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α'-corrected black holes in String Theory

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Abstract: We consider the well-known solution of the Heterotic Superstring effective action to zeroth order in α' that describes the intersection of a fundamental string with momentum and a solitonic 5-brane and which gives a 3-charge, static, extremal, supersymmetric black hole in 5 dimensions upon dimensional reduction on T^5. We compute explicitly the first-order in α' corrections to this solution, including SU(2) Yang-Mills fields which can be used to cancel some of these corrections and we study the main properties of this α'-corrected solution: supersymmetry, values of the near-horizon and asymptotic charges, behavior under α'-corrected T-duality, value of the entropy (using Wald formula directly in 10 dimensions), existence of small black holes etc. The value obtained for the entropy agrees, within the limits of approximation, with that obtained by microscopic methods. The α' corrections coming from Wald's formula prove crucial for this result.

Keywords: Black Holes in String Theory, Black Holes, Superstrings and Heterotic Strings

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Contents

1 Introduction

Ever since Strominger and Vafa's computation of the microscopic entropy of an extremal, static, 3-charge black hole in 5 dimensions [1], showing perfect agreement at first order with the macroscopic (Bekenstein-Hawking) entropy, there has been a keen interest in going beyond this approximation both at the microscopic and macroscopic levels.

Going beyond the first approximation at the macroscopic level involves considering corrections to the superstring field theory effective action and finding solutions of the corresponding equations of motion valid to the required approximation level that describe black holes. Then one needs to use an entropy formula such as Wald's [2, 3] to take into account the corrections to the action and not just the corrected geometry of the solution.

Independently of their origin (string or worldsheet loops) the corrections to the superstring effective action are terms of higher order in the curvatures and take a very complicated form, specially after compactification. Thus, no successful attempts to solving the corrected equations of motion for black holes have been made so far and researchers in this field have adopted different strategies to simplify the problem: either considering only a number of tractable corrections (the Gauss-Bonnet term is one of them) which may appear integrated in the structure (the prepotential) of an otherwise normal, quadratic, $\mathcal{N} = 2, d = 4$ supergravity (see, e.g. ref. [4] and references therein) or by dealing only with the near-horizon solution through different approaches (see e.g. ref. [5] and references therein).

In both cases it is argued that the most important corrections are being captured, basically because the expected result is found, but a definite proof is not available. Dealing with near-horizon geometries, for instance, leads to the problem of finding the total, asymptotic charges of the black holes which occur in the mass formula and some of the corrections to the entropy are attributed to the difference between near-horizon and total charges which, actually, are not known. Furthermore, the calculation of the entropy is also affected by the lack of knowledge of the complete action, even if the near-horizon geometry is known (by hypothesis).

The fact that, in general, the microscopic entropies are reproduced by these methods can only be regarded as circumstantial evidence of their validity. Only the explicit knowledge of the complete (near-horizon to infinity) α'-corrected black hole solutions and the subsequent calculation of the entropy using the full action can clarify the situation.

In this paper we carry out this program for the same 3-charge 5-dimensional black hole considered by Strominger and Vafa in the context of the Heterotic Superstring effective action, to first order in α': we find the explicit α' corrections to all the fields of the solution and then we apply Wald's formula to the complete action obtaining an unambiguous answer that reproduces the microscopic result found in ref. [6]. As we will show, this is possible because we carry out all the calculations directly in 10 dimensions and, for these black-hole solutions all the α' corrections are the Laplacian of a function which provides the correction to the harmonic functions of the zeroth-order solution.

We have found it convenient to add a SU(2) instanton field to Strominger and Vafa's solution because, as we will see, it can be used at pleasure to make arbitrarily small or cancel identically many of the α' corrections. This cancellation takes place not just at the level of the field strengths and curvatures, but also at the level of the Chern-Simons term via a mechanism that we will explain in full detail in a coming publication [7].

Since the corrections associated to the gauge fields have the same form as those associated to the curvature of the torsionful spin connection, it also helps us to better understand the latter and the nature of the so-called *symmetric 5-brane*, found in ref. [8] which is known to be an exact solution of the Heterotic Superstring effective action to all orders in α'.

This paper is organized as follows: in section 2 we review the Heterotic Superstring effective action, its fermionic supersymmetry transformations and its equations of motion to $\mathcal{O}(\alpha')$. Since most of this work will be carried out in 10-dimensional language, this section sets the basis and the conventions for the rest of the paper. In section 3 we propose a 10-dimensional ansatz for the α'-corrected solution that reduces to the Strominger and Vafa's 3-charge black hole when $\alpha' = 0$, we show that it preserves 4 out of 16 supercherges (section 3.1), plug it into the equations of motion of the previous section, and solve for the undetermined functions. In section 4 we start the study of the α'-corrected solution by computing the numbers of branes that source the solution and trying to understand their relation with the total, asymptotic charges of the fields. In order to gain a better understanding of this point, in section 5 we explore the behavior under T-duality of this solution using the α'-corrected Buscher T-duality rules proposed in ref. [9]. Both the solution and the T-duality rules pass the test.[1] In section 6 we study the α'-corrected geometry of the 5-dimensional black hole that one obtains by compactification of the solution on a T^5. Finding the form of all the 5-dimensional fields is very complicated (it requires performing the compactification of the corrected action), except for the metric and the dilaton, which are the 5-dimensional fields that interest us the most. This allows us to find under which conditions there is a regular horizon and compute the area of the horizon (the entropy of the zeroth-order solution) and the mass of the solution. Then, in section 7 we compute the corrections to the entropy using Wald's formula in 10-dimensional form. We find two possible corrections to first-order in α', one of which vanishes identically due to the very special properties of the 10-dimensional near-horizon geometry [11]. The α'-corrected entropy reproduces the expected result once the difference in conventions have been taken into account. In section 8 we study the issue of the existence of *small black holes* with classical vanishing area. Finally, in section 9 we study the limits under which the solution can be considered a good first-order in α' approximation to an exact solution of the full Heterotic Superstring effective action. Section 10 contains our conclusions.

2 The Heterotic Superstring effective action to $\mathcal{O}(\alpha')$

The Heterotic Superstring effective action to $\mathcal{O}(\alpha')$ can be written in the string frame in the following concise form [12]:[2]

$$S = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|g|}\, e^{-2\phi} \left\{ R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!} H^2 - \frac{1}{2} T^{(0)} \right\} . \tag{2.1}$$

Let us now review the definition of the different terms that appear in it. First of all, ϕ is the dilaton field and the vacuum expected value of e^{ϕ} is the Heterotic Superstring coupling constant g_s. The 10-dimensional Newton constant $G_N^{(10)}$ is given in terms the string length ℓ_s (with $\alpha' = \ell_s^2$) and g_s by

$$G_N^{(10)} = 8\pi^6 g_s^2 \ell_s^8 . \tag{2.2}$$

R is the Ricci scalar of the string-frame metric $g_{\mu\nu}$. $T^{(0)}$ is one of the three "T-tensors" associated to α' corrections and which are defined as

$$\begin{aligned} T^{(4)} &\equiv 6\alpha' \left[F^A \wedge F^A + R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a \right] , \\ T^{(2)}{}_{\mu\nu} &\equiv 2\alpha' \left[F^A{}_{\mu\rho} F^A{}_\nu{}^\rho + R_{(-)\,\mu\rho}{}^a{}_b R_{(-)\,\nu}{}^{\rho\,b}{}_a \right] , \\ T^{(0)} &\equiv T^{(2)\,\mu}{}_\mu . \end{aligned} \tag{2.3}$$

[1] In ref. [10], essentially the same α'-corrected T-duality rules have been used to show the invariance of the temperature and entropy of the BTZ black hole in a simplified model.

[2] We follow the conventions of ref. [13] for the spin connection and curvature and for the gamma matrices. See also ref. [14].

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In these definitions, $R_{(-)}{}^a{}_b$ is one of the two Lorenz curvature 2-forms $R_{(\pm)}{}^a{}_b$ of the two *torsionful spin connection 1-forms* $\Omega_{(\pm)}{}^a{}_b$ that can be constructed by combining the Levi-Civita spin connection ω^{ab} 1-form with a torsion piece proportional to the Kalb-Ramond field strength H. F^A is the SU(2) Yang-Mills field strength and H is the Kalb-Ramond field strength 3-form. All these objects are defined by

$$\Omega_{(\pm)}{}^a{}_b = \omega^a{}_b \pm \frac{1}{2} H_\mu{}^a{}_b dx^\mu \,, \tag{2.4}$$

$$R_{(\pm)}{}^a{}_b = d\Omega_{(\pm)}{}^a{}_b - \Omega_{(\pm)}{}^a{}_c \wedge \Omega_{(\pm)}{}^c{}_b \,, \tag{2.5}$$

$$F^A = dA^A + \frac{1}{2}\epsilon^{ABC} A^B \wedge A^C \,, \tag{2.6}$$

$$H = dB + 2\alpha' \left(\omega^{\rm YM} + \omega^{\rm L}_{(-)}\right) . \tag{2.7}$$

In the definition of H, $\omega^{\rm YM}$ and $\omega^{\rm L}_{(-)}$ are, respectively, the Yang-Mills and Lorentz Chern-Simons terms

$$\omega^{\rm YM} = dA^A \wedge A^A + \frac{1}{3}\epsilon^{ABC} A^A \wedge A^B \wedge A^C \,, \tag{2.8}$$

$$\omega^{\rm L}_{(\pm)} = d\Omega_{(\pm)}{}^a{}_b \wedge \Omega_{(\pm)}{}^b{}_a - \frac{2}{3}\Omega_{(\pm)}{}^a{}_b \wedge \Omega_{(\pm)}{}^b{}_c \wedge \Omega_{(\pm)}{}^c{}_a \,. \tag{2.9}$$

Then, the Bianchi identity of H is

$$dH - \frac{1}{3}T^{(4)} = 0 \,. \tag{2.10}$$

The above action contains an infinite number of implicit α' corrections which arise due to the recursive way in which H is defined: H depends on the Lorentz Chern-Simons form of $\omega^L_{(-)}$, which depends on $\Omega_{(-)}$, which, in its turn, is defined in terms of H. At the order at which we are working, it is enough to keep in the definitions of $\Omega_{(\pm)}$ only the terms of zeroth order in α', that is

$$\Omega_{(\pm)}{}^a{}_b = \omega^a{}_b \pm \frac{1}{2} H^{(0)a}_\mu{}_b dx^\mu \,, \quad \text{where} \quad H^{(0)} \equiv dB \,. \tag{2.11}$$

Furthermore we will ignore all the α'^2 terms in the action.

The equations of motion that follow from this action are very complicated and, in order to deal with them, we proceed as in section 3 of ref. [15]: we separate the variations with respect to each field $(g_{\mu\nu}, B_{\mu\nu}, \phi, A^A_\mu)$ into those corresponding to the *explicit* occurrences of the fields in the action (i.e. when they do not appear in $\Omega_{(-)}{}^a{}_b$) and those corresponding to *implicit* occurrences via $\Omega_{(-)}{}^a{}_b$:

$$\begin{aligned}
\delta S &= \frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta S}{\delta B_{\mu\nu}}\delta B_{\mu\nu} + \frac{\delta S}{\delta A^A{}_\mu}\delta A^A{}_\mu + \frac{\delta S}{\delta \phi}\delta\phi \\
&= \left.\frac{\delta S}{\delta g_{\mu\nu}}\right|_{\rm exp.}\delta g_{\mu\nu} + \left.\frac{\delta S}{\delta B_{\mu\nu}}\right|_{\rm exp.}\delta B_{\mu\nu} + \left.\frac{\delta S}{\delta A^A{}_\mu}\right|_{\rm exp.}\delta A^A{}_\mu + \frac{\delta S}{\delta \phi}\delta\phi \\
&\quad + \frac{\delta S}{\delta \Omega_{(-)}{}^a{}_b}\left(\frac{\delta \Omega_{(-)}{}^a{}_b}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta \Omega_{(-)}{}^a{}_b}{\delta B_{\mu\nu}}\delta B_{\mu\nu} + \frac{\delta \Omega_{(-)}{}^a{}_b}{\delta A^A{}_\mu}\delta A^A{}_\mu\right) .
\end{aligned} \tag{2.12}$$

Written in this way, we can then make use of the lemma proven in section 3 of ref. [12]: $\delta S/\delta\Omega_{(-)}{}^{a}{}_{b}$ is proportional to α' and to the zeroth-order equations of motion of $g_{\mu\nu}, B_{\mu\nu}$ and ϕ plus terms of higher order in α'. Thus, for any solution of the zeroth-order equations which is exact or up to terms of order α', these terms are, at least, of order α'^2 and can be safely ignored for our purposes.

The variations with respect to the explicit occurrences of the fields are, after some manipulations

$$\begin{aligned} R_{\mu\nu} - 2\nabla_\mu\partial_\nu\phi + \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} - T^{(2)}{}_{\mu\nu} &= 0\,, &(2.13)\\ (\partial\phi)^2 - \frac{1}{2}\nabla^2\phi - \frac{1}{4\cdot 3!}H^2 + \frac{1}{8}T^{(0)} &= 0\,, &(2.14)\\ d\left(e^{-2\phi}\star H\right) &= 0\,, &(2.15)\\ \alpha' e^{2\phi}\mathfrak{D}_{(+)}\left(e^{-2\phi}\star F^A\right) &= 0\,, &(2.16)\end{aligned}$$

where $\mathfrak{D}_{(+)}$ is the exterior derivative covariant with respect to the SU(2) group and with respect to the torsionful connection $\Omega_{(+)}$, that is

$$e^{2\phi}d\left(e^{-2\phi}\star F^A\right) + \epsilon^{ABC}A^B\wedge\star F^C + \star H\wedge F^A = 0\,. \tag{2.17}$$

The three non-trivial zeroth-order equations can be obtained from these by setting $\alpha' = 0$. This eliminates the Yang-Mills fields, the T-tensors and the Chern-Simons terms in H.

We are also going to need the supersymmetry transformation laws of the gravitino ψ_μ, dilatino λ and gaugini χ^A for vanishing fermions, to find the unbroken supersymmetries of the field configurations under study. These are given by

$$\begin{aligned} \delta_\epsilon\psi_\mu &= \nabla^{(+)}_\mu\epsilon \equiv \left(\partial_\mu - \frac{1}{4}\not{\Omega}_{(+)\,\mu}\right)\epsilon\,, &(2.18)\\ \delta_\epsilon\lambda &= \left(\not{\partial}\phi - \frac{1}{12}\not{H}\right)\epsilon\,, &(2.19)\\ \alpha'\delta_\epsilon\chi^A &= -\frac{1}{4}\alpha'\not{F}^A\epsilon\,. &(2.20)\end{aligned}$$

In these expressions H includes the Chern-Simons terms, which provide the first α' corrections.

