Acoustic Vibration Problem for dissipative fluids

Felipe Lepe*, Salim Meddahi†, David Mora‡
and Rodolfo Rodríguez§

Abstract

In this paper we analyze a finite element method for solving a quadratic eigenvalue problem derived from the acoustic vibration problem for a heterogeneous dissipative fluid. The problem is shown to be equivalent to the spectral problem for a noncompact operator and a thorough spectral characterization is given. The numerical discretization of the problem is based on Raviart-Thomas finite elements. The method is proved to be free of spurious modes and to converge with optimal order. Finally, we report numerical tests which allow us to assess the performance of the method.

Keywords: Spectral problems, dissipative fluid, finite elements, error estimates.

AMS Subject Classification: 65N25, 65N30, 76M10

1 Introduction

This paper deals with the numerical approximation of an acoustic dissipative fluid system. This kind of problem has attracted much interest, since it is frequently encountered in engineering applications (see, for instance, [5, 20, 32]). One typical example is to achieve optimal designs that reduce noise and vibrations in fluid-structure systems like cars, aircraft or dams.

Although dissipation is usually neglected in standard acoustics, modeling this phenomenon is essential to study the effect of noise reduction techniques. Indeed, in most real situations, damping mechanisms that transform mechanical energy into heat do exist. Sometimes these mechanisms are based on surface damping arising from viscoelastic materials placed on the boundary of the propagation domain. In these cases, the dissipative effects are typically included in the model by means of a surface impedance in the boundary conditions (see, for instance, [4, 6, 7]). The present paper addresses damping when it arises in the propagation media itself due to friction and heat

*Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile and CI^{2}MA, Universidad de Concepción, Concepción, Chile. (`flepe@ubiobio.cl`). This author was partially supported by Proyecto Plan Plurianual 2016-2020 Universidad del Bío-Bío (Chile) and CONICYT fellowship (Chile).

†Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, Calvo Sotelo s/n, Oviedo, España. (`salim@uniovi.es`). This author was partially supported by Spain's Ministry of Economy Project MTM2017-87162-P (Spain)

‡Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile and CI^{2}MA, Universidad de Concepción, Concepción, Chile. (`dmora@ubiobio.cl`). This author was partially supported by CONICYT-Chile through FONDECYT project 1140791, by DIUBB through project 171508 GI/VC, Universidad del Bío-Bío (Chile) and by CONICYT-Chile through the project AFB170001 of the PIA Program: Concurso Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal.

§CI^{2}MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile. (`rodolfo@ing-mat.udec.cl`). This author was partially supported by CONICYT-Chile through the project AFB170001 of the PIA Program: Concurso Apoyo a Centros Cientficos y Tecnológicos de Excelencia con Financiamiento Basal.

conduction. A general approach to this topic can be found in the books by Landau and Lifshitz [22], Morse [31], and Pierce [37], all of which include extensive bibliographic references on the subject.

This paper focus on computing the (complex) vibration frequencies and modes of an acoustic dissipative fluid system within a rigid cavity. One motivation for considering this problem is that it constitutes a stepping stone towards the more challenging goal of devising numerical approximations for coupled systems involving fluid-structure interaction between viscous fluids and solid structures. The natural model for the fluid system should be based on the Stokes equations for compressible fluids. However, since in real applications the viscosity is typically very small, the resulting problem turns out a singular perturbation of that for an inviscid fluid. This fact leads to a kind of dilemma, since appropriate finite elements for the Stokes equations introduce spurious modes in the limit case of a vanishing viscosity, whereas the finite elements that avoid such spectral pollution fail when applied to the Stokes equation.

To circumvent this drawback, we resort to an alternative model based on a curl-free displacement formulation (see [8] for the derivation of a similar model in the time domain from basic mechanical laws). Let us remark that in principle the fluid displacement does not need to be curl-free. However, since the viscosity term due to vorticity is typically very small, except perhaps near the walls of the enclosure, it may be neglected in the interior of the enclosure and eventually modeled as a wall impedance on its boundary (see [32] for a similar model).

This curl-free displacement formulation for a viscous fluid leads to a quadratic eigenvalue problem (QEP), as it happens in [4]. However, the resulting problem involves additional challenges related to the fact that the essential spectrum does not reduce to a single point as in [4]. In fact, in this case, we can only prove that the essential spectrum is well separated from the physically relevant eigenvalues when the viscosity is sufficiently small (as it happens in practice). On the other hand, the associated solution operator is not regularizing. Because of this, we need to split it into two terms for the numerical analysis. One of them is dealt with the techniques from [4], but the spectral approximation analysis for the other is new.

As is shown below (cf. Remark 2.1), the vibration frequencies and modes of a viscous homogeneous irrotational fluid within a rigid cavity can be obtained without actually solving a QEP. In fact, these frequencies can be algebraically computed from those of the analogous inviscid fluid, whose approximation is nowadays a well known subject (see, for instance, [5]). However, this is not the case for a heterogeneous fluid and this is the reason why we choose this as our model problem. In particular, we consider the QEP arising from the acoustic vibration problem for a dissipative fluid system that consists of two homogeneous viscous immiscible fluids contained in a rigid cavity.

QEP has many applications in the study of vibration for solid systems, acoustic fluids, electrical circuits etc., where the damping effects are involved. A state of art for the QEP up to the beginning of this century can be found in [39]. However, there are not many works with a rigorous mathematical framework in the context of the numerical approximation of the eigenvalue problem of a partial differential operator involving damping. The first article proving this type of results is [4], where the authors have considered a displacement formulation for a fluid in a rigid cavity with absorbing walls. The theory of non-compact operators of [16] is used to obtain error estimates with minor modifications due the non-conformity.

On the other hand, alternative formulations for the absorbing wall problem have been studied in the Engineering literature. For instance, a formulation for the QEP in terms of the fluid pressure has been proposed in [23]. This type of formulation leads to a rational eigenvalue problem, for which different algorithms to compute the vibration frequencies have been introduced. Alternatively, two formulations one based on the fluid pressure and the other on the fluid displacement, have been considered in [14], where an improved Arnoldi algorithm have been proposed to solve the discrete problem. On the other hand, an application of the damping effects in electromechanical-thermoelastic systems is presented in [1]. Moreover, an a-posteriori estimator for a QEP contextualized in the

photonic crystal applications is proposed in [17]. Nevertheless, all the previous mentioned studies present different numerical technologies to solve the QEP, but without a rigorous mathematical analysis. Such a rigorous analysis is present instead in the recent paper [27], where an efficient multiscale technique based on localized orthogonal decompositions to solve discrete generic damped vibration problems has been proposed (see also [26]).

In the present paper, we consider a displacement based variational formulation of the transmission eigenproblem resulting from our physical model. This approach leads to a QEP, which is transformed into an equivalent double-size linear eigenvalue problem that fits within the functional framework for nonselfadjoint and noncompact bounded operators. At the continuous level, we follow [21] to obtain an appropriate spectral characterization. Next, we propose an $H(\mathrm{div})$-conforming mixed finite element approximation of the problem and adapt the abstract spectral approximation theory for noncompact operators developed in [15, 16] to prove that the spectrum is correctly approximated and to obtain error estimates.

The discrete analysis relies partly on the techniques used in the Raviart-Thomas mixed approximation of the **grad**-div eigenvalue problem. This spectral problem emerged in the study of coupled fluid-solid systems in [3] (see also [28] for a similar setting in elasticity). The **grad**-div spectral problem is posed in $H(\mathrm{div})$ and it is closely related to the Maxwell eigenvalue problem, which is formulated in terms of the **curl**-**curl** operator in $H(\mathbf{curl})$ (see [30, 10, 13]). Although the two spectral problems have been initially studied in isolation from each other, a common framework becomes now clear thanks to the language of differential forms and the approach based on the finite element exterior calculus provided by [2] (see also [11, Part 4]).

With X representing either $H(\mathrm{div})$ or $H(\mathbf{curl})$, we let K be the kernel of the div or the **curl** operator in the corresponding space. If we denote by X_h a conforming mixed finite element approximation of X and consider $K_h := K \cap X_h$, then the main tool in the proof of the correct spectral approximation relays on the construction of a projector P onto $K^\perp$ such that

$$\lim_{h\to 0} \|I - P\|_{\mathcal{L}(K_h^\perp, X)} = 0. \tag{1.1}$$

Here $K^\perp$ stands for L^2-orthogonal complement of K in X and $K_h^\perp$ represents the L^2-orthogonal complement of K_h in X_h.

To our knowledge, this tool has first been introduced in the $H(\mathrm{div})$-setting in [3] and more recently in [28, Lemma 4.2]. The same tool appeared independently in the context of the Maxwell eigenvalue problem; see for example [18, Lemma 4.5] where the projector P is named the Hodge mapping. A related abstract approach is considered in [12] where the so-called GAP property is introduced. It is shown in [12] that the GAP property is equivalent to (1.1) when $K_h \subset K$ (as in the cases considered here). It is also shown that the GAP property implies the discrete compactness property which was used in [13] as another tool to study edge element approximation of the Maxwell eigenproblem. See also the discussion given in [11, Section 19] regarding the topic in the language of differential forms.

The negative impact that material parameters have on the regularity of the solution of the boundary value problem complicates the analysis in this common framework (see [18, Remark 13]). Here, we follow the lines of the methodology presented originally in [3] and use the information about interface singularities of solutions of the Neumann boundary value problem for $\mathrm{div}(\kappa\,\mathbf{grad})$ with κ piecewise constant, to obtain (1.1) for the eigenproblem considered in this paper.

The paper is organized as follows: in Section 2, we introduce the spectral problem and the corresponding variational formulation, which leads to a quadratic eigenvalue problem. We introduce an auxiliary unknown to transform the quadratic eigenvalue problem into a linear one. Moreover, we introduce the corresponding solution operator for the spectral problem. In Section 3, we provide a thorough spectral characterization of the solution operator, based on the theory developed in [21]. We also consider the limit problem (i.e., the case when the viscosity vanishes) and the relation

between the solutions of the dissipative and non-dissipative problems. In Section 4, we introduce a finite element discretization using Raviart-Thomas elements for both fluids and imposing the continuity of the corresponding normal components on the interface. We analyze the discrete spectral problem analogously as in the continuous case and introduce the corresponding discrete solution operator. We use the abstract theory from [15] to prove the convergence. We also prove error estimates for our problem by adapting the arguments from [4]. Finally, in Section 5, we report some numerical tests which allow us to asses the performance of the proposed method.

Throughout the paper, Ω is a generic Lipschitz bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$, with outer unit normal vector $\boldsymbol{n}$. We denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth function compactly supported in Ω. For $r \geq 0$, $\|\cdot\|_{r,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $\mathrm{H}^r(\Omega)$ or $\mathrm{H}^r(\Omega)^d$ with the convention $\mathrm{H}^0(\Omega) := \mathrm{L}^2(\Omega)$. We also define the Hilbert space $\mathrm{H}(\mathrm{div};\Omega) := \{\boldsymbol{v} \in \mathrm{L}^2(\Omega)^d : \ \mathrm{div}\, \boldsymbol{v} \in \mathrm{L}^2(\Omega)\}$, whose norm is given by $\|\boldsymbol{v}\|^2_{\mathrm{div},\Omega} := \|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{div}\, \boldsymbol{v}\|^2_{0,\Omega}$, and its subspace $\mathrm{H}_0(\mathrm{div};\Omega) := \{\boldsymbol{v} \in \mathrm{H}(\mathrm{div};\Omega) : \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

Finally, C represents a generic constant independent of the discretization parameters, which may take different values at different places.

