Universidad de Oviedo

Programa de Doctorado en Economía y Empresa

TESIS DOCTORAL:

Aplicaciones del análisis de eficiencia y de redes complejas a redes logísticas y de dominancia

Applications of efficiency and complex network analysis to logistics and dominance networks

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PREFACE

This Thesis is presented as a compendium of five publications. The structure of the document follows the recommendations stated in the University of Oviedo regulations regarding the format of a Thesis to be presented at this institution.

The major purpose of this Thesis is to contribute to the knowledge of *Data Envelopment Analysis* (DEA). Data Envelopment Analysis is a technique that aims to evaluate the efficiency of homogeneous productive units and to determine benchmarks to guide inefficient units to achieve the *Efficient Frontier*. Several contributions to DEA are sought in this Thesis.

One of the main goals of this Thesis is to develop a new approach to analyse the dominance relationships that appear between the assessed units in a DEA context. This approach is a methodology based on *Complex Network Analysis*.

Complex Network Analysis (CNA) is a traditional technique to characterise the relationships among the elements of a system. It has a powerful visualisation tool that can be applied to a DEA context. The first methodological approach introduced here is called *Dominance Network Analysis* (DNA). Since it is based on CNA, it is flexible enough to represent the different relationships usually studied in DEA.

In DEA there are many different models designed to evaluate the relative technical efficiency, but also the economic efficiency. The proposed DNA metholodgy is capable of evaluating both relationships at the same time, as described in three of the papers included in this Thesis, using different datasets to show the potential of this new approach.

Another main goal of this Thesis is to develop a new concept to determine intermediate benchmarks that the inefficient units can achieve in the technology seeking the Efficient Frontier. The new concept, called *Gradient-based stepwise benchmarking paths*, is based on the Efficiency Field Potential (EFP). This approach determines consecutive intermediate benchmarks, depending on the position of the evaluated *Decision Making Unit* (DMU).

The last contribution of this Thesis is to explore the potential of DEA in a project team assessment context. In teamwork, one of the most difficult problems is to measure the contribution of each member to the team. However, if this contribution were known, there should exist a relationship between the efficiency of the team members and the efficiency of their developed projects. So, the efficiency of each team member could be estimated based on the efficiency of their results.

This Thesis is divided into five Chapters. The first one, the Introduction, introduces the issues studied in this Thesis and the basic literature, which was used to support the research undertaken and developed. The second Chapter defines the objectives of the Thesis, while the results are shown in the third Chapter, and the conclusions are gathered in the fourth Chapter. Apart from that, there are three appendices completing this documentation. In the first one the list of the five papers making up this Thesis can be found. All these publications are included in the second appendix, and the third one presents a report with the impact factors of the journals in which the mentioned publications were published.

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(CNA)   Complex Network Analysis

(CRS)   Constant Return Scale

(DEA)   Data Envelopment Analysis

(DMU)   Decision Making Unit

(DNA)   Dominance Network Analysis

(EES)   Efficiency Equipotential Surface

(EFP)   Efficiency Field Potential

(EFV)   Efficiency Field Vector

(FDH)   Free Disposal Hull

(GSS)   Gradient Stepsize

(IBT)   Intermediate Benchmarking Targets

(ID)   Improvement Dimensions

(MILP)   Mixed Integer Linear Programming

(MIP)   Measure Inefficiency Proportions

(RAM)   Russell Adjusted Measure

(SBI)   Slack Based Inefficiency

(SEIP)   Scale Efficiency Improvement Program

(SOM)   Self-Organizing Map

(TEIP)   Technical Efficiency Improvement Program

(UBT)   Ultimate Benchmarking Targets

(VRS)   Variable Return Scale

(WAM)   Weighted Additive Model

1. INTRODUCTION

The successful results of a company in the market depend on achieving a competitive advantage, its strategy in the market, and also the image that its customers have about the firm, its products and its competitors. The technical process used by a company has a direct effect on its strategy, and it could represent a competitive advantage over its competitors. The importance of the efficiency of the technical process is based on the limited resources needed by the firm to produce its outputs: the more valuable and scarce they are, the more expensive it is to achieve them. As a consequence, the management of resources and the efficiency of the process benchmarked against its competitors is critical for the survival of the company.

There are several ways to make a difference in the market, the most common being offering new products, improving their quality, or reducing the price of existing ones by reducing costs. When a company wants to develop an efficient production process, its ultimate goal is to achieve the largest amount of outputs possible using the minimum quantity of resources. The cost of a productive factor depends mainly on its accessibility and how it is managed. All these productive factors are ultimately managed by the human factor, arguably the most valuable and critical asset of the company. The performance of each company depends on its management capacity.

In the last 60 years, a considerable research effort has been undertaken to evaluate the efficiency of the production units and how resources could be used to achieve

better results. When a production process is assessed, it is important to do it not only technically, but also from the management perspective. A production process is technically more efficient than others if it requires fewer resources for the number of finished goods, while retaining the same level of quality.

On the other hand, the projects that a company accomplish must be managed to achieve a better position in the market against its competitors. From a project management perspective, the good performance of the team affects the outcome of the project and one of the main factors affecting the good performance of the team is the individual contribution of each member. However, measuring the contribution and efficiency of each member is a difficult task. Even so, the research by Wiest, Porter and Ghiselli (1961) about the relationship between team performance and the individual performance of each member of the team, states that a combination of the individual scores is a good prediction of the team performance.

Data Envelopment Analysis (DEA) is a powerful methodology to evaluate the relative efficiency of productive processes by comparing them with other homogeneous productive processes. In the last 30 years, a huge variety of applications has been developed. As reported by Liu et al. (2013), the main fields where this growth in applications could be observed are banking, health care, transportation, education, agriculture and farming. This technique allows the identification of the feasible technological frontier regarding the information related to the resources and finished goods.

DEA not only offers the possibility to evaluate the performance of each unit but also the stepwise improvement path. The inefficient units could improve their performance in multiple directions, either improving the reduction of inputs or increasing production. Once the direction is defined, the potential improvement that it could achieve depends on its resources and the available technology.

DEA can evaluate a huge amount of units and analyse multidimensional variables; however, it focuses on computing individual efficiency scores and targets. It could be interesting to assess the efficiency of the dataset as a whole by establishing a dominance relationship between the assessed units to integrate the situation of each

one compared to all the others. Therefore, another technique is needed that could determine these kinds of relationships, visualise them and also analyse the structure of all the relationships of the units as a whole.

Today there are multiple scientific areas studying systems as a whole. Any researched process or system could be considered to be the result of multiple interactions between different variables or items. For the sake of simplicity, following the willingness to understand a process or system, there is a reduction in the number of variables that could be taken into account. However, in different fields the current tendency is to develop more integrationist models to justify non-linear phenomena and understand dynamic processes. Several methodologies try to consolidate more variables to provide an answer to how a system works, whether its behaviour is linked to the structure of the network, which are the effect of that structure —Multi-agent Systems and CNA can be listed among these methodologies. CNA has been traditionally used to represent flows of information, material or relationships between different agents. It is very well-known modelling in sociology, but in the last 25 years there has been a spectacular growth in its application to other disciplines, such as Linguistics, Engineering, Biology, Economics (as in da Fontoura Costa et al., 2011).

The main aim of CNA is to research the interaction of the system, then analyse and characterise the resultant network. These analyses are used to model networks, understand how they were created, reveal the hidden structures and predict the future effects of the structure in the behaviour of the networks. The research on complex networks determines, for instance, how diseases are spread, how information could flow more quickly through a network, how vulnerable a network is to random or target attack. Apart from these analyses, CNA provides a flexible and powerful visualisation tool to represent the system.

The general aim of this Thesis is to contribute to further research on the field of efficiency analysis and also provide a new tool to evaluate the situation of the assessed units based on the dominance relationships between those units. These issues open a fruitful field of research that could be applied to logistics and other different fields.

In the rest of the Chapter, a review of the literature which was used to develop the objectives of this Thesis is introduced.

1.1 Data Envelopment Analysis

DEA is a well-known non-parametric technique that can be used to gauge the relative efficiency of homogeneous and comparable DMU. This technique assesses the efficiency using the number of sources (inputs) which are used by each DMU and the results that it achieves (outputs). The amount of the input i and the output k of the unit j are defined by x_{ij} and y_{kj} respectively. One unit is more efficient than another one if, when they are compared, the latter has potential input reductions while maintaining the output levels, or potential output increases while maintaining the input levels. The efficiency of a DMU is defined by an efficiency score based on the potential improvements, the orientation of the used DEA model (i.e. if the aim is to find input improvements, output improvements, or both), the metric (e.g. radial, non-radial, additive, etc.), and the technology assumptions (*Constant Return Scale* (CRS), *Variable Return Scale* (VRS), *Free Disposal Hull* (FDH), etc.).

Figure 1 shows a small dataset with seven DMUs having one input and one output. The three different technologies that can be considered in DEA are represented: the dotted line represents the technology FDH where the observed DMU is only feasible; the dashed line represents the technology VRS, which considers that a linear combination of the observed units is feasible; finally, the continuous line represents the CRS technology which considers that the observed units are scalable. The three arrows represent the three different orientations that a DMU can use to achieve the Efficient Frontier: the discontinuous arrow represents the input orientation, which seeks an input improvement while keeping the output level, the double arrow represents the output orientation, seeking to maximise the output without reducing the inputs; finally the continuous arrow represents the orientation used in the additive model which seeks the furthest efficient DMU in the Efficient Frontier.
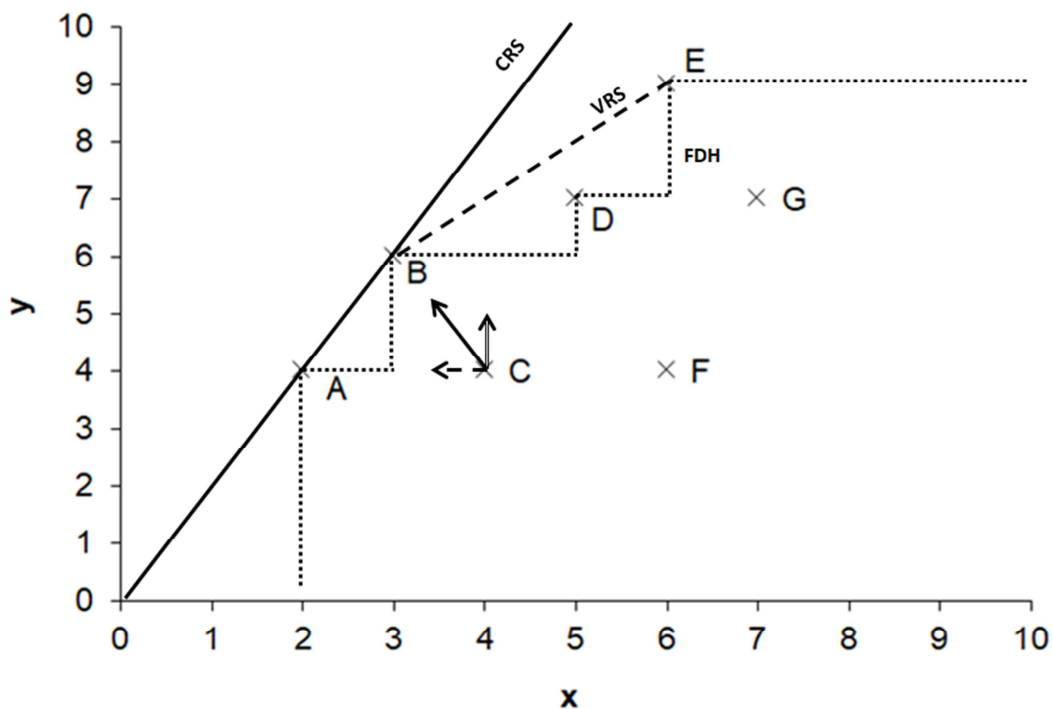
Figure 1: Visualisation of the Efficient Frontier and the direction of improvement for the inefficient DMUs

1.1.1 Relevant Additive Models in DEA

DEA is used to evaluate the inefficiency of the DMUs belonging to the technology, thus determining their distance to the Efficient Frontier. There are different ways to measure this distance depending on the orientation of the model. There are three main categories: it could be *input oriented* where the aim is to find input improvement; *output oriented* if the aim is to seek output improvement; and *additive models* if both variables are to be improved.

Apart from that, there are two different ways to evaluate the performance of a DMU, i.e. it could be seen from a technical or economic point of view. The *technical* perspective follows the criterion related to the efficiency of the production process. However, a firm can change the input and output quantities to achieve a better *economic* performance, depending on the input/output price units. As a consequence, the economic performance could be assessed in relation to profit, cost or the revenue of the firms.

In this section, diverse models which were used to develop this Thesis are described, including three well-known technical additive models from both a technical and economic perspective. Based on these models, inefficiency or efficiency metrics are developed to measure the inefficiency between two DMUs or between a DMU and the Efficient Frontier.

In these models, n DMUs are assessed. Each DMU consumes m inputs $X_j = (x_{1j}, \dots, x_{mj})$ and produces s outputs $Y_j = (y_{1j}, \dots, y_{sj})$. As described above, the additive model seeks both input and output improvements, so the difference of each variable from the Efficient Frontier is measured to determine the potential improvement of each DMU. These differences are measured with input slacks (h_i^-) and output slacks (h_k^+). So, all efficient DMUs which determine the Efficient Frontier are identified with zero slacks. For these DMUs there is no possible improvement without worsening any of these DMUs as defined by the "*Pareto-Koopmans*" definition.

The three models defined from the technical perspective are *Measure Inefficiency Proportions* (MIP), *Russell Adjusted Measure* (RAM) and the *Weighted Additive model.*

MIP and RAM models are described in Cooper, Park and Pastor (1999). The MIP model represents the potential improvement of each DMU with respect to its position. It is described as [1]-[4] (DMU 0 being the assessed DMU). So, the inefficiency value is a proportional improvement of the assessed DMU.

$$Max \qquad \text{MIP} = \sum_{i=1}^{m} \frac{h_{i0}^-}{x_{i0}} + \sum_{k=1}^{s} \frac{h_{k0}^+}{y_{k0}} \qquad [1]$$

s.t.

$$x_{i0} = \sum_{j=1}^{n} x_{ij}\lambda_j + h_{i0}^- \qquad i = 1, \dots, m \qquad [2]$$

$$y_{k0} = \sum_{j=1}^{n} y_{kj}\lambda_j - h_{k0}^+ \qquad k = 1, \dots, s \qquad [3]$$

$$0 \leq \lambda_j, \; h_{i0}^-, h_{k0}^+ \; \forall i, r, j \tag{4}$$

On the other hand, RAM normalises the slacks of each DMU before maximising them. In contrast to MIP, RAM is independent of the position of the assessed DMU. After normalising, the value obtained in the model depends only on the distance of the inefficient DMU to the Efficient Frontier (eqs. [5]-[8])

$$Max \quad RAM = \frac{1}{m+s} \left(\sum_{i=1}^{m} \frac{h_{i0}^-}{R_i^-} + \sum_{k=1}^{s} \frac{h_{k0}^+}{R_k^+} \right) \tag{5}$$

s.t.

$$x_{i0} = \sum_{j=1}^{n} x_{ij} \lambda_j + h_{i0}^- \quad i = 1, \dots, m \tag{6}$$

$$y_{k0} = \sum_{j=1}^{n} y_{kj} \lambda_j - h_{k0}^+ \quad k = 1, \dots, s \tag{7}$$

$$0 \leq \lambda_j, \; h_{i0}^-, h_{k0}^+ \; \forall i, r, j \tag{8}$$

The normalisation is done with the range of the variables [9]-[10]. These ranges represent the maximum possible value of inefficiency and offer a fixed reference to determine the inefficiency of the DMUs.

$$R_i^- = \max_j \{x_{ij}\} - \min_j \{x_{ij}\} \tag{9}$$

$$R_k^+ = \max_k \{y_{kj}\} - \min_k \{y_{kj}\} \tag{10}$$

The previous models consider only the information related to the position of the DMUs. However there are other models which take into account the relative importance of each input and output. The following model [11]-[14] is described in Cooper et al. (2011). It is a weighted additive model that measures the technical inefficiency of a given DMU 0 (TI_0). Each input and output has an associated weight,

$W^- = (w_1^-, \dots, w_m^-)$ for the inputs and $W^+ = (w_1^+, \dots, w_s^+)$ for the outputs. The weights represent the relative importance of each variable.

$$TI_0 = \sum_{i=1}^{m} w_i^- h_{i0}^- + \sum_{k=1}^{m} w_k^+ h_{k0}^+ \qquad [11]$$

s.t.

$$\sum_{j=1}^{n} x_{ij}\lambda_j + h_{i0}^- \leq x_{i0} \qquad i = 1, \dots, m \qquad [12]$$

$$\sum_{j=1}^{n} y_{kj}\lambda_j - h_{k0}^+ \geq y_{k0} \qquad k = 1, \dots, s \qquad [13]$$

$$0 \leq \lambda_j, \ h_{i0}^-, h_{k0}^+ \ \forall i, r, j \qquad [14]$$

From the economic perspective, as described above, there are three different concepts to analyse: profit, revenue and cost. To evaluate the economic consequences of the amount of the consumed inputs and the produced outputs, it is necessary to have two more parameters (input price unit q_i, and output price unit p_k). It is important to highlight that economic efficiency is based on technical efficiency and any DMU cannot be economically efficient without being technically efficient.

Cooper et al. (2011) develop a new profit inefficiency measure to establish profit as the term to evaluate economic efficiency. This model is based on the Weighted Additive model. One Weighted Additive model is used to evaluate the achievable optimal level profit of the assessed DMUs (eqs. [15]-[19])

$$\Pi(\overline{q}, \overline{p}) = \text{Max} \ \sum_k p_k \hat{y}_k - \sum_i q_i \hat{x}_i \qquad [15]$$

s.t.

$$\sum_j \lambda_j \, x_{ij} = \hat{x}_i \qquad \forall \, i \qquad\qquad [16]$$

$$\sum_j \lambda_j \, y_{kj} = \hat{y}_k \qquad \forall \, k \qquad\qquad [17]$$

$$\sum_j \lambda_j = 1 \qquad\qquad [18]$$

$$\lambda_j \in \{0,1\} \quad \forall \, j \qquad\qquad [19]$$

So, the profit inefficiency, PI_0 of a DMU 0 is given by eq. [20], i.e., the normalised difference between the optimal level of profit and the current profit of the assessed DMU.

The main advantage of the proposed model is that it is homogeneous of degree zero because it is normalised by the minimum ratio of market prices to its relative weights.

$$PI_0 = \dfrac{\Pi(\overline{q},\overline{p}) - \left(\sum_k p_k \, y_{k0} - \sum_i q_i \, x_{i0} \right)}{\min\left\{ \dfrac{q_1}{w_1^x}, \dfrac{q_2}{w_2^x}, \ldots, \dfrac{q_m}{w_m^x}, \dfrac{p_1}{w_1^y}, \dfrac{p_2}{w_2^y}, \ldots, \dfrac{p_s}{w_s^y} \right\}} \qquad [20]$$

Cooper et al. (2011) also prove that $PI_0 \geq TI_0$. Due to this fact, economic inefficiency can be decomposed into technical inefficiency and allocative inefficiency (AI), the latter being the difference between the economic inefficiency and the technical inefficiency [21].

$$PI = TI + AI \hspace{4cm} [21]$$

Due to the PI and TI being based on slacks, the Pareto-Koopmans definition is applied, so there is no risk that the projected DMUs on the production frontier belong to the non-efficient frontier. All the efficient units are identified with slacks zero and belong to the Efficient Frontier.

As mentioned above, revenue can be used to determine the economic efficiency. It is a similar approach to Profit Inefficiency, but with an output orientation. Aparicio et al. (2013) develop revenue inefficiency and also decompose it into technical inefficiency and allocative inefficiency. However, a similar approach based on input orientation and cost function evaluates the economic perspective from the cost term.

All the models are defined for CRS technology; however, they can be applied to other technologies, such as VRS adding to the models $\sum_j \lambda_j = 1$, or in the case of considering FDH technology, adding $\lambda_j = \{0,1\}$.

1.1.2   Intermediate Target and Stratification

In the same way that there are multiple assumptions to determine the feasible technology or the orientation that an inefficient DMU can follow to achieve the Efficient Frontier, multiple methodologies are developed to determine the stepwise benchmarking path which guides the DMU to its desirable position in the Efficient Frontier.

Two different kinds of benchmarks define the intermediate steps of an inefficient DMU to its desirable position. *Intermediate Benchmarking Targets* (IBT) comprise the benchmarks of the assessed DMU until the Efficient Frontier, while *Ultimate Benchmarking Targets* (UBT) guide the DMU along the Efficient Frontier to its desirable position.

The efficiency of a unit is assessed with respect to the DMUs that are used to assess its performance. Based on this concept called *Context-Dependent* DEA, Seiford and Zhu

(2003) define attractiveness and progress measures concerning DMUs of the evaluation context and also a stratification process. The stratification process categorises the DMU in layers depending on their efficiency and attractiveness. The DMUs which belong to layers with lower values are more efficient, the DMUs of the layer 0 being the efficient units of the technology. Layer 1 is built by analysing the technology after removing the nodes of the previous layers, and so on for the successive layers. All DMUs which belong to a layer have the same level of attractiveness.

In the case that the efficient benchmark of an inefficient DMU is a distant target for the inefficient units, several methodologies identify intermediate targets based on similarity of DMUs. Other methodologies consider that the DMUs can only achieve benchmarks which are below a certain distance, so determine a bounded stepwise path.

Estrada et al. (2009) define a *Proximity-Based Target Selection process*. This process uses a Self-Organizing Map (SOM) to cluster DMUs in neighbourhoods that consume similar inputs in order to determine the potential intermediate benchmark of inefficient DMUs. Moreover, the intermediate benchmarks that comprise the shortest path to the Efficient Frontier are selected by the *Reinforcement Learning algorithm*. This algorithm determines the intermediate benchmarking of an inefficient DMU between its neighbours by limiting the efficiency score that a DMU can achieve.

Lozano and Villa (2010) propose two models in a VRS case to define successive intermediate targets: one model, called the *Technical Efficiency Improvement Program* (TEIP) for the inefficient units to achieve the Efficient Frontier, and another one called the *Scale Efficiency Improvement Program* (SEIP) for efficient units to achieve the scale efficiency. Both models determine the spaced intermediate targets by fixing an upper bound.

In the same direction, Lim et al. (2011) define stepwise benchmarking paths for inefficient units to achieve the Efficient Frontier, but selecting existing DMUs as intermediate targets based on the concept of context-dependent. So, an intermediate target would be an inefficient DMU but more attractive than the inefficient assessed

DMU, taking into account the three criteria of attractiveness, progress and infeasibility, after clustering DMUs into different layers.

Park, Bae and Lim (2011) propose a target selection method to take into account the similarity using the *k-means clustering algorithm*. Instead of determining the size of the stepwise benchmarking path depending on the number of stratified layers, the target selection method selects the cluster which has the highest similarity with the assessed DMU and determines the next benchmarking target of the DMU with the highest efficiency in the selected cluster.

Another stepwise benchmarking method, which takes into account not only the similarity between nodes, but also the efficiency gaps, is proposed by Ghahraman and Prior (2016). This method reduces the risk of failure of achieving out-of-reach efficiency targets. After measuring the efficiency scores with a DEA model, a directed weighted network is created where all the inefficient units point to a better performer and the weight of each link takes into account the input changes and the efficiency gap covered. *Dijkstra's* shortest path algorithm is used to calculate the optimal stepwise benchmarking path for each inefficient unit.

Table 1 shows a summary of the main characteristics of the different approaches. In particular, for each approach the table shows the type of IBT, and UBT considered, whether stratification is used, whether the benchmarking path is computed over a benchmark network (whose nodes are the DMUs and whose edges indicate the possible steps that can be taken to form the efficiency improvement path), whether bounds on the stepsizes are considered, whether the DMUs are clustered, the similarity criteria considered for selecting each IBT, and whether the method suffers zigzagging (i.e. moving in inverse directions in successive steps). The final column shows some specific features of the methods.

Among the specific features of some of the methods, it is remarkable that a preference structure can be considered to select the UBT, the consideration of a fixed cost for carrying out each benchmarking step, or computing a decision tree from the DMU stratification to try to identify the differences in input and output ranges in two

successive layers. The extension of stepwise benchmarking to centralized DEA and to systems with two stages in series is also remarkable.

Table 1. Summary of existing stepwise efficiency improvement approaches

Reference	IBT	UBT	Stratification	Benchmark network	Stepsize constraints	Clustering	Similarity criteria	Zigzagging	Other features
Hong et al. (1999)	Existing DMUs	Existing DMUs	Yes (Tiers)	No	No	SOM (inputs only)	Same cluster	Yes	Decision tree for tier classification
Lozano and Villa (2005)	Feasible operating points	Efficient operating points	No	No	Yes (on the change of each variable)	No	Efficiency improvement	No	CRS
Estrada et al. (2009)	Existing DMUs	Existing DMUs	No	No	Yes (on efficiency change)	SOM (inputs only)	SOM distance	Yes	Reinforcement Learning
Sharma and Yu (2009)	Existing DMUs	Existing DMUs	Yes (Tiers)	No	No	SOM (inputs only)	Same cluster	Yes	
Sharma and Yu (2010)	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	No	No	No	Attractiveness and progress	Yes	Decision tree for attribute prioritization
Lozano and Villa (2010)	Feasible operating points	MPSS efficient operating points	No	No	Yes (on the change of each variable)	No	Efficiency improvement	No	VRS
Park et al. (2011)	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	No	No	k-means	Inter-cluster distance	Yes	
Lim et al. (2011)	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	No	Yes (joint linear constraints)	No	Attractiveness, progress and Infeasibility	Yes	
Suzuki and Nijkamp (2011)	Feasible operating points	Efficient operating points	Yes (Context dependent DEA)	No	No	No	Distance friction minimization	Yes	

Reference	IBT	UBT	Stratification	Benchmark network	Stepsize constraints	Clustering	IBT selection criteria	Zigzagging	Other features
Park et al. (2012a)	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	Yes	No	No	Resource improvement, directional proximity (inputs)	Yes	Shortest Path
Park et al. (2012b)	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	No	No	SOM (inputs only)	SOM distance, directional proximity (inputs)	Yes	Preference structure
Park et al. (2012c)	Existing DMUs	Existing DMU	Yes (Context dependent DEA)	No	No	No	Least distance measure	Yes	Resource priority analysis
Khodakarami et al. (2014)	Feasible operating points	MPSS efficient operating points	No	No	Yes (on the change of each variable)	No	Ray average productivity	Yes	Extension to two stage systems
Park et al. (2014)	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	Yes	No	k-means (XE matrix)	Same cluster, Least distance measure	Yes	Shortest Path
Fang (2015)	Feasible operating points	Efficient operating points	No	No	Yes (on efficiency change)	No	Efficiency improvement	No	Centralized DEA
Park and Sung (2016)	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	Yes	No	k-means (XE matrix)	Same cluster	Yes	
Ghahraman & Prior (2016)	Existing DMUs	Existing DMUs	No	Yes	Yes (on the change of each variable)	Network components	Euclidean distance (normalised inputs), efficiency change	Yes	Shortest Path, Fixed cost

Notes: IBT=Intermediate Benchmark Targets, UBT=Ultimate Benchmark Targets, CRS=Constant Returns to Scale, VRS=Variable Returns to Scale, MPSS=Most Productive Scale Size, SOM=Self-Organizing Map, XE=Cross-efficiency

1.2 Complex Network Analysis

As mentioned above, Complex Network methodology is used to represent systems. Nodes and links compose each network, the nodes or vertices represent the entities of the system, and the links or edges the relationships among them. A huge variety of different fields, apart from sociology, have developed applications of CNA in their areas to use an integrationist approach. One of the main purposes of this technique is to visualise the relationship in the system and measure the defined properties in CNA in order to understand the system as a whole, as communities and as individual units.
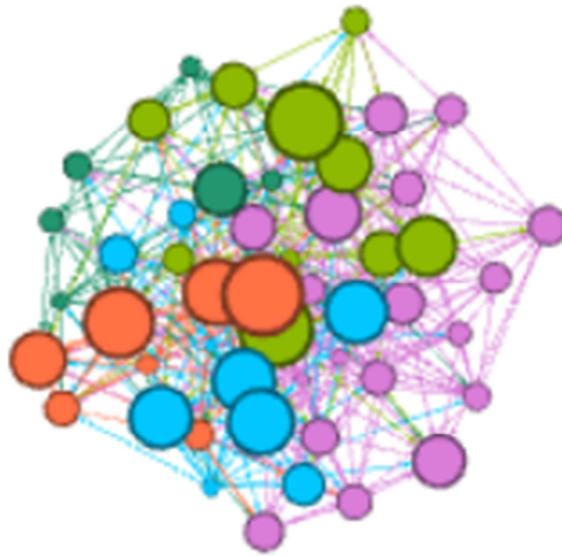
Figure 2: Visualisation of a random generated network

1.2.1 Kind of Networks and Basic Properties

There are different ways to establish a relationship between the nodes of the network. The links can be directed or undirected; as a result, the network is a *directed* or *undirected graph* respectively. Each link always has an associated weight, so the network is a *weighted graph*, but in the case that all the links are equal to one, the network is considered to be an *unweighted graph*.

Therefore, a network can represent a system with different kinds of relationships, in which case *multiplex networks* are used to evaluate each different relationship (type of link). An edge is usually a connection between two nodes, but in the case that it connects more than two vertices, these are called hyperedges. Moreover, the network which contains them is called a *hypergraph*.

Another kind of network is the *bipartite network*, which has two types of nodes connected by links and all the links connect nodes of a different type. So, there is no link that connects two nodes of the same type. All these relationships among the nodes are gathered in the adjacency matrix or the incidence matrix; both matrices provide the same information although their structure is different.

The length between two nodes is the sum of the weights links that create the route to connect them. There are different paths to achieve one node from another one; however, the shortest one is called the *geodesic path*. There are multiple routes to achieve a node from another node and each route could have a different length. If the distance between two nodes independently on the considered route is the same, then the network has the *additivity property*. Another highlighted property is the *transitivity property*, if in all the situations where there is a link between nodes i and j and another one between j and h, another link connects the nodes i and h.

1.2.2 Metrics

When measuring complex networks there are different metrics. Each one is classified depending on from which level the network is being assessed. There are three kinds of levels: one from a nodes perspective as the fundamental part of the network; another one from a cluster of nodes; and another one from a network perspective.

One of the main metrics at the nodes level is the *degree,* which measures the number of links that have their origin and destination in the same node, as well as the *strength* whose value is the sum of the edges that are linked to the node. It is important to highlight that for directed networks these metrics can consider the direction of the edges if they have the origin or the destination of the node.

Although the degree is a metric at the node level, the degree of the nodes can be observed at the network level with the *degree distribution* or with the *degree-degree correlation*. The degree distribution is the probability distribution of the degrees over the whole network and the degree-degree correlation is used to establish a relationship between the degree of a node and the average degree of its neighbour.

Apart from that, in the case that the purpose of the analysis is to study the position of the node, clustering coefficient and centrality metrics can be used. The *clustering coefficient* of a node determines if the neighbours of a node are connected between them. *Centrality metrics* determine the importance of each node, but this can be considered from different point of view. Specifically, *betweenness centrality* determines the proportion of the number of geodesic paths that pass through it, while *eigenvector centrality* analyses the importance of a node depending on the importance of its neighbours. Apart from those, *alpha-centrality* is also a centrality metric with the same interpretation as eigenvector centrality, but alpha-centrality being the generalized formula.

Therefore, two nodes can be compared to study if they are equivalents, based on their structure, meaning that they are linked to the same nodes. To study this equivalency the *Pearson correlation coefficient* or *Jaccard's similarity coefficient* can be used.

As a cluster of nodes, the nodes can be joined in communities or components. In communities or clusters, a criterion is established to define which attribute must have a group of nodes to be considered as a community. The criterion is usually based on how they are connected, for example, if the network has more links than in a random generated network or there are overlapping small connected structures.

Another way to gather nodes is with the component. A component is a subgraph of the network whose nodes are connected so that each node can achieve any other node of the subgraph. Depending on the direction of the links, whether it is considered or not, the component is a strongly or weakly connected component respectively. Apart from that, based on the *structural similarity coefficient* a *hierarchical clustering algorithm* can be used to identify groups of nodes that have a given degree of structural similarity among them.

From the network perspective, some of the main metrics are *diameter*, *geodesic path*, *average path*, *network efficiency* and *density*. The diameter represents the longest geodesic path of the network, while the average path length calculates the average of the geodesic path of the network. The network efficiency is the inverse harmonic mean of the path lengths and the density, which determines how many links are in a network by taking into account the maximum number of links that could exist. Apart from that, if high connected nodes tend to be linked to other high connected nodes, then it is considered that the network has an assortative property, or disassortative in the case that they are connected to low connected nodes.

1.2.3   Classical Models

In addition to measuring a real network, different models can be used with certain properties to generate networks to compare with the real ones (Newman, 2003). Amongst others, we can find *Random Graphs*, *Small Worlds* or *Scale-Free models*. The first of these was developed by Erdös and Rényi (1959). This model creates random links with a certain probability. The Small-Worlds model was defined by Watts and Strogatz (1998), and created from regular networks reconnecting several links randomly. This network has as its special properties high clustering and short average path length, which are so frequent in Sociology. Another well-known model is the Scale-Free model developed by Barabási and Albert (1999), having as a specific attribute its high connectivity between high connected nodes due to its continuous growth with nodes that prefer to be connected to high connected nodes.

1.2.4   Applications of Complex Network Analysis

CNA is a versatile technique to analyse a system and has been used by different applications. Liu et al. (2009) rank 40 Internet companies with a network-based approach after evaluating the units in DEA. To rank the DMUs, a directed weighted network is created where the links represent the endorsement between DMUs, and the weight of these relationships is based on the lambda values of the DEA calculations

after evaluating all the possible input/output combinations. The ranking is created with a centrality measure called *eigenvector centrality*. A similar network approach is used by Liu and Lu (2010) to evaluate and rank the research and development performance of Taiwan's government-supported research institutes, taking into account that the performance is assessed in two-stages. One stage is for the technology development, and the other for the technology diffusion. Ho et al. (2014) analyse the technology transfer efficiencies in the US with a similar approach. Liu and Lu (2012) apply an analogous network-based approach, but applying alpha centrality to each stage used by DEA to evaluate a real-world problem of banks.

However, these approaches sum all the lambda values obtained in the applied DEA analysis to all the combinations of inputs and outputs. This assumption considers that all the inputs and outputs have the same weights and are substitutable. Apart from that, all the links point to the efficient peers. So, links point to a better performer only if it is an efficient DMU. After announcing these concerns, Ghahraman and Prior (2016) created a network-based approach completely different from analysing benchmarks and outliers.

Ghahraman and Prior (2016), as mentioned above, proposed a method to achieve the optimal stepwise benchmarking path based on Dijkstra's shortest path. But also it uses a centrality metric called alpha-centrality and a clustering technique to determine the presence of outliers of specialized units. The alpha-centrality metric is used to determine the benchmarking potential of each DMU depending not only on the situation and the benchmark status of the assessed DMU, but also on the situation of their neighbours and status. On the other hand, the clustering technique is used to determine outliers by gathering all the DMUs into a cluster that can be a benchmark unit between them, due to the relative improvement being lower than a certain threshold.

2. OBJECTIVES

As stated in the previous chapter, the goal of this Thesis is to contribute to further research on the field of efficiency analysis, as well as providing a new tool to evaluate the situation of the assessed units based on the dominance relationships between the units.

Specially, the main objectives of this research are:

O1. To explore and formalise a tool to develop the concept of a dominance network where dominance relationships between assessed units could be analysed and visualised.

O2. To apply the developed tool to the concepts of profit and revenue efficiency.

O3. To introduce the concept of efficiency potential in DEA.

O4. To explore the use of DEA in project management contexts.

As mentioned above, in pursuing these objectives, five papers were published covering those goals. The references of these papers ([A] to [E]) can be found in Appendix 1.

Each publication covers at least one of the goals stated above. In this chapter a general overview of those objectives and papers is presented. A report on each one can be found in Appendix 2.

O1.To explore and formalise a tool to develop the concept of a dominance network where dominance relationships between assessed units could be analysed and visualised.

With the purpose of developing the concept of a dominance network, three research papers were published ([A], [B] and [C]). All these papers introduce the methodology called Dominance Network Analysis (DNA), which applies the techniques developed for complex network to efficiency assessment.

DNA creates a network to manage the dominance relationships between the assessed DMUs. As the dominance relationship is understood here, a link is created between two nodes having its origin in the inefficient DMU and its destination in a more efficient DMU. As a result a directed acyclic network called the Dominance Network is defined. The weight of the links is the relative inefficiency of the connected DMUs. Thanks to the transitivity and additivity property of the network, the minimum and maximum distance of any inefficient DMU to the Efficient Frontier can easily be measured.

DNA allows the classification of each node, depending on its dominance relationship and position in the network. Apart from classification methods commonly used in DEA such as stratification, others can be used based on CNA, such as gathering the nodes into communities, components, or clusters based on the similarity of nodes. Both CNA and DEA are useful to define different metrics to characterise the network at different levels. These metrics vary, depending on the position of the DMUs and show the relative position in the network, such as clustering coefficient, betweenness centrality, etc. Others, such as degree-degree correlation shows to which kind of nodes the nodes are linked. Degree-degree correlation is used to evaluate the relationships of the nodes based on how they are linked to other nodes depending on their degree. Assortativity is observed if high connected nodes are linked to other high connected ones, or disassortativity when high connected nodes are linked to low connected ones.

Apart from the analysis and visualisation, Dominance Networks can be used to make partial rankings using multi-criteria decision making. The computed ranking

disaggregates the layers, and ranks of the nodes according to a defined preference relation.

DNA offers a valuable integrated DEA CNA framework. DEA has different restrictions to plot DMUs' multidimensional input and output vectors; this handicap is solved by the Dominance Network. The visualisation tools of CNA are really powerful due to the filters linked to it.

O2. To apply the developed tool to the concepts of profit and revenue efficiency.

The proposed method in the previous objective is so flexible that it can stand different kinds of relationships. DEA, as mentioned in the Introduction, considers that the DMUs can be analysed from an economic perspective apart from the technical one.

An enhanced DNA is proposed to consider both kinds of relationships. Technical efficiency considers the input consumption and the output production while economic efficiency uses the cost of the inputs and the price of the outputs. Moreover, the analysis of economic and technical inefficiency leads to the concept of Allocative Inefficiency which measures the difference between both relative performances. As mentioned in DEA, the economic perspective can be analysed using the concepts of profit, cost or revenue. The utilization of each concept to analyse the DMUs from the economic perspective depends on the availability of data and the criteria of the analyst.

As a result of this enhanced approach, a multiplex directed weighted acyclic network is created to represent technical and economic dominance relationships. The created network has two Efficient Frontiers, one for the technical relationship and the other for the economic relationship. The distance between them is measured by the concept of allocative inefficiency, which will be represented by hyperedges.

The relationships between these two relationships and the properties of the created network are studied, but due to the definition of the links, it is expected that the economic relationship between two nodes only exists if there is a technical

relationship between them. It is also expected that the existence of an economic relationship is linked to the concept of scalar potential.

O3.To introduce the concept of efficiency potential in DEA.

The Introduction shows that there is an extensive literature about the stepwise benchmarking paths. However, in this Thesis a new approach is introduced based on the gradient of an EFP; the EFP is defined in such a way that the fewer inputs consumed and the more outputs produced by DMUs, the lower its EFP value.

Each DMU, depending on its position, has a scalar EFP and a negative gradient which defines the *Efficiency Field Vector* (EFV), and all the DMUs that have the same scalar EFP are over the same *Efficiency Equipotential Surface* (EES). The EFVs are perpendicular to the EES and define the direction of an inefficient DMU to achieve the Efficient Frontier.

The EFP definition can be accommodated to dimensionless inputs and outputs, but it has to keep the principles of strong monotonicity, efficiency achievement, and the lower values being linked to the efficient DMUs.

Based on these concepts, two models are defined to determine the intermediate steps that an inefficient DMU has to take to achieve the Efficient Frontier; also, another model is developed to define the movement of the efficient DMU along the Efficient Frontier to achieve the point of minimum potential.

O4.To explore the use of DEA in project management contexts.

A classical problem of the literature on project management is to determine the contribution of the team members to the projects on which they have worked. However, if the contribution of each member were known, theoretically there should be a relationship between the efficiency of the team members and the efficiency of the projects in which they were involved.

One of the objectives of this Thesis is to determine if DEA could be applied to determine the performance of each of the team members. If there is a relationship

between the performance of the team members and their results, the performance of each member of the team could be assessed based on the performance of the projects on which he/she has worked.

3. RESULTS

As stated in Chapter 2, there were four objectives to be developed in five research papers. In this chapter, a summary of the achieved results in each of these research papers are summarized. As mentioned before, all the publications are available in Appendix 2.

3.1 Efficiency assessment using network analysis tools[1]

As mentioned in the previous chapter, this paper introduces in detail the Dominance Network methodology to analyse and visualise multidimensional datasets. This methodology, based on CNA and DEA, represents the dominance relationship and its relative inefficiency performance among the assessed DMUs. The created network represents all the possible benchmarking paths that any inefficient DMU has to take to achieve the Efficient Frontier.

This paper presents multiple metrics that could be applied to dominance networks at different levels, from the node level to the network level, and also defines different filters that could be applied to the visualisation of the network.

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[1] Lozano, S. and Calzada-Infante, L., "Efficiency assessment using network analysis tools" *Journal of the Operational Research Society* (2018) doi: https://doi.org/10.1080/01605682.2017.1409866

The proposed methodology is applied to a small dataset of the literature found in Lim et al. (2011). This dataset is composed of 12 DMUs with two inputs and one constant output. The final network (see Figure 3) is composed of the DMUs as nodes (D) and directed links (E), which show the dominance relationship. There is a link between the DMU r and DMU j if DMU r consumes more inputs than the DMU j for the same level of outputs or if the DMU r produces fewer outputs than the DMU j for the same level of inputs. It is said that DMU j dominates DMU r, D(r) being all the nodes which dominate DMU j. Therefore, the weight of the links between the two DMUs is defined by eq. [22].

$$e_{rj} = \begin{cases} 0 & \text{if } j \notin D(r) \\ \sum_i \dfrac{x_{ir} - x_{ij}}{g_i^x} + \sum_k \dfrac{y_{kj} - y_{kr}}{g_k^y} & \text{if } j \in D(r) \end{cases} \qquad [22]$$
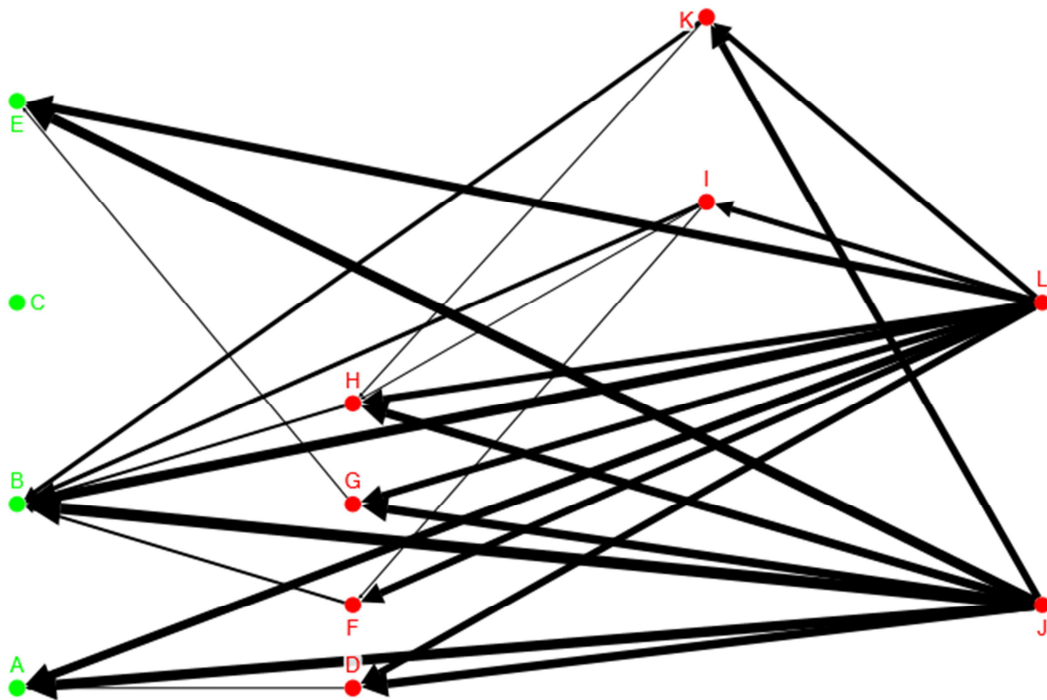
Figure 3: Dominance Network applied to the Lim et al. (2011) dataset

The weight of the links represents the relative inefficiency between the assessed DMUs, which is based on the RAM DEA model. The width of the links is proportional to the relative inefficiency between the DMUs of origin and destination.

As can be observed in Figure 3, all the nodes of the network which are not the origin for any link are efficient. The Efficient Frontier is composed of four DMUs and DMU (C) is an outlier because it does not have any relationship with the rest of the nodes.

DNA allows the classification of each node depending on its dominance relationship, defining which nodes are dominated (D^{-1}), which dominated nodes are only dominated by the efficient DMU j $(D^{--1}(j))$. Therefore, all the nodes that are not the origin of any link compose the Efficient Frontier (D^*). Apart from that, each node is associated with a layer whose value is the maximum number of intermediate benchmarks that could have a DMU before achieving the Efficient Frontier.

In addition, different classifications commonly used in DEA or CNA can be applied. From the DEA perspective, a stratification process can be applied, so the nodes can be categorised into layers, in layer 0 are all the efficient nodes and the rest of the nodes are in layers with higher values, depending on the number of steps they have to take before achieving the Efficient Frontier. On the other hand, CNA classification methods provide a different point of view. For example splitting the nodes into components can easily check in which group each DMU is compared. The components that are composed of only one node are isolated nodes; they are called outliers because they cannot be compared with the rest of the nodes, i.e., these DMUs neither dominate any other nor are dominated.

Moreover, different metrics at node, layer, component and network level are defined, some of them based on CNA metrics such as clustering coefficient, betweenness centrality, density, diameter, degree of the nodes, etc. In addition, other metrics are based on DEA metrics such as the *inefficiency measure* (e_r^{max}) which measures the maximum distance of the inefficient DMU r to the Efficient Frontier; the *benchmarking potential* (κ_j) as a measure of the advantage of an efficient DMU j over its dominated DMUs; and the distance of the *closest dominating efficient node j* (τ_r^{min}). Therefore, there are other, different metrics to measure the composition of each level and also its relationship with the efficient DMUs and the rest of the nodes.

This analysis not only shows the potential of the defined metrics but also the potential of the visualisation of multidimensional datasets where filters can be applied. The filters increase the utility of CNA visualisation, so the networks can be represented by certain properties allowing certain parts of them to be highlighted.

Some of the filters visualise edges or nodes that have a particular characteristic over or under a certain threshold. However, other visualisations typical of CNA can be used such as bipartite graph filtering to visualise only the relationships between the efficient and inefficient DMUs. This kind of filter can be applied only for certain inefficient or efficient DMUs in order to visualise the relationship to a certain DMU. Another two classical visualisations of CNA are the *ego-network filter* and the *skeletonisation filter*. The former shows all the relationships linked to a certain node, while the latter removes all the transitive arcs showing a network more clearly without losing information.
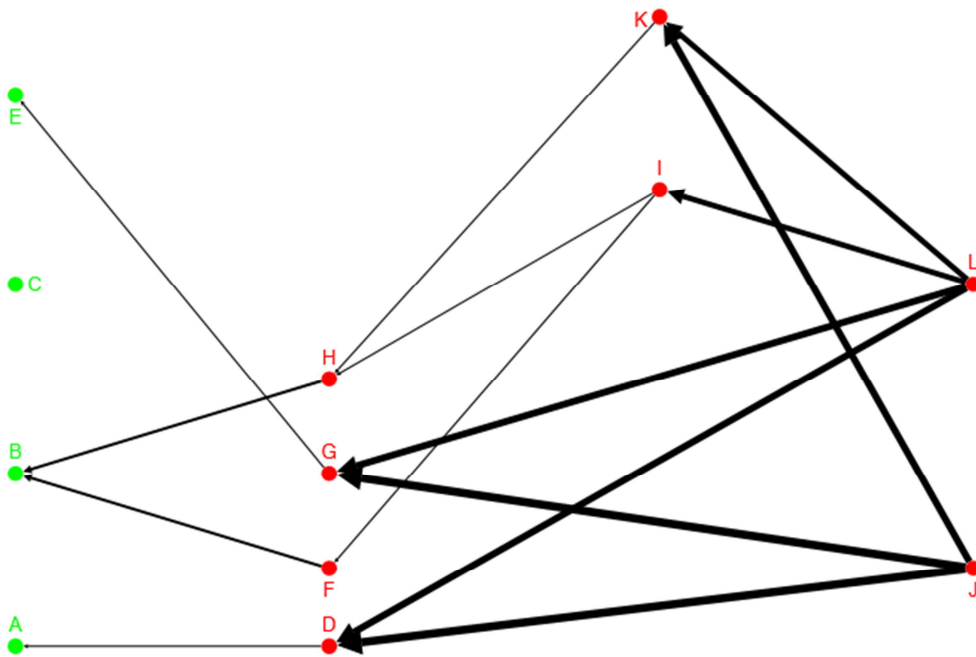
Figure 4: The skeletonisation filtering of the Dominance Network applied to the Lim et al. (2011) dataset

Figure 4 is the result of applying the skeletonisation to the dominance network in Figure 3. This filter allows a simplified representation of the network without losing

information due to the additivity and transitivity properties which are implied in the network.

To test the robustness of the proposed approach, different direction vectors are considered to calculate the normalisation. Four direction vectors are considered:

- $g = \left(g_i^x, g_k^y \right)$

- $g^{average} = \left(x_i^{aver}, y_k^{aver} \right)$

- $g^{range} = \left(R_i, \hat{R}_k \right)$

- $g^{median} = \left(x_i^{median}, y_k^{median} \right)$

After creating and analysing the dominance network for each direction vector, the obtained results show a high correlation between them. The absolute magnitude of the applied metrics differs for different direction vectors, but their relative values are totally consistent.

Apart from studying the impact of a change in the direction vector, the sensibility of the network after introducing an additional efficient or inefficient DMU is also studied. The network topology changes a little and affects the DMUs that are linked to the new DMU. The position of the DMU can affect the number of geodesic paths, the layers which are linked to the DMUs, etc., but by adding just a single DMU does not change the network topology too much. However, the observed changes that appear in Dominance Networks can also be observed in DEA, due to the efficiency score of the DMUs, depending on their position.

3.2   Analysing Olympic Games through dominance networks[2]

The aim of this application paper is to assess and rank the performance of the countries in the Olympic Games taking into account their capacity and the dominance

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[2] Calzada-Infante, L. and Lozano, S., "Analysing Olympic Games through dominance networks", *Physica A*, 462 (2016) pp:1215-1230

relationships within their performance. The advantage of this method over other traditional simplified ranking methods, such as lexicographic order, is to consider the capabilities of the countries. This point is considered in many DEA models that have been applied to Olympic Games datasets. However, establishing a rank based on a DEA model, such as the *Integer-Valued DEA (IDEA)* model defined in Wu et al. (2010), only considers the distance to the Efficient Frontier, while the proposed approach considers all the dominance relationships established by the position of all the DMUs in the space.

In the proposed approach, two inputs (GDP and Populations (POP)) and one output (number of medals) are used to evaluate the performance of the countries as is traditionally done in DEA.

With these variables, a weighted directed network is created where the nodes are the countries and the links are the dominance relationships between them. A directed link from country r to country j exists if country j consumes fewer resources to produce the same or higher level of outputs or if country j produces more outputs consuming the same of inputs as country r. The weight of these links represents the weighted difference of medals achieved between the linked countries (eq. [23]). In this analysis the weighted coefficient used are $v_B = 1; v_S = a; v_G = a^2$.

$$
\begin{aligned}
w_{rj} &= v_G \cdot \left(Gold_j - Gold_r \right) + v_S \cdot \left(Silver_j - Silver_r \right) + v_B \cdot \left(Bronze_j - Bronze_r \right) = \\
&= v_G \cdot \Delta Gold_{jr} + v_S \cdot \Delta Silver_{jr} + v_B \cdot \Delta Bronze_{jr}
\end{aligned}
\tag{23}
$$

After creating the network for three different values for the parameter a, minor changes could be observed, so the value of $a = 2$ is used to evaluate the performance of the countries.

The network, which has a layered structure, has more than one component. Some of them are outliers which cannot be benchmarked against other countries. Most of the countries belong to the giant weakly connected component. All the efficient countries are easily identified because they belong to layer 0. The layer is linked to the distance

of the performers to the Efficient Frontier and it is tied to the in- and out- strength. Figure 6 shows to which layer each country belongs.

Some metrics based on CNA are used in this publication, such as strength, clustering or betweenness centrality, among others. The value of these metrics varies, depending on the position of the DMUs in the network, and shows the relative position of each node. Other metrics such as the average length and the network diameter indicate the overall magnitude of these performance differences.

Apart from that, after applying degree-degree correlation, assortativity is detected in the case of the average in-degree of the average in-degree of in-nearest neighbours and also for the average out-degree of the average out-degree of out-nearest neighbours. This kind of assortativity means that nodes with high in-degree tend to receive links from nodes with high in-degree and nodes with high out-degree tend to connect to nodes with high out-degree. On the other hand, disassortativity is observed in the average out-degree of in-nearest neighbours and the average in-degree of out-nearest neighbours, which means that nodes with high in-degree tend to receive links from nodes with low out-degree and nodes with high out-degree tend to connect to nodes with low in-degree.

A hierarchical clustering algorithm is used to identity groups of nodes that have a given degree of structural similarity among them based on Jaccard's coefficient. To achieve these clusters the Single Linkage Clustering Algorithm is applied. Three groups have exactly the same structural similarity, which means they have the same incoming and outgoing links, which in turn means that they dominate and are dominated by the same countries. Bahrain (BH) and Trinidad & Tobago (TT) are structurally similar to France (FR) and Italy (IT) and also to Tajikistan (TJ) and Kyrgyzstan (KG).