3 α' corrections to the $d = 10$ Heterotic Superstring background

We are interested in the following 10-dimensional field configuration

$$\begin{aligned} ds^2 &= \frac{2}{\mathcal{Z}_-}du\left(dv - \frac{1}{2}\mathcal{Z}_+du\right) - \mathcal{Z}_0(d\rho^2 + \rho^2 d\Omega^2_{(3)}) - dy^i dy^i\,, \qquad i = 1,\ldots,4\,,\\ H &= d\mathcal{Z}_-^{-1}\wedge du\wedge dv - \frac{\rho^3\mathcal{Z}_0'}{8}\sin\theta d\theta\wedge d\psi\wedge d\phi\,,\\ A^A &= -\frac{\rho^2}{(\kappa^2+\rho^2)}v^A_L\,,\\ e^{-2\phi} &= e^{-2\phi_\infty}\frac{\mathcal{Z}_-}{\mathcal{Z}_0}\,, \end{aligned} \tag{3.1}$$

where the functions $\mathcal{Z}_{+,-,0}$ are assumed to be of the form

$$\mathcal{Z}_0 = 1 + \frac{\mathcal{Q}_0}{\rho^2} + \alpha' f_0(\rho)\,, \qquad\qquad \mathcal{Z}_\pm = 1 + \frac{\mathcal{Q}_\pm}{\rho^2} + \alpha' f_\pm(\rho)\,, \tag{3.2}$$

where, in their turn, $f_{\pm,0}$ are functions of ρ to be determined. Observe that all the functions in this ansatz depend on the radial coordinate ρ of a $\mathbb{R}^4$ space, which is adequate for single, static, branes and black holes. The connections and curvatures for this ansatz are computed in appendix A in a slightly more general form, using Cartesian coordinates x^m with $x^m x^m = \rho^2$.

When the undetermined functions $f_{+,-,0}$ and the SU(2) gauge field are set to zero, this field configuration is a well known 1/4 supersymmetric solution of the zeroth-order equations of the Heterotic Superstring effective action [16, 17] describing an intersection or superposition of

1. *Solitonic (S)* or *Neveu-Schwarz (NS)* 5-branes [8, 18],[3] lying in the directions u, v, $y^1, \cdots, y^4$. The $\mathbb{R}^4$ space parametrized by the coordinates x^m, $m = 1, \cdots, 4$ is their transverse space and it is the common transverse space of the whole solution. They are described by the function $\mathcal{Z}_0$ and their charge is represented by $\mathcal{Q}_0$ at this order.

2. A *fundamental string (F1)* lying in the directions u, v and smeared over the rest of the S5-branes' worldvolume directions y^i, $i = 1, \cdots, 4$. It is described by the function $\mathcal{Z}_-$ and its charge (winding number) is represented by $\mathcal{Q}_-$ at this order.

3. A *gravitational pp-wave (W)* carrying momentum along the v direction (i.e. along the F1). It is described by the function $\mathcal{Z}_-$ and its charge (momentum) is represented by $\mathcal{Q}_+$ at this order. The interchange between $\mathcal{Q}_+$ and $\mathcal{Q}_-$ under T-duality at zeroth order in α' corresponds to the interchange between winding and momentum of the F1.

Upon dimensional reduction over a T^5, this solution gives a static,extremal, 3-charge, 1/2 supersymmetric black hole in $\mathcal{N} = 1, d = 5$ supergravity, which is dual to the one studied by Strominger and Vafa in ref. [1].[4]

In ref. [22] we considered the addition of the above SU(2) gauge field, which is nothing but a BPST instanton, in the context of Heterotic Supergravity. Heterotic Supergravity is just $\mathcal{N} = 1, d = 10$ supergravity coupled to vector supermultiplets and can be obtained from the Heterotic Superstring effective action in eq. (2.1) by eliminating all the terms containing the torsionful spin connection $\Omega_{(-)}$. Thus, it only contains part of the α' terms of the Heterotic Superstring effective action. However, it is exactly invariant under supersymmetry [12], which makes it easier to use supersymmetric solution-generating techniques and, indeed, using these techniques in $\mathcal{N} = 1, d = 5$ gauged supergravity it was shown that with f_0 given by

$$f_0(\rho) = 8\frac{\rho^2 + 2\kappa^2}{(\rho^2 + \kappa^2)^2}\,, \tag{3.3}$$

[3] We feel more inclined to use the name S5-branes.
[4] See also refs. [19–21].

and $f_+ = f_- = 0$ the above field configuration is an exact supersymmetric solution which, upon dimensional reduction over a T^5 gives a static, extremal, 3-charge, 1/2 supersymmetric black hole in $\mathcal{N} = 1, d = 5$ supergravity with non-Abelian hair [22–25].

To be more precise, $f_0(\rho)$ is defined up to an arbitrary harmonic function. In eq. (3.3) the harmonic function has been chosen so as to make $f_0(\rho)$ regular at $\rho = 0$ while keeping the normalization of $\mathcal{Z}_0$ at infinity. We will always use the same convention to choose the arbitrary harmonic functions that can be added to $f_{0,+,-}(\rho)$. With this convention, the only $1/\rho^2$ pole in $\rho \to 0$ limit of the α'-corrected $\mathcal{Z}_0$ is the original term $\mathcal{Q}_0/\rho^2$ where $\mathcal{Q}_0$ is proportional to the number of S5-branes [8, 18].

Further shifts by harmonic functions can always be absorbed into a redefinition of $\mathcal{Q}_0$ Observe that, in the $\rho \to \infty$ limit, the coefficient of the $1/\rho^2$ term is not $\mathcal{Q}_0$ but $\mathcal{Q}_0 + 8\alpha'$. The difference is due to the contribution of the BPST instanton which sources a "gauge 5-brane" [26, 27] which in its turn increases to the total charge of the NS 6-form $\tilde{B}$ dual to the Kalb-Ramond 2-form B measured at infinity. In this case, the difference between these two quantities, number of S5-branes and total 5-brane charge at infinity, has a simple explanation in terms of a delocalized gauge 5-brane but, as we are going to see, other α' corrections lead to very similar differences between "near-horizon" and "asymptotic" (total) charges which do not have a (known) similar, simple, interpretation.

The fact that the α' corrections associated to the torsionful spin connection $\Omega_{(-)}$ have the same structure as those associated to the gauge fields should not come as a surprise: the theory treats the Yang-Mills and the torsionful spin connection on exactly the same footing [28] and the curvature of the latter occurs as that of another non-Abelian gauge field sourcing the Einstein equations. The main difference is that the torsionful spin connection is not an independent field and, furthermore, its "kinetic term" occurs in the action with the wrong sign.

Thus, on general grounds, one expects additional α' corrections in $f_{+,-,0}(\rho)$ similar to eq. (3.3), with opposite sign and depending on $\mathcal{Q}_+, \mathcal{Q}_-, \mathcal{Q}_0$ instead of κ. These corrections cannot be assigned to something like a "gravitational 5-brane", as far as we know, but they are similarly delocalized and they will generically contribute to the total charges at infinity. This may give rise to the problem of how to count the number of branes through the computation of the charge.

Remarkably enough, when the instanton field is included together with the rest of first-order α' corrections, some of of these contributions to the total charge disappear completely and the total charge at infinity has the same value as the "near-horizon" ($\rho \to 0$) charge, as it happens at zeroth order in α'. Actually, since, according to the previous discussion, the structure of those corrections is the same as that of those associated to the Yang-Mills fields we can cancel them against each other, eliminate completely the first-order α' corrections and (probably, we conjecture) all the higher order corrections. It is likely that the addition of more general gauge fields can be used to solve this problem for all charges and also to, eventually, cancel all the α' corrections [29].

We will discuss this issue at length in section 4.

Right now our goal is to determine the functions $f_{+,-,0}(\rho)$ so that the above field configuration is a solution of the Heterotic Superstring effective action to first order in α'

(i.e. up to terms of $\mathcal{O}(\alpha'^2)$). However, before doing it, we are going to show that these field configurations preserve 1/4 of the supersymmetries for any value of the functions $\mathcal{Z}_+, \mathcal{Z}_-, \mathcal{Z}_0$ and for any Yang-Mills field strength which is self-dual in the 4-dimensional space $\mathbb{R}^4$ transverse to the S5-branes to first order in α'.

3.1 Unbroken supersymmetries of the ansatz

Using the Zehnbein basis and results in appendix A for the torsionful spin connection $\Omega_{(+)}$, the different components of the supersymmetry transformation rules eqs. (2.18)–(2.20) take the following form for our ansatz:

$$\delta_\epsilon \psi_+ = \left[\partial_+ + \frac{1}{4}\frac{\mathcal{Z}_-\partial_m \mathcal{Z}_+}{\mathcal{Z}_0^{1/2}}\Gamma^m\Gamma^+\right]\epsilon\,, \tag{3.4}$$

$$\delta_\epsilon \psi_- = \left[\partial_- + \frac{1}{4}\frac{\partial_m \log \mathcal{Z}_-}{\mathcal{Z}_0^{1/2}}\Gamma^m\Gamma^+\right]\epsilon\,, \tag{3.5}$$

$$\delta_\epsilon \psi_m = \left[\partial_m + \frac{1}{8}\frac{\partial_q \log \mathcal{Z}_0}{\mathcal{Z}_0^{1/2}}(\mathbb{M}^+_{qm})_{np}\Gamma^{np}(1-\tilde{\Gamma})\right]\epsilon\,, \tag{3.6}$$

$$\delta_\epsilon \psi_i = \partial_i \epsilon\,, \tag{3.7}$$

$$\delta_\epsilon \lambda = -\frac{1}{2\mathcal{Z}_0^{1/2}}\Gamma^m\left[\partial_m \log \mathcal{Z}_-\Gamma^-\Gamma^+ - \partial_m \log \mathcal{Z}_0(1-\tilde{\Gamma})\right]\epsilon\,, \tag{3.8}$$

$$\alpha'\delta_\epsilon \chi^A = -\frac{1}{8\mathcal{Z}_0^{1/2}}\alpha' \not{F}^A(1-\tilde{\Gamma})\epsilon\,, \tag{3.9}$$

where $\tilde{\Gamma} \equiv \Gamma^{2345}$ is the chirality matrix in the $\mathbb{R}^4$ space transverse to the S5-branes. All these transformations vanish identically for constant spinors satisfying the constraints

$$\tilde{\Gamma}\epsilon = +\epsilon\,, \qquad \Gamma^+\epsilon = 0\,, \tag{3.10}$$

which reduce the number of independent components to 1/4 of the 16.

3.2 Explicit computation of the α' corrections

We just have to plug the supersymmetric configuration eq. (3.1) in the equations of motion (2.13)–(2.16) as well as in the Bianchi identity eq. (2.10) and try to solve them for $f_{+,-,0}(\rho)$. Our ansatz assumes implicitly that no more components of the metric are necessary to this order and that its structure and symmetries will remain intact. Only the functions associated to the different branes can receive corrections. These assumptions based in our experience with the non-Abelian black hole of ref. [22] will prove correct, as we are going to see.

The terms of order α' in eqs. (2.13)–(2.16) are proportional to the T-tensors defined in eq. (2.3), which were computed for this ansatz in ref. [22]. They are explicitly given by[5]

$$\hat{T}^{(4)} \sim \alpha' \left[\frac{\kappa^4}{(\kappa^2+\rho^2)^4} - \frac{\mathcal{Q}_0^2}{(\mathcal{Q}_0+\rho^2)^4}\right] d\rho\rho^3 \wedge \sin\theta d\theta \wedge d\Psi \wedge d\phi\,, \tag{3.11}$$

$$\hat{T}^{(2)}{}_{uu} = -\alpha' \frac{32\mathcal{Q}_-\mathcal{Q}_+\rho^4\left[\mathcal{Q}_0^2 + \mathcal{Q}_0\left(\mathcal{Q}_- + 3\rho^2\right) + \mathcal{Q}_-^2 + 3\mathcal{Q}_-\rho^2 + 3\rho^4\right]}{\left(\mathcal{Q}_0+\rho^2\right)^4\left(\mathcal{Q}_-+\rho^2\right)^4}\,, \tag{3.12}$$

$$\hat{T}^{(2)}{}_{ij} = \alpha'\delta_{ij}\frac{48\rho^2}{\left(\mathcal{Q}_0+\rho^2\right)^5}\left[\mathcal{Q}_0^2 - \frac{\kappa^4\left(\tilde{\mathcal{Q}}_0+\rho^2\right)^4}{(\kappa^2+\rho^2)^4}\right], \tag{3.13}$$

$$\begin{aligned}\hat{T} = -\alpha' &\frac{192\rho^4}{(\kappa^2+\rho^2)^4\left(\mathcal{Q}_0+\rho^2\right)^6}\left[\kappa^8\mathcal{Q}_0^2 + 4\kappa^6\mathcal{Q}_0^2\rho^2\right.\\ &\left.-\kappa^4\left(\mathcal{Q}_0^4 + 4\mathcal{Q}_0^3\rho^2 + 4\mathcal{Q}_0\rho^6 + \rho^8\right) + 4\kappa^2\mathcal{Q}_0^2\rho^6 + \mathcal{Q}_0^2\rho^8\right]\,.\end{aligned} \tag{3.14}$$

Observe that, while all the scalar invariants that one can construct with these T-tensors, and which occur in the action, depend on the parameters $\mathcal{Q}_0$ and κ^2 only, the component $\hat{T}^{(2)}{}_{uu}$, which occurs in the equations of motion, depends on $\mathcal{Q}_+, \mathcal{Q}_-$ and $\mathcal{Q}_0$ but not on κ^2. $\hat{T}^{(2)}{}_{uu}$ vanishes identically at $\rho = 0$, where we expect the horizon to be, and it also vanishes asymptotically at $\rho \to \infty$, but it is relevant at finite values of ρ. Thus, arguments solely based on the behavior of the scalar invariants as functions of $\mathcal{Q}_0$ and κ^2 miss completely this correction. Furthermore, this correction disappears if one considers near-horizon geometries only.