2 The model problem

We take as our model problem the case of two immiscible fluids within a rigid cavity. Let Ω_i with $i = 1, 2$ be the polygonal (in the 2D case) or polyhedral (in the 3D case) Lipschitz domains occupied by each of the fluids. Let ρ_i be the corresponding densities, ν_i the fluid viscosities, and c_i the acoustic speeds, which we consider all constant, ρ_i and c_i strictly positive and ν_i non negative. We denote by $\boldsymbol{n}_i$ the outward unit normal vectors corresponding to each subdomain. We define $\Omega := (\overline{\Omega}_1 \cup \overline{\Omega}_2)^\circ$, $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$, and $\Gamma_i := \partial\Omega_i \cap \partial\Omega$, $i = 1, 2$. We assume that each domain Ω_i as well as Ω are simply connected (see Figure 1).

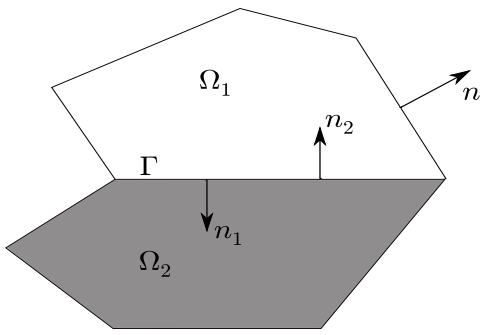

Figure 1: 2D sketch of the polygonal domains for the fluids.

We consider small displacements of a compressible viscous fluid at rest neglecting convective terms. The equation of motion derived from the Navier-Stokes equation reads

$$\rho_i \ddot{\boldsymbol{U}}_i = 2\nu_i \Delta \dot{\boldsymbol{U}}_i - \nabla P_i \quad \text{in } \Omega_i$$

where $\boldsymbol{U}_i$ denotes the fluid displacement and P_i the pressure fluctuation in the domain Ω_i, $i = 1, 2$. The dot represents derivation with respect to time. (See [8] and [37] for a more detailed derivation).

Moreover, since the fluid is compressible, we consider the isentropic relation

$$P_i + \rho_i c_i^2 \operatorname{div} \boldsymbol{U}_i = 0 \qquad \text{in } \Omega_i. \tag{2.2}$$

Since we are considering irrotational fluids, we assume that $\mathbf{curl}\, \boldsymbol{U}_i = \mathbf{0}$. Hence, considering the identity $\Delta \dot{\boldsymbol{U}}_i = \nabla(\operatorname{div} \dot{\boldsymbol{U}}_i) - \mathbf{curl}(\mathbf{curl}\, \dot{\boldsymbol{U}}_i)$, we conclude that $\Delta \dot{\boldsymbol{U}}_i = \nabla(\operatorname{div} \dot{\boldsymbol{U}}_i)$. Then, the equations of our model problem are the following:

$$\rho_1 \ddot{\boldsymbol{U}}_1 - 2\nu_1 \nabla(\operatorname{div} \dot{\boldsymbol{U}}_1) + \nabla P_1 = \mathbf{0} \quad \text{in } \Omega_1 \times (0,T), \tag{2.3}$$

$$P_1 + \rho_1 c_1^2 \operatorname{div} \boldsymbol{U}_1 = 0 \quad \text{in } \Omega_1 \times [0,T], \tag{2.4}$$

$$\rho_2 \ddot{\boldsymbol{U}}_2 - 2\nu_2 \nabla(\operatorname{div} \dot{\boldsymbol{U}}_2) + \nabla P_2 = \mathbf{0} \quad \text{in } \Omega_2 \times (0,T), \tag{2.5}$$

$$P_2 + \rho_2 c_2^2 \operatorname{div} \boldsymbol{U}_2 = 0 \quad \text{in } \Omega_2, \times [0,T], \tag{2.6}$$

$$\boldsymbol{U}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{U}_2 \cdot \boldsymbol{n}_2 = 0 \quad \text{on } \Gamma \times [0,T], \tag{2.7}$$

$$(2\nu_1 \operatorname{div} \dot{\boldsymbol{U}}_1 + P_1) - (2\nu_2 \operatorname{div} \dot{\boldsymbol{U}}_2 + P_2) = 0 \quad \text{on } \Gamma \times (0,T), \tag{2.8}$$

$$\boldsymbol{U}_1 \cdot \boldsymbol{n}_1 = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.9}$$

$$\boldsymbol{U}_2 \cdot \boldsymbol{n}_2 = 0 \quad \text{on } \Gamma_2 \times (0,T). \tag{2.10}$$

Let us remark that a similar argument leads exactly to the same equations in 2D.

Multiplying equations (2.3) and (2.5) by a test function $\boldsymbol{v} \in \mathrm{H}_0(\operatorname{div};\Omega)$, integrating by parts, and using the boundary conditions and the transmission conditions on Γ, we obtain

$$\int_\Omega \rho \ddot{\boldsymbol{U}} \cdot \boldsymbol{v} + 2 \int_\Omega \nu \operatorname{div} \dot{\boldsymbol{U}} \operatorname{div} \boldsymbol{v} - \int_\Omega P \operatorname{div} \boldsymbol{v} = 0 \quad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div},\Omega), \tag{2.11}$$

where

$$\boldsymbol{U} := \begin{cases} \boldsymbol{U}_1 & \text{in } \Omega_1, \\ \boldsymbol{U}_2 & \text{in } \Omega_2, \end{cases} \quad P := \begin{cases} P_1 & \text{in } \Omega_1, \\ P_2 & \text{in } \Omega_2, \end{cases} \quad \nu := \begin{cases} \nu_1 & \text{in } \Omega_1, \\ \nu_2 & \text{in } \Omega_2, \end{cases}$$

$$\rho := \begin{cases} \rho_1 & \text{in } \Omega_1, \\ \rho_2 & \text{in } \Omega_2, \end{cases} \quad \text{and } c := \begin{cases} c_1 & \text{in } \Omega_1, \\ c_2 & \text{in } \Omega_2. \end{cases}$$

Using (2.4) and (2.6) we eliminate P in (2.11) and write

$$\int_\Omega \rho \ddot{\boldsymbol{U}} \cdot \boldsymbol{v} + 2 \int_\Omega \nu \operatorname{div} \dot{\boldsymbol{U}} \operatorname{div} \boldsymbol{v} + \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{U} \operatorname{div} \boldsymbol{v} = 0 \qquad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div},\Omega). \tag{2.12}$$

The *vibration modes* of this problem are complex solutions of the form $\boldsymbol{U}(\boldsymbol{x},t) = e^{\lambda t}\boldsymbol{u}(\boldsymbol{x})$ with $\lambda \in \mathbb{C}$. Looking for this kind of solutions leads to the following quadratic eigenvalue problem:

Problem 1. Find $\lambda \in \mathbb{C}$ and $\mathbf{0} \neq \boldsymbol{u} \in \mathrm{H}_0(\operatorname{div};\Omega)$ such that

$$\lambda^2 \int_\Omega \rho \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + 2\lambda \int_\Omega \nu \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = 0 \qquad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div};\Omega).$$

Let us remark that in absence of viscosity (i.e., $\nu = 0$) we are left with the free vibration problem of two inviscid fluids in contact (whose numerical approximation has not been analyzed either). The eigenvalues λ^2 of such a problem are negative real numbers (as will be proved below), so that λ are purely imaginary, namely, $\lambda = \pm i\omega$ with ω being the so called *natural vibration frequencies*, which correspond to periodic in time solutions $\boldsymbol{U}(\boldsymbol{x},t) = e^{-i\omega t}\boldsymbol{u}(\boldsymbol{x})$ of the time domain problem. This is the reason why, for $\nu = 0$, Problem 1 is usually written as follows: Find $\omega > 0$ and $\mathbf{0} \neq \boldsymbol{u} \in \mathrm{H}_0(\operatorname{div};\Omega)$ such that

$$\int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = \omega^2 \int_\Omega \rho \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div};\Omega). \tag{2.13}$$

In the applications, ν is typically very small. As we will show below, in such a case there are eigenvalues λ of Problem 1 that lie close to $\pm i\omega$ with ω being a natural vibration frequency (i.e., a solution of (2.13)). Actually, we will prove below that those λ converge to $\pm i\omega$ as $\|\nu\|_{\infty,\Omega}$ goes to zero. On solving Problem 1, the aim is to compute the eigenvalues λ close to the smallest natural vibration frequencies $\omega > 0$, which are the most relevant in the applications.

Remark 2.1. In the case of a homogeneous viscous fluid, ρ, c and ν are constant in the whole Ω. Then, Problem 1 can be written as

$$\lambda^2 \int_\Omega \rho \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + \frac{2\lambda\nu + \rho c^2}{\rho c^2} \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = 0 \qquad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div}, \Omega).$$

Hence, in such a case, $(\lambda, \boldsymbol{u})$ is an eigenpair of Problem 1 if and only if $-\frac{\lambda^2 \rho c^2}{2\lambda\nu + \rho c^2} = \omega^2$ with $(\omega, \boldsymbol{u})$ being a solution to problem (2.13). Therefore, for a homogeneous viscous fluid, λ can be algebraically computed from the solution of (2.13) as follows:

$$\lambda = \frac{-\nu\omega^2 \pm \sqrt{\nu^2\omega^4 - \rho^2 c^4 \omega^2}}{\rho c^2}.$$

We denote $\mathcal{H} := \mathrm{L}^2(\Omega)^d$ endowed with the weighted inner product

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{H}} := \int_\Omega \rho \boldsymbol{v} \cdot \overline{\boldsymbol{w}}$$

and $\mathcal{V} := \mathrm{H}_0(\operatorname{div}; \Omega)$ with the inner product

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{V}} := \int_\Omega \rho \boldsymbol{v} \cdot \overline{\boldsymbol{w}} + \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{v} \operatorname{div} \overline{\boldsymbol{w}}.$$

Notice that the inner products in $\mathcal{H}$ and $\mathcal{V}$ induce norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{V}}$ on each of these spaces equivalent to the classical $\mathrm{L}^2(\Omega)^d$ and $\mathrm{H}(\operatorname{div}; \Omega)$ norms, respectively. Therefore, when it might be convenient, we will use these classical norms.

Clearly $\lambda = 0$ is an eigenvalue of Problem 1 with associated eigenspace

$$\mathcal{K} = \mathrm{H}_0(\operatorname{div}^0, \Omega) := \{\boldsymbol{v} \in \mathrm{H}_0(\operatorname{div}; \Omega) : \ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}.$$

We define:

$$\mathcal{G} := \mathcal{K}^{\perp_{\mathcal{V}}} = \{\boldsymbol{v} \in \mathcal{V} : (\boldsymbol{v}, \boldsymbol{w})_{\mathcal{V}} = 0 \quad \forall \boldsymbol{w} \in \mathcal{K}\}.$$

Since $\mathcal{K}$ is a closed subspace of $\mathcal{V}$, clearly $\mathcal{V} = \mathcal{G} \oplus \mathcal{K}$. Notice that $\mathcal{G}$ and $\mathcal{K}$ are also orthogonal in the $\mathcal{H}$ inner product. Hence,

$$\mathcal{G} = \{\boldsymbol{v} \in \mathcal{V} : (\boldsymbol{v}, \boldsymbol{w})_{\mathcal{H}} = 0 \ \ \forall \boldsymbol{w} \in \mathcal{K}\}.$$

The following result brings a characterization of the space $\mathcal{G}$.