The differences between the in- and out- strength has been identified as useful to gauge each country's performance. Figure 5 shows this difference for each country and its relationship to the ratio of their GDP to POP. So, a partial ranking of the countries using an analogy between the in- and out- strength and the positive and negative outranking flows is computed in Promethee I.
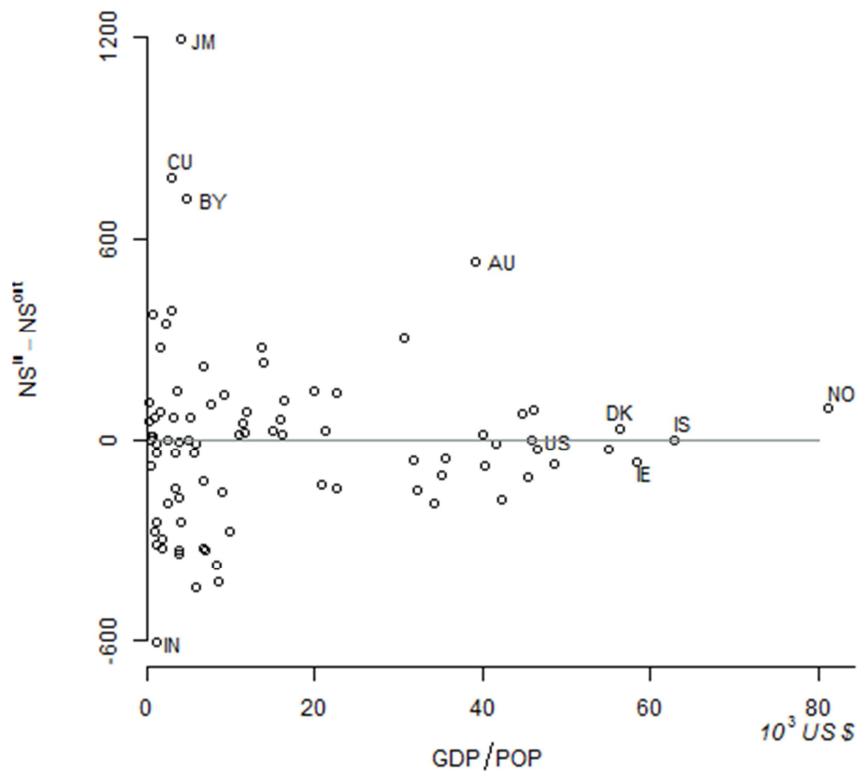
Figure 5: In-out strength difference versus per capita GDP (a=2)

This partial ranking is compared with the lexicographic and IDEA ranking defined by Wu et al. (2010), and Lexicographic ranking. The latter does not consider the resources that each country has, something that IDEA and the proposed method do. When IDEA and the achieved partial ranking are compared, the proposed method shows that it has more discriminant power than IDEA because the proposed method only ranks four countries in the first position while IDEA ranks 15.
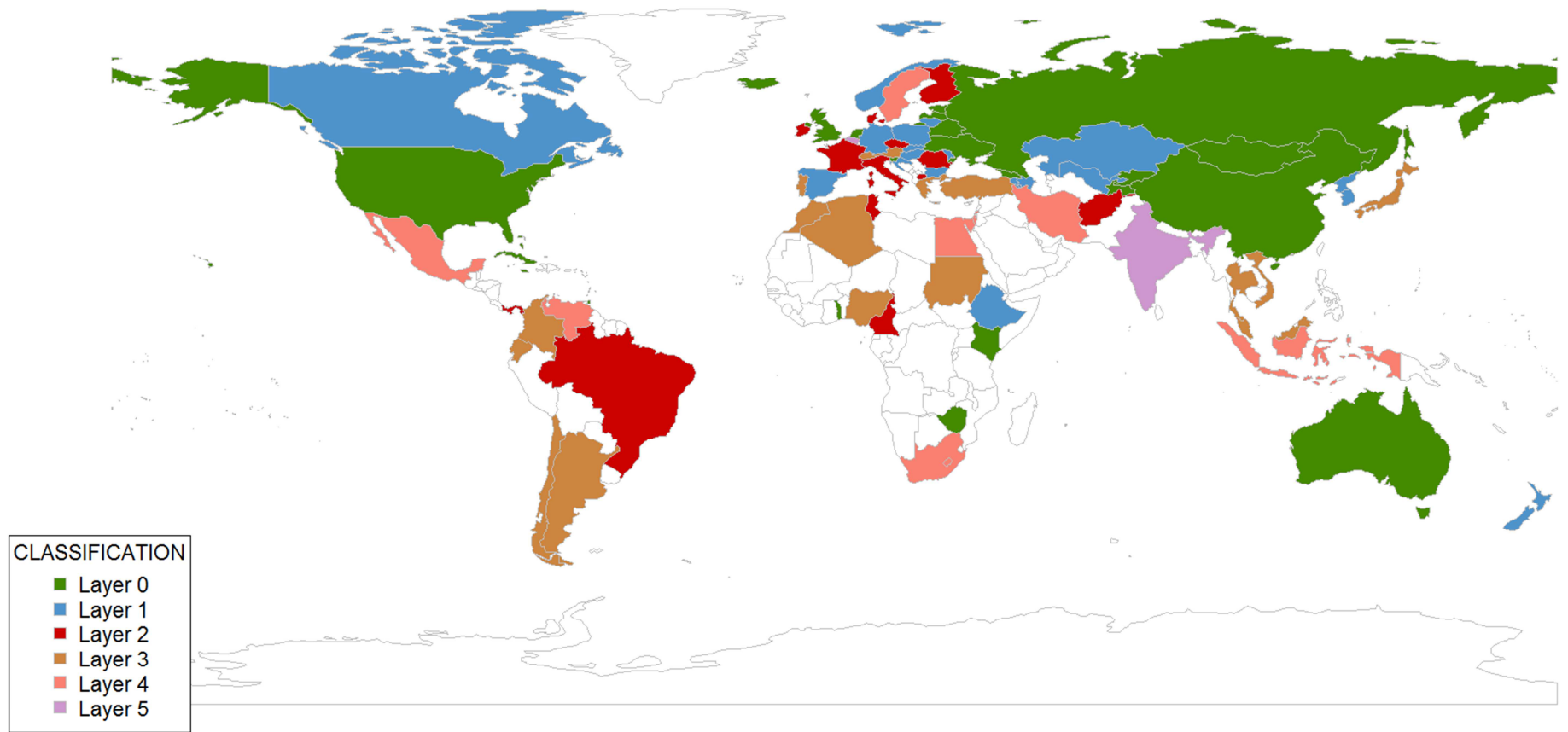
Figure 6: Geographical distribution of layer structure (a=2)

3.3 Dominance network analysis of economic efficiency[3]

The aim of this paper is to assess the technical, economic and allocative efficiency of a set of DMUs using DNA. One DMU could be technically efficient, but not economically efficient because of its input/output unit prices not being competitive.

In this research paper, a novel way of assessing the economic and technical performance of DMUs using complex network tools is proposed. Technical efficiency leads to technical dominance applied in the publications in §3.1 and §3.2; on the other hand economic efficiency leads to economic dominance based on cost, revenue and profit terms.

In this paper a multiplex directed weighted acyclic network integrates both technically inefficient relationships and economically inefficient relationships. The nodes of the network are linked by arcs, representing both the relationships along them. As explained above, technical dominance relationship exists if two nodes are comparable and one node is more efficient than the other. In this case, one link that represents the dominance relationship in technical terms points to the better performer and the length of this link measures the relative technical inefficiency between them. The same is applicable for economically inefficient relationship.

The metrics used to determine the length of both kinds of relationships are based on additive models. The metric in eq. [24] measures the technical inefficiency in the case that both inputs and outputs are known, while the metrics [25] and [26] evaluate the economic inefficiency. The former uses the profit concept and the latter, the revenue concept. The application of one concept of revenue or profit depends on the availability of the data. As a consequence of these metrics, the network has the properties of additivity apart from the essential transitivity property explained above.

$$e_{rj} = \frac{1}{m+s} \cdot \left(\sum_{i=1}^{m} \frac{x_{ir} - x_{ij}}{c_i^x} + \sum_{k=1}^{s} \frac{y_{kj} - y_{kr}}{c_k^y} \right) \qquad [24]$$

[3] Lozano, S. and Calzada-Infante, L., "Dominance network analysis of economic efficiency", *Expert Systems With Applications*, 82 (2017) pp:53-66

$$e_{rj}^{PI} = \frac{\sum\limits_k p_k \, y_{kj} - \sum\limits_i q_i \, x_{ij} - \left(\sum\limits_k p_k \, y_{kr} - \sum\limits_i q_i \, x_{ir} \right)}{\min\left\{ \dfrac{q_1}{w_1^x}, \dfrac{q_2}{w_2^x}, ..., \dfrac{q_m}{w_m^x}, \dfrac{p_1}{w_1^y}, \dfrac{p_2}{w_2^y}, ..., \dfrac{p_s}{w_s^y} \right\}}$$ [25]

$$e_{rj}^{RI} = \frac{\sum\limits_k p_k \, y_{kj} - \sum\limits_k p_k \, y_{kr}}{\min\left\{ \dfrac{p_1}{w_1^y}, \dfrac{p_2}{w_2^y}, ..., \dfrac{p_s}{w_s^y} \right\}}$$ [26]

As described in Chapter 1, the economic inefficiency between two DMUs is greater than the technical inefficiency between them and the difference between these inefficiencies is measured in DEA by allocative inefficiency. In DNA, Allocative
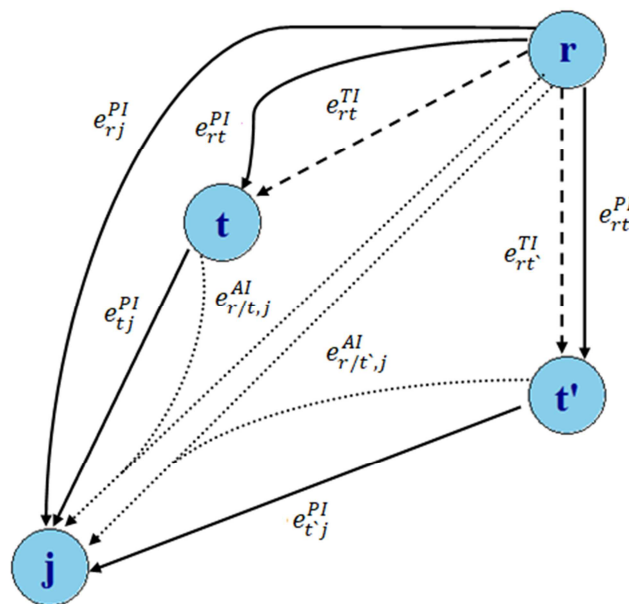
Figure 7: Example of TI and PI edges (discontinuous and continuous arrow, respectively) and AI hyperedges (dotted arrow).

Inefficiency is represented by hyperedges which link inefficient DMUs with economically efficient DMUs as an economic benchmark and technically efficient DMUs as technical benchmarks as can be observed in Figure 7. The length of these

hyperedges determines the distance between the technically Efficient Frontier and the economically Efficient Frontier through the connected technical benchmark.

The enhanced proposed approach is applied to two datasets with the purpose of showing the stated proposition in the paper. The first dataset is from Aparicio et al. (2015) which evaluates 11 DMUs with two inputs and one constant output, and the second dataset evaluates 57 bank branches with two inputs and four outputs obtained from Silva Portela and Thanassoulis (2005).
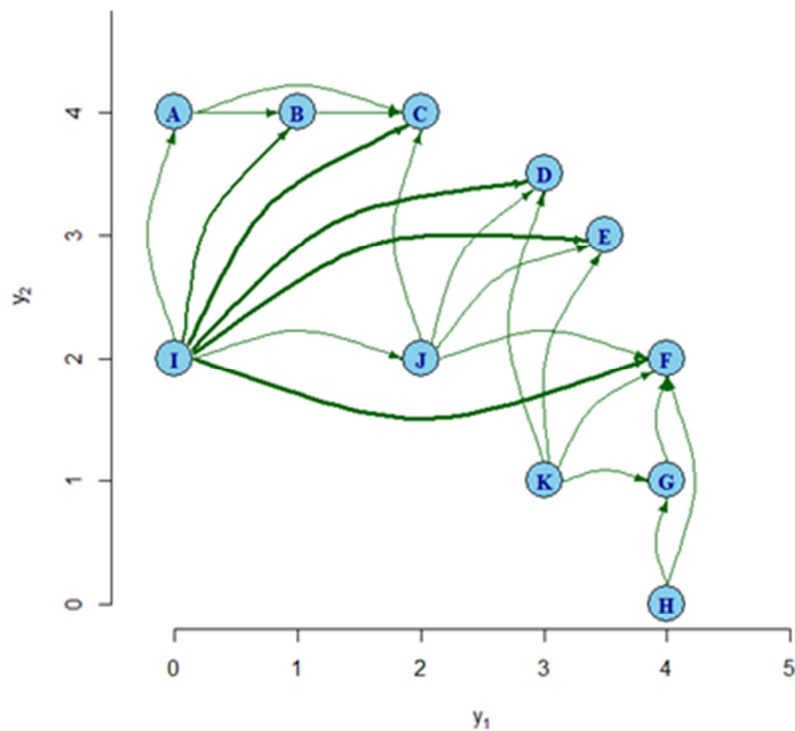
Figure 8: Complete graph with technical inefficient relation of the small size dataset

The network created to evaluate the small dataset considers the concept of revenue to measure the economic efficiency. The network can be observed in Figure 8 and Figure 9. Figure 8 shows the technical dominance relationships, while Figure 9 shows the economic ones. In this latter figure, the isorevenues and their scalar potential can also be appreciated. The DMUs located over the same isorevenues have the same level of income so there could not be any economic dominance relationships among them. In

both figures the width of the links is proportional to the relative inefficiency that they represent.

This dataset has four technically efficient DMUs but only one of them is economically efficient. The network is composed of four layers and as the highest layer is composed of only one node I, this means that the furthest distance of DMU (I) is the diameter of the network. Moreover, for all those inefficient DMUs with only one TI efficient benchmark $e_r^{TI,max} = \tau_r^{TI,min}$ while, for the rest of the inefficient DMUs, $e_r^{TI,max} > \tau_r^{TI,min}$.
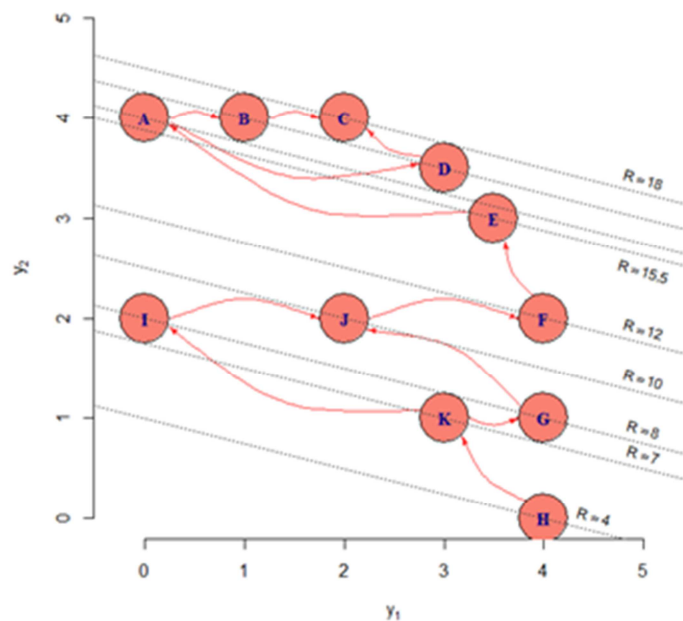
Figure 9: Skeleton filter over the graph with revenue inefficiency relationship of the small-size dataset

The second dataset evaluates different bank branches, and in this case the economic relationship is based on profit. The technically inefficient relationship of the units is composed of one giant weakly connected component and 16 isolated nodes. These isolated nodes are mavericks from a DEA point of view. The structure of the giant weakly connected component is composed of three layers. The layer 0 with all the technically efficient nodes is composed of 42 nodes, but only one of them is

economically efficient. Looking at the economic relationships, it is observed there is no more than one node per layer, due to all the DMUs having different levels of profit. Observing the results of the applied metrics, it is easy to check how the in-strength is a measure correlated with the good performance of the DMUs, and their higher values are linked to a better performer. The skeleton of the profit inefficient relationship can be observed in Figure 10, as the width of the links are proportional to the relative inefficiency, the highest gap between each pair of nodes is between nodes B17 and B53.
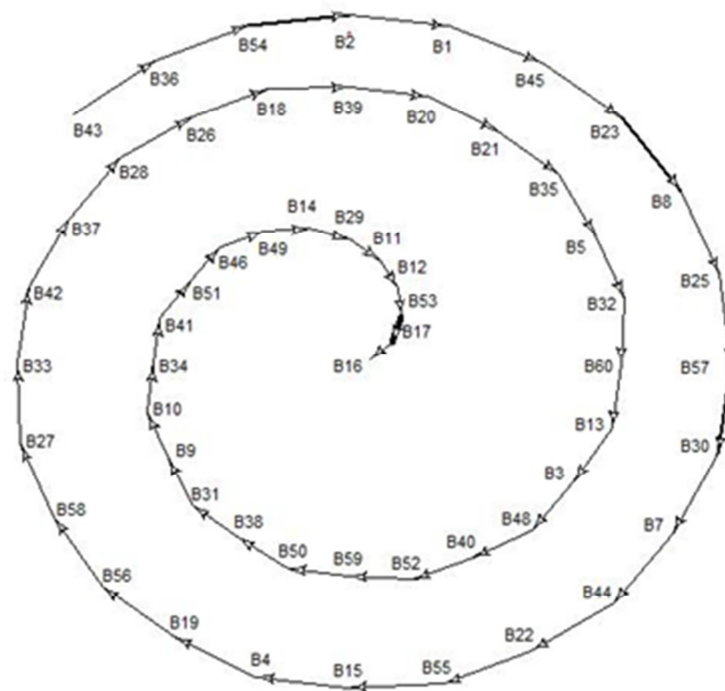
Figure 10: Spiral layout of the skeleton of the PI relationship of the bank branches' dataset

In both datasets it can be observed that only when there is a technical relationship between two DMUs, an economic relationship could exist between them. In addition, the economic inefficiency is always greater than or equal to the technical inefficiency. All the DMUs have an economic scalar potential linked to their position and based on this scalar potential appears the economic relationships.

3.4   Computing gradient-based stepwise benchmarking paths[4]

In this research paper, a new stepwise benchmarking approach is designed based on the gradient of an Efficiency Field Potential (EFP). The EFP is defined such that the fewer inputs are consumed and the more outputs are produced by DMUs, the lower its EFP value.

Each DMU, depending on its position, has a scalar EFP and a negative gradient which defines the Efficiency Field Vector (EFV), and all the DMUs which have the same scalar EFP are over the same Efficiency Equipotential Surface (EES). The EFV are perpendicular to the EES and define the direction of an inefficient DMU to achieve the Efficient Frontier. Figure 11 visualises these concepts in three bidimensional plots where the arrows represent EFV and the dotted lines represents the EES.

Two models are proposed to determine the continuous stepwise path of the inefficient DMUs: *Improvement Dimensions* (ID) and *Gradient Stepsize* (GSS). The ID model is *Mixed Integer Linear Programming* (MILP) and determines which input/output dimensions can be improved in step t if a minimum improvement could be achieved in those dimensions. Using as an input the output of the ID model, the GSS model defines the next position of the assessed DMU. This model is a *Goal Programming model*, where the goal is the desired stepsize, which is limited by the maximum step that can be achieved by that DMU.

Once the DMU has achieved the Efficient Frontier, the efficient DMU can be moved to the area of minimum EFP, with the non-linear optimization model based on EFP definition.

An extension of the proposed approach is also developed, in which it is described how to apply a preference structure to the methodology and also how undesirable outputs and non-discretional variables could be considered.

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[4] Lozano, S. and Calzada-Infante, L., "Computing gradient-based stepwise benchmarking paths", *Omega* (2017) doi: https://doi.org/10.1016/j.omega.2017.11.002
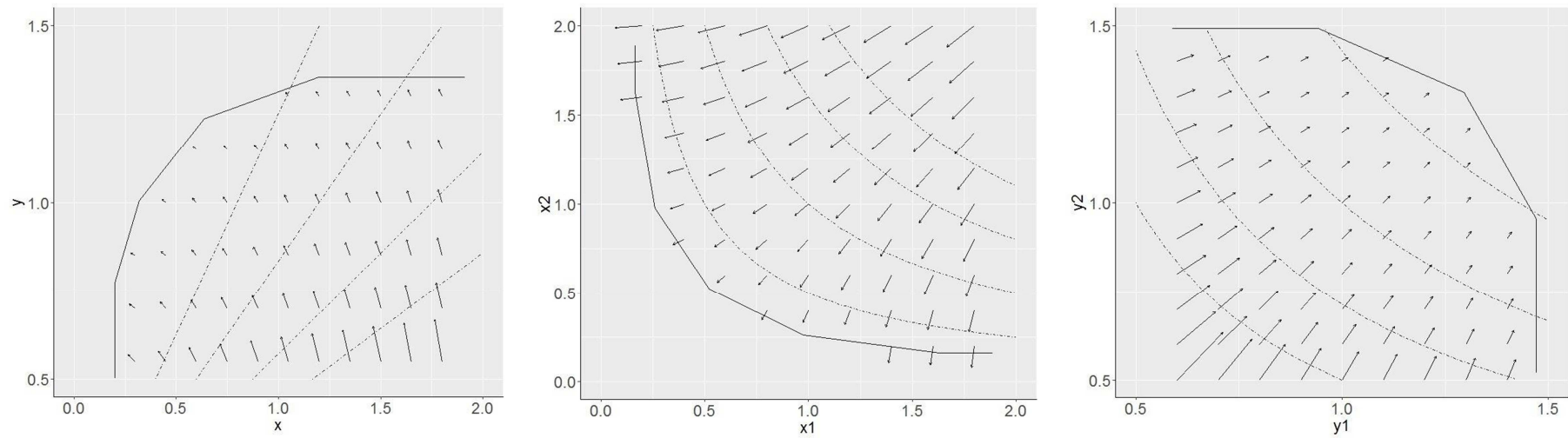
Figure 11: Efficiency vector fields for three bidimensional cases. (The left figure shows a dataset with one input and one output, the centre figure shows a dataset with two inputs and one constant output, and the right figure shows a dataset with one constant input and two outputs)

The new approach is applied to an organic farming dataset, which is composed of 26 DMUs, three inputs (Fuel consumption, Carbon input and Nitrogen input, where Carbon input is non-discretional) and three outputs (Yield fresh matter, Net Primary Production and CO_2 emissions, the latter being undesirable).

The proposed approach offers good results regarding the relationship between the Slack-Based Inefficiency (SBI) score and EFP values. There is a positive correlation between both values and the DMUs having large SBI reductions are the ones that also achieve the largest EFP reduction.

The proposed approach is flexible because it allows the stepsize to be changed, and also a preference structure to be incorporated. Moreover, it is effective in avoiding zigzagging, but it cannot handle either integer inputs or outputs, or operating points with zero inputs or zero outputs. However, regarding the latter, it could be solved using a linear definition of EFP.

3.5   Assessing individual performance based on the efficiency of projects[5]

Assessing the performance of individual team members is a complex task. In this paper a new approach based on DEA has been proposed to determine the individual performance using as the start point the relative efficiency of the projects in which they are involved, and considering that the contribution of each team member to the project is known. A two-step approach is proposed to calculate the performance of the team members.

The first step of the model is to evaluate the relative efficiency of the projects using an input-oriented DEA model, taking into account the cost, duration and difficulty of the projects as input and their revenue as output.

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[5] Adenso-Díaz, B., Lozano, S., Gutiérrez, E., Calzada, L. and García, S., "Assessing individual performance based on the efficiency of projects" *Computers & Industrial Engineering*, 107 (2017) pp:280-288

After evaluating the efficiency of each project J (θ_J), a linear regression is carried out to determine the performance of each member of the teamwork. The linear regression [27] uses the relative efficiency of the project as the dependent variable and the contribution of each member to the project as the independent variable. The error terms (ε_J) are assumed to be independent and normally distributed with mean zero and unknown standard deviation, because it is reasonable to believe that the performance of each individual will vary from one project to another.

$$\theta_J = \sum_i \alpha_i Contrib_{iJ} + \varepsilon_J \tag{27}$$

After analysing the linear regression analysis, the performance of each member i (α_i) is normalised using the maximum performance of the members of the team.

A large number of instances have been randomly generated to validate the proposed approach. These instances have considered a variety of scenarios based on four factors: the number of people working in each project (F1), the variability of the contributions of the individuals involved in a project (F2), the variability of the individuals' efficiencies (F3) and the noise (F4), catching all those events that affect the performance of the projects but cannot be attributed to each team member's performance.

For validation, the estimated performance is compared to the real performance. The correlation between both variables decreases when the team size increases and when the variability in the individual's efficiency has homogeneous groups of high-performance employees, as can be seen in Figure 12.
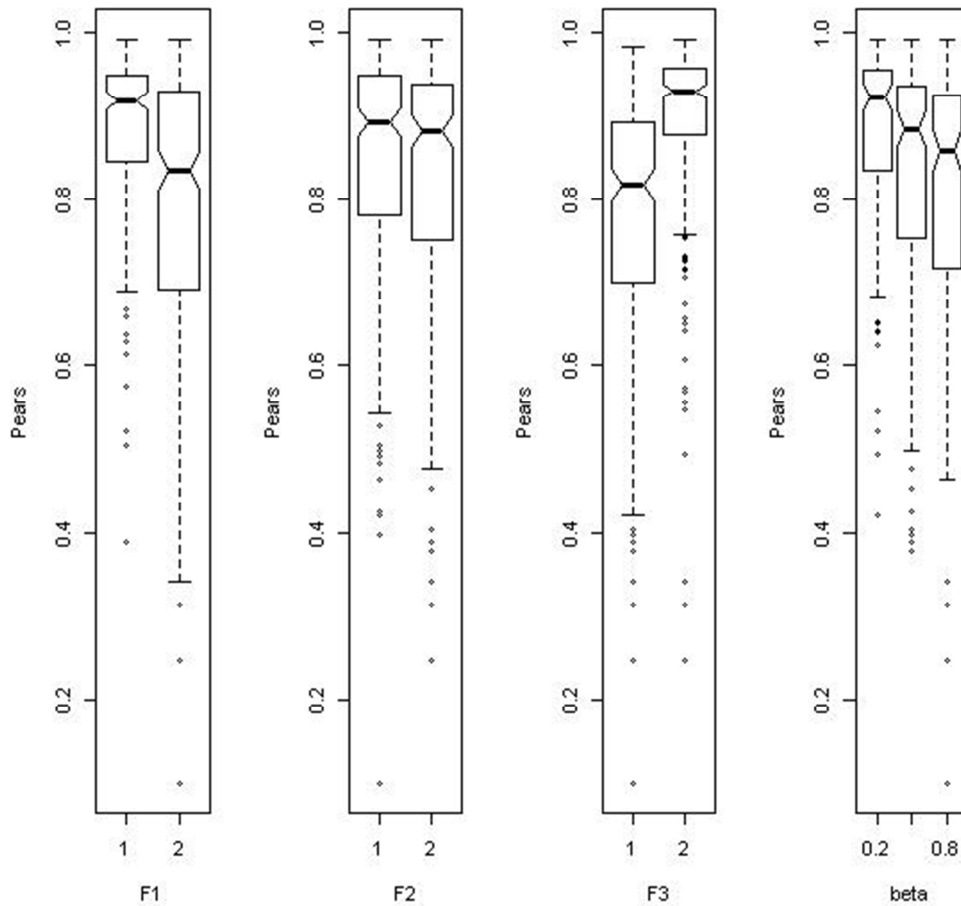
Figure 12: Boxplots of Pearson correlation between estimated relative performances and true individual efficiencies for the different factors.

After testing the proposed approach, this paper evaluates in a real case the individual performance of 10 programmers and analysts from a Spanish software engineering company who have been involved in 46 projects.

After calculating the relative efficiencies of the projects and the multilinear regression analysis, it can be observed that there is a good fit of the linear regression specification with the R-squared value of 0.856. All the independent variables except one are significant at the 0.1 level, which is not significant given the linear model specification considered. All the employees had a significant and negative coefficient.

4. CONCLUSIONS

In this Thesis a new methodology called Dominance Network Analysis (DNA) was defined. This methodology has been implemented to complement and enhance the traditional numerical assessment of DEA, creating a DEA-CNA framework to analyse the dataset at different levels, assessing the dataset as a whole considering an FDH technology and an additive normalised-slacks efficiency measure. The filtering and visualisation capabilities of these tools are very valuable for the large multidimensional dataset. It is easy to identify for each inefficient unit all the possible stepwise paths, i.e., the sequences of targets to achieve the Efficient Frontier, and how far they are from it.

Therefore, it is also easy to determine the dominance relationships, who is dominated by whom and the potential improvement of each step. The Dominance Network visualisation features are very helpful to perceive and assess the relative performance level of the different assessed units.

One application case was related to the performance of the countries that participated in the Olympic Games in Beijing 2008. After applying this methodology, different global and node-specific metrics provide a complete characterisation of the performance differences between the assessed countries. Once the position of each country is determined with respect to the rest of the DMU and has provided a complete characterisation of the performance between the different countries, a partial ranking

has been computed using the analogy with Promethee I. This gives an assessment of the relative performance of the different countries, which considers all the relationships between the countries and makes use of all the information available.

The advantage of this method over other traditional, simplified ranking methods such as lexicographic order is to consider the capabilities of the countries. This point is considered in many DEA models, e.g. the Integer-Valued DEA model. However, DEA models only consider the distance to the Efficient Frontier while the proposed approach considers all the dominance relationships established by the position of all the DMUs in the space.

Moreover, an enhanced approach of DNA was developed to evaluate the DMUs from an economic perspective, apart from the technical perspective. This approach has been applied to the bank field. A multiplex acyclic directed network is created. This network is composed of two Efficient Frontiers, one per relationship. The distance between both Efficient Frontiers is measured by Allocative Inefficiency hyperedges. This new approach is applied to a bank dataset and has several interesting relationships which were defined as propositions in the research paper.

Both approaches are interesting not only in assessing the efficiency of the individual DMUs but also to capture the implicit dependencies between them through the dominance relationships. This fact provides new insights that complement the conventional DEA analysis.

Therefore the DNA methodology, which appears as an innovative approach to determine the stepwise benchmarking paths based on efficiency gradient fields is defined. New concepts are defined: Efficiency Field Potential, Efficiency Equipotential Surfaces and Efficiency Field Vector. Each position of the DMUs is associated with an EFP where the smaller the input consumption and the larger the output production, the smaller the EFP. The negative EFP gradient defines the direction that the inefficient DMUs have to follow to achieve the Efficient Frontier.

This benchmarking approach is comprised of three models to achieve the minimum EFP. Two of them define a stepwise benchmarking path to achieve the Efficient

Frontier with a bounded stepsize along the negative EFP direction, and the third model determines the minimum EFP to which the efficient DMU could aspire. The proposed model can be used with preference structures, non-discretionary variables and undesirable outputs.

This approach has been applied to two datasets, an illustrative example and an organic farms dataset. The illustrative example shows the potential of the proposed approach, while in the organic farms case it is proved that the new stepwise benchmarking approach can be used for datasets with undesired outputs and non-discretional variables.

The last contribution of this Thesis is a two-step approach to evaluate the performance of team members in the context of project management, based on the performance of the projects on which they have worked. The first step evaluates the relative efficiency of the projects in order to calculate the relative performance of each member by using a multilinear regression analysis. The normalised coefficient indicates the relative performance of the individual and three cases are identified, depending on whether the variable is significant, in which case the coefficient can be positive or negative; or non-significant, which means that there is no contribution to the project. This performance score allows the ranking of individuals. This methodology was successfully applied to a software company involved in many development projects.

4.1 Further Research

Regarding the approach of DNA, it would be interesting to extend this approach to other technologies studied in DEA, such as CRS and VRS. Moreover, other inefficiency measures can be used in this approach. It could be interesting to determine how the properties of the network, and also the sensibility of the resulting network based on the considered metric, change despite the risk of losing additivity property.

In this Thesis, the efficiency of DMUs in a certain period as a fixed image has been studied. However, in DEA the DMUs could be assessed in different periods of time and

their evolution over time could be controlled. It would be interesting to apply this principle to the DNA and consider multi-period input-output data.

Regarding the gradient-based stepwise benchmarking path approach, it could be interesting to define an additive EFP in order to be able to evaluate operating points with zero inputs or zero outputs.

Finally, regarding the assessment of the individual performance, other DEA models could be applied to evaluate the performance of the projects, avoiding the slacks of the radial model used. Moreover, it could be interesting to evaluate the individual performance of team members with other regression specifications, (sub-linear for instance), although the linear regression has given rather high goodness of fit indexes.

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APPENDIX 1. LIST OF PAPERS PUBLISHED UNDER THE SCOPE OF THIS THESIS

[A] Lozano, S. and **Calzada-Infante, L.**, "Efficiency assessment using network analysis tools" *Journal of the Operational Research Society* (2018) https://doi.org/10.1080/01605682.2017.1409866

[B] **Calzada-Infante, L**. and Lozano, S., "Analysing Olympic Games through dominance networks", *Physica A*, 462 (2016) pp:1215-1230

[C] Lozano, S. and **Calzada-Infante, L.,** "Dominance network analysis of economic efficiency", *Expert Systems With Applications*, 82 (2017) pp:53-66

[D] Lozano, S. and **Calzada-Infante, L.**, "Computing gradient-based stepwise benchmarking paths", *Omega* (2017) https://doi.org/10.1016/j.omega.2017.11.002

[E] Adenso-Díaz, B., Lozano, S., Gutiérrez, E., **Calzada, L.** and García, S., "Assessing individual performance based on the efficiency of projects" *Computers & Industrial Engineering*, 107 (2017) pp:280-288

APPENDIX 2. PAPERS

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Efficiency assessment using network analysis tools

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ABSTRACT

In this paper, some of the network analysis techniques generally used for complex networks are applied to efficiency assessment. The proposed approach is units invariant and allows the computation of many interesting indexes, such as node specificity, benchmarking potential, clustering coefficient, betweenness centrality, components and layers structure, in- and out-degree distributions, etc. It also allows the visualisation of the dominance relationships within the data-set as well as the potential benchmarks and the gradual improvement paths from inefficient nodes. A number of useful filters (bipartite subgraph, ego networks, threshold networks, skeletonisation, etc.) can be applied on the network in order to highlight and focus on specific subgraphs of interest. The proposed approach provides a new perspective on efficiency analysis, one that allows not only to focus on the distance to the efficient frontier and potential targets of individual units but also to study the data-set as a whole, with its component and layer structure, its overall dominance density, etc.

1. Introduction

Data Envelopment Analysis (DEA) is a well-known non-parametric technique that can be used to assess the relative efficiency of homogeneous (i.e., comparable) Decision-Making Units (DMUs). Each DMU consumes certain amounts of inputs in order to produce certain amount of outputs. The aim is to detect inefficiencies, i.e., potential reductions of the inputs maintaining the output levels or potential output increases maintaining the input consumption levels. Based on these potential improvements an efficiency score can be computed. There are different DEA models, depending on the orientation (whether to seek mainly inputs improvements or output improvements or both), on the metric used for the improvements (e.g., radial, non-radial, additive, etc.) and the technology assumptions (Constant Returns to Scale, Variable Returns to Scales, Free Disposal Hull, etc.). With respect to DEA applications, they are numerous and span all sectors (see, e.g., Cooper, Seiford, & Zhu, 2004; Liu, Lu, Lu, & Lin, 2013a).

The approach proposed in his paper is related to some research topics in DEA, such as DMU layers, intermediate benchmarks, and closest targets. Thus, Seiford and Zhu (2003) showed how to identify the successive layers of DMUs in a sample, a process that can also be called stratification. Thus, the efficient DMUs belong to layer 0 while layer 1 is formed by the DMUs that would be efficient if those in the previous layer are removed, and

so on for the successive layers. Introducing the concept of context-dependent DEA, Seiford and Zhu (2003) also defined attractiveness and progress measures for each DMU with respect to the different DMU layers.

Intermediate targets refer to the advantages of developing gradual improvement paths to the efficient frontier (e.g., Lozano & Villa, 2005, 2010). Lim, Bae, and Lee (2011) proposed a stepwise benchmarking path based on stratification and using a composite index of the attractiveness, progress, and feasibility of each efficiency improvement step. The efficiency measurements are based on the Range-Adjusted Measure (RAM) DEA model (Cooper, Park, & Pastor, 1999). Park, Bae, and Lim (2011) use, in addition to stratification, DMU clusters computed using k-means algorithm on the DMU input vectors. The successive intermediate benchmarks are thus chosen among those DMUs in the cluster with highest similarity to that to which the current DMU belongs. Park, Bae, and Lim (2012a) also use stratification but the benchmarking path selection uses two criteria, namely minimise resource improvement and maximise improvement direction proximity. Park, Bae, and Lim (2012b) uses the decision-maker's preferences to select the ultimate target and, for the selection of the intermediate benchmarks, a weighted sum of a measure of direction improvement proximity and a measure of resource consumption proximity. The latter is computed as the distance in the grid that results from quantising

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the input vectors using a Self-Organising Map (SOM). Estrada, Song, Kim, Namn, and Kang (2009) also uses SOM applied on input vectors but they propose computing the optimal path to the frontier through a reinforced learning algorithm that uses efficiency improvement as the reward of each step, thus effectively computing a shortest path to the efficient frontier. More recently, Ghahraman and Prior (2016) have proposed a gradual improvement method that computes the optimal stepwise benchmarking paths from the inefficient DMUs to the efficient ones using Dijkstra's shortest path on a benchmarking network whose edges have a length that depends on the input similarity and the efficiency gap between the nodes linked. The idea is to use as successive intermediate targets operating points that are not very dissimilar in their input endowments and in their efficiency level. The method also considers maximum allowed change in inputs in each step.

The issue of closest target and minimum distance to the efficient frontier (EF) has been also the subject of much research. Thus, González and Álvarez (2001) proposes to use an input-specific contraction measure that computes the sum of input contractions required to reach the EF. Silva Portela, Borges, and Thanassoulis (2003) presents a procedure to compute closest targets using a non-oriented DEA model and studies the cases of convex and non-convex technologies, the latter a.k.a. Free Disposal Hull (FDH). Aparicio, Ruiz, and Sirvent (2007) proposes different mathematical programming models for computing the closest target, depending on how closeness is measured. Baek and Lee (2009) presents a method based on a Least-Distance Measure to obtain the shortest projection from a DMU to the EF. Aparicio and Pastor (2013, 2014a) propose an output Russell measure and an output-oriented strong Hölder distance function, respectively, both based on an extended facet production possibility set, that have the strong monotonicity property and that allow the computation of closest targets. López-Espín, Aparicio, Giménez, and Pastor (2014) introduce a new approach based on metaheuristics in order to determine least distances. Aparicio, Mahlberg, Pastor, and Sahoo (2014) uses the Principle of Least Action and a minimisation version of the Measure of Inefficiency Proportions (MIP) to compute the closest projection on the frontier and, hence, minimum total effort needed by to achieve technical efficiency. Aparicio and Pastor (2014b) proposes a model based on weighted Euclidean distance to obtain the closest targets. An, Pang, Chen, and Liang (2015) uses an enhanced Russell measure to compute closest targets in the presence of undesirable outputs. Last but not least, two recent papers deal with the minimum distance to the EF in the case of FDH technology, namely Ebrahimnejad, Shahverdi, Balf, and Hatefi (2013) and Fukuyama, Hougaard, Sekitani, and Shi (2016).

Consider now the also very fruitful field of Complex Networks Analysis (CNA), which has witnessed in the last decade a tremendous growth, both in terms of theory developments and of applications. The basic concepts of these techniques are presented and explained in a number of textbooks and survey papers (e.g., Jackson, 2010; Newman, 2003; Wasserman & Faust, 1994) while some interesting applications include transportation networks (e.g., Derrible & Kennedy, 2010; Lordan, Sallan, & Simo, 2014, Zanin & Lillo, 2013), power grid networks (Pagani & Aiello, 2013), financial markets (e.g., Nobi, Maeng, Ha, & Lee, 2014; Oh, 2014), etc.

Apart from the research carried out by Ghahraman and Prior (2016) mentioned above, CNA and DEA has been jointly used in Lee, Seo, Choe, and Kim (2012) to study collaboration network patterns and research performance of public research institutes and in Lin and Tan (2014) to study the relationship between the centrality and the performance of the individuals in an organisation. The bibliometric studies of DEA literature in Liu et al. (2013a), Liu, Lu, Lu, and Lin (2013b), Liu, Lu, and Lu (2016) and Lampe and Hilgers (2015) also use CNA and clustering techniques, applied, in this case, to citation and co-citation networks. CNA tools have also been proposed to increase the discrimination power of DEA. Thus, the network-based approach in Liu, Lu, Yang, and Chuang (2009) uses the intensity variables (the lambda values as they are generally termed in DEA) to build a weighted, directed network, computing eigenvector centrality measures to rank the efficient DMUs. To populate the network arcs, different input/output combinations are considered. Liu and Lu (2010, 2012) and Ho, Liu, Lu, and Huang (2014) extend the network-based approach to the case of two stage systems, building a separate network and computing a DMU ranking for each stage and computing the relative strength of each input/output factor for each efficient DMU. Different DEA models can be used to compute the corresponding lambda values for each stage. This network-based approach has been applied in different contexts, such as internet companies, banks, research institutes, and universities.

Recently, Calzada-Infante and Lozano (2016) proposed a CNA approach to assess the performance of countries in the Olympic Games in Beijing 2008, taking into account the number of medals won and the countries' population and GDP. That paper introduced the concept of dominance network, whose usefulness for efficiency assessment is further explored and formalised in this paper. Lozano and Calzada-Infante (2017) have applied dominance network analysis to profit and revenue efficiency, i.e., to cases in which the input and output unit prices are known. In this paper, a methodology for globally assessing the efficiency of a sample of DMUs considering an FDH technology and using a CNA approach is proposed. Specifically, a dominance

network is built using edges whose weight is equal to the relative efficiency distance between the two DMUs, measured as the sum of the relative input reductions and output increases from the DMU origin of the link to the DMU destination of the link. This weighted, directed network allows computing a number of interesting CNA indexes and measures for the individual nodes, for its connected components, for the different layers and for the network as a whole. Also, the visualisation of the network, which can be enhanced through a series of selective filters, provides additional information into the performance and benchmarking paths of the inefficient DMUs, about the benchmarking popularity of the efficient DMUs, etc.

The structure of the paper is the following. In Section 2, the network construction process is presented and some basic CNA sets and indexes are defined. In Section 3, additional CNA measures as well as a number visualisation filters are proposed. The proposed methodology is illustrated in Section 4. Finally, Section 5 summarises and concludes.

2. Network construction and basic CNA sets and indexes

2.1. Network construction and notation

Consider a production process that uses m inputs to produce s outputs. There is a set D of DMUs that corresponds to different observed instances of this production process. The number of DMUs is n, i.e., $|D| = n$. We are interested in assessing the performance of these DMUs in terms of their relative efficiency in carrying out the production process. Let $x_j = \left(x_{1j}, x_{2j}, \ldots x_{mj} \right)$ and $y_j = \left(y_{1j}, y_{2j}, \ldots y_{sj} \right)$ denote the input and output vectors, respectively, of DMU j. Let $D(r) = \left\{ j \neq r : x_{ij} \leq x_{ir} \ \forall i, \ y_{kj} \geq y_{kr} \ \forall k \right\} \subset D$ be the set of DMUs that weakly dominate DMU r.

The proposed dominance network (D, E) has D as the set of nodes and $E = \left\{ (r, j) : r \in D \wedge j \in D(r) \right\}$ as the corresponding set of edges. The term dominance comes from the fact that the network has an arc between two DMUs r and j if and only if DMU j dominates DMU r. Let us define the weight of an edge (r, j) as the relative efficiency improvement obtained when moving from r to j, using a given normalisation vector $g = \left(g_i^x, g_k^y \right)$, i.e.,

$$e_{rj} = \begin{cases} 0 & \text{if } j \notin D(r) \\ \sum_i \frac{x_{ir} - x_{ij}}{g_i^x} + \sum_k \frac{y_{kj} - y_{kr}}{g_k^y} & \text{if } j \in D(r) \end{cases} .$$

Note that this definition is units invariant. For the normalisation vectors, different possibilities may be considered. One option is to use $g = \left(x_i^{\text{aver}}, y_k^{\text{aver}} \right)$ where $x_i^{\text{aver}} = \frac{1}{n} \sum_{j \in D} x_{ij} \ \forall i$ and $y_k^{\text{aver}} = \frac{1}{n} \sum_{j \in D} y_{kj} \ \forall k$ are the average value of the different inputs and outputs, respectively.

Alternatively, the normalisation vector can use the range of the input and output variables $R_i = x_i^{\text{max}} - x_i^{\text{min}} \ \forall i$ and $\hat{R}_k = y_k^{\text{max}} - y_k^{\text{min}} \ \forall k$, respectively, i.e., $g = \left(R_i, \hat{R}_k \right)$. This units-and-translation-invariant alternative is especially interesting in case some of the data are negative. Note however that, as pointed out by one of the reviewers, using $g = \left(x_i^{\text{aver}}, y_k^{\text{aver}} \right)$ or $g = \left(R_i, \hat{R}_k \right)$ has the drawback of making the efficiency score dependent on the whole sample, including the inefficient units, when it is clear that these units do not affect the efficient frontier.

Note, before proceeding further, that the edges have the transitivity property, i.e., if there exist the arcs (r, p) and (p, j) then there exist also the arc (r, j). This occurs because $p \in D(r) \wedge j \in D(p) \Rightarrow j \in D(r)$. Moreover, the weight of this transitive arc is equal to the sum of the corresponding weights, i.e., $e_{rp} + e_{pj} = e_{rj} \quad \forall p, j \in D(r), j \in D(p)$. This additivity property derives from the fact that the normalisation vector used to normalise the input and output changes in the definition of the edge weight is the same for all edges. Moreover, the additivity property implies that all the paths that connect two nodes r and j have the same weight, equal to that of the arc that directly links them. This means that all the paths that connect two nodes are geodesics, i.e., shortest paths.

Using the notation
$$\delta \left(e_{rj} \right) = \begin{cases} 1 & e_{rj} > 0 \Leftrightarrow (r, j) \in E \\ 0 & \text{otherwise} \end{cases}$$ to indicate whether the arc (r, j) exists, we can define the associated unweighted, undirected network (D, A) whose incidence matrix is

$$a_{rj} = a_{jr} = \begin{cases} 1 & \text{if } \exists \ edge \ (r, j) \ or \ \exists \ edge \ (j, r) \Leftrightarrow (r, j) \in E \vee (j, r) \in E \\ 0 & otherwise \end{cases} .$$

Given two nodes r and j, we will say that they are connected in the graph (D, E) if there is a path that connects them in the underlying undirected graph (D, A). The graph (D, E) can have one or more maximally connected components. Each component c is formed by a set of nodes $D_c \subseteq D$ such j and r are connected $\forall j, r \in D_c$ and $r \in D_c \wedge j$ connected with $r \Rightarrow j \in D_c$. Note that a component c is actually formed by the nodes in D_c plus the arcs that link them $E_c = \left\{ (r, j) \in E : r \in D_c \wedge j \in D_c \right\} \subseteq E$. In other words, a component c is a subgraph (D_c, E_c). Note, in passing, that $D_c \cap D_{c'} = \emptyset \quad \forall c \neq c', \bigcup_c D_c = D$ and $\sum_c |D_c| = |D| = n$.

Let $D^* = \left\{ r : D(r) = \emptyset \right\} \subseteq D$ be the set of efficient nodes and $D^*(r) = \begin{cases} \{r\} & \text{if } r \in D^* \\ D(r) \cap D^* & \text{if } r \notin D^* \end{cases} \subseteq D^*$ the set of efficient benchmarks of node r. The set of efficient nodes of component c is just $D_c^* = \left\{ j : j \in D_c \cap D^* \right\} = \bigcup_{r \in D_c} D^*(r) \subseteq D^*$.

In order to compute the in-degree of a node, we will need to consider the set

$D^{-1}(r) = \left\{ j \neq r : x_{ij} \geq x_{ir} \; \forall i \quad y_{kj} \leq y_{kr} \; \forall k \right\} \subset D$

which corresponds to the DMUs that are dominated by DMU r. Let $E^{-1}(j) = \left\{ (r, j) : r \in D^{-1}(j) \right\}$ be the set of edges that are incident to node j as. Also let $D^{--1}(j) = \left\{ r : r \in D^{-1}(j) \wedge D(r) \cap D^* = \{j\} \right\} \subseteq D^{-1}(j)$ be the set of nodes for which j is their only efficient benchmark. Let $E(r) = \left\{ (r, j) : j \in D(r) \right\}$ be the set of edges that go out of node r and $E^*(r) = \left\{ (r, j) : j \in D^*(r) \right\} \subseteq E(r)$ the set of direct edges from node r to the EF.

Let NL_c be the number of layers of component c. Let $c(r)$ denotes the component to which node r belongs and let

$$\lambda(r) = \begin{cases} 0 & \text{if } D(r) = \emptyset \\ 1 + \max\limits_{j \in D(r)} \lambda(j) & \text{otherwise} \end{cases}$$

be the layer, within that component, to which node r belongs. Let $L_{cq} = \left\{ r \in D_c : \lambda(r) = q \right\}$ be set of nodes that belong to layer q of component c. In particular, $L_{c0} = D_c^*$, i.e., the layer 0 of a component is formed by the efficient nodes that belong to that component.

2.2. Basic CNA indexes

Once the necessary definitions and notations have been defined, we are ready to compute some basic CNA indexes. Since the meaning and interestingness of most of these indexes is clear and to keep the length of the paper within a reasonable limit, we will not comment them in detail. Some indexes are computed at the node level while others correspond to the layer, the component or the network level. Thus, at the node level we may compute:

In-degree of node r: $d_r^{in} = |D^{-1}(r)|$
Out-degree of node r: $d_r^{out} = |D(r)|$
Specificity of node r: $\eta_r = |D(r)| + |D^{-1}(r)| = d_r^{in} + d_r^{out}$
Hub index of node r: $\gamma_r = |D(r)| \cdot |D^{-1}(r)| = d_r^{in} \cdot d_r^{out}$
Number of
    efficient benchmarks of node r: $|D^*(r)|$
Maximum distance of node r to the EF: $e_r^{max} = \max\limits_{j \in D^*(r)} e_{rj}$
Distance of node r to closest
    dominating efficient node: $\tau_r^{min} = \min\limits_{j \in D^*(r)} e_{rj}$

Before proceeding further, and in order to avoid confusion, let us make clear that the last indicator, τ_r^{min}, just computes the distance to the closest efficient node that dominates node r. This should not be confused with the minimum distance of node r to the efficient frontier, a measure which can computed using specific DEA approaches (e.g. Fukuyama et al., 2016). The purpose of computing τ_r^{min} is to complement the conventional inefficiency measure e_r^{max}, which aims at improving inputs and outputs as much as possible and therefore looks for the most distant efficient node that dominates node r. The idea behind the minimum distance to the efficient frontier (and also behind τ_r^{min}) is that it should be easier (i.e., less demanding) for a node r to move towards a

closer efficient target than to a more distant one. The difference is that τ_r^{min} is restricted to the efficient nodes that dominate node r while the minimum distance to the efficient frontier may sometimes identify as closest efficient target an efficient node that does not dominate node r. In other words, τ_r^{min} is no substitute for a proper measure of minimum distance to EF (like, e.g., the one proposed in Fukuyama et al., 2016), which requires a relatively complex calculation and that can, in any case, be computed to supplement the proposed CNA DEA approach.

In addition, for efficient nodes $j \in D^*$ we may compute:

Benchmarking count: $|D^{-1}(j)| = d_j^{in}$
Benchmarking necessity: $|D^{--1}(j)|$
Benchmarking potential: $\kappa_j = \sum\limits_{r \in D^{-1}(j)} e_{rj}$

Inefficiency radius: $\sigma_j = \max\limits_{r \in D^{-1}(j)} e_{rj}$

Superefficiency index: $\varphi_j = \left| \left\{ r : D(r) = \{j\} \wedge \lambda(r) = 1 \right\} \right|$

At the layer level, we may compute:

Percentage of component c nodes that form layer q: $v_{cq} = \frac{|L_{cq}|}{|D_c|}$

At the component level, we may compute:

Size of component c (as number of nodes): $|D_c|$

Size of component c (as % of nodes): $\alpha_c = \frac{|D_c|}{|D|}$

Size of component c (as number of edges): $NE_c = |E_c|$

Size of component (as % of edges): $\hat{\alpha}_c = \frac{NE_c}{NE}$

Average degree: $d_c^{aver} = \frac{1}{|D_c|} \sum\limits_{r \in D_c} d_r^{in} = \frac{1}{|D_c|} \sum\limits_{r \in D_c} d_r^{out}$

Density: $\rho_c = \begin{cases} 1 & \text{if } |D_c| = 1 \\ \frac{d_c^{aver}}{|D_c| - 1} & \text{otherwise} \end{cases}$

Average distance of component c to EF: $\theta_c^{aver} = \frac{1}{|D_c|} \sum\limits_{r \in D_c} e_r^{max}$

Diameter of component c: $\Delta_c = \max\limits_{r, j \in D_c} e_{rj}$

Efficient nodes percentage of component c: $\pi_c = \frac{|D_c^*|}{|D^*|} = \frac{|D^* \cap D_c|}{|D^*|}$

Percentage of efficient nodes in component c: $\hat{\pi}_c = \frac{|D_c^*|}{|D_c|} = \frac{|D^* \cap D_c|}{|D_c|}$

At the network level, we may compute:

Total number of edges: $NE = \sum\limits_r |D(r)|$

Number of components: NC

Network diameter: $\Delta = \max\limits_{(r, j) \in E} e_{rj} = \max\limits_{r \in D_c} e_r^{max} = \max\limits_{j \in D_c^*} \sigma_j$

Average distance to EF: $\theta^{aver} = \sum\limits_c \frac{|D_c| \cdot \theta_c^{aver}}{|D|} = \frac{1}{n} \sum\limits_{r \in D} e_r^{max}$

Percentage of efficient nodes: $\hat{\pi} = \frac{|D^*|}{|D|}$

In order to get better acquainted with these definitions, let us consider two extreme scenarios: one in which all DMUs are efficient and another in which only one DMU is efficient. If all DMUs are efficient, then $D^* = D$, $D(r) = \emptyset \; \forall r, d_r^{in} = d_r^{out} = 0 \; \forall r$. In that case there would be as many components as DMUs and their size would be minimal, i.e., $NC = |D| = n$ and $|D_c| = 1 \; \forall c$. We also have that $e_r^{max} = 0 \; \forall r \Rightarrow \Delta_c = 0 \; \forall c$, i.e., the diameter of all the components, and therefore of the network, would be zero, as it would also be the density of the

components $\rho_c = 0$ $\forall c$. The percentage of efficient nodes and the average efficiency of each component would be $\hat{\pi}_c = \theta_c^{\text{aver}} = 1$ $\forall c$, implying $\hat{\pi} = \theta^{\text{aver}} = 1$. The node specificity of every DMU would be $\eta_r = 0$ as it would also be their inefficiency radius $\sigma_j = 0$ $\forall j$.