Let us consider, first, the Yang-Mills fields. It can be seen that, given the structure of the fields in our ansatz, independently of the actual values of the $\mathcal{Z}$-functions, the α'-corrected Yang-Mills equation eq. (2.16) is satisfied automatically provided that F^A is self-dual in the 4-dimensional Euclidean space transverse to the S5-branes, that is, $\star_{(4)}F^A = +F^A$, which is a property of our ansatz.

Next, we consider the Bianchi identity eq. (2.10) for the 3-form H in eq. (3.1). Substituting the ansatz the identity takes the form[6]

$$\begin{aligned}&-\frac{\alpha'}{8\rho^3}\frac{d}{d\rho}\left(\rho^3\frac{df_0}{d\rho}\right) d\rho\rho^3 \wedge \sin\theta d\theta \wedge d\Psi \wedge d\phi = \\ &24\alpha'\left[\frac{\kappa^4}{(\kappa^2+\rho^2)^4} - \frac{\mathcal{Q}_0^2}{(\mathcal{Q}_0+\rho^2)^4}\right] d\rho\rho^3 \wedge \sin\theta d\theta \wedge d\Psi \wedge d\phi + \mathcal{O}(\alpha'^2)\,.\end{aligned} \tag{3.15}$$

This leads to the following equation for f_0

$$\frac{d}{d\rho}\left(\rho^3\frac{df_0}{d\rho}\right) = -192\rho^3\left[\frac{\kappa^4}{(\kappa^2+\rho^2)^4} - \frac{\mathcal{Q}_0^2}{(\mathcal{Q}_0+\rho^2)^4}\right]\,, \tag{3.16}$$

[5] $\hat{T}^{(4)}$ is computed explicitly in appendix A.

[6] We do not simplify the common factors in the left- and right-hand sides.

that can be integrated immediately, giving[7]

$$f_0 = 8\left[\frac{\rho^2+2\kappa^2}{(\rho^2+\kappa^2)^2} - \frac{\rho^2+2\mathcal{Q}_0}{(\rho^2+\mathcal{Q}_0)^2}\right] + \frac{c_0}{\rho^2} + d_0\,. \tag{3.17}$$

Here c_0 and d_0 are integration constants corresponding to the arbitrary shift by a harmonic function of $\mathcal{Z}_0$ discussed at the beginning of this section. There we also convened to choose c_0 so that f_0 has no $1/\rho^2$ poles in the $\rho \to 0$ limit and d_0 so that f_0 vanishes asymptotically to preserve the asymptotic normalization of the full metric. This has already been done in the expression above, which is finite in the $\rho \to 0$ limit and vanishes in the $\rho \to \infty$ limit. Therefore, $c_0 = d_0 = 0$

Observe that, if $\mathcal{Q}_0 = 0$, the second term in eq. (3.17) should not be there at all. However, the above expression gives a spurious $-1/\rho^2$ pole when $\mathcal{Q}_0 = 0$. Thus, we will have to treat the cases $\mathcal{Q}_0 = 0$ and $\mathcal{Q}_0 \neq 0$ independently. The same is also true for the $\kappa = 0$ case, since in this limit the instanton just gives an Abelian-like contribution that can be interpreted as 8 S5-branes.[8] Then we may simply reabsorb these 8 additional S5-branes into $\mathcal{Q}_0$.

The first term in f_0 is just the one in eq. (3.3) and is associated to the $F^A \wedge F^A$ term. The second is associated to the $R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a$ term and has exactly the same structure because, as we said, $\Omega_{(-)}$ behaves exactly as another gauge field. The presence of two terms with the same structure but opposite signs ensures that the coefficient of the $1/\rho^2$ pole in the $\rho \to 0$ limit is the same as the coefficient of the $1/\rho^2$ term in the $\rho \to \infty$ limit: the contribution of the gauge 5-brane to the charge of $\tilde{B}$ is cancelled by another contribution which cannot be assigned to any known brane. Typically, the latter is the only α' correction considered in the literature in the context of black holes, where uysually non-Abelian fields are not introduced.

The presence of two corrections with the same functional form but opposite signs not only suppresses the difference between "near-horizon" and asymptotic, total charge of $\tilde{B}$: setting $\kappa^2 = \mathcal{Q}_0$ the whole first-order α' correction vanishes identically. With this identification between the instanton size parameter and the S5-brane charge, the component of the full solution described by $\mathcal{Z}_0$ is nothing but the so-called *symmetric 5-brane*, found in ref. [8], which is known to be an exact solution of the Heterotic Superstring effective action to all orders in α'. Finding the symmetric 5-brane solution in this form sheds new light on its origin and meaning. Of course, the complete solution has additional fields which give rise to some α' corrections of their own even if $\kappa^2 = \mathcal{Q}_0$, via $\hat{T}^{(2)}{}_{uu}$.

Finally, notice that if $\mathcal{Q}_0 \gg \kappa^2$ the second term is irrelevant compared to the first one, except in the asymptotic limit $\rho \to \infty$, where both are comparable. In ref. [22] this fact was used to argue that the solution of Heterotic Supergravity that includes the instanton suffered only small α' corrections to first order. In section 9 we will study the issue of α' and other corrections from a more general point of view.

[7]This result is obtained in appendix A.1 in a more transparent way. The integrability of this equation is due to a set of very interesting properties of this class of ansatzs that will be explored in more generality in ref. [7].

[8]The Yang-Mills field becomes pure gauge in this limit except at $\rho = 0$.

Next, let us consider the equation of motion of B, eq. (2.15). It yields the following equation for f_-

$$\frac{d}{d\rho}\left(\rho^3\frac{df_-}{d\rho}\right)=0\,, \tag{3.18}$$

which means that $f_-(\rho)$ is just a harmonic function, which we absorb into a redefinition of $\mathcal{Q}_-$ according to our general prescription. Therefore $\mathcal{Z}_-$ does not receive any first-order α' corrections.

Now we can turn our attention to the Einstein equations eq. (2.13). We have checked that with the present configuration all of them are satisfied up to $\mathcal{O}(\alpha'^2)$ except for the uu one, which gives the following equation for f_+:

$$\frac{1}{\rho^3}\frac{d}{d\rho}\left(\rho^3\frac{df_+}{d\rho}\right)=-\frac{128\mathcal{Q}_+\mathcal{Q}_-\left(\mathcal{Q}_0^2+\mathcal{Q}_-^2+3\mathcal{Q}_0\rho^2+3\mathcal{Q}_-\rho^2+3\rho^4+\mathcal{Q}_0\mathcal{Q}_-\right)}{(\mathcal{Q}_0+\rho^2)^3(\mathcal{Q}_-+\rho^2)^3}\,. \tag{3.19}$$

where the right-hand side is proportional to $\hat{T}^{(2)}{}_{uu}$. This equation is solved by

$$f_+(\rho)=-\frac{16\mathcal{Q}_+\mathcal{Q}_-}{\rho^6\mathcal{Z}_0^{(0)}\mathcal{Z}_-^{(0)}}\,, \tag{3.20}$$

up to an arbitrary harmonic function c_+/ρ^2 to be chosen according to our prescription.

In the $\rho\to 0$ limit the above $f_+(\rho)$ diverges as $-16\mathcal{Q}_+\mathcal{Q}_0^{-1}/\rho^2$ if $\mathcal{Q}_0\neq 0$. Then, we choose $c_+=+16\mathcal{Q}_+\mathcal{Q}_0^{-1}$ and we are left with

$$f_+(\rho)=\frac{16\mathcal{Q}_+(\rho^2+\mathcal{Q}_0+\mathcal{Q}_-)}{\mathcal{Q}_0(\rho^2+\mathcal{Q}_0)(\rho^2+\mathcal{Q}_-)}\,, \tag{3.21}$$

which has the same structure as $f_0(\rho)$ without the corrections associated to the instanton.[9] Thus, since in this case there is no contribution to the gauge fields that could cancel this α' correction, the "near-horizon" charge and the total, asymptotic charge associated to $\mathcal{Z}_+$ (total momentum) are different and we are faced with the problem of deciding which of them represents the momentum of the string. We will discuss this issue in section 4.

If $\mathcal{Q}_0=0$, $f_+\sim 1/\rho^4$ when $\rho\to 0$ and there is no need to shift it by a harmonic function.

Finally, one can check that the dilaton equation is satisfied up to $\mathcal{O}(\alpha'^2)$ terms.

Summarizing the results of this section, we have constructed a solution of the Heterotic String effective action to first order in α' of the form given in eq. (3.1) with the $\mathcal{Z}$ functions given, for $\mathcal{Q}_0\neq 0$ by

$$\mathcal{Z}_0=\mathcal{Z}_0^{(0)}+8\alpha'\left[\frac{\rho^2+2\kappa^2}{(\rho^2+\kappa^2)^2}-\frac{\rho^2+2\mathcal{Q}_0}{(\rho^2+\mathcal{Q}_0)^2}\right]+\mathcal{O}(\alpha'^2)\,, \tag{3.22}$$

$$\mathcal{Z}_-=\mathcal{Z}_-^{(0)}+\mathcal{O}(\alpha'^2)\,, \tag{3.23}$$

$$\mathcal{Z}_+=\mathcal{Z}_+^{(0)}+16\alpha'\frac{\mathcal{Q}_+(\rho^2+\mathcal{Q}_0+\mathcal{Q}_-)}{\mathcal{Q}_0(\rho^2+\mathcal{Q}_0)(\rho^2+\mathcal{Q}_-)}+\mathcal{O}(\alpha'^2)\,, \tag{3.24}$$

[9] Up to a factor of 2 it is identical to it if we set $\mathcal{Q}_-=\mathcal{Q}_0$.

and for $\mathcal{Q}_0 = 0$ by

$$\begin{aligned}
\mathcal{Z}_0 &= 1 + 8\alpha' \frac{\rho^2 + 2\kappa^2}{(\rho^2+\kappa^2)^2} + \mathcal{O}(\alpha'^2)\,, && (3.25)\\
\mathcal{Z}_- &= \mathcal{Z}_-^{(0)} + \mathcal{O}(\alpha'^2)\,, && (3.26)\\
\mathcal{Z}_+ &= \mathcal{Z}_+^{(0)} - 16\alpha' \frac{\mathcal{Q}_+\mathcal{Q}_-}{\rho^4(\rho^2+\mathcal{Q}_-)} + \mathcal{O}(\alpha'^2)\,, && (3.27)
\end{aligned}$$

where $\mathcal{Z}_0^{(0)}, \mathcal{Z}_\pm^{(0)}$ are the pieces of the functions $\mathcal{Z}_0^{(0)}, \mathcal{Z}_\pm^{(0)}$ of zeroth order in α', namely the harmonic functions in $\mathbb{E}^4$

$$\mathcal{Z}_{0,+,-}^{(0)} = 1 + \frac{\mathcal{Q}_{0,+,-}}{\rho^2}\,. \tag{3.28}$$

We would like to stress at this point that the $\mathcal{O}(\alpha'^2)$ terms that we have ignored in the equations of motion derived from the action eq. (2.1) are proportional to products of the Chern-Simons 3-forms that occur in in H. We will discuss in detail in section 9 when it is justified to disregard these terms as well as the rest of the terms of higher-order in α' and in the string coupling constant that enter in the Heterotic Superstring Effective action so, rather than just a solution to the first-order equations of motion of the Heterotic Superstring Effective action, we can consider that this is a first-order solution of the full effective action with second-order corrections in α' and one- and higher-loop corrections which are negligible when compared with the above solution.

Let us close this section by commenting the relation of these solutions to the ones described in [30], which were also argued to be exact solutions at first order in α'. Those can be obtained from eqs. (3.22) by removing the 1's from the harmonic functions $\mathcal{Z}_{0,+,-}^{(0)}$, but this has the effect of removing as well all the α' corrections except the one due to the SU(2) instanton. The reason is that at zeroth order in α' such solutions are just $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$, for which $\hat{R}_{(-)}$ vanishes identically [11] — see also section 7. Hence, only corrections coming from the Yang-Mills fields appear in that case.

4 The α'-corrected charges

Before doing any explicit calculation, it is good to have a more qualitative discussion on the meaning of the charges that we are going to calculate.

As we have discussed in the previous section, the α' corrections introduce delocalized terms in the the fields which, generically, give contributions to the total charges of the fields computed at spatial infinity. The term in $\mathcal{Z}_0$ associated to the presence of the BPST instanton (let us ignore the second one due to the curvature of the torsionful spin connection) contributes to the total charge at infinity of the NS 6-form $\tilde{B}$ dual to the Kalb-Ramond 2-form B and its contribution, which can be explained in terms of a *gauge 5-brane* [26, 27] is equivalent to that of 8 S5-branes:

$$\mathcal{Z}_0 \overset{\rho\to\infty}{\sim} 1 + (\mathcal{Q}_0 + 8\alpha')/\rho^2 + \cdots \tag{4.1}$$

If we are interested in finding how many S5-branes there are in the background this contribution to the total charge must be taken into account and we could say that $\mathcal{Q}_0 = N_{\mathrm{S5}}\alpha'$.

Alternatively, one can look at the "near-horizon" charge which will be determined by the coefficient of the $1/\rho^2$ pole in the $\rho \to 0$ limit. By convention, this is always the coefficient in $\mathcal{Z}^{(0)}$, $\mathcal{Q}_0$:

$$\mathcal{Z}_0 \overset{\rho\to 0}{\sim} \mathcal{Q}_0/\rho^2 + \cdots \tag{4.2}$$

Now, let us take into account the second term in f_0 associated to the $R_{(-)} \wedge R_{(-)}$ term. This term contributes to the total charge at infinity as -8 S5-branes:

$$\mathcal{Z}_0 \overset{\rho\to\infty}{\sim} 1 + (\mathcal{Q}_0 + 8\alpha' - 8\alpha')/\rho^2 + \cdots \tag{4.3}$$

and, therefore, the total charge and the "near-horizon charge" which is always given by eq. (4.2) are both equal to $\mathcal{Q}_0$ in this case. We do not know of any delocalized extended object to which this negative contribution to the charge can be attributed to but the net effect is that we do not need to worry about the different contributions to this charge.

Now we can try to use more rigorous definitions (which, of course, will give the same result).