Lemma 2.1. *There holds*

$$\mathcal{G} = \frac{1}{\rho} \nabla(\mathrm{H}^1(\Omega)) \cap \mathcal{V}.$$

Proof. We will prove this result by checking the double inclusion. Let $\boldsymbol{v} \in \mathcal{G}$. Then, for all $\boldsymbol{\psi} \in \mathcal{D}(\Omega)^d$, since $\mathbf{curl}\, \boldsymbol{\psi} \in \mathcal{K}$, we have

$$0 = \int_\Omega \rho\, \mathbf{curl}\, \boldsymbol{\psi} \cdot \overline{\boldsymbol{v}} = \int_\Omega \boldsymbol{\psi} \cdot \mathbf{curl}\, (\rho \overline{\boldsymbol{v}}).$$

Thus, $\mathbf{curl}\,(\rho\boldsymbol{v}) = \mathbf{0}$ in Ω. Since Ω is simply connected, this implies that there exists $\varphi \in \mathrm{H}^1(\Omega)$ such that $\rho\boldsymbol{v} = \nabla\varphi$. Hence, $\boldsymbol{v} \in \frac{1}{\rho}\nabla(\mathrm{H}^1(\Omega)) \cap \mathcal{V}$. Conversely, let $\boldsymbol{v} \in \frac{1}{\rho}\nabla(\mathrm{H}^1(\Omega)) \cap \mathcal{V}$ and $\boldsymbol{w} \in \mathcal{K}$. Let $\varphi \in \mathrm{H}^1(\Omega)$ be such that $\boldsymbol{v} = \frac{1}{\rho}\nabla\varphi$. Then,

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{H}} = \int_\Omega \rho \left(\frac{1}{\rho}\nabla\varphi\right) \cdot \overline{\boldsymbol{w}} = -\int_\Omega \varphi \operatorname{div} \overline{\boldsymbol{w}} + \int_{\partial\Omega} \varphi(\overline{\boldsymbol{w}} \cdot \boldsymbol{n}) = 0.$$

Therefore, $\boldsymbol{v} \in \mathcal{G}$. The proof is complete. □

In what follows we prove additional regularity for the functions in $\mathcal{G}$ on each subdomain. From now on, s will denote a positive number such that the following lemma holds true.

Lemma 2.2. *There exists $s > 0$ (with s depending on ρ, Ω_1 and Ω_2) such that, for all $\boldsymbol{v} \in \mathcal{G}$, $\boldsymbol{v} \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$ and*

$$\|\boldsymbol{v}\|_{s,\Omega_1} + \|\boldsymbol{v}\|_{s,\Omega_2} \leq C\|\operatorname{div} \boldsymbol{v}\|_{0,\Omega}, \tag{2.14}$$

where C is a positive constant independent of $\boldsymbol{v}$.

Proof. According to Lemma 2.1, there exists $\varphi \in \mathrm{H}^1(\Omega)$ such that $\boldsymbol{v} = \frac{1}{\rho}\nabla\varphi$. Consequently, $\varphi \in \mathrm{H}^1(\Omega)/\mathbb{C}$ is the unique solution of the following well-posed Neumann problem:

$$\begin{aligned} \operatorname{div}\left(\frac{1}{\rho}\nabla\varphi\right) &= \operatorname{div} \boldsymbol{v} \quad \text{in } \Omega, \\ \frac{1}{\rho}\frac{\partial\varphi}{\partial\boldsymbol{n}} &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Hence, in the 3D case, the result follows from [34, Lemma 2.20], while in the 2D case, it follows by applying [34, Lemma 4.3]. (See [36] for more details.) □

Remark 2.2. The above lemma establishes the existence of a regularity exponent $s > 0$ that will play a role in the error estimates of the numerical method proposed in this paper. In spite of the fact that we refer to [34] in the proof of this lemma, the value of s that arises from this reference is far from being optimal, since it is valid for global regularity in $\mathrm{H}^s(\Omega)^d$ and for any arbitrary geometrical setting of the subdomains Ω_1 and Ω_2. In most of the applications, the subdomains at which ρ is constant are similar to those shown in Figure 1. In such a case, a detailed analysis of how this exponent depends on the geometry of the domain and on the coefficient ρ can be found in [25] for the 2D case and in [24] for 3D problems (see also [9, 33]). Let us remark that, although the analyses of these references is for problems with Dirichlet boundary conditions, similar results hold true for Neumann boundary conditions as in our case (see [25, Remark III.5.2]). In particular, for instance, Lemma 2.2 holds true for $s = 1$ in the example reported as Test 1 in Section 5 (see Figure 2).

From the physical point of view, the time domain problem (2.12) is dissipative in the sense that its solution should decay as t increases. The latter happens if and only if the so called *decay rate*, $\mathrm{Re}(\lambda)$, is negative. The following result shows that this is the case in our formulation.

Lemma 2.3. *Let $(\lambda, \boldsymbol{u}) \in \mathbb{C} \times \mathcal{V}$ be a solution of Problem 1. If $\lambda \neq 0$, then $Re(\lambda) < 0$.*

Proof. The proof is similar to that of Lemma 2.1 of [4]. □

Remark 2.3. Any eigenpair $(\lambda, \boldsymbol{u})$ of Problem 1 satisfies

$$\lambda^2 \int_\Omega \rho\boldsymbol{u} \cdot \overline{\boldsymbol{v}} + \int_\Omega (2\lambda\nu + \rho c^2) \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}.$$

Since the coefficients are constant in each subdomain, if $2\lambda\nu + \rho c^2 \neq 0$ in Ω_i, by testing with $\boldsymbol{v} \in \mathcal{D}(\Omega_i)^d$ we obtain that $\operatorname{div} \boldsymbol{u}|_{\Omega_i} \in \mathrm{H}^1(\Omega_i)$, $i = 1, 2$. On the other hand, if $2\lambda\nu + \rho c^2 = 0$ in Ω_i ($i = 1$ or 2), then, for $\lambda \neq 0$, by testing again with $\boldsymbol{v} \in \mathcal{D}(\Omega_i)^d$, we obtain that $\boldsymbol{u} = \boldsymbol{0}$ in Ω_i. Thus, in any case, $\operatorname{div} \boldsymbol{u}|_{\Omega_i} \in \mathrm{H}^1(\Omega_i)$, $i = 1, 2$.

For the theoretical analysis it is convenient to transform Problem 1 into a linear eigenvalue problem. With this aim we introduce the new variable $\widehat{\boldsymbol{u}} := \lambda \boldsymbol{u}$, as usual in quadratic problems, and the space $\widetilde{\mathcal{V}} := \mathcal{V} \times \mathcal{H}$ endowed with the corresponding product norm, which carry us to the following:

Problem 2. Find $\lambda \in \mathbb{C}$ and $\boldsymbol{0} \neq (\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \widetilde{\mathcal{V}}$ such that

$$\int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = \lambda \left(-2 \int_\Omega \nu \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} - \int_\Omega \rho \widehat{\boldsymbol{u}} \cdot \overline{\boldsymbol{v}} \right) \qquad \forall \boldsymbol{v} \in \mathcal{V}, \tag{2.15}$$

$$\int_\Omega \rho \widehat{\boldsymbol{u}} \cdot \overline{\widehat{\boldsymbol{v}}} = \lambda \int_\Omega \rho \boldsymbol{u} \cdot \overline{\widehat{\boldsymbol{v}}} \qquad \forall \widehat{\boldsymbol{v}} \in \mathcal{H}. \tag{2.16}$$

We observe that $\lambda = 0$ is an eigenvalue of Problem 2 and its associated eigenspace is $\widetilde{\mathcal{K}} := \mathcal{K} \times \{0\}$. Let $\widetilde{\mathcal{G}}$ be the orthogonal complement of $\widetilde{\mathcal{K}}$ in $\mathcal{V} \times \mathcal{H}$. Notice that $\widetilde{\mathcal{G}} = \mathcal{G} \times \mathcal{H}$.

We introduce the sesquilinear continuous form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}},$$

and the sesquilinear continuous forms $\widetilde{a}, \widetilde{b} : \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}$ defined as follows:

$$\widetilde{a}((\boldsymbol{u}, \widehat{\boldsymbol{u}}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) := \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + \int_\Omega \rho \widehat{\boldsymbol{u}} \cdot \overline{\widehat{\boldsymbol{v}}},$$

$$\widetilde{b}((\boldsymbol{u}, \widehat{\boldsymbol{u}}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) := -2 \int_\Omega \nu \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} - \int_\Omega \rho \widehat{\boldsymbol{u}} \cdot \overline{\boldsymbol{v}} + \int_\Omega \rho \boldsymbol{u} \cdot \overline{\widehat{\boldsymbol{v}}}.$$

In what follows we prove that $a(\cdot, \cdot)$ and $\widetilde{a}(\cdot, \cdot)$ are elliptic in $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, respectively.

Lemma 2.4. *The sesquilinear form $a : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ is $\mathcal{G}$-elliptic and, consequently, $\widetilde{a} : \widetilde{\mathcal{G}} \times \widetilde{\mathcal{G}} \to \mathbb{C}$ is $\widetilde{\mathcal{G}}$-elliptic.*

Proof. The proof is a direct consequence of Lemma 2.2. □

Let $\boldsymbol{T} : \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}$ be the bounded linear operator defined by $\boldsymbol{T}(\boldsymbol{f}, \boldsymbol{g}) := (\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \widetilde{\mathcal{G}}$, where $(\boldsymbol{u}, \widehat{\boldsymbol{u}})$ is the unique solution of the following problem:

$$\widetilde{a}((\boldsymbol{u}, \widehat{\boldsymbol{u}}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) = \widetilde{b}((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) \qquad \forall (\boldsymbol{v}, \widehat{\boldsymbol{v}}) \in \widetilde{\mathcal{G}}.$$

It is easy to check that

$$\widehat{\boldsymbol{u}} = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.17}$$

and

$$\int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = -2 \int_\Omega \nu \operatorname{div} \boldsymbol{f} \operatorname{div} \overline{\boldsymbol{v}} - \int_\Omega \rho \boldsymbol{g} \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}. \tag{2.18}$$

As a consequence of the above equalities, we have that $\mu = 0$ is an eigenvalue of $\boldsymbol{T}$ with associated eigenspace $\{\boldsymbol{0}\} \times \mathcal{G}^{\perp_{\mathcal{H}}}$, which is nontrivial since $\mathcal{G}^{\perp_{\mathcal{H}}} \supset \mathcal{K}$. The following lemma shows that the nonzero eigenvalues of $\boldsymbol{T}$ are exactly the reciprocals of the nonzero eigenvalues of Problem 2 with the same corresponding eigenfunctions.