On the contrary, if there is only one efficient DMU $D^* = \{J\}$ then there would be just one big component of size n, i.e., $NC = 1$ y $D_1 = D$. Its diameter would be the inefficiency of the most inefficient DMU, $\Delta = \Delta_1 = \max_{r \neq J} e_{rJ} = \sigma_J$. The percentage of efficient nodes would be minimal $\hat{\pi} = \hat{\pi}_1 = \frac{1}{n}$. The in-degree, the specificity and the benchmarking necessity of the efficient DMU J would be maximal $\eta_J = d_J^{\text{in}} = |D^{-1}(J)| = n - 1$ and this DMU would be the only efficient benchmark for all other DMUs, i.e., $D^*(r) = \{J\}$ $\forall r$. For each inefficient DMU r its distance to EF, i.e., e_r^{max} would be the weight of the only arc in its set of direct edges to the frontier $E^*(r) = \{(r, J)\}$ $\forall r$.

As we can see, the above CNA indexes provide a rather informative picture of the data-set as a whole and of the different DMUs individually. Thus, for example, looking at $\hat{\pi}$ we may see the discriminant power of DEA for this specific data-set. For each of the efficient DMUs $j \in D^*$ we can, for example, look at $|D^{-1}(j)|$ and check whether that DMU is the only efficient target for some of the inefficient DMUs. Or we may look at its inefficiency radius σ_j to estimate the magnitude of the efficiency worsening that some observed inefficient DMU has experienced.

For an inefficient DMU r, we may check the layer it belongs to and, looking at $|D^*(r)|$, see if there are more than one dominating efficient benchmarks to choose from. It may happen that one of those alternative efficient targets may involve an efficiency improvement τ_r^{min} lower than the inefficiency measure e_r^{max}. We may look at the sets $D(r)$ and $D^{-1}(r)$ as the neighbourhoods of DMU r, i.e., units that are similar from the point of view of their input consumption and output production patterns. Actually the sum of the cardinality of both neighbourhoods is proposed as a measure of the specificity of the input and output bundles of a DMU. Thus, a node with $\eta_r = 0$ is sort of an outlier, i.e., a single-node component not related in any way with the rest of the observations.

We may see if the whole data-set is connected or there are different clusters of observations as indicated by the different connected components. In that case, each component is independent of the others, which means that the set of DMUs can be partitioned into groups of observations that can be assessed separately. Thus, each component c has a different size, its own layer structure $\nu_{cq} = \frac{|L_{cq}|}{|D_c|}$, its own diameter Δ_c, its own share of efficient DMUs $\hat{\pi}_c$, its own density ρ_c, its own average efficiency θ_c^{aver}, etc.

One may argue that some of these indexes are already available in a conventional DEA assessment. That

is correct. However, the advantages of the proposed approach is that not only it provides a much richer information set (see next section for additional CNA measures) but it does so in a simple and intuitive way, within an integrated DEA CNA framework. Add to that its visualisation capabilities (see next section) and you get a very powerful analysis tool. Moreover, while conventional DEA assessment focuses on computing individual efficiency scores and targets, the proposed DEA CNA approach analyses the data-set at different levels, i.e. not only at the individual level but also at the layer, component and at the global network-wide level. Upon reflection we may reach the conclusion that assessing the data-set as a whole makes sense because, although probably not sufficiently emphasised in general, the efficiency scores computed by DEA are always relative, meaning that they depend on the specific data-set used. Therefore, it may be interesting to look at and to assess the efficiency of the data-set as a whole.

2.3. Managerial/efficiency assessment implications

In this section, the interpretation and the usefulness of the proposed CNA DEA approach is discussed. Some of the measures are quite simple and intuitive, like the percentage of efficient DMUs $\hat{\pi}$. Most others (such as, in- and out-degree, diameter, etc.) are direct application of standard CNA measures, they have a specific meaning in the DEA context considered. Thus, the in-degree of a DMU r is the number of DMUs it dominates while its out-degree is the number of DMUs that dominate r. The total degree of a DMU r (i.e., the sum of the in- and out-degrees) has been labelled as specificity index η_r since it reflects the number of DMUs with input/output mixes similar to that of DMU r. Note that low values of that index imply high specificity, i.e., not many similar DMUs. The hub index of a node r γ_r is a measure of how many times r lies on the path from an inefficient DMU dominated by r to a (possibly efficient) DMU that dominates r. This measure, similar to the betweenness centrality β_r, gives an idea of the potential of a DMU as intermediate step in the improvement paths of inefficient DMUs.

The number of efficient benchmarks of a DMU r $|D^*(r)|$ measures the options it has when projected onto the efficient frontier. This flexibility allows DMU r to choose among different efficient targets in $D^*(r)$, all of which weakly dominate r. Actually, the inefficiency measure e_r^{max} represents the distance to the one among those potential efficient targets that is farthest from DMU r while τ_r^{min} represent the distance to the one that is closest.

The benchmark count of an efficient DMU j κ_j, which coincides with its in-degree, corresponds to the number of DMUs it dominates and measures the number of inefficient DMUs that may potentially choose j as target. Its

benchmark potential (a.k.a. as in-strength) measures the total efficiency improvement obtained if all the units it dominates are projected onto it. The inefficiency radius σ_j of an efficient DMU j is the inefficiency of the most inefficient of the DMUs that j dominates. The superefficiency index φ_j of an efficient DMU j is the number of inefficient DMUs that would be labelled efficient, were not for the fact that they are dominated by DMU j. Thus, as in conventional DEA, if DMU j was deleted from the sample those DMUs would be assessed as efficient.

The layer structure and its composition, given by indexes $\lambda(r)$ and v_{cq}, are also very interesting to determine the degree of quantisation of the inefficiency level of the DMUs in the sample. The number of components in the sample indicates the number of communities or clusters within the sample. Some of the components, however, may contain a single efficient DMU, which we may call mavericks. Other connected components may have a large size and contain many DMUs that are related between them through dominance relations. The DMUs in each component define a cluster of operating points that lie in a certain region within the production possibility set. The number of components informs therefore of the number of such distinct regions that exist in the sample. Analogously, the size of the components measured in terms of nodes gives an idea of how populated those regions are while their size measured by the number of edges gives an idea of their level of interdependence/connectivity in terms of dominance. The density of a component is also a measure of the relative frequency of dominance between the DMUs in that component. The higher the density, the more frequent is that a DMU dominates or is dominated by another. Since dominance implies inefficiency, the more pervasive dominance is, the more inefficiency exist and therefore the higher the room for efficiency improvement in the sample.

The average distance of a component c to the EF θ_c^{aver} represents the average of the distances of the DMUs in the component. If θ_c^{aver} is small then the DMUs in this region are not very inefficient. In other words, the component corresponds to a region that is close to the EF. On the contrary, if θ_c^{aver} is large then there must be DMUs in that component that are far from the EF. In other words, a significant fraction of the DMUs in that component are rather inefficient. The diameter of a component Δ_c indicates how inefficient the most inefficient of the DMUs in that component is, i.e., how far from the EF the region corresponding to that component stretches.

The efficient nodes percentage of a component c π_c indicates the fraction of the EF that belongs to that component. This informs about the partition of the EF among the components, i.e., the intersection between each operating region and the EF. The percentage of efficient nodes in component c $\hat{\pi}_c$ indicates whether the efficient DMUs represent a small or a large fraction of that component, i.e., whether most or just a few of the DMUs in that component are efficient.

Finally, note that some of the indexes assess individual nodes while others refer to the whole set of DMUs. The latter differs from conventional DEA and provides a holistic view of the set of DMUs that may be particularly useful when assessing the efficiency of a specific sector or industry.

3. Additional CNA measures and network visualisation

3.1. Additional CNA measures

Apart from the basic indexes presented in Section 2, a number of additional CNA measures can be computed. Thus, at the component level we may define:

In-degree distribution of component c: $H_c^{\text{in}}(h) = \frac{|\{r \in D_c : d_r^{\text{in}} = h\}|}{|D_c|}$

Out-degree distribution of component c: $H_c^{\text{out}}(h) = \frac{|\{r \in D_c : d_r^{\text{out}} = h\}|}{|D_c|}$

Distribution of distance to frontier for component c: $F_c(e) = \frac{|\{r \in D_c : e_r^{\max} \leq e\}|}{|D|}$

Distribution of edge weight for component c: $G_c(e) = \frac{|\{(r,j) \in E : r \in D_c \ \wedge \ j \in D_c \ \wedge \ e_{rj} \leq e\}|}{NE}$

Similar information can be obtained at the network level:

In-degree distribution: $H^{\text{in}}(h) = \frac{|\{r \in D : d_r^{\text{in}} = h\}|}{|D|}$

Out-degree distribution: $H^{\text{out}}(h) = \frac{|\{r \in D : d_r^{\text{out}} = h\}|}{|D|}$

Distribution of distance to frontier: $F(e) = \sum_c F_c(e)$

Distribution of edge weight: $G(e) = \sum_c G_c(e)$

Note that the above definitions imply that $F_c(0) = G_c(0) = \frac{|D_c^*|}{|D|} \quad \forall c$ and $F_c(\Delta_c) = \frac{|D_c|}{|D|} \quad \forall c$. Similarly, $F(0) = G(0) = \frac{|D^*|}{|D|} = \hat{\pi}$ and $F(\Delta) = 1$.

An interesting CNA measure of local cohesiveness is the *clustering coefficient of a node j*, which is the percentage of its alters (i.e. those in $D^{-1}(j) \cup D(j)$) that are also linked between them, i.e.,

$$CC_j = \begin{cases} 0 & \text{if } |D^{-1}(j) \cup D(j)| \leq 1 \\ \dfrac{|D^{-1}(j)| \cdot |D(j)| + \sum\limits_{r \in D^{-1}(j)} |D^{-1}(j) \cap D(r)| + \sum\limits_{r \in D(j)} |D(j) \cap D^{-1}(r)|}{\binom{|D^{-1}(j)| + |D(j)|}{2}} & \text{otherwise} \end{cases}$$

The *average clustering coefficient of a component c* is, hence, $CC_c^{\text{aver}} = \frac{1}{|D_c|} \sum\limits_{j \in D_c} CC_j$.

Another interesting CNA measure is the *betweenness centrality of a node p* which is related to its status as an intermediate benchmark. It can be computed as the relative frequency with which it occurs that node p is in the shortest path between any two other nodes r and j. Mathematically

$$\beta_p = \sum_{\substack{r,j \in D \\ r \neq j \\ r \neq p \neq j}} \frac{\varsigma(r,j|p)}{\varsigma(r,j)}$$

where $\varsigma(r,j)$ is the number of shortest paths between r and j and $\varsigma(r,j|p)$ is the number of those shortest paths that go through node p. Clearly, the betweenness centrality

of an efficient node is 0. Also, $\beta_p = 0$ $\forall p: D^{-1}(p) = 0$. In other words, $|D(p)| = 0 \lor |D^{-1}(p)| = 0 \Rightarrow \beta_p = 0$ i.e., a node cannot have this type of centrality if no arcs enter or go out of it.

In addition to the above CNA measures it might be interesting to define the eigenvector centrality of a node, which is related to its importance/prestige. It seems, at least it has occurred in all the experiments carried out, that the edge weight matrix corresponding to E has only the null eigenvalue (of multiplicity n) and that its eigenvectors are just the canonical unit vectors corresponding to the efficient DMUs. That would imply that the eigenvalue centrality would be zero for inefficient nodes and one for efficient nodes, which means that, in our case, this CNA measure is not very informative.

3.2. Network visualisation

In addition to the information supplied by the quantitative CNA indexes and measures defined, the proposed approach has a powerful visualisation capability. Thus, while it is normally not possible to directly plot the DMUs multidimensional input and output vectors, it is possible to draw the proposed dominance network and distinguish the different components, the efficient nodes (from which no edges leave), the in- and out-degree of the different nodes, their respective neighbourhoods, the potential efficient benchmarks of a DMU and, in particular, its closest target, etc. All this, and more, can be perceived by looking at the graphical representation of the network. This is a very useful feature of the proposed approach. However, for large data-sets, the network may contain a large number of nodes and the corresponding graphical representation may be cluttered. That is why it is good to consider some visualisation filters that can help select appropriate subgraphs whose reduced size allow better focus on the aspects of interest. The following is a non-exhaustive list of potential visualisation filters that can be helpful.

3.2.1. Upper threshold filter
This filter would generate a subgraph in which all the arcs above a given threshold are removed. The resulting graph would thus be (D, E') where $E' = \left\{ (r, j) \in E: 0 < e_{rj} \leq \varepsilon \right\}$. This allows focusing on the shorter links, which represent more accessible efficiency improvements. Given an inefficient DMU $r \in D \backslash D^*$ and assuming that the threshold does not exceed a certain value ($\varepsilon \leq \varepsilon_r^{\max} = \max\limits_{j \in D(r) \cup \{r\}} \min\limits_{j' \in D(j)} e_{jj'}$) we are guaranteed that the result subgraph contains at least a path connecting r with the EF. Moreover, the weight of each of the links in that path is below the given threshold.

3.2.2. Lower threshold filter
This filter is analogous and would generate a subgraph in which all the arcs below a given threshold are removed.

The resulting graph would thus be (D, E') where $E' = \left\{ (r, j) \in E: e_{rj} \geq \hat{\varepsilon} \right\}$. However, the aim of this filter is to focus on the larger links and therefore concentrate on the most significant efficient improvements.

3.2.3. Bipartite graph filter
This filter would only keep in the result graph (D, E') those arcs that directly link the inefficient DMUs $r \in D \backslash D^*$ with their corresponding efficient benchmarks $D^*(r) \subset D^*$, i.e.,

$$E' = \left\{ (r, j) \in E: r \in D \backslash D^* \land j \in D^*(r) \right\} = \bigcup_{r \in D \backslash D^*} E^*(r).$$

The name of the filter comes from the fact that the result subgraph is bipartite. The aim of this filter is to focus on the distance to the efficient benchmarks of the inefficient DMUs as well as on the benchmarking count, potential and necessity, as well as the inefficiency radius, of the efficient DMUs.

3.2.4. Efficient benchmarks filter
This filter is similar to the bipartite graph filter but specific for each inefficient DMU. Thus, given an inefficient DMU $r \in D \backslash D^*$ the result graph $(D^*(r), E^*(r))$ only includes those arcs that link it with its efficient benchmarks.

3.2.5. Benchmarking reach filter
This filter is similar to the bipartite graph filter but specific for each efficient DMU. Thus, given an efficient DMU $j \in D^*$ the result graph $\left(D^{-1}(j), E^{-1}(j) \right)$ only includes those arcs that go into this node. In that way, the benchmarking count, potential and reach of the selected efficient DMU is clearly perceptible as it is also its inefficiency radius.

3.2.6. High specificity filter
This filter would keep in the result subgraph (D', E') only nodes with an specificity level above a certain value, i.e. $D' = \left\{ r \in D: \eta_r \geq z \right\}$, and the arcs that enter or leave those nodes, i.e., $E' = \left\{ (r, j): r \in D' \land j \in D(r) \right\} \cup \left\{ (r, j): j \in D' \land r \in D^{-1}(j) \right\}$ This filter would focus on highly connected nodes.

3.2.7. Low specificity filter
This filter is similar to the previous one but would keep in the result subgraph (D', E') only nodes with an specificity level below a certain value and the arcs that enter or leave those nodes, i.e., $D' = \left\{ r \in D: \eta_r \leq z \right\}$, $E' = \left\{ (r, j): r \in D' \land j \in D(r) \right\} \cup \left\{ (r, j): j \in D' \land r \in D^{-1}(j) \right\}$ This filter would retain low connected nodes.

3.2.8. Ego network filter
This filter focuses on a specific node p and consists of the subgraph defined by p and its alters $D(p) \cup D^{-1}(p)$, i.e., $\left(D^{\text{ego}}(p), E^{\text{ego}}(p) \right)$, where $D^{\text{ego}}(p) = \left\{ p \right\} \cup D(p) \cup D^{-1}(p)$ and $E^{\text{ego}}(p) = E^{-1}(p) \cup E(p) \cup \left\{ (r, j) \in E: r, j \in D(p) \cup D^{-1}(p) \right\}$.

3.2.9. Skeletonisation filter

This filters remove transitive arcs, i.e., those arcs $\{(r, j): \exists p \in D(r) \cap D^{-1}(j)\}$ Therefore, the skeleton subgraph (D, E^S), where $E^S = \{(r, j): D(r) \cap D^{-1}(j) = \emptyset\}$, allows the visualisation of the basic dominance relationships between the nodes. Since the transitivity of such dominance relationships are implied, this filter significantly reduces the number of edges thus reducing the clutter and facilitating the stratification of the DMUs in the data-set.

As we can see, there are many potential filters that can be devised, which represent a significant enhancement of the visualisation capabilities of the proposed approach. In addition, implementing these filters is relatively straightforward since existing network CNA software (e.g., Pajek, Gephi, UCINET, NodeXL, etc.) define a visibility attribute for each edge, which allows showing only those that meet specific conditions.

4. Illustration

In this section, and in order to illustrate the proposed approach, we will apply it to the 12 DMUs, 2 inputs + 1 constant output data-set in Lim et al. (2011). In first place, we will carry out the analysis assuming a direction vector $g = (g_i^x, g_k^y) = (1, 1, 1)$. To test the robustness of the proposed approach, we will consider also other possible direction vectors. In third place, we will consider the existence of an additional DMU distinguishing two cases depending on whether the new DMU is efficient or not. Figure 1 shows the input consumption of the original 12 DMUs (labelled A–L). The plot also shows the two DMUs (namely M and M') corresponding to input/output vectors $(x, y) = (6, 6, 1)$ and $(x, y) = (4, 4, 1)$, respectively, that will be considered in Section 4.3. Note that in this simple data-set with only two inputs and a constant output, it is easy to visualise the operation points of DMUs and see the dominance relationships. That is not possible in the general, multidimensional case.
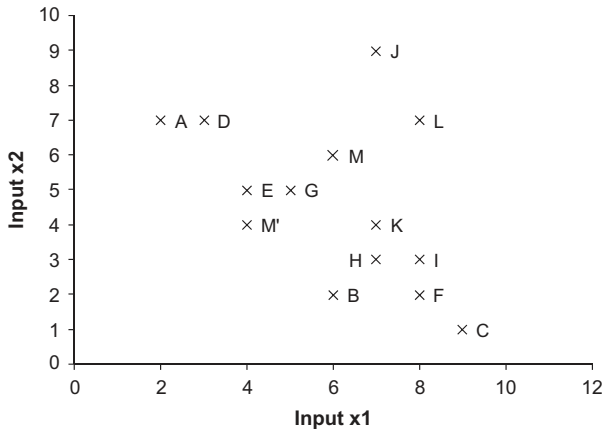
4.1. Results for direction vector $g^{(1,1,1)}$

Table 1 shows the edge weight matrix computed with direction vector $g^{(1,1,1)} = (g_i^x, g_k^y) = (1, 1, 1)$. Table 2 shows some of the proposed CNA DEA indexes for that direction vector. Note that $|D^*| = 4$, namely $D^* = \{A, B, C, E\}$. There a total of $NE = 25$ edges. The network diameter is $\Delta = 8.0$. The average distance to the frontier is $\theta^{aver} = 2.250$. Note that only nodes J and L have more than one efficient benchmark. Both nodes belong to the furthermost layer from the EF. They are also the ones with the highest specificity index. The nodes with the largest hub index are, however, H and K. This is also reflected in their having the largest betweenness centrality coefficients. Table 2 also shows the CNA indexes associated with the efficient units. Note that node B is more necessary for benchmarking than the other three efficient nodes. It also has the highest benchmarking potential, the largest inefficiency radius and the largest superefficieny index. In the bottom part of Table 2, some component-level CNA indexes for component 1 are shown. That component contains 11 out of the 12 nodes and all 25 edges. Its density is around 23%. Three out of the 11 nodes are efficient. Actually, the component contains 3 out of the 4 efficient nodes in the data-set. The component has a high clustering coefficient, which is basically a consequence of the transitivity of the dominance relations. The component has four layers, with more than half of the nodes in the first two layers (labelled layers 0 and 1).

Figure 1. Input vectors for Lim et al. (2011) data-set (DMUs A–L). DMUs M and M' are additionally considered in Section 3.4.

Table 1. Edge weights matrix for Lim et al. (2011) data-set.

DMU	A	B	C	D	E	F	G	H	I	J	K	L
A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
B	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
D	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
E	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
F	0.0	2.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
G	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
H	0.0	2.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
I	0.0	3.0	0.0	0.0	0.0	1.0	0.0	1.0	0.0	0.0	0.0	0.0
J	7.0	8.0	0.0	6.0	7.0	0.0	6.0	6.0	0.0	0.0	5.0	0.0
K	0.0	3.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0
L	6.0	7.0	0.0	5.0	6.0	5.0	5.0	5.0	4.0	0.0	4.0	0.0

Table 2. CNA DEA indexes for Lim et al. (2011) data-set and direction vector $g^{(1,1,1)}$.

DMU r	e_r^{max}	τ_r^{min}	d_r^{in}	d_r^{out}	η_r	γ_r	$|D^*(r)|$	$\lambda(r)$	CC_r	β_r
A	0.0	0.0	3	0	3	0	1	0	0.667	0.000
B	0.0	0.0	6	0	6	0	1	0	0.600	0.000
C	0.0	0.0	0	0	0	0	1	0	0.000	0.000
D	1.0	1.0	2	1	3	2	1	1	0.667	1.000
E	0.0	0.0	3	0	3	0	1	0	0.667	0.000
F	2.0	2.0	2	1	3	2	1	1	1.000	0.583
G	1.0	1.0	2	1	3	2	1	1	0.667	1.000
H	2.0	2.0	4	1	5	4	1	1	0.700	1.708
I	3.0	3.0	1	3	4	3	1	2	0.833	1.208
J	8.0	7.0	0	7	7	0	3	3	0.238	0.000
K	3.0	3.0	2	2	4	4	1	2	0.833	1.583
L	7.0	6.0	0	9	9	0	3	3	0.250	0.000

DMU j	$|D^{-1}(j)|$	$|D^{--1}(j)|$	κ_j	σ_j	φ_j
A	3	1	14.0	7.0	1
B	6	4	25.0	8.0	2
C	0	0	0.0	0.0	0
E	3	1	14.0	7.0	1

a_1	11/12	θ_1^{aver}	2.455	CC_1^{aver}	0.647
$\hat{\alpha}_1$	1.000	Δ_1	8.0	$NL_1 = 4$	
d_1^{aver}	2.273	π_1	3/4	$v_{10} = 3/11$ $v_{12} = 2/11$	
ρ_1	0.227	$\hat{\pi}_1$	3/11	$v_{11} = 4/11$ $v_{13} = 2/11$	
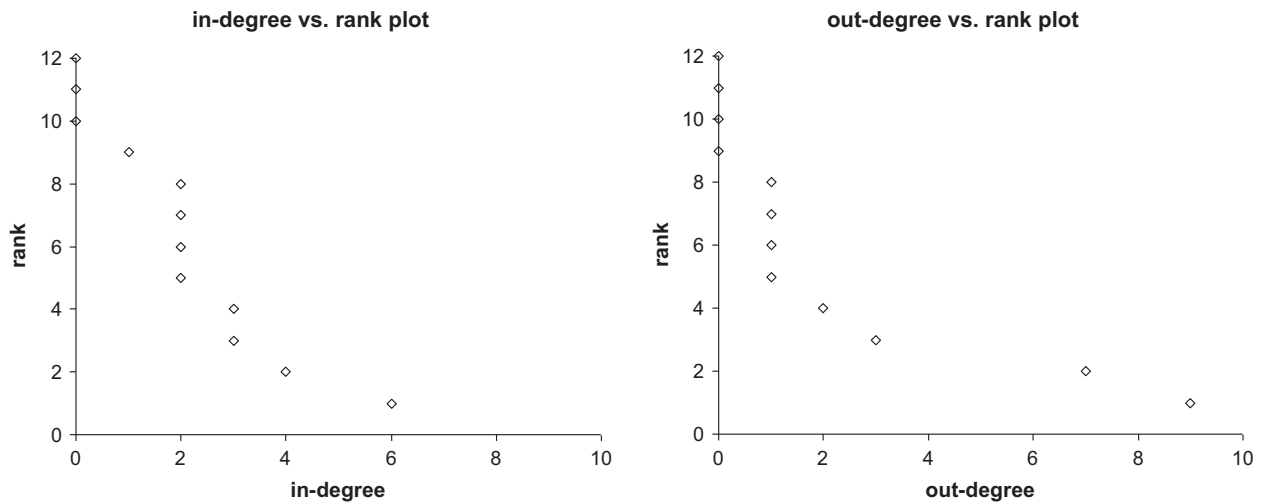
Figure 2. In- and out-degree rank plots for Lim et al. (2011) data-set.
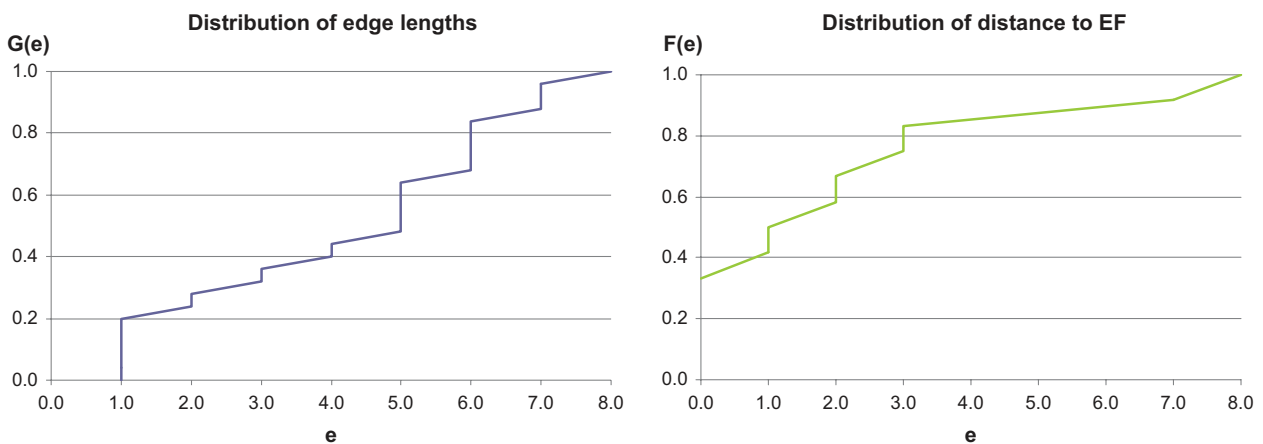
Figure 3. Distribution of edge weights and distance to EF for Lim et al. (2011) data-set.
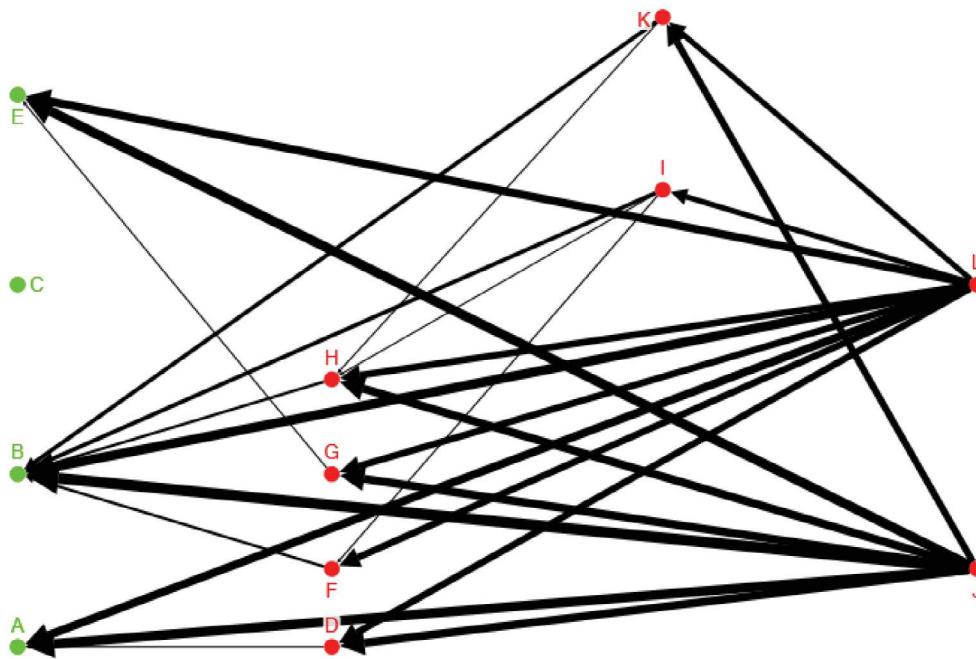
Figure 4. Full network for Lim et al. (2011) data-set.
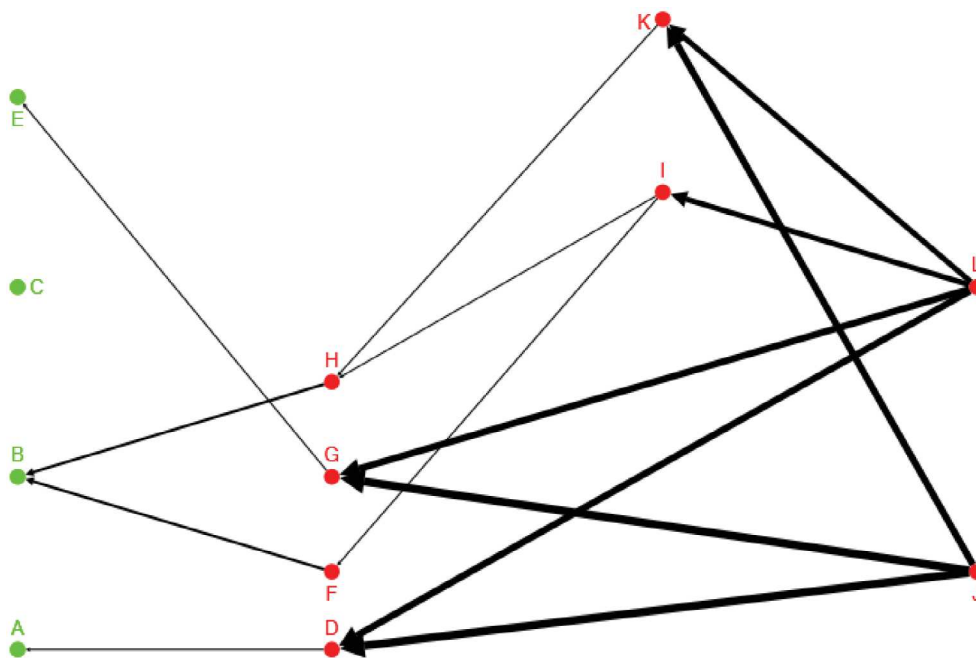
Figure 5. Skeleton subgraph for Lim et al. (2011) data-set.

Figure 2 shows the in- and out-degree rank plots. To produce this plot, the nodes are arranged in decreasing order of their in- or out-degree. That means that the nodes with the largest in- or out-degree are given rank 1 while the nodes with smallest in- or out-degree are assigned rank n. The plot just shows the rank of all nodes with each in- or out-degree value.

Figure 3 shows the distribution of the distance to the EF and the distribution of the edge weight. The maximum in-degree is 6 while the maximum out-degree is 9. The average of both in- and out-degree distributions is, of course, the same (2.08 = 25 edges/12 nodes). With

respect to Figure 3, it can be seen that slightly less than 50% of the edges have a weight below 5.0 and that around 75% of the nodes have a distance to the EF below 3.0.

Figure 4 shows the network, drawn using NodeXL, a free, powerful and easy-to-use Excel template for exploring network graphs (Hansen, Schneiderman, & Smith, 2011). Efficient nodes have been coloured green and inefficient ones red. It can be seen that there is a big component (let us label it $c = 1$) that contains all nodes except node C. The transitivity of dominance relationships is clearly visible. Note that the width of the any
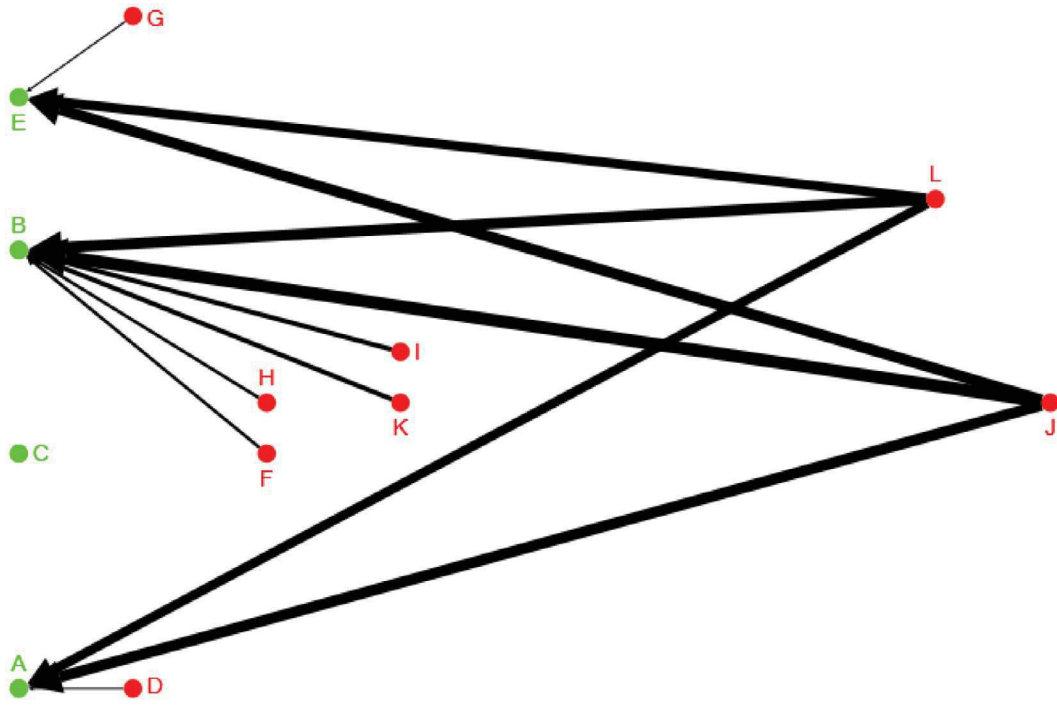
Figure 6. Bipartite subgraph for Lim et al. (2011) data-set.

arc (r, j) shown in the graph is proportional to its edge weight value e_{rj}.

Figure 5 shows the result of the skeletonisation filter. The reduction in clutter is noticeable but the basic dominance structure is kept intact. Dominance transitivity is implicit. The layer structure is also clearly visible.

Figure 6 shows the bipartite subgraph in which, for each inefficient DMU, only the edges towards efficient DMUs are kept. The horizontal axis corresponds to the inefficiency score, i.e., the more to the right, the more inefficient a DMU is. In this graph, the benchmarking potential and the inefficiency radius of each efficient DMU can be readily appreciated. Also, the density of this bipartite subgraph gives an idea of the degree of overlap of the operating regions dominated by each efficient unit. In this case, the density is $12/(4 \times 8) = 0.375$. Since each inefficient node must be linked to at least one efficient node the minimum density is $8/(4 \times 8) = 0.25$. The actual density is relatively low (i.e., close to the minimum density) due to the fact that, except for the layer 3 DMUs J and L, the rest of inefficient DMUs are dominated by just one efficient DMU.

4.2. Results for alternative direction vectors

Since some of the proposed CNA DEA indexes depend on the edge weights, which in turn depend on the direction vector g used, a sensitivity analysis of the results for different definitions of the direction vector has been carried. Thus, apart from the case $g^{(1, 1, 1)} = (1, 1, 1)$ reported above, three other different cases have been considered: $g^{average} = \left(x_i^{aver}, y_k^{aver}\right), g^{range} = \left(R_i, \hat{R}_k\right)$ and $g^{median} = \left(x_i^{median}, y_k^{median}\right)$, which correspond to use the

average, range, and median, respectively, for each input and output dimension. Table 3 shows the correlation coefficients of the results obtained in each case for the following indexes: $e_r^{max}, \tau_r^{min}, \kappa_j$ and σ_j. Detailed results are shown in Appendix 1. It can be seen that the correlations are very high. The table also shows the corresponding values for the indexes $\theta_c, \tau_c,$ and Δ_c for the large connected component (labelled $c = 1$). Although, of course, the absolute magnitude of these measures differ for the different direction vectors (which represent different

Table 3. Comparison of CNA DEA indexes corresponding to different direction vectors.

	$g^{(1, 1, 1)}$	$g^{average}$	g^{range}	g^{median}
Correl. e_r^{max}				
$g^{(1, 1, 1)}$	1.000	0.998	0.999	0.996
$g^{average}$	–	1.000	0.995	1.000
g^{range}	–	–	1.000	0.993
g^{median}	–	–	–	1.000
Correl. τ_r^{min}				
$g^{(1, 1, 1)}$	1.000	0.996	1.000	0.991
$g^{average}$	–	1.000	0.994	0.999
g^{range}	–	–	1.000	0.988
g^{median}	–	–	–	1.000
Correl. κ_j				
$g^{(1, 1, 1)}$	1.000	0.996	0.999	0.991
$g^{average}$	–	1.000	0.991	0.999
g^{range}	–	–	1.000	0.985
g^{median}	–	–	–	1.000
Correl. σ_j				
$g^{(1, 1, 1)}$	1.000	0.991	0.998	0.981
$g^{average}$	–	1.000	0.980	0.998
g^{range}	–	–	1.000	0.966
g^{median}	–	–	–	1.000
θ_1	2.455	0.480	0.325	0.466
τ_1	25.000	4.390	3.393	4.048
Δ_1	8.000	1.690	1.018	1.698
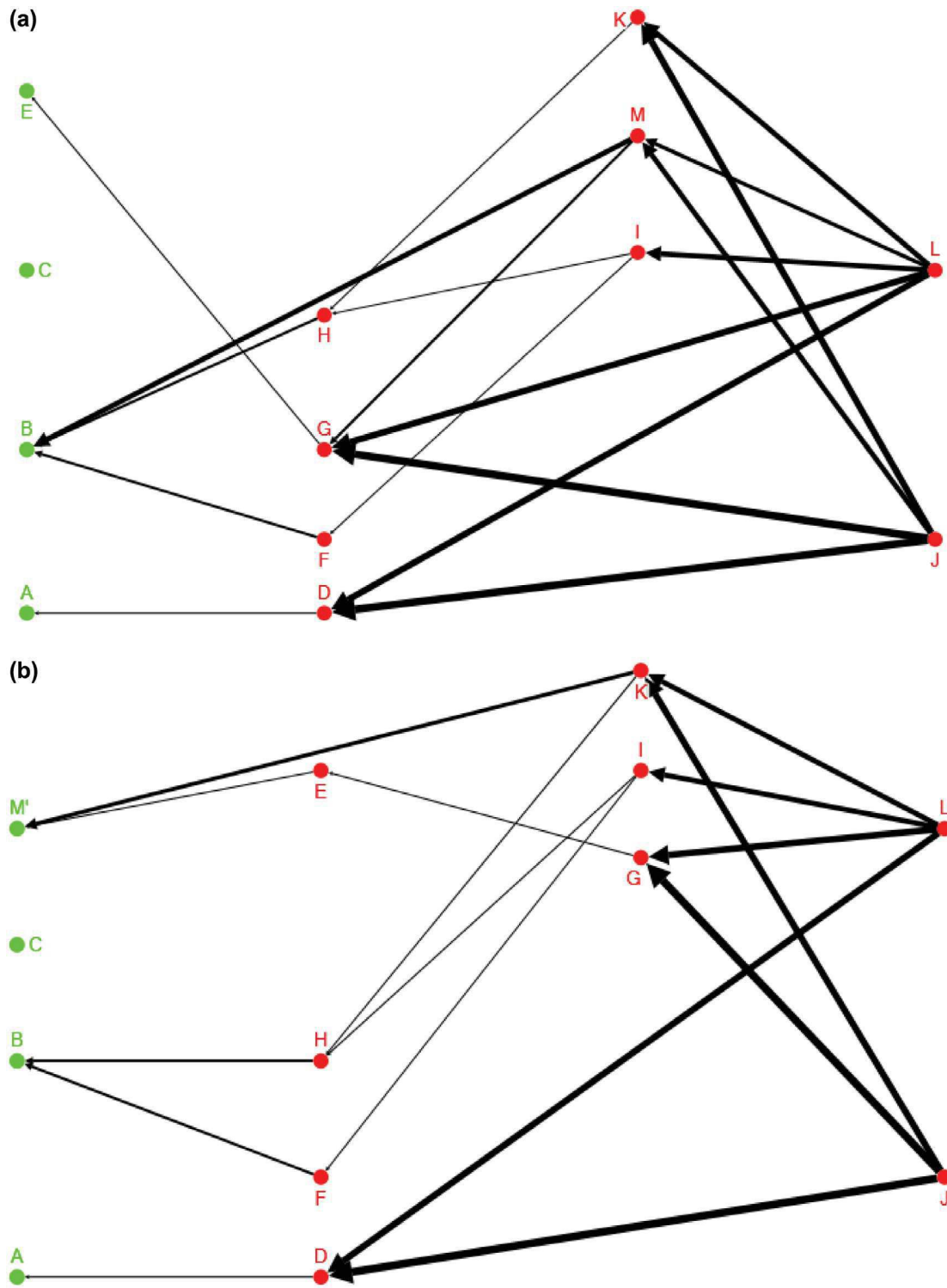
Figure 7. Skeleton for Lim et al. (2011) data-set plus new DMU M (panel a) and *M'* (panel *b*).

ways of normalising the input and output changes) their relative values are totally consistent. These results lead us to claim that it is not necessary to do the calculations with different normalisation vectors because the efficiency assessment carried out by the proposed CNA DEA approach will basically draw the same overall picture.

4.3. *Impact of an additional DMU*

In order to study the effects of considering an additional DMU using the proposed approach, in this section we will assume that the data-set includes the original 12 DMUs (A–L) plus a new one. We consider two cases. In one case, the new DMU (labelled M) has input/output vector (6, 6, 1) and therefore is not efficient.

Table 4. CNA DEA results for Lim et al. (2011) data-set plus new DMU M.

DMU r	e_r^{max}	τ_r^{min}	d_r^{in}	d_r^{out}	η_r	γ_r	$|D^*(r)|$	$\lambda(r)$	CC_r	β_r
A	0.0	0.0	3	0	3	0	1	1	0.667	0.000
B	0.0	0.0	7	0	7	0	1	1	0.524	0.000
C	0.0	0.0	0	0	0	0	1	1	0.000	0.000
D	1.0	1.0	2	1	3	2	1	1	0.667	1.000
E	0.0	0.0	4	0	4	0	1	1	0.833	0.000
F	2.0	2.0	2	1	3	2	1	1	1.000	0.556
G	1.0	1.0	3	1	4	3	1	1	0.833	1.500
H	2.0	2.0	4	1	5	4	1	1	0.700	1.567
I	3.0	3.0	1	3	4	3	1	1	0.833	1.167
J	8.0	7.0	0	8	8	0	3	3	0.286	0.000
K	3.0	3.0	2	2	4	4	1	1	0.833	1.456
L	7.0	6.0	0	10	10	0	3	3	0.267	0.000
M	4.0	3.0	2	3	5	6	2	2	0.700	2.311

DMU j	$|D^{-1}(j)|$	$|D^{--1}(j)|$	κ_j	σ_j	φ_j
A	3	1	14.0	7.0	1
B	7	4	29.0	8.0	2
C	0	0	0.0	0.0	0
E	4	1	17.0	7.0	1

a_1	12/13	θ_1^{aver}	2.583	CC_1^{aver}	0.664
$\hat{\alpha}_1$	1.000	Δ_1	8.0	$NL_1 = 4$	
d_1^{aver}	2.500	π_1	3/4	$v_{10} = 3/12 \quad v_{12} = 3/12$	
ρ_1	0.227	$\hat{\pi}_1$	3/12	$v_{11} = 4/12 \quad v_{13} = 2/12$	

Table 5. CNA DEA results for Lim et al. (2011) data-set plus new DMU M'.

DMU r	e_r^{max}	τ_r^{min}	d_r^{in}	d_r^{out}	η_r	γ_r	$|D^*(r)|$	$\lambda(r)$	CC_r	β_r
A	0.0	0.0	3	0	3	0	1	0	0.667	0.000
B	0.0	0.0	6	0	6	0	1	0	0.600	0.000
C	0.0	0.0	0	0	0	0	1	0	0.000	0.000
D	1.0	1.0	2	1	3	2	1	1	0.667	1.000
E	1.0	1.0	3	1	4	3	1	1	0.833	1.300
F	2.0	2.0	2	1	3	2	1	1	1.000	0.583
G	2.0	2.0	2	2	4	4	1	2	0.833	1.800
H	2.0	2.0	4	1	5	4	1	1	0.700	1.708
I	3.0	3.0	1	3	4	3	1	2	0.833	1.208
J	8.0	7.0	0	8	8	0	3	3	0.286	0.000
K	3.0	3.0	2	3	5	6	1	2	0.700	1.983
L	7.0	6.0	0	10	10	0	3	3	0.267	0.000
M'	0.0	0.0	5	0	5	0	1	0	0.700	0.000

DMU j	$|D^{-1}(j)|$	$|D^{--1}(j)|$	κ_j	σ_j	φ_j
A	3	1	14.0	7.0	1
B	6	4	25.0	8.0	2
C	0	0	0.0	0.0	0
M'	5	1	15.0	6.0	1

a_1	12/13	θ_1^{aver}	2.417	CC_1^{aver}	0.659
$\hat{\alpha}_1$	1.000	Δ_1	8.0	$NL_1 = 4$	
d_1^{aver}	2.500	π_1	3/4	$v_{10} = 3/12 \quad v_{12} = 3/12$	
ρ_1	0.227	$\hat{\pi}_1$	3/12	$v_{11} = 4/12 \quad v_{13} = 2/12$	

Conversely, in the second case, the new DMU (labelled M') has input/output vector (4, 4, 1) and is efficient (see Figure 1). Figure 7 shows the skeleton of the corresponding dominance networks for the cases of adding DMU M, respectively M'.

Note that DEA always carries out a relative efficiency assessment, which means that the efficiency score and efficiency status of the DMUs can change depending on the set of DMUs considered. The same occurs with the proposed CNA DEA approach, i.e., the dominance network and its corresponding CNA DEA indexes depend on the set of DMUs considered. Table 4 shows the results of adding DMU M to the original set of DMUs. It can be seen that, since M is inefficient, the inefficiency score e_r^{max} of the original DMUs does not change, nor does τ_r^{min}. The network topology changes a little bit, affecting the degree distribution and the clustering coefficients of some nodes, specifically those that are linked by the new DMU M. The betweenness centrality of the nodes also change since the total number

of shortest paths between the nodes in the network has increased. The benchmarking potential of the two efficient DMUs that dominate M has increased, by 4.0 in the case of DMU B (which is farther from M) and by 3.0 in the case of DMU E (which is closer). These numbers correspond, respectively, to the distances e_r^{max} and τ_r^{min} of DMU M. The layer structure does not change much. All original nodes stay in their corresponding layers with the new DMU assigned to layer 2. Some aggregate indexes for component one (like the average degree d_1^{aver} or the average inefficiency score θ_1^{aver}) also change. The average clustering coefficient of component 1 also increases a little bit.

Table 5 shows the results for adding DMU M' to the original set of DMUs. It can be seen that in this case, since M' is efficient, the inefficiency score e_r^{max} of some of the original DMUs changes. In particular, DMU E, which originally was efficient, is not any more. The clustering and betweenness coefficients of some of the original DMUs also change. Note how the CNA DEA indexes of the new DMU are completely different to those that were observed for DMU M in the previous case. Actually, since M' is efficient, it has an associated benchmarking potential and inefficiency radius. Also note that, as before, adding just a single DMU does not change the network topology much. Two of the original nodes though, namely, DMUs E and G, have moved to a deeper layer.

5. Conclusions

In this paper, a novel way of assessing the efficiency of a set of DMUs using CNA tools has been presented. The approach considers an FDH technology and an additive normalised-slacks efficiency measure. This new methodology complements and enhances the traditional numerical assessment of DEA with, among other features, powerful filtering and visualisation capabilities, which are very valuable, especially for large multidimensional data-sets. Thus, for example, with this tool it is easy to identify a sequence of targets or stepwise path to the efficient frontier. It is also rather easy to measure and visualise the potential of each efficient DMU as a benchmark for the inefficient units. Similarly, from the point of view of the inefficient DMUs, it is easy to assess the effort required to target the different efficient DMUs that dominate it. In addition, other indicators that have not been considered in DEA previously, such as betweenness centrality, clustering coefficient or specificity can now be computed. It is also possible to study the effects of the network growth (e.g., the impact of adding new DMUs to the data-set). In summary, the idea explored in this research is to build upon the strengths and versatility of both DEA and CNA.

The proposed CNA DEA approach allows the computation of a number of interesting quantitative measures that together give a very complete picture of the performance of the individual DMUs and of the data-set as a whole, something which, for example, may be of interest in a centralised DEA context. This distinctive feature of the approach, i.e., its use of a systemic perspective that is interested not only in assessing the efficiency of the individual DMUs but also to capture the implicit (through the dominance relations) dependencies between them, provides new insights that complement the conventional DEA analysis. Take, for example, the existence of components that represent independent groups of DMUs, the characterisation of the layer structure of each component, the distribution of the distance to the frontier of the different DMUs or of the weights of the different edges, etc. In this respect, the proposed approach provides an integrated framework that allows processing and comparing the results of different data-sets. Also very interesting is the possibility of analysing the dynamic features of the network when applied to multi-period input–output data. Similarly, metafrontier analysis can also be studied from this network perspective.

In spite of the many strengths of the proposed approach, there are some limitations as well as many issues that need further research. The most important point is to try to extend the analysis to convex technologies (CRS and VRS). The case of multiple technologies (e.g., Lozano, 2014) may also be considered. The proposed CNA indexes can be computed for any dominance network, not being dependent on the assumption of an FDH technology. The DEA technology assumed has an effect on the own dominance network definition, that only considers the observed DMU. Thus, in order to extend the approach to the CRS and VRS cases, additional nodes corresponding to the projections of the observed DMUs on those two frontiers may have to be introduced. In this way, the VRS and CRS frontiers can be sampled and included in the dominance network. In the case of FDH it is simpler, as the efficient frontier is formed by a subset of the observed DMUs and it is therefore perfectly included in the proposed dominance network.

Extending the approach to other efficiency measures would also be interesting. In particular, it might be interesting to study the effect of computing edges weights using the MIP since in that case not all paths connecting two nodes would be geodesics. There are more venues of research that open up, like processing different real-world data-sets (of which there are many in the DEA literature) and try to find regularities in their CNA measures (e.g., if the existence of one large component is the rule), using graph mining techniques to find clusters of DMUs, etc. What is clear is that the CNA

approach provides a fresh perspective on the problem of efficiency assessment.

Disclosure statement

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Appendix

Table 6 shows some CNA DEA indexes for $g^{average}$, g^{range} and g^{median}. Only indexes that depend on the direction vector chosen are shown. As indicated in the main text, the topology of the dominance network, its component and layer structure and many features (like component and layer structure, degree distributions, specificity, hub index, clustering and betweenness centrality coefficients, etc.), however, are the same independently of the direction vector.

Table 6. CNA DEA indexes for Lim et al. (2011) data-set and different direction vectors.

DMU r	e_r^{max}			τ_r^{min}			κ_j			σ_j		
	$g^{average}$	g^{range}	g^{median}	$g^{average}$	g^{range}	g^{median}	$g^{average}$	g^{range}	g^{median}	$g^{average}$	g^{range}	g^{median}
A	0.000	0.000	0.000	0.000	0.000	0.000	2.382	1.964	2.159	1.247	0.964	1.159
B	0.000	0.000	0.000	0.000	0.000	0.000	4.951	3.286	4.841	1.690	1.018	1.698
C	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
D	0.162	0.143	0.143	0.162	0.143	0.143	–	–	–	–	–	–
E	0.000	0.000	0.000	0.000	0.000	0.000	2.606	1.893	2.476	1.359	0.929	1.317
F	0.324	0.286	0.286	0.324	0.286	0.286	–	–	–	–	–	–
G	0.162	0.143	0.143	0.162	0.143	0.143	–	–	–	–	–	–
H	0.380	0.268	0.365	0.380	0.268	0.365	–	–	–	–	–	–
I	0.543	0.411	0.508	0.543	0.411	0.508	–	–	–	–	–	–
J	1.690	1.018	1.698	1.247	0.929	1.159	–	–	–	–	–	–
K	0.599	0.393	0.587	0.599	0.393	0.587	–	–	–	–	–	–
L	1.415	0.911	1.397	0.973	0.821	0.857	–	–	–	–	–	–

Analysing Olympic Games through dominance networks

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HIGHLIGHTS

- A dominance network is built based on the medals won by each country, its population and its GDP.
- Complex networks analysis can be performed on this weighted, directed network.
- Network is transitive, acyclic and layered.
- Global and node-specific measures are computed and analysed (efficiency, clustering, betweenness).
- Beijing 2008 data are used to illustrate the approach.

ARTICLE INFO

ABSTRACT

The aim of this paper is to assess the results/performance of countries in the Olympic Games, taking into account their size and resources. A complex network analysis approach is proposed. The first step is to build the dominance network, which is a weighted directed graph in which nodes represent the participating nations and the arc length between any two nations measures the weighted difference in the number of medals won by both countries. An arc from a country to another b exists only if the latter has won more medals than the former and, in addition, it is smaller in population and in terms of GDP. In other words, an arc between two nodes exists if the origin nation performs worse than the destination when, given the population and GDP of both countries, it should have performed better (or at least equally). This dominance network has transitive links and a layered structure and, apart from being visualized, it can be characterized using different complex network measures. The results of the Beijing 2008 Olympic Games are used to illustrate the proposed approach.