In order to compute 5-brane charge we need to use NS 6-form $\tilde{B}$ dual to the Kalb-Ramond 2-form B. The equation of motion of the latter can be written in the form

$$d\left(e^{-2\phi} \star H + \mathcal{O}(\alpha')\right) = 0\,, \tag{4.4}$$

where, according to the discussion in section 2 the $+\mathcal{O}(\alpha')$ terms are related to the zeroth-order equations of motion. Locally, the equation of motion is solved by

$$e^{-2\phi} \star H + \mathcal{O}(\alpha') \equiv d\tilde{B}\,, \quad \Rightarrow \quad H = e^{2\phi} \star \tilde{H}\,, \quad \text{with} \quad \tilde{H} \equiv d\tilde{B} + \mathcal{O}(\alpha')\,. \tag{4.5}$$

The 6-form equation of motion can be obtained from the Bianchi identity of H eq. (2.10)

$$d(e^{2\phi} \star \tilde{H}) - 2\alpha' \left(F^A \wedge F^A + R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a\right) = 0\,, \tag{4.6}$$

and, if we couple the system to N_{S5} solitonic 5-branes lying in the directions $\frac{1}{2}(u+v)$, $y^1, \cdots, y^4$, it takes the form[10]

$$d(\star e^{2\phi} \tilde{H}) - 2\alpha' \left(F^A \wedge F^A + R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a\right) = 4\pi^2 \alpha' N_{S5} \star_{(4)} \delta^{(4)}(\rho)\,. \tag{4.8}$$

This identity means that the expression in the left-hand side is sensitive to the $1/\rho^2$ poles in the $\rho \to 0$ limit and, therefore, the "near-horizon charge" $\mathcal{Q}_0$ essentially counts the number of S5-branes in the background, as we explained before. This can be checked explicitly

[10] Here we have used the normalization of the Heterotic Superstring effective action in eq. (2.1), the normalization of the Wess-Zumino term of the S5-branes $N_{S5} T_{S5}\, g_s^2 \int \phi_* \tilde{B}$ and the values of the 10-dimensional Newton constant eq. (2.2) and the S5-brane tension in terms of the string length $\ell_s^2 = \alpha'$ and the string coupling constant g_s, which are given by

$$T_{S5} = \frac{1}{(2\pi \ell_s)^5 \ell_s g_s^2}\,. \tag{4.7}$$

by using the form eq. (A.27) for the Bianchi identity in the above expression, and the conclusion is that[11]

$$\mathcal{Q}_0 = N_{S5}\ \alpha' \,. \tag{4.9}$$

If, instead of the number of S5-branes, we wanted to calculate the total 5-brane charge at infinity, we should move the α' terms to the right-hand side

$$d(\star e^{2\phi}\tilde{H}) = 4\pi^2\alpha' N_{S5} \star_{(4)} \delta^{(4)}(\rho) - 2\alpha' \left(F^A \wedge F^A + R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a\right) , \tag{4.10}$$

and integrate over the 4-dimensional transverse space to the 5-branes. The total charge would be $(N_{\text{S5}} + 8N_{\text{G5}} - 8N_{\text{U5}})\alpha'$ where N_{G5} is the number of gauge 5-branes and is equal to the instanton number of the gauge field and N_{U5} is the number of "unknown 5-branes" associated to the torsionful spin connection $\Omega_{(-)}$ and is equal to its instanton number too. In our solution $N_{\text{U5}} = 1$, and, is we include the SU(2) instanton, $N_{\text{G5}} = 1$. Then, the total 5-brane charge is, again, given by eq. (4.9).

Finally, observe that, in the end, the α' terms in the Bianchi identity are simply those in eq. (3.15) and, as discussed above, only the $1/\rho^2$ poles in f_0 give contributions to the δ-function.

Let us now move to the fundamental string charge (winding number), described by $\mathcal{Z}_-$ which is not affected by α' corrections. Repeating the discussion at the beginning of this section we would conclude that the "near-horizon charge" and the total charge at infinity should both be equal to $\mathcal{Q}_-$ which, in its turn, should be proportional to the winding number.

In fact, following ref. [22] if we have N_{F1} fundamental strings lying in the direction $\frac{1}{2}(u-v)$ we have

$$T_{F1}N_{F1} = \frac{g_s^2}{16\pi G_N^{(10)}} \int_{V^8} d\left(\star e^{-2\phi}H + \mathcal{O}(\alpha')\right) , \qquad \text{where} \qquad T_{F1} = \frac{1}{2\pi\alpha'} , \tag{4.11}$$

where $\mathcal{O}(\alpha')$ are terms associated to the zeroth-order equations of motion, as we have discussed, and where V^8 is the space transverse to worldsheet parametrized by u and v, whose boundary is the product $\mathrm{T}^4 \times S^3_\infty$. The $\mathcal{O}(\alpha')$ terms do not contribute to this integral for the same reason they do not introduce α'-corrections in $\mathcal{Z}_-$, which remains a harmonic function whose pole is the sole contribution to the above integral (see eq. (3.18)). Therefore, using Stokes' theorem and the value of volume of the T^4, $(2\pi\ell_S)^4$, we get again

$$\mathcal{Q}_- = \ell_s^2 g_s^2 N_{F1} \,. \tag{4.12}$$

Following the same reasoning, the strings' momentum can be found by just looking at the coefficient of the $1/\rho^2$ pole in $\mathcal{Z}_+$ which we have denoted, according to the general convention, by $\mathcal{Q}_+$:

$$\mathcal{Q}_+ = \frac{g_s^2 \ell_s^4}{R_z^2} N_W \,. \tag{4.13}$$

[11]When $\kappa = 0$ there is an additional contribution to the pole equivalent to 8 S5-branes that, as we said before, we will simply absorb into a redefinition of $\mathcal{Q}_0$ so the above identification will always hold.

However, in this case, the total momentum at infinity is different because there is a first-order in α' delocalized contribution in $f_+(\rho)$ which is not cancelled by the Yang-Mills field's contribution:

$$\mathcal{Z}_+ \overset{\rho\to\infty}{\sim} 1 + \mathcal{Q}_+ \left(1 + 16\alpha'/\mathcal{Q}_0\right)/\rho^2 + \cdots \tag{4.14}$$

If $\mathcal{Q}_0$ is small, the difference between the string's momentum, which we have argued should be measure in the near-horizon limit, and the total, asymptotic momentum, which is assumed to be the momentum of the string in some of the literature, can be large and with important physical consequences, as we are going to see in section 8.

5 α'-corrected T-duality

In ref. [22] we arrived to the relation between $\mathcal{Q}_+$ and $N_{\rm W}$ eq. (4.13) via a T-duality transformation of the solution, which is commonly understood to interchange momentum and winding of a fundamental string wrapped on a circle. We can call

$$N'_{\rm F1} = N_{\rm W}\,, \qquad N'_{\rm W} = N_{\rm F1}\,, \tag{5.1}$$

the "microscopic T-duality rules". However, these microscopic T-duality rules come form the study of the Heterotic String spectrum on $\mathbb{M}^{1,8}\times S^1$, in absence of any other background field, but the system under consideration contains a non-perturbative S5-brane wrapped around the T-duality direction and it is conceivable that the string spectrum and the microscopic T-duality rules eq. (5.1), which should be supplemented by

$$g'_s = g_s\ell_s/R_x\,, \qquad R'_x = \ell_s^2/R_x\,, \tag{5.2}$$

suffer α' corrections.

In order to clarify this point we are going to perform a T-duality transformation of the solution in the direction of propagation of the wave $x \equiv \frac{1}{2}(u-v)$ using the α'-corrected Buscher T-duality rules of ref. [9] (for $\mu,\nu \neq \underline{x}$):

$$\begin{aligned}
g'_{\mu\nu} &= g_{\mu\nu} + \left[g_{\underline{xx}}G_{\underline{x}\mu}G_{\underline{x}\nu} - 2G_{\underline{xx}}G_{\underline{x}(\mu}g_{\nu)\underline{x}}\right]/G^2_{\underline{xx}}\,, \\
B'_{\mu\nu} &= B_{\mu\nu} - G_{\underline{x}[\mu}G_{\nu]\underline{x}}/G_{\underline{xx}}\,, \\
g'_{\underline{x}\mu} &= -g_{\underline{x}\mu}/G_{\underline{xx}} + g_{\underline{xx}}G_{\underline{x}\mu}/G^2_{\underline{xx}}\,, \qquad & B'_{\underline{x}\mu} &= -B_{\underline{x}\mu}/G_{\underline{xx}} - G_{\underline{x}\mu}/G_{\underline{xx}}\,, \\
g'_{\underline{xx}} &= g_{\underline{xx}}/G^2_{\underline{xx}}\,, & e^{-2\phi'} &= e^{-2\phi}|G_{\underline{xx}}|\,, \\
A'^A_{\underline{x}} &= -A^A_{\underline{x}}/G_{\underline{xx}}\,, & A'^A_{\mu} &= A^A_{\mu} - A^A_{\underline{x}}G_{\underline{x}\mu}/G_{\underline{xx}}\,,
\end{aligned} \tag{5.3}$$

where $G_{\mu\nu}$ is defined by

$$G_{\mu\nu} \equiv g_{\mu\nu} - B_{\mu\nu} - 2\alpha'\left\{A^A_{\mu}A^A{}_{\nu} + \Omega_{(-)\,\mu}{}^a{}_b\Omega_{(-)\,\nu}{}^b{}_a\right\}\,. \tag{5.4}$$

Notice that these α'-corrected T-duality transformations are only well-defined if $G_{\underline{xx}} \neq 0$. This corresponds to the first-order deformation of the non-vanishing radii condition at zeroth-order, which is $g_{\underline{xx}} \neq 0$. This issue becomes relevant for the exotic solutions presented in section 8, for which it is not possible to apply the transformation.

We only need the components $G_{\underline{xx}}, G_{\mu\underline{x}}, G_{\underline{x}\mu}$. Taking into account that, in terms of the coordinates t, x, x^m, y^i, the metric and the Kalb-Ramond 2-form (given in eq. (A.29)) take the form[12]

$$
\begin{aligned}
ds^2 &= \frac{(2-\mathcal{Z}_+)}{\mathcal{Z}_-}dt^2 - \frac{(2+\mathcal{Z}_+)}{\mathcal{Z}_-}dx^2 - 2\frac{\mathcal{Z}_+}{\mathcal{Z}_-}dtdx - \mathcal{Z}_0 dx^m dx^m - dy^i dy^i\,, &(5.5)\\
B &= -\frac{2}{\mathcal{Z}_-}dt\wedge dx + \frac{1}{4}\mathcal{Q}_0\cos\theta d\varphi\wedge d\psi\,, &(5.6)
\end{aligned}
$$

that $A^A_{\underline{x}} = 0$ and

$$
\begin{aligned}
\Omega_{(-)\underline{x}}{}^a{}_b\Omega_{(-)\underline{x}}{}^b{}_a &= \Omega_{(-)t}{}^a{}_b\Omega_{(-)\underline{x}}{}^b{}_a = \Omega_{(-)\underline{x}}{}^a{}_b\Omega_{(-)t}{}^b{}_a = 2\mathcal{Z}_0^{-1}\mathcal{Z}_-^{-2}\partial_m\mathcal{Z}_+\partial_m\mathcal{Z}_-\\
&= -\frac{1}{2\mathcal{Z}_-}\left(f_+(\rho) - \frac{16\mathcal{Q}_+\mathcal{Q}_-}{\rho^2}\right)\,, \qquad (5.7)
\end{aligned}
$$

the only non-vanishing components of $G_{\mu\nu}$ we are interested in are given by

$$
\begin{aligned}
G_{\underline{xx}} &= -\mathcal{Z}_-^{-1}\left[(2+\mathcal{Z}_+) - \alpha'\left(f_+(\rho) - \frac{16\mathcal{Q}_+}{\mathcal{Q}_0\rho^2}\right)\right]\\
&= -\mathcal{Z}_-^{-1}\left[2 + \mathcal{Z}_+^{(0)} + \frac{16\mathcal{Q}_+}{\mathcal{Q}_0\rho^2}\right]\,, &(5.8)\\
G_{t\underline{x}} &= \frac{(2-\mathcal{Z}_+)}{\mathcal{Z}_-} - 2\alpha'\mathcal{Z}_0^{-1}\mathcal{Z}_-^{-2}\partial_m\mathcal{Z}_+\partial_m\mathcal{Z}_-\\
&= \mathcal{Z}_-^{-1}\left[2 - \left(\mathcal{Z}_+^{(0)} + \frac{16\mathcal{Q}_+}{\mathcal{Q}_0\rho^2}\right)\right]\,, &(5.9)\\
G_{\underline{x}t} &= G_{\underline{xx}}\,, &(5.10)
\end{aligned}
$$

where f_+ is the function given in eq. (3.21) and $\mathcal{Z}_+^{(0)}$ is the piece of $\mathcal{Z}_+$ of zeroth order in α' defined in eq. (3.28).

Observe that $\mathcal{Z}_+^{(0)}$ always occurs in the combination

$$
\hat{\mathcal{Z}}_+^{(0)} = 1 + \frac{\hat{\mathcal{Q}}_+}{\rho^2}\,, \qquad \hat{\mathcal{Q}}_+ \equiv \mathcal{Q}_+\left(1 + 16\alpha'/\mathcal{Q}_0\right)\,, \tag{5.11}
$$

where, in view of eq. (4.14) $\hat{\mathcal{Q}}_+$ is the total, asymptotic, momentum.