Lemma 2.5. *There holds that $(\mu, (\boldsymbol{u}, \widehat{\boldsymbol{u}}))$ is an eigenpair of $\boldsymbol{T}$ (i.e. $\boldsymbol{T}(\boldsymbol{u}, \widehat{\boldsymbol{u}}) = \mu(\boldsymbol{u}, \widehat{\boldsymbol{u}})$) with $\mu \neq 0$ if and only if $(\lambda, (\boldsymbol{u}, \widehat{\boldsymbol{u}}))$ is a solution of Problem 2 with $\lambda = 1/\mu \neq 0$.*

Proof. The proof is similar to that of Lemma 2.3 from [4]. □

3 Spectral Characterization

The goal of this section is to characterize the spectrum of the solution operator $\boldsymbol{T}$. Since the inclusion $\mathrm{H}_0(\mathrm{div};\Omega) \hookrightarrow \mathrm{L}^2(\Omega)^d$ is not compact, it is easy to check from (2.17) that $\boldsymbol{T}$ is not compact either. However, we will show that the essential spectrum, has to lie in a small region of the complex plane, well separated from the isolated eigenvalues which, according to Lemma 2.5, correspond to the solutions of Problem 2. With this aim, we will resort to the theory described in [21] to decompose appropriately $\boldsymbol{T}$. Let $\boldsymbol{T}_1, \boldsymbol{T}_2 : \mathcal{G} \to \mathcal{G}$ be the operators given by

$$\boldsymbol{T}_1\boldsymbol{f} = \boldsymbol{u}_1 \in \mathcal{G} : \quad a(\boldsymbol{u}_1, \boldsymbol{v}) = 2\int_\Omega \nu \,\mathrm{div}\,\boldsymbol{f}\,\mathrm{div}\,\overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}, \tag{3.1}$$

$$\boldsymbol{T}_2\boldsymbol{g} = \boldsymbol{u}_2 \in \mathcal{G} : \quad a(\boldsymbol{u}_2, \boldsymbol{v}) = \int_\Omega \rho \boldsymbol{g} \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}. \tag{3.2}$$

It is easy to check that these operators are self-adjoint with respect to $a(\cdot,\cdot)$. Moreover $\boldsymbol{T}_1$ is non-negative and $\boldsymbol{T}_2$ is positive with respect to $a(\cdot,\cdot)$ (namely, $a(\boldsymbol{T}_1\boldsymbol{v}, \boldsymbol{v}) \geq 0$ $\forall \boldsymbol{v} \in \mathcal{G}$ and $a(\boldsymbol{T}_2\boldsymbol{v}, \boldsymbol{v}) > 0$ $\forall \boldsymbol{v} \in \mathcal{G}$, $\boldsymbol{v} \neq \boldsymbol{0}$). Moreover, we have the following result.

Lemma 3.1. *The operator $\boldsymbol{T}_2 : \mathcal{G} \to \mathcal{G}$ is compact.*

Proof. Since $a(\cdot,\cdot)$ is $\mathcal{G}$-elliptic (cf. Lemma 2.4), applying Lax-Milgram's Lemma, we know that problem (3.2) is well posed and has a unique solution $\boldsymbol{u}_2 \in \mathcal{G}$. Moreover, according to Lemma 2.2, we know that there exists $s > 0$ such that $\boldsymbol{u}_2 \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$. On the other hand, notice that (3.2) also holds for $\boldsymbol{v} \in \mathcal{K}$, since in such a case $a(\boldsymbol{u}_2, \boldsymbol{v}) = 0 = \int_\Omega \rho \boldsymbol{g} \cdot \overline{\boldsymbol{v}}$ for $\boldsymbol{g} \in \mathcal{G}$. Hence, since $\mathcal{V} = \mathcal{G} \oplus \mathcal{K}$, we have that

$$a(\boldsymbol{u}_2, \boldsymbol{v}) = \int_\Omega \rho \boldsymbol{g} \cdot \overline{\boldsymbol{v}} \quad \forall \boldsymbol{v} \in \mathcal{V}.$$

Then, by testing this equation with $\boldsymbol{v} \in \mathcal{D}(\Omega)^d \subset \mathcal{V}$, we have that $-\nabla(\rho c^2 \,\mathrm{div}\,\boldsymbol{u}_2) = \rho \boldsymbol{g}$ in Ω, so that $\rho c^2 \,\mathrm{div}\,\boldsymbol{u}_2 \in \mathrm{H}^1(\Omega)$. Therefore, since ρ and c are positive constants in each subdomain Ω_1 and Ω_2, we have that $\mathrm{div}\,\boldsymbol{u}_2|_{\Omega_i} \in \mathrm{H}^1(\Omega_i)$, $i = 1, 2$. Since the inclusions $\{v \in \mathrm{L}^2(\Omega) : v|_{\Omega_i} \in \mathrm{H}^1(\Omega_i),\ i = 1, 2.\} \subset \mathrm{L}^2(\Omega)$ and $\mathrm{H}^s(\Omega_1 \cup \Omega_2)^d \subset \mathrm{L}^2(\Omega)^d$, are compact, we derive that $\boldsymbol{T}_2$ is compact too. □

The operator $\boldsymbol{T}$ can be written in terms of the operators $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ given above as follows:

$$\boldsymbol{T} = \begin{pmatrix} -\boldsymbol{T}_1 & -\boldsymbol{T}_2 \\ \boldsymbol{I} & \boldsymbol{0} \end{pmatrix}.$$

Moreover, by defining as in [21] the operators

$$\boldsymbol{S} := \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}_2^{1/2} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{H} := \begin{pmatrix} -\boldsymbol{T}_1 & -\boldsymbol{T}_2^{1/2} \\ \boldsymbol{T}_2^{1/2} & \boldsymbol{0} \end{pmatrix},$$

we have that $\boldsymbol{ST} = \boldsymbol{HS}$. We note that the eigenvalues of $\boldsymbol{T}$ and $\boldsymbol{H}$ and their algebraic multiplicities coincide. Moreover the corresponding Jordan chains have the same length. In fact, let $\{\boldsymbol{x}_k\}_{k=1}^r$ be a Jordan chain associated with the eigenvalue μ of $\boldsymbol{T}$. Then, using the identities above, we observe that

$$\boldsymbol{HSx}_k = \boldsymbol{STx}_k = \boldsymbol{S}(\mu \boldsymbol{x}_k + \boldsymbol{x}_{k-1}) = \mu \boldsymbol{Sx}_k + \boldsymbol{Sx}_{k-1}, \quad k = 1, \ldots, r.$$

This shows that $\{\boldsymbol{Sx}_k\}_{k=1}^r$ is a Jordan chain of $\boldsymbol{H}$ of the same length. Actually, the whole spectra of $\boldsymbol{T}$ and $\boldsymbol{H}$ coincide as is shown in the following result, which has been proved in Lemma 3.2 of [4].

Lemma 3.2. *There holds*

$$\mathrm{Sp}(\boldsymbol{T}) = \mathrm{Sp}(\boldsymbol{H}).$$

Moreover, $\mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{T}) = \mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{H})$*, too.*

The operator $\boldsymbol{H}$ can be written as the sum of a self-adjoint operator $\boldsymbol{B}$ and a compact operator $\boldsymbol{C}$:

$$\boldsymbol{H} = \boldsymbol{B} + \boldsymbol{C} \quad \text{with} \quad \boldsymbol{B} := \begin{pmatrix} -\boldsymbol{T}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \quad \text{and} \quad \boldsymbol{C} := \begin{pmatrix} \boldsymbol{0} & -\boldsymbol{T}_2^{1/2} \\ \boldsymbol{T}_2^{1/2} & \boldsymbol{0} \end{pmatrix}.$$

Then, applying the classical Weyl's Theorem (see [38]), we have that $\mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{H}) = \mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{B})$ and the rest of the spectrum $\mathrm{Sp}_{\mathrm{disc}}(\boldsymbol{H}) := \mathrm{Sp}(\boldsymbol{H}) \setminus \mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{H})$ consists of isolated eigenvalues with finite algebraic multiplicity. Moreover, $\mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{B}) = \mathrm{Sp}_{\mathrm{ess}}(-\boldsymbol{T}_1) \cup \{0\}$.

Our next goal is to show that the essential spectrum of $\boldsymbol{T}_1$ must lie in a small region of the complex plane. Actually, we will localize the whole spectrum of $\boldsymbol{T}_1$. With this aim, we analyze separately for which values $\mu \in \mathbb{C}$, the operator $(\mu \boldsymbol{I} - \boldsymbol{T}_1)$ is not necessarily one-to-one and for which values it is not necessarily onto.

- If $(\mu \boldsymbol{I} - \boldsymbol{T}_1)$ is not one-to-one, then there exists $\boldsymbol{f} \in \mathcal{G}$, $\boldsymbol{f} \neq \boldsymbol{0}$, such that $\boldsymbol{T}_1 \boldsymbol{f} = \mu \boldsymbol{f}$, namely,

  $$\mu \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{f} \operatorname{div} \overline{\boldsymbol{v}} = 2 \int_\Omega \nu \operatorname{div} \boldsymbol{f} \operatorname{div} \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}.$$

  Then, testing with $\boldsymbol{v} = \boldsymbol{f}$ and using that in each subdomain the coefficients ρ and c are positive, we deduce that

  $$\mu = \frac{2 \int_\Omega \nu \, |\operatorname{div} \boldsymbol{f}|^2}{\int_\Omega \rho c^2 \, |\operatorname{div} \boldsymbol{f}|^2}$$

  (we recall that for $\boldsymbol{0} \neq \boldsymbol{f} \in \mathcal{G}$, $\int_\Omega |\operatorname{div} \boldsymbol{f}|^2 > 0$ because of Lemma 2.2). Hence,

  $$\mu \in \left[\frac{2 \min_\Omega\{\nu\}}{\max_\Omega\{\rho c^2\}}, \frac{2 \max_\Omega\{\nu\}}{\min_\Omega\{\rho c^2\}} \right].$$

- On the other hand, $(\mu \boldsymbol{I} - \boldsymbol{T}_1)$ is onto if and only if for any $\boldsymbol{g} \in \mathcal{G}$ there exists $\boldsymbol{f} \in \mathcal{G}$ such that $\boldsymbol{T}_1 \boldsymbol{f} = \mu \boldsymbol{f} - \boldsymbol{g}$, which from (3.1) reads

  $$\int_\Omega \rho c^2 \operatorname{div} \boldsymbol{g} \operatorname{div} \overline{\boldsymbol{v}} = \int_\Omega (-2\nu + \mu \rho c^2) \operatorname{div} \boldsymbol{f} \operatorname{div} \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}.$$

  By writing $\mu = \alpha + \beta i$ with $\alpha, \beta \in \mathbb{R}$, the equation above reads:

  $$\int_\Omega (-2\nu + \alpha \rho c^2 + \rho c^2 \beta i) \operatorname{div} \boldsymbol{f} \operatorname{div} \overline{\boldsymbol{v}} = \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{g} \operatorname{div} \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}.$$

  We observe that for all $\beta \neq 0$ the problem above has a solution and hence the operator $(\mu \boldsymbol{I} - \boldsymbol{T}_1)$ is onto. On the other hand, if $\beta = 0$, then μ has to be real. In such a case, the operator $\boldsymbol{T}_1$ will still be onto when $(-2\nu + \mu \rho c^2)$ has the same sign in the whole domain Ω. This happens at least in two cases:

  (i) when $\mu > \dfrac{2 \max_\Omega\{\nu\}}{\min_\Omega\{\rho c^2\}}$, in which case $-2\nu + \mu \rho c^2 > 0$,

  (ii) when $\mu < \dfrac{2 \min_\Omega\{\nu\}}{\max_\Omega\{\rho c^2\}}$, in which case $-2\nu + \mu \rho c^2 < 0$.

  Therefore, if $(\mu \boldsymbol{I} - \boldsymbol{T}_1)$ is not onto, then $\mu \in \left[\frac{2 \min_\Omega\{\nu\}}{\max_\Omega\{\rho c^2\}}, \frac{2 \max_\Omega\{\nu\}}{\min_\Omega\{\rho c^2\}} \right]$, too.