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1. Introduction

Undoubtedly the Olympic Games are one of the most important sports events, followed by billions of people all over the world. Almost all countries take part and the medal count they obtain can be used to rank and assess their performance. The most common ranking method is lexicographic order. The idea is to rank countries based on their number of gold medals. Ties are broken ordering by silver medals, and if ties persist, then they are broken ordering by bronze medals. The lexicographic order is not a sophisticated ranking method. Precisely, because of its simplicity it is the one used by journalists and also, for example, in Wikipedia. It has, however, two drawbacks. One is that it uses a non-compensatory criterion so that one gold medal is worth more than any number of silver or bronze medals. The other is that it does not take into

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account the attributes of the different countries. Thus, it is to be expected that more populated countries should win more medals, since they are more likely to have athletes that excel at any given sport. However, that does not always occur. Thus, for example, India does not normally have its fair share of medals. It can be reasonably argued that economic prowess also plays a role and, in fact, richer countries have more means to support elite athletes and provide them with appropriate training facilities. Again, this is only a general rule and there are rich countries (such as, for example, Saudi Arabia) that do not win as many medals as they might. Nevertheless, these two explanatory variables (namely, population and Gross Domestic Product (GDP) or, alternatively, GDP per capita) are often used to assess (and also to forecast) the performance of nations in the Olympic Games (Lozano et al. [1], Bernard and Busse [2], Wu and Liang [3], Vagenas and Vlachokyriakou [4], etc.). The mathematical methodology most commonly used is Data Envelopment Analysis (DEA) and there are actually a fair number of DEA applications to the Olympic Games [1,5–11,3,12]. All these DEA approaches involve solving some optimization model, generally of the Linear Programming type.

Instead of solving an optimization model in this paper, a different approach will be used. It is based on applying network analysis (NA) to a dominance network built with the data on population, GDP and the number of gold, silver and bronze medals won by the different countries. NA is a well established paradigm that has taken off in the last 25 years, with a spectacular growth of its theoretical corpus and with an even bigger growth in terms of applications. As regards, NA theory, there are a number of good textbooks on the subject (e.g. Refs. [13,14]). As regards applications, they span many different sectors, including transportation (e.g. Ref. [15]), economics (e.g. Ref. [16]), supply networks (e.g. Ref. [17]), etc. In particular, NA has been applied to study soccer teams (e.g. Ref. [18]), tennis (e.g. Ref. [19]), cricket [20,21], baseball (e.g. Ref. [22]), etc. However, to the best of our knowledge no NA approach has been applied to assess the performance of participating nations at the Olympic Games as proposed in this paper. Admittedly, the proposed NA approach is more complex and powerful than lexicographic order or than methods that compute and analyse simple weighted ratios (like, e.g. the weighted number of medals per capita or per GDP monetary unit) but it allows a more complete assessment. It is also more effective, as it can integrate more information than these simpler methods. Last, but not least, NA provides also superior visualization capabilities.

The structure of the paper is the following. In Section 2 some basic NA concepts and measures are briefly reviewed. In Section 3 how to build the dominance network associated with the Olympic Games is explained and the structure and features of this type of network are discussed. Section 4 uses the Beijing 2008 Summer Olympic Games to illustrate the proposed approach. Finally, Section 5 summarizes and concludes.

2. Some basic NA concepts and measures

Consider a directed weighted network $N(V, E)$ where V is the set of nodes and E is the set of directed arcs between them. The weight of each arc (i, j) is denoted as w_{ij}. The density of the network, i.e. the ratio of the actual number of arcs to the maximum number of arcs that the network might have with this number of nodes, is a global measure that applies to the whole network. It can also be applied at the network component level. Defining connected components in a directed network is more complex than in an undirected network but for the purpose of this paper we will use the concept of weakly connected components, which are the largest subsets of nodes that are connected (i.e. there is a path between them) on the associated undirected network. Connected components represent independent, i.e. not connected among themselves, subgraphs, for which specific network measures can be computed. Thus, if the network $N(V, E)$ has C connected components given by the node partition $\{V_1, V_2, \ldots, V_C\}$, then the corresponding subnetworks are $N_c(V_c, E_c)$ $c = 1, 2, \ldots, C$ where $E_c = \{(i, j) \in E | i, j \in V_c\}$ represents the set of arcs between the nodes in V_c. In particular, there can be components formed by just a single node (called an isolate).

Although in- and out-degrees, d_r^{in} and d_r^{out} are local measures, the corresponding degree distribution is a global network feature. The structural and dynamic properties of a network can be quite different depending on their degree distribution as this affects processes such as diffusion and characteristics such as resilience [23]. For a weighted directed network, since the arcs have different weights, more important than the node degree are its in- and out-strengths $\text{NS}_r^{\text{in}} = \sum_{j \in V_r^{\text{in}}} w_{jr}$ and $\text{NS}_r^{\text{out}} = \sum_{j \in V_r^{\text{out}}} w_{rj}$.

Another local property that has a global effect on the structure and the working of a network is the clustering coefficient of a node, which is the fraction of its neighbours that are also directly connected. Defining the in- and out-neighbourhood of a node r as $V_r^{\text{in}} = \{j \in V : (j, r) \in E\}$ and $V_r^{\text{out}} = \{j \in V : (r, j) \in E\}$ the clustering coefficient of node r is

$$
\text{CC}_r = \begin{cases} 0 & \text{if } \left| V_r^{\text{in}} \cup V_r^{\text{out}} \right| \leq 1 \\ \dfrac{\displaystyle\sum_{j \in V_r^{\text{in}} \cup V_r^{\text{out}}} \sum_{\substack{p \in V_r^{\text{in}} \cup V_r^{\text{out}} \\ p \neq j}} \delta_{jp}}{\left| V_r^{\text{in}} \cup V_r^{\text{out}} \right| \cdot \left(\left| V_r^{\text{in}} \cup V_r^{\text{out}} \right| - 1 \right)} & \text{otherwise} \end{cases}
$$

where $\delta_{jp} = \begin{cases} 1 & \text{if } (j, p) \in E \\ 0 & \text{otherwise} \end{cases}$ indicates whether or not the directed arc (j, p) exists. The above CC can be corrected for degree–degree correlation as suggested by Soffer and Vázquez [24]. In any case, the average clustering coefficient of the network is $\text{CC}^{\text{aver}} = \frac{1}{|V|} \sum_{r \in V} \text{CC}_r$.

Other interesting NA measures are the betweenness centrality of the nodes (which measure the frequency with which they lie in the shortest path between any two nodes in the network), the average path length (which is the average of the geodesic distances between the nodes) the network diameter (which is the maximum geodesic distance between any two nodes), the closeness centrality of a node r (which is the inverse of the average geodesic distance of a node to the other nodes) or the network efficiency (which measures the overall capability of the network to exchange information or transmit signals along the shortest paths).

Finally, many real-world networks show degree–degree correlations. Thus, plotting the degree-dependent average-nearest-neighbour degree versus the corresponding degree variable, positive correlations (a.k.a. assortativity) or negative correlations (a.k.a. disassortativity) can be found.

Also, there may be nodes in a network that are structurally similar, i.e. they have incoming links from more or less the same nodes and have outgoing links to the same countries. The structural similarity between two nodes can be measured using, for example, Jaccard similarity coefficient. Based on this structural similarity coefficient a hierarchical clustering algorithm can be used to identify groups of nodes that have a given degree of structural similarity among them.

3. Proposed NA of Olympic Games

In order to apply NA to Olympic Games the first step is to build the corresponding dominance network. The set of nodes V corresponds to the set of countries that won any medal. A directed arc between two countries r and j exists if r is dominated by j, which means that having j less population and less GDP than r it won, on a weighted basis, more medals than r. The weighted difference in the number of medals won by the two countries will be the weight associated with that arc. Mathematically, the set of arcs corresponds to $E = \{(r, j) : \text{Pop}_j \leq \text{Pop}_r \ \wedge \ \text{GDP}_j \leq \text{GDP}_r \ \wedge \ w_{rj} > 0\}$ where the arc weights are computed as

$$w_{rj} = v_G \cdot \left(\text{Gold}_j - \text{Gold}_r\right) + v_S \cdot \left(\text{Silver}_j - \text{Silver}_r\right) + v_B \cdot \left(\text{Bronze}_j - \text{Bronze}_r\right)$$
$$= v_G \cdot \Delta\text{Gold}_{jr} + v_S \cdot \Delta\text{Silver}_{jr} + v_B \cdot \Delta\text{Bronze}_{jr}.$$

The weighting coefficients v_G, v_S and v_B above represent the relative value of gold, silver and bronze medals, respectively. These relative values imply a compensation scheme so that, for example, one gold medal is worth $\frac{v_G}{v_S}$ silver medals or $\frac{v_G}{v_B}$ bronze medals.

The dominance network $N(V, E)$ defined above has two interesting characteristics. One is the transitivity of the arcs, i.e. $(r, j) \in E \ \wedge \ (j, p) \in E \Rightarrow (r, p) \in E$, which derives from the transitivity of the dominance relationship considered. Moreover, it is easy to see that

$$w_{rp} = v_G \cdot \Delta\text{Gold}_{pr} + v_S \cdot \Delta\text{Silver}_{pr} + v_B \cdot \Delta\text{Bronze}_{pr}$$
$$= v_G \cdot \left(\Delta\text{Gold}_{pj} + \Delta\text{Gold}_{jr}\right) + v_S \cdot \left(\Delta\text{Silver}_{pj} + \Delta\text{Silver}_{jr}\right) + v_B \cdot \left(\Delta\text{Bronze}_{pj} + \Delta\text{Bronze}_{jr}\right)$$
$$= v_G \cdot \Delta\text{Gold}_{pj} + v_S \cdot \Delta\text{Silver}_{pj} + v_B \cdot \Delta\text{Bronze}_{pj} + v_G \cdot \Delta\text{Gold}_{jr} + v_S \cdot \Delta\text{Silver}_{jr} + v_B \cdot \Delta\text{Bronze}_{jr}$$
$$= w_{rj} + w_{jp}.$$

The above additive property means that the length of the shortest path between any two connected nodes is equal to the sum of the weights of the arcs in the path and equal to the weight that directly connects the origin and the end of the path. In other words, in this network all the paths between any two nodes have equal length and all of them are therefore geodesics (i.e. shortest paths).

The network $N(V, E)$ may have different weakly connected components. In particular, it may be possible that a component is formed by a single node, provided that it is not dominated by any other node and does not dominate any other nodes. These isolates may be called outliers.

Another important property of this network is that each connected component c has a hierarchical, i.e. layered, structure. Thus, we can distinguish a subset of nodes that dominate some nodes but are not dominated by any other node. We can call these nodes Dominating and denote this subset as $V_c^D \subset V_c$. At the other extreme we can find a subset of nodes that are dominated by other nodes but do not dominate any other nodes. Let us call these nodes dominated and denote this subset as $V_c^d \subset V_c$. The rest of the nodes are such that they dominate some nodes and at the same time are dominated by other nodes. Let us denote this subset as $V_c^{Dd} = V_c \setminus \left(V_c^D \cup V_c^d\right)$. We can assign each node to a layer depending on their position in the chain of dominance relationships. Thus, the nodes in V_c^D have $d_r^{\text{out}} = 0$, i.e. they have no outgoing arcs, since they are not dominated by any node. Let us assign them to layer 0. The layer of the rest of nodes of the component can be computed recursively using

$$\lambda(r) = \begin{cases} 0 & \text{if } r \in V_c^D \\ 1 + \max_{j:(r,j)\in E_c} \lambda(j) & \text{otherwise} \end{cases}$$

i.e. each node belongs to the layer that follows the deepest layer to which those nodes that dominate it belong. The number of layers in a component is thus $\lambda_c^{\max} = 1 + \max_{r \in V_c} \lambda(r)$.

The average length within a component c represents the average weighted difference in the number of medals won by the countries that belong to that component, i.e. $\alpha_c^{\text{aver}} = \frac{1}{|V_c| \cdot (|V_c| - 1)} \cdot \sum_{(r,j) \in E_c} w_{rj}$; analogously, the diameter, the component, the maximum of those weighted differences, i.e. $\alpha_c^{\max} = \max_{(r,j) \in E_c} w_{rj}$.

Since the proposed dominance network is not fully connected, we can compute the in- and out-closeness centrality of a node r as $\kappa_r^{\text{in}} = \frac{|V_r^{\text{in}}|}{\sum_{j \in V_r^{\text{in}}} w_{jr}} = \frac{|V_r^{\text{in}}|}{\text{NS}_r^{\text{in}}}$ and $\kappa_r^{\text{out}} = \frac{|V_r^{\text{out}}|}{\sum_{j \in V_r^{\text{out}}} w_{rj}} = \frac{|V_r^{\text{out}}|}{\text{NS}_r^{\text{out}}}$, respectively. They represent the average weighted medal difference of country r with respect to those that it dominates (in-closeness) or that dominate it (out-closeness). Thus, the in- and out-closeness centrality of country r just corresponds to the inverse of the average in- and out-strengths NS_r^{in} and NS_r^{out}, respectively, which indicate whether the weighted difference in the number of medals with respect to the countries that r dominates or that dominate r, respectively, is large or small. Good performing countries have high in-strength and low out-strength, and correspondingly, low out-closeness centrality and high in-closeness centrality.

The network efficiency, $\eta = \frac{1}{|V| \cdot (|V| - 1)} \sum_{(r,j) \in A} \frac{1}{w_{rj}}$, represents in our case a measure of the overall weighted differences in the number of medals between the countries that form the network. A lower efficiency means that the weighted differences in medals are large while the opposite occurs when the overall weighted medal differences are small.

As regards the nodes degrees, the in- and out-degree of a node r represent the number of nodes that r dominates and that dominate r, respectively. Denoting as $d(r)$ and $D(r)$ respectively the corresponding sets, i.e. $d(r) = \{j : (j, r) \in E\}$ and $D(r) = \{j : (r, j) \in E\}$, we have $d_r^{\text{in}} = |d(r)|$ and $d_r^{\text{out}} = |D(r)|$. Note also that isolates have zero in- and out-degrees $d_r^{\text{in}} = d_r^{\text{out}} = 0$. Similarly, the nodes in the layer 0 of a component c also have out-degree zero, i.e. $d_r^{\text{out}} = 0 \ \forall r \in V_{c(r)}^D$. That is because those nodes have, by definition, no nodes that dominate them, i.e. $D(r) = \varnothing$, i.e. there are no nodes that dominate them. In our application, we can say that the countries with zero out-degree (and consequently, zero out-strength) are the best performing countries. They are the benchmarks, the role models, for all those countries that they dominate. Although outliers are also, in that sense, best performers, they do not have this benchmarking function, as there are no countries dominated by them. It may be interesting to record which countries belong to these two categories. To that end, let us define the sets $V^{\text{outliers}} = \{r \in V : d_r^{\text{in}} = d_r^{\text{out}} = 0\}$, $V^{\text{benchmarks}} = \{r \in V : d_r^{\text{out}} = 0 \ \wedge \ d_r^{\text{in}} > 0\}$ and $V^{\text{best-performers}} = V^{\text{outliers}} \cup V^{\text{benchmarks}}$.

The average degree of a component d_c^{aver} represents, in our application, the number of countries that a country dominates (or is dominated by) on average. The network density, in turn, indicates the probability that a country dominates or is dominated by another. The in- and out-degree distributions give an idea of how frequent and extended the dominance relationships are. Thus, it may happen that the in-degree is centred around the average value or, alternatively, it may have a large dispersion, with many countries dominating (or being dominated by) a few countries and a few countries dominating (or being dominated by) many.

As regards degree–degree correlations, since the dominance network $N(V, E)$ is directed, the following average nearest-neighbour degrees can be computed for each node r:

- average in-degree of out-nearest neighbours (AIDONN)

$$\text{AIDONN}_r = \frac{1}{|V_r^{\text{out}}|} \cdot \sum_{j \in V_r^{\text{out}}} k_j^{\text{in}}$$

- average in-degree of in-nearest neighbours (AIDINN)

$$\text{AIDINN}_r = \frac{1}{|V_r^{\text{in}}|} \cdot \sum_{j \in V_r^{\text{in}}} k_j^{\text{in}}$$

- average out-degree of out-nearest neighbours (AODONN)

$$\text{AODONN}_r = \frac{1}{|V_r^{\text{out}}|} \cdot \sum_{j \in V_r^{\text{out}}} k_j^{\text{out}}$$

- average out-degree of in-nearest neighbours (AODINN)

$$\text{AODINN}_r = \frac{1}{|V_r^{\text{in}}|} \cdot \sum_{j \in V_r^{\text{in}}} k_j^{\text{out}}.$$

The corresponding degree-dependent averages correspond to averaging any of these measures for all the nodes that have a given degree, i.e.

$$\text{AIDONN}(k^{\text{out}}) = \frac{1}{|V| \cdot P(k^{\text{out}})} \cdot \sum_{r : k_r^{\text{out}} = k^{\text{out}}} \text{AIDONN}_r$$

$$\text{AIDINN}(k^{\text{in}}) = \frac{1}{|V| \cdot P(k^{\text{in}})} \cdot \sum_{r : k_r^{\text{in}} = k^{\text{in}}} \text{AIDINN}_r$$

$$\text{AODONN}(k^{\text{out}}) = \frac{1}{|V| \cdot P\left(k^{\text{out}}\right)} \cdot \sum_{r:k_r^{\text{out}}=k^{\text{out}}} \text{AODONN}_r$$

$$\text{AODINN}(k^{\text{in}}) = \frac{1}{|V| \cdot P\left(k^{\text{in}}\right)} \cdot \sum_{r:k_r^{\text{in}}=k^{\text{in}}} \text{AODINN}_r.$$

An important aspect of NA is the possibility of representing graphically the network and visualizing the relationships it contains. Due to the arc transitivity property of the proposed Olympic Games dominance network, the clustering coefficient is relatively high due to the existence of many connections. Therefore, apart from drawing the whole network, with all its arcs, it may be interesting to draw two simplified but representative subnetworks. One is the network skeleton, which is formed by removing from the network all transitive arcs, i.e. all the arcs in $E^{\text{transitive}} = \{(r, j) : \exists p \in D(r) \cap d(j)\}$. An equivalent form of expressing this is $E^{\text{skeleton}} = E \setminus E^{\text{transitive}} = \{(r, j) : \neg \exists p \in V \text{ such that } (r, p) \in E \wedge (p, j) \in E\}$. Note that this skeletonization does not imply any loss of information since the transitive arcs are implicit and can be readily recovered.

The second subnetwork that may be interesting to draw is the bipartite graph that contains only the arcs between the non-dominated nodes $V^{\text{best-performers}}$ and the dominated nodes $V^{\text{dominated}} = V \setminus V^{\text{best-performers}}$. The arcs in this subnetwork correspond to the maximum weighted differences in the number of medals between a country and those that dominate it. More precisely, this subgraph only retains, for each country, the arcs towards the best-performing countries that dominate it. The weights of those arcs represent, therefore, the corresponding medal difference, i.e. the effort that a country should have to make in order to reach the same level as one of its benchmarks. The arcs from one country to other countries that may dominate but that are not best-performers do not appear in this subnetwork as they do not represent maximum medal differences, i.e. they do not represent maximum improvement potential.

Another type of subnetwork that may be of interest is the subgraph centred on a node, a.k.a. the ego network of the node. This is formed by the node itself plus all the countries which it dominates and that dominate it. This subgraph allows studying the situation and performance of a specific country. Thus, it is not the same that a country has many incoming arcs and a few (or none) outgoing arcs as the opposite structure. Also, the weights of the corresponding incoming and outgoing arcs provide information about the separation (in terms of performance) between a country and those that it dominates or that dominate it. Mathematically, the ego network on node r is $N(V(r), E(r))$ where $V(r) = \{r\} \cup D(r) \cup d(r)$ and $E(r) = \{(r, j) : j \in D(r)\} \cup \{(j, r) : j \in d(r)\} \cup \{(j, k) : j \in d(r) \; k \in D(r)\}$.

4. NA of Beijing 2008 summer Olympic games

In this section the proposed approach will be illustrated by the Olympic Games that took place in Beijing in 2008. The data about population, GDP and number of Gold, Silver and Bronze medals have been extracted from Ref. [11]. In order to weight the differences in medals between the countries we have considered weight coefficients of the form $v_B = 1$; $v_S = a$; $v_G = a^2$ for some $a \geq 1$. Table 1 shows some global information about the resulting dominance network for several values of parameter a, which corresponds to varying the relative value between the different types of medals. Note that the dominance network changes with the value of parameter a. Thus, the number of links, and consequently the network density, grows slightly as a increases. The average clustering (corrected by degree–degree correlation) is high, due to the transitivity of the links, and also increases with a. As can be expected, the average path length and diameter increase as a increases. The network efficiency, on the other hand, decreases as a increases.

It can be seen from Table 1 that although the values of all network measures depend on a, the changes are either minor or, in the case of the average path length, the diameter and the network efficiency, predictable and logical. That is why the results that will be shown from now on correspond to a specific value of parameter a (namely $a = 2$). Thus, Table 2 shows, for $a = 2$, the joint distribution of the in- and out-degrees. The countries with the zero out-degree row are those that belong to the set of non-dominated nodes $V^{\text{best-performers}}$. In particular, those in the cell zero in- and zero out-degree correspond to the set of isolates V^{outliers}. The countries with zero in-degree are those countries that do not dominate any other. In general, a high out-degree can be considered an indication of inferior performance. Note also that, although some countries dominate many others and some countries are dominated by many others, it is not frequent that both situations coincide.

Fig. 1 shows the distribution of the in-, out- and total degree. The vertical axis represents the inverse cumulative probability $\Pr(\geq k) = \Pr(\text{degree} \geq k)$. The in-, out- and total degree are distributed approximately linearly. Note that the maximum in-degree is around 50 while the maximum out-degree is somewhat smaller (around 40). Because nodes with large in-degree do not have large out-degrees (and vice versa) the maximum total degrees is also around 50.

Table 3 shows the layer distribution of the network. Since when a country dominates another, the latter must belong to a higher numbered layer, the lower the layer the better the performance. In particular, by definition, the countries in layer 0 are those that belong to the set of non-dominated nodes $V^{\text{best-performers}}$. This includes the three isolates in $V^{\text{outliers}} = \{CN, TG, US\}$. No countries are dominated by these outliers because for that to happen they would need to have a larger population and larger GDP, something which, at least in the case of United States (US) and China (CN), is not feasible. They are also non-dominated since, at least in the case of CN and US, it is unlikely that smaller countries (with less population and less GDP) win more medals. Other countries in layer 0 include Jamaica (JM), Cuba (CU), Australia (AU), Russia (RU), Great Britain (GB), among others. These countries are known for traditionally winning a relatively high number of medals, having, in

Table 1
Global information of dominance network for different medals weightings.

	$a = 1$		$a = 2$		$a = 3$	
	Network	Giant Compon.	Network	Giant Compon.	Network	Giant Compon.
# Components	5	–	4	–	4	–
Layers	5	5	5	5	6	6
Nodes	86	82	86	83	86	83
Links	675	675	734	734	755	755
Density	0.092	0.102	0.100	0.108	0.103	0.111
Aver. clustering	0.313	0.329	0.369	0.382	0.391	0.405
Aver. path length	–	5.539	–	11.375	–	20.856
Diameter	–	57	–	134	–	251
Network efficiency	0.038	0.042	0.027	0.029	0.018	0.019
Isolates	CN, TG, US, IS		CN, TG, US		CN, TG, US	

Table 2
Joint distribution of in- and out-degree ($a = 2$).

Out-degree		0	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	>40
	>40	ZA	–	–	–	–	–	–	–	–	–
	36–40	IN, VE	–	–	–	–	–	–	–	–	–
	31–35	EG	MY	–	–	–	–	–	–	–	–
	26–30	IR, VN	DZ, CL CO, MX, SD	–	–	–	–	–	–	–	–
	21–25	NG	ID, MA	–	–	–	–	–	–	–	–
	16–20	BE, GR, IL	EC	–	–	–	–	–	–	–	–
	11–15	AT	AR, PT, SG, TH, TR	–	–	–	–	–	–	–	–
	6–10	AF, BR, JP	IE, SE, CH	–	TN, MK	–	–	–	–	–	–
	1–5	CA, MU, MD, ES	ET, FR, IT, KR, PL, DE	CM, DK, FI, NO	CZ, DO, PA, RO, UZ	BG, HR HU, KZ, KP	AM, AZ, LT, SK	NZ	–	–	–
	0	CN, TG, US	IS, NL, RU, GB	AU, KG, TJ, UA	BS, BH, KE, TT	EE, ZW	CU, LV, SI	BY	–	GE, MN	JM
		0	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	>40
		In-degree									

Table 3
Layer distribution ($a = 2$).

Layer no.	0	1	2	3	4	5
# countries	24	20	14	18	8	2
Countries	AU; BH; BS; BY; CN; CU; EE; GB; GE; IS; JM; KE; KG; LV; MN; NL; RU; SI; TG; TJ; TT; UA; US; ZW	AM; AZ; BG; CA; DE; ES; ET; HR; HU; KP; KR; KZ; LT; MD; MU; NO; NZ; PL; SK; UZ	AF; BR; CM; CZ; DK; DO; FI; FR; IE; IT; PA; RO; TN; MK	AR; AT; CH; CL; CO; DZ; EC; GR; JP; MA; MY; NG; PT; SD; SG; TH; TR; VN	EG; ID; IL; IR; MX; SE; VE; ZA	BE; IN
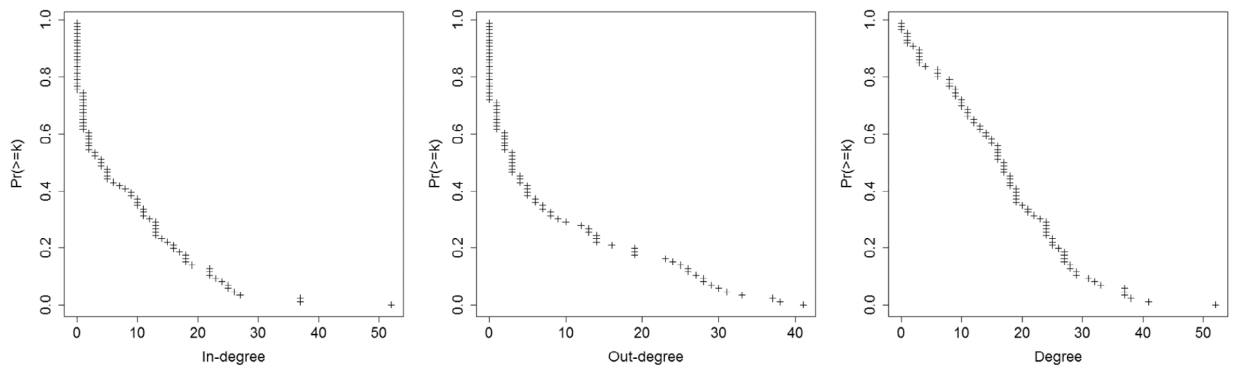
Fig. 1. Inverse cumulative probability of in-, out- and total degree ($a = 2$).

some cases, a small population and or a small GDP. The presence of some countries in this selected group of benchmark countries may not be expected, such as, for example, Slovenia (SI), Bahrain (BH) or Kyrgyzstan (KG). This shows the ability of the proposed dominance network approach to identify good performers (relative to their population and GDP) that may go undetected by the naked eye.
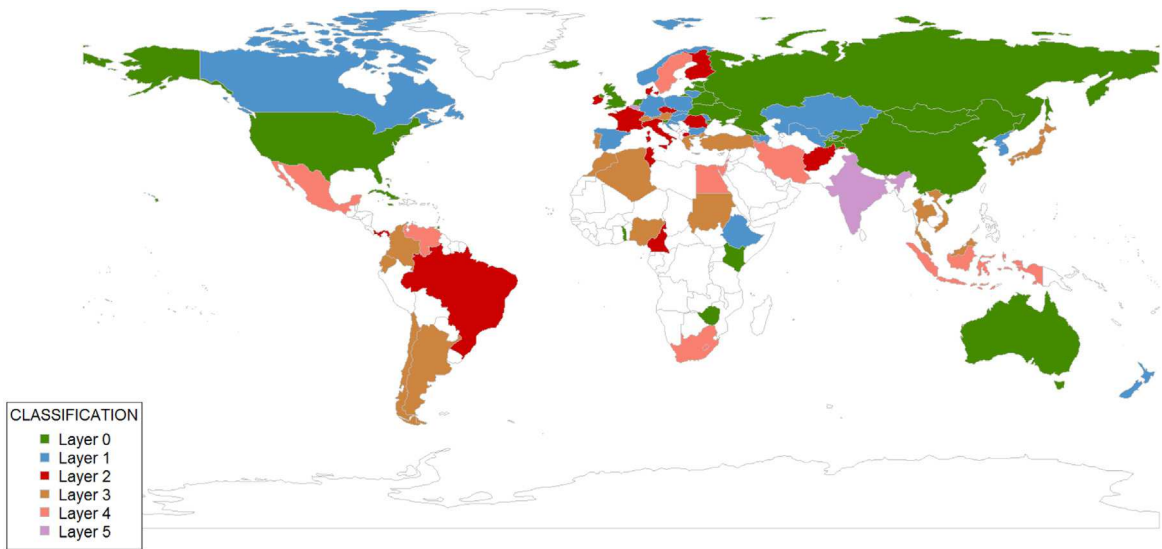
Fig. 2. Geographical distribution of layer structure ($a = 2$).
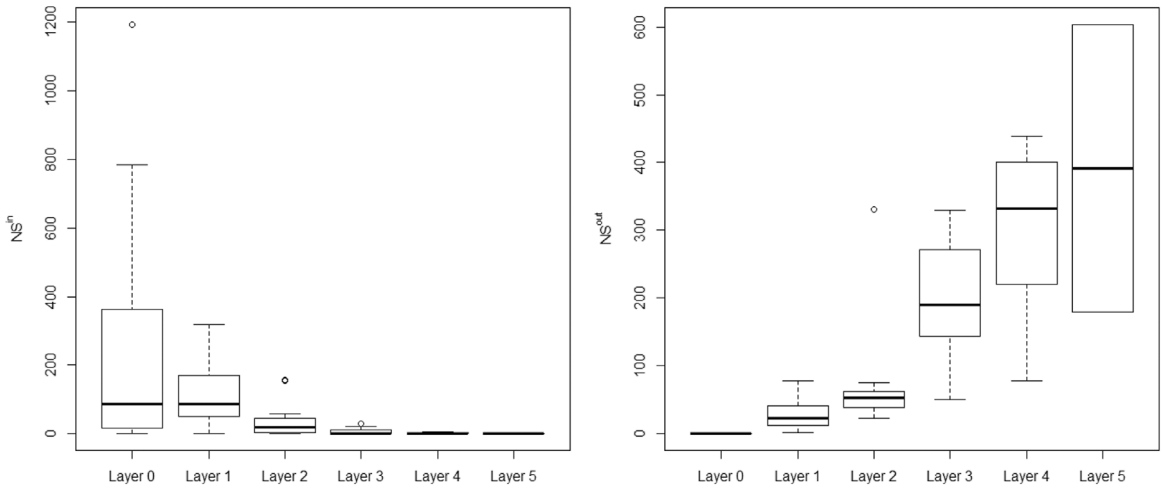
Fig. 3. Boxplots of in- and out-strength ($a = 2$).

On the other hand, in general, the higher the layer, the worse tends to be the country's performance. Looking, therefore, at the deeper layers we find countries that, relative to their population and GDP, do not earn a high number of medals. That is what seems to happen to countries such as India (IN), Belgium (BE), Egypt (EG), Indonesia (ID), Israel (IL), Iran (IR), Mexico (MX), Sweden (SE), Venezuela (VE) and South Africa (ZA). Looking back at Table 2 we find that these countries are dominated by many other smaller and poorer countries, as many as 40 other countries in the case of South Africa and more than 15 in the case of Belgium and Israel.

Fig. 2 shows the geographical distribution of the countries in the different network layers drawn with Rworldmap R package [25]. The strongest performers span the five continents and include big countries (such as the US, China and Russia) but also smaller ones (such as Cuba (CU), Iceland (IS) or Togo (TG)). In general, the more developed countries tend to belong to lower layers. Some countries in Central and South America and Asia and many in Africa have a white colour, indicating that they did not win any medals at the Beijing 2008 Olympics.

Fig. 3 shows the boxplots of the in- and out- strengths (NS^{in} and NS^{out}, respectively) of the countries in each layer. There is a clear upward trend for the out-strength and also, downwards and less intense, for the in-strength. In other words, the higher the layer to which a node belongs, the larger its out-strength tends to be and the lower its in-strength.

Fig. 4 shows the joint distribution of the in- and out-strength (NS^{in} and NS^{out}, respectively). Countries with zero out-strength are those in layer 0, i.e. those in the set of non-dominated nodes $V^{best\text{-}performers}$. As we move from down up and from right to left, the performance worsens. The corresponding lines with equal values of the difference between in- and out-strength have been drawn as reference. The distribution of the points has a similar shape to that of the entries
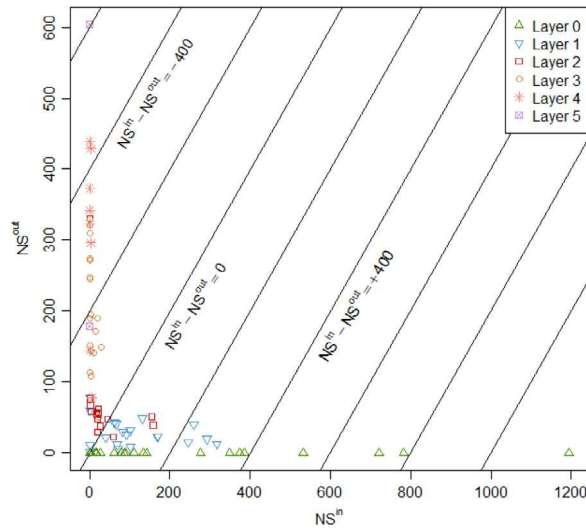
Fig. 4. Joint distribution of in- and out-strength ($a = 2$).
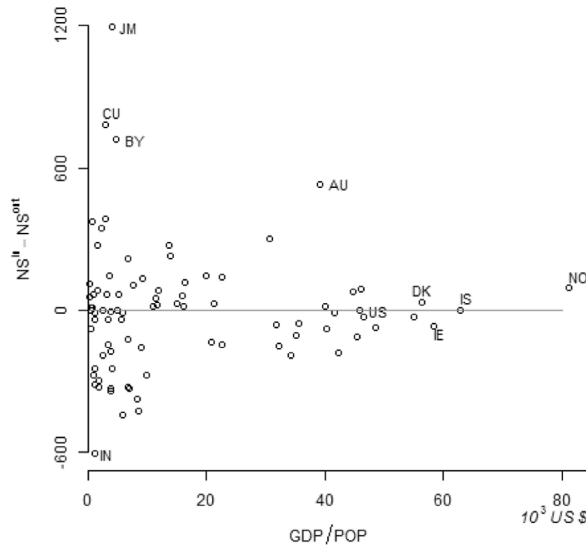
Fig. 5. In–out strength difference versus per capita GDP ($a = 2$).

in Table 2, with many countries concentrated along the horizontal line at zero height and along the vertical line with zero x value. The rest have low x and y values. High x/high y points do not exist.

Fig. 5 shows the difference between the in- and out- strength versus the per capita GDP. Positive values are an indication of good performance, and negative values mean a poor performance. Some sample countries have been labelled. It can be seen that there is more variability in performance among poorer countries, with some of them (such as Jamaica, Belarus (BY) or Cuba (CU)) performing very well while others (such as India (IN)) performing badly. Richer countries have positive or negative values for this difference but the range of values is smaller.

Fig. 6 shows the boxplots of the clustering coefficient (corrected by degree–degree correlation) and betweenness centrality (CC_r and β_r, respectively) of the countries in each layer. In both cases it seems that the nodes in the middle layers have a higher value than those in the lowest and highest layers. In particular, the nodes in the first and the last layer all have zero betweenness centrality. The betweenness coefficient indicates the importance of a country as a reference for the countries which are dominated by it.

Fig. 7 shows the clustering coefficient (corrected by degree–degree correlation) versus the total degree. Note that both axes are in log scale. A small negative correlation can be perceived, which is an indication of the existence of a hierarchical structure, so that the more connected nodes form a lower relative number of triangles with their neighbours [26].

Fig. 8 shows the degree–degree correlations between nodes. It can be seen that there is assortativity (as indicated by a positive slope) in the case of the average in-degree of in-nearest-neighbours (AIDINN) and the average out-degree of
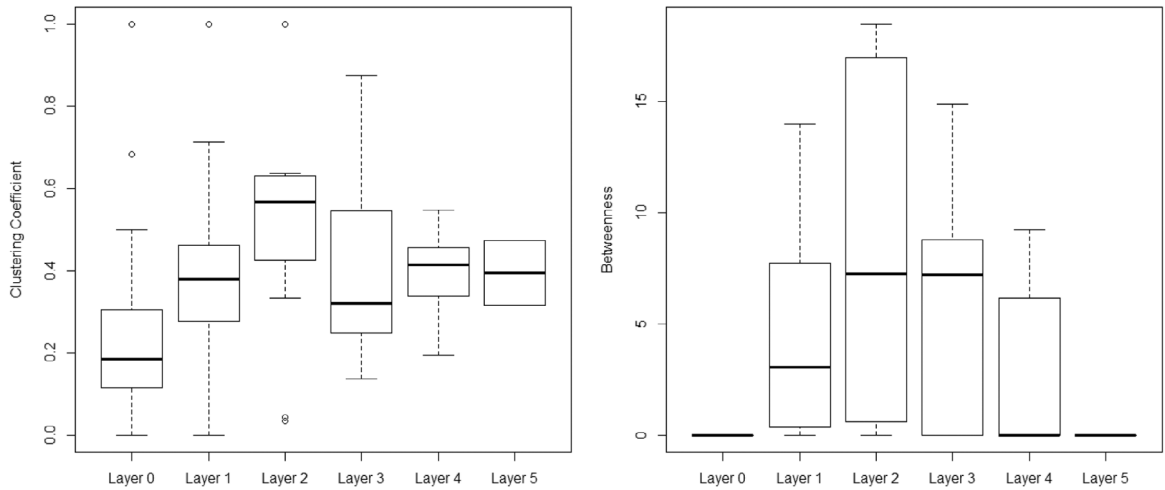
Fig. 6. Boxplots of clustering and betweenness coefficients ($a = 2$).
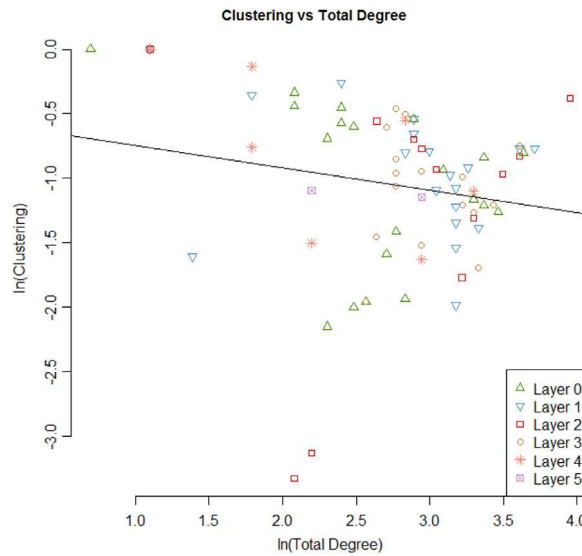
Fig. 7. Clustering coefficient versus total degree (log log scale, $a = 2$).

out-nearest neighbours (AODONN). This means that nodes with high in-degree tend to receive links from nodes with high in-degree and nodes with high out-degree tend to connect to nodes with high out-degree. On the other hand, there is disassortativity (as indicated by a negative slope) in the average out-degree of in-nearest neighbours (AODINN) and the average in-degree of out-nearest neighbours (AIDONN) cases, which means that nodes with high in-degree tend to receive links from nodes with low out-degree and nodes with high out-degrees tend to connect to nodes with low in-degree.

Fig. 9 shows the skeleton of the dominance network. The network layout reflects its layered structure. Although useful, the layered structure does not tell the whole story. Thus, the width of the links, which indicates their weights, should also be taken into account. For example, the wide link between Brazil (BR) and Russia (RU) means that RU has won a significantly higher number of medals than BR, in spite of RU not having a larger population and GDP than BR. Curiously, as the figure shows, and as shown in Tables 2 and A.1, that is the only country that RU dominates. Most other benchmark countries, such as Zimbabwe (ZW), Ukraine (UA), Tajikistan (TJ), etc. dominate more countries and thus receive more incoming links. We mentioned before that high layer countries are usually dominated by many other, better-performing countries. This is not shown in Fig. 9 because this is the skeleton network in which, to reduce clutter, the transitive connections (i.e. links that close triangles) have been removed. Except for some noticeably wider links, more links represent a small medals difference between the two corresponding countries. The transitive links, not shown in the skeleton network, would have more width.

As an example of ego networks, Fig. 10 shows the network centred on Switzerland (CH). It can be seen that this country belongs to a middle layer, receiving links from five countries in higher layers and sending links to six countries in lower layers. The weights of the links indicate the weighted difference in performance with respect to Switzerland's in and out
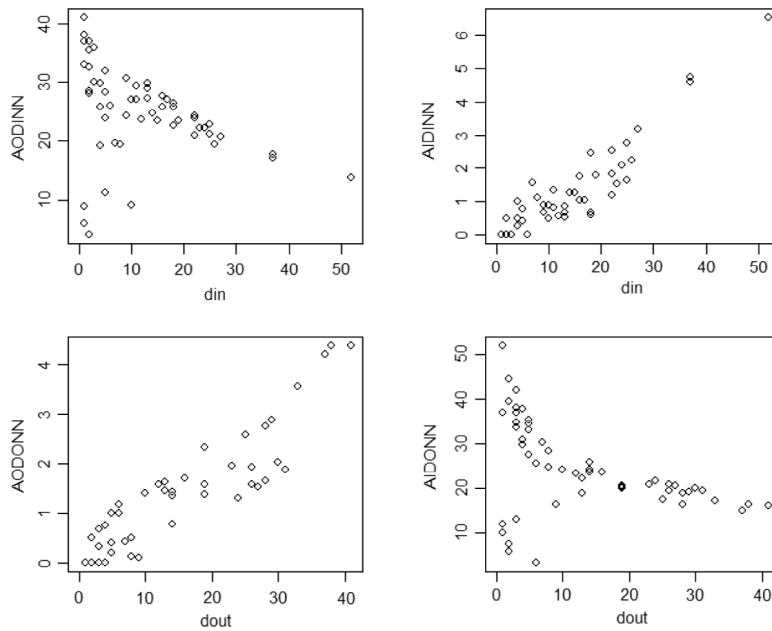
Fig. 8. Degree-dependent average nearest-neighbour degree versus degree ($a = 2$).
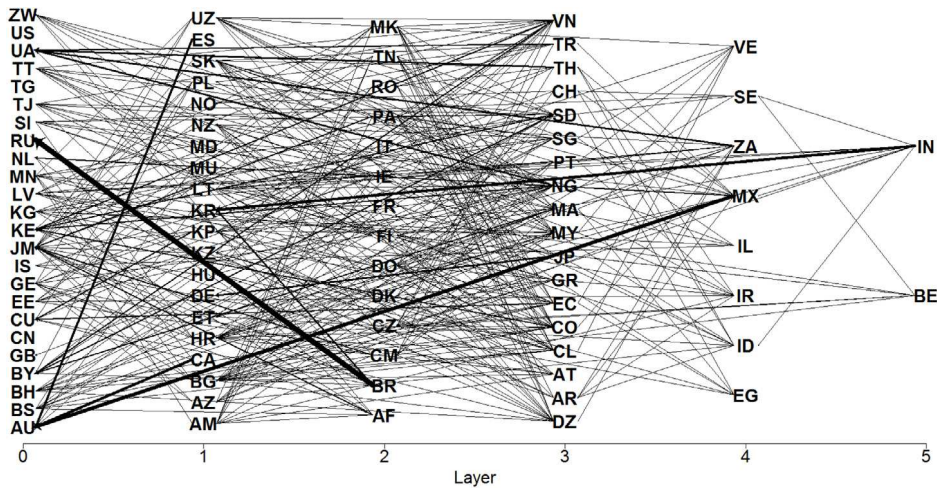
Fig. 9. Skeleton of dominance network ($a = 2$).

neighbours. In particular, the weight of its links with the best performing countries in layer 0 (namely, Jamaica (JM) and Georgia (GE)) indicates the potential improvement in the number of medals that the country might achieve if it performed as well as those layer 0 countries. Note that the in-strength is lower than the out-strength. Note also the high level of clustering (CC = 0.77), which indicates the existence of many dominance relationships also among its in and out neighbours. This country also has a high betweenness centrality, indicating that it can be considered as a country with an intermediate performance level, so that, although it is dominated by some better-performing countries, it can be used as an intermediate target for other worse performing countries. Note in this regard that although a country may be dominated by several other countries, the difficulty and effort level involved in achieving the same results as one of those dominating countries vary greatly, depending on the corresponding link length. Therefore, it may seem natural for a country to set a gradual improvement path so that its first steps represent more realistic improvement targets.

Table 4 shows the results of applying the first steps of the Single Linkage Clustering Algorithm (SLCA) to the Jaccard structural similarity coefficient between the different nodes of the network. Recall that structural similarity measures to what extent two nodes have the same incoming and outgoing vectors. Thus, in particular, the nodes in each of the three groups of countries that cluster at the 1000 threshold correspond to countries that have exactly the same incoming and outgoing links, i.e. countries that dominate the same countries and are dominated by exactly the same countries. Specifically,
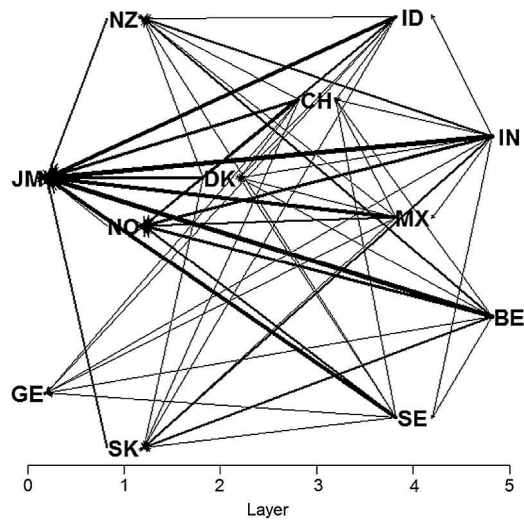
Fig. 10. Ego network of Switzerland (CH) ($a = 2$).

Table 4
SLCA hierarchical clustering of Jaccard structural similarity coefficients ($a = 2$).

Threshold	Clusters
1.000	{TT, BH}, {IT, FR}, {KG, TJ}
0.897	{TT, BH}, {IT, FR}, {KG, TJ}, {ID, MX}, {VN, SD}
0.895	{TT, BH}, {IT, FR}, {KG, TJ}, {ID, MX}, {VN, SD}, {TN, MK}
0.889	{TT, BH}, {IT, FR}, {KG, TJ}, {ID, MX}, {VN, SD}, {TN, MK}, {TH, AR}
0.875	{TT, BH}, {IT, FR}, {KG, TJ}, {ID, MX}, {VN, SD}, {TN, MK}, {TH, AR}, {MY, CL}
0.871	{TT, BH}, {IT, FR}, {KG, TJ}, {ID, MX}, {VN, SD}, {TN, MK}, {TH, AR}, {MY, CL}, {CO, DZ}
0.857	{TT, BH}, {IT, FR}, {KG, TJ}, {ID, MX}, {VN, SD}, {TN, MK}, {TH, AR}, {MY, CL}, {CO, DZ, MA}
0.846	{TT, BH}, {IT, FR}, {KG, TJ}, {ID, MX}, {VN, SD}, {TN, MK}, {LT, LV}, {TH, AR}, {MY, CL}, {CO, DZ, MA}
0.842	{TT, BH}, {IT, FR}, {KG, TJ}, {ID, MX}, {VN, SD}, {TN, MK}, {LT, LV}, {TH, AR}, {MY, CL}, {CO, DZ, MA}, {GR, BE}
0.833	{TT, BH}, {IT, FR}, {KG, TJ, BS}, {ID, MX}, {VN, SD}, {TN, MK}, {LT, LV}, {TH, AR}, {MY, CL}, {CO, DZ, MA}, {GR, BE}
0.826	{TT, BH}, {IT, FR}, {KG, TJ, BS}, {ID, MX}, {VN, SD}, {TN, MK}, {LT, LV}, {TH, AR}, {MY, CL}, {CO, DZ, MA}, {GR, BE}, {EE, AM}
0.821	{TT, BH}, {IT, FR}, {KG, TJ, BS}, {ID, MX}, {VN, SD}, {TN, MK}, {LT, LV}, {TH, AR}, {MY, CL}, {CO, DZ, MA}, {GR, BE}, {EE, AM}, {SK, NZ}
0.813	{TT, BH, PA}, {IT, FR}, {KG, TJ, BS}, {ID, MX}, {VN, SD}, {TN, MK}, {LT, LV}, {TH, AR}, {MY, CL}, {CO, DZ, MA}, {GR, BE}, {EE, AM}, {SK, NZ}
0.800	{TT, BH, PA}, {IT, FR}, {KG, TJ, BS}, {ID, MX}, {VN, SD}, {DO, TN, MK}, {LT, LV, BG}, {TH, AR}, {MY, CL, CO, DZ, MA}, {GR, BE}, {EE, AM}, {SK, NZ}

Bahrain (BH) and Trinidad & Tobago (TT) are structurally similar and so are France (FR) and Italy (IT) and also Tajikistan (TJ) and Kyrgyzstan (KG). The fact that the two structurally similar countries are sometimes so close geographically is remarkable. More importantly, the fact that they are similar in socioeconomic terms indicates that structural similarity in the dominance network is maximal for countries which are similar in terms of population and GDP and, in addition, have a similar performance in the Olympics. As the difference in population and GDP between two countries or in terms of medals won increases their structural similarity in the network decreases. Relaxing the level of similarity required in the SLCA, new groups of progressively somewhat lower levels of structural similarity groups are formed. We have stopped the hierarchical clustering at the 0.800 threshold, which still represents a relatively high level of similarity. At that level, the largest cluster has size 5 and is formed by Malaysia (MY), Chile (CL), Colombia (CO), Algeria (AZ) and Morocco (MA). As it can be seen in Table 3 all these countries belong to the same layer (namely layer 3). They also have similar in- and out-degrees, as indicated in Table 2. Other, smaller clusters are, for example, {Lithuania (LT), Latvia (LV), Bulgaria (BG)} (in this case LV belongs to layer 0 while LT and BG belong to layer 1), {Indonesia (IN), Mexico (MX)} (layers 5 and 4, respectively, and both with small in-degree and large out-degree), {Greece (GR), Belgium (BE)} (layers 5 and 3, respectively, and both with zero in-degree and intermediate out-degree), {Thailand (TH), Argentina (AR)} (both in layer 3 and both with low in-degree and small/intermediate out-degrees), {Vietnam (VN), Sudan (SD)} (both in layer 3 and both with small in-degree and large out-degree), {Slovakia (SK), New Zealand (NZ)} (both in layer 1 and both with large in-degree and small out-degree), etc.

Finally, if we establish an analogy between the in- and out-strength (NS^{in} and NS^{out}, respectively) with the positive and negative outranking flows, respectively, used in the multi-criteria decision making Promethee I method [27], we can compute a partial ranking of the countries. This NA ranking does not correspond to applying Promethee I as a stand-alone method. Promethee I requires defining, for each criteria, a preference function that reflects the subjective preferences of the Decision Maker (DM). From those preferences, outgoing and incoming (a.k.a. positive and negative, respectively) outranking flows from/to each alternative are computed. Based on these outgoing and incoming outranking flows, Promethee I ranks the alternatives using a certain criterion. Since in our case no DM exists, we cannot apply that method directly. What we
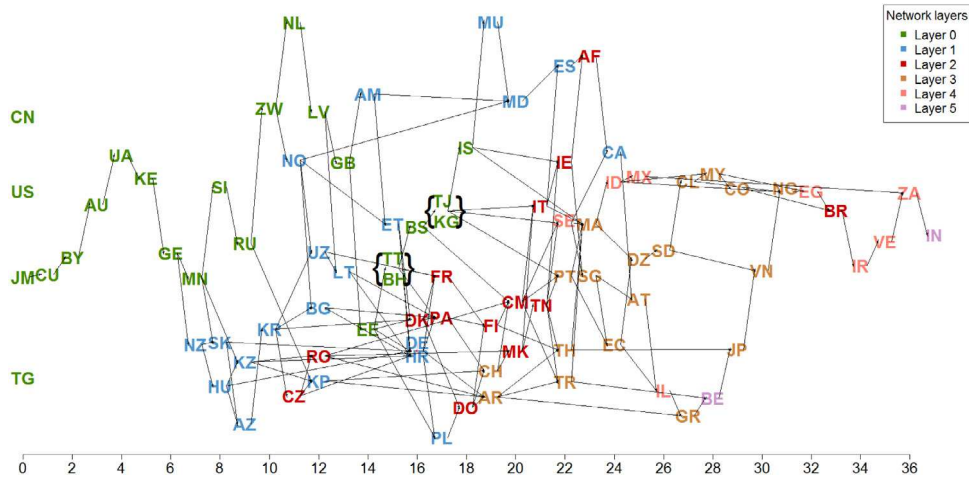
Fig. 11. Partial ranking of countries using Promethee I criteria on in- and out-strength ($a = 2$).

have done is to use the same criterion that Promethee I uses to rank the alternatives based on the outgoing and incoming outranking flows but applying it to the in- and out-strengths of our dominance network.

Fig. 11 shows the results of using the Promethee I ranking criteria on the in- and out-strength for Beijing 2008. It can be noted that the position of the countries in the partial ranking is very related to the layer to which they belong. The computed partial ranking disaggregates the layers and ranks the corresponding countries according to the defined preference relation. Note that a preference relation between two countries a and b (shown as an edge between them in Fig. 11) exists if, and only if:

$$\left(NS_a^{in} > NS_b^{in} \text{ and } NS_a^{out} < NS_b^{out}\right) \quad \text{or}$$

$$\left(NS_a^{in} = NS_b^{in} \text{ and } NS_a^{out} < NS_b^{out}\right) \quad \text{or}$$

$$\left(NS_a^{in} > NS_b^{in} \text{ and } NS_a^{out} = NS_b^{out}\right)$$

where NS_a^{in} and NS_a^{out} (respectively, NS_b^{in} and NS_b^{out}) are the in- and out-strength of node a (respectively, node b). Analogously, an indifferent relationship between two countries a and b (shown as the two countries between curly brackets in Fig. 11) exists if, and only if:

$$NS_a^{in} = NS_b^{in} \quad \text{and} \quad NS_a^{out} = NS_b^{out}.$$

The partial ranking of the countries obtained by NA using this Promethee I analogy confirms the findings observed in the other figures. Thus, for example, Egypt (EG), Brazil (BR), Iran (IR), Venezuela (VE), South Africa (ZA) and India (IN) did not perform very well in Beijing 2008 while, apart from the outliers (China (CN), United States (US) and Togo (TG)) other countries, such as Jamaica (JM), Cuba (CU), Belarus (BY), Australia (AU) or Ukraine (UA) performed very well, always relative to their respective population and GDP. Note how this method also identifies as indifferent Tajikistan (TJ) and Kyrgyzstan (KG), which as was seen above, are structurally equivalent. The same happens with Trinidad & Tobago (TT) and Bahamas (BH). Note also that Switzerland (CH), whose ego network was shown above, occupies a middle position in this ranking.