Applying straightforwardly the above T-duality rules gives the following solution

$$
\begin{aligned}
ds'^2 &= \frac{2}{(2+\hat{\mathcal{Z}}_+^{(0)})}d\tilde{u}\left[d\tilde{v} - \frac{1}{2}(\mathcal{Z}_- + \alpha' f'_-)d\tilde{u}\right] - \mathcal{Z}_0 dx^m dx^m - dy^i dy^i\,,\\
B' &= \frac{1}{(2+\hat{\mathcal{Z}}_+^{(0)})}d\tilde{u}\wedge d\tilde{v} + \frac{1}{4}\mathcal{Q}_0\cos\theta d\varphi\wedge d\psi\,, \qquad (5.12)\\
A'^A &= A^A\,,\\
e^{-2\phi'} &= e^{-2\phi_\infty}\frac{(2+\hat{\mathcal{Z}}_+^{(0)})}{\mathcal{Z}_0}\,,
\end{aligned}
$$

[12]Observe that here it is not possible to shift away the harmonic $-\frac{16\mathcal{Q}_+\mathcal{Q}_-}{\rho^2}$ pole. Its presence here is the root of the microscopic T-duality rules that we are going to obtain.

where

$$f'_-(\rho) \equiv -\frac{16\mathcal{Q}_+\mathcal{Q}_-}{\rho^6 \mathcal{Z}_0^{(0)}(2+\hat{\mathcal{Z}}_+^{(0)})}\,, \tag{5.13}$$

and where we have defined the light-cone coordinates

$$\tilde{v} \equiv 2t\,, \qquad \tilde{u} \equiv x'\,. \tag{5.14}$$

Observe that, at this order in α', we can replace $\mathcal{Q}_+$ by $\hat{\mathcal{Q}}_+$ in f'_-:

$$f'_-(\rho) \equiv -\frac{16\hat{\mathcal{Q}}_+\mathcal{Q}_-}{\rho^6 \mathcal{Z}_0^{(0)}(2+\hat{\mathcal{Z}}_+^{(0)})}\,, \tag{5.15}$$

and, then, rewrite the combination

$$\mathcal{Z}_- + \alpha' f'_- = 1 + \frac{\mathcal{Q}_-(1-16\alpha'/\mathcal{Q}_0)}{\rho^2} + 16\alpha' \frac{\mathcal{Q}_-(3\rho^2+\mathcal{Q}_0+3\hat{\mathcal{Q}}_+)}{(\rho^2+\mathcal{Q}_0)(3\rho^2+\hat{\mathcal{Q}}_+)} + \mathcal{O}(\alpha'^2)\,, \tag{5.16}$$

or

$$\mathcal{Z}_- + \alpha' f'_- = \hat{\mathcal{Z}}_- + 16\alpha' \frac{\hat{\mathcal{Q}}_-(3\rho^2+\mathcal{Q}_0+3\hat{\mathcal{Q}}_+)}{(\rho^2+\mathcal{Q}_0)(3\rho^2+\hat{\mathcal{Q}}_+)} + \mathcal{O}(\alpha'^2)\,, \tag{5.17}$$

where we have defined

$$\hat{\mathcal{Z}}_- \equiv 1 + \frac{\hat{\mathcal{Q}}_-}{\rho^2}\,, \qquad \hat{\mathcal{Q}}_- \equiv \mathcal{Q}_-(1-16\alpha'/\mathcal{Q}_0)\,. \tag{5.18}$$

Thus, the T-dual configuration, including the first-order α'-corrections, can be obtained by replacing everywhere in the original solution

$$\begin{aligned} \mathcal{Z}_-^{(0)\prime} &= 2 + \hat{\mathcal{Z}}_+^{(0)}\,, \\ \mathcal{Z}_+^{(0)\prime} &= \hat{\mathcal{Z}}_-^{(0)}\,. \end{aligned} \tag{5.19}$$

Since the constant part of the function $\mathcal{Z}_+$ in the original configuration can be shifted via coordinate transformations $v \to au$ for any constant a,[13] we conclude that the net effect of the T-duality transformation at the level of near-horizon charges is

$$\begin{aligned} \mathcal{Q}'_- &= \hat{\mathcal{Q}}_+ = \mathcal{Q}_+(1+16\alpha'/\mathcal{Q}_0)\,, \\ \mathcal{Q}'_+ &= \hat{\mathcal{Q}}_- = \mathcal{Q}_-(1-16\alpha'/\mathcal{Q}_0)\,. \end{aligned} \tag{5.20}$$

At first sight, these transformation rules are inconsistent with the relations between the charges $\mathcal{Q}_{+,-}$ and the winding and momentum numbers $N_{\mathrm{F1}}, N_{\mathrm{W}}$ in eqs. (4.12) and (4.13) and the microscopic T-duality rules eqs. (5.1) and (5.2), but there are some encouraging signs. For instance, this transformation is an involution to $\mathcal{O}(\alpha'^2)$ as long as $16\alpha'/\mathcal{Q}_0 < 1$:

$$\mathcal{Q}''_\mp = \mathcal{Q}'_\pm(1 \pm 16\alpha'/\mathcal{Q}_0) = \mathcal{Q}_\mp[1-(16\alpha'/\mathcal{Q}_0)^2] \sim \mathcal{Q}_\mp + \mathcal{O}(\alpha'^2)\,. \tag{5.21}$$

[13] It can also be eliminated by T-dualizing in a slightly different direction [7].

Then, using eqs. (4.12) and (4.13), the transformations eqs. (5.20) and the T-duality transformation of the moduli eq. (5.2), which is still valid in the α'-corrected context,[14] we arrive at the following microscopic T-duality transformations that replace eq. (5.1) in this context:

$$N'_{\mathrm{F1}} = N_{\mathrm{W}}(1 + 16/N_{\mathrm{S5}})\,, \qquad N'_{\mathrm{W}} = N_{\mathrm{F1}}(1 - 16/N_{\mathrm{S5}})\,, \tag{5.22}$$

and which are involutive to second order in $1/N_{\mathrm{S5}}$ if $N_{\mathrm{S5}} \gg 16$.

The correctness of these rules cannot be showed using the effective field theory methods used in this paper. It should be mentioned that, had we adopted the point of view that the asymptotic $\hat{\mathcal{Q}}_+ = g_s^2 \ell_s^4 N_W / R_z^2$, the rules eq. (5.1) would still hold. However, since $\mathcal{Q}_+ = \hat{\mathcal{Q}}_+(1 - 16\alpha'/\mathcal{Q}_0)$, it can become negative for small values of N_{S5}, giving rise to 5-dimensional black holes with regular horizon and negative or vanishing mass. These pathological solutions disappear if $N_{\mathrm{S5}} \gg 1$ because the first-order α' corrections become very small. We will discuss in section 9 if it is necessary to impose this condition or not.

6 α' corrections to the 5-dimensional non-Abelian black hole solution

When we compactify the Heterotic Superstring Effective action to first order in α' on a T^5 we get a very complicated action with higher-order terms in curvatures which is very difficult to work with. The definitions of some gauge fields are also affected by the presence of the Chern-Simons term of the torsionful spin connection $\Omega_{(-)}$. However, we can just focus on the metric and the two scalar fields of the 5-dimensional solution (the 5-dimensional dilaton field ϕ and the Kaluza-Klein scalar of the $6 \to 5$ compactification, k), which are obtained from the 10-dimensional one exactly as in absence of α' corrections and take the form [22]

$$\begin{aligned} ds^2 &= f^2 dt^2 - f^{-1}(d\rho^2 + \rho^2 d\Omega^2_{(3)})\,, \\ e^{2\phi} &= e^{2\phi_\infty} \frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, \\ k &= k_\infty (f\mathcal{Z}_+)^{3/4}\,, \end{aligned} \tag{6.1}$$

where ϕ_∞ and k_∞ are the asymptotic values of ϕ and k, the metric function f is given by

$$f^{-3} = \mathcal{Z}_0\, \mathcal{Z}_+\, \mathcal{Z}_-\,, \tag{6.2}$$

and the $\mathcal{Z}$ functions take the form given in eqs. (3.22) and (3.25).

Observe that the α' corrections of $\mathcal{Z}_0$ cancel identically in the $\rho \to \infty$ limit, unless $\mathcal{Q}_0 = 0$, in which case only the term associated to the Yang-Mills field contributes. The value of its contribution in that limit is independent of the value of κ but, according to the previous discussions, when $\kappa = 0$ this contribution must be understood as that of 8 S5-branes and we will simply absorb them into $\mathcal{Q}_0 = 0$.

Taking these considerations and conventions into account, and expressing all the 5-dimensional constants in terms of the 10-dimensional ones using eqs. (2.2), (4.9), (4.12)

[14] They follow from eqs. (5.3) by restoring the radius of the x direction so that $g_{\underline{xx}} \sim (R_z/\ell_s)^2$ at infinity.

and (4.13) the mass of this family of black-hole solutions is given by

$$\begin{aligned}
M &= \frac{\pi}{4G_N^{(5)}}\left[\mathcal{Q}_0 + \mathcal{Q}_+(1+16\alpha'/\mathcal{Q}_0) + \mathcal{Q}_-\right] \\
&= \frac{R_z}{g_s^2\ell_s^2}N_{\rm S5} + \frac{R_z}{\ell_s^2}N_{\rm F1} + \frac{1}{R_z}N_{\rm W}(1+16/N_{\rm S5})\,, \quad \text{for} \quad \mathcal{Q}_0 \neq 0\,, \qquad (6.3) \\
M &= \frac{\pi}{4G_N^{(5)}}\left[8\alpha' + \mathcal{Q}_+ + \mathcal{Q}_-\right] \\
&= 8\frac{R_z}{g_s^2\ell_s^2} + \frac{R_z}{\ell_s^2}N_{\rm F1} + \frac{1}{R_z}N_{\rm W} \qquad \text{for} \quad \mathcal{Q}_0 = 0\,. \qquad (6.4)
\end{aligned}$$

The mass depends on the total, asymptotic charges and, therefore, written in terms of the numbers of branes ("near-horizon charges"), contains additional terms from the delocalized fields.

The area of the horizon, which will give the leading contribution to the entropy, as we will see in section 7, is given by

$$\begin{aligned}
A_{\rm H} &= 2\pi^2\sqrt{\mathcal{Q}_0\mathcal{Q}_+\mathcal{Q}_-}\,, \qquad \text{for} \quad \mathcal{Q}_0 \neq 0\,, \qquad (6.5)\\
A_{\rm H} &= 2\pi^2\sqrt{-16\alpha'\mathcal{Q}_+\mathcal{Q}_-}\,, \qquad \text{for} \quad \mathcal{Q}_0 = 0\,.
\end{aligned}$$

In the $\mathcal{Q}_0 = 0$ case one of the two non-vanishing charges has to be negative for the horizon to exist at all. If $\mathcal{Q}_- < 0$ then $\mathcal{Z}_-$ will vanish at $\rho^2 = |\mathcal{Q}_-|$. If $\mathcal{Q}_- < 0$ the vanishing of $\mathcal{Z}_+$ depends on the values of $\mathcal{Q}_+$ and $\mathcal{Q}_-$ and we will explores the different possibilities in section 8 even though the near-horizon geometry is singular in $d = 10$.

In the next section we consider other possible contributions to the entropy.

7 BH entropy

In order to find the entropy, we would need to compactify the action down to 5 dimensions and use there Wald's entropy formula [2, 3]

$$S = -2\pi\int_{\rm H} d^3x\sqrt{|h|}\,\frac{\partial\mathcal{L}_{(5)}}{\partial R_{abcd}}\epsilon_{ab}\epsilon_{cd}\,, \qquad (7.1)$$

where h is determinant of the 3-dimensional metric induced on the horizon $ds^2_{\rm H}$, ϵ_{ab} is the binormal to the bifurcation surface, normalized as $\epsilon_{ab}\epsilon^{ab} = -2$, $\mathcal{L}_{(5)}$ is the 5-dimensional Lagrangian and R_{abcd} is the 5-dimensional Riemann tensor.[15]

This formula is valid for diff-invariant theories. The 10-dimensional action eq. (2.1) is, by construction, exactly diff-invariant to first order in α', and so would be the 5-dimensional theory that follows from the direct compactification to 5 dimensions. Therefore, Wald's formula can be applied to it and no terms such as those considered in ref. [31] need to be added.

[15]All the indices in this expression run from 0 to 4. 10-dimensional indices will be distinguished with hats in this section.

Compactifying the α'-corrected action is a very involved calculation that, quite understandably, we would like to avoid carrying out. Thus, we try a different strategy, directly applying this formula to the 10-dimensional action.

First of all, we have to identify the part of the 10-dimensional Riemann curvature that corresponds to the 5-dimensional one. The decomposition of the 10-dimensional metric in 5-dimensional variables is given by [22]

$$d\hat{s}^2 = e^{\phi-\phi_\infty}\left[(k/k_\infty)^{-2/3}ds^2 - k^2\mathcal{A}^2\right] - dy^i dy^i\,, \tag{7.2}$$

where $\mathcal{A}$ is the 1-form

$$\mathcal{A} \equiv dz + \frac{k_\infty^{1/3}}{\sqrt{12}}(A^1 + A^2)\,, \tag{7.3}$$

and A^1, A^2 are certain 5-dimensional vector fields.

If we decompose the 10-dimensional flat and curved indices as, respectively, $\hat{a} = a\,, z\,, i$ and $\hat{\mu} = \mu\,, \underline{z}\,, \underline{i}$, the, Fünfbein $e^a{}_\mu$ is related to the components $\hat{e}^a{}_\mu$ of the Zehnbein $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$ by

$$\hat{e}^a{}_\mu = e^{(\phi-\phi_\infty)/2}(k/k_\infty)^{-1/3}e^a{}_\mu\,, \tag{7.4}$$

so the 5-dimensional Riemann curvature R_{abcd} is related to the $\hat{R}_{abcd}$ components of the 10-dimensional Riemann curvature $\hat{R}_{\hat{a}\hat{b}\hat{c}\hat{d}}$ by

$$\hat{R}_{abcd} = e^{-(\phi-\phi_\infty)}(k/k_\infty)^{2/3}R_{abcd} + \dots\,. \tag{7.5}$$

Furthermore, the 10-dimensional Riemann curvature enters the curvature tensor of the torsionful spin connection $\hat{R}_{(-)\,\hat{a}\hat{b}\hat{c}\hat{d}}$ in this way

$$\hat{R}_{(-)\,\hat{a}\hat{b}\hat{c}\hat{d}} = \hat{R}_{\hat{a}\hat{b}\hat{c}\hat{d}} - \hat{\nabla}_{[\hat{a}}\hat{H}_{\hat{b}]\hat{c}\hat{d}} + \frac{1}{2}\hat{H}_{[\hat{a}|\hat{c}\hat{e}}\hat{H}_{|\hat{b}]\hat{d}}{}^{\hat{e}}\,, \tag{7.6}$$

so

$$\hat{R}_{(-)\,abcd} = e^{-(\phi-\phi_\infty)}(k/k_\infty)^{2/3}R_{abcd} + \dots\,. \tag{7.7}$$

Taking into account these relations, Wald's entropy formula eq. (7.1) can be rewritten in terms of the 10-dimensional Lagrangian and the 10-dimensional Riemann tensor for the family of metrics under consideration as[16]

$$S = -2\pi\int_{\mathrm{H}\times\mathrm{S}^1\times\mathrm{T}^4} d^8\hat{x}\,\frac{\sqrt{|\hat{g}|}}{\sqrt{f}}\,e^{-(\phi-\phi_\infty)}(k/k_\infty)^{2/3}\frac{\partial\mathcal{L}_{(10)}}{\partial\hat{R}_{abcd}}\epsilon_{ab}\epsilon_{cd}\,, \tag{7.9}$$

where $|\hat{g}|$ is the absolute value of the full 10-dimensional metric and we we are integrating over the co-dimension 2 surface $\mathrm{H}\times\mathrm{S}^1\times\mathrm{T}^4$, and where the binormal ϵ_{ab} is intrinsically

[16]The reason why the metric function appears explicitly is because it is the optimal way of taking into account the rescalings the action goes through in the dimensional reduction. We can write, for these metrics,

$$\sqrt{|h|}\mathcal{L}_{(5)} = \frac{\sqrt{|g|}}{\sqrt{f}}\mathcal{L}_{(5)} = \frac{\sqrt{|\hat{g}|}}{\sqrt{f}}\,\mathcal{L}_{(10)}\,, \tag{7.8}$$

because the Lagrangian density is the same in any dimension.