Now we are in position to write the following spectral characterization of the solution operator $\boldsymbol{T}$.

Theorem 3.1. *The spectrum of* $\boldsymbol{T}$ *consists of*

$$\mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{T}) = \mathrm{Sp}(-\boldsymbol{T}_1) \cup \{0\}$$

with

$$\mathrm{Sp}(\boldsymbol{T}_1) \subset \left[\frac{2\min_\Omega\{\nu\}}{\max_\Omega\{\rho c^2\}}, \frac{2\max_\Omega\{\nu\}}{\min_\Omega\{\rho c^2\}}\right]$$

and $\mathrm{Sp}_{\mathrm{disc}}(\boldsymbol{T}) := \mathrm{Sp}(\boldsymbol{T}) \setminus \mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{T})$*, which is a set of isolated eigenvalues of finite algebraic multiplicity.*

Proof. As a consequence of the classical Weyl's Theorem (see [38]) and Lemma 3.2,

$$\mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{T}) = \mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{H}) = \mathrm{Sp}_{\mathrm{ess}}(\boldsymbol{B}) = \mathrm{Sp}_{\mathrm{ess}}(-\boldsymbol{T}_1) \cup \{0\},$$

whereas the inclusion follows from the above analysis. □

In what follows, we will show that for ν small enough some of the eigenvalues of $\boldsymbol{T}$ are well separated from its essential spectrum. With this end, given $\boldsymbol{f} \in \mathcal{G}$, by testing (3.1) with $\boldsymbol{v} = \boldsymbol{u}_1 \in \mathcal{G}$ and using the the definition of $a(\cdot,\cdot)$, we have that

$$\min_\Omega\{\rho c^2\}\|\boldsymbol{u}_1\|^2_{\mathrm{div},\Omega} \le a(\boldsymbol{u}_1, \boldsymbol{u}_1) \le 2\|\nu\|_{\infty,\Omega}\|\operatorname{div}\boldsymbol{f}\|_{0,\Omega}\|\boldsymbol{u}_1\|_{\mathrm{div},\Omega}.$$

Therefore $\|\boldsymbol{T}_1\|_{\mathcal{L}(\mathcal{G}\times\mathcal{G})} \to 0$ as $\|\nu\|_{\infty,\Omega}$ goes to zero. Consequently, $\boldsymbol{H}$ converges in norm to the operator

$$\boldsymbol{H}_0 := \begin{pmatrix} \boldsymbol{0} & -\boldsymbol{T}_2^{1/2} \\ \boldsymbol{T}_2^{1/2} & \boldsymbol{0} \end{pmatrix}$$

as $\|\nu\|_{\infty,\Omega}$ goes to zero. Thus, from the classical spectral approximation theory (see [19]), the isolated eigenvalues of $\boldsymbol{H}$ converge to those of $\boldsymbol{H}_0$.

Since the isolated eigenvalues of $\boldsymbol{H}$ and $\boldsymbol{T}$ coincide (cf. Lemma 3.2), in order to localize those of $\boldsymbol{T}$, we begin by characterizing those of $\boldsymbol{H}_0$. Let μ be an isolated eigenvalue of $\boldsymbol{H}_0$ and $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \mathcal{G}\times\mathcal{G}$ the corresponding eigenfunction. It is easy to check that

$$\boldsymbol{H}_0\begin{pmatrix}\boldsymbol{u}\\ \widehat{\boldsymbol{u}}\end{pmatrix} = \mu\begin{pmatrix}\boldsymbol{u}\\ \widehat{\boldsymbol{u}}\end{pmatrix} \quad\Longleftrightarrow\quad \boldsymbol{T}_2\boldsymbol{u} = -\mu^2\boldsymbol{u} \quad\text{and}\quad \boldsymbol{T}_2^{1/2}\boldsymbol{u} = \mu\widehat{\boldsymbol{u}}. \tag{3.3}$$

Since $\boldsymbol{T}_2$ is compact, self-adjoint, and positive, its spectrum consists of a sequence of positive eigenvalues that converge to zero and 0 itself. Notice that the spectrum of $\boldsymbol{T}_2$ is related with the solution of the eigenvalue problem (2.13). In fact, this problem has 0 as an eigenvalue with corresponding eigenspace $\mathcal{K}$. The rest of the eigenvalues ω^2 are strictly positive and the corresponding eigenfunctions $\boldsymbol{u} \in \mathcal{K}^{\perp_\nu} =: \mathcal{G}$, so that they are also solutions of the following problem: Find $\omega > 0$ and $\boldsymbol{u} \in \mathcal{G}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = \omega^2 \int_\Omega \rho\boldsymbol{u}\cdot\overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}.$$

Clearly $(\omega^2, \boldsymbol{u})$ is an eigenpair of the above problem with $\omega > 0$ if and only if $\boldsymbol{T}_2\boldsymbol{u} = \frac{1}{\omega^2}\boldsymbol{u}$. Thus, by virtue of (3.3), we have that the eigenvalues of $\boldsymbol{H}_0$ are given by $\pm i/\omega$ and hence they are purely imaginary.

Now we are in a position to establish the following result.

Theorem 3.2. *For each isolated eigenvalue* $\pm i/\omega$ *of* $\boldsymbol{H}_0$ *of algebraic multiplicity* m*, let* $r > 0$ *be such that the disc* $D_r := \{z \in \mathbb{C} : \ |z \mp i/\omega| < r\}$ *intersects* $\mathrm{Sp}(\boldsymbol{H}_0)$ *only in* $\pm i/\omega$*. Then, there exists* $\delta > 0$ *such that if* $\|\nu\|_{\infty,\Omega} < \delta$*, there exist* m *eigenvalues of* $\boldsymbol{T}$*,* $\mu_1, \ldots, \mu_m$*, (repeated according to their respective algebraic multiplicities) lying in the disc* D_r*. Moreover,* $\mu_1, \ldots, \mu_m \to \frac{i}{\omega}$ *as* $\|\nu\|_{\infty,\Omega}$ *goes to zero.*

As claimed above, the eigenvalues of $\boldsymbol{T}$ that are relevant in the applications, are those which are close to $\pm i/\omega$ for the smallest positive vibration frequencies ω of (2.13). According to the above theorem, these eigenvalues are well separated from the real axis and, hence, from the essential spectrum of $\boldsymbol{T}$ (cf. Theorem 3.1).

4 Spectral Approximation

In this section, we propose and analyze a finite element method to approximate the solutions of Problem 1. With this end, we introduce appropriate discrete spaces. Let $\{\mathcal{T}_h(\Omega)\}_{h>0}$ be a family of regular partitions of Ω such that $\mathcal{T}_h(\Omega_i) := \{T \in \mathcal{T}_h : T \subset \overline{\Omega}_i\}$ are partitions of Ω_i, $i = 1, 2$. We introduce the lowest-order Raviart-Thomas finite element space:

$$\mathcal{V}_h := \{\boldsymbol{v} \in \mathcal{V} : \boldsymbol{v}|_T(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x},\ \boldsymbol{a} \in \mathbb{R}^d,\ b \in \mathbb{R},\ \boldsymbol{x} \in T\}.$$

The discretization of Problem 1 reads as follows:

Problem 3. Find $\lambda_h \in \mathbb{C}$ and $\mathbf{0} \neq \boldsymbol{u}_h \in \mathcal{V}_h$ such that

$$\lambda_h^2 \int_\Omega \rho \boldsymbol{u}_h \cdot \overline{\boldsymbol{v}}_h + 2\lambda_h \int_\Omega \nu \operatorname{div} \boldsymbol{u}_h \operatorname{div} \overline{\boldsymbol{v}}_h + \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u}_h \operatorname{div} \overline{\boldsymbol{v}}_h = 0 \qquad \forall \boldsymbol{v}_h \in \mathcal{V}_h.$$

We proceed as we did in the continuous case and introduce a new discrete variable $\widehat{\boldsymbol{u}}_h := \lambda_h \boldsymbol{u}_h$ to rewrite the problem above in the following equivalent form:

Problem 4. Find $\lambda_h \in \mathbb{C}$ and $\mathbf{0} \neq (\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in \mathcal{V}_h \times \mathcal{V}_h$ such that

$$\int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u}_h \operatorname{div} \overline{\boldsymbol{v}}_h = \lambda_h \left(-2 \int_\Omega \nu \operatorname{div} \boldsymbol{u}_h \operatorname{div} \overline{\boldsymbol{v}}_h - \int_\Omega \rho \widehat{\boldsymbol{u}}_h \cdot \overline{\boldsymbol{v}}_h \right) \qquad \forall \boldsymbol{v}_h \in \mathcal{V}_h,$$
$$\int_\Omega \rho \widehat{\boldsymbol{u}}_h \cdot \overline{\widehat{\boldsymbol{v}}}_h = \lambda_h \int_\Omega \rho \boldsymbol{u}_h \cdot \overline{\widehat{\boldsymbol{v}}}_h \qquad \forall \widehat{\boldsymbol{v}}_h \in \mathcal{V}_h.$$

We observe that $\lambda_h = 0$ is an eigenvalue of this problem and its associated eigenspace is $\widetilde{\mathcal{K}}_h := \mathcal{K}_h \times \{0\}$ with $\mathcal{K}_h := \mathcal{K} \cap \mathcal{V}_h$ being the eigenspace of $\lambda_h = 0$ in Problem 3. At the beginning of Section 5, we will show that Problem 4 is well posed, in the sense that it is equivalent to a generalized matrix eigenvalue problem with a symmetric positive definite right-hand side matrix.

We introduce the well known Raviart-Thomas interpolation operator, $\Pi_h : \mathcal{V} \cap \mathrm{H}^r(\Omega_1 \cup \Omega_2)^d \to \mathcal{V}_h$, $r \in (0, 1]$ (see [29]), for which there hold the approximation result

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{0,\Omega} \leq Ch^r (\|\boldsymbol{v}\|_{r,\Omega_1} + \|\boldsymbol{v}\|_{r,\Omega_2} + \|\operatorname{div} \boldsymbol{v}\|_{0,\Omega}) \tag{4.1}$$

and the commuting diagram property

$$\operatorname{div}(\Pi_h \boldsymbol{v}) = \mathcal{P}_h(\operatorname{div} \boldsymbol{v}), \tag{4.2}$$

where

$$\mathcal{P}_h : \mathrm{L}^2(\Omega) \to \mathcal{U}_h := \{v_h \in \mathrm{L}^2(\Omega) :\ v_h|_T \in \mathcal{P}_0(T)\ \ \forall T \in \mathcal{T}_h\}$$

is the standard L^2-orthogonal projector. Then, for any $r \in (0, 1]$ we have that

$$\|q - \mathcal{P}_h q\|_{0,\Omega} \leq Ch^r \|q\|_{r,\Omega} \qquad \forall q \in \mathrm{H}^r(\Omega). \tag{4.3}$$

Let $\mathcal{G}_h$ be the orthogonal complement of $\mathcal{K}_h$ in $\mathcal{V}_h$, and $\widetilde{\mathcal{G}}_h := \mathcal{G}_h \times \mathcal{G}_h \subset \widetilde{\mathcal{V}} = \mathcal{V} \times \mathcal{H}$ endowed with the corresponding product norm. Note that $\mathcal{G}_h \nsubseteq \mathcal{G}$ and hence $\widetilde{\mathcal{G}}_h \nsubseteq \widetilde{\mathcal{G}}$.