Table 5 compares the ranking obtained by the proposed NA approach with those of the lexicographic order (LEX) and integer DEA (IDEA) [11]. Ties are assigned the same (average) rank. It can be seen that the NA rank is generally closer to that of IDEA than to the lexicographic rank. This is confirmed by a non-parametric test, namely Spearman rank-order correlation. The corresponding correlation coefficients (all of them with p-value < 0.001) are 0.689 for LEX vs. IDEA, 0.551 for LEX vs. NA and 0.824 for IDEA vs. NA. That is not surprising since both IDEA and NA use information about population and GDP while LEX ignores this information and simply ranks the countries as per their medal counts. However, while IDEA ranks 15 countries in the first position, NA only ranks 4 in the first position. In this sense it can be argued that NA has more discriminant power than IDEA.

5. Conclusions

In this paper an NA approach to assess and visualize the performance of the different countries participating in the Olympic Games has been presented. The proposed approach takes into account the relative value of the different types of medal as well as the size and wealth of the different countries in terms of population and GDP, respectively. From the data, a dominance network is built reflecting all the instances in which a smaller and poorer country wins more medals (i.e. has a higher weighted medal count) than a more populated and richer country. The weight of an arc between two countries

Table 5
Rankings obtained by lexicographic order (LEX), integer DEA (IDEA) and NA.

Code	LEX	IDEA	NA	Code	LEX	IDEA	NA
AF	83	83	63	KZ	29	29	17
DZ	67.5	76.5	69	KE	15	8	9
AR	34.5	49	48.5	KP	33	40	26
AM	79	48	31.5	KR	7	17	19.5
AU	6	8	7	KG	67.5	29	42
AT	63	71	69	LV	45	43	26
AZ	39	43	17	LT	58	50	29.5
BS	67.5	8	37.5	MY	74.5	80.5	75.5
BH	53.5	8	34	MU	83	60	48.5
BY	16	16	6	MX	36	62	69
BE	47.5	63	75.5	MD	83	70	52
BR	23	54.5	82	MN	31.5	8	11.5
GB	4	8	29.5	MA	67.5	72	63
BG	42.5	46	26	NL	12	21.5	22
CM	53.5	57	52	NZ	26	26	11.5
CA	19	41	66	NG	61	69	80
CL	74.5	79	73.5	NO	22	19.5	22
CN	1	8	2.5	PA	53.5	57	42
CO	67.5	78	77.5	PL	20	35.5	42
HR	57	52	37.5	PT	47.5	61	58.5
CU	28	8	5	RO	17	29	26
CZ	24	33	22	RU	3	8	17
DK	30	38.5	37.5	SK	25	29	14
DO	47.5	52	45.5	SI	41	32	14
EC	74.5	74	66	ZA	74.5	82	85
EG	83	85	81	SG	74.5	74	63
EE	47.5	24	31.5	ES	14	34	58.5
ET	18	19.5	34	SD	74.5	74	71.5
FI	44	52	48.5	SE	56	54.5	58.5
FR	10	21.5	42	CH	34.5	46	48.5
GE	27	24	10	TJ	67.5	38.5	42
DE	5	18	37.5	TH	31.5	46	58.5
GR	59	68	73.5	TG	83	8	2.5
HU	21	24	14	TT	60	35.5	347
IS	74.5	8	45.5	TN	53.5	57	54.5
IN	50	80.5	86	TR	37	59	58.5
ID	42.5	65	66	UA	11	8	8
IR	51	66	83	US	2	8	2.5
IE	63	64	58.5	UZ	40	43	26
IL	83	84	71.5	VE	83	86	84
IT	9	29	54.5	VN	74.5	76.5	79
JM	13	8	2.5	MK	63	67	52
JP	8	37	77.5	ZW	38	8	19.5

corresponds to the distance in terms of performance between the two countries, measured by the corresponding weighted difference in the number of medals each of them won.

This Olympic Games dominance network may have more than one component, some of which may represent outlier countries which cannot be benchmarked against any other country. Most of the countries, however, belong to a giant, weakly-connected component which has a layered structure. The non-dominated nodes, assigned to layer 0, are the best performing countries that act as benchmarks for the other countries. In general, the deeper the layer to which a country belongs, the more distant it is from the best-performers and the more countries outperform it. The in- and out-degree of a node tell us this and the corresponding in- and out-degree distributions inform about the overall performance differences between the countries for the given Olympic Games. The average length and the network diameter indicate the overall magnitude of these performance differences. Similarly, the network density and network efficiency can also be used to gauge the overall number and size of the performance differences. Other node-specific measures, such as in- and out-strength, in- and out-closeness centrality, clustering coefficient and betweenness centrality, allow the characterization of the position of each node in the network. The difference between in- and out-strength has been identified as particularly useful to gauge a country's performance. Also a partial ranking of the countries using an analogy between the in- and out-strength and the positive and negative outranking flows in Promethee I can be computed.

The proposed NA approach has been applied to the Beijing 2008 Olympic Games, and it was found that the density, average clustering, average path length and diameter all increased with the value of the parameter that weights the three medal types (Gold vs. Silver vs. Bronze). The network efficiency, which is the inverse harmonic mean of the path lengths, changed in the opposite direction. The layer composition classifies the countries so that layer 0 corresponds to the best performers and the higher the layer the worse the performance. There are up to five layers with half of the countries belonging to the first two layers. The in-degree and in-strength decrease with the layer number and the opposite occurs

Table A.1
Node-specific network measures for Beijing 2008 dominance network ($a = 2$).

Code	Country	d_r^{in}	d_r^{out}	NS_r^{in}	NS_r^{out}	κ_r^{in}	κ_r^{out}	$\lambda(r)$	CC_r	β_r
AF	Afghanistan	0	8	0	76	–	0.11	2	0.04	0.00
DZ	Algeria	1	26	1	245	1	0.11	3	0.28	5.39
AR	Argentina	5	12	29	149	0.17	0.08	3	0.60	14.87
AM	Armenia	22	1	71	6	0.31	0.17	1	0.38	7.20
AU	Australia	10	0	532	0	0.02	–	0	0.50	0.00
AT	Austria	0	14	0	114	–	0.12	3	0.23	0.00
AZ	Azerbaijan	22	2	170	23	0.13	0.09	1	0.34	9.29
BS	Bahamas	12	0	17	0	0.71	–	0	0.13	0.00
BH	Bahrain	13	0	26	0	0.5	–	0	0.14	0.00
BY	Belarus	27	0	721	0	0.04	–	0	0.31	0.00
BE	Belgium	0	19	0	179	–	0.11	5	0.32	0.00
BR	Brazil	0	9	0	331	–	0.03	2	0.04	0.00
GB	Britain	2	0	77	0	0.03	–	0	1.00	0.00
BG	Bulgaria	18	3	101	32	0.18	0.09	1	0.34	5.11
CM	Cameroon	9	5	17	55	0.53	0.09	2	0.57	8.24
CA	Canada	0	3	0	78	–	0.04	1	–	0.00
CL	Chile	1	27	1	274	1	0.1	3	0.25	5.47
CN	China	0	0	0	0	–	–	0	–	0.00
CO	Colombia	1	30	1	330	1	0.09	3	0.30	7.17
HR	Croatia	18	4	66	41	0.27	0.1	1	0.39	13.98
CU	Cuba	25	0	783	0	0.03	–	0	0.30	0.00
CZ	Czech Rep.	14	4	158	39	0.09	0.1	2	0.52	17.46
DK	Denmark	8	3	59	23	0.14	0.13	2	0.64	7.99
DO	Dominican Rep.	13	5	45	48	0.29	0.1	2	0.50	12.77
EC	Ecuador	2	19	2	144	1	0.13	3	0.39	8.55
EG	Egypt	0	33	0	323	–	0.1	4	0.38	0.00
EE	Estonia	19	0	62	0	0.31	–	0	0.22	0.00
ET	Ethiopia	5	1	69	12	0.07	0.08	1	0.47	0.18
FI	Finland	6	5	19	48	0.32	0.1	2	0.56	4.09
FR	France	1	2	19	29	0.05	0.07	2	1.00	0.62
GE	Georgia	37	0	349	0	0.11	–	0	0.44	0.00
DE	Germany	1	2	41	22	0.02	0.09	1	1.00	0.64
GR	Greece	0	16	0	152	–	0.11	3	0.24	0.00
HU	Hungary	18	2	292	20	0.06	0.1	1	0.45	10.10
IS	Iceland	3	0	3	0	1	–	0	–	0.00
IN	India	0	37	0	604	–	0.06	5	0.47	0.00
ID	Indonesia	1	25	3	296	0.33	0.08	4	0.40	6.66
IR	Iran	0	29	0	342	–	0.08	4	0.30	0.00
IE	Ireland	1	8	2	67	0.5	0.12	2	0.33	0.52
IL	Israel	0	19	0	144	–	0.13	4	0.20	0.00
IT	Italy	1	2	4	59	0.25	0.03	2	1.00	0.62
JM	Jamaica	52	0	1194	0	0.04	–	0	0.68	0.00
JP	Japan	0	6	0	190	–	0.03	3	0.88	0.00
KZ	Kazakhstan	16	3	260	40	0.06	0.07	1	0.46	12.17
KE	Kenya	15	0	374	0	0.04	–	0	0.20	0.00
KP	Korea, P. Rep.	16	3	131	49	0.12	0.06	1	0.39	8.29
KR	Korea, Rep.	5	1	169	23	0.03	0.04	1	0.70	2.26
KG	Kyrgyzstan	10	0	14	0	0.71	–	0	0.12	0.00
LV	Latvia	24	0	86	0	0.28	–	0	0.21	0.00
LT	Lithuania	22	2	82	30	0.27	0.07	1	0.30	6.62
MY	Malaysia	1	31	1	322	1	0.1	3	0.28	8.71
MU	Mauritius	0	1	0	2	–	0.5	1	–	0.00
MX	Mexico	1	28	3	429	0.33	0.07	4	0.43	9.26
MD	Moldova	0	1	0	11	–	0.09	1	–	0.00
MN	Mongolia	37	0	277	0	0.13	–	0	0.46	0.00
MA	Morocco	2	23	3	195	0.67	0.12	3	0.37	11.41
NL	Netherlands	4	0	89	0	0.04	–	0	0.20	0.00
NZ	New Zealand	26	1	317	13	0.08	0.08	1	0.33	5.11
NG	Nigeria	0	24	0	310	–	0.08	3	0.14	0.00
NO	Norway	7	1	102	8	0.07	0.12	1	0.64	0.59
PA	Panama	13	3	26	38	0.5	0.08	2	0.43	6.54
PL	Poland	4	4	61	43	0.07	0.09	1	0.71	2.26
PT	Portugal	3	14	10	141	0.3	0.1	3	0.45	8.68
RO	Romania	11	5	155	51	0.07	0.1	2	0.63	18.49
RU	Russia	1	0	134	0	0.01	–	0	–	0.00
SK	Slovakia	23	1	245	15	0.09	0.07	1	0.26	2.62
SI	Slovenia	25	0	142	0	0.18	–	0	0.17	0.00

Table A.1 (*continued*)

Code	Country	d_r^{in}	d_r^{out}	NS_r^{in}	NS_r^{out}	κ_r^{in}	κ_r^{out}	$\lambda(r)$	CC_r	β_r
ZA	South Africa	0	41	0	439	–	0.09	4	0.46	0.00
SG	Singapore	2	14	2	108	1	0.13	3	0.35	8.80
ES	Spain	0	1	0	60	–	0.02	1	–	0.00
SD	Sudan	1	26	1	248	1	0.1	3	0.27	7.30
SE	Sweden	2	10	6	78	0.33	0.13	4	0.55	5.69
CH	Switzerland	5	6	21	50	0.24	0.12	3	0.77	10.32
TJ	Tajikistan	10	0	14	0	0.71	–	0	0.12	0.00
TH	Thailand	4	13	19	190	0.21	0.07	3	0.57	13.87
TG	Togo	0	0	0	0	–	–	0	–	0.00
TT	Trinidad & Tobago	13	0	26	0	0.5	–	0	0.14	0.00
TN	Tunisia	11	7	21	62	0.52	0.11	2	0.58	16.97
TR	Turkey	2	13	15	172	0.13	0.08	3	0.55	6.05
UA	Ukraine	9	0	386	0	0.02	–	0	0.22	0.00
US	United States	0	0	0	0	–	–	0	–	0.00
UZ	Uzbekistan	13	3	92	26	0.14	0.12	1	0.38	3.56
VE	Venezuela	0	38	0	373	–	0.1	4	0.45	0.00
VN	Vietnam	0	28	0	271	–	0.1	3	0.18	0.00
MK	Macedonia (FYRM)	11	7	21	57	0.52	0.12	2	0.58	18.38
ZW	Zimbabwe	17	0	110	0	0.15	–	0	0.14	0.00

with the out-degree and out-strength. The clustering coefficient and the betweenness centrality, on the other hand, are maximum in the middle layers. There is slightly negative correlation between the clustering coefficient and the total degree (indicating a hierarchical structure) and assortativity between in-degrees of the linked nodes as well as between their out-degrees and disassortativity between the in-degree of the source nodes and the out-degree of the destination nodes and between the out-degree of the source nodes and the in-degree of the destination nodes. Note that many of the above results can be explained (and sometimes expected) from the way the dominance network is constructed and from the meaning of the characterization measures considered. Thus, for example, transitivity contributes to clustering. Also, the layer structure means that the higher the layer, the worse the performance and that betweenness centrality is zero at the extreme layers. However, although these are general assertions that can be made ex-ante, the proposed NA can quantify these effects as well as other results that were not so obvious.

A partial ranking of the countries has been computed using the analogy with Promethee I and this gives an assessment of the relative performance of the different countries. As a general rule, apart from some outliers (United States (US) and China (CN) because of their large GDP and population, respectively, and Togo (TG), on the other side, for its small population), small and/or poor countries that win many medals (such as Jamaica (JM), Cuba (CU) or Belarus (BY)) come out on top of the ranking while large and/or relatively rich countries that do not win many medals (such as Japan (JP), Brazil (BR), Iran (IR), South Africa (ZA) or India (IN)) are ranked at the bottom. In addition to the data analysis possibilities that the proposed dominance network approach offers, the visualization capabilities that come with it must be emphasized. The whole network, but also its skeleton or the ego network of specific countries, can be rendered and the performance of each country can be ascertained from its relative position in the network.

Summarizing, this paper has shown that global and node-specific NA measures provide a complete characterization of the performance differences between the different countries that participate in the Olympic Games. Although the proposed NA approach may seem too sophisticated compared with simpler methods based on lexicographic order or on weighted ratios, our claim is that those methods do not capture all the relationships between the countries, among other reasons because they do not make use of all the information available. Thus, for example, lexicographic order only considers the number of medals won by each country, ignoring its population and economic prowess. The proposed methodology, based on the concept of dominance network, is a rigorous and more general approach, which can accommodate additional explanatory variables (both of the type the larger, the better and the smaller, the better). Also, NA visualization features are very helpful to perceive and assess the relative performance level of the different countries. Our conclusion is, thus, that NA is a helpful tool to process and visualize this type of complex multidimensional data. The analysis developed in this paper can also be used in different Olympics Games in order to determine how the performance of the countries has evolved. As another topic for further research, we would like to apply NA to Olympic and world championships records.

Acknowledgements

Appendix

Table A.1 shows detailed values of node-specific measures, namely in- and out-degree, in- and out-strength, in- and out-closeness centrality, layer, clustering coefficient and betweenness centrality.

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Dominance network analysis of economic efficiency

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ABSTRACT

This paper proposes an enhanced Dominance Network (DN) to assess the technical, economic and allocative efficiency of a set of Decision Making Units (DMUs). In a DN, the nodes represent DMUs and the arcs correspond to dominance relationships between them. Two types of dominance relationship are considered: technical and economic. The length of a technical dominance arc between two nodes is a weighted measure of the input and output differences between the two DMUs. The length of an economic dominance arc between two nodes corresponds to the cost, revenue or profit difference between them (depending on whether only unit input prices, unit output prices or both are known). The proposed dominance network is a multiplex network with two relations; the structure of both relations is similar. Thus, both of them are layered and their arcs have transitivity and additivity properties. However, since technical dominance implies economic dominance but not the reverse, economic dominance is more common and has a deeper structure. It may also have an underlying potential field so that the length of the arcs between any two nodes depends on the difference in their potentials and the direction of the arcs depends on the sign of that difference. Allocative inefficiencies can also be gauged on this DN. Complex network measures can be used to characterize and study this type of DN.

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1. Introduction

Non-parametric Frontier Analysis methods (of which Data Envelopment Analysis, DEA, is probably the best known) assess the relative efficiency of DMUs. These methods see the DMUs as production processes that transform inputs into outputs. Moreover, they are data-driven so that only the data on the input consumption and output production of each DMU are required. DEA approaches have been extensively studied and applied as can be seen in the existing textbooks on the subject (e.g. Cooper, Seiford, & Zhu, 2004, 2006; Färe, Grosskopf, & Lovell, 1985, 1994).

Recently, Calzada-Infante and Lozano (2016) used DN to assess the medal winning performance of nations at Olympic Games. The corresponding DN is characterized using different complex network measures, such as node strength, clustering coefficient, betweenness centrality, degree-degree correlation, etc. This paper proposes an enhanced DN analysis that includes not only technical efficiency but also economic (cost, revenue or profit) efficiency. Technical efficiency (TE) basically measures the maximum input reduction and output increase that a given DMU may achieve. Economic efficiency, on the other hand, refers to the maximum cost reduction or revenue or profit increase that a given DMU may achieve. TE assessment only requires input and output data while economic efficiency assessment also requires knowing the outputs and inputs unit prices.

Since TE assessment does not use price information, the projection of a DMU is done considering only the operating points that dominate it in technical terms, i.e. in terms of the input consumption and output production. This leads to the technical dominance criterion. Economic efficiency, on the other hand, makes use of price information, allowing certain inputs to be increased or certain outputs to be decreased if that leads to an overall cost, revenue or profit improvement, depending on whether we are considering cost, revenue or profit efficiency. In any case, economic efficiency leads to an economic dominance criterion which is different from that of technical dominance and it is based on the corresponding cost, revenue or profit. Hence, in the proposed approach we consider a DN with two different dominance relations, one based on TE and the other based on economic efficiency.

The proposed enhanced DN approach is a novel way of assessing the economic performance of DMUs using complex networks tools. Some researchers have previously applied complex network analysis in a DEA context but their approach was not based on the DN concept. Thus, for example, complex network analysis has been proposed to rank efficient DMUs (Ho, Liu, Lu, & Huang, 2014;

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Leem & Chun, 2015; Liu & Lu, 2010; Liu, Lu, Yang, & Chuang, 2009, 2014). In these approaches, the network considered is weighted and directed and it is constructed based on the optimal intensity variables (commonly known as lambda parameters) computed using an envelopment DEA model, sometimes with different input-output specifications. The nodes of this network are the DMUs and the arc weights are the optimal values of the intensity variables. If only one input-output specification is used, the resulting network is bipartite, with the arcs going from each inefficient DMU to the efficient DMUs in its peer group. Two different centrality measures are proposed to rank the efficient DMUs.

Ghahraman and Prior (2015) use a different approach aimed at selecting an optimal stepwise benchmark path from an inefficient operating unit (OU) to the efficient frontier. Their network considers that the nodes are the DMUs but this time an arc between two nodes r and j exists if, and only if, OU j has a higher efficiency score than OU r. The corresponding arc weight is computed using a weighted measure that takes into account the input similarities between the two DMUs, their efficiency gap (modified using an exponential penalty function) and a fixed cost for each link. They compute shortest paths on this network as well as clusters of DMUs (based on the maximum input and output percentage changes in a single step). They also use complex network analysis to discriminate between efficient and intermediate DMUs, clustering the DMUs and identifying possible outliers and specialized units.

The DN approach first used in Calzada-Infante and Lozano (2016) is different from those commented on above as it constructs a DN based on the dominance relationships between the DMUs. Thus, an arc between a node r and a node j exists if, and only if, DMU j (weakly) dominates DMU r, i.e. if DMU j consumes less (or at most the same amount of) inputs and produces more (or at least an equal amount of) outputs than DMU r. Calzada-Infante and Lozano (2016) use what can be called a Normalized Additive Inefficiency (NAI) metric to compute the arc weights and implicitly assumes a basic Free Disposal Hull (FDH) technology. However, other DEA metrics and other DEA technologies can also be considered.

In this paper, an enhanced DN is considered for the cases in which, in addition to the input and output data, unit input prices, unit output prices or both, are known. In those cases, in addition to the technical dominance relationships, economic dominance relationships can be defined. Thus, the relationships between the DMUs are analysed from two different points of view: technical and economic. The resulting DN integrates both relations and its analysis, using complex network tools, provides an innovative approach to economic efficiency assessment. Thus, while in DEA the TE score measures the distance to the frontier, i.e. the difference between a DMU and its efficient benchmark, the DN approach also considers the existence of links between any two inefficient DMUs if one dominates the other in technical terms. The length of the corresponding arc measures the relative inefficiency between them, i.e. the difference in their respective TE. Similarly, while in DEA the economic efficiency of a DMU refers to the maximum cost, revenue or profit improvement for a given DMU, in the enhanced DN approach the lengths of the arcs of the economic efficiency relation correspond to the difference in cost, revenue or profit between the origin and the destination nodes. By following a directed path along the DN, successive improvements in TE or in economic efficiency (depending on the dominance relation considered) can be obtained. Thus, the proposed approach allows a graphical and quantitative representation of the TE and economic efficiency of the DMUs in the sample and of the possible improvement paths that can be followed.

This double graphical plus quantitative feature of the proposed DN approach is useful because most DEA problems involve multi-

ple inputs and outputs observations whose direct visualization in a multidimensional space is not possible. Particularly interesting is the DN approach in those cases in which the DMUs belong to the same organization (e.g. bank branches, retail stores, bus routes, etc) because this analysis tool allows a global perspective of the problem, i.e. a systemic view of the dataset, at the same time that the relative performance and relative position of each individual DMU within the whole is ascertained. Another situation in which this tool may be useful is in a competing DMU scenario (e.g. airlines, mutual funds, etc) in which the visualization capability of DN allows a sort of efficiency positioning map of the different DMUs so that a strategic analysis of the technical and economic efficiency status and relative position of each DMU may be assessed.

The structure of this paper is the following. In Section 2 a review of the literature is carried out. In Section 3 the proposed approach is presented and discussed. Section 4 illustrates the proposed approach on a simple dataset while Section 5 presents a real world application in the banking sector. Section 6 summarizes and concludes.

2. Literature review

In Section 2.1 the DN approach used in Calzada-Infante and Lozano (2016) is reviewed while Section 2.2 reviews the relevant profit efficiency decomposition literature.

2.1. DN analysis of Calzada-Infante and Lozano (2016)

Let

$D = \{1, 2, ..., n\}$ set of DMUs

$j, r, p = 1, 2, \ldots, n$ indexes on DMUs

$i = 1, 2, \ldots, m$ index on input dimensions

$k = 1, 2, \ldots, s$ index on output dimensions

$x_j = (x_{1j}, x_{2j}, ..., x_{mj})$ input vector of DMU j

$y_j = (y_{1j}, y_{2j}, ..., y_{sj})$ output vector of DMU j

A dominance relationship exists between a DMU r and a DMU j if the latter weakly dominates the former, i.e. $x_{ij} \leq x_{ir}$ for each i and $y_{kj} \geq y_{kr}$ for each k, with at least one of the inequalities being strict. Let $D(r) = \{j : j \text{ dominates } r\}$, the set of DMUs that weakly dominate a DMU r. A DMU r is non-dominated, hence efficient, if $D(r) = \emptyset$. Let $D^* = \{r : D(r) = \emptyset\}$, the set of non-dominated DMUs.

Calzada-Infante and Lozano (2016) build a DN (D,E) whose vertexes are the DMUs and whose directed edges correspond to the dominance relationships, i.e. $E = \{(r, j) : j \in D(r)\}$. The length (i.e. the weight) of each edge can be computed using the following NAI metric

$$e_{rj} = \frac{1}{m+s} \cdot \left(\sum_{i=1}^{m} \frac{x_{ir} - x_{ij}}{c_i^x} + \sum_{k=1}^{s} \frac{y_{kj} - y_{kr}}{c_k^y} \right) \tag{1}$$

where c_i^x and c_k^y are slacks normalizing constants. These normalizing constants can be, for example, the range or the mean value.

Based on the DN thus defined, Calzada-Infante and Lozano (2016) propose different measures to assess the efficiency of each DMU and also of the whole sample. Thus, for example, the out-degree of each node, $d_r^{out} = |D(r)|$, is the number of DMUs that dominate it. Defining $D^{-1}(r) = \{f : r \in D(f)\}$,

the in-degree of a node, $d_r^{in} = |D^{-1}(r)|$, is the number of DMUs that it dominates. The in- and out-strengths, $s_r^{in} = \sum_{f \in D^{-1}(r)} e_{fr}$ and $s_r^{out} = \sum_{j \in D(r)} e_{rj}$, measure the intensity of the dominance relationships of a DMU. The efficient benchmarks of a DMU are $D^*(r) = \{ \begin{array}{ll} \{r\} & if\ r \in D^* \\ D(r) \cap D^* & if\ r \notin D^* \end{array} \}$. An inefficiency score for each DMU can be computed as the distance to its farthest efficient benchmark $e_r^{max} = \max\limits_{j \in D^*(r)} e_{rj}$. Also, the distance to the closest efficient benchmark of a DMU can be computed, $\tau_r^{min} = \min\limits_{j \in D^*(r)} e_{rj}$.

DN are directed acyclic graphs and, therefore, have a layer structure with the layer of each node determined recursively using $\lambda(r) = \{ \begin{array}{ll} 0 & if\ D(r) = \emptyset \\ 1 + \max\limits_{j \in D(r)} \lambda(j) & otherwise \end{array}$.

Another interesting property of the DN is that the arcs are transitive, i.e. $r \in D(p) \wedge j \in D(r) \Rightarrow j \in D(p)$. This transitivity allows a simplified representation of the DN, called its skeleton, in which all transitive edges have been removed and are implicit. Also, for the NAI metric (1) the following additivity property holds: $r \in D(p)$ and $j \in D(r) \Rightarrow e_{pj} = e_{pr} + e_{rj}$.

In addition to the above local DN measures, some global DN measures can also be defined:

Average degree (average number of DMUs that dominate a given DMU=average number of DMUs that a given DMU dominates): $d^{aver} = \frac{1}{|D|} \sum_{r \in D} d_r^{in} = \frac{1}{|D|} \sum_{r \in D} d_r^{out}$

Network density (number of arcs divided by maximum possible number of arcs): $\rho = \frac{d^{aver}}{|D|-1}$

Network diameter (maximum distance between any two nodes): $\Delta = \max\limits_{(r,j) \in E} e_{rj}$

Calzada-Infante and Lozano (2016) also compute other measures, such as clustering coefficient (CC_r), betweenness centrality (BC_r), etc. that are not going to be used in this paper. For the meaning, mathematical expressions and algorithms for computing these measures commonly used in complex network analysis, the interested reader is referred to Fagiolo (2007) and Brandes (2008).

2.2. Relevant profit efficiency decomposition approaches

There exist many DEA papers dealing with possible ways to decompose profit, revenue and cost efficiency (e.g. Aparicio, Borras, Pastor, & Vidal, 2015; Färe, Fukuyama, Grosskopf, & Zelenyuk, 2015; Silva Portela & Thanassoulis, 2005, 2007). The idea is to relate the TE of a DMU with its economic efficiency. As we said before, TE is restricted to look for operating points that consume less input and produce more output while economic efficiency is not thus constrained. Hence economic efficiency is generally less than the TE and can be decomposed into a TE component and an allocative efficiency (AE) component that captures the difference between the two.

In what follows we will review those profit efficiency decomposition approaches that are most relevant to this research, which are Cooper, Pastor, Aparicio, and Borras (2011) and Aparicio, Borras, Pastor, and Vidal (2013), which use a weighted additive model to measure TE. Let:

q_i: unit price of input i
p_k: unit price of output k
w_i^x: weight that represents the relative importance of input i (from the decision maker's point of view)
w_k^y: weight that represents the relative importance of output k (from the decision maker point of view)
s_i^-: slack (i.e. potential improvement) of input i
s_k^+: slack (i.e. potential improvement) of output k

Cooper et al. (2011) use the following weighted additive model to measure the technical inefficiency (TI_0) of a given DMU 0

$$TI_0 = Max \quad \sum_i w_i^x s_i^- + \sum_k w_k^y s_k^+ \tag{2}$$

s.t.

$$\sum_j \lambda_j x_{ij} = x_{i0} - s_i^- \qquad \forall i \tag{3}$$

$$\sum_j \lambda_j y_{kj} = y_{k0} + s_k^+ \qquad \forall k \tag{4}$$

$$\sum_j \lambda_j = 1 \tag{5}$$

$$s_i^- \geq 0 \quad \forall i \qquad s_k^+ \geq 0 \quad \forall k \tag{6}$$

$$\lambda_j \geq 0 \quad \forall j \tag{7}$$

Note that (5) and (7) imply that the above model assumes Variable Return Scale (VRS). For the FDH case assumed in this paper, the only change to be made in the formulation is the substitution of (7) by (7′)

$$\lambda_j \in \{0,1\} \quad \forall j \tag{7′}$$

To compute the profit inefficiency (PI_0) of a given DMU 0, Cooper et al. (2011) use

$$PI_0 = \frac{\Pi(\bar{q}, \bar{p}) - \left(\sum_k p_k y_{k0} - \sum_i q_i x_{i0} \right)}{\min \left\{ \frac{q_1}{w_1^x}, \frac{q_2}{w_2^x}, ..., \frac{q_m}{w_m^x}, \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, ..., \frac{p_s}{w_s^y} \right\}} \tag{8}$$

where $\bar{q} = (q_1, ..., q_m)$, $\bar{p} = (p_1, ..., p_s)$ and $\Pi(\bar{q}, \bar{p})$ represents the optimal level of profit given by

$$\Pi(\bar{q}, \bar{p}) = Max \quad \sum_k p_k \hat{y}_k - \sum_i q_i \hat{x}_i$$

s.t.

$$\sum_j \lambda_j x_{ij} \leq \hat{x}_i \qquad \forall i$$

$$\sum_j \lambda_j y_{kj} \geq \hat{y}_k \qquad \forall k \tag{9}$$

$$\sum_j \lambda_j = 1$$

$$\lambda_j \in \{0,1\} \quad \forall j$$

The Allocative Inefficiency (AI_0) of DMU 0 is defined as the residual component, i.e. $AI_0 = PI_0 - TI_0$. Cooper et al. (2011) show that $PI_0 \geq TI_0$ and therefore $AI_0 \geq 0$.

For revenue efficiency decomposition, Aparicio et al. (2013) use a similar approach but considering an output orientation. Thus, for example, the Revenue Inefficiency (RI) can be measured as

$$RI_0 = \frac{R(x_0, \bar{p}) - \sum_k p_k y_{k0}}{\min \left\{ \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, ..., \frac{p_s}{w_s^y} \right\}} \tag{10}$$

where

$$R(x_0, \bar{p}) = Max \quad \sum_k p_k \hat{y}_k$$

$s.t.$

$$\sum_j \lambda_j x_{ij} \leq x_{i0} \qquad \forall i$$

$$\sum_j \lambda_j y_{kj} \geq \hat{y}_k \qquad \forall k \tag{11}$$

$$\sum_j \lambda_j = 1$$

$$\lambda_j \in \{0, 1\} \quad \forall j$$

Analogously, the output-oriented TI_0 can be computed by solving

$$TI_0 = Max \quad \sum_k w_k^y s_k^+$$

$s.t.$

$$\sum_j \lambda_j x_{ij} \leq x_{i0} \qquad \forall i$$

$$\sum_j \lambda_j y_{kj} = y_{k0} + s_k^+ \qquad \forall k \tag{12}$$

$$\sum_j \lambda_j = 1$$

$$s_k^+ \geq 0 \quad \forall k \quad \lambda_j \in \{0, 1\} \quad \forall j$$

As before, the allocative efficiency can be computed as the difference $AI_0 = RI_0 - TI_0$. Aparicio et al. (2013) show that $RI_0 \geq TI_0$ and therefore $AI_0 \geq 0$.

3. Enhanced DN for economic efficiency assessment

The DN analysis proposed in Calzada-Infante and Lozano (2016) only considers technical efficiency dominance relationships. Let us use the superscript TI (which stands for Technical Inefficiency) to distinguish this type of dominance relationship so that all the measures that refer to this TI relation will be superscripted TI. Thus, for example, the arc lengths defined in (1) will be labelled e_{rj}^{TI}, the set of DMUs that dominate r in a technical efficiency sense are denoted as $D^{TI}(r) = \{j : x_{ij} \leq x_{ir} \, \forall i \text{ and } y_{kj} \geq y_{kr} \, \forall k \text{ and } j \neq r\}$, the set of TI arcs are $E^{TI} = \{(r, j) : j \in D^{TI}(r)\}$, the set of efficient (i.e. non-dominated) DMUs are D^{TI*}, the distance of a DMU to its closest and farthest technical efficient benchmark are $\tau_r^{TI,min}$ and $e_r^{TI,max}$, respectively, the in- and out-degrees of the TI relation are $d_r^{TI,in}$ and $d_r^{TI,out}$, respectively, the average TI degree is $d^{TI,aver} = \frac{1}{|D|} \sum_{r \in D} d_r^{TI,in} = \frac{1}{|D|} \sum_{r \in D} d_r^{TI,out}$, the network density of the TI relation is $\rho^{TI} = \frac{d^{TI,aver}}{|D|-1}$, the network diameter of the TI relation is $\Delta^{TI} = \max_{(r, j) \in E} e_{rj}^{TI}$, etc.

It is important to note that if we define the slacks normalizing constant of the NAI metric so that $w_i^x = \frac{1}{(m+s) \cdot c_i^x}$ and $w_k^y = \frac{1}{(m+s) \cdot c_k^y}$, then (1) is consistent with the weighted additive TI measure (2), leading to measuring the length of the TI arc from r to $j \in D^{TI}(r)$ as

$$e_{rj}^{TI} = \sum_i w_i^x \cdot (x_{ir} - x_{ij}) + \sum_k w_k^y \cdot (y_{kj} - y_{kr}) \tag{13}$$

Note also that, instead of NAI, other linear and non-linear efficiency metrics could be used to define the TI edge lengths. For example:

Russell Graph Measure (RGM) (Färe et al., 1985)

$$e_{rj}^{TI,RGM} = \frac{1}{m+s} \cdot \left(\sum_{i=1}^m \frac{x_{ij}}{x_{ir}} + \sum_{k=1}^s \frac{y_{kr}}{y_{kj}} \right) \tag{14}$$

Measure of Inefficiency Dominance (MID) (Bardhan, Bowlin, Cooper, & Sueyoshi, 1996)

$$e_{rj}^{TI,MID} = \frac{1}{m+s} \cdot \left(\sum_{i=1}^m \frac{x_{ir} - x_{ij}}{x_{ir}} + \sum_{k=1}^s \frac{y_{kj} - y_{kr}}{y_{kj}} \right) \tag{15}$$

Measure of Inefficiency Proportions (MIP) (Cooper & Tone, 1997)

$$e_{rj}^{TI,MIP} = \frac{1}{m+s} \cdot \left(\sum_{i=1}^m \frac{x_{ir} - x_{ij}}{x_{ir}} + \sum_{k=1}^s \frac{y_{kj} - y_{kr}}{y_{kr}} \right) \tag{16}$$

Enhanced Russell Graph Measure (ERM) (Pastor, Ruiz, & Sirvent, 1999), a.k.a. Slacks-Based Measure of Efficiency (SBM) (Tone, 2001)

$$e_{rj}^{TI,ERM} = \frac{\frac{1}{m} \cdot \sum_{i=1}^m \frac{x_{ij}}{x_{ir}}}{\frac{1}{s} \cdot \sum_{k=1}^s \frac{y_{kj}}{y_{kr}}} = \frac{1 - \frac{1}{m} \cdot \sum_{i=1}^m \frac{x_{ir} - x_{ij}}{x_{ir}}}{1 + \frac{1}{s} \cdot \sum_{k=1}^s \frac{y_{kj} - y_{kr}}{y_{kr}}} \tag{17}$$

Geometric Distance Function (GDF) (Silva Portela & Thanassoulis, 2005, 2007)

$$e_{rj}^{TI,GDF} = \frac{\left(\prod_k \frac{y_{kj}}{y_{kr}} \right)^{1/s}}{\left(\prod_i \frac{x_{ij}}{x_{ir}} \right)^{1/m}} = \left(\prod_k \frac{y_{kj}}{y_{kr}} \right)^{1/s} \left(\prod_i \frac{x_{ir}}{x_{ij}} \right)^{1/m} \tag{18}$$

In the same way that NAI has the additivity property, MID has the triangular inequality property $r \in D(p) \text{ and } j \in D(r) \Rightarrow e_{pj}^{TI,MID} \leq e_{pr}^{TI,MID} + e_{rj}^{TI,MID}$ and GDF has the multiplicative property $r \in D(p) \text{ and } j \in D(r) \Rightarrow e_{pj}^{TI,GDF} = e_{pr}^{TI,GDF} \cdot e_{rj}^{TI,GDF}$. However, although studying the different DN derived from using these other TI metrics is an interesting research topic, it is outside the scope of this paper since they do not lend themselves to the revenue and profit decomposition of Cooper et al. (2011) and Aparicio et al. (2013) that are applied in this paper. Actually that is the main reason for choosing the NAI metric, apart from it being simpler than other metrics.

In addition to the TI dominance relation, which is analogous to the one used in Calzada-Infante and Lozano (2016), this article goes further and defines a PI dominance relation by directing an arc from DMU r to DMU j if, and only if, $\sum_k p_k y_{kr} - \sum_i q_i x_{ir} < \sum_k p_k y_{kj} - \sum_i q_i x_{ij}$. From this we can define the sets $D^{PI}(r) = \{j : j \text{ PI dominates } r\}$ and the set of arcs of the PI dominance relation $E^{PI} = \{(r, j) : j \in D^{PI}(r)\}$. Like TI dominance, PI dominance is transitive.

To be consistent with (8), the length of the PI arcs is computed as

$$e_{rj}^{PI} = \frac{\sum_k p_k y_{kj} - \sum_i q_i x_{ij} - \left(\sum_k p_k y_{kr} - \sum_i q_i x_{ir} \right)}{\min \left\{ \frac{q_1}{w_1^x}, \frac{q_2}{w_2^x}, \dots, \frac{q_m}{w_m^x}, \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, \dots, \frac{p_s}{w_s^y} \right\}} \tag{19}$$

This means that, in the same way as the TI relation, the lengths of the PI edges have the additivity property, i.e. $r \in D^{PI}(p) \wedge j \in D^{PI}(r) \Rightarrow e_{pj}^{PI} = e_{pr}^{PI} + e_{rj}^{PI}$.

Therefore, the proposed enhanced DN consists of the nodes representing the DMUs and among these nodes there exist two types of arc: TI and PI. A TI arc between a DMU r and a DMU j indicates that the origin of the arc is dominated in a TI sense, i.e. DMU j consumes less (or at most equal) of each of the inputs and produces more (or at least equal) of each of the outputs than DMU r does. A PI arc between DMU r and DMU j simply means that the profit of DMU j is greater than that of DMU r; nothing is said about the

amounts of inputs consumed or the amounts of outputs produced by each DMU.

Proposition 1. $E^{TI} \subseteq E^{PI}$

Proof. (see Appendix)

Therefore, whenever there is a TI arc between two nodes then there is also a PI arc between those two nodes. Moreover, as the following proposition shows, the length of the PI arc is never smaller than that of the TI arc.

Proposition 2. $e_{rj}^{PI} \geq e_{rj}^{TI} \quad \forall (r, j) \in E^{TI}$

Proof. (see Appendix)

Analogous to the DN measures for the TI relation, DN measures for the PI relation can be computed. However, some peculiarities exist. Thus, for example, although the set of profit efficient (i.e. PI non-dominated) DMUs $D^{PI,*} = \{r : D^{PI}(r) = \emptyset\}$ may contain more than one DMU, in that case, as the following proposition shows, all of them must have the same profit.

Proposition 3. $\sum_k p_k y_{kj} - \sum_i q_i x_{ij} = \Pi(\bar{q}, \bar{p}) \quad \forall j \in D^{PI,*}$

Proof. (see Appendix)

Corollary 1. $PI_j = 0 \quad \forall j \in D^{PI,*}$

The following proposition shows that the distance from a DMU to the PI efficient set is a unique value, i.e.

Proposition 4. $e_r^{PI,\max} = \max\limits_{j \in D^{PI,*}(r)} e_{rj}^{PI} = \min\limits_{j \in D^{PI,*}(r)} e_{rj}^{PI} = \tau_r^{PI,\min} = PI_r \quad \forall r$

Proof. (see Appendix)

The PI in-strength of a DMU r is $s_r^{PI,in} = \sum_{f \in D^{PI,-1}(r)} e_{fr}^{PI}$, where $D^{PI,-1}(r) = \{f : r \in D^{PI}(f)\}$ is the set of DMUs that it dominates. This PI in-strength corresponds to the increase in profit that would result if those DMUs it dominates raised their profit level to that of DMU r. In particular, for any PI efficient DMU, this PI in-strength would be equal to the overall increase in profit that results if all DMUs become PI efficient, i.e.

Corollary 2. $s_r^{PI,in} = \sum_{p \in D^{PI,-1}(r)} e_r^{PI,\max} = \sum_{p \in D} PI_r = \frac{|D| \cdot \Pi(\bar{q}, \bar{p}) - \sum_{p \in D} [\sum_k p_k y_{kp} - \sum_i q_i x_{ip}]}{\min\{\frac{q_1}{w_1^x}, \frac{q_2}{w_2^x}, ..., \frac{q_m}{w_m^x}, \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, ..., \frac{p_s}{w_s^y}\}} \quad \forall r \in D^{PI,*}$

PI dominance is rather common. Actually there exist at least as many PI arcs as TI arcs, i.e. $|E^{PI}| \geq |E^{TI}|$ as shown in Proposition 1. The overall network density is, in any case, the sum of those due to the two relations, i.e. $\rho = \rho^{TI} + \rho^{PI}$.

Also the layers in the PI relation computed using $\lambda^{PI}(r) = \begin{cases} 0 & \text{if } D^{PI}(r) = \emptyset \\ 1 + \max\limits_{j \in D^{PI}(r)} \lambda^{PI}(j) & \text{otherwise} \end{cases}$ represent a partial ordering of the DMUs so that there are as many layers as different profit values for the set of DMUs. This layer structure is, therefore, deeper than in the case of the TI relation. Generally the layers of the PI relation contain only one node. Two nodes j and j' belong to the same layer if, and only if, $\sum_k p_k y_{kj} - \sum_i q_i x_{ij} = \sum_k p_k y_{kj'} - \sum_i q_i x_{ij'}$.

Another interesting feature of the PI relation is that it is based on a scalar potential

$$PI(\bar{q}, \bar{p}, \bar{x}_r, \bar{y}_r) = \frac{\Pi(\bar{q}, \bar{p}) - \left(\sum_k p_k y_{kr} - \sum_i q_i x_{ir}\right)}{\min\left\{\frac{q_1}{w_1^x}, \frac{q_2}{w_2^x}, ..., \frac{q_m}{w_m^x}, \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, ..., \frac{p_s}{w_s^y}\right\}} \quad (20)$$

$$= \frac{1}{\alpha} \cdot \left[\Pi(\bar{q}, \bar{p}) - \left(\sum_k p_k y_{kr} - \sum_i q_i x_{ir}\right)\right] \quad (20)$$

Proposition 5. The PI relation results from the underlying scalar potential $PI(\bar{q}, \bar{p}, \bar{x}, \bar{y})$. Moreover, the PI arc lengths equal the potential differences among the nodes, i.e. $e_{rj}^{PI} = PI(\bar{q}, \bar{p}, \bar{x}_r, \bar{y}_r) - PI(\bar{q}, \bar{p}, \bar{x}_j, \bar{y}_j) = PI_r - PI_j$

Proof. (see Appendix)

The above results allow us to picture the PI relation as embedded in a potential field so that each operating point $(\bar{x}, \bar{y})$ PI-dominates (and therefore has PI edges towards) every other operating point with lower potential. The lowest potential nodes are precisely those in the set $D^{PI,*}$ and their PI potential, using Proposition 3 and (20), is zero, i.e. $PI(\bar{q}, \bar{p}, \bar{x}_j, \bar{y}_j) = 0 \quad \forall j \in D^{PI,*}$. The corresponding gradient network (see Toroczkai, Kozma, Bassler, Hengartner, & Korniss, 2008) would be formed by the arcs linking each DMU with those in the set $D^{PI,*}$, which is equivalent to the result obtained with the bipartite graph filter applied to the PI relation. This filter selects only the arcs between PI efficient and PI inefficient DMUs.

The proposed enhanced DN considers two relations, TI and PI. From them, and taking into account the PI decomposition $PI_0 = TI_0 + AI_0$, an AI hyper-relation can be derived. The set of hyperedges of the AI hyper-relation is $H^{AI} = \{(r/t, j) : r \in D \wedge t \in D^{TI,*}(r) \wedge j \in D^{PI,*}\}$. However, given any DMU r ($r \in D$), let t be any of its TI efficient benchmarks (i.e. $t \in D^{TI,*}(r) = D^{TI}(r) \cap D^{TI,*}$) and j any of the PI efficient DMUs (i.e. $j \in D^{PI,*}$). A directed AI hyperedge (r/t,j) can be defined whose length is

$$e_{r/t,j}^{AI} = e_{rj}^{PI} - e_{rt}^{TI} \quad (21)$$

Note that the existence of the RI arc (r,j) is guaranteed because the PI efficient DMUs in $j \in D^{PI,*}$ PI-dominate all others. Note also that for $r \in D^{TI,*}$, we have $D^{TI,*}(r) = \{r\}$ and, therefore, the hyperedge (r/t,j) reduces to a normal edge (r/r,j) with length $e_{r/r,j}^{AI} = e_{rj}^{PI} = PI_r$, i.e. the AI value of a TI efficient DMU is equal to its PI value.

Interestingly, if a DMU r is inefficient and has more than one TI efficient benchmark, its AI value depends not only on the TI inefficient DMU r but also on the chosen TI efficient benchmark. Thus, DMU r might be projected onto a closer (hence, easier to reach) TI efficient benchmark, thus leading to a larger AI value or it might be projected onto a further TI efficient benchmark, in which case the TI reduction would be larger and the AI value lower.

This is illustrated in Fig. 1, where DMU r is assumed to be TI dominated by two TI efficient DMUs t and t', with the latter closer than the former, i.e. $e_{rt'}^{TI} \leq e_{rt}^{TI}$. Then, if r is projected onto t, it would have to reduce its inputs and increase its outputs by $x_{ir} - x_{it}$ and $y_{kt} - y_{kr}$, respectively, which represents a weighted efficiency improvement effort of e_{rt}^{TI}. In that case, the remaining AI would be $e_{r/t,j}^{AI} = e_{rj}^{PI} - e_{rt}^{TI} = PI_r - e_{rt}^{TI}$. If, however, t' is chosen as the TI efficient benchmark, then the reductions in inputs and outputs would be $x_{ir} - x_{it'}$ and $y_{kt'} - y_{kr}$, respectively, leading to a smaller weighted efficiency improvement effort of $e_{rt'}^{TI}$ but a higher AI value $e_{r/t',j}^{AI} = e_{rj}^{PI} - e_{rt'}^{TI} = PI_r - e_{rt'}^{TI}$.

Proposition 6. $PI_r - e_r^{TI,\max} \leq e_{r/t,j}^{AI} \leq PI_r - \tau_r^{TI,\min} \quad \forall (r/t, j) \in H^{AI}$
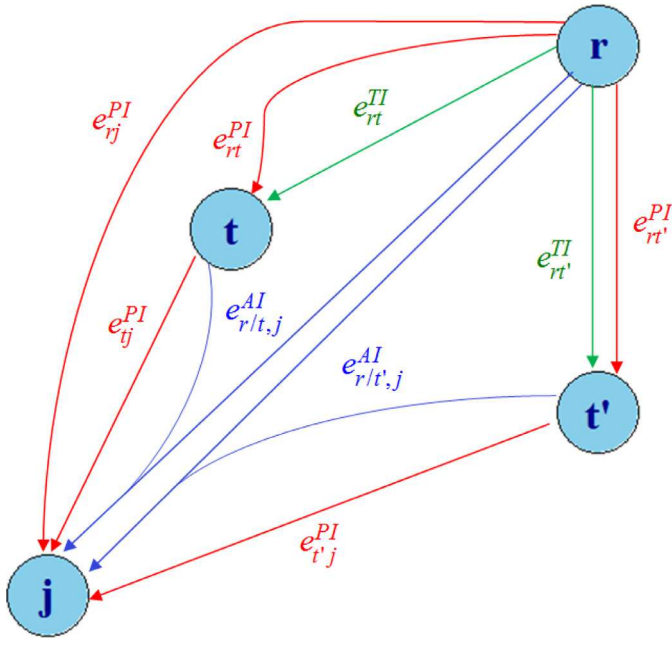
Fig. 1. Example of TI and PI edges (green and red, respectively) and AI hyperedges (blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proof. (see Appendix)

In the special case that $w_i^x = \frac{q_i}{\alpha} \ \forall i \quad w_k^y = \frac{p_k}{\alpha} \ \forall k$, i.e. if the weights of the input and output slacks in (13) are proportional to the input and output unit prices, then the length of the TI edges coincides with that of the PI edges given by (19), i.e. $e_{rj}^{TI} = e_{rj}^{PI} = PI_r - PI_j$. In that case, the length of the AI hyperedges $(r/t, j) \in H^{AI}$ would be

$$e_{r/t,j}^{AI} = e_{rj}^{PI} - e_{rt}^{TI} = e_{rj}^{PI} - e_{rt}^{PI} = PI_r - PI_j - (PI_r - PI_t) = PI_t - PI_j = PI_t \quad (22)$$

where Proposition 5 and Corollary 1 have been taken into account. This means that the AI value of a hyperedge $(r/t, j) \in H^{AI}$ only depends on (actually, it is equal to) the PI value of the TI efficient benchmark t. In other words, for any TI inefficient DMU projected onto a TI efficient benchmark t, its corresponding AI would be equal to the PI of that TI efficient benchmark.

Finally, although the above enhanced DN discussion has assumed that both input and output unit prices are available and therefore a PI analysis can be carried out, the approach can also be applied in the case that only input unit prices or output unit prices are known. There are some differences, however. Thus, for example, if output unit prices are known (but not input unit prices) then:

- Although the TI relation would contain the same arcs as before (i.e. E^{TI}), the length of the TI arcs would be computed using (23) instead of (13).

$$e_{rj}^{TI} = \sum_k w_k^y \cdot \left(y_{kj} - y_{kr} \right) \quad (23)$$

- Instead of PI dominance, the following RI dominance would be defined $D^{RI}(r) = \{ j : \bar{x}_j \leq \bar{x}_r \wedge \sum_k p_k y_{kr} < \sum_k p_k y_{kj} \}$ leading to the set of arcs of the RI dominance relation $E^{RI} = \{ (r, j) : j \in D^{RI}(r) \}$.

Table 1
Illustrative small-size dataset.

DMU j	x_j	y_{1j}	y_{2j}	$\sum_k p_k y_{kj}$	$R(x_j, \bar{p})$	RI_j
A	1	0	4	16	18	2
B	1	1	4	17	18	1
C	1	2	4	18	18	0
D	1	3	3.5	17	18	1
E	1	3.5	3	15.5	18	2.5
F	1	4	2	12	18	6
G	1	4	1	8	18	10
H	1	4	0	4	18	14
I	1	0	2	8	18	10
J	1	2	2	10	18	8
K	1	3	1	7	18	11

- To be consistent with (10), the lengths of these RI dominance arcs would be defined as

$$e_{rj}^{RI} = \frac{\sum_k p_k y_{kj} - \sum_k p_k y_{kr}}{\min \left\{ \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, ..., \frac{p_s}{w_s^y} \right\}} \quad (24)$$

- Propositions 1 and 2 would still apply, i.e. $E^{TI} \subseteq E^{RI}$ and $e_{rj}^{RI} \geq e_{rj}^{TI} \quad \forall (r, j) \in E^{TI}$.

- Propositions 3, 4 and 5 would no longer apply. The reason is that, unlike (9), (11) depends on the DMU being projected, more specifically, on its input vector. That means that the RI of the two DMUs r and j are measured as the difference between their respective revenue and the maximum revenue each one could obtain, given its input vector, which is generally different. Therefore, the non-RI dominated DMUs, i.e. those in the set $D^{RI,*} = \{ r : D^{RI}(r) = \emptyset \}$, do not need to have the same revenue. Each has the maximum revenue given its input vector, which may differ from one DMU to another. Similarly, the RI relation does not have an underlying scalar potential equivalent to (20). If we define

$$RI(\bar{p}, \bar{x}_r, \bar{y}_r) = \frac{R(\bar{x}_r, \bar{p}) - \sum_k p_k y_{kr}}{\min \left\{ \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, ..., \frac{p_s}{w_s^y} \right\}} \quad (25)$$

then the fact that $RI(\bar{p}, \bar{x}_r, \bar{y}_r) > RI(\bar{p}, \bar{x}_j, \bar{y}_j)$ does not necessarily imply the existence of an arc between r and j. Moreover, even if such an arc existed the expression $e_{rj}^{RI} = RI(\bar{p}, \bar{x}_r, \bar{y}_r) - RI(\bar{p}, \bar{x}_j, \bar{y}_j)$ would not hold in general, but only if $R(\bar{x}_r, \bar{p}) = R(\bar{x}_j, \bar{p})$.

4. Illustration of the proposed approach on a small dataset

In order to illustrate the proposed approach, a small single-input, two-output dataset from Aparicio et al. (2015) will be used. The single input is constant and only the output unit prices are taken into account. Specifically, $p_1 = 1$ and $p_2 = 4$. Let us assume, however, that the output slacks weights are equal, i.e. $w_1^y = w_2^y = 1$. Hence $\alpha = \min\{\frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}\} = 1$.