5-dimensional. In the Vielbein basis, though, ϵ_{ab} has the same components both in the 5-dimensional and in the 10-dimensional basis.

Let us apply this formula to the different pieces of the 10-dimensional action that contain the 10-dimensional Riemann tensor, manifestly, or via $\hat{R}_{(-)}$.

Applied to the Riemann-Hilbert term, we have

$$\frac{\partial \mathcal{L}_{(10)}}{\partial \hat{R}_{abcd}} = \frac{1}{16\pi G_N^{(10)}} e^{-2(\hat{\phi}-\hat{\phi}_\infty)} \hat{\eta}^{ac} \hat{\eta}^{bd} , \tag{7.10}$$

$$\frac{\sqrt{|\hat{g}|}}{\sqrt{f}} = \frac{1}{8} k_\infty e^{+3(\hat{\phi}-\hat{\phi}_\infty)} (k/k_\infty)^{-2/3} (\rho^6 f^{-3})^{1/2} \sin\theta , \tag{7.11}$$

where, evidently $\hat{\eta}^{ab} = \eta^{ab}$. Then, taking the $\rho \to 0$ limit, substituting in the formula and integrating over the 5 compact dimensions whose coordinates take values in $[0, 2\pi\ell_s)$, and over the 3-sphere, and using $k_\infty = R_z/\ell_s$, we get the zeroth-order contribution to the entropy

$$S^{(0)} = \frac{A_{\rm H}}{4G_N^{(5)}} , \tag{7.12}$$

where $A_{\rm H}$ is the area of the horizon, computed in eq. (6.5) and where the 5-dimensional Newton constant is

$$\frac{1}{G_N^{(5)}} = \frac{(2\pi\ell_s)^4 (2\pi R_z)}{G_N^{(10)}} . \tag{7.13}$$

Using the result eq. (6.5) and the relations between the 5- and 10-dimensional constants, this zeroth-order contribution is, in terms of the brane numbers

$$S^{(0)} = 2\pi \sqrt{N_{\rm S5} N_{\rm F1} N_{\rm W}} , \tag{7.14}$$

which is the classical, zeroth-order in α' result.

There are two terms that contribute to Wald's formula at first order in α' through the occurrence of $\hat{R}_{(-)}$: the kinetic term of the Kalb-Ramond field, whose field strength contains $\hat{R}_{(-)}$ in the Lorentz-Chern-Simons term, and in the $\hat{T}^{(2)}$ tensor. Let us start with the latter.

The contribution of the $\hat{T}^{(0)}$ tensor term of the action to Wald's formula is clearly proportional to $\hat{R}_{(-)}$. However, when evaluated on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$, $\hat{R}_{(-)}$ vanishes identically [11]. It is easy to prove this fact explicitly: the Riemann tensor takes the form

$$\hat{R}_{\hat{a}\hat{b}\hat{c}\hat{d}} = \left(-\frac{2}{L^2} \hat{g}_{\hat{a}[\hat{c}} \hat{g}_{\hat{d}]\hat{b}} , \frac{2}{L^2} \hat{g}_{\hat{a}[\hat{c}} \hat{g}_{\hat{d}]\hat{b}} , 0 \right) , \tag{7.15}$$

in a more or less obvious notation in which each factor corresponds, respectively, to AdS_3, S^3 and T^4, and L is the common radius of AdS_3 and of the sphere. Only the indices corresponding to those subspaces are active in each factor, but we will not introduce new indices to keep the notation as simple as possible.

On the other hand, the 3-form field strength can be put in the form

$$\hat{H} = \frac{2}{L} \left(-d\Pi_3 + dV_3 \right) , \tag{7.16}$$

where $d\Pi_3$ is the volume form of the AdS$_3$ factor (with unit radius) and dV_3 the volume form of S^3 (of unit radius too). Then, $\hat{H}$ is covariantly constant, $\hat{\nabla}\hat{H} = 0$, and we can see that, in the same notation,

$$\hat{H}_{[\hat{a}|\hat{c}\hat{e}}\hat{H}_{|\hat{b}]\hat{d}}{}^{\hat{e}} = \left(+\frac{4}{L^2}\hat{g}_{\hat{a}[\hat{c}}\hat{g}_{\hat{d}]\hat{b}}\,, -\frac{4}{L^2}\hat{g}_{\hat{a}[\hat{c}}\hat{g}_{\hat{d}]\hat{b}}\,, 0\right)\,, \tag{7.17}$$

which implies, according to the definition eq. (7.6) $\hat{R}_{(-)\,\hat{a}\hat{b}\hat{c}\hat{d}} = 0$. Since AdS$_3 \times$ S$^3 \times$ T^4 is the near-horizon of extremal black holes as the ones we are considering, we conclude that for these extremal black holes the $\hat{T}^{(0)}$-tensor term in the action does not contribute to Wald's entropy formula.

Then, the only possible first-order contribution comes from

$$\begin{aligned}
S^{(1)} &= -2\pi \int d^8\hat{x}\frac{\sqrt{|\hat{g}|}}{\sqrt{f}}\, e^{-(\phi-\phi_\infty)}(k/k_\infty)^{2/3}\frac{\partial}{\partial \hat{R}_{abcd}}\left\{\frac{1}{2\cdot 3!}\frac{e^{-2(\hat{\phi}-\hat{\phi}_\infty)}}{16\pi G_N^{(10)}}\hat{H}^2\right\}\epsilon_{ab}\epsilon_{cd} \\
&= -\frac{1}{48 G_N^{(10)}}\int d^8\hat{x}\frac{\sqrt{|\hat{g}|}}{\sqrt{f}}\, e^{-3(\phi-\phi_\infty)}(k/k_\infty)^{2/3}\hat{H}^{\hat{e}\hat{f}\hat{g}}\frac{\partial \hat{H}_{\hat{e}\hat{f}\hat{g}}}{\partial \hat{R}_{abcd}}\epsilon_{ab}\epsilon_{cd} \\
&= \frac{\alpha'}{8G_N^{(10)}}\int d^8\hat{x}\frac{\sqrt{|\hat{g}|}}{\sqrt{f}}\, e^{-3(\phi-\phi_\infty)}(k/k_\infty)^{2/3}\hat{H}^{ab\hat{g}}\hat{\Omega}_{(-)\,\hat{g}}{}^{cd}\epsilon_{ab}\epsilon_{cd}\,.
\end{aligned} \tag{7.18}$$

The binormal has the following components:[17] $\epsilon_{0\sharp} = 1$, where $e^\sharp = f^{-1/2}d\rho$ and, therefore,

$$S^{(1)} = \frac{\alpha'}{2G_N^{(10)}}\int d^8\hat{x}\frac{\sqrt{|\hat{g}|}}{\sqrt{f}}\, e^{-3(\phi-\phi_\infty)}(k/k_\infty)^{2/3}\hat{H}^{0\sharp\hat{g}}\hat{\Omega}_{(-)\,\hat{g}}{}^{0\sharp}\,. \tag{7.19}$$

In appendix A we have computed explicitly the components of $\hat{H}$ (eq. (A.6)) using the Zehnbein basis eq. (A.2), but this is not the basis related by a simple rescaling to the Fünfbein basis in which $\epsilon_{0\sharp} = 1$. The relation is

$$\begin{aligned}
\hat{e}^0 &= \frac{1}{2}\sqrt{\mathcal{Z}_+\mathcal{Z}_-}\,\hat{e}^+ + \frac{1}{\sqrt{\mathcal{Z}_+\mathcal{Z}_-}}\,\hat{e}^-\,, \\
\hat{e}^1 &= \frac{1}{2}\sqrt{\mathcal{Z}_+\mathcal{Z}_-}\hat{e}^+ - \frac{1}{\sqrt{\mathcal{Z}_+\mathcal{Z}_-}}\hat{e}^-\,, \\
\hat{e}^\sharp &= \frac{x^m}{\rho}\hat{e}^m\,,
\end{aligned} \tag{7.20}$$

and leads to

$$\hat{H}^{0\sharp\hat{g}} = \frac{1}{2}\delta^{\hat{g}}{}_-\sqrt{\mathcal{Z}_+\mathcal{Z}_-}\frac{x^m}{\rho}\hat{H}^{+m-} + \delta^{\hat{g}}{}_+\frac{1}{\sqrt{\mathcal{Z}_+\mathcal{Z}_-}}\frac{x^m}{\rho}\hat{H}^{-m+}\,, \tag{7.21}$$

$$\hat{\Omega}_{(-)\,\hat{g}}{}^{0\sharp} = \delta_{\hat{g}}{}^+\left(\frac{1}{2}\sqrt{\mathcal{Z}_+\mathcal{Z}_-}\frac{x^m}{\rho}\hat{\Omega}_{(-)\,+}{}^{+m} + \frac{1}{\sqrt{\mathcal{Z}_+\mathcal{Z}_-}}\frac{x^m}{\rho}\hat{\Omega}_{(-)\,+}{}^{-m}\right)\,, \tag{7.22}$$

$$\begin{aligned}
\hat{H}^{0\sharp\hat{g}}\hat{\Omega}_{(-)\,\hat{g}}{}^{0\sharp} &= -\frac{x^m}{\rho}\hat{H}_{+-m}\frac{x^n}{\rho}\left(\frac{1}{2}\hat{\Omega}_{(-)\,+-n} + \frac{1}{\mathcal{Z}_+\mathcal{Z}_-}\hat{\Omega}_{(-)\,++n}\right) \\
&= \frac{1}{2\mathcal{Z}_0}\partial_\rho \log \mathcal{Z}_-\left(\partial_\rho \log\mathcal{Z}_- + \partial_\rho \log \mathcal{Z}_+\right)\,.
\end{aligned} \tag{7.23}$$

[17] The global sign is irrelevant, as it appears twice in the formula.

Observe that, in the near-horizon ($\rho \to 0$) limit, $\partial_\rho \log \mathcal{Z}_- \left(\partial_\rho \log \mathcal{Z}_- + \partial_\rho \log \mathcal{Z}_+\right) \sim 1/\rho^2$, and the above term will only be finite if, in the same limit, $\mathcal{Z}_0 \sim 1/\rho^2$, i.e. if $\mathcal{Q}_0 \neq 0$. Nevertheless, what really matters is the $\rho \to 0$ limit of the product of this term with $(\rho^6 f^{-3})^{1/2}$, and this limit is finite if the separate limits of the two factors are finite (this is what happens when all the charges are finite) or when $\mathcal{Q}_0 = \mathcal{Q}_+ = 0$, a case in which there is no classical horizon. For small black holes, this contribution will be divergent.

Then, plugging this result into the above expression for $S^{(1)}$ and evaluating it for $\mathcal{Q}_0 \neq 0$, we get

$$S^{(1)} = +\frac{8\alpha'}{\mathcal{Q}_0} S^{(0)} , \tag{7.24}$$

and, to first order in α', and for $\mathcal{Q}_0 \neq 0$ the entropy is given by

$$\begin{aligned} S &= 2\pi\sqrt{N_{\rm S5} N_{\rm F1} N_{\rm W}}\,(1 + 8/N_{\rm S5}) \\ &\sim 2\pi\sqrt{(N_{\rm S5}+16) N_{\rm F1} N_{\rm W}}\,, \quad \text{for} \quad N_{\rm S5} \gg 16\,. \end{aligned} \tag{7.25}$$

8 Small black holes

Another potentially interesting feature of these α'-corrected solutions which has been observed in the literature before,[18] is the emergence of regular horizons in certain configurations with only two non-vanishing charges which, in our case, must be $\mathcal{Q}_+$ and $\mathcal{Q}_-$. For $\mathcal{Q}_0 = 0$, the area of the horizon is given by the second equation in (6.5), which we rewrite here for the sake of convenience:

$$A_{\rm H} = 2\pi^2\sqrt{-16\alpha' \mathcal{Q}_+ \mathcal{Q}_-}\,. \tag{8.1}$$

This expression can be real and finite, and $\mathcal{Z}_- > 0\ \forall\, \rho$, if $\mathcal{Q}_+ < 0$ and $\mathcal{Q}_- > 0$. Now we have to study if there are values of these constants for which $\mathcal{Z}_+(\rho) > 0\ \forall\, \rho$, making the 5-dimensional metric completely regular. This function can be written in the form

$$\mathcal{Z}_+ = 1 - \frac{|\mathcal{Q}_+|}{\rho^2}\left[1 - \frac{16\alpha' \mathcal{Q}_-}{\rho^2(\rho^2 + \mathcal{Q}_-)}\right] + \mathcal{O}(\alpha'^2)\,. \tag{8.2}$$

It is not difficult to see that there are values of $\mathcal{Q}_+ <$ and $\mathcal{Q}_- > 0$ for which the regularity condition is satisfied. The orange-shaded region in figure 1 corresponds to the values of $\mathcal{Q}_+, \mathcal{Q}_-$ for which the black holes have a regular horizon due to the α' corrections. The blue-shaded area corresponds to the small black holes with $|\mathcal{Q}_+| \geq \mathcal{Q}_-$, which have negative or vanishing mass.

For $\mathcal{Q}_- \gg \alpha'$ it is possible to see that the condition on the other charge is $0 > \mathcal{Q}_+ > -64\alpha'$. Thus, the small black holes are confined to the region of small $\mathcal{Q}_+$.

9 Range of validity of the solution

So far we have studied the solutions ignoring whether they are really good solutions of the complete Heterotic Superstring effective action to first order in α' and to zeroth order in the string coupling constant, everywhere.

[18] See, e.g. refs. [32, 33], the review ref. [4] and references therein.