The following result provides estimates for the terms in the Helmholtz decomposition of functions in $\mathcal{G}_h$. Let us recall that, here and thereafter, $s > 0$ denotes the optimal regularity exponent such that Lemma 2.2 holds true.

Lemma 4.1. *For any $\boldsymbol{v}_h \in \mathcal{G}_h$, there exists $\xi \in \mathrm{H}^1(\Omega)$ and $\chi \in \mathcal{K}$ such that*

$$\boldsymbol{v}_h = \frac{1}{\rho}\nabla\xi + \chi$$

with $\frac{1}{\rho}\nabla\xi \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$, and the following estimates holds

$$\left\|\frac{1}{\rho}\nabla\xi\right\|_{s,\Omega_1} + \left\|\frac{1}{\rho}\nabla\xi\right\|_{s,\Omega_2} \leq C\|\operatorname{div}\boldsymbol{v}_h\|_{0,\Omega} \qquad \textit{and} \qquad \|\chi\|_{0,\Omega} \leq Ch^s\|\operatorname{div}\boldsymbol{v}_h\|_{0,\Omega}.$$

Proof. The proof follows by repeating the arguments of the proof of Lemma 4.1 from [4], taking care of the presence of the discontinuous coefficient ρ. □

Remark 4.1. An alternative proof for the previous lemma is based on Lemma 5.10 from [2], taking into account the regularity that has been proved in Lemma 2.2 of the present paper.

The following result is a direct consequence of Lemma 4.1.

Lemma 4.2. *The sesquilinear form $a : \mathcal{G}_h \times \mathcal{G}_h \to \mathbb{C}$ is $\mathcal{G}_h$-elliptic, with ellipticity constant not depending on h. Consequently, $\widetilde{a} : \widetilde{\mathcal{G}}_h \times \widetilde{\mathcal{G}}_h \to \mathbb{C}$ is $\widetilde{\mathcal{G}}_h$-elliptic uniformly in h.*

Now, we are in position to introduce the discrete version of the operator $\boldsymbol{T}$. Let $\boldsymbol{T}_h : \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}$ be defined by $\boldsymbol{T}_h(\boldsymbol{f},\boldsymbol{g}) := (\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ with $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in \widetilde{\mathcal{G}}_h$ being the solution of

$$\widetilde{a}((\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h), (\boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h)) = \widetilde{b}((\boldsymbol{f},\boldsymbol{g}), (\boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h)) \qquad \forall (\boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in \widetilde{\mathcal{G}}_h.$$

It is easy to check that $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) = \boldsymbol{T}_h(\boldsymbol{f},\boldsymbol{g})$ if and only if

$$\widehat{\boldsymbol{u}}_h = \mathcal{P}_{\mathcal{G}_h}\boldsymbol{f}, \tag{4.4}$$

where $\mathcal{P}_{\mathcal{G}_h}$ is the $\mathcal{H}$-orthogonal projection onto $\mathcal{G}_h$, and $\boldsymbol{u}_h \in \mathcal{G}_h$ solves

$$\int_\Omega \rho c^2 \operatorname{div}\boldsymbol{u}_h \operatorname{div}\boldsymbol{v}_h = -2\int_\Omega \nu \operatorname{div}\boldsymbol{f} \operatorname{div}\overline{\boldsymbol{v}}_h - \int_\Omega \rho\boldsymbol{g}\cdot\overline{\boldsymbol{v}}_h \qquad \forall \boldsymbol{v}_h \in \mathcal{G}_h. \tag{4.5}$$

Since $\boldsymbol{T}_h(\widetilde{\mathcal{V}}) \subset \widetilde{\mathcal{G}}_h$, there holds $\mathrm{Sp}(\boldsymbol{T}_h) = \mathrm{Sp}(\boldsymbol{T}_h|_{\widetilde{\mathcal{G}}_h}) \cup \{0\}$ (cf. [3, Lemma 4.1]). Thus, we will restrict our attention to $\boldsymbol{T}_h|_{\widetilde{\mathcal{G}}_h}$.

As claimed above, at the beginning of Section 5, Problem 4 will be shown to be equivalent to a well posed generalized matrix eigenvalue problem. This problem has $\lambda_h = 0$ as an eigenvalue with corresponding eigenspace $\widetilde{\mathcal{K}}_h$. The rest of the eigenvalues are related with the spectrum of $\boldsymbol{T}_h|_{\widetilde{\mathcal{G}}_h}$ according to the following lemma.

Lemma 4.3. *There holds that $(\mu_h, (\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h))$ is an eigenpair of $\boldsymbol{T}_h|_{\widetilde{\mathcal{G}}_h}$ with $\mu_h \neq 0$ if and only if $(\lambda_h, (\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h))$ is a solution of Problem 4 with $\lambda_h = 1/\mu_h$.*

Proof. The proof follows essentially as that of Lemma 2.5, by using the fact that $\mathcal{V}_h \times \mathcal{V}_h = \widetilde{\mathcal{G}}_h \oplus (\mathcal{K}_h \times \mathcal{K}_h)$. □

Our next goal is to show that any isolated eigenvalue of $\boldsymbol{T}$ with algebraic multiplicity m is approximated by exactly m eigenvalues of $\boldsymbol{T}_h$ (repeated according to their respective algebraic multiplicities) and that spurious eigenvalues do not arise. With this end, we will adapt to our problem the theory from [4], which in turn use arguments introduced in [15, 16] to deal with non compact operators. From now on, let $\mu \in \mathrm{Sp}_{\mathrm{disc}}(\boldsymbol{T})$, $\mu \neq 0$, be a fixed isolated eigenvalue of finite

algebraic multiplicity m. Let $\mathcal{E}$ be the invariant subspace of $\boldsymbol{T}$ corresponding to μ. Our analysis will be based on proving the following two properties:

$$\text{P1.}\quad \|\boldsymbol{T}-\boldsymbol{T}_h\|_h := \sup_{\mathbf{0}\neq(\boldsymbol{f}_h,\boldsymbol{g}_h)\in\widetilde{\mathcal{G}}_h} \frac{\|(\boldsymbol{T}-\boldsymbol{T}_h)(\boldsymbol{f}_h,\boldsymbol{g}_h)\|_{\widetilde{\mathcal{V}}}}{\|(\boldsymbol{f}_h,\boldsymbol{g}_h)\|_{\widetilde{\mathcal{V}}}} \to 0 \quad \text{as } h\to 0;$$

$$\text{P2.}\quad \forall(\boldsymbol{v},\widehat{\boldsymbol{v}})\in\mathcal{E},\ \inf_{(\boldsymbol{v}_h,\widehat{\boldsymbol{v}}_h)\in\widetilde{\mathcal{G}}_h} \|(\boldsymbol{v},\widehat{\boldsymbol{v}})-(\boldsymbol{v}_h,\widehat{\boldsymbol{v}}_h)\|_{\widetilde{\mathcal{V}}} \to 0 \quad \text{as } h\to 0.$$

Let $(\boldsymbol{f}_h,\boldsymbol{g}_h)\in\widetilde{\mathcal{G}}_h$ and $(\boldsymbol{u},\widehat{\boldsymbol{u}}) := \boldsymbol{T}(\boldsymbol{f}_h,\boldsymbol{g}_h)$. From (2.18), we can write $\boldsymbol{u}=\boldsymbol{u}_1+\boldsymbol{u}_2$ with $\boldsymbol{u}_1,\boldsymbol{u}_2\in\mathcal{G}$ satisfying

$$\boldsymbol{u}_1\in\mathcal{G}:\quad \int_\Omega \rho c^2 \operatorname{div}\boldsymbol{u}_1 \operatorname{div}\overline{\boldsymbol{v}} = -2\int_\Omega \nu \operatorname{div}\boldsymbol{f}_h \operatorname{div}\overline{\boldsymbol{v}} \qquad \forall\boldsymbol{v}\in\mathcal{G}, \tag{4.6}$$

and

$$\boldsymbol{u}_2\in\mathcal{G}:\quad \int_\Omega \rho c^2 \operatorname{div}\boldsymbol{u}_2 \operatorname{div}\overline{\boldsymbol{v}} = -\int_\Omega \rho\boldsymbol{g}_h\cdot\overline{\boldsymbol{v}} \qquad \forall\boldsymbol{v}\in\mathcal{G}. \tag{4.7}$$

The following result states some properties of the solutions of the problems above.

Lemma 4.4. *For $(\boldsymbol{f}_h,\boldsymbol{g}_h)\in\widetilde{\mathcal{G}}_h$, let $(\boldsymbol{u},\widehat{\boldsymbol{u}}) := \boldsymbol{T}(\boldsymbol{f}_h,\boldsymbol{g}_h)$ and consider the decomposition $\boldsymbol{u}=\boldsymbol{u}_1+\boldsymbol{u}_2$ as above. Hence, $\boldsymbol{u}_1,\boldsymbol{u}_2\in \mathrm{H}^s(\Omega_1\cup\Omega_2)^d$, $\operatorname{div}\boldsymbol{u}_1\in\mathcal{U}_h$, $\operatorname{div}\boldsymbol{u}_2\in\mathrm{H}^{1+s}(\Omega_1\cup\Omega_2)$, and the following estimates hold true*

$$\|\boldsymbol{u}_1\|_{s,\Omega_1}+\|\boldsymbol{u}_1\|_{s,\Omega_2}\le C\|\boldsymbol{f}_h\|_{\mathrm{div},\Omega}, \tag{4.8}$$

$$\|\boldsymbol{u}_2\|_{s,\Omega_1}+\|\boldsymbol{u}_2\|_{s,\Omega_2}+\|\operatorname{div}\boldsymbol{u}_2\|_{1+s,\Omega_1}+\|\operatorname{div}\boldsymbol{u}_2\|_{1+s,\Omega_2}\le C\|\boldsymbol{g}_h\|_{\mathrm{div},\Omega}. \tag{4.9}$$

Proof. Since $\boldsymbol{u}_1\in\mathcal{G}$, due to Lemma 2.2 we have that $\boldsymbol{u}_1\in\mathrm{H}^s(\Omega_1\cup\Omega_2)^d$ and $\|\boldsymbol{u}_1\|_{s,\Omega_1}+\|\boldsymbol{u}_1\|_{s,\Omega_2}\le C\|\operatorname{div}\boldsymbol{f}_h\|_{0,\Omega}$. Moreover, note that (4.6) also holds for $\boldsymbol{v}\in\mathcal{K}$ and hence for all $\boldsymbol{v}\in\mathcal{V}$. Then, we write

$$\int_\Omega (\rho c^2\operatorname{div}\boldsymbol{u}_1+2\nu\operatorname{div}\boldsymbol{f}_h)\operatorname{div}\overline{\boldsymbol{v}}=0 \qquad \forall\boldsymbol{v}\in\mathcal{V}.$$

Thus, taking test functions in $\mathcal{D}(\Omega)^d\subset\mathcal{V}$ we have $\nabla(\rho c^2\operatorname{div}\boldsymbol{u}_1+2\nu\operatorname{div}\boldsymbol{f}_h)=0$. Since ρ,c,ν and $\operatorname{div}\boldsymbol{f}_h$ are piecewise constant, we have that $\operatorname{div}\boldsymbol{u}_1$ is piecewise constant as well; namely, $\operatorname{div}\boldsymbol{u}_1\in\mathcal{U}_h$.