Table 1 shows the input and outputs of each DMU j, its corresponding revenue $\sum_k p_k y_{kj}$, the maximum revenue it could obtain, given its input endowment $R(\bar{x}_j, \bar{p})$, and its revenue inefficiency RI_0 as per (10). Note that there is only one RI efficient DMU, namely DMU C.

Fig. 2 shows the TI relation of the proposed DN. The width of each edge (r,j) is proportional to its length e_{rj}^{TI}. It can be seen that the set of TI efficient DMUs, i.e. those with $d_r^{TI,out} = \emptyset$, is $D^{TI,*} = \{C, D, E, F\}$. The density of this relation is 0.191 (i.e. the actual number of TI edges is 19.1% of the total number of edges that might exist). Table 2 shows some DN measures of this TI relation. Thus, for each DMU r, the table shows its in and out degree
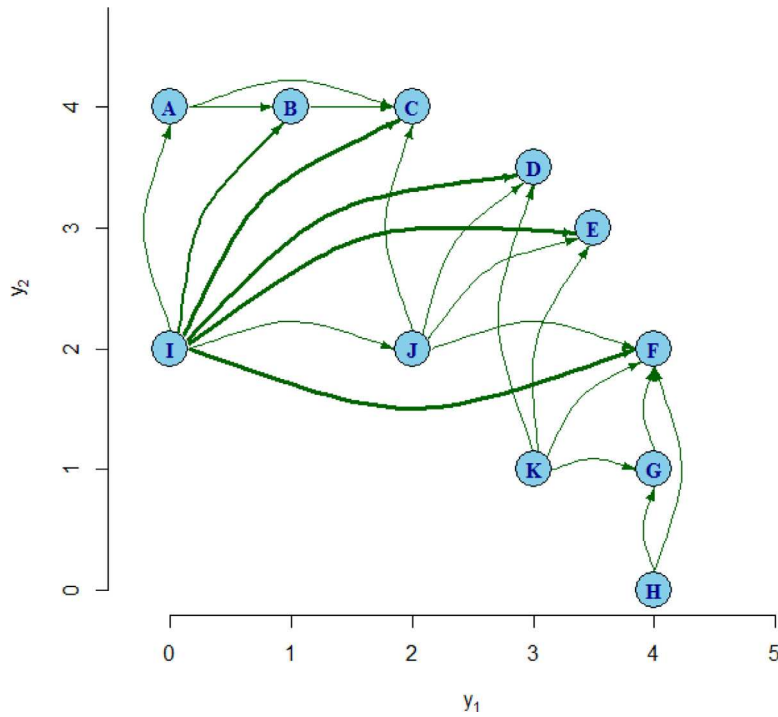
Fig. 2. TI relation of illustrative small-size dataset.

Table 2
Some DN measures of the TI relation for the illustrative small-size dataset.

DMU r	d_r^{out}	d_r^{in}	s_r^{out}	s_r^{in}	$D^{TI,*}(r)$	$e_r^{TI,max}$	$\tau_r^{TI,min}$	$\lambda(r)$
A	2	1	3	2	{C}	2	2	2
B	1	2	1	4	{C}	1	1	1
C	0	4	0	9	{C}	0	0	0
D	0	3	0	9.5	{D}	0	0	0
E	0	3	0	9.5	{E}	0	0	0
F	0	5	0	11	{F}	0	0	0
G	1	2	1	2	{F}	1	1	1
H	2	0	3	0	{F}	2	2	2
I	7	0	24	0	{C,D,E,F}	4.5	4	3
J	4	1	9	2	{C,D,E,F}	2.5	2	1
K	4	0	8	0	{D,E,F}	2.5	2	2
Aver.	1.9	1.9	4.5	4.5	–	1.4	1.3	–

d_r^{in} and d_r^{out}, its in- and out-strength s_r^{in} and s_r^{out}, its set of efficient benchmarks $D^{TI,*}(r)$, its distance to the farthest and closest efficient benchmarks $e_r^{TI,max}$ and $\tau_r^{TI,min}$ and its layer $\lambda(r)$.

The TI efficient DMUs are those which have $d_r^{out} = 0$, which is C, D, E and F. Looking at $D^{TI,*}(r)$ it can be noted that some TI inefficient DMUs, such as A, B, G and H, have only one TI efficient benchmark while others, such as I, J and K, have several. For the former $e_r^{TI,max} = \tau_r^{TI,min}$ while, for the latter, $e_r^{TI,max} > \tau_r^{TI,min}$. The in- and out-strengths of different nodes represent the length of all incoming and outgoing TI arcs, respectively. Hence, the out-strength of the TI efficient nodes is zero while their in-strength can be considered a measure of their benchmarking potential.

Note that Table 2 also showed the layer of each node. The layer can be more clearly seen in Fig. 3. Note that there are four layers (labelled 0 to 3). The layer distribution is the following: four (i.e. 36.4%) of the nodes lie in layer 0, three (i.e. 27.3%) in each of layers 1 and 3 and one node (9.1%) in layer 3. Note that, as before, the width of each edge (r,j) is proportional to its length e_{rj}^{TI}.

Panel a) of Fig. 4 shows the complete RI relation. The width of each edge (r,j) is proportional to its length e_{rj}^{RI}. Panel b) shows the corresponding skeleton subgraph, which results from removing
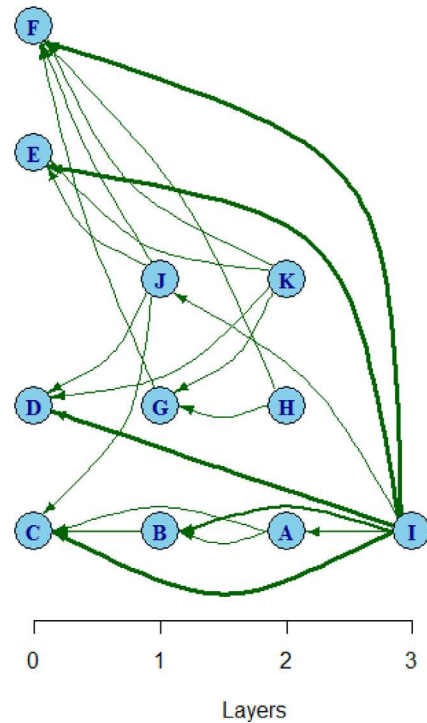
Fig. 3. Layered layout of the TI relation of the illustrative small-size dataset.

the transitive edges. Isorevenue lines are also shown in panel b. They are straight lines in this two-output dataset. In the general multi-output case they would be isorevenue hyperplanes. Panel c) of Fig. 4 shows the layered layout of the RI skeleton. If two nodes lie on the same isorevenue line, e.g. DMUs B and D, then there is no RI edge between them. Otherwise there is an edge from the node with less revenue to the one with more revenue. The skeleton only contains the edges from the nodes on an isorevenue line
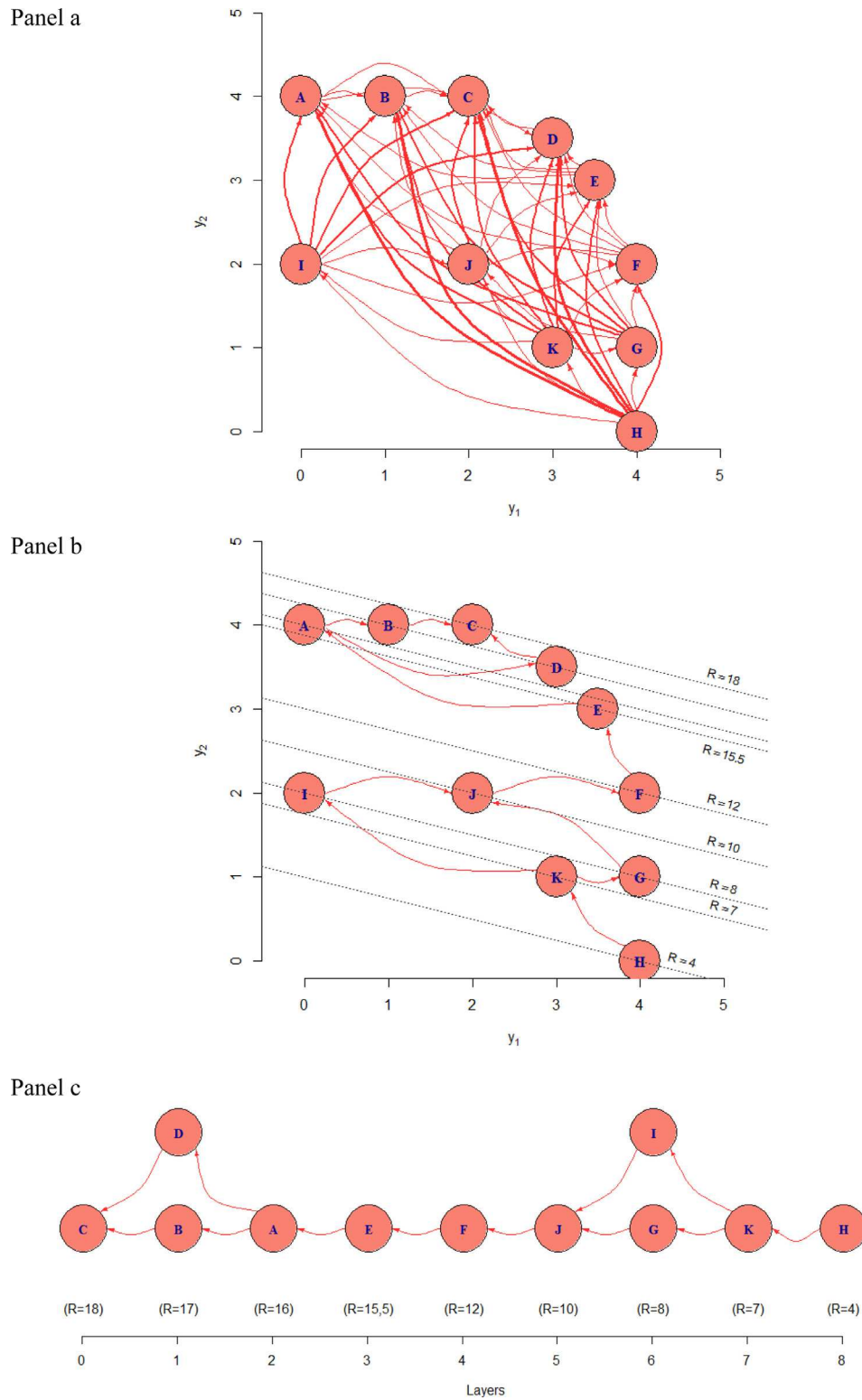
Fig. 4. RI relation of the small-size dataset (complete graph, skeleton and layered skeleton).

to the nodes on the contiguous upward isorevenue line. The number of RI layers is equal to the number of different isorevenue lines passing through the DMUs. Below each layer in panel c) the corresponding revenue level is shown. Note that the deep, streamlined structure observed for this specific dataset (typical of the existence of an underlying potential) occurs because all DMUs consume the same amount of input. For the general RI case, no such potential

exists and therefore it may happen that there is no RI dominance between two DMUs, even though one has more revenue than the other. As indicated in Section 3, for a DMU j to RI dominate a DMU r it must happen not only that $\sum_k p_k y_{kr} < \sum_k p_k y_{kj}$ but also that $\bar{x}_j \leq \bar{x}_r$.

Table 3 shows some DN measures of the RI relation. We can see that, although Proposition 4 does not hold for general RI scenarios,

Table 3
Some DN measures of the RI relation for the illustrative small-size dataset.

DMU r	d_r^{out}	d_r^{in}	s_r^{out}	s_r^{in}	$D^{RI,*}(r)$	$e_r^{RI,max}$	$\lambda(r)$
A	3	7	4	47.5	{C}	2	2
B	1	8	1	55.5	{C}	1	1
C	0	10	0	65.5	{C}	0	0
D	1	8	1	55.5	{C}	1	1
E	4	6	6	44	{C}	2.5	3
F	5	5	23.5	23	{C}	6	4
G	7	2	49.5	5	{C}	10	6
H	10	0	88.5	0	{C}	14	8
I	7	2	49.5	5	{C}	10	6
J	6	4	35.5	13	{C}	8	5
K	9	1	58.5	3	{C}	11	7
Aver.	4.8	4.8	28.8	28.8	–	6.0	–

Table 4
AI hyper-relation for the illustrative small-size dataset.

$r \in D$	$t \in D^{TI,*}(r)$	$j \in D^{RI,*}$	e_{rj}^{RI}	e_{rt}^{TI}	$e_{r/t,j}^{AI}$
A	C	C	2	2	0
B	C	C	1	1	0
C	C	C	0	0	0
D	D	C	1	0	1
E	E	C	2.5	0	2.5
F	F	C	6	0	6
G	F	C	10	1	9
H	F	C	14	2	12
I	C	C	10	4	6
	D	C	10	4.5	5.5
	E	C	10	4.5	5.5
	F	C	10	4	6
J	C	C	8	2	6
	D	C	8	2.5	5.5
	E	C	8	2.5	5.5
	F	C	8	2	6
K	D	C	11	2.5	8.5
	E	C	11	2.5	8.5
	F	C	11	2	9

since in this dataset all DMUs have the same inputs, the minimum and maximum distance from any DMU to the PI efficient DMUs coincide. It would do so anyway because there is a single PI-efficient DMU, namely DMU C. A total of eight layers exist with most of the layers composed of a single DMU and the layers numbered, as mentioned above, in decreasing order of the DMU revenues. Since each DMU is connected to the DMUs in all lower layers, the edge density of the RI relation is rather high (0.482). The in- and out-degrees of the RI relation are larger than those of the TI relation as expected from Proposition 1. The average distance to the PI ef-

ficient frontier is 6 while the RI diameter, which is equal to the inefficiency radius of DMU C, is 14.

Finally, Table 4 shows the AI hyper-relation. As indicated in Section 3, when a DMU has more than one TI efficient benchmark, the length of its AI hyperedges varies depending on the TI efficient

Table 5
Some DN measures of the TI relation for the bank branches dataset.

DMU r	d_r^{out}	d_r^{in}	s_r^{out}	s_r^{in}	$D^{TI,*}(r)$	$e_r^{TI,max}$	$\tau_r^{TI,min}$	$\lambda(r)$
B36	13	0	34,644.1	0.0	{B2;B4;B7;B8;B25;B27;B28;B33;B37;B44;B50; B55;B57}	4468.6	945.7	1
B22	10	0	20,042.4	0.0	{B5;B10;B20;B29;B32; B38;B50;B52;B53;B59}	3438.4	1014.3	1
B45	8	0	22,679.0	0.0	{B4;B20;B28;B29;B37; B50;B55}	4274.3	1723.8	2
B39	4	0	6237.9	0.0	{B10;B34;B46;B49}	2225.4	885.7	1
B42	4	0	6921.0	0.0	{B14;B38;B46;B49}	2363.5	848.5	1
B30	3	0	4227.6	0.0	{B20;B28;B37}	1582.2	1234.0	1
B18	2	1	2765.7	2473.6	{B29;B50}	1800.8	964.9	1
B26	2	0	2988.6	0.0	{B50;B53}	2225.8	762.8	1
B15	1	0	1806.3	0.0	{B34}	1806.3	1806.3	1
B19	1	0	890.9	0.0	{B20}	890.9	890.9	1
B21	1	0	656.7	0.0	{B38}	656.7	656.7	1
B48	1	0	1414.9	0.0	{B14}	1414.9	1414.9	1
B56	1	0	733.3	0.0	{B20}	733.3	733.3	1
B58	1	0	831.7	0.0	{B20}	831.7	831.7	1
B9	1	0	470.5	0.0	{B46}	470.5	470.5	1
B10	0	2	0.0	3746.3	{B10}	0.0	0.0	0
B14	0	2	0.0	3241.6	{B14}	0.0	0.0	0
B2	0	1	0.0	945.7	{B2}	0.0	0.0	0
B20	0	6	0.0	8590.2	{B20}	0.0	0.0	0
B25	0	1	0.0	1479.7	{B25}	0.0	0.0	0
B27	0	1	0.0	3343.1	{B27}	0.0	0.0	0
B28	0	3	0.0	7582.6	{B28}	0.0	0.0	0
B29	0	3	0.0	8886.4	{B29}	0.0	0.0	0
B32	0	1	0.0	1477.0	{B32}	0.0	0.0	0
B33	0	1	0.0	2717.2	{B33}	0.0	0.0	0
B34	0	2	0.0	2692.0	{B34}	0.0	0.0	0
B37	0	3	0.0	8114.9	{B37}	0.0	0.0	0
B38	0	3	0.0	3196.7	{B38}	0.0	0.0	0
B4	0	2	0.0	4477.6	{B4}	0.0	0.0	0
B44	0	1	0.0	2329.6	{B44}	0.0	0.0	0
B46	0	3	0.0	4096.9	{B46}	0.0	0.0	0
B49	0	2	0.0	4588.9	{B49}	0.0	0.0	0
B5	0	1	0.0	1014.3	{B5}	0.0	0.0	0
B50	0	5	0.0	11,610.2	{B50}	0.0	0.0	0
B52	0	1	0.0	1693.5	{B52}	0.0	0.0	0
B53	0	2	0.0	5664.3	{B53}	0.0	0.0	0
B55	0	2	0.0	5560.7	{B55}	0.0	0.0	0
B57	0	1	0.0	1668.9	{B57}	0.0	0.0	0
B59	0	1	0.0	2032.8	{B59}	0.0	0.0	0
B7	0	1	0.0	2738.0	{B7}	0.0	0.0	0
B8	0	1	0.0	1348.0	{B8}	0.0	0.0	0
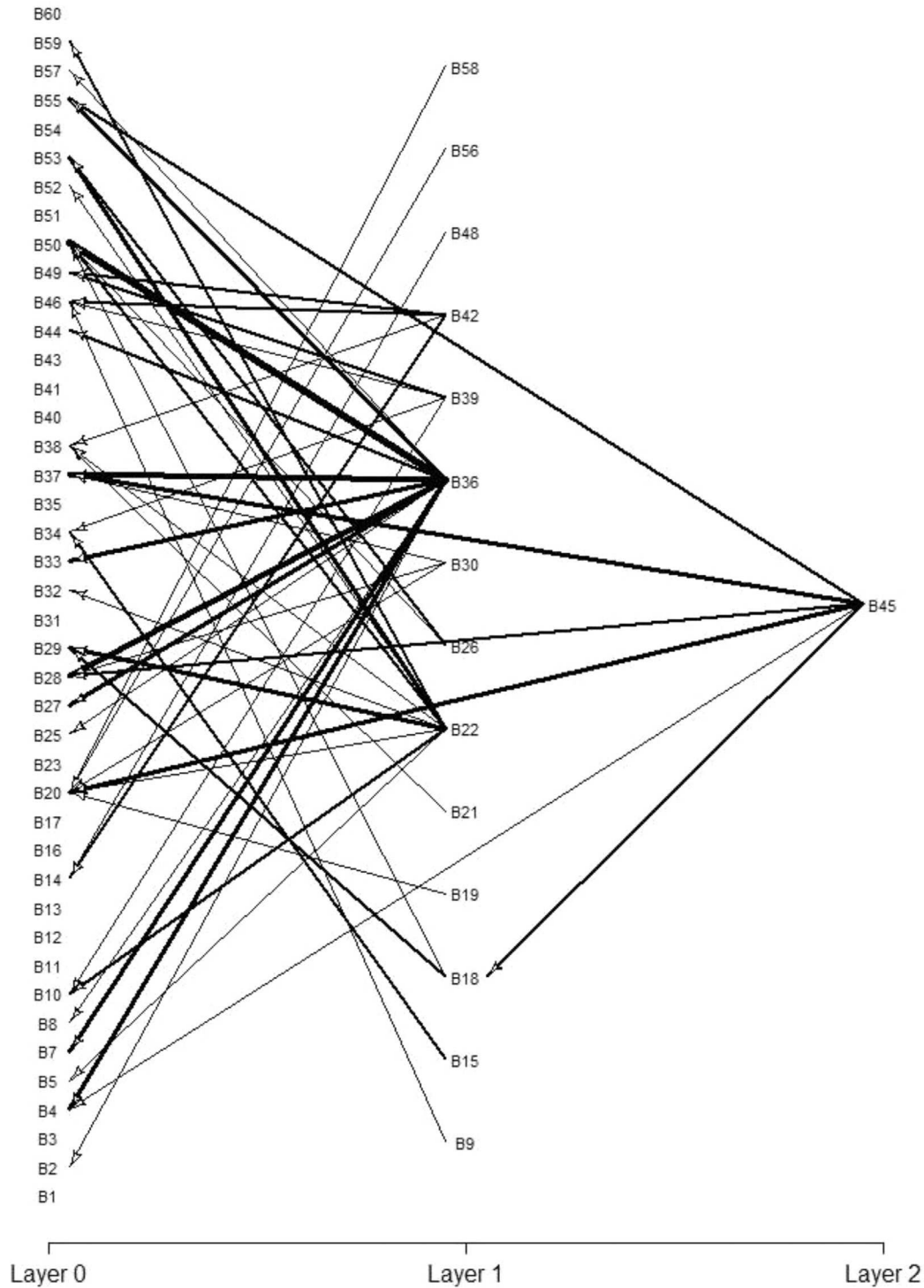
Fig. 5. Layered layout of the TI relation of bank branches dataset.

benchmark considered. In any case, the AI hyper-relation captures the difference between the RI distance to the RI projection and the TI distance to the TI projection. Note that, since $D^{RI,*} = \{C\}$ for this dataset, the RI projection is always node C. The TI projections of each TI inefficient node r can be any of the nodes in its $D^{TI,*}(r)$ set. The length of corresponding hyperedges can be computed in accordance with (18). The AI value of the four TI efficient DMUs,

namely, C, D, E and F, are equal to their RI value, i.e. their RI distance to DMU C, which is the only RI efficient DMU.

5. Application to a bank branches dataset

In this section the proposed enhanced DN approach is applied to a sample of 57 bank branches which was originally studied in Silva Portela and Thanassoulis (2005). The inputs considered are

Table 6
Some DN measures of the PI relation for the bank branches dataset.

| DMU r	Profit$_r$	$\lambda(r)$	$e_r^{PI.\max}$	s_r^{in}	DMU r	Profit$_r$	$\lambda(r)$	$e_r^{PI.\max}$	s_r^{in}
B16	649.2	0	0.0	3269,315.2	B39	341.2	29	55,014.4	616,762.2
B17	642.1	1	1278.2	3197,733.2	B18	340.8	30	55,082.5	614,923.7
B53	532.4	2	20,856.6	2120,926.4	B26	335.3	31	56,053.5	589,678.5
B12	515.5	3	23,875.6	1957,897.6	B28	327.5	32	57,456.2	554,611.3
B11	509.2	4	25,001.0	1898,252.0	B37	325.9	33	57,730.7	548,024.2
B29	503.0	5	26,122.4	1839,939.9	B42	321.7	34	58,498.3	530,367.5
B14	497.6	6	27,083.4	1790,929.9	B33	299.1	35	62,527.7	441,720.7
B49	476.5	7	30,843.1	1602,941.7	B27	298.2	36	62,688.6	438,342.9
B46	470.1	8	31,984.3	1547,024.9	B58	279.2	37	66,080.6	370,502.9
B51	468.8	9	32,213.7	1536,011.4	B56	274.8	38	66,856.8	355,754.3
B41	467.4	10	32,479.6	1523,515.5	B19	265.0	39	68,610.3	324,191.5
B34	458.1	11	34,125.8	1447,792.6	B4	262.0	40	69,146.9	315,069.2
B10	450.3	12	35,525.8	1384,792.7	B15	259.0	41	69,692.9	306,332.9
B9	423.5	13	40,319.5	1173,868.6	B55	257.7	42	69,910.8	303,065.6
B31	415.8	14	41,684.8	1115,160.3	B22	246.5	43	71,915.7	274,996.4
B38	413.9	15	42,029.2	1100,694.0	B44	231.0	44	74,680.2	239,058.5
B50	407.1	16	43,232.6	1051,353.7	B7	229.7	45	74,927.0	236,096.5
B59	380.9	17	47,913.5	864,120.6	B30	203.9	46	79,523.4	185,535.9
B52	377.5	18	48,517.3	840,571.4	B57	170.4	47	85,500.6	125,763.6
B40	374.0	19	49,145.3	816,706.5	B25	151.1	48	88,955.1	94,673.4
B48	364.6	20	50,835.6	754,167.7	B8	145.8	49	89,899.2	87,120.5
B3	364.2	21	50,905.9	751,636.8	B23	113.6	50	95,656.5	46,819.4
B13	355.4	22	52,475.3	696,707.5	B45	104.0	51	97,361.1	36,592.0
B60	354.1	23	52,695.1	689,234.7	B1	102.0	52	97,713.6	34,829.7
B32	348.1	24	53,783.5	653,314.8	B2	93.1	53	99,316.4	28,418.3
B5	347.9	25	53,813.5	652,357.3	B54	55.5	54	106,034.1	8265.1
B35	346.0	26	54,158.6	641,657.1	B36	37.7	55	109,211.2	1910.9
B21	343.9	27	54,527.3	630,597.6	B43	27.0	56	111,122.2	0.0
B20	342.8	28	54,722.1	624,947.4	–	–	–	–	–
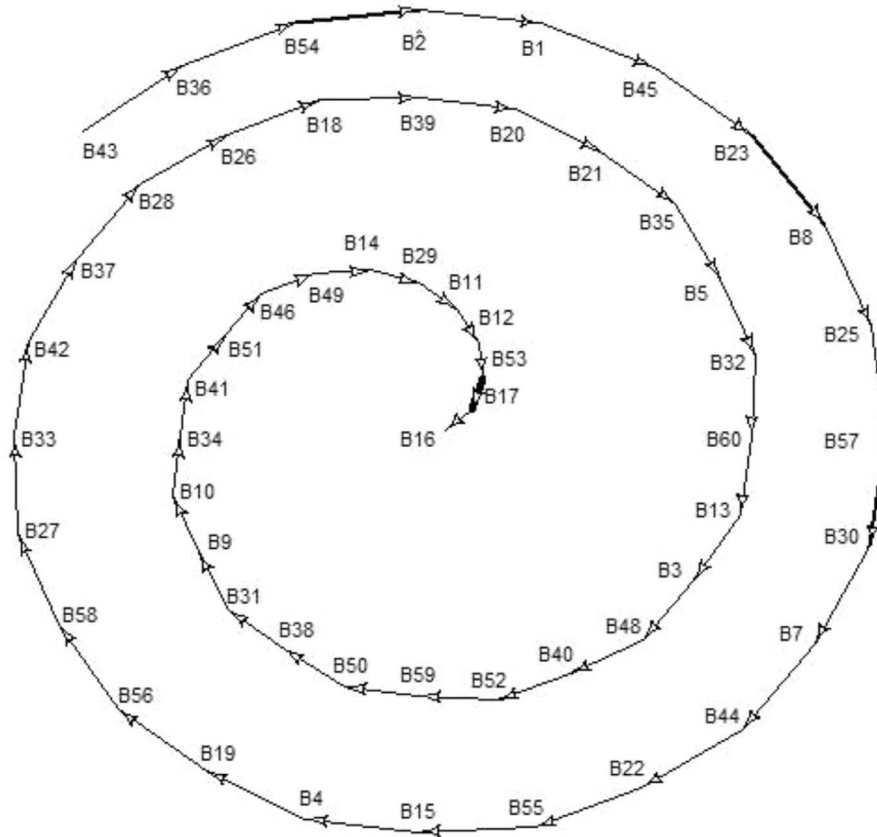
Fig. 6. Spiral layout of the skeleton of the PI relation of the bank branches dataset.

staff costs and supply costs while the outputs are value of current accounts, value of other resources, value of credit by bank and value of credit by associates. The unit input prices are unity (as the inputs are already measured in monetary terms) while the unit output prices are 0.0408, 0.0118, 0.0056 and 0.0157, respectively. These unit output prices correspond to net interest rates. The reader is referred to Silva Portela and Thanassoulis (2005) for more information on these data and the bank to which they belong.

The first step is building the TI relation. Unity weights $w_i^x = 1 \; \forall i$ $w_k^y = 1 \; \forall k$ have been used in (13). The TI relation has a three layer structure, with 42 DMUs in layer 0, 14 in layer 1 and 1 in layer 2. Table 5 shows the basic DN measures of that relation for the different nodes. The technical inefficient bank branches are shown first (in decreasing order of their d_r^{out}), followed by all the technical efficient ones, except the existing 16 isolates, i.e. nodes that have neither incoming nor outgoing links. These isolates are also efficient and generally correspond to what are called maverick in DEA terminology, i.e. DMUs with a specific input-output mix different from those of any other DMU. The TI relation is sparse, with a density of 0.016, which corresponds to a total number of edges of 53. Among the efficient DMUs, the one with the largest in-strength (i.e. the largest benchmarking potential) is DMU B50. Although the TI relation has two layers there is only one DMU (namely B45) in the deepest layer. Some of the TI inefficient DMUs are dominated by more than one efficient DMU thus having the possibility to choose between alternative benchmarks. The values $e_r^{TI,\max}$ (respectively $\tau_r^{TI,\min}$) represents the maximum (respectively minimum) distance to those alternative benchmarks.

Fig. 5 shows the TI relation using a layered layout. The edge width is proportional to its weight e_{rj}^{TI}. Note the TI efficient isolates. The benchmarking potential of the other TI efficient nodes can also be assessed by the number and width of their incoming edges. The alternative benchmarks for each TI inefficient DMU are also clear. In the case of DMU B45, apart from the possibility of being projected directly onto any of its benchmarks, there is also the possibility of following two stepwise improvement paths through inefficient DMU B18.

As regards the PI relation, Table 6 shows some DN measures. The DMUs are shown in decreasing order of their profit. Each DMU belongs to a different PI layer so that $Profit_r$ and s_r^{in} decrease with the layer while $e_r^{PI,\max}$ increases with the layer. Note that $e_r^{PI,\max}$ corresponds to the increase in PI potential (and hence in profits) that DMU r could have if it operated as the maximum profit DMU (namely B16). Similarly, s_r^{in} represents the total increase in PI potential if all DMUs dominated by r achieved their profit performance. In particular, for B16, this value provides information about the total profit increase that the system would achieve if all units performed optimally.

Fig. 6 shows a spiral layout of the skeleton of the PI relation. The skeleton is a way of reducing the clutter of the PI relation by removing the transitive arcs, maintaining them implicitly. Note that the layers increase from the inside to the outside of the spiral, going from the maximum profit DMU B16 to the least profit DMU B43. The edge width is proportional to the corresponding weight e_{rj}^{PI} and hence to the profit difference between the DMUs r and j.

Finally, Table 7 shows the hyperedges of the AI hyper-relation. Only the hyperedges involving TI inefficient DMUs are shown. This table shows, for each TI inefficient DMU, all possible two-step paths where the first step takes the DMU to the technical efficiency frontier (removing technical inefficiency) and the second step takes it further into profit efficiency (by removing allocative inefficiency). Note that e_{rj}^{PI} represents the total PI improvement, e_{rt}^{TI} represents the length of the TI step and the AI component $e_{r/t.j}^{AI}$ corresponds to the difference.

Table 7
AI hyper-relation for bank branches dataset.

$r \in D$	$t \in D^{TI,*}(r)$	$j \in D^{RI,*}$	e_{rt}^{PI}	e_{rt}^{TI}	$e_{r/t.j}^{AI}$
B9	B46	B16	40,319.5	470.5	39,849.0
B15	B34	B16	69,693.0	1806.3	67,886.6
B18	B29	B16	55,082.5	1800.8	53,281.8
	B50	B16	55,082.5	964.9	54,117.6
B19	B20	B16	68,610.3	890.9	67,719.4
B21	B38	B16	54,527.3	656.7	53,870.6
B22	B5	B16	71,915.7	1014.4	70,901.4
	B10	B16	71,915.7	2363.7	69,552.0
	B20	B16	71,915.7	1544.5	70,371.2
	B29	B16	71,915.7	2811.3	69,104.4
	B32	B16	71,915.7	1477.0	70,438.7
	B38	B16	71,915.7	1691.5	70,224.2
	B50	B16	71,915.7	1975.4	69,940.3
	B52	B16	71,915.7	1693.5	70,222.2
	B53	B16	71,915.7	3438.4	68,477.3
B23	B59	B16	95,656.5	2032.8	93,623.7
B26	B50	B16	56,053.5	762.8	55,290.7
	B53	B16	56,053.5	2225.8	53,827.7
B30	B20	B16	79,523.4	1582.2	77,941.2
	B28	B16	79,523.4	1234.0	78,289.4
	B37	B16	79,523.4	1411.4	78,112.0
B36	B2	B16	109,211.2	945.7	108,265.6
	B4	B16	109,211.2	2753.8	106,457.4
	B7	B16	109,211.2	2738.0	106,473.3
	B8	B16	109,211.2	1348.0	107,863.2
	B25	B16	109,211.2	1479.7	107,731.5
	B27	B16	109,211.2	3343.1	105,868.1
	B28	B16	109,211.2	3689.4	105,521.9
	B33	B16	109,211.2	2717.2	106,494.0
	B37	B16	109,211.2	3866.8	105,344.4
	B44	B16	109,211.2	2329.6	106,881.6
	B50	B16	109,211.2	4468.6	104,742.6
	B55	B16	109,211.2	3295.4	105,915.8
	B57	B16	109,211.2	1668.9	107,542.3
B39	B10	B16	55,014.5	1382.6	53,631.8
	B34	B16	55,014.5	885.7	54,128.8
	B46	B16	55,014.5	1744.2	53,270.3
	B49	B16	55,014.5	2225.4	52,789.0
B42	B14	B16	58,498.3	1826.7	56,671.6
	B38	B16	58,498.3	848.5	57,649.8
	B46	B16	58,498.3	1882.3	56,616.1
	B49	B16	58,498.3	2363.5	56,134.9
B45	B4	B16	97,361.1	1723.8	95,637.3
	B20	B16	97,361.1	3007.6	94,353.5
	B28	B16	97,361.1	2659.3	94,701.8
	B29	B16	97,361.1	4274.3	93,086.7
	B37	B16	97,361.1	2836.7	94,524.4
	B50	B16	97,361.1	3438.5	93,922.6
	B55	B16	97,361.1	2265.3	95,095.8
B48	B14	B16	50,835.6	1414.9	49,420.7
B56	B20	B16	66,856.8	733.3	66,123.5
B58	B20	B16	66,080.6	831.7	65,248.9

6. Conclusions

Calzada-Infante and Lozano (2016) considers a DN based on the inputs and outputs of a set of DMUs using a number of measures and indexes to characterize and assess the performance of the individual DMUs as well as of the whole sample. This paper presents an enhanced DN approach that can be applied to the case in which the inputs and output unit prices are available and therefore, in addition to a technical efficiency assessment, an economic and allocative efficiency assessment can be carried out. To that end a new PI relation on the DN is defined. In the same way as the TI relation, the PI relation is transitive and layered, and the corresponding edge lengths have the additivity property. Moreover, TI dominance implies PI dominance and when a TI and a PI edge exist between two nodes, the length of the PI edge is never smaller than that of the TI edge. DN measures of the PI relation, such as edge density,

diameter, in- and out-degrees, in- and out-strengths, etc. can be computed.

Interestingly, unlike the TI relation, the PI relation has an underlying scalar potential, so that each possible operating point has a PI potential and would have PI edges towards any other operating point with less PI potential, and the length of those links would be equal to the corresponding PI potential difference.

A characteristic feature of the PI relation, derived from the existence of the underlying PI potential, is the typical streamlined layer structure in which all the operating points in a given layer have the same PI potential and therefore there are no connections between them, i.e. they are not neighbors. The edges, however, connect the nodes in each layer with the nodes in all the succeeding layers, which have lower potential. Layer 0 consists of the nodes with maximum profit and, therefore, with zero potential.

As a reflection of the conventional DEA PI decomposition into a TI and a residual AI component, the proposed DN approach defines an AI hyper-relation that captures the difference between the PI and TI relations. An inherent feature of this AI hyper-relation is that it depends not only on the TI inefficient DMU being projected but also on the TI efficient benchmark chosen. Thus, the TI distance to a closer TI efficient benchmark leads to a higher AI component, while projection onto a further TI efficient benchmark leaves a smaller AI residual component. It has been shown that bounds on the lengths of the AI hyperedges can be computed using the maximum and minimum TI distances $e_r^{TI,max}$ and $\tau_r^{TI,min}$, respectively. Also, the length of the AI hyperedges for the special case of input and output slacks weights proportional to input and output prices have been determined.

When only the input or the output unit prices are known, then only a cost or revenue efficiency, respectively, can be carried out. In those cases, a Cost Inefficiency (CI) or an RI relation, respectively, can be defined. The corresponding DN analysis is similar (e.g. the transitivity and additivity properties hold and the same DN measures can be computed) but there is a significant difference in that, in those cases, an underlying potential is not guaranteed to exist.

Two different datasets have been used to illustrate the proposed approach. The proposed enhanced DN with its TI and RI/PI relations has been plotted and characterized. The transitivity property suggests using the skeleton subgraph to represent the complete TI or PI relation, thus leading to a less cluttered plot. As shown in these illustrations, particularly convenient is the layer layout of the DN.

This paper has shown the applicability and usefulness of enhanced DN analysis. DN analysis can be further extended, for example, to datasets involving multiple periods, or to DMUs with a network DEA internal structure. Also, while this paper has used an NAI (i.e. a weighted additive) metric, the research question of how the DN measures are affected when other efficiency metrics are considered is left for further research.

Acknowledgements

Appendix

Proposition 1. $E^{TI} \subseteq E^{PI}$

Proof. Looking at the definition of the two sets of arcs we have only to show that $D^{TI}(r) \subseteq D^{PI}(r)$ $\forall r$. And, in fact, that is the case since, $\forall r \in D$

$j \in D^{TI}(r) \Rightarrow x_{ij} \leq x_{ir} \forall i$ $y_{kj} \geq y_{kr} \forall k$ with at least one of the inequalities being strict

$\Rightarrow \sum_i q_i x_{ij} \leq \sum_i q_i x_{ir}$ $\sum_k p_k y_{kj} \geq \sum_k p_k y_{kr}$ with at least one of the inequalities being strict

$$\Rightarrow \sum_k p_k y_{kj} - \sum_i q_i x_{ij} > \sum_k p_k y_{kr} - \sum_i q_i x_{ir} \Rightarrow j \in D^{PI}(r).$$

Proposition 2. $e_{rj}^{PI} \geq e_{rj}^{TI}$ $\forall (r, j) \in E^{TI}$

Proof. Let us define $\alpha = \min\{\frac{q_1}{w_1^x}, \frac{q_2}{w_2^x}, ..., \frac{q_m}{w_m^x}, \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, ..., \frac{p_s}{w_s^y}\}$ then $\alpha \leq \frac{q_i}{w_i^x}$ $\forall i$ $\alpha \leq \frac{p_k}{w_k^y}$ $\forall k \Rightarrow w_i^x \leq \frac{q_i}{\alpha}$ $\forall i$ $w_k^y \leq \frac{p_k}{\alpha}$ $\forall k$. Hence

$$e_{rj}^{PI} = \frac{\sum_k p_k y_{kj} - \sum_i q_i x_{ij} - \left(\sum_k p_k y_{kr} - \sum_i q_i x_{ir}\right)}{\min\left\{\frac{q_1}{w_1^x}, \frac{q_2}{w_2^x}, ..., \frac{q_m}{w_m^x}, \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, ..., \frac{p_s}{w_s^y}\right\}}$$

$$= \frac{1}{\alpha} \cdot \left[\sum_k p_k \left(y_{kj} - y_{kr}\right) - \sum_i q_i \left(x_{ij} - x_{ir}\right)\right]$$

$$\geq \sum_k w_k^y \left(y_{kj} - y_{kr}\right) - \sum_i w_i^x \left(x_{ij} - x_{ir}\right) = e_{rj}^{TI}.$$

Proposition 3. $\sum_k p_k y_{kj} - \sum_i q_i x_{ij} = \Pi(\bar{q}, \bar{p})$ $\forall j \in D^{PI,*}$

Proof. The PI relation is based on the profit obtained by each DMU so that arcs always go from the DMU with less profit to the DMU with more profit. We have that $j, j' \in D^{PI,*} \Rightarrow \sum_k p_k y_{kj} - \sum_i q_i x_{ij} = \sum_k p_k y_{kj'} - \sum_i q_i x_{ij'}$ since otherwise one of them would be PI dominated and therefore could not belong to $D^{PI,*}$. Therefore, all the DMUs in $D^{PI,*}$ have the same profit, which is the maximum profit among all the DMUs. This maximum profit has to be equal to $\Pi(\bar{q}, \bar{p})$ because:

(a). if $\Pi(\bar{q}, \bar{p}) < \max_{j \in D}\{\sum_k p_k y_{kj} - \sum_i q_i x_{ij}\}$ then denoting $j_* = \arg\max_{j \in D}\{\sum_k p_k y_{kj} - \sum_i q_i x_{ij}\}$ we would obtain that $(\bar{x}_{j_*}, \bar{y}_{j_*})$ is an operating point that is feasible in (9) and has a larger objective function than the optimum, which is a contradiction.

(b). if $\Pi(\bar{q}, \bar{p}) > \max_{j \in D}\{\sum_k p_k y_{kj} - \sum_i q_i x_{ij}\}$ then denoting by j^* the benchmark used in the optimal solution of (9), i.e. $j_* = \{j \in D : \lambda_j^* = 1\}$ we would obtain that $\Pi(\bar{q}, \bar{p}) = \sum_k p_k \sum_j \lambda_j^* y_{kj} - \sum_i q_i \sum_{j \in J^*} \lambda_j^* x_{ij} = \sum_k p_k y_{kj_*} - \sum_i q_i x_{ij_*} >> \arg\max_{j \in D}\{\sum_k p_k y_{kj} - \sum_i q_i x_{ij}\}$

thus reaching, again, a contradiction.

Hence $\Pi(\bar{q}, \bar{p}) = \arg\max_{j \in D}\{\sum_k p_k y_{kj} - \sum_i q_i x_{ij}\} = \sum_k p_k y_{kj} - \sum_i q_i x_{ij}$ $\forall j \in D^{PI,*}$ $\square$

Proposition 4. $e_r^{PI,max} = \max_{j \in D^{PI,*}(r)} e_{rj}^{PI} = \min_{j \in D^{PI,*}(r)} e_{rj}^{PI} = \tau_r^{PI,min} = PI_r$ $\forall r$

Proof. By (1), Proposition 3 and (8) it follows that

$$
e_{rj}^{PI} = \frac{\sum_k p_k y_{kj} - \sum_i q_i x_{ij} - \left(\sum_k p_k y_{kr} - \sum_i q_i x_{ir} \right)}{\min \left\{ \frac{q_1}{w_1^x}, \frac{q_2}{w_2^x}, \ldots, \frac{q_m}{w_m^x}, \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, \ldots, \frac{p_s}{w_s^y} \right\}}
$$

$$
= \frac{\Pi(\bar{q}, \bar{p}) - \left(\sum_k p_k y_{kr} - \sum_i q_i x_{ir} \right)}{\min \left\{ \frac{q_1}{w_1^x}, \frac{q_2}{w_2^x}, \ldots, \frac{q_m}{w_m^x}, \frac{p_1}{w_1^y}, \frac{p_2}{w_2^y}, \ldots, \frac{p_s}{w_s^y} \right\}} = PI_r \quad \forall j \in D^{PI,*}
$$

Hence, the PI distance from r to any PI efficient DMU (if there is more than one) is exactly the same and coincides with its PI score. □

Proposition 5. The PI relation results from an underlying scalar potential $PI(\bar{q}, \bar{p}, \bar{x}, \bar{y})$. Moreover, the PI arc lengths are equal to the potential differences between the nodes, i.e. $e_{rj}^{PI} = PI(\bar{q}, \bar{p}, \bar{x}_r, \bar{y}_r) - PI(\bar{q}, \bar{p}, \bar{x}_j, \bar{y}_j) = PI_r - PI_j$

Proof. The existence of an underlying potential field means that each node r has a potential $PI(\bar{q}, \bar{p}, \bar{x}_r, \bar{y}_r)$ so that there exist arcs from nodes with higher potential to nodes with lower potential. In our case the potential corresponds to the profit inefficiency (20). Hence

$$
PI(\bar{q}, \bar{p}, \bar{x}_r, \bar{y}_r) - PI\left(\bar{q}, \bar{p}, \bar{x}_j, \bar{y}_j \right)
$$

$$
= \frac{1}{\alpha} \cdot \left[\Pi(\bar{q}, \bar{p}) - \left(\sum_k p_k y_{kr} - \sum_i q_i x_{ir} \right) \right]
$$

$$
- \frac{1}{\alpha} \cdot \left[\Pi(\bar{q}, \bar{p}) - \left(\sum_k p_k y_{kj} - \sum_i q_i x_{ij} \right) \right]
$$

$$
= \frac{1}{\alpha} \cdot \left[\left(\sum_k p_k y_{kj} - \sum_i q_i x_{ij} \right) - \left(\sum_k p_k y_{kr} - \sum_i q_i x_{ir} \right) \right]
$$

$$
= e_{rj}^{PI}
$$

Proposition 6. $PI_r - e_r^{TI,\max} \leq e_{r/t,j}^{AI} \leq PI_r - \tau_r^{TI,\min} \quad \forall (r/t, j) \in H^{AI}$

Proof. On the one hand, e_{rt}^{TI} is bounded by $e_r^{TI,\max}$ and $\tau_r^{TI,\min}$ as $\tau_r^{TI,\min} = \min_{t' \in D^{TI,*}(r)} e_{rt'}^{TI} \leq e_{rt}^{TI} \leq \max_{t' \in D^{TI,*}(r)} e_{rt'}^{TI} = e_r^{TI,\max} \quad \forall r \in D \setminus D^{TI,*} \forall t \in D^{TI,*}(r)$.

On the other hand, from Proposition 5 and Corollary 1, we have that $e_{rj}^{PI} = PI_r - PI_j = PI_r \quad \forall j \in D^{PI,*}$. Hence, from (18), the result follows. □

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Computing gradient-based stepwise benchmarking paths☆

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ABSTRACT

In this paper, a new stepwise benchmarking approach is presented. It is based on the concept of efficiency field potential given by a continuous and differentiable function that decreases monotonously as the amount of inputs consumed is reduced and the amount of outputs produced is increased. A gradient-based stepwise efficiency improvement method is proposed and the graphical interpretation of the continuous gradient-based trajectories is shown. A minimum potential DEA model is also formulated. The proposed approach is units invariant and can take into account preference structure, non-discretionary variables and undesirable outputs. The proposed method has been applied to an organic farming dataset.

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1. Introduction

Data Envelopment Analysis (DEA) is a well-known non-parametric methodology for assessing the relative efficiency of a set of Decision Making Units (DMUs). From the inputs consumption and outputs production of the observed DMUs, and using a few axioms such as convexity and free disposability, DEA infers a Production Possibility Set (PPS) (a.k.a. DEA technology) which contains all the feasible operating points. The corresponding non-dominated subset is the Efficient Frontier (EF). Conventional DEA models project the inefficient DMUs on the EF using an orientation (input, output, directional distance, etc.) and a metric (radial, non-radial, slacks-based, etc.) (see, e.g., [15,16,45]). Note, however, that some DEA models (like radial, directional distance, hyperbolic, etc) do not necessarily exhaust all input and output slacks and hence may compute targets that are only weakly efficient.

Since conventional DEA models aim at reducing inputs and/or increasing outputs as much as possible, they tend to compute efficient targets that are as "far" from the observed DMU as possible. This makes those targets harder to achieve since the improvements in terms of the corresponding input reductions and output increases may be significant.

One way to alleviate this distant target problem is to compute the closest efficient targets. There is an abundant literature on the subject, dating back to Frei and Harker [20], which used the Euclidean distance to the strongly efficient frontier. A weighted Euclidean distance to the strongly efficient frontier has been used by Baek and Lee [12], Amirteimoori and Kordrostami [1] and Aparicio and Pastor [8]. Other authors have used other ways of measurement the similarity/closeness between the DMU and the potential efficient targets (e.g. [13,42]). Other approaches include Gonzalez and Alvarez [22], which propose a modified version of the input-oriented Russell efficiency measure, and Aparicio et al. [11], which use an Enhanced Russell Graph Measure (ERGM, a.k.a. Slacks-Based Measure, SBM) together with a characterization of the Pareto-efficient frontier based on the set of extreme efficient DMUs. Additional papers dealing with least distance target computation are Pastor and Aparicio [38], Ando et al. [2,3], Aparicio and Pastor [7,9] and Aparicio et al. [6]. The reader is referred to Aparicio [4] and Aparicio et al. [5,10] for recent developments in the field as well as an up-to-date review of the literature on this topic.

Another line of research, which is the one followed in this paper, is to compute a stepwise improvement path so that a number of Intermediate Benchmark targets (IBTs) are computed, leading to an Ultimate Benchmark target (UBT) on the EF. There are two types of stepwise efficiency improvement method: those that use the existing DMUs as IBT and UBT and those that compute IBT and UBT belonging to the PPS and EF respectively, but not necessarily coincident with any of the existing DMUs. The second group is composed of just a few approaches, basically Lozano and Villa [28,29], Suzuki and Nijkamp [44], Khodakarami et al. [25] and Fang [19]. The first group is more numerous and, in most cases, uses the stratification approach proposed in the Context-Dependent (CD) DEA approach of Seiford and Zhu [39] which identifies successive

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layers of DMUs and computes so-called attractiveness and progress measures for each DMU with respect to the different DMU layers.

Table 1 shows a summary of the main characteristics of the different approaches. In particular, for each approach the table shows the type of IBT and UBT considered, whether stratification is used, whether the benchmarking path is computed over a benchmark network (whose nodes are the DMUs and whose edges indicate the possible steps that can be taken to form the efficiency improvement path), whether bounds on the stepsizes are considered, whether the DMUs are clustered, the similarity criteria considered for selecting each IBT and whether the method suffers from zigzagging (i.e. moving in inverse directions in successive steps). The final column shows some specific features of the methods.

Note that most methods have used stratification to segment the sample and clustering to group similar DMUs based mainly on their inputs (although sometimes on inputs and outputs or on cross-efficiency scores). Self-Organizing Maps (SOMs) and k-means are the clustering algorithms generally used. Those methods that consider only the existing DMUs use clustering and other mechanisms (such as directional similarity) to try to compute consistent benchmarking paths (so that if an input is reduced in one step it is not increased in the next) but in general they cannot prevent zigzagging from happening.

Note that some (but not all) methods limit the amount of inputs and outputs changes allowed in each step. Those methods that build a benchmark network usually compute the sequence of targets solving a shortest path problem. In order to select the next IBT along the benchmarking path, different criteria have been considered. Thus, they can be required to belong to the same cluster as the original DMU being projected or they should not be far (in terms of the inter-cluster distance or in terms of SOM distance) from the previous IBT. Using the attractiveness and progress measures computed by the CD, DEA stratification has also been proposed. The distance between the current IBT and the candidates for the next IBT is often taken into account as it is also the change in efficiency between those DMUs.

Among the specific features of some of the methods, we have the possibility of using a preference structure to select the UBT, the consideration of a fixed cost for carrying out each benchmarking step or computing a decision tree from the DMU stratification to try to identify the differences in inputs and output ranges in two successive layers. The extension of stepwise benchmarking to centralized DEA and to systems with two stages in series is also remarkable.

In this paper, a completely new stepwise benchmarking approach that uses the gradient of an efficiency field potential (EFP) is presented. Thus, each feasible operating point is assigned a scalar EFP so that the negative gradient of that potential defines an efficiency field vector (EFV). The proposed stepwise efficiency improvement path is computed by moving a discrete stepsize along the direction of this EFV. Since the negative gradient of the EFP always points to the less-input/more-output subspace, a step in that direction monotonously reduces the inputs and increases the outputs so that after a finite number of steps an EF target is reached. The continuous version of the method corresponds to moving along the efficiency field lines perpendicular to the efficiency equipotential surfaces and defines rather interesting trajectories, which can be easily visualized in the case of a single input as well as in the case of two inputs and a constant output or two outputs and a constant input. The proposed approach is units invariant and can accommodate a preference structure as well as non-discretionary variables and undesirable outputs.

The EFP and EFV concepts introduced in this paper are original contributions. The closest relatives we have found in the DEA literature are the Geometric Distance Function (GDF) used in, for example, Silva Portela and Thanassoulis [43], the dominance network

profit potential in Lozano and Calzada-Infante [30] and the two-dimensional gradient line approach in Maital and Vaninsky [31]. The proposed approach is, however, completely different, differing from such approaches in its purpose and in the methodology and concepts used. Thus, Silva Portela and Thanassoulis [43] use GDF mainly as way of measuring technical efficiency. i.e. measuring the distance from an observed DMU to its target. Although the functional form of GDF is also multiplicative, they use geometric averages and they neither define a potential function on the PPS as this paper does, nor do they study stepwise efficiency improvements. On the other hand, Lozano and Calzada-Infante [30] use the concept of profit potential to designate the profit associated to each operating point. They do that in the context of a dominance network where the nodes represent the DMU and the arcs go from lower profit nodes to larger profit nodes. Those dominance networks are then studied using complex network analysis tools. Finally, Maital and Vaninsky [31] compute so-called gradient lines on a two-dimensional section of the PPS determined by a simultaneous radial input reduction and radial output expansion. With their approach they are able to determine locally optimal proportional change in inputs and outputs using the information provided by a single DMU.

The structure of the paper is the following. In Section 2 the efficiency field potential, efficiency equipotential surface, efficiency field vector and gradient-based trajectories are presented. In Section 3 the proposed gradient-based stepwise benchmarking approach is formulated and illustrated on a simple two-dimensional dataset. Section 4 presents some extensions of the proposed approach. Section 5 presents an application of the proposed approach while Section 6 summarizes and concludes.

2. Efficiency field potential and efficiency field vector

Let us consider that we have a set of n DMUs which consume m inputs and produce s outputs. Let I and O represent the set of inputs and outputs, respectively. Let $x_j = (x_{1j}, x_{2j}, ... x_{mj})$ and $y_j = (y_{1j}, y_{2j}, ... y_{sj})$ the input and output vectors, respectively, of DMU j. Using the conventional DEA methodology, the following Variable Returns to Scale (VRS) PPS can be inferred from the observations

$$T^{VRS} = \left\{ (x, y) \in \mathbb{R}^{m+s} : \exists \lambda_j \geq 0 \ \sum_{j=1}^{n} \lambda_j = 1 \ \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i \ \forall i \right.$$
$$\left. \times \sum_{j=1}^{n} \lambda_j y_{kj} \geq y_k \ \forall k \right\} \tag{1}$$

Using the average of the different input and output dimensions x_i^{aver} and y_k^{aver} appropriate dimensionless inputs and output vector for the observed DMUs can be computed

$$\hat{x}_{ij} = \frac{x_{ij}}{x_i^{aver}} \ \forall i \forall j \qquad \hat{y}_{kj} = \frac{y_{kj}}{y_k^{aver}} \ \forall k \forall j \tag{2}$$

There is a one to one correspondence between the VRS PPS T^{VRS} and the corresponding VRS PPS defined using the dimensionless input and output vectors

$$\hat{T}^{VRS} = \left\{ (\hat{x}, \hat{y}) \in \mathbb{R}^{m+s} : \exists \lambda_j \geq 0 \ \sum_{j=1}^{n} \lambda_j = 1 \ \sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \hat{x}_i \ \forall i \right.$$
$$\left. \times \sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k \ \forall k \right\} \tag{3}$$

Thus, if $\hat{x}_i = \frac{x_i}{x_i^{aver}} \ \forall i \quad \hat{y}_k = \frac{y_k}{y_k^{aver}} \ \forall k$ then $(x, y) \in T^{VRS} \Leftrightarrow (\hat{x}, \hat{y}) \in \hat{T}^{VRS}$.

Table 1
Summary of existing stepwise efficiency improvement approaches.

Reference	IBT	UBT	Stratification	Benchmark network	Stepsize constraints	Clustering	Similarity criteria	Zigzagging	Other features
Hong et al. [24]	Existing DMUs	Existing DMUs	Yes (Tiers)	No	No	SOM (inputs only)	Same cluster	Yes	Decision tree for tier classification
Lozano and Villa [28]	Feasible operating points	Efficient operating points	No	No	Yes (on the change of each variable)	No	Efficiency improvement	No	CRS
Estrada et al. [18]	Existing DMUs	Existing DMUs	No	No	Yes (on efficiency change)	SOM (inputs only)	SOM distance	Yes	Reinforcement Learning
Sharma and Yu [40]	Existing DMUs	Existing DMUs	Yes (Tiers)	No	No	SOM (inputs only)	Same cluster	Yes	
Sharma and Yu [41]	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	No	No	No	Attractiveness and progress	Yes	Decision tree for attribute prioritization
Lozano and Villa [29]	Feasible operating points	MPSS efficient operating points	No	No	Yes (on the change of each variable)	No	Efficiency improvement	No	VRS
Park et al. [32]	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	No	No	k-means	Inter-cluster distance	Yes	
Lim et al. [27]	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	No	Yes (joint linear constraints)	No	Attractiveness, progress and Infeasibility	Yes	
Suzuki and Nijkamp [44]	Feasible operating points	Efficient operating points	Yes (Context dependent DEA)	No	No	No	Distance friction minimization	Yes	
Reference	IBT	UBT	Stratification	Benchmark network	Stepsize constraints	Clustering	IBT selection criteria	Zigzagging	Other features
Park et al. [33]	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	Yes	No	No	Resource improvement, directional proximity (inputs)	Yes	Shortest Path
Park et al. [34]	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	No	No	SOM (inputs only)	SOM distance, directional proximity (inputs)	Yes	Preference structure
Park et al. [35]	Existing DMUs	Existing DMU	Yes (Context dependent DEA)	No	No	No	Least distance measure	Yes	Resource priority analysis
Khodakarami et al. [25]	Feasible operating points	MPSS efficient operating points	No	No	Yes (on the change of each variable)	No	Ray average productivity	Yes	Extension to two stage systems
Park et al. [36]	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	Yes	No	k-means (XE matrix)	Same cluster, Least distance measure	Yes	Shortest Path
Fang [19]	Feasible operating points	Efficient operating points	No	No	Yes (on efficiency change)	No	Efficiency improvement	No	Centralized DEA
Park and Sung [37]	Existing DMUs	Existing DMUs	Yes (Context dependent DEA)	Yes	No	k-means (XE matrix)	Same cluster	Yes	
Ghahraman & Prior [21]	Existing DMUs	Existing DMUs	No	Yes	Yes (on the change of each variable)	Network components	Euclidean distance (normalized inputs), efficiency change	Yes	Shortest Path, Fixed cost

Notes: IBT = Intermediate Benchmark Targets, UBT = Ultimate Benchmark Targets, CRS = Constant Returns to Scale, VRS = Variable Returns to Scale, MPSS = Most Productive Scale Size, SOM = Self-Organizing Map, XE = Cross-efficiency

For each feasible operating point $(\hat{x}, \hat{y})$ with $\hat{x} > 0$ and $\hat{y} > 0$ we can assign the following strictly positive EFP

$$P(\hat{x}, \hat{y}) = \frac{\prod_{i=1}^{m} \hat{x}_i}{\prod_{k=1}^{s} \hat{y}_k} \qquad (4)$$

It is clear that the less input an operating point consumes and the more output it produces, the lower its EFP. Note also that, since the dimensionless input and output vectors of the average DMU have all components equal to one, its associated EFP is also one, i.e. $P(\hat{x}^{aver}, \hat{y}^{aver}) = P(\mathbf{1}, \mathbf{1}) = 1$. In spite of these and other interesting features that will be commented below, it must be acknowledged the above definition of the EFP is somewhat arbitrary. As it has been discussed in the literature (e.g. [14,17]) other functional forms as well as other ways of normalizing the variables and removing their dimensions can be devised, with the different alternatives having different properties as regards invariance, strong/weak monotonicity, separability, etc. Hence, although the proposed approach uses the EFP definition given in (4), in principle it may be adapted to work also with other alternative EFP specifications. The resulting efficiency improvement path for other EFP specifications would of course be different, but its main properties (i.e. strong monotonicity and efficiency achievement) can be maintained.

The Efficiency Equipotential Surfaces (EESs) correspond to

$$\frac{\prod_{i=1}^{m} \hat{x}_i}{\prod_{k=1}^{s} \hat{y}_k} = \gamma \Leftrightarrow \prod_{k=1}^{s} \hat{y}_k = \gamma^{-1} \prod_{i=1}^{m} \hat{x}_i \qquad (5)$$

More importantly, the EFP has an associated EFV given by the negative EFP gradient

$$E(\hat{x}, \hat{y}) = -\nabla P(\hat{x}, \hat{y})$$
$$= \left(-\frac{\partial P}{\partial \hat{x}_1}, -\frac{\partial P}{\partial \hat{x}_2}, ..., -\frac{\partial P}{\partial \hat{x}_m}, -\frac{\partial P}{\partial \hat{y}_1}, -\frac{\partial P}{\partial \hat{y}_2}, ..., -\frac{\partial P}{\partial \hat{y}_s} \right) \qquad (6)$$

The corresponding partial derivatives can be easily computed and, interestingly, can be expressed in terms of the EFP since

$$\frac{\partial P}{\partial \hat{x}_i} = \frac{\prod_{i' \neq i} \hat{x}_{i'}}{\prod_k \hat{y}_k} = \frac{P(\hat{x}, \hat{y})}{\hat{x}_i} \quad \forall i$$

$$\frac{\partial P}{\partial \hat{y}_k} = -\frac{1}{\hat{y}_k^2} \frac{\prod_i \hat{x}_i}{\prod_{k' \neq k} \hat{y}_{k'}} = -\frac{P(\hat{x}, \hat{y})}{\hat{y}_k} \quad \forall k \qquad (7)$$

Note that the components of the EFV

$$E(\hat{x}, \hat{y}) = \left(-\frac{P(\hat{x}, \hat{y})}{\hat{x}_1}, -\frac{P(\hat{x}, \hat{y})}{\hat{x}_2}, ..., -\frac{P(\hat{x}, \hat{y})}{\hat{x}_m}, \frac{P(\hat{x}, \hat{y})}{\hat{y}_1}, \right.$$
$$\left. \frac{P(\hat{x}, \hat{y})}{\hat{y}_2}, ..., \frac{P(\hat{x}, \hat{y})}{\hat{y}_s} \right) \qquad (8)$$

always point towards the less input/more output region. It is also clear that at each feasible operating point $(\hat{x}, \hat{y})$, $E(\hat{x}, \hat{y})$ is perpendicular to the corresponding EES. Fig. 1 shows the EFV for three special cases that can be shown in a bidimensional plot. These three cases correspond to a single input and a single output (case XY), two inputs and a constant output (case XX1) and two outputs and constant input (case 1YY).

In the XY case, the EESs are straight lines that pass through the origin. The Efficiency Field Lines (EFLs), which are tangent to the EFV (and hence perpendicular to the EES), correspond to circles centred at the origin.

In the XX1 case the EESs are rectangular hyperboles whose EP decreases as they approach the origin. The corresponding EFLs are also hyperboles which are symmetrical around the bisector line $\hat{x}_1 = \hat{x}_2$. The case 1YY is similar, with the only difference being that the EFP of the EES decreases as the hyperboles move away from the origin.

The mathematical expressions for the EFL correspond to the following differential equation, which represents moving along the direction of the negative EFP gradient

$$\left(\frac{d\hat{x}}{dt}, \frac{d\hat{y}}{dt} \right) = -\nabla P(\hat{x}, \hat{y}) \qquad (9)$$

Solving this partial differential equation with boundary condition $(\hat{x}_0, \hat{y}_0)$ leads to the following three groups of quadratic surfaces

$$\hat{x}_i^2 - \hat{x}_{i'}^2 = \hat{x}_{i0}^2 - \hat{x}_{i'0}^2 \quad \forall i \neq i'$$
$$\hat{y}_k^2 - \hat{y}_{k'}^2 = \hat{y}_{k0}^2 - \hat{y}_{k'0}^2 \quad \forall k \neq k' \qquad (10)$$
$$\hat{x}_i^2 + \hat{y}_k^2 = \hat{x}_{i0}^2 + \hat{y}_{k0}^2 \quad \forall i, k$$

Of the first group of $m \cdot (m-1)/2$ equations in (10), only $m-1$ are linearly independent. Similarly, only $s-1$ equations out of the $s \cdot (s-1)/2$ equations in the second group are linearly independent. And only 1 equation from the third group is linearly independent. This means that the total number of linearly independent equations is $m+s-1$ which means that (10) defines a one-dimensional subspace. The EFLs are curves that result from the intersection of this equation system. Thus, for example, in the XY case seen above, (10) reduces to

$$\hat{x}^2 + \hat{y}^2 = \hat{x}_0^2 + \hat{y}_0^2 \qquad (11)$$

which corresponds to the circular EFLs that can be noticed in panel a) of Fig. 1. Similarly, in the XX1 case the EFLs (10) reduce to

$$\hat{x}_1^2 - \hat{x}_2^2 = \hat{x}_{10}^2 - \hat{x}_{20}^2 \qquad (12)$$

which for $\hat{x}_{10} \neq \hat{x}_{20}$ are hyperboles and for $\hat{x}_{10} = \hat{x}_{20}$ corresponds to the bisector line $\hat{x}_1 = \hat{x}_2$. Finally, in the 1YY case, the EFLs (10) reduce to

$$\hat{y}_1^2 - \hat{y}_2^2 = \hat{y}_{10}^2 - \hat{y}_{20}^2 \qquad (13)$$

which, again, are hyperboles for $\hat{y}_{10} \neq \hat{y}_{20}$ and the bisector line $\hat{y}_1 = \hat{y}_2$ if $\hat{y}_{10} = \hat{y}_{20}$.

3. Proposed gradient-based stepwise benchmarking approach

3.1. Stepwise efficiency improvement path

As shown in the previous section, the EFV always points towards less input and more output. Following such direction leads to a strictly increasing efficiency. The trajectories defined by the EFL (10) correspond to the continuous path that results from following the EFV as per (9). However, changing the inputs in a continuous fashion is not practical and may not even be implementable. That is why we propose a stepwise benchmarking approach which is a discrete version of that continuous efficiency improvement trajectory.

Assume that a stepwise efficiency improvement path from DMU 0 is to be computed. DMU 0 will the starting point, i.e. the step $t = 0$, of efficiency improvement path $(\hat{x}^0, \hat{y}^0) = (\hat{x}_0, \hat{y}_0)$. Assuming that $\hat{x}^t > 0$ and $\hat{y}^t > 0$, the corresponding EFP and EFV $P(\hat{x}^t, \hat{y}^t)$ and $E(\hat{x}^t, \hat{y}^t)$, respectively, can easily be computed. In each iteration two DEA models are solved. The first one is labelled the Improvement Dimensions (ID) DEA model and determines the input and output dimensions that can be improved in that step. Let

Data

$(\hat{x}^t, \hat{y}^t)$ Current (i.e. step t) IBT

$E(\hat{x}^t, \hat{y}^t)$ EFV at current IBT

ε Small value stepsize along EFV from current IBT

Decision variables

λ_j linear combination variables used to compute a feasible operating point
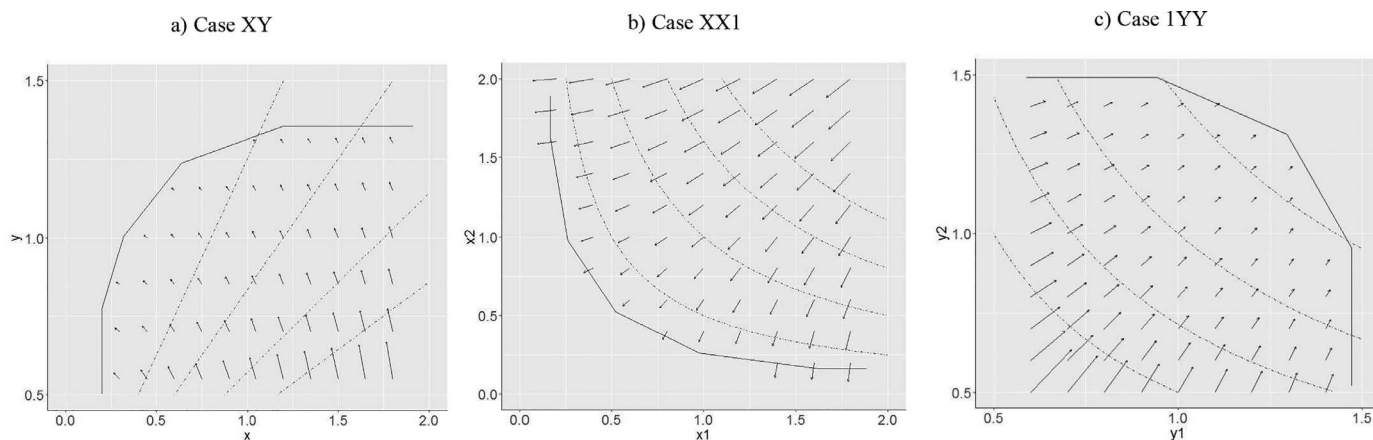
a) Case XY

b) Case XX1

c) Case 1YY

Fig. 1. Efficiency vector fields for three bidimensional cases.

u_i binary variable indicating whether input dimension i can be improved moving in the EFV direction

v_k binary variable indicating whether output dimension k can be improved moving in the EFV direction

<u>ID DEA model (iteration t)</u>

$$Max \quad \sum_{i=1}^{m} u_i + \sum_{k=1}^{s} v_k \qquad (14)$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \hat{x}_i^t + \varepsilon \cdot \frac{E_i(\hat{x}^t, \hat{y}^t)}{\|E(\hat{x}^t, \hat{y}^t)\|} \cdot u_i \quad \forall i \qquad (15)$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k^t + \varepsilon \cdot \frac{E_k(\hat{x}^t, \hat{y}^t)}{\|E(\hat{x}^t, \hat{y}^t)\|} \cdot v_k \quad \forall k \qquad (16)$$

$$\sum_{j=1}^{n} \lambda_j = 1 \qquad (17)$$

$$\lambda_j \geq 0 \quad j = 1, 2, ..., n \quad u_i \in \{0, 1\} \forall i \quad v_k \in \{0, 1\} \forall k \qquad (18)$$

This easy-to-solve Mixed Integer Linear Programming (MILP) model identifies all the input and output dimensions that can be improved in step t along the EFV direction. Note that the above model is feasible. Thus, the current IBT $(\hat{x}^t, \hat{y}^t)$ corresponds to a feasible solution with $u_i = 0 \forall i \quad v_k = 0 \forall k$. The model is also bounded as it is the sum of a finite number of binary variables.

As regards parameter ε, its purpose is to detect whether we have reached the efficient frontier and, if not, which input and output dimensions can be improved. The stepwise benchmarking path computed does not depend on the exact value chosen, provided it is small enough. In principle, assuming that the current intermediate benchmark target (IBT) is not too close to the weak efficiency frontier any small positive value amount $\varepsilon > 0$ can be used. Only if the IBT is very close to the weak efficiency frontier, i.e. one or more inputs or amounts may improve but just by a very small amount, precaution has to be taken to choose ε small enough to detect that improvement in those dimensions is possible. Using a value equal to the precision level of the most precise of the input and output variables should be safe as, in that case, even if the possibility of improving a certain input or output dimension would go undetected, the magnitude of such improvement would be negligible and, for all practical purposes, the weak efficiency frontier had been reached. For example, if the border of the PPS along a certain output dimension corresponds to a value of 10.0

and the current IBT has reached a value very close to that, let us say 9.9, then the question is how to be able to detect that there is still a small margin to continue improving that output. Let us assume that the precision with which that variable is measured (in the observed data) is 0.1, i.e. one decimal position. If $\varepsilon > 0.1$ then the model would not detect that this output dimension can be improved because there is no feasible operating point with an output equal to $9.9 + \varepsilon > 10.0$. Hence it should be $\varepsilon \leq 0.1$. If, instead, the output precision of the variable were two decimal positions and the current IBT had an output value of 9.92 then $\varepsilon = 0.1$ would not detect that there is still some small margin for improvement as $9.92 + \varepsilon > 10.00$ would get us out of the PPS. In this second case it would have to be $\varepsilon \leq 0.01$ if we want to detect potential improvements of that size. In the above reasoning we have not taken into account the fact that in the ID DEA model (14)–(18) ε multiplies the size of the normalized gradient component, which is lower than one, which means that a value of ε somewhat larger than the precision value could be used, but it is better to be on the safe side and that is why we suggest ε equal to the precision level. Note that, in any case, the value of ε can be an issue only if the IBT is very close to the border of the PPS, something which occurs infrequently. Much more frequent is that the IBT exactly reaches the border of the PPS in a given step.

Given the optimal solution of the above model, the improvement dimensions sets I_t^- and O_t^+ can be determined as follows

$$I_t^- = \{i : u_i = 1\} \qquad O_t^+ = \{k : v_k = 1\} \qquad (19)$$

If $I_t^- = O_t^+ = \emptyset$, then $(\hat{x}^t, \hat{y}^t)$ is technically efficient and therefore there is no need to solve the corresponding gradient stepsize (GSS) DEA model. Otherwise, we would solve a GSS DEA model that computes the next IBT along the negative gradient direction. To control the stepsize, the unit vector in the direction pointed by the EFV is used. The model follows the negative EFP gradient direction and, if advancing the given stepsize Δ is feasible, then that would be the next IBT. Otherwise, the model advances as much as possible. Let

Data

I_t^- Input dimensions that can be improved in step t
O_t^+ Output dimensions that can be improved in step t
$(\hat{x}^t, \hat{y}^t)$ Current (i.e. step t) IBT
$E(\hat{x}^t, \hat{y}^t)$ EFV at current IBT
Δ Desired stepsize value

Decision variables

λ_j linear combination variables used to compute a feasible operating point
$(\hat{x}^{t+1}, \hat{y}^{t+1})$ Next (i.e. step t + 1) IBT

δ Computed stepsize value

δ^-, δ^+ Negative and positive deviations, respectively, from the desired stepsize value

GSS DEA model (iteration t)

$$Min \quad \delta^- + \delta^+ \tag{20}$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \hat{x}_i^{t+1} = \hat{x}_i^t + \delta \cdot \frac{E_i(\hat{x}^t, \hat{y}^t)}{\|E(\hat{x}^t, \hat{y}^t)\|} \quad \forall i \in I_t^- \tag{21}$$

$$\sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \hat{x}_i^{t+1} = \hat{x}_i^t \quad \forall i \notin I_t^- \tag{22}$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k^{t+1} = \hat{y}_k^t + \delta \cdot \frac{E_k(\hat{x}^t, \hat{y}^t)}{\|E(\hat{x}^t, \hat{y}^t)\|} \quad \forall k \in O_t^+ \tag{23}$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k^{t+1} = \hat{y}_k^t \quad \forall k \notin O_t^+ \tag{24}$$

$$\sum_{j=1}^{n} \lambda_j = 1 \tag{25}$$

$$\delta + \delta^- - \delta^+ = \Delta \tag{26}$$

$$\lambda_j \geq 0 \quad j = 1, 2, \ldots, n \quad \delta \geq 0 \quad \delta^- \geq 0 \quad \delta^+ \geq 0 \tag{27}$$

This linear programming (LP) model moves along the direction of the unit EFV $\frac{E(\hat{x}^t, \hat{y}^t)}{\|E(\hat{x}^t, \hat{y}^t)\|}$ (ignoring those components corresponding to input and output dimensions that cannot be improved if moving in that direction) from the current IBT $(\hat{x}^t, \hat{y}^t)$. In principle, the model tries to move the given stepsize Δ. However, if that is not feasible then the next IBT $(\hat{x}^{t+1}, \hat{y}^{t+1})$ corresponds to moving as much as possible along the EFV direction until any of the dimensions in the sets I_t^- and O_t^+ cannot improve more. Note that this is a simple Goal Programming model where the goal is the desired stepsize Δ and δ^- and δ^+ are the negative and positive deviations respectively. The achievement function (20) penalizes both types of deviation. Note also that the above model is always feasible and bounded. This can be seen taking into account that the current IBT $(\hat{x}^t, \hat{y}^t)$ is a feasible solution with an associated objective function value of Δ (corresponding to a value of $\delta = \delta^+ = 0$ and $\delta^- = \Delta$). Therefore, if giving the whole Δ step size keeps us within the PPS then it is feasible. Otherwise some input and/or output dimensions reach their PPS limit and no further improvement in those dimensions can be pursued. In other words, some input and/or output dimensions may limit the advancement along the negative gradient direction and prevent the step size from reaching its desired value Δ, thus determining the maximum feasible step size.

If the next IBT $(\hat{x}^{t+1}, \hat{y}^{t+1})$ has been computed, a new iteration (step $t+1$) is carried out. If the new IBT is technically efficient, the ID DEA model will not find any feasible improvement dimensions and that IBT is the UBT. Conversely, if the new IBT is not technically efficient, then it is still possible to continue improving efficiency along certain input and output dimensions that the corresponding ID DEA model will determine.

Note that there does not exist an optimal value for the Δ parameter. That parameter controls the stepsize, i.e. the amount of input and output changes in each step of the stepwise efficiency improvement program. That depends on the DMU, and specifically on how fast it wishes to achieve efficiency. A high value of the Δ parameter allows larger input and output changes in each

step, which means that the efficient frontier can be reached in few steps. On the contrary, a small value of the Δ parameter means that smaller input and output changes are allowed in each step and therefore more steps will be required to achieve efficiency.

It should be emphasized that the computed stepwise benchmarking path and, hence the UBT, depends on the value of Δ chosen. However, we see this parameter dependence more as a plus than as a con. Thus, the proposed approach has the flexibility/degree of freedom to allow the analyst, together with the DMU, try several values of the Δ parameter and select the efficiency improvement path that best suits its interests and its capacities. In other words, when designing a stepwise benchmarking approach, having the possibility of choosing among different UBT may be advantageous, provided they are all efficient. In any case, it is more important to correctly determine the amount of change, i.e. the effort and likelihood of success, involved in each step of the efficiency improvement path than the exact final UBT to aim for.

A possibility, kindly indicated by one reviewer, consists in using a different value of Δ for each DMU. Also, the value of Δ could be modified (e.g. reduced) from one step to the next, which can be justified as at first large efficiency improvements may be easier to carry out while at latter stages additional improvements may be harder to achieve.

Finally, note that the proposed efficiency improvement path follows the direction of the gradient (actually the negative gradient) and hence always points to reducing inputs and increasing outputs. This means that the UBT of a given DMU 0 will always dominate it. The minimum EFP (MEFP) model formulated below computes the feasible operating point with minimum EFP. That operating point does not generally dominate a given DMU 0. In other words, although the UBT of the efficiency improvement path has lower EFP than the starting DMU 0, it is not normally a MEFP operating point. Moving to such MEFP operating point once the UBT has been reached can be conceived (as a continuation of the proposed efficiency improvement path) but then, unlike what occurs along the efficiency improvement path, in this case the changes required will necessarily involve either input increases or output reductions. This is because we are talking about moving from one Pareto efficient point to another. In any case, the final efficient, minimum EFP target can be computed by solving the following non-linear optimization model

MEFP DEA model

$$Min \quad \sum_{i=1}^{m} \log(\hat{x}_i) - \sum_{k=1}^{s} \log(\hat{y}_k) \tag{28}$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \hat{x}_i \quad \forall i \in I \tag{29}$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k \quad \forall k \in O \tag{30}$$

$$\sum_{j=1}^{n} \lambda_j = 1 \tag{31}$$

$$\lambda_j \geq 0 \quad j = 1, 2, \ldots, n \tag{32}$$

An LP version of the above model cannot be formulated as long as the non-linear definition of the EFP given in (4) is considered. Although this is a drawback, note that this MEFP DEA model plays a minor role in the proposed path approach. Moreover, it is possible to use an additive-type definition for the EFP which would render the corresponding MEFP DEA model a linear program. In any case, the MEFP model is not actually used in the computation

Table 2
Two output/constant input dataset from Cooper et al. [[16], Chapter 1].

DMU j	A	B	C	D	E	F	G
x_j	1	1	1	1	1	1	1
y_{1j}	1	2	3	4	4	5	6
y_{2j}	5	7	4	3	6	5	2
$\hat{x}_j$	1	1	1	1	1	1	1
$\hat{y}_{1j}$	0.2800	0.5600	0.8400	1.1200	1.1200	1.4000	1.6800
$\hat{y}_{2j}$	1.0938	1.5313	0.8750	0.6563	1.3125	1.0938	0.4375
$P(\hat{x}_j, \hat{y}_{1j}, \hat{y}_{2j})$	3.2653	1.1662	1.3605	1.3605	0.6803	0.6531	1.3605

Table 3
Gradient-based stepwise efficiency improvement path for DMU A ($\Delta = 0.1$).

Step t	$\hat{x}^t$	$\hat{y}_1^t$	$\hat{y}_2^t$	I_t^-	O_t^+	$P(\hat{x}^t, \hat{y}_1^t, \hat{y}_2^t)$	$\frac{\partial P}{\partial \hat{x}}$	$\frac{\partial P}{\partial \hat{y}_1}$	$\frac{\partial P}{\partial \hat{y}_2}$	θ^{BCC-O}
0	1	0.2800	1.0938	∅	{1, 2}	3.2653	3.2653	−11.6618	−2.9854	0.714
1	1	0.3735	1.1177	∅	{1, 2}	2.395	2.3955	−6.4137	−2.1433	0.730
2	1	0.4629	1.1476	∅	{1, 2}	1.8825	1.8825	−4.0668	−1.6405	0.759
3	1	0.5481	1.1819	∅	{1, 2}	1.5436	1.5436	−2.8162	−1.3060	0.798
4	1	0.6293	1.2196	∅	{1, 2}	1.3028	1.3028	−2.0701	−1.0682	0.837
5	1	0.7069	1.2596	∅	{1, 2}	1.1230	1.1230	−1.5887	−0.8916	0.878
6	1	0.7811	1.3013	∅	{1, 2}	0.9838	0.9838	−1.2594	−0.7560	0.918
7	1	0.8524	1.3441	∅	{1, 2}	0.8729	0.8729	−1.0240	−0.6494	0.958
8	1	0.9209	1.3875	∅	{1, 2}	0.7826	0.7826	−0.8498	−0.5640	0.998
9	1	0.9235	1.3892	∅	∅	0.7794	0.7794	−0.8440	−0.5610	1.000

Table 4
Gradient-based stepwise efficiency improvement path for DMU A ($\Delta = 0.2$).

Step t	$\hat{x}^t$	$\hat{y}_1^t$	$\hat{y}_2^t$	I_t^-	O_t^+	$P(\hat{x}^t, \hat{y}_1^t, \hat{y}_2^t)$	$\frac{\partial P}{\partial \hat{x}}$	$\frac{\partial P}{\partial \hat{y}_1}$	$\frac{\partial P}{\partial \hat{y}_2}$	θ^{BCC-O}
0	1	0.2800	1.0938	∅	{1, 2}	3.2653	3.2653	−11.6618	−2.9854	0.714
1	1	0.4670	1.1416	∅	{1, 2}	1.8757	1.8757	−4.0166	−1.6430	0.757
2	1	0.6369	1.2111	∅	{1, 2}	1.2964	1.2964	−2.0354	−1.0704	0.834
3	1	0.7911	1.2922	∅	{1, 2}	0.9782	0.9782	−1.2365	−0.7570	0.915
4	1	0.9325	1.3788	∅	{1, 2}	0.7778	0.7778	−0.8341	−0.5641	0.996
5	1	0.9390	1.3832	∅	∅	0.7699	0.7699	−0.8199	−0.5566	1.000

of the stepwise efficiency improvement path. It has been formulated so that the corresponding minimum efficiency potential can be used as a reference value for the efficiency potentials of the observed DMUs and of the computed IBT. What is important is that, of the two main models used in the iterative process for computing the stepwise benchmarking path, the ID DEA model (14)–(18) is MILP and the GSS DEA model (20)–(27) is LP, both of which can be easily solved using any common optimization package (e.g. LINGO or CPLEX).

3.2. Illustrative example

In this section the seven DMUs, two outputs/constant input dataset in Cooper et al. [[16], Chapter 1] shown in Table 2, is considered. DMUs B, E, F and G are efficient. Table 2 also shows the corresponding dimensionless input and output vectors (computed using $x^{aver} = 1$, $y_1^{aver} = 3.5714$ and $y_2^{aver} = 4.5714$) as well as the associated efficiency potentials.

Table 3 shows the ten-step efficiency improvement path for DMU A computed using the iterative approach described in Section 3.1 using stepsizes $\varepsilon = 0.0001$ and $\Delta = 0.1$. For each step the current operating point $(\hat{x}^t, \hat{y}^t)$, the improvement dimensions I_t^- and O_t^+, the EFP $P(\hat{x}^t, \hat{y}^t)$ and its gradient, are shown. Also, for reference, the radial output efficiency score of each operating point is shown. Table 4 shows the alternative shorter (five-step) efficiency improvement path computed for the same DMU but using a stepsize $\Delta = 0.2$. Note that every step moving along the EFV direction leads to increasing both outputs while the input stays constant. Hence, the efficiency potential decreases in each step and the radial efficiency θ^{BCC-O} increases. The end of the efficiency improvement paths is found when $I_t^- = O_t^+ = \emptyset$ which indicates a

technically efficient operating point. As expected, in the case of $\Delta = 0.2$, the improvements are larger than in the case $\Delta = 0.1$. Also, rather interestingly, the UBT is different in both cases, with the $\Delta = 0.2$ UBT having a slightly lower potential (0.7699 versus 0.7794).

Table 5 shows the efficiency improvement paths for the three inefficient DMUs (using $\Delta = 0.2$) expressed in the original units of measurement. Fig. 2 shows these stepwise efficiency improvement paths. In all cases, the two outputs and the radial θ^{BCC-O} efficiency increase monotonously and the final UBT lies on the technical efficiency frontier. Note that for DMUs C and D it only takes six steps to reach the EF. None of these UBTs, however, corresponds to the MEFP operating point. For this dataset, the MEFP operating point corresponds to DMU F, which is the feasible operating point at which the efficiency equipotential surface of minimum potential is tangent to the PPS.

4. Extensions of the proposed approach

In this section some possible extensions of the proposed approach are presented. Although, in order to simplify exposition they are treated independently, the corresponding features can be present simultaneously.

4.1. Preference structure

If there is a preference structure or value judgment on the importance of the different inputs and outputs, this can be taken into account in the proposed approach. Thus, let α_i^x and α_k^y be the weights that reflect such relative importance and let us assume that $\sum_{i \in I} \alpha_i^x + \sum_{k \in O} \alpha_k^y = 1$. In this case the EFP can be defined as

Table 5
Efficiency improvement path for DMUs A, C and D ($\Delta = 0.2$) in original units of measurement.

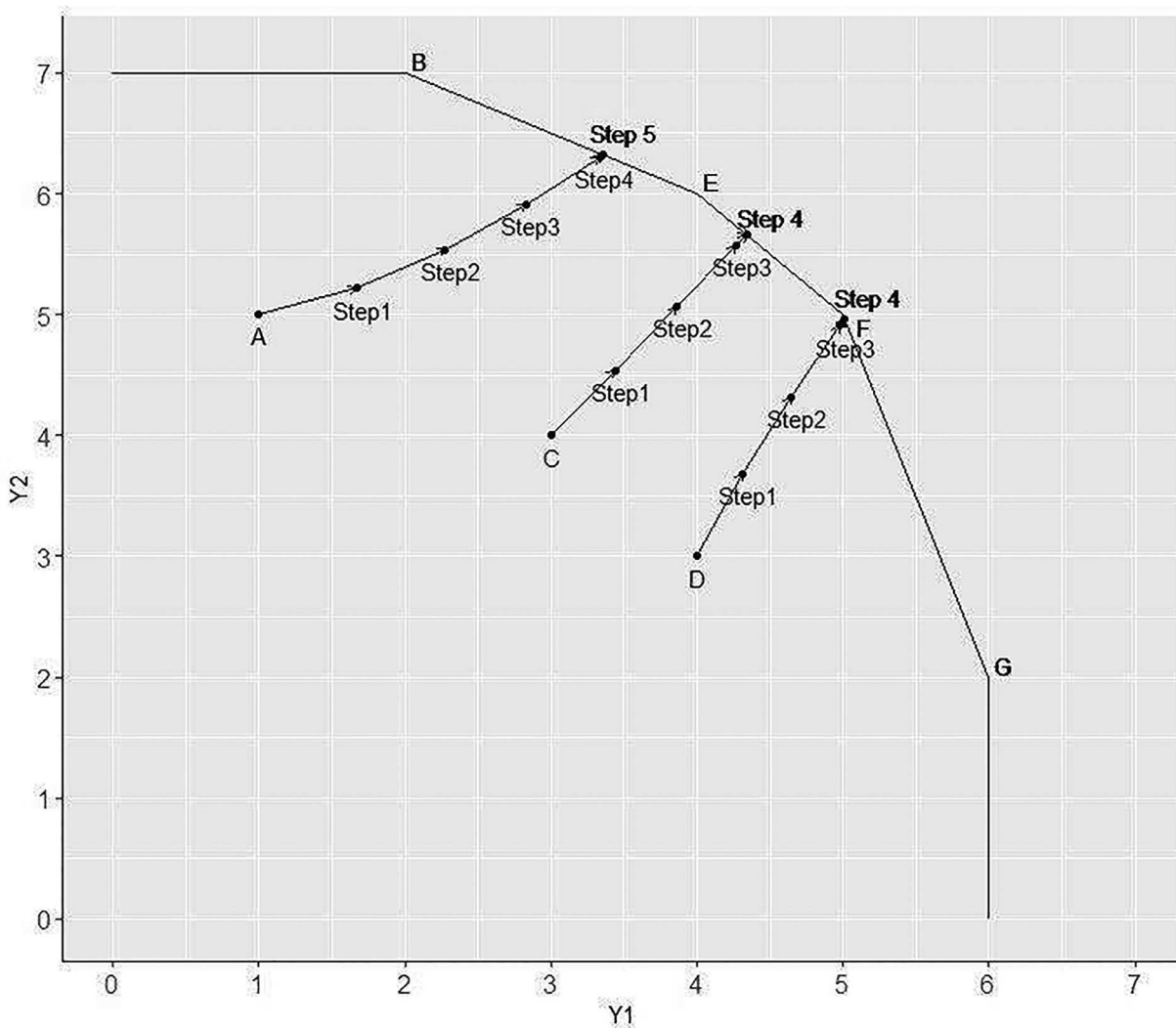|	DMU A				DMU C				DMU D			
Step t	x^t	y_1^t	y_2^t	θ^{BCC-O}	x^t	y_1^t	y_2^t	θ^{BCC-O}	x^t	y_1^t	y_2^t	θ^{BCC-O}
0	1	1	5	0.714	1	3	4	0.700	1	4	3	0.750
1	1	1.6678	5.2188	0.757	1	3.4407	4.5415	0.798	1	4.3142	3.6864	0.831
2	1	2.2747	5.5366	0.834	1	3.8624	5.0650	0.893	1	4.6436	4.3180	0.912
3	1	2.8254	5.9073	0.915	1	4.2667	5.5700	0.984	1	4.9772	4.9057	0.992
4	1	3.3304	6.3030	0.996	1	4.3391	5.6609	1.000	1	5.0121	4.9638	1.000
5	1	3.3537	6.3232	1.000	–	–	–	–	–	–	–	–

Fig. 2. Gradient-based stepwise efficiency improvement paths for DMUs A, C and D ($\Delta = 0.2$).

$$P(\hat{x}, \hat{y}) = \frac{\prod_{i=1}^{m}(\hat{x}_i)^{\alpha_i^x}}{\prod_{k=1}^{s}(\hat{y}_k)^{\alpha_k^y}} \qquad (33)$$

whose gradient is

$$\frac{\partial P}{\partial \hat{x}_i} = \frac{\alpha_i^x \cdot (\hat{x}_i)^{\alpha_i^x - 1} \cdot \prod_{i' \neq i}(\hat{x}_{i'})^{\alpha_{i'}^x}}{\prod_{k \in O}(\hat{y}_k)^{\alpha_k^y}} = \frac{\alpha_i^x \cdot P(\hat{x}, \hat{y})}{\hat{x}_i} \quad \forall i \qquad (34)$$

$$\frac{\partial P}{\partial \hat{y}_k} = -\frac{\alpha_k^y \cdot (\hat{y}_k)^{-\alpha_k^y - 1} \cdot \prod_{i \in I}(\hat{x}_i)^{\alpha_i^x}}{\prod_{k' \neq k}(\hat{y}_{k'})^{\alpha_{k'}^y}} = -\frac{\alpha_k^y \cdot P(\hat{x}, \hat{y})}{\hat{y}_k} \quad \forall k \qquad (35)$$

Compared with (7) it can be seen that the basic change is that the components of the EFP gradient have to be multiplied by the preference weights α_i^x and α_k^y. This has to be taken into account in constraints (21) and (23) of the GSS DEA model. The ID DEA model is unaffected, while the objective function of MEFP DEA model also changes to

$$Min \quad \sum_{i=1}^{m} \alpha_i^x \cdot \log(\hat{x}_i) - \sum_{k=1}^{s} \alpha_k^y \cdot \log(\hat{y}_k) \qquad (36)$$

4.2. Non-discretionary variables

Let I^{ND} and O^{ND} be the sets of non-discretionary inputs and outputs, respectively. Let $I^D = I \backslash I^{ND}$ and $O^D = O \backslash O^{ND}$ and $m' = |I^D|$ and

$s' = |O^D|$ be the number of discretionary inputs and discretionary outputs, respectively.

The ID DEA model changes slightly as the non-discretionary dimensions are not considered as candidates for improvement. Hence

Modified ID DEA model (iteration t)

$$Max \quad \sum_{i \in I^D} u_i + \sum_{k \in O^D} v_k \tag{37}$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \hat{x}_i^t + \varepsilon \cdot \frac{E_i(\hat{x}^t, \hat{y}^t)}{\|E(\hat{x}^t, \hat{y}^t)\|} \cdot u_i \quad \forall i \in I^D \tag{38}$$

$$\sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \hat{x}_i^t \quad \forall i \in I^{ND} \tag{39}$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k^t + \varepsilon \cdot \frac{E_k(\hat{x}^t, \hat{y}^t)}{\|E(\hat{x}^t, \hat{y}^t)\|} \cdot v_k \quad \forall k \in O^D \tag{40}$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k^t \quad \forall k \in O^{ND} \tag{41}$$

$$\sum_{j=1}^{n} \lambda_j = 1 \tag{42}$$

$$\lambda_j \geq 0 \quad j = 1, 2, ..., n \quad u_i \in \{0, 1\} \, \forall i \in I^D \quad v_k \in \{0, 1\} \, \forall k \in O^D \tag{43}$$

The improvement dimensions sets are now defined as

$$I_t^- = \{i \in I^D : u_i = 1\} \qquad O_t^+ = \{k \in O^D : v_k = 1\} \tag{44}$$

With respect to the GSS DEA model, there is no change in the formulation. The MEFP DEA model also requires some minor changes as shown below. In particular, since the MEFP operating target of a DMU is constrained by the corresponding values of the non-discretionary inputs and outputs, that MEFP depends on the DMU being projected.

Modified MEFP DEA model

$$Min \quad \sum_{i \in I^D} \log(\hat{x}_i) - \sum_{k \in O^D} \log(\hat{y}_k)$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \hat{x}_i \quad \forall i \in I^D$$

$$\sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \hat{x}_{i0} \quad \forall i \in I^{ND}$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k \quad \forall k \in O^D \tag{45}$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_{k0} \quad \forall k \in O^{ND}$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0 \quad j = 1, 2, ..., n$$

4.3. Undesirable outputs

Let B be the set of undesirable outputs and $p = |B|$ the number of undesirable outputs. Let z_{bj} be the amount of undesirable output

b produced by DMU j and $\hat{z}_{bj} = \frac{z_{bj}}{z_b^{aver}}$ the corresponding dimensionless value. The EFP can then be defined as

$$P(\hat{x}, \hat{y}, \hat{z}) = \frac{\prod_{i=1}^{m} \hat{x}_i \cdot \prod_{b=1}^{p} \hat{z}_b}{\prod_{k=1}^{s} \hat{y}_k} \tag{46}$$

which means that the components of the EFV are

$$E(\hat{x}, \hat{y}, \hat{z}) = -\nabla P(\hat{x}, \hat{y}, \hat{z})$$

$$= \left(-\frac{P(\hat{x}, \hat{y}, \hat{z})}{\hat{x}_1}, ..., -\frac{P(\hat{x}, \hat{y}, \hat{z})}{\hat{x}_m}, \frac{P(\hat{x}, \hat{y}, \hat{z})}{\hat{y}_1}, ..., \frac{P(\hat{x}, \hat{y}, \hat{z})}{\hat{y}_s}, \right.$$

$$\left. -\frac{P(\hat{x}, \hat{y}, \hat{z})}{\hat{z}_1}, ..., -\frac{P(\hat{x}, \hat{y}, \hat{z})}{\hat{z}_p} \right) \tag{47}$$

Assuming weak disposability of the undesirable outputs and using the approach in Kuosmanen [26], the corresponding modified models are

Modified ID DEA model (iteration t)

$$Max \quad \sum_{i=1}^{m} u_i + \sum_{k=1}^{s} v_k + \sum_{b=1}^{p} w_b$$

s.t.

$$\sum_{j=1}^{n} (\lambda_j + \mu_j) \cdot \hat{x}_{ij} \leq \hat{x}_i^t + \varepsilon \cdot \frac{E_i(\hat{x}^t, \hat{y}^t, \hat{z}^t)}{\|E(\hat{x}^t, \hat{y}^t, \hat{z}^t)\|} \cdot u_i \quad \forall i$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k^t + \varepsilon \cdot \frac{E_k(\hat{x}^t, \hat{y}^t, \hat{z}^t)}{\|E(\hat{x}^t, \hat{y}^t, \hat{z}^t)\|} \cdot v_k \quad \forall k$$

$$\sum_{j=1}^{n} \lambda_{jr} \hat{z}_{bj} = \hat{z}_b^t + \varepsilon \cdot \frac{E_k(\hat{x}^t, \hat{y}^t, \hat{z}^t)}{\|E(\hat{x}^t, \hat{y}^t, \hat{z}^t)\|} \cdot w_b \quad \forall b$$

$$\sum_{j=1}^{n} (\lambda_j + \mu_j) = 1$$

$$\lambda_j \geq 0 \quad \mu_j \geq 0 \quad \forall j \quad u_i \in \{0, 1\} \, \forall i \quad v_k \in \{0, 1\} \, \forall k \quad w_b \in \{0, 1\} \, \forall b \tag{48}$$

$$I_t^- = \{i : u_i = 1\} \qquad O_t^+ = \{k : v_k = 1\} \qquad B_t^- = \{b : w_b = 1\} \tag{49}$$

Modified GSS DEA model (iteration t)

$$Min \quad \delta^- + \delta^+$$

s.t.

$$\sum_{j=1}^{n} (\lambda_j + \mu_j) \cdot \hat{x}_{ij} \leq \hat{x}_i^{t+1} = \hat{x}_i^t + \delta \cdot \frac{E_i(\hat{x}^t, \hat{y}^t, \hat{z}^t)}{\|E(\hat{x}^t, \hat{y}^t, \hat{z}^t)\|} \quad \forall i \in I_t^-$$

$$\sum_{j=1}^{n} (\lambda_j + \mu_j) \cdot \hat{x}_{ij} \leq \hat{x}_i^{t+1} = \hat{x}_i^t \quad \forall i \notin I_t^-$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k^{t+1} = \hat{y}_k^t + \delta \cdot \frac{E_k(\hat{x}^t, \hat{y}^t, \hat{z}^t)}{\|E(\hat{x}^t, \hat{y}^t, \hat{z}^t)\|} \quad \forall k \in O_t^+$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k^{t+1} = \hat{y}_k^t \quad \forall k \notin O_t^+$$

$$\sum_{j=1}^{n} \lambda_j \hat{z}_{bj} = \hat{z}_b^{t+1} = \hat{z}_b^t + \delta \cdot \frac{E_b(\hat{x}^t, \hat{y}^t, \hat{z}^t)}{\|E(\hat{x}^t, \hat{y}^t, \hat{z}^t)\|} \quad \forall b \in B_t^-$$

$$\sum_{j=1}^{n} \lambda_j \hat{z}_{bj} = \hat{z}_b^{t+1} = \hat{z}_b^t \quad \forall b \notin B_t^-$$

$$\sum_{j=1}^{n} (\lambda_j + \mu_j) = 1$$

Table 6
Efficient DMUs and average DMU of organic farming dataset.

DMU	Fuel consumption	Total C input	Total N input	Yield fresh matter	NPP	CO_2 emissions	$P(\hat{x}^t, \hat{y}^t, \hat{z}^t)$
O1	39.30	472.18	46.40	1880.0	3109.96	414.36	0.8000
O5	49.91	118.84	25.87	1500.0	2657.88	497.30	0.2509
O6	48.86	104.24	23.34	1200.0	2300.97	473.18	0.2671
O10	55.65	384.70	40.60	3000.0	4442.42	494.81	0.4231
O11	56.52	1297.82	94.23	3000.0	4442.42	88.18	0.5997
O12	55.49	228.24	32.23	2500.0	3847.57	523.94	0.2915
O15	43.14	141.08	45.24	1800.0	3201.68	701.83	0.4396
O17	43.52	209.54	29.90	2000.0	3460.38	480.56	0.2483
O20	55.85	469.99	43.26	2600.0	3541.75	348.11	0.5628
O21	42.67	372.38	50.23	2800.0	3747.01	384.58	0.3835
O22	48.91	123.70	25.55	1600.0	2776.85	557.43	0.2543
O25	33.90	1252.08	57.13	1500.0	2813.62	50.36	0.3794
Aver. DMU	48.00	514.21	45.10	1853.3	3047.20	386.96	1.0000

$$\delta + \delta^- - \delta^+ = \Delta$$
$$\lambda_j \geq 0 \quad \mu_j \geq 0 \quad j = 1, 2, \ldots, n \quad \delta \geq 0 \quad \delta^- \geq 0 \quad \delta^+ \geq 0 \qquad (50)$$

Modified MEFP DEA model

$$Min \quad \sum_{i=1}^{m} \log\left(\hat{x}_i\right) - \sum_{k=1}^{s} \log\left(\hat{y}_k\right) + \sum_{b=1}^{p} \log\left(\hat{z}_b\right)$$

$$s.t.$$

$$\sum_{j=1}^{n} \left(\lambda_j + \mu_j\right) \cdot \hat{x}_{ij} \leq \hat{x}_i \quad \forall i \in I$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{kj} \geq \hat{y}_k \quad \forall k \in O \qquad (51)$$

$$\sum_{j=1}^{n} \lambda_j \hat{z}_{bj} = \hat{z}_b \quad \forall b \in B$$

$$\sum_{j=1}^{n} \left(\lambda_j + \mu_j\right) = 1$$

$$\lambda_j \geq 0 \quad \mu_j \geq 0 \quad j = 1, 2, \ldots, n$$

5. Application to organic farming benchmarking

The proposed approach has been applied to a dataset consisting of 26 organic farms in Southern Spain. The unit of analysis is 1 ha. of land of each of the 26 farms. Using a slacks-based inefficiency measure (SBI), an efficiency assessment of the DMUs has been carried out in Gutiérrez et al. [23], which found that 12 DMUs were technically efficient. In this section we will compute, for each inefficient DMU, a stepwise benchmarking path using the proposed gradient-based approach. Note that the dataset has three inputs, namely Fuel consumption (in l/ha), Total Carbon input (in kg C/ha) and Total Nitrogen input (in kg N/ha), of which the Total C input is considered non-discretionary. There are also three outputs, namely Yield fresh matter (in Mg/ha), Net Primary Production (excl. fresh matter yield) (NPP) (in Mg dry matter/ha) and CO_2 emissions (in kg CO_2eq/ha), the latter being an undesirable output.

Table 6 shows the inputs and outputs of the 12 efficient DMUs. In addition to these observed DMUs, the virtual average DMU is shown. For each operating point, its EFP is shown in the last column. Note that, since the efficient DMUs occupy different positions in the input/output space, their associated EFP values also differ, although they are low in general (often much lower than the EFP value of the average DMU, which, by definition, is equal to unity). This is so because, on the one hand, EFP is lower the lower the input consumption, the lower the undesirable output production and the higher the desirable output production and, on the other hand, efficient DMUs tend to consume fewer input, produce more desirable output and less undesirable output. Hence, their EFP is

expected to be low. Their EFP might be possibly reduced a bit but not through technical efficiency improvement, as they are already technical efficient. In other words, their outputs can be increased but only if their inputs consumption and CO_2 emissions also increase. Similarly, their input consumption and or their CO_2 emissions can be decreased, but only if their outputs also decrease.

Table 7 shows, the inputs and outputs for each of the 14 inefficient DMUs, as well as those of the UBT of the proposed stepwise efficiency improvement path computed using $\varepsilon = 0.0001$ and $\Delta = 0.2$. The number of steps required, the EFP of both operating points and their efficiency score computed using the SBI DEA model of Gutiérrez et al. [23] are also shown. Note that the number of steps of the efficiency improvement paths varies from 3 to 9, and depends on how far the initial DMU is from the EF. Although, as indicated above, the same value of Δ has been used for all DMUs, that is not compulsory. The value of Δ used can vary from one DMU to another so that the computation of the stepwise benchmarking path can be adapted to the circumstances and wishes of each DMU. In particular, the amount of input and output changes a DMU may be willing to implement may depend on their inefficiency level and on the speed with which it wishes to reach technical efficiency.

Note that in all cases the UBT of the efficiency improvement paths as well as the MEFP target are technical efficient. The UBT has a lower potential than the initial DMU, although higher than the MEFP target. Note also that the change in the inputs, outputs and undesirable output along the efficiency improvement path is monotonous, while moving from the UBT to the MEFP generally involves increasing some inputs (and possible also some outputs) or decreasing some outputs (generally decreasing the inputs and the undesirable output as well). The changes from the UBT to the MEFP may also involve input substitution effects (i.e. increasing some inputs while reducing others) or some output substitution (i.e. increasing some outputs at the expense of others). In the absence of information about input and output prices, the advantages of such changes cannot be assessed. That is unlike the changes computed by the efficiency improvement path, which never lead to increasing the inputs or the undesirable output nor to decreasing the desirable outputs and thus are always guaranteed not only to increase efficiency but also to increase profit.

Fig. 3 shows the SBI score of the DMUs versus their corresponding EFP. The efficient DMUs have SBI equal to zero and therefore lie on the horizontal axis. As mentioned before, these efficient DMUs have different EFP values but, in general, they are lower than those of the inefficient DMUs. For the latter, there is a positive correlation between their SBI and EFP values.

Panel a) of Fig. 4 shows the amount of EFP and SBI reduction from the observed DMU to the UBT of its efficiency improvement path. Recall that those UBT are efficient (i.e. have SBI equal zero)

Table 7
Summary of stepwise efficiency improvement path of the inefficient DMUs.

|	Fuel consump.	Total C input	Total N input	Yield	NPP	CO_2 emissions	$P(\hat{x}^t, \hat{y}^t, \hat{z}^t)$	SBI score
DMU O2	52.48	36.96	307.35	2000.00	3252.72	488.50	0.587	0.117
UBT (Step 3)	48.00	31.96	307.35	2151.12	3503.88	461.00	0.378	0.000
MEFP O2	50.29	35.87	307.35	2558.40	4008.75	488.52	0.346	0.000
DMU O3	49.66	38.98	324.24	1400.00	2538.91	457.68	1.060	0.281
UBT (Step 5)	43.00	31.37	324.24	1761.77	3059.96	406.62	0.432	0.000
MEFP O3	42.92	44.22	324.24	2563.50	3662.27	412.95	0.355	0.000
DMU O4	47.75	48.31	526.26	1200.00	2104.91	271.49	1.711	0.289
UBT (Step 4)	42.42	43.67	526.26	1706.23	2556.93	208.61	0.611	0.000
MEFP O4	41.14	51.44	526.26	2572.60	3583.74	326.12	0.516	0.000
DMU O7	44.69	40.64	373.53	1000.00	2063.03	462.66	1.995	0.471
UBT (Step 7)	41.40	37.44	373.53	1734.45	2968.36	381.77	0.563	0.000
MEFP O7	42.66	50.24	373.53	2798.30	3745.79	384.14	0.385	0.000
DMU O8	45.20	46.60	485.83	2000.00	3252.72	444.70	0.917	0.127
UBT (Step 9)	41.60	43.35	485.83	2260.84	3462.53	364.51	0.535	0.000
MEFP O8	41.54	51.12	485.83	2632.35	3626.64	341.48	0.484	0.000
DMU O9	46.73	45.90	472.98	1880.00	3109.96	456.10	1.038	0.166
UBT (Step 8)	41.68	41.37	472.98	2188.84	3442.79	375.41	0.533	0.000
MEFP O9	41.67	51.02	472.98	2651.34	3640.27	346.36	0.473	0.000
DMU O13	51.07	36.26	318.49	2000.00	3460.38	498.89	0.557	0.098
UBT (Step 3)	46.44	32.34	318.49	2210.26	3648.11	477.89	0.372	0.000
MEFP O13	51.06	36.56	318.49	2622.00	4071.21	489.42	0.357	0.000
DMU O14	47.27	47.88	516.38	1000.00	1899.65	291.75	2.353	0.420
UBT (Step 5)	40.84	42.31	516.38	1563.25	2477.07	222.18	0.671	0.000
MEFP O14	41.23	51.36	516.38	2587.20	3594.22	329.87	0.509	0.000
DMU O16	45.20	46.18	500.09	2000.00	3460.38	454.86	0.899	0.114
UBT (Step 5)	42.76	41.89	500.09	2400.77	3588.81	409.74	0.558	0.000
MEFP O16	41.40	51.23	500.09	2611.27	3611.51	336.06	0.496	0.000
DMU O18	56.36	48.44	1119.50	1500.00	2412.81	154.74	1.713	0.121
UBT (Step 7)	42.85	47.29	1119.50	1562.49	2542.41	128.33	0.961	0.000
MEFP O18	35.63	53.23	1119.50	1326.74	2488.64	44.54	0.375	0.000
DMU O19	53.11	59.86	477.46	2500.00	3439.12	373.85	0.865	0.111
UBT (Step 5)	44.60	54.73	477.46	2726.39	3701.43	334.40	0.506	0.000
MEFP O19	41.62	51.05	477.46	2644.72	3635.52	344.66	0.477	0.000
DMU O23	49.61	39.08	247.59	1125.00	2211.74	448.77	1.135	0.345
UBT (Step 5)	43.04	31.57	247.59	1655.48	2858.31	389.34	0.363	0.000
MEFP O23	46.15	32.22	247.59	2217.23	3673.71	483.66	0.287	0.000
DMU O24	43.34	73.89	1431.67	2000.00	3252.72	78.04	0.721	0.066
UBT (Step 4)	42.91	71.91	1431.67	2013.81	3305.32	62.61	0.545	0.000
MEFP O24	33.90	57.13	1431.67	1500.00	2813.62	50.36	0.434	0.000
DMU O26	41.79	49.60	1093.36	1200.00	2425.56	164.42	1.678	0.147
UBT (Step 3)	39.38	47.80	1093.36	1462.53	2593.18	124.54	0.886	0.000
MEFP O26	35.97	52.46	1093.36	1292.58	2424.56	43.40	0.375	0.000
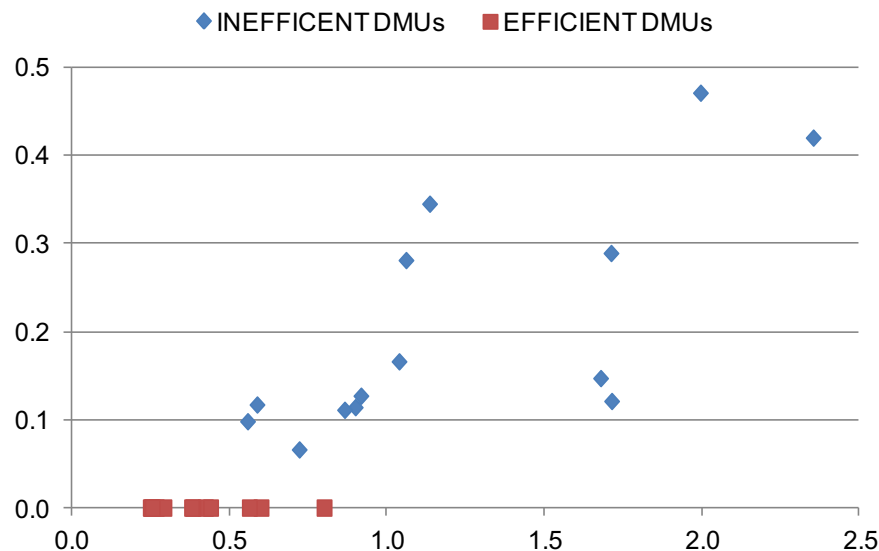
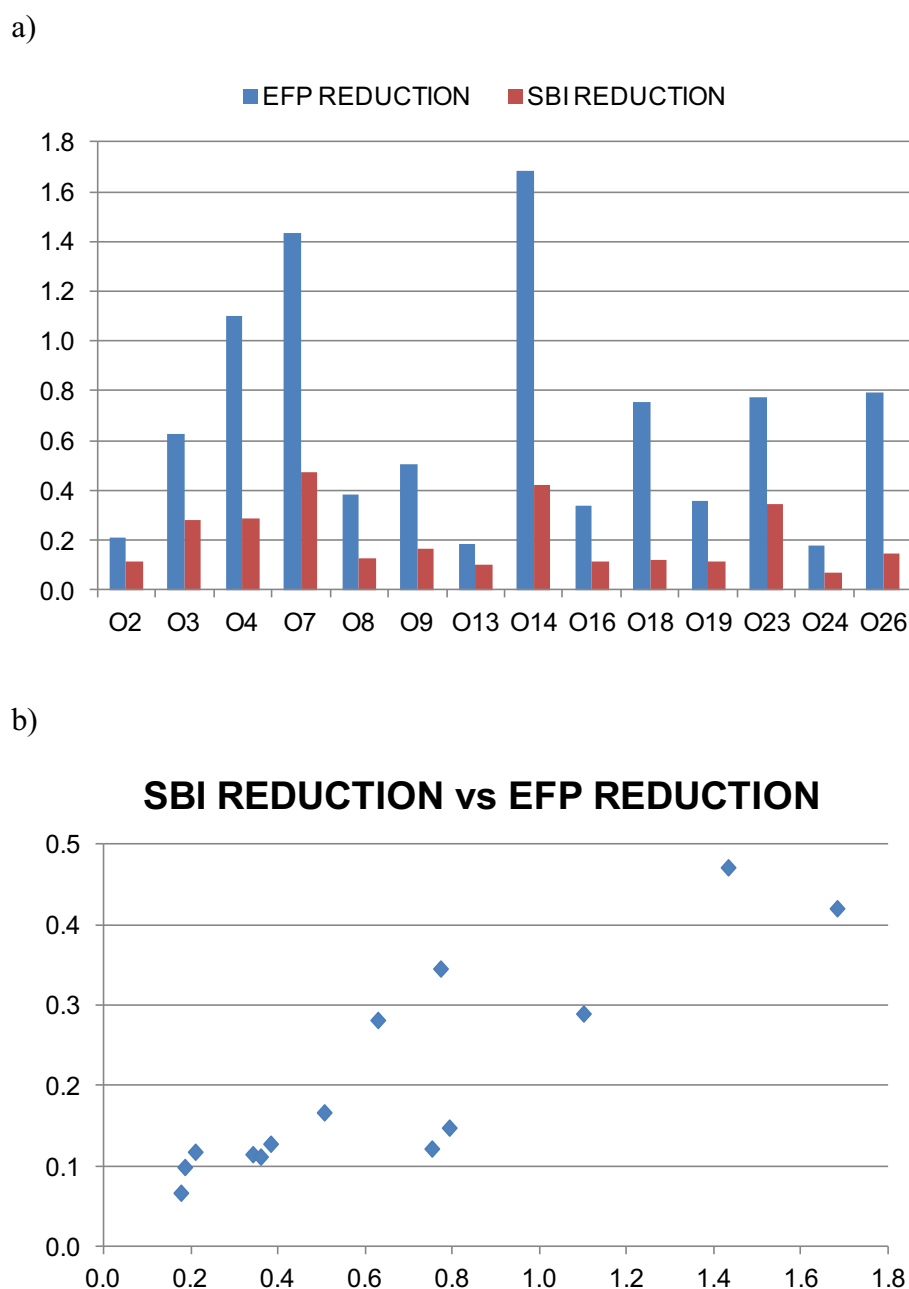**Fig. 3.** SBI and EFP of efficient and inefficient DMUs.

a)

b)

Fig. 4. SBI and EFP reductions achieved along the different efficiency improvement paths.

which means that the SBI reduction of each DMU is equal to its corresponding SBI score. Note that the DMUs that have large SBI reductions (like O14, O7, O23, O3 or O4) are also the ones that also achieve the largest EFP reduction. This positive correlation between SBI and EFP reductions is more noticeable in panel b).

6. Conclusions

In this paper, a new stepwise benchmarking approach is presented. It is based on innovative concepts such as efficiency field potential, efficiency equipotential surfaces and efficiency field vector. The idea is to associate an EFP to each feasible operating point so that the smaller the inputs consumption and the larger the outputs production, the smaller the EFP. Moreover, the negative EFP gradient represents the direction of maximum efficiency improvement and always leads to input reductions and outputs increases.

Such an EFP gradient is easy to compute and changes from one operating point to another. A stepwise gradual efficiency improvement approach is proposed by moving a bounded stepsize along the negative EFP gradient direction. Care has to be taken not to step out of the PPS. The computed stepwise benchmarking path depends on the amount of input and output changes allowed in each step. This allows the proposed approach to adapt to situations in which a DMU is close to (or far from) the efficient frontier and is willing to achieve small (or large) input and output improvements in each step. A different bound on the amount of inputs and output changes can be used for each DMU or for the different steps of the efficiency improvement programs. All this adds flexibility to the proposed approach, thus increasing its applicability.

The proposed approach can also incorporate a preference structure reflecting the relative importance of the different inputs and outputs. With the appropriate modifications, it can also handle

non-discretionary variables and undesirable outputs. Actually, the proposed approach has been applied to an organic farming dataset with non-discretionary inputs and undesirable outputs. The usefulness of the proposed approach for computing bounded stepwise efficiency improvements that end at an efficient operating point has been shown. The length of the stepwise benchmarking paths depends on the distance to the EF from which they start and on the stepsize bound. Different stepsize bounds generally lead to different UBTs.

Although the proposed approach is rather intuitive and effective, it has some limitations. Thus, it cannot handle integer inputs or outputs. Also, the EFP is undefined for those operating points which have zero consumption of a certain input or zero production of a certain output. Dealing with these zero data occurrences is a topic for further research. One possible solution, kindly pointed out by one of the reviewers, is to consider a linear (i.e. additive) definition of the EFP. That would lead to EES that are hyperplanes and EFL that are straight lines (with negative slope for inputs and positive for outputs). In principle it seems possible, and it is a topic for further research, to reformulate the proposed approach for this type of additive EFP. Actually, an additive EFP would be very appropriate in case that the input and output prices were known since then the EFP would be equivalent to the profit function. Such type of profit potential was already considered in Lozano and Calzada-Infante [30] in the context of dominance networks and using it to compute profit improvement paths would be possible. Alternatively, instead of the local information provided by the efficiency potential gradient, the profit improvement direction proposed in Zofio et al. [46] can be used to determine a stepwise profit improvement path.

Acknowledgment

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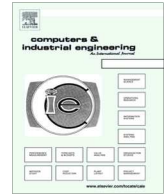
Assessing individual performance based on the efficiency of projects

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ABSTRACT

Assessing the performance of each of the team members in any project is a difficult task since in most cases the outcomes mask the individual contributions. In this paper a two-step approach is proposed to estimate the relative performance of the members of the organization who have participated in the different projects carried out. Firstly, a DEA model evaluates the efficiency of the projects, whose results are used later on in a regression model to assess the performance of each individual. The proposed approach is validated using 480 instances identifying the factors that may affect the accuracy of the results. The application to a case study involving a real-world dataset of 46 software development projects carried out by a ten-member team is also presented. The results show that the proposed approach estimates, on average, the unknown true efficiency of the individuals rather well and is robust against data noise.

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1. Introduction

Managing projects represents a big challenge from many different aspects. Not only the technical difficulty of the tasks to be performed is to be dealt with, trying to sort out all the problems that may appear, but also the complexity of interacting groups of people, from different backgrounds in many cases, adds further issues to the project execution and success.

When different people are involved in the development of a project, one of the problems that arise is how to measure the performance of each team member with respect to the project outcome. The nature of the contributions is usually different; different people work in different stages of the projects, interacting in complex ways, with different criticality in the tasks performed. With all these factors in the scenario, on most occasions it is impossible to measure accurately how much of the project outcome could be credited to each participant, whether in order to reward their work or simply to record their performance evolution.

Extensive research has been carried out dealing with team project issues, such as team membership, diversity and performance (Mathieu, Maynard, Rapp, & Gilson, 2008; Rubino, Avery, & Volpone, 2014). However, the assessment of the performance of each member to the global project outcome is a problem that has not been sufficiently studied and with no clear solution so far.

In this paper we try to shine a light on the problem of evaluating each member's performance using the technical efficiency of the projects in which they were involved. When an efficient team member has been participating in different projects, it could be expected that the performance of those projects would be higher than if the team member had lower performance efficiency. Based on that hypothesis, by using Data Envelopment Analysis (DEA) the efficiency of the projects are evaluated, and then a regression model taking into account the contribution of each member computes an estimation of the individual contribution of each person.

The structure of the paper is the following: in the next section a short review of the literature is presented, introducing in more detail the problem to be tackled; Section 3 presents the modelling details of the proposed approach, while Section 4 provides the results of some computational experiments carried out to validate and gauge the accuracy of the proposed approach. Section 5 comments on the results of the application of the proposed approach to a real case study, and Section 6 summarizes the main results of this research.

2. Problem description

Most of the works dealing with team effectiveness (Kozlowski & Ilgen, 2006) still consider the IPO (input-process-output) model, where the input describes antecedents of team interaction (characteristics, team-level factors, contextual and organization factors), which interact to drive processes that transform the inputs into outcomes (i.e., results and by-products valued by their performance: quality, quantity, satisfaction, commitment...). Note that there is not only one way to assess the individual quality of the work done and different performance metrics can be used. For

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instance, Gamble and Hale (2013) consider four facets when assessing individual performance: the contribution (direct participation and involvement), the influence (how an individual directs the team's progress), impact (relationship between what they do and their results) and impression (how well team members acknowledge the efforts of the others).

Although the IPO model has been valuable in the study of effectiveness, more recently it has been adapted and extended in different ways. For instance, Tekleab, Quigley, and Tesluk (2009) incorporate the length of time that the teams are interacting and other mediator variables to better understand what is happening in the process part of the model. However, the quantitative evaluation of individuals when they are working in groups is quite difficult, in part due to the lack of information on the daily evolution of the project, but also because in most cases the outcomes mask individuals' contributions.

Perhaps the most well-known case, in which huge interest has been shown about this evaluation problem, is in higher education when assessing the work developed by groups of students. In order to expose students to real-world situations, it is quite usual for them to be challenged to work in groups on relatively large projects (Coppit, 2006) that later are assessed and have an impact on the individual marks of each member. However, to assign a mark at the end of the project, the teacher will find it difficult to determine the actual contribution of each team member in the final work, given that most of the tasks are developed independently, without the presence of the teacher.

In most cases, the easiest solution is to assign the same grade to each member (Alden, 2011) as an emulation of the real world where the success or failure of a project is the same for each participant (for instance, in a sports competition all team members receive identical medals). However, it is well known that this procedure does not discourage the unproductive student who benefits from the work of his/her peers. In fact, different typical behaviours have been characterized (Coppit, 2006), by analyzing how these group members interact, as follows: the hitchhiker (who looks to maximize his mark while minimizing his work), the underachiever (who minimizes his work and is happy with a lower grade), the procrastinator (willing to work but delivering his part just immediately prior to the deadline), the dilettante (who is involved in many tasks but only superficially and never delivers a result), among others. All these behaviours together undermine the morale of the team and definitely affect the quality of the results as the system is not seen as fair by the more efficient members.

There is also some literature analyzing ways to deal with this problem in higher education, in many cases related to business or engineering teaching (Gamble & Hale, 2013; Mathews, 1994; Salama & Ahmed, 2011). The most usual way to look for a fairer mark is to add a peer evaluation to the global assessment of the outcome, given that those who really know what the individual contribution of each member has been are the ones who have worked on the inside. Different approaches have been used for rating the student's contribution (Wang & Vollstedt, 2014) including team journals (recording the effective distribution of effort among the members) or computerized team evaluation using different software. However many authors (Kennedy, 2005; Wang & Vollstedt, 2014) have raised doubts about its accuracy, reporting that when peer evaluation is used to obtain some extra points (merit pay) teams frequently distribute the merit pay uniformly – again making the evaluation system unfair.

In general, the management performance assessment of a project has been widely studied in the literature (Qureshi, Warraich, & Hijazi, 2009; Wateridge, 1995), considering new factors beyond the traditional "iron triangle" of time, cost and quality. Wateridge (1995) considers that the quality should be seen not only from the clients' point of view, but also from the perception of all the stakeholders involved in the project.

If assessing team performance is a task that is not simple to carry out, linking overall performance with individual proficiency results is even harder. In the 60s, Wiest, Porter, and Ghiselli (1961) identified three main points around this relationship: how the team performance compares with the performance expected on the basis of adding the members' proficiency; how well the global performance can be predicted from individuals' performance; and whether the group performance can be better explained by the performance of the best and worst members.

Results show that regarding the first point, summation of individual contributions exceeds the performance when working as a group; in relation to the third point, it seems that depending on the flexibility of the tasks that the members must carry out, there is a higher correlation with the most proficient member or with the least (in this latter case, when the tasks are more rigidly assigned). Wiest et al. (1961) also found that when the proficiencies of the members are similar, it is more likely that they form an efficient team. Also Campion, Medsker, and Higgs (1993) studied the relationship between the group characteristics and their effectiveness, finding that heterogeneity is not positively related to effectiveness, which means that a variety of skills should be present in the group, but all members must be similarly skilled. Other researchers have studied how diversity affects group performance. For instance Jackson and LePine (2003) observed that grouping low and high performers could provoke a negative reaction from high performers if the low performance is attributed to low conscientiousness.

In relation to the number of projects in which an individual should be involved, Chan (2014) studied the Multiple Team Membership problem, finding that as the number of projects in which a person is simultaneously participating increases, his/her performance improves due to an enrichment in his innovation process; however, beyond a certain number of projects, performance starts to decrease and this implies a fragmentation in his/her attention due to continuous switches in the work.

In this paper we return to the second point raised by Wiest et al. (1961), but looking for the reverse equation: we are not concerned with estimating the performance of a team, given the individual performances, but the opposite: given the team performance, to assess how good the performance of each single individual is. To do this, first we need to assess the good performance of a project, and DEA has been identified as an appropriate tool for the purpose.

DEA is a non-parametric methodology for assessing the relative efficiency of a number of comparable entities, generally termed Decision Making Units (DMUs). The methodology assumes that DMUs consume inputs (i.e. resources) and produces outputs (e.g. revenue). DEA identifies and uncovers sources of inefficiency by seeing if it is possible for each DMU to attain their current output level with a reduced amount of inputs. The larger the reduction of inputs deemed feasible, the lower the efficiency. Those DMUs for which no potential input reductions are feasible are labelled as efficient. There is extensive literature on DEA, both in terms of theory (e.g. Cooper, Seiford, & Tone, 2006) and application (e.g. Cooper, Seiford, & Zhu, 2004). In particular, DEA has been used for evaluating projects in project portfolio selection (e.g. Tavana, Keramatpour, Santos-Arteaga, & Ghorbaniane, 2015), benchmarking software development projects (e.g. Pai, Subramanian, & Pendharkar, 2015; Sudhaman & Thangavel, 2015), contractor selection in procurement projects (e.g. Yang, Wang, Wang, & Ma, 2015), appraising construction and engineering projects (Abbasian-Hosseini, Hsiang, Leming, & Liu, 2014; Caulfield, Bailey, & Mullarkey, 2013), and for managing new product development projects (Donthu & Unal, 2014), etc. Moreover, regarding the evaluation of working teams, DEA (more specifically Network DEA) has been used also to assess the performance of individuals in work-

groups when some of the outcomes are shared, i.e. attributable to the workgroup (Liang, Chen, Zha, & Hu, 2015).

Other researchers have used DEA to evaluate which parts of a process are more important and their efficient performance, for instance, in the evaluation of solutions of genetic algorithms in order to determine which operators are decisive and which parameters' value should be taken into account to obtain efficient solutions (Lu & Yu, 2012); with a similar aim, (Lu, 2015) has analysed solutions of genetic algorithms considering a robust DEA model, in order to take into account the uncertainties of the outputs due to the probabilistic operators.

3. Proposed approach

The approach proposed in this paper assumes that a certain organization has carried out a number of projects for each of which the project cost, its duration, difficulty and revenue are recorded. Moreover, for each project the individuals who participated in the project team and their relative contribution to the project are available. With all that information, an assessment of the performance of each member of the organization involved in these projects can be carried out using the following two-step approach.

The first step is to compute the relative efficiency of the different projects. To that end, DEA will be used. Fig. 1 shows the inputs and outputs considered.

Let,

$j=1,2,\ldots,n$	Index on projects
$Cost_j$	Cost incurred in project j
$Duration_j$	Duration of project j
$Difficulty_j$	Difficulty of project j
$Revenue_j$	Revenue obtained from project j
J	Index of a specific project whose relative efficiency is to be assessed
λ_j	DEA auxiliary variable associated to project j
θ_J	Relative efficiency of project J

The specific DEA model used is the following

$$\text{Min} \quad \theta_J$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \, Cost_j \leqslant \theta_J \, Cost_J$$

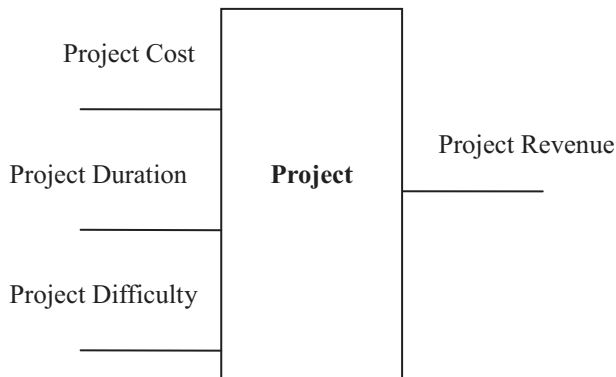$$\sum_{j=1}^{n} \lambda_j \, Duration_j \leqslant \theta_J \, Duration_J$$

Fig. 1. DEA inputs and outputs considered.

$$\sum_{j=1}^{n} \lambda_j \, Difficulty_j \leqslant Difficulty_J \tag{1}$$

$$\sum_{j=1}^{n} \lambda_j \, Revenue_j \geqslant Revenue_J$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geqslant 0 \; \forall j \quad \theta_J \text{ free}$$

This is an input-oriented, radial-efficiency, Variable Returns to Scale (VRS) DEA model of the BCC-I type (Banker, Charnes, & Cooper, 1984). The Difficulty input is considered non-discretionary (see Banker & Morey, 1986). The idea behind this DEA model is to see by how much the Cost and Duration of project J could have been reduced, given the Difficulty and Revenue of project J. The cost, duration and revenue of a project are objective and easily measurable. As regards the difficulty of a project, this is a subjective variable and it is the manager responsible for the projects the one that is in a better position to gauge it.

An efficiency score of $\theta_J = 1$ indicates that project J is relatively efficient, while if $\theta_J < 1$ then it would be possible, according to the data on the sample of projects, to have incurred a lower cost (i.e. $\theta_J Cost_J$) and to have finished the project in less time (i.e. $\theta_J Duration_J$).

The second step of the proposed approach consists of carrying out a linear regression using as the dependent variable the efficiency of a project and as independent variables the relative contributions of the different members of the organizations. Thus, let $Contribution_{ij}$ be the overall relative contribution of individual i to project j; this/her contribution reflects both this dedication to the project and the importance of his/her work for the success of the project. One way to operationalize this variable (that simplifies the four facets defined by Gamble & Hale, 2013, previously mentioned) is to compute the sum, extended to the tasks realized by the given individual within the given project, of the product of the percentage of his/her man × hours dedication on that task multiplied by a coefficient (between zero and 1) that reflects the criticality of that task. The first factor, i.e. the dedication, is objective and easily measurable. The second factor, i.e. the criticality of the task for the project success, is more subjective but the manager responsible for each project should be able to estimate it. We will assume that the task criticality coefficients for each project sum up 1.0, so that $\sum_{i=1}^{m} Contribution_{ij} = 1 \quad \forall j$ where m is the number of members of the organization involved in the projects. Of course, not all individuals will be involved in all the projects and therefore $Contribution_{ij} = 0$ if individual i does not take part in project j. Finally note that including total contribution as an input in the DEA model would not make any sense given the normalization of those variables.

Therefore, the model specification considered for the linear regression is

$$\theta_J = \sum_i \alpha_i \cdot Contrib_{iJ} + \varepsilon_J \tag{2}$$

where θ_J are the relative efficiencies of the different projects computed by using the DEA model of step 1, α_i are the parameters to be estimated for each individual i and ε_J, the error term, that represents the regression error, i.e. the amount of the dependent variable not explained by the regressors considered. The error terms ε_J are assumed to be mutually independent and normal distributed with mean zero and unknown standard deviation σ. By normalizing the coefficients computed using the above linear regression model, an

estimation of the relative performance of each individual in the organization can be computed

$$\text{Performance}_i = \frac{\alpha_i}{\max_k \alpha_k} \qquad (3)$$

The rationale behind the proposed approach is that if an individual has a high participation in projects with high efficiency scores then he/she should be credited and assigned a good relative performance score. On the other hand, individuals who participate in low efficiency projects receive a low relative performance score. We can also see it the other way around, i.e. if an individual is a good performer then the projects in which he/she participates should tend to be efficient, especially if the other members of the project team are also good performers. Bad performers, on the other hand, tend to increase both cost and duration (and therefore lower the efficiency) of the projects in which they participate. Note that since the proposed approach computes a sort of average performance of each individual in the different projects it participates, it is reasonable to assume that the performance of the individuals may vary from one project to another and that is why the regression specification includes an error term that captures those random fluctuations.

A final remark about the proposed approach is that the regression analysis of step 2 identifies some of the regressors as significant, while others may not be found significant at a given significance level. For each non significant variable this means that the null hypothesis cannot be rejected (at the given significance level) and that its corresponding coefficient α_i (and, therefore, Performance$_i$) is zero. For the significant variables the corresponding coefficients α_i are generally positive which means that the contributions of those individuals to the projects in which they have participated translate into an increase in efficiency of those projects. But, in principle, it is also possible that a significant variable may have an estimated coefficient $\alpha_i < 0$. This should be interpreted as being that the contributions of the individual to the projects in which he/she participates reduces the projects' efficiency, i.e. he/she subtracts more than adds to the projects' efficiency. This is an undesirable situation that does not occur frequently but that the proposed approach can detect in case it happens. In any case, the computed relative performance scores lead to a ranking of the performance of the different individuals and this may be of utmost importance for management.

4. Experimental framewok and results

In order to validate and test the accuracy of our model to evaluate individuals' efficiency, a set of experiments have been carried out. The idea is to generate a large number of instances (group of projects) for which we know all the relevant data (contribution of each team member, duration, cost and revenue of each project, dependent on the – unknown for the model – performance level of each individual).

Therefore the procedure consists of randomly generating "true" performance levels (labelled individual efficiencies **ef**$_i$), randomly assigning individuals to some projects j with randomly generated contribution levels **cont**$_{ij}$ (so that $\Sigma_i \text{cont}_{ij} = 1 \ \forall j$) and calculate for each project its data partially based on these variables. Thus, in our experiment,

- The difficulty of a project **Difficulty**$_j$ is a random variable in the range [0.5, 1.5]
- The expected duration of a project **ed**$_j$ is a random variable in the range [4, 36] months multiplied by the difficulty of the project **Difficulty**$_j$. Thus, a project with a larger difficulty should take longer than a project with a lower difficulty. The real

duration of a project, **Duration**$_j$, is calculated as the expected duration **ed**$_j$ divided by an efficiency factor **ef**$_j$ = Σ_i ef$_i$ ∗ cont$_{ij}$ + γ, with γ a random uniform variable in the interval $[-\beta, \beta]$.
- The cost of a project **Cost**$_j$ is calculated as the number of people assigned to the project times the real duration of the project.
- The revenue of a project **Revenue**$_j$ is calculated as the number of people assigned to the project, times **ed**$_j$, times a profit factor (1.25, in our case).

Note that the way that the project data are generated involves the efficiency of the projects and that these were linked to the efficiency of the individuals. This was not done to produce favourable results but because it is more reasonable than if we generate the duration and cost of the projects without any regards to the supposedly known efficiency of the individuals. The rationale of the proposed approach is that efficient individuals perform efficiently in the projects they participate. If the project data are generated randomly then there would be no link between the project efficiencies and the efficiencies of the individuals, i.e. inefficient individuals might develop an efficient project and vice versa. Our assumption is that is unlikely to happen.

In order to test the behaviour of our model in a variety of scenarios, we identified four factors that could affect its performance. By introducing them into the procedure for the generation of the instances, it will be possible to assess how relevant these factors are for the accuracy of the obtained results. The factors considered are the following:

- F1. *Number of people working in each project.* It may be expected that bigger teams could be less exposed to the influence of individual members. Therefore we considered a low level (F1.1) with 30% of all the individuals in the department involved in each project, and a high level (F1.2) with 60%. Assuming that the number of individuals considered is ten, this means that three or six people are involved in each project depending on the level of this factor.
- F2. *The variability of the contributions of the individuals involved in a project.* A low level of variability (F2.1), with (unnormalized) contributions in the range [0.35, 0.65] will capture the situation of quite homogeneous projects (with everybody involved to a similar degree) while the high level (F2.2) will allow a dispersed degree of contributions in the project (in the range [0.2, 0.9]).
- F3. *The variability of the individuals' efficiencies.* Homogenous members could mean less individual influence on the project results. Therefore, in our case a low level (F3.1) means a quite homogeneous group of high-performance employees, everyone in the range [0.75, 1.0], while a high level (F3.2) means a larger dispersion in the performance level of the personnel (in the range [0.3, 1.0]). We remark on how important it is in the literature dealing with team formation, i.e. the homogeneity of the team members (Jackson & LePine, 2003). Note also that, although the individual efficiencies are generated within the given intervals, a certain number of individuals (one, two or three, with equal probability) are assigned an individual efficiency **ef**$_i$ = 1.
- F4: Three levels of the amount of *noise in the data* $\beta \in \{0.2; 0.5; 0.8\}$ have been considered. This noise represents events that can affect the execution of the projects (their duration, cost, etc.) but cannot be attributed to the team members' performance.

In our experiments, for each of the $2^3 \times 3$ factor level combinations of the four factors considered, 20 instances were generated. This gives a total of $8 \times 20 \times 3 = 480$ instances. Each instance was processed using the proposed approach. Thus, the relative effi-

Table 1
Results of regression assumptions' tests for the different combination of factors' levels (% of test rejections for each of the four tests).

	F4.1	F4.2	F4.3
F1.1 F2.1 F3.1	5/45/5/0	0/15/10/0	5/45/15/0
F1.1 F2.1 F3.2	5/20/10/0	0/25/10/0	5/45/10/0
F1.1 F2.2 F3.1	10/40/0/0	5/25/0/0	0/20/0/0
F1.1 F2.2 F3.2	0/35/0/0	0/15/0/0	0/30/0/0
F1.2 F2.1 F3.1	0/45/0/0	0/30/10/0	0/50/10/0
F1.2 F2.1 F3.2	5/30/5/0	0/15/5/0	5/25/0/0
F1.2 F2.2 F3.1	5/35/0/0	0/30/0/0	5/50/5/0
F1.2 F2.2 F3.2	5/35/5/0	5/20/0/0	5/40/0/0

Note: Regression assumption tests include Linearity[a]/Normality[b]/Homoscedasticity[c]/Autocorrelation[d].

[a] Rejections of the null hypothesis of linearity at 0.05 level of significance (Rainbow test).

[b] Rejections of the null hypothesis of normality at 0.05 level of significance (Shapiro-Wilk test).

[c] Rejections of the null hypothesis of homoscedasticity at 0.05 level of significance (studentized Breusch-Pagan test).

[d] Rejections of the null hypothesis of autocorrelation of error terms at 0.05 level of significance (Durbin-Watson test).

ciency of the 50 projects of each instance was computed using model (1). These were then used as dependent variables in a regression analysis as per Eq. (2). The linear regression models were estimated using the R statistical software (R Core Team, 2015).

In order to test whether linear regression assumptions hold, we have carried out linearity, normality, homoscedasticity and autocorrelation tests. For those independent variables found to be significant (at the 0.05 level), the corresponding relative performance score (3) was computed. As regards the regression assumptions,

Table 1 shows that the only one that is rejected with certain frequency is normality.

Fig. 2 shows the boxplots of the Pearson correlation coefficients between the estimated relative performance of the individuals and their true efficiency. In general, the correlations are high, with average values above 0.8. The correlation seems to decrease when the team size increases and when the variability in the efficiency of individuals increases. When the variability in the contributions of the team members increases and when the random noise increases, the correlation also seems to decrease but not as much. This indicates that the proposed approach is robust with respect to the noise level present in the data.

Fig. 3 shows the scatter plots for the 24 factor level combinations of the "true" individual efficiencies used to generate the corresponding instances and the relative performance scores computed by the proposed approach. The bisector line of each scatter plot represents a perfect coincidence. Taking into account that the data have noise, it is not surprising that there are small errors in the estimations computed by the proposed approach.

Regarding the confidence interval for the average ratio of the estimated performance score to the "true" individual efficiency, Table 2 shows these results. As can be observed, these confidence intervals are generally around unity, although sometimes they are slightly above or below this value. This means that the proposed approach on average estimates the true efficiencies with a rather small upward or downward bias.

We have also carried out parametric and non-parametric tests for comparing the difference between the estimated and true efficiency of the individuals and see if the difference is significant for some factor levels combinations. The results are shown in Table 3. Previous to the inference analysis, assumptions underlying the paired mean test are evaluated, i.e., normality and homogeneity
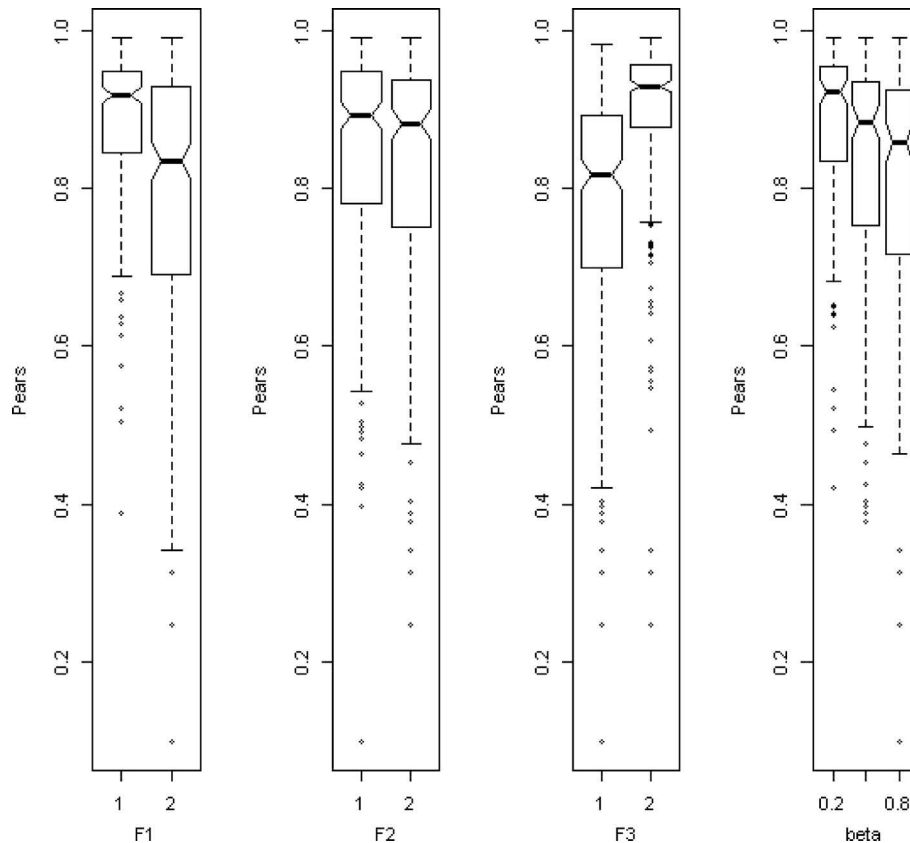

Fig. 2. Boxplots of Pearson correlation between estimated relative performances and true individual efficiencies, for the different factors.
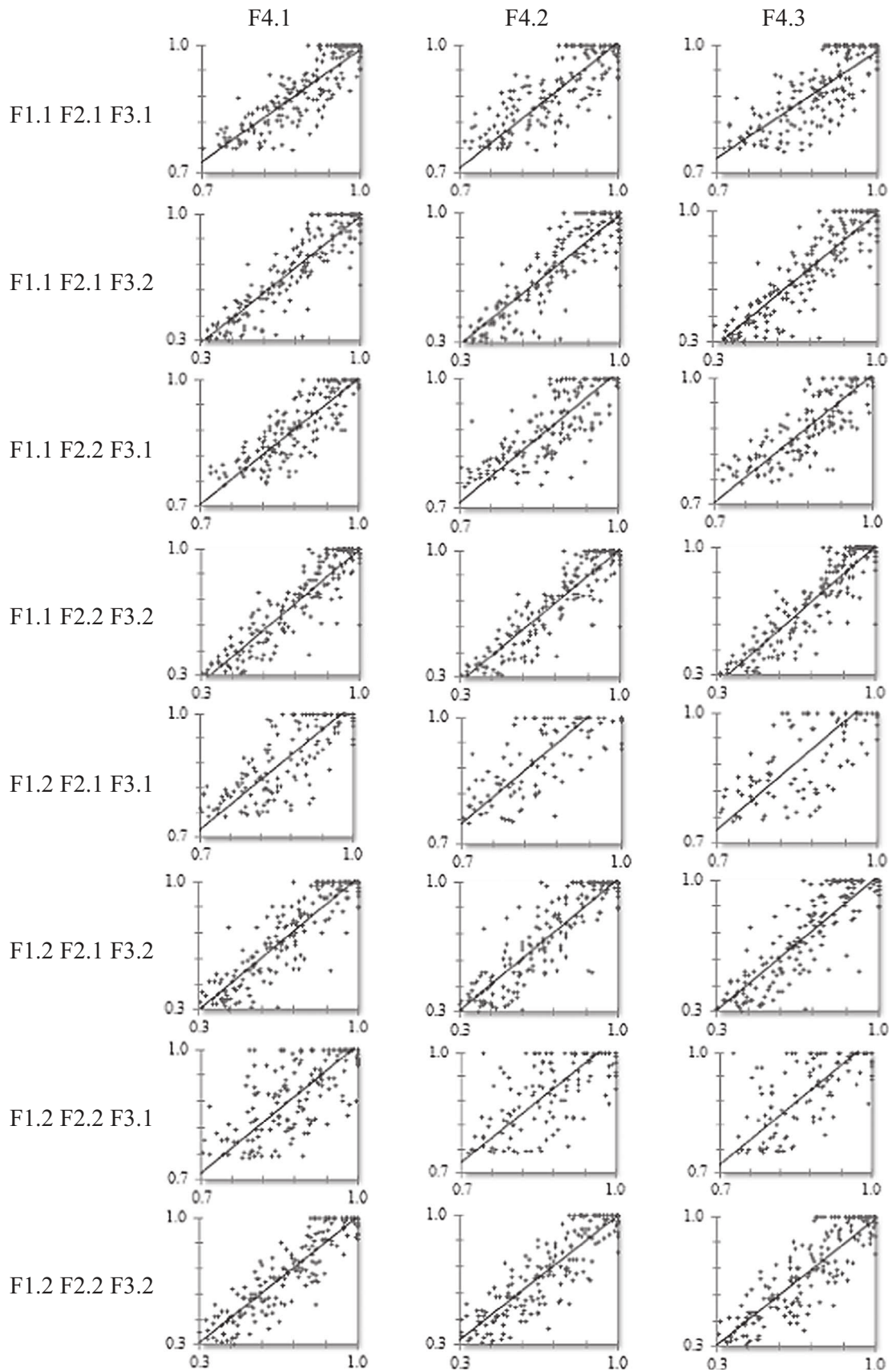
Fig. 3. Scatter plot of estimated performance score (Y axis) versus true individual efficiency (X axis). Each table cell corresponds to one of the $2^3 \times 3$ factor level combinations.

Table 2
Bootstrap confidence interval for the average of the ratio of relative performance score to true individual efficiency.

	F4.1	F4.2	F4.3
F1.1 F2.1 F3.1	(0.989, 1.003)	(0.981, 0.996)	(0.987, 1.004)
F1.1 F2.1 F3.2	(1.018, 1.071)	(1.015, 1.070)	(1.027, 1.084)
F1.1 F2.2 F3.1	(0.988, 1.001)	(0.976, 0.994)	(0.986, 1.002)
F1.1 F2.2 F3.2	(1.018, 1.063)	(1.009, 1.058)	(1.027, 1.073)
F1.2 F2.1 F3.1	(0.960, 0.980)	(0.920, 0.946)	(0.933, 0.961)
F1.2 F2.1 F3.2	(0.986, 1.036)	(0.981, 1.040)	(0.985, 1.047)
F1.2 F2.2 F3.1	(0.973, 0.993)	(0.953, 0.977)	(0.969, 0.993)
F1.2 F2.2 F3.2	(0.983, 1.027)	(0.981, 1.029)	(0.994, 1.044)

Note: Central 95% confidence interval calculated using pivotal bootstrap procedure (1000 replications).

of correlated variances tests. Although for different combination of factor levels paired data showed departures from non-normality and homogeneity of variances due to the skewed nature of the distributions, the parametric t-test procedure is robust to small departures of normality, based on large sample theory (Lehman, 1999). Nevertheless, when efficiencies did not follow normality criteria according to the Shapiro-Wilk test and/or homogeneity of variances according Pitman-Morgan test, the non-parametric Wilcoxon-rank test was performed. As it can be seen in the table, although for some combinations there is statistical difference between the actual and estimated efficiency, according to the confidence interval these differences are rather narrow. The results show that in most combinations of factors levels there is insufficient evidence to suggest a difference between actual and estimated efficiency at a significance level of 0.05.

Therefore, based on these experiments, we conclude that, in general, our model is able to "guess" quite accurately the real efficiency of each individual, in most of the cases. Also, among the four factors considered, there is no specific factor that is so influential that it makes the model behave incorrectly.

5. Case study

Now that the validity of the proposed approach has been established in the previous section, in this section it is applied to a case study for which the individuals' performances are unknown and have to be estimated. The dataset corresponds to 46 real software development projects carried out by a small Spanish software engineering company. The workforce in this department consists of 10 programmers and analysts who have participated, with different contribution levels, in those projects. Fig. 4 shows a snapshot of the projects dataset. Note that, as requested by the company the cost and revenue data have been distorted by multiplying both by a constant factor, which is equivalent to expressing them in a virtual monetary unit, instead of expressing them in euros. This has no effect on the proposed approach because the DEA model used is units-invariant.

Fig. 5 shows the efficiency scores of the different projects versus their corresponding difficulty level. The efficiency scores are not very high in general, with a number of projects below 0.4. Note that the efficiency of the projects does not seem to be correlated with their difficulty. Thus, for each difficulty level there are efficient and inefficient projects, although it is true that for the highest difficulty level there are more inefficient than efficient projects, while for the lowest difficulty level the opposite occurs.

The projects efficiencies are input to the regression analysis which leads to the results shown in Table 4. Note that the R-squared value is rather high (0.856), indicating a good fit of the

Table 3
Parametric and non-parametric paired tests for comparing the equality of central tendency of estimated and true individual efficiency.

Id	Combination of factor levels	Normality test[a]	Homogeneity of correlated variances test[b]	Paired t-test[c]	95% confidence interval[d]	Paired non-parametric t-test[e]	Is there, on average, difference between estimated and true individual efficiency?
1	F1.1 F2.1 F3.1 F4.1	W = 0.981**	t = 0.80	t = 1.66	[−9 · 10⁻³; 0.01]	W = 7900	No
2	F1.1 F2.1 F3.1 F4.2	W = 0.955**	t = 3.27**	t = −1.81	[−0.03; 1 · 10⁻³]	W = 7728	No
3	F1.1 F2.1 F3.1 F4.3	W = 0.982*	t = 1.04	t = 1.87	[−3 · 10⁻⁴; 0.01]	W = 8125	No
4	F1.1 F2.1 F3.2 F4.1	W = 0.951**	t = 1.29**	t = −1.61	[−0.02; 2 · 10⁻³]	W = 8053	No
5	F1.1 F2.1 F3.2 F4.2	W = 0.993	t = 0.85	t = 6.10**	[0.02; 0.03]	–	Yes
6	F1.1 F2.1 F3.2 F4.3	W = 0.980**	t = 1.30**	t = 1.81	[−1 · 10⁻³; 0.03]	W = 9123	No
7	F1.1 F2.2 F3.1 F4.1	W = 0.991	t = 1.05	t = 3.93**	[8 · 10⁻³; 0.03]	–	Yes
8	F1.1 F2.2 F3.1 F4.2	W = 0.992	t = 1.17	t = 1.43	[−3 · 10⁻³; 0.02]	–	No
9	F1.1 F2.2 F3.1 F4.3	W = 0.987	t = 0.92	t = 3.29**	[4 · 10⁻³; 0.01]	–	Yes
10	F1.1 F2.2 F3.2 F4.1	W = 0.964**	t = 1.21**	t = −1.58	[−0.02; 2 · 10⁻³]	W = 7786	No
11	F1.1 F2.2 F3.2 F4.2	W = 0.990	t = 0.92	t = 4.11**	[8 · 10⁻³; 0.02]	–	Yes
12	F1.1 F2.2 F3.2 F4.3	W = 0.951**	t = 1.29**	t = −0.96	[−0.02; 6 · 10⁻³]	W = 8713	No
13	F1.2 F2.1 F3.1 F4.1	W = 0.992	t = 0.61**	t = 10.55**	[0.05; 0.07]	W = 15,593**	Yes
14	F1.2 F2.1 F3.1 F4.2	W = 0.982*	t = 1.28**	t = 1.79	[−1 · 10⁻³; 0.03]	W = 8624	No
15	F1.2 F2.1 F3.1 F4.3	W = 0.989	t = 0.877	t = 6.21**	[0.02; 0.04]	–	Yes
16	F1.2 F2.1 F3.2 F4.1	W = 0.994	t = 1.15	t = 1.30	[−4 · 10⁻³; 0.02]	–	No
17	F1.2 F2.1 F3.2 F4.2	W = 0.991	t = 1.09	t = 1.66	[−1.1 · 10⁻³; 0.01]	–	No
18	F1.2 F2.1 F3.2 F4.3	W = 0.969**	t = 1.27**	t = −2.39*	[−0.03; −2 10⁻³]	W = 7364	Yes
19	F1.2 F2.2 F3.1 F4.1	W = 0.992	t = 1.01	t = 1.87	[−3 · 10⁻³; 0.01]	–	No
20	F1.2 F2.2 F3.1 F4.2	W = 0.956**	t = 1.36**	t = −2.07*	[−0.03; −6 · 10⁻⁴]	W = 7999	No
21	F1.2 F2.2 F3.1 F4.3	W = 0.993	t = 0.63**	t = 7.90**	[0.03; 0.06]	W = 14,542	Yes
22	F1.2 F2.2 F3.2 F4.1	W = 0.980**	t = 1.33	t = 1.61	[−0.01; 0.02]	W = 8863	No
23	F1.2 F2.2 F3.2 F4.2	W = 0.994	t = 1.02	t = 3.85***	[0.01; 0.03]	–	Yes
24	F1.2 F2.2 F3.2 F4.3	W = 0.989	t = 1.18*	t = 0.13	[−0.01; 0.01]	W = 9075	No

Sample size per each combination of factor levels: 200 observations.
* indicate significance at the 5% level (p-value < 0.05).
** indicate significance at the 1% level (p-value < 0.01).
[a] Shapiro Wilk Normality test for difference between actual efficiency and estimated efficiency.
[b] Pitman-Morgan test.
[c] Depends on the case: equal variances are assumed or equal variances are not assumed (Yuen-Welch test).
[d] Mean of differences (two-tailed).
[e] Wilcoxon signed rank test with continuity correction.

| PROJECT | \multicolumn{10}{c}{CONTRIBUTION TO THE PROJECT} | DURATION | COST | REVENUE | DIFFICULTY |

PROJECT	1	2	3	4	5	6	7	8	9	10	DURATION	COST	REVENUE	DIFFICULTY
1	1	15		14			5	39	7	19	12	29800	31184	4
2	3	7	2	22						66	24	87484	90762	5
3	4			5	65	9				17	24	85667	150000	3
4			28				72				4	6180	140	3
5	10	15	65				8	2			24	60750	30000	2
6		89	6				5				8	26191	30952	2
7		51	47							2	3	9058	12240	2
8	60	3	6							22	3	38700	51128	2

Fig. 4. Snapshot of projects' data of case study, with 10 individuals involved at a different level of contribution in each project.
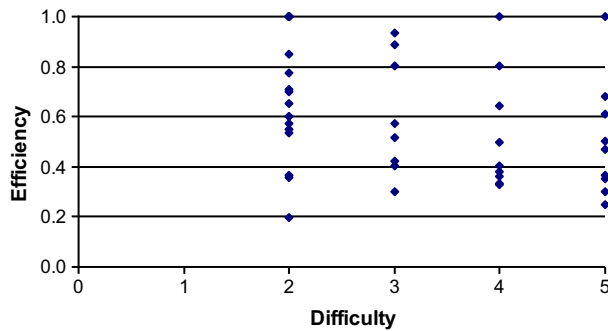

Fig. 5. Projects' efficiency versus difficulty.

Table 4
Multiple linear regression analysis results for the real case study.

Employee i	Estimated α_i	Standard error	Relative performance score (Eq. (3))	Rank
1	0.634***	0.141	0.510	7
2	0.542***	0.119	0.436	9
3	0.708***	0.165	0.570	4
4	0.053	0.316	0.043	10
5	0.643*	0.364	0.517	6
6	1.243**	0.459	1.000	1
7	0.878***	0.213	0.706	3
8	0.569*	0.304	0.458	8
9	1.019*	0.555	0.820	2
10	0.686***	0.234	0.552	5

Note: Model estimated with OLS. n = 46 observations; *RMSE* = 0.254 with 36 degrees of freedom; $R^2_{adj} = 0.856$.
* p-value < 0.1.
** p-value < 0.05.
*** p-value < 0.01.

linear regression specification. All independent variables except one are significant at the 0.1 level. With respect to the contribution of Employee 4, the results show that it is not significant, given the linear model specification considered. Note also that none of the employees had a significant and negative coefficient.

6. Conclusions

This paper has addressed the issue of estimating the performance of individuals when they work in teams. The rationale behind the proposed approach is that when the team performs well then individual members should be credited and vice versa. Thus, the first step is to assess the performance of the teams. This is done using DEA to estimate the relative efficiency scores of the projects carried out in the company. These project efficiencies are computed based on some basic data about each project such

as its duration, incurred cost, difficulty and generated revenue. The second step is to carry out a regression analysis of the projects' efficiencies using the corresponding team members' contributions. The normalized coefficients of the contribution of each individual indicate their relative performance. We distinguish three cases depending on whether the variable is significant (in which case the coefficient can be positive or even negative) or non-significant (which is equivalent to a zero coefficient). The estimated relative individual performances also allow the ranking of individuals.

The proposed approach has been validated using randomly generated synthetic datasets that covered all combinations of four factors, namely the team sizes, the variability of the team members' contributions, the variability of team members' efficiencies and the noise present in the data. The two factors that seem to affect the degree of accuracy of the proposed approach more are the team sizes and the variability of the individuals' efficiencies. The method seems to be robust with respect to increasing noise.

In addition, an application of the proposed approach to a case study has been presented. The situation studied corresponds to a software development company and the dataset involves 46 projects carried out in the last few years. The results show the usefulness of the proposed approach, which is able to assess the individuals' performances manifested through the efficiency of the projects in which they participate as teams.

About the limitations of the present study, we can note that the regression analysis carried out assumes a linear regression specification. Although this model specification has given rather high goodness of fit indexes, other regression models (e.g. sub-linear) could be tried. Also, the DEA model used computes a radial efficiency measure, which can leave some input and output slacks. Hence, alternatively a non-radial or slacks-based efficiency measure may be used.

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APPENDIX 3. REPORT LISTING THE IMPACT FACTORS OF THE PAPERS

The following list has been prepared with the data from the Journal Citation Reports from Clarivate Analytics.

[A] <u>Journal of the Operational Research Society</u>

ISSN: 0160-5682
Publisher: PALGRAVE MACMILLAN LTD
Categories: Operations Research & Management Science

Year	Impact Factor	Operations Research & Management Science	
		Rank	*Quartile*
2016	1.077	56/83	Q3
2015	1.225	41/82	Q2

Table A3-1: Rank of Journal of the Operational Research Society

[B] Physica A: <u>Statistical Mechanics and its Applications</u>

ISSN: 0378-4371
Publisher: ELSEVIER SCIENCE BV
Categories: Physics, Multidisciplinary and Physics

Year	Impact Factor	Physics, Multidisciplinary	
		Rank	Quartile
2016	2.243	18/79	Q1
2015	1.785	23/79	Q2

Table A3-2: Rank of Physica A: Statistical Mechanics and its Applications

[C] <u>Expert Systems with Applications</u>

ISSN: 0957-4174
Publisher: PERGAMON-ELSEVIER SCIENCE LTD
Categories: Computer Science, Artificial Intelligence; Engineering, Electrical & Electronic and Operations Research & Management Science

Year	Impact Factor	Computer Science, Artificial Intelligence		Engineering, Electrical & Electronic		Operations Research & Management Science	
		Rank	Quartile	Rank	Quartile	Rank	Quartile
2016	3.928	18/133	Q1	37/262	Q1	3/83	Q1
2015	2.981	19/130	Q1	27/257	Q1	6/82	Q1

Table A3-3: Rank of Expert Systems with Applications

[D] <u>Omega – International Journal of Management Science</u>

ISSN: 0305-0483
Publisher: PERGAMON-ELSEVIER SCIENCE LTD
Categories: Operations Research & Management Science and Management

Year	Impact Factor	Operations Research & Management Science	
		Rank	*Quartile*
2016	4.029	2/83	Q1
2015	3.962	2/82	Q1

Table A3-4: Rank of Omega International Journal of Management Science

[E] <u>Computers & Industrial Engineering</u>

ISSN: **0360-8352**
Publisher: PERGAMON-ELSEVIER SCIENCE LTD
Categories: Computer Science, Interdisciplinary Applications and Engineering, Industrial

Year	Impact Factor	Computer Science, Interdisciplinary Applications		Engineering, Industrial	
		Rank	*Quartile*	*Rank*	*Quartile*
2016	2.623	28/105	Q2	9/44	Q1
2015	2.086	27/104	Q2	9/44	Q1

Table A3-5: Rank of Computers & Industrial Engineering