2(\Gamma)}
$$

where $\bar{u}_{h_j}$ is the solution of (P_{h_j}) and $I_{h_j} : C(\Gamma) \to U_{h_j}$ is the nodal Lagrange interpolation operator. If the exact solution is singular at the point $x_0 = (0,0)$, we use standard quasi-interpolation.
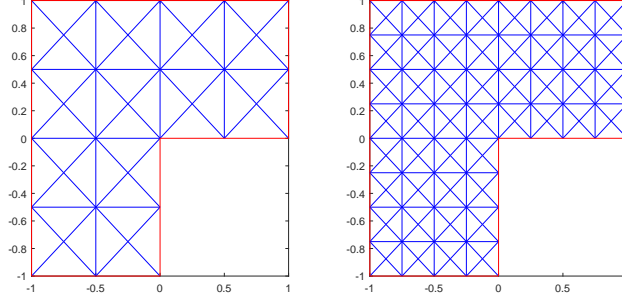
FIGURE 3. Family of quasi-uniform meshes which is not $O(h^2)$-irregular
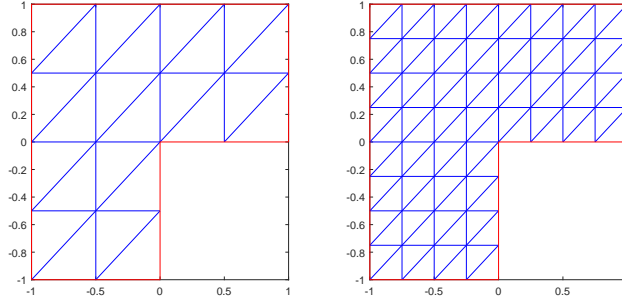
FIGURE 4. Family of quasi-uniform $O(h^2)$-irregular meshes

Since we are using a dyadic refinement strategy, we have that $h_{j+1} = h_j/2$ and we can measure the Experimental Order of Convergence at level $j = 2, \ldots, J$ as

$$EOC_j = \log_2 e_{j-1} - \log_2 e_j.$$

It is to be expected that EOC_j converges to the Theoretical Order of Convergence (TOC) as $j \to \infty$, so for every problem we report on $EOC := EOC_J$ and compare with the corresponding TOC.

Let (ρ, θ) denote the usual polar coordinates in $\mathbb{R}^2$ and define Ω as the interior of the convex hull of the set of points $\{(0,0), (1,0), (\cos(\omega_1), \sin(\omega_1))\}$ if $\pi/3 \leq \omega_1 \leq \pi/2$ and $\Omega = \{(x_1, x_2) \in (-1,1) \times (-1,1) : 0 < \theta < \omega_1\}$ for $\pi/2 < \omega_1 < 2\pi$. We will consider the following cases

1. $\bar{\varphi} = r^\lambda \sin(\lambda\theta)(\sin(\omega_1)(x_1 - 1) + (1 - \cos(\omega_1))x_2)$ if $\omega_1 \leq \pi/2$,
2. $\bar{\varphi} = r^\lambda \sin(\lambda\theta)(1 - x_1)(1 - x_2)$ if $\pi/2 < \omega_1 \leq 3\pi/4$,
3. $\bar{\varphi} = r^\lambda \sin(\lambda\theta)(1 - x_1^2)(1 - x_2)$ if $3\pi/4 < \omega_1 \leq 5\pi/4$,
4. $\bar{\varphi} = r^\lambda \sin(\lambda\theta)(1 - x_1^2)(1 - x_2^2)$ if $5\pi/4 < \omega_1 < 2\pi$,

where we have tested the value $\lambda = \lambda_1$ for $\pi/3 \leq \omega_1 < 2\pi$, and the worst-case $\lambda = 2\lambda_1$ for $\pi < \omega_1 < 2\pi$. Straightforward calculations show that $\Delta\bar{\varphi} \in H^t(\Omega) \cap W^{1,p^*}(\Omega)$ for all $t < \lambda - 1$ and some $p^* > 2$. Also $\partial_\nu\bar{\varphi} \in H^s(\Gamma)$ for all $s < \lambda - 1/2$. Hence, for an unconstrained problem $\bar{u} \in H^s(\Gamma)$ for all $s < \lambda - 1/2$, which implies that $\bar{y} \in H^t(\Omega)$ for all $t < \lambda_1$ and therefore $y_\Omega = \bar{y} + \Delta\bar{\varphi} \in H^t(\Omega)$ for all $t < \lambda_1 - 1$.