On the other hand, since $\boldsymbol{u}_2\in\mathcal{G}$, by applying Lemma 2.2 again we have that $\boldsymbol{u}_2\in\mathrm{H}^s(\Omega_1\cup\Omega_2)^d$ and $\|\boldsymbol{u}_2\|_{s,\Omega_1}+\|\boldsymbol{u}_2\|_{s,\Omega_2}\le C\|\boldsymbol{g}_h\|_{0,\Omega}$. To prove additional regularity for $\operatorname{div}\boldsymbol{u}_2$, we use Lemma 4.1 to write $\boldsymbol{g}_h=\frac{1}{\rho}\nabla\xi+\chi$ with $\chi\in\mathcal{K}$, $\frac{1}{\rho}\nabla\xi\in\mathrm{H}^s(\Omega_1\cup\Omega_2)^d$ and $\|\frac{1}{\rho}\nabla\xi\|_{s,\Omega_1}+\|\frac{1}{\rho}\nabla\xi\|_{s,\Omega_2}\le C\|\operatorname{div}\boldsymbol{g}_h\|_{0,\Omega}$. Moreover, since ρ is constant in each subdomain Ω_i, also $\nabla\xi|_{\Omega_i}\in\mathrm{H}^s(\Omega_i)^d$, $i=1,2$. Then, from (4.7) we have that

$$\int_\Omega \rho c^2\operatorname{div}\boldsymbol{u}_2\operatorname{div}\overline{\boldsymbol{v}} = -\int_\Omega \nabla\xi\cdot\overline{\boldsymbol{v}} \qquad \forall\boldsymbol{v}\in\mathcal{G}.$$

Since the above equation trivially holds for $\boldsymbol{v}\in\mathcal{K}$ too, it holds for all $\boldsymbol{v}\in\mathcal{V}$. Then, by testing it with $\boldsymbol{v}\in\mathcal{D}(\Omega)^d$ we have that $\nabla(\rho c^2\operatorname{div}\boldsymbol{u}_2)=-\nabla\xi\in\Omega$. Therefore, by restricting to Ω_i, $i=1,2$, we have that $\nabla(\rho c^2\operatorname{div}\boldsymbol{u}_2|_{\Omega_i})=-\nabla\xi|_{\Omega_i}\in\mathrm{H}^s(\Omega_i)^d$. Since ρ and c are piecewise constant, we conclude that $\operatorname{div}\boldsymbol{u}_2|_{\Omega_i}\in\mathrm{H}^{1+s}(\Omega_1\cup\Omega_2)$, and

$$\|\operatorname{div}\boldsymbol{u}_2\|_{1+s,\Omega_1}+\|\operatorname{div}\boldsymbol{u}_2\|_{1+s,\Omega_2}\le C\|\nabla\xi\|_{0,\Omega}\le C\|\operatorname{div}\boldsymbol{g}_h\|_{0,\Omega}.$$

Hence, we conclude the proof. □

We consider a similar decomposition in the discrete case. For $(\boldsymbol{f}_h,\boldsymbol{g}_h)\in\widetilde{\mathcal{G}}_h$, let $(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h) := \boldsymbol{T}_h(\boldsymbol{f}_h,\boldsymbol{g}_h)$. We write $\boldsymbol{u}_h=\boldsymbol{u}_{1h}+\boldsymbol{u}_{2h}$ with $\boldsymbol{u}_{1h}$ and $\boldsymbol{u}_{2h}$ satisfying

$$\boldsymbol{u}_{1h}\in\mathcal{G}_h:\quad \int_\Omega \rho c^2\operatorname{div}\boldsymbol{u}_{1h}\operatorname{div}\overline{\boldsymbol{v}}_h = -2\int_\Omega \nu\operatorname{div}\boldsymbol{f}_h\operatorname{div}\overline{\boldsymbol{v}}_h \qquad \forall\boldsymbol{v}_h\in\mathcal{G}_h, \tag{4.10}$$

and

$$\boldsymbol{u}_{2h} \in \mathcal{G}_h : \quad \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u}_{2h} \operatorname{div} \overline{\boldsymbol{v}}_h = -\int_\Omega \rho \boldsymbol{g}_h \cdot \overline{\boldsymbol{v}}_h \qquad \forall \boldsymbol{v}_h \in \mathcal{G}_h. \tag{4.11}$$

These are the finite element discretization of problems (4.6) and (4.7), respectively, and the following error estimates hold true.

Lemma 4.5. *Let $(\boldsymbol{f}_h, \boldsymbol{g}_h) \in \widetilde{\mathcal{G}}_h$. Let $\boldsymbol{u}_1, \boldsymbol{u}_2$ be the solutions of problems* (4.6) *and* (4.7)*, respectively, and $\boldsymbol{u}_{1h}, \boldsymbol{u}_{2h}$ those of problems* (4.10) *and* (4.11)*, respectively. Then, the following estimates hold true:*

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_{\operatorname{div},\Omega} \le Ch^s \|\boldsymbol{f}_h\|_{\mathcal{V}}, \tag{4.12}$$

$$\|\boldsymbol{u}_2 - \boldsymbol{u}_{2h}\|_{\operatorname{div},\Omega} \le Ch^s \|\boldsymbol{g}_h\|_{\mathcal{H}}. \tag{4.13}$$

Proof. Since $\mathcal{G}_h \nsubseteq \mathcal{G}$, we will resort to the second Strang Lemma, which for problems (4.6) and (4.10) reads as follows:

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_{\operatorname{div},\Omega} \le C \left[\inf_{\boldsymbol{v}_h \in \mathcal{G}_h} \|\boldsymbol{u}_1 - \boldsymbol{v}_h\|_{\operatorname{div},\Omega} + \sup_{\boldsymbol{0} \neq \boldsymbol{v}_h \in \mathcal{G}_h} \frac{a(\boldsymbol{u}_1 - \boldsymbol{u}_{1h}, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\operatorname{div},\Omega}} \right]. \tag{4.14}$$

Because of Lemma 4.4, $\Pi_h \boldsymbol{u}_1$ is well defined. Since $\Pi_h \boldsymbol{u}_1 \in \mathcal{V}_h = \mathcal{G}_h \oplus \mathcal{K}_h$, there exists $\widetilde{\boldsymbol{u}}_{1h} \in \mathcal{G}_h$ and $\check{\boldsymbol{u}}_h \in \mathcal{K}_h$ such that $\Pi_h \boldsymbol{u}_1 = \widetilde{\boldsymbol{u}}_{1h} + \check{\boldsymbol{u}}_h$. Then, since $\boldsymbol{u}_1 - \widetilde{\boldsymbol{u}}_{1h}$ is orthogonal to $\check{\boldsymbol{u}}_h$, we observe that

$$\begin{aligned}
\|\boldsymbol{u}_1 - \widetilde{\boldsymbol{u}}_{1h}\|_{\mathcal{V}}^2 &\le \|\boldsymbol{u}_1 - \widetilde{\boldsymbol{u}}_{1h}\|_{\mathcal{V}}^2 + \|\check{\boldsymbol{u}}_h\|_{\mathcal{V}}^2 \\
&= \|(\widetilde{\boldsymbol{u}}_{1h} - \boldsymbol{u}_1) + \check{\boldsymbol{u}}_h\|_{\mathcal{V}}^2 = \|\boldsymbol{u}_1 - \Pi_h \boldsymbol{u}_1\|_{\mathcal{V}}^2 \\
&\le C \left(\|\boldsymbol{u}_1 - \Pi_h \boldsymbol{u}_1\|_{0,\Omega}^2 + \|\operatorname{div} \boldsymbol{u}_1 - \operatorname{div}(\Pi_h \boldsymbol{u}_1)\|_{0,\Omega}^2 \right).
\end{aligned}$$

The first term on the right hand side above is bounded as follows:

$$\|\boldsymbol{u}_1 - \Pi_h \boldsymbol{u}_1\|_{0,\Omega} \le Ch^s (\|\boldsymbol{u}_1\|_{s,\Omega_1} + \|\boldsymbol{u}_1\|_{s,\Omega_2} + \|\operatorname{div} \boldsymbol{u}_1\|_{0,\Omega}) \le Ch^s \|\boldsymbol{f}_h\|_{\mathcal{V}},$$

where we have used (4.1), (4.8), and the fact that $\|\operatorname{div} \boldsymbol{u}_1\|_{0,\Omega} \le C \|\operatorname{div} \boldsymbol{f}_h\|_{0,\Omega}$, which in turn follows from (4.6) by taking $\boldsymbol{v} = \boldsymbol{f}_h$. On the other hand, the second term vanishes because of (4.2) since $\operatorname{div} \boldsymbol{u}_1 \in \mathcal{U}_h$ (cf. Lemma 4.4). Hence, $\|\boldsymbol{u}_1 - \widetilde{\boldsymbol{u}}_{1h}\|_{\operatorname{div},\Omega} \le Ch^s \|\boldsymbol{f}_h\|_{\mathcal{V}}$, which allows us to control the approximation term in (4.14).

For the consistency term, it is enough to recall that (4.6) holds for all $\boldsymbol{v} \in \mathcal{V}$. Then, by using (4.10), it is easy to check that $a(\boldsymbol{u}_1 - \boldsymbol{u}_{1h}, \boldsymbol{v}_h) = 0$ for all $\boldsymbol{v}_h \in \mathcal{G}_h \subset \mathcal{V}$. From this, the Strang estimate for $\|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_{\operatorname{div},\Omega}$ reads as follows:

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_{\operatorname{div},\Omega} \le C \inf_{\boldsymbol{v}_h \in \mathcal{G}_h} \|\boldsymbol{u}_1 - \boldsymbol{v}_h\|_{\operatorname{div},\Omega} \le Ch^s \|\boldsymbol{f}_h\|_{\mathcal{V}}.$$

Thus (4.12) holds true.

To prove (4.13), we resort again to the second Strang Lemma:

$$\|\boldsymbol{u}_2 - \boldsymbol{u}_{2h}\|_{\operatorname{div},\Omega} \le C \left[\inf_{\boldsymbol{v}_h \in \mathcal{G}_h} \|\boldsymbol{u}_2 - \boldsymbol{v}_h\|_{\operatorname{div},\Omega} + \sup_{\boldsymbol{0} \neq \boldsymbol{v}_h \in \mathcal{G}_h} \frac{a(\boldsymbol{u}_2 - \boldsymbol{u}_{2h}, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\operatorname{div},\Omega}} \right]. \tag{4.15}$$

Since $\boldsymbol{u}_2 \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$ (cf. Lemma 4.4), we have that $\Pi_h \boldsymbol{u}_2$ is well defined. We proceed as above and write $\Pi_h \boldsymbol{u}_2 = \widetilde{\boldsymbol{u}}_{2h} + \check{\boldsymbol{u}}_h$ with $\widetilde{\boldsymbol{u}}_{2h} \in \mathcal{G}_h$ and $\check{\boldsymbol{u}}_h \in \mathcal{K}_h$ to obtain

$$\|\boldsymbol{u}_2 - \widetilde{\boldsymbol{u}}_{2h}\|_{\operatorname{div},\Omega} \le C \left[\|\boldsymbol{u}_2 - \Pi_h \boldsymbol{u}_2\|_{0,\Omega} + \|\operatorname{div} \boldsymbol{u}_2 - \operatorname{div}(\Pi_h \boldsymbol{u}_2)\|_{0,\Omega} \right]. \tag{4.16}$$

For the first term on the right hand side above, (4.1) and Lemma 4.4 yield

$$\|\boldsymbol{u}_2 - \Pi_h \boldsymbol{u}_2\|_{0,\Omega} \le Ch^s(\|\boldsymbol{u}_2\|_{s,\Omega_1} + \|\boldsymbol{u}_2\|_{s,\Omega_2} + \|\operatorname{div} \boldsymbol{u}_2\|_{0,\Omega}) \le Ch^s \|\boldsymbol{g}_h\|_{\mathcal{H}}.$$

For the second term, we have from (4.3) and from Lemma 4.4 again

$$\begin{aligned}\|\operatorname{div} \boldsymbol{u}_2 - \operatorname{div} \Pi_h \boldsymbol{u}_2\|_{0,\Omega}^2 &= \|\operatorname{div} \boldsymbol{u}_2 - \mathcal{P}_h(\operatorname{div} \boldsymbol{u}_2)\|_{0,\Omega}^2, \\ &\le Ch(\|\operatorname{div} \boldsymbol{u}_2\|_{1,\Omega_1} + \|\operatorname{div} \boldsymbol{u}_2\|_{1,\Omega_2}) \le Ch\|\boldsymbol{g}_h\|_{\mathcal{H}}.\end{aligned}$$

Hence, $\|\boldsymbol{u}_2 - \widetilde{\boldsymbol{u}}_{2h}\|_{\operatorname{div},\Omega} \le Ch^s\|\boldsymbol{g}_h\|_{\mathcal{H}}$, which allows us to bound the approximation term in (4.15).