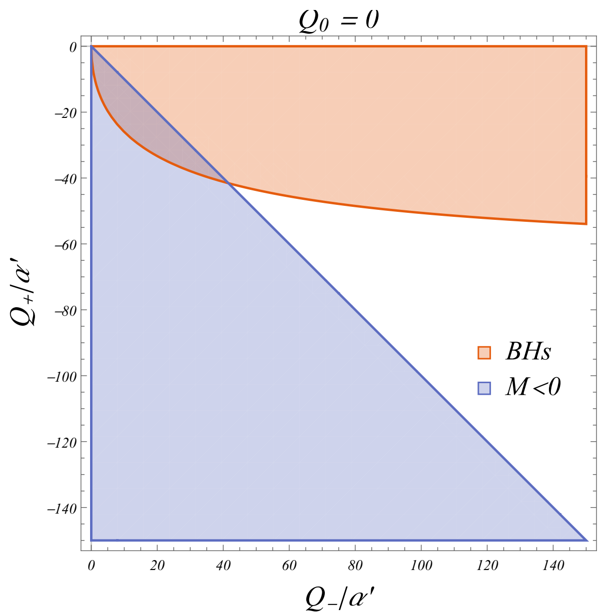

Figure 1. Location in $\mathcal{Q}_+$-$\mathcal{Q}_-$ charge space of the small black holes ($\mathcal{Q}_0 = 0$) whose horizon area is rendered finite due to the α' corrections in their geometry. .

Let us start with the possible loop corrections. These will be small if

$$e^{\phi} = e^{\phi_\infty} \sqrt{\mathcal{Z}_0/\mathcal{Z}_-}\,, \tag{9.1}$$

whose vacuum expectation value gives the string coupling constant, is small. For $\mathcal{Q}_0 \neq 0$, it is easy to see that, at spatial infinity, this requires

$$e^{\phi_\infty} = g_s \ll 1\,, \tag{9.2}$$

while at the horizon (and also at intermediate values of ρ) this requires

$$\mathcal{Q}_- \lesssim \mathcal{Q}_0\,, \quad \text{or} \quad N_{\rm F1} \gg N_{\rm S5}\,. \tag{9.3}$$

For $\mathcal{Q}_0 = 0$, the dilaton vanishes at $\rho = 0$ and there is no need to impose any more conditions.

Another important condition that the solution must satisfy is that the radius of compactification of the 6th dimension, measured in ℓ_s units by $|g_{zz}|^{1/2}$ in the 10-dimensional string frame

$$|g_{zz}|^{1/2} = k e^{\frac{1}{2}(\phi-\phi_\infty)} = k_\infty \sqrt{\left|\frac{\mathcal{Z}_+}{\mathcal{Z}_-}\right|}\,, \tag{9.4}$$

is much larger than the self-dual[19] radius $\sim \ell_s$, at which new massless modes appear in the string spectrum that invalidate the effective action we have used because they have not been taken into account in it. At infinity, this condition, $|g_{zz}|^{1/2} \gg 1$, translates into

$$k_\infty \gg 1\,, \quad \Rightarrow \quad R_z \gg \ell_s\,. \tag{9.5}$$

[19]Self-dual under T-duality.

If $\mathcal{Q}_0 = 0$, $|g_{zz}|^{1/2}$ diverges in the $\rho \to 0$ limit and, again, no conditions must be imposed on the remaining charges. If $\mathcal{Q}_0 \neq 0$, we find the following condition

$$\mathcal{Q}_+ \gtrsim \mathcal{Q}_- \,, \quad \Rightarrow \quad N_{\mathrm{W}} \gg N_{\mathrm{F1}} \,. \tag{9.6}$$

All these conditions can be summarized into

$$N_{\mathrm{W}} \gg N_{\mathrm{F1}} \gg N_{\mathrm{S5}} \,. \tag{9.7}$$

For the case $\mathcal{Q}_0 = 0$ the only conditions that need to be satisfied are those affecting the moduli, namely $g_s \ll 1$, $k_\infty > 1$. These solutions, however, have many other problems: their metrics are singular at $\rho = 0$ in $d = 10$, to start with and the reason why they are regular in $d = 5$ is that the compactification radius is also singular there.

Finally, we must find if and when the solution of the first-order in α' equations that we have obtained can be considered a first-order in α' approximation to a solution of the full Heterotic Superstring effective action. Clearly, this happens if and when the higher-order corrections to the $\mathcal{Z}$-functions are very small, compared with the first-order solution.

It is not easy to assess the relevance of higher-order corrections without actually computing them, which becomes increasingly difficult. Since the higher-order corrections of the action and equations of motion are expected to contain powers of the first-order corrections, many of them codified in the so-called "$\hat{T}$-tensors" and in the Chern-Simons terms present in $\hat{H}$, it is reasonable to expect that the higher-order corrections will be smaller than the first-order corrections if the first-order corrections are small enough. Since the first-order corrections are proportional to the $\hat{T}$-tensors and to the Chern-Simons terms, they will be small if the later are also small. Actually, a necessary criterion for a supersymmetric solution to be exact to all orders in α' is the vanishing of $\hat{T}$-tensors and the Chern-Simons terms [15].

The origin of the T-tensors is the need to supersymmetrize the Yang-Mills and Lorentz Chern-Simons terms. There may be other terms in the action with a different origin such as the well-known $\zeta(3)R^4$ term, but very little is known about them. When $\mathcal{Q}_0 \neq 0$, it is usually argued that these terms as well as other invariants occur in the action as inverse powers of N_{S5}, once the factors of α' have been taken into account. The consequence is that $\mathcal{Q}_0$ is usually taken to be large so N_{S5} is very large.

However, we would like to stress that it is not enough to study the scalar invariants constructed from the curvature or from the T-tensors because, as discussed in the paragraph following eqs. (3.11)–(3.14), some components of the curvature and of the T tensors that occur in the equations of motion and source the first-order α' corrections such as $\hat{T}^{(2)}{}_{uu}$ disappear in the scalar invariants. Thus, even if all the curvature invariants vanish, one can expect non-vanishing α' corrections to the solutions such as those occurring in $\mathcal{Z}_+$.

The 3 $\hat{T}$-tensors are defined in eq. (2.3) and their values, computed for this kind of solutions in ref. [22] to $\mathcal{O}(\alpha'^2)$, are given in eqs. (3.11)–(3.14). It is convenient to analyze the corrections for the cases $\mathcal{Q}_0 \neq 0$ and $\mathcal{Q}_0 = 0$ separately.

When $\mathcal{Q}_0 \neq 0$, all the components of these tensors, except for $\hat{T}^{(2)}{}_{uu}$, as well as the combined Yang-Mills- and Lorentz-Chern-Simons terms, become arbitrarily small for

$\kappa^2 \sim \mathcal{Q}_0$. In fact, in agreement with this, the correction to $\mathcal{Z}_0$, which we write here for convenience

$$8\alpha' \left[\frac{\rho^2 + 2\kappa^2}{(\rho^2 + \kappa^2)^2} - \frac{\rho^2 + 2\mathcal{Q}_0}{(\rho^2 + \mathcal{Q}_0)^2} \right] , \tag{9.8}$$

also becomes arbitrarily small in this limit, independently of the value of $\mathcal{Q}_0$, which is usually necessary to assume very large.

For $\kappa^2 = \mathcal{Q}_0$ all the components of the T-tensors, except for $\hat{T}^{(2)}{}_{uu}$, and the correction of $\mathcal{Z}_0$ vanish identically. As mentioned before, if we set $\mathcal{Q}_+ = \mathcal{Q}_- = 0$ we recover the so-called "symmetric 5-brane" solution of ref. [8] which has been argued to be an exact solution to all orders in α' ($\hat{T}^{(2)}{}_{uu} = 0$). In the general case we have to consider the effects of the non-vanishing $\hat{T}^{(2)}{}_{uu}$. At first order, it sources the uu component of the Einstein equations, which only affect $\mathcal{Z}_+$. At higher order, it cannot occur in any invariant, as we have explained. It can only appear multiplied by invariants sourcing the same component of the Einstein equations. Those invariants can be made as small as wanted with $\kappa^2 \sim \mathcal{Q}_0$ and, therefore, since the first-order correction of $\mathcal{Z}_+$ ($f_+(\rho)$ in eq. (3.21)) is regular everywhere, it is reasonable to expect that the higher-order corrections will also be finite but much smaller.

The conclusion, thus, is that for $\mathcal{Q}_0 \neq 0$, and $\mathcal{Q}_0 \gg \alpha'$, taking $\kappa^2 \sim \mathcal{Q}_0$ we get a very good approximation to an exact solution of the Heterotic String effective action.

For $\mathcal{Q}_0 = 0$ (small black holes) the corrections associated to the gauge 5-brane are finite and at higher orders, multiplied by higher powers of α', much smaller, but cannot be completely cancelled. We can simply remove the gauge 5-brane to simplify the problem, eliminating these corrections. The main problem, though, is the correction in $\mathcal{Z}_+$ associated to $\hat{T}^{(2)}{}_{uu}$, which diverges at the horizon and which will diverge there at higher orders even if we multiply it by small numbers, as long as they are non zero. The divergence of the first-order correction, by itself, only indicates that the zeroth-order solution is not to be trusted at the horizon. The first-order solution can be trusted if the rest of the corrections vanish which, according to the previous discussions, may happen if we remove the gauge 5-brane.

Nevertheless, as we have pointed out before, the small black-hole solutions are singular in 10 dimensions and the α' correction to their entropy seems to be divergent. Furthermore, their T-dual is singular because $\mathcal{Q}'_- = -|\mathcal{Q}_+| < 0$ and $\mathcal{Z}'_-$ with vanish at $\rho^2 = |\mathcal{Q}_+|$. These properties suggest that these solutions, which may have negative mass in $d = 5$, are not good solutions of Heterotic String Theory.

10 Discussion

In this paper we have computed explicitly the first α' corrections to a 3-charge 5-dimensional black hole to which we have added an SU(2) Yang-Mills instanton, and we have studied some of the effects that these corrections have on the geometry, entropy and mass of the solutions. We have also studied the effect of an α'-corrected T-duality transformation in the α'-corrected solution, testing simultaneously the validity of our solution and of the T-duality rules proposed, long time ago, in ref. [9]. Studying the effect of these

α'-corrected T-duality transformations requires the knowledge of α'-corrected solutions, which is very scarce in the literature.

The fact that the corrections can be computed explicitly is, by itself, a remarkable fact. The computability of the corrections to the $\mathcal{Z}_0$ function is due to the surprisingly simple form of the Bianchi identity for the configurations we have considered: a linear combination of Laplacians, a "coincidence" that can be generalized to more complicated supersymmetric configurations [7].

Finding the α' corrections to the S5-brane solution in presence of a gauge 5-brane has also allowed us to gain better understanding of the symmetric 5-brane solution found in ref. [8].

Furthermore, we have shown how the α' corrections to the entropy of the 5-dimensional black holes can be computed using Wald's formula directly in 10-dimensional language. Our calculation is very clean and transparent and shows the relevance of the Lorentz-Chern-Simons term in the corrections and the irrelevance of the curvature-squared terms (which was already known since ref. [11]). Our results concerning the invariance under α'-corrected T-duality (up to interchange of numbers of branes) of the family of solutions considered here, implies the invariance of the α'-corrected entropy formula under the same transformations, in agreement with the results of ref. [10].

Of course, we must compare our results with other results about higher-order α' corrections to supersymmetric black-hole solutions in the literature.[20]

Most of the work done in this field deals with solutions to ungauged 4- and 5-dimensional $\mathcal{N} = 2$ (8-supercharge) supergravities obtained via Calabi-Yau compactifications from M-theory or type II theories and (at least some of) the 1st-order in α' corrections are said to be effectively encoded in corrections to the prepotential (in $d = 4$), for instance. This obscures the origin of the corrections, which may or may not represent all the corrections one finds in higher dimensions (see, e.g. [34]), and makes it very difficult to say anything about the relevance of corrections of orders higher than 1. Furthermore, the absence of non-Abelian fields forbids the use of the "symmetric" mechanism we have used to make very small or cancel many of the corrections and argue the validity of our solution.[21] Finally, it is unclear where the relevant contribution of 10-dimensional Lorentz-Chern-Simons term to the first-order corrections is to be found in 4 or 5 dimensions. Thus, comparing our results with those obtained within this approach is very difficult.

Some work has also been done using a 10-dimensional approach to the computation of the corrections in Heterotic Superstring Theory,[22] but only near-horizon geometries were studied,[23] while we have studied and computed the corrections to the full black-hole geometry from infinity to the horizon. While we have concluded that the parameters of interest are the "near-horizon charges", because they count the number of string-theory

[20] See, for instance, refs. [4, 5] and references therein.

[21] It has been explored with Abelian fields, though. See ref. [35] and references therein. An early use of this mechanism applied to a configuration related to that studied here can be found in ref. [36], but it does not have enough charges to be a regular extremal black hole in lower dimensions. In a forthcoming publication we will show the relation between that configuration and the one studied here [7].

[22] See ref. [5] and references therein.

[23] They have also been used in the context of the entropy-functional approach [37].

objects sourcing the solution, of course, the total charges, measured at infinity and these constants are related, and the relation can be computed explicitly in our α'-corrected solutions because they describe both regions. Writing the entropy or the mass in terms of one or the other is a matter of choice, but, after they are written in terms of numbers of branes and other quantized quantities that are expected to be strictly positive, one expects the solution to have sensible physical properties. It is not hard to see that, when all the charges are different from zero, if the asymptotic charges were identified with the quantized charges, it would be possible to find negative-mass solutions.

The value found for the α'-corrected entropy, eq. (7.25) seems to disagree with the value of the microscopic entropy computed in ref. [6] as written in ref. [5], but the value of α' in that reference is 8 times ours and, therefore, they coincide, although the route followed to arrive at the same result is totally different.

The fact that the α' corrections associated to the torsionful spin connection have the "wrong sign" as compared with those of the Yang-Mills fields is clearly the source of some of this pathological behavior, already hinted at by the results of ref. [38], in which the α'-corrected black holes were shown to be repulsive. The addition of Yang-Mills fields can correct some of these effects, making some of the α' corrections very small or zero, but not all of them. It is, however, likely, that a more general kind of Yang-Mills fields which give rise to non-Abelian dyons in 5 dimensions can cancel all of them. Work in this direction is in progress [7].

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A Connection, torsionful spin connection etc.