If the problem is constrained, then $\bar{u} \in L^\infty(\Gamma)$ and therefore $\bar{y} \in W^{1,p^*}(\Omega)$ for all $2 < p^* < p_D$ and $y_\Omega \in W^{1,p^*}(\Omega)$ for some $p^* > 2$.

Notice that for the case $5\pi/4 < \omega_1 < 3\pi/2$ we have that $\omega_6 = 2\pi - \omega_1 \in (\pi/2, \pi)$ and for the case $7\pi/4 < \omega_1 < 2\pi$ we have $\omega_7 = 5\pi/2 - \omega_1 \in (\pi/2, \pi)$, so when we choose $\lambda = \lambda_1$ in the definition of $\bar{\varphi}$ and solve a constrained problem, the leading singular exponent to be taken into account should be, respectively, λ_6 or λ_7. Nevertheless, the exact adjoint state has been chosen in such a way that in the first case $c_{6,m} = 0$ and in the second case $c_{7,m} = 0$ for $m = 1, 2, 3$, so for this example we need not take this into account. For instance, for $\omega = 11\pi/8$, $\omega_6 = 5\pi/8 > \pi/2$, and hence $\Lambda = 1.6$. So the observed order the convergence for a superconvergent mesh predicted by Theorem 5.3 is (almost) 1.1. Nevertheless, the observed order of convergence is 1.5. There is not an unexplained superconvergence phenomenon. It is simply that in this very example, $c_{1,6} = 0$, so a careful rewriting of the proofs for this precise case, would show order almost $3/2$.

We fix $\nu = 1$. For constrained problems, we will consider $a = -1/\lambda_1$ and $b = 1$. We choose a such that the asymptotic behavior of the error shows up for the mesh sizes used. If $|a|$ were too big, the problem would behave like an unconstrained one for our meshes; on the other hand, were $|a|$ too small, we would be approximating an optimal control very similar to a constant and the experimental orders of convergence would be too high for our meshes.

Graphs with the experimental results can be found in figures 5 and 6. In all the graphs, the horizontal axis represents the biggest angle of the domain, while the vertical axis shows the order of convergence. Theoretical results have been plotted with lines (green solid for superconvergent meshes and blue dashed for non-superconvergent meshes). Experimental results have been plotted with dots (green circles for superconvergent meshes and blue asterisks for non-superconvergence). As at the end of Section 1, we will denote $\mathcal{O} = 1$ in general and $\mathcal{O} = 3/2$ for superconvergent meshes, and neglect the arbitrary small ε.

In Figure 5 we have the results for general unconstrained problems. The theoretical order of convergence is $\min\{\mathcal{O}, \lambda - 1/2\}$, cf. Theorem 4.1.

In Figure 6 we have the results for constrained problems The order of convergence will be $\min\{\mathcal{O}, \Lambda - 1/2, 2\lambda\}$ cf. Theorem 5.3. For convex domains, we have $\Lambda = \lambda$, so the results are like the ones for unconstrained problems. In the left picture, we have the data of the experiment in the generic case, where, for the nonconvex domains studied, $\Lambda = \min\{\lambda_j : \lambda_j > 1\} = 2$. In the right picture, we have the data for the worst-case experiment with a skew-symmetric solution. In this case, $\Lambda = 2\lambda$.

It is remarkable that experimental results are quite in agreement with theoretical estimates.

REFERENCES

[1] T. Apel, M. Mateos, J. Pfefferer and A. Rösch, On the regularity of the solutions of Dirichlet optimal control problems in polygonal domains, *SIAM J. Control Optim.*, **53** (2015), 3620–3641.

[2] T. Apel, S. Nicaise and J. Pfefferer, Discretization of the Poisson equation with non-smooth data and emphasis on non-convex domains, *Numer. Methods Partial Differential Equations*, **32** (2016), 1433–1454.

[3] T. Apel, J. Pfefferer and A. Rösch, Finite element error estimates on the boundary with application to optimal control, *Mathematics of Computation*, **84** (2015), 33–70.