For the consistency term, given $\boldsymbol{v}_h \in \mathcal{G}_h$ we apply Lemma 4.1 to write $\boldsymbol{v}_h = \frac{1}{\rho}\nabla\xi + \chi$ with $\frac{1}{\rho}\nabla\xi \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$, $\chi \in \mathcal{K}$, and $\|\chi\|_{0,\Omega} \le Ch^s\|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega}$. Then, from (4.7) we have

$$a(\boldsymbol{u}_2, \boldsymbol{v}_h) = \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u}_2 \operatorname{div} \overline{\boldsymbol{v}}_h = \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u}_2 \operatorname{div}\left(\frac{1}{\rho}\nabla\overline{\xi}\right) = \int_\Omega \boldsymbol{g}_h \cdot \nabla\overline{\xi}.$$

On the other hand, from (4.11),

$$a(\boldsymbol{u}_{2h}, \boldsymbol{v}_h) = \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u}_{2h} \operatorname{div} \overline{\boldsymbol{v}}_h = \int_\Omega \rho \boldsymbol{g}_h \cdot \overline{\boldsymbol{v}}_h = \int_\Omega \boldsymbol{g}_h \cdot \nabla\overline{\xi} + \int_\Omega \rho \boldsymbol{g}_h \cdot \overline{\chi}.$$

Therefore,

$$a(\boldsymbol{u}_2 - \boldsymbol{u}_{2h}, \boldsymbol{v}_h) = -\int_\Omega \rho \boldsymbol{g}_h \cdot \overline{\chi} \le Ch^s\|\boldsymbol{g}_h\|_{0,\Omega}\|\boldsymbol{v}_h\|_{\operatorname{div},\Omega}$$

and, hence,

$$\sup_{\boldsymbol{0}\neq\boldsymbol{v}_h\in\mathcal{G}_h} \frac{a(\boldsymbol{u}_2 - \boldsymbol{u}_{2h}, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\operatorname{div},\Omega}} \le Ch^s\|\boldsymbol{g}_h\|_{0,\Omega},$$

which allows us to complete the proof. □

Now, we are in a position to establish the following result.

Lemma 4.6. *Property P1 holds true. Moreover,*

$$\|\boldsymbol{T} - \boldsymbol{T}_h\|_h \le Ch^s.$$

Proof. For $(\boldsymbol{f}_h, \boldsymbol{g}_h) \in \widetilde{\mathcal{G}}_h$, let $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) := \boldsymbol{T}(\boldsymbol{f}_h, \boldsymbol{g}_h)$ and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) := \boldsymbol{T}_h(\boldsymbol{f}_h, \boldsymbol{g}_h)$. From (2.17) and (4.4) we have that $\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h = \boldsymbol{f}_h - \mathcal{P}_{\mathcal{G}_h}\boldsymbol{f}_h = 0$. Hence, by writing $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$ and $\boldsymbol{u}_h = \boldsymbol{u}_{1h} + \boldsymbol{u}_{2h}$ as in Lemma 4.5, we have from this lemma

$$\|\boldsymbol{T} - \boldsymbol{T}_h\|_h \le \sup_{\boldsymbol{0}\neq(\boldsymbol{g}_h,\boldsymbol{f}_h)\in\widetilde{\mathcal{G}}_h} \frac{\|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_{\operatorname{div},\Omega} + \|\boldsymbol{u}_2 - \boldsymbol{u}_{2h}\|_{\operatorname{div},\Omega}}{\|(\boldsymbol{f}_h, \boldsymbol{g}_h)\|_{\widetilde{\mathcal{V}}}} \le Ch^s.$$

Thus, we conclude the proof. □

Our next goal is to prove property P2. With this aim, first we will prove the following additional regularity result,.

Lemma 4.7. *Let* $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \mathcal{E}$. *Then,* $\boldsymbol{u}, \widehat{\boldsymbol{u}} \in \mathcal{G} \subset \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$, $\operatorname{div} \boldsymbol{u}, \operatorname{div} \widehat{\boldsymbol{u}} \in \mathrm{H}^{1+s}(\Omega_1 \cup \Omega_2)$, *and*

$$\|\boldsymbol{u}\|_{s,\Omega_1} + \|\boldsymbol{u}\|_{s,\Omega_2} + \|\operatorname{div} \boldsymbol{u}\|_{1+s,\Omega_1} + \|\operatorname{div} \boldsymbol{u}\|_{1+s,\Omega_2} \le C\|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}}, \tag{4.17}$$

$$\|\widehat{\boldsymbol{u}}\|_{s,\Omega_1} + \|\widehat{\boldsymbol{u}}\|_{s,\Omega_2} + \|\operatorname{div} \widehat{\boldsymbol{u}}\|_{1+s,\Omega_1} + \|\operatorname{div} \widehat{\boldsymbol{u}}\|_{1+s,\Omega_2} \le C\|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}}. \tag{4.18}$$

Proof. We prove the above inequalities for all the generalized eigenfunctions of $\boldsymbol{T}$. Let $\{(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k)\}_{k=1}^p$ be a Jordan chain of the operator $\boldsymbol{T}$ associated with μ. Then, $\boldsymbol{T}(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k) = \mu(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k) + (\boldsymbol{u}_{k-1}, \widehat{\boldsymbol{u}}_{k-1})$, $k = 1, \ldots, p$, with $(\boldsymbol{u}_0, \widehat{\boldsymbol{u}}_0) = \boldsymbol{0}$. We will use an induction argument on k. Assume that $\boldsymbol{u}_{k-1}$ and $\widehat{\boldsymbol{u}}_{k-1}$ belong to $\mathcal{G}$ and satisfy (4.17) and (4.18), respectively (which obviously hold for $k = 1$). First note that, because of the boundedness of $\boldsymbol{T}$, we have

$$\|(\boldsymbol{u}_{k-1}, \widehat{\boldsymbol{u}}_{k-1})\|_{\widetilde{\mathcal{V}}} \leq C\|(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k)\|_{\widetilde{\mathcal{V}}}. \tag{4.19}$$

On the other hand, by using (2.17) and (2.18) we have that

$$\mu\widehat{\boldsymbol{u}}_k + \widehat{\boldsymbol{u}}_{k-1} = \boldsymbol{u}_k \quad \text{in } \Omega \tag{4.20}$$

and that $\mu\boldsymbol{u}_k + \boldsymbol{u}_{k-1} \in \mathcal{G}$ satisfies

$$\int_\Omega \rho c^2 \operatorname{div}(\mu\boldsymbol{u}_k + \boldsymbol{u}_{k-1}) \operatorname{div} \overline{\boldsymbol{v}} = -2\int_\Omega \nu \operatorname{div} \boldsymbol{u}_k \operatorname{div} \overline{\boldsymbol{v}} - \int_\Omega \rho\widehat{\boldsymbol{u}}_k \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}.$$

Hence, $\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k \in \mathcal{G}$.

We observe that the equation above also holds for any $\boldsymbol{v} \in \mathcal{K}$. Then,

$$\int_\Omega \rho c^2 \operatorname{div}(\mu\boldsymbol{u}_k + \boldsymbol{u}_{k-1}) \operatorname{div} \overline{\boldsymbol{v}} = -2\int_\Omega \nu \operatorname{div} \boldsymbol{u}_k \operatorname{div} \overline{\boldsymbol{v}} - \int_\Omega \rho\widehat{\boldsymbol{u}}_k \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{V}.$$

Thus, considering test functions in $\mathcal{D}(\Omega)^d \subset \mathcal{V}$ we obtain

$$\nabla((\mu\rho c^2 + 2\nu) \operatorname{div} \boldsymbol{u}_k) = \rho\widehat{\boldsymbol{u}}_k - \nabla(\rho c^2 \operatorname{div} \boldsymbol{u}_{k-1}). \tag{4.21}$$

Let us assume that $\mu\rho c^2 + 2\nu \neq 0$ in both Ω_1 and Ω_2 (we discuss the other case at the end of the proof). Hence, since ρ, c, and ν are constant in each Ω_i, $\rho_i\widehat{\boldsymbol{u}}_k - \nabla(\rho_i c_i^2 \operatorname{div} \boldsymbol{u}_{k-1}) \in \mathrm{L}^2(\Omega_i)^d$, $\operatorname{div} \boldsymbol{u}_k|_{\Omega_i} \in \mathrm{H}^1(\Omega_i)$, and

$$\|\operatorname{div} \boldsymbol{u}_k\|_{1,\Omega_i} \leq C\left(\|\operatorname{div} \boldsymbol{u}_{k-1}\|_{1,\Omega_i} + \|(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k)\|_{\widetilde{\mathcal{V}}}\right), \qquad i = 1, 2.$$

Now, since $\boldsymbol{u}_k \in \mathcal{G}$, due to Lemma 2.2 we have that $\boldsymbol{u}_k \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$. Then, from (2.14) and the previous estimate we have

$$\|\boldsymbol{u}_k\|_{s,\Omega_1} + \|\boldsymbol{u}_k\|_{s,\Omega_2} \leq C\left(\|\operatorname{div} \boldsymbol{u}_{k-1}\|_{1,\Omega_1} + \|\operatorname{div} \boldsymbol{u}_{k-1}\|_{1,\Omega_2} + \|(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k)\|_{\widetilde{\mathcal{V}}}\right). \tag{4.22}$$

On the other hand, from (4.20) we obtain

$$\|\widehat{\boldsymbol{u}}_k\|_{s,\Omega_i} \leq \frac{1}{\mu}\left(\|\widehat{\boldsymbol{u}}_{k-1}\|_{s,\Omega_i} + \|\boldsymbol{u}_k\|_{s,\Omega_i}\right), \qquad i = 1, 2, \tag{4.23}$$

and, from (4.21),

$$\|\operatorname{div} \boldsymbol{u}_k\|_{1+s,\Omega_i} \leq C(\|\widehat{\boldsymbol{u}}_k\|_{s,\Omega_i} + \|\operatorname{div} \boldsymbol{u}_{k-1}\|_{1+s,\Omega_i}), \qquad i = 1, 2. \tag