In this appendix we are going to compute explicitly the connections and curvatures of the ansatz eq. (3.1). While that ansatz is spherically symmetric in a 4-dimensional space, it is more convenient to do some of the computations using a slightly more general ansatz and then particularize to spherical symmetry.

Thus, here, we are interested in 10-dimensional metrics of the form

$$ds^2 = \frac{2}{\mathcal{Z}_-} du \left[dv - \frac{1}{2}\mathcal{Z}_+ du \right] - \mathcal{Z}_0 dx^m dx^m - dy^i dy^i \,, \tag{A.1}$$

where $m,n,i,j = 1,2,3,4$ and the functions $\mathcal{Z}_\pm, \mathcal{Z}_0, H$ are functions on the first 4-dimensional space with coordinates x^m. Thus, the metric is independent of the light-cone coordinates u, v and of the 4 spatial coordinates y^i.

A simple choice of Zehnbein is

$$e^+ = \mathcal{Z}_-^{-1} du\,, \quad e^- = dv - \frac{1}{2}\mathcal{Z}_+ du\,, \quad e^m = \mathcal{Z}_0^{1/2}\, dx^m\,, \quad e^i = dy^i\,, \tag{A.2}$$

and the inverse basis is

$$e_+ = \mathcal{Z}_- \left(\partial_u + \frac{1}{2}\mathcal{Z}_+ \partial_v\right)\,, \quad e_- = \partial_v\,, \quad e_m = \mathcal{Z}_0^{-1/2}\partial_m\,, \quad e_i = \partial_i\,, \tag{A.3}$$

where $\partial_m \equiv \partial_{\underline{m}}$ and $\partial_i \equiv \partial_{\underline{i}}$.

Using the structure equation $de^a = \omega^a{}_b \wedge e^b$ we find that the non-vanishing components of the spin connection are given by

$$\begin{aligned} \omega_{-+m} &= \omega_{+-m} = \omega_{m+-} = \tfrac{1}{2}\mathcal{Z}_0^{-1/2}\partial_m \log \mathcal{Z}_-\,, \quad \omega_{+m+} = -\tfrac{1}{2}\mathcal{Z}_-\mathcal{Z}_0^{-1/2}\partial_m \mathcal{Z}_+\,, \\ \omega_{mnp} &= \mathcal{Z}_0^{-3/2}\delta_{m[n}\partial_{p]}\mathcal{Z}_0\,. \end{aligned} \tag{A.4}$$

We are also interested in 3-form field strengths of the general form

$$H = du \wedge dv \wedge d\mathcal{Z}_-^{-1} + \star_{(4)} d\mathcal{Z}_0\,, \tag{A.5}$$

where $\star_{(4)}$ is the Hodge dual in the first 4-dimensional space with the orientation $\varepsilon^{\sharp 123} = +1$. Their non-vanishing flat components are

$$H_{m+-} = -\mathcal{Z}_0^{-1/2}\partial_m \log \mathcal{Z}_-\,, \qquad H_{mnp} = \mathcal{Z}_0^{-1/2}\,\varepsilon_{mnpq}\partial_q \log \mathcal{Z}_0\,. \tag{A.6}$$

Then, the non-vanishing flat components of torsionful spin connection $\Omega_{(-)abc} \equiv \omega_{abc} - \frac{1}{2}H_{abc}$ are

$$\begin{aligned} \Omega_{(-)++m} &= \tfrac{1}{2}\mathcal{Z}_-\mathcal{Z}_0^{-1/2}\partial_m \mathcal{Z}_+\,, & \Omega_{(-)+-m} &= \mathcal{Z}_0^{-1/2}\partial_m \log \mathcal{Z}_-\,, \\ \Omega_{(-)m+-} &= \Omega_{(-)+-m}\,, & \Omega_{(-)mnp} &= \mathcal{Z}_0^{-1/2}(\mathbb{M}^-_{mq})_{np}\partial_q \log \mathcal{Z}_0\,, \end{aligned} \tag{A.7}$$

and those of the torsionful spin connection $\Omega_{(+)abc} \equiv \omega_{abc} + \frac{1}{2}H_{abc}$ are given by

$$\begin{aligned} \Omega_{(+)++m} &= \tfrac{1}{2}\mathcal{Z}_-\mathcal{Z}_0^{-1/2}\partial_m \mathcal{Z}_+\,, & \Omega_{(+)-+m} &= \mathcal{Z}_0^{-1/2}\partial_m \log \mathcal{Z}_-\,, \\ \Omega_{(+)mnp} &= \mathcal{Z}_0^{-1/2}(\mathbb{M}^+_{mq})_{np}\partial_q \log \mathcal{Z}_0\,, \end{aligned} \tag{A.8}$$

where the 4×4 matrices $\mathbb{M}^\pm_{mq}$ are the self- and anti-self-dual parts of the generators of SO(4):

$$(\mathbb{M}_{mq})_{np} = (\mathbb{M}_{np})_{mq} \equiv 2\delta_{n[m}\delta_{q]p}\,, \qquad \mathbb{M}^\pm_{mq} \equiv \frac{1}{2}\left(\mathbb{M}_{mq} \pm \frac{1}{2}\varepsilon_{mqrs}\mathbb{M}_{rs}\right)\,. \tag{A.9}$$

The components with curved indices are given by

$$\begin{aligned} \Omega_{(-)\underline{m}+-} &= \partial_m \log \mathcal{Z}_-\,, & \Omega_{(-)u-m} &= -\mathcal{Z}_0^{-1/2}\partial_m \mathcal{Z}_-^{-1}\,, \\ \Omega_{(-)u+m} &= \tfrac{1}{2}\mathcal{Z}_0^{-1/2}\partial_m \mathcal{Z}_+\,, & \Omega_{(-)\underline{m}np} &= (\mathbb{M}^-_{mq})_{np}\partial_q \log \mathcal{Z}_0\,, \end{aligned} \tag{A.10}$$

and

$$\begin{aligned} \Omega_{(+)u+m} &= \tfrac{1}{2}\mathcal{Z}_0^{-1/2}\partial_m \mathcal{Z}_+\,, & \Omega_{(+)v+m} &= \mathcal{Z}_0^{-1/2}\partial_m \log \mathcal{Z}_-\,, \\ \Omega_{(+)mnp} &= (\mathbb{M}^+_{mq})_{np}\partial_q \log \mathcal{Z}_0\,, \end{aligned} \tag{A.11}$$

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A.1 Solving the Bianchi identity for H

Observe that $\Omega_{(-)\underline{m}np}$ coincides with the form of the 't Hooft ansatz for SU(2) Yang-Mills multi-instanton solutions using SO(4) indices.[24] Furthermore, this is the only piece of $\Omega_{(-)\mu ab}$ that contributes to the Lorentz-Chern-Simons term:

$$\omega^{\rm L}_{(-)} = d\Omega_{(-)mn}\wedge\Omega_{(-)nm} + \frac{2}{3}\Omega_{(-)mn}\wedge\Omega_{(-)np}\wedge\Omega_{(-)pm} = \star_{(4)}d(\partial\log\mathcal{Z}_0)^2\,. \tag{A.12}$$

Then

$$R_{(-)}{}^{a}{}_{b}\wedge R_{(-)}{}^{b}{}_{a} = d\omega^{\rm L}_{(-)} = d\star_{(4)}d(\partial\log\mathcal{Z}_0)^2 = -\partial_m\partial_m(\partial\log\mathcal{Z}_0)^2 d^4x\,, \tag{A.13}$$

where d^4x is the volume form of $\mathbb{E}^4$. To obtain this expression we have used the local connection $\Omega_{(-)\underline{m}np}$ given in (A.10), which is well defined in $\mathbb{R}^4$ except at the pole of $\mathcal{Z}_0$ at $\rho=0$, where it becomes singular. Since the quantity computed in (A.13) is gauge invariant, the result obtained is valid everywhere except at this isolated point, which is not covered by our local connection. Evaluating explicitly the right hand side, at zeroth-order in α', we obtain

$$-\partial_m\partial_m(\partial\log\mathcal{Z}^{(0)}_0)^2 = \partial_m\partial_m\left[4\frac{\rho^2+2\mathcal{Q}_0}{(\rho^2+\mathcal{Q}_0)^2} - \frac{4}{\rho^2}\right] = \partial_m\partial_m\left[4\frac{\rho^2+2\mathcal{Q}_0}{(\rho^2+\mathcal{Q}_0)^2}\right] - 4\delta^{(4)}(\rho)\,. \tag{A.14}$$

While the first term in that expression is a continuous, regular function, the second term just introduces a pointlike singularity at $\rho=0$ that, according to the preceding discussion, should be interpreted as spurious.

Since the components of the 4-form $R_{(-)}{}^{a}{}_{b}\wedge R_{(-)}{}^{b}{}_{a}$ are continuous, at this stage it is clear that at this order in α' we have[25]

$$R_{(-)}{}^{a}{}_{b}\wedge R_{(-)}{}^{b}{}_{a} = \partial_m\partial_m\left[4\frac{\rho^2+2\mathcal{Q}_0}{(\rho^2+\mathcal{Q}_0)^2}\right]dx^4 \equiv -\partial_m\partial_m\left[(\partial\log\mathcal{Z}^{(0)}_0)^2\right]_{\backslash\odot} d^4x\,, \tag{A.15}$$

where $\mathcal{Z}^{(0)}_0$ is the piece of $\mathcal{Z}_0$ which is of zeroth order in α', which is the harmonic function in $\mathbb{E}^4$ defined in eq. (3.28). Here we have introduced the symbols $\{\backslash\odot\}$ to indicate that the (harmonic) singular term should be removed from the term within squared brackets.

It is convenient to use the 't Hooft ansatz with SO(4) indices for the gauge field as well. We can write it in the form

$$A = \mathbb{M}^{-}_{mp}\partial_p\log\mathcal{Z}_{\rm YM}dx^m\,, \tag{A.16}$$

where $\mathcal{Z}_{\rm YM}$ is the harmonic function on $\mathbb{E}^4$

$$\mathcal{Z}_{\rm YM} = 1 + \frac{\kappa^2}{\rho^2}\,. \tag{A.17}$$

[24] The same is true, with opposite self-duality, for $\Omega_{(+)\underline{m}np}$, but we will focus on $\Omega_{(-)\underline{m}np}$ only, because it is the one whose Chern-Simons 3-form and curvature occur in the equations of motion.

[25] This result is also obtained by performing a (singular) local Lorentz transformation that would render the torsionful spin connection regular at $\rho=0$, in virtue of the removable singularity theorem of Uhlenbeck ref. [39].

Using the result obtained for the $\omega^L_{(-)}$,[26]

$$\begin{aligned}\omega^{\rm YM} &= -\star_{(4)}\, d(\partial\log\mathcal{Z}_{\rm YM})^2\,, \qquad\qquad (A.22)\\ F^A\wedge F^A &= d\omega^{\rm YM} = \partial_m\partial_m\left[(\partial\log\mathcal{Z}_{\rm YM})^2\right]_{\backslash\odot} d^4x\,, \qquad\qquad (A.23)\end{aligned}$$

where, following the same reasoning as before, the singular contribution must be removed. Thus, taking into account the general form of the 3-form H in eq. (A.5), the Bianchi identity of the 3-form field strength eq. (2.10) can be written in the form

$$-\partial_m\partial_m\left\{\mathcal{Z}_0+2\alpha'\left[(\partial\log\mathcal{Z}_{\rm YM})^2-(\partial\log\mathcal{Z}_0^{(0)})^2\right]_{\backslash\odot}\right\}=0\,, \qquad (A.24)$$

The above equation is solved by

$$\mathcal{Z}_0=\mathcal{Z}_0^{(0)}+2\alpha'\left[(\partial\log\mathcal{Z}_0^{(0)})^2-(\partial\log\mathcal{Z}_{\rm YM})^2\right]_{\backslash\odot}+\mathcal{O}(\alpha'^2)\,, \qquad (A.25)$$

where we have used that $\mathcal{Z}_0=\mathcal{Z}_0^{(0)}+\mathcal{O}(\alpha')$. In the language of section 3,

$$f_0(\rho)=2\left[(\partial\log\mathcal{Z}_0^{(0)})^2-(\partial\log\mathcal{Z}_{\rm YM})^2\right]_{\backslash\odot}=8\left[\frac{\rho^2+2\kappa^2}{(\rho^2+\kappa^2)^2}-\frac{\rho^2+2\mathcal{Q}_0}{(\rho^2+\mathcal{Q}_0)^2}\right]\,, \qquad (A.26)$$

which is the same result as in eq. (3.17). Upon substitution in the Bianchi identity, it reduces to the Laplacian of a harmonic function on $\mathbb{E}^4$:

$$-\partial_m\partial_m\mathcal{Z}_0=0\,. \qquad (A.27)$$

As usual, this equation is not satisfied at the singularities of the harmonic function and the corresponding δ-functions will give contributions to the S5-brane charge (see eq. (4.8)).

Using these results in the definition of H eq. (2.7) we arrive at the following equation for the Kalb-Ramond 2-form B:

$$dB=d\left[\mathcal{Z}_-^{-1}du\wedge dv\right]+\star_{(4)}d\mathcal{Z}_0^{(0)}\,. \qquad (A.28)$$

The integrability condition is satisfied if $\mathcal{Z}_0^{(0)}$ is harmonic in $\mathbb{E}^4$ and for the value in eq. (3.28), it is given by

$$B=\mathcal{Z}^{-1}du\wedge dv+\frac{1}{4}\mathcal{Q}_0\cos\theta d\varphi\wedge d\psi\,, \qquad (A.29)$$

and receives no α'-corrections to this order.

[26]We have to take into account that the anti-self-dual SO(4) generators have the normalization

$$\mathrm{Tr}(\mathbb{M}^-_{mn}\mathbb{M}^-_{pq})=-2(\mathbb{M}^-_{mn})_{pq}\,,\qquad [\mathbb{M}^-_{0i},\mathbb{M}^-_{0j}]=\varepsilon_{ijk}\mathbb{M}^-_{0k}\,,\qquad i=1,2,3\,. \qquad (A.18)$$

Therefore, using the above representation,

$$\begin{aligned}F &= dA+A\wedge A\,, \qquad\qquad (A.19)\\ \omega^{\rm YM} &= -\mathrm{Tr}\left[dA\wedge A+\frac{2}{3}A\wedge A\wedge A\right]\,, \qquad\qquad (A.20)\\ F^A\wedge F^A &= -\mathrm{Tr}F\wedge F\,. \qquad\qquad (A.21)\end{aligned}$$

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