[4] C. Bacuta, J. Bramble and J. Xu, Regularity estimates for elliptic boundary value problems in Besov spaces, *Mathematics of Computation*, **72** (2003), 1577–1595.
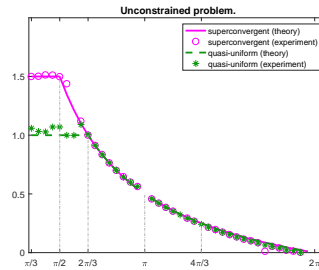
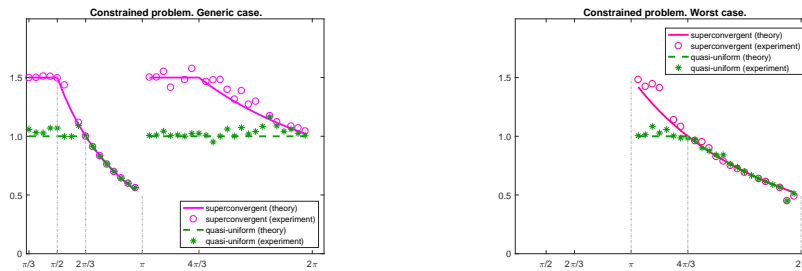FIGURE 5. Unconstrained problems. Experimental orders of convergence vs biggest angle.

FIGURE 6. Constrained problems. Experimental orders of convergence vs biggest angle. Left: generic case. Right: worst case.

[5] R. Bank and J. Xu, Asymptotically exact a posteriori error estimators, part I: Grids with superconvergence, *SIAM Journal on Numerical Analysis*, **41** (2003), 2294–2312.

[6] S. Bartels, C. Carstensen and G. Dolzmann, Inhomogeneous Dirichlet conditions in a priori and a posteriori finite element error analysis, *Numerische Mathematik*, **99** (2004), 1–24.

[7] M. Berggren, Approximations of very weak solutions to boundary-value problems, *SIAM J. Numer. Anal.*, **42** (2004), 860–877 (electronic).

[8] E. Casas and J.-P. Raymond, Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations, *SIAM J. Control Optim.*, **45** (2006), 1586–1611 (electronic).

[9] E. Casas and F. Tröltzsch, Error estimates for linear-quadratic elliptic control problems, in *Analysis and optimization of differential systems*, Springer, 2003, 89–100.

[10] P. Ciarlet, Basic error estimates for elliptic problems, in *Handbook of Numerical Analysis* (eds. P. Ciarlet and J. Lions), vol. II. Finite Element Methods (Part 1), North-Holland, 1991, 17–352.

[11] M. Costabel, Boundary integral operators on Lipschitz domains: Elementary results, *SIAM J. Math. Anal.*, **19** (1988), 613–626.

[12] K. Deckelnick, A. Günther and M. Hinze, Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains, *SIAM J. Control Optim.*, **48** (2009), 2798–2819.

[13] A. Demlow, D. Leykekhman, A. Schatz and L. Wahlbin, Best approximation property in the W^1_∞ norm for finite element methods on graded meshes, *Mathematics of Computation*, **81** (2012), 743–764.

[14] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, vol. 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[15] T. Horger, J. Melenk and B. Wohlmuth, On optimal L^2- and surface flux convergence in FEM, *Computing and Visualization in Science*, **16** (2013), 231–246.

[16] M. Mateos, Optimization methods for Dirichlet control problems, *to appear in Optimization*, https://arxiv.org/abs/1701.07619.

[17] M. Mateos and I. Neitzel, Dirichlet control of elliptic state constrained problems, *Comput. Optim. Appl.*, **63** (2016), 825–853.

[18] S. May, R. Rannacher and B. Vexler, Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems, *SIAM Journal on Control and Optimization*, **51** (2013), 2585–2611.

[19] J. Melenk and B. Wohlmuth, Quasi-optimal approximation of surface based lagrange multipliers in finite element methods, *SIAM Journal on Numerical Analysis*, **50** (2012), 2064–2087.

[20] S. Nazarov and B. A. Plamenevsky, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, vol. 13 of De Gruyter Expositions in Mathematics, Walter de Gruyter & Co., Berlin, 1994.

[21] J. Nečas, *Direct Methods in the Theory of Elliptic Equations*, Corrected 2nd edition, Monographs and Studies in Mathematics, Springer Berlin Heidelberg, 2012.

[22] J. Pfefferer, *Numerical analysis for elliptic Neumann boundary control problems on polygonal domains*, PhD thesis, Universität der Bundeswehr München, 2014, `http://athene.bibl.unibw-muenchen.de:8081/node?id=92055`.

[23] R. Rannacher and R. Scott, Some optimal error estimates for piecewise linear finite element approximations, *Mathematics of Computation*, **38** (1982), 437–445.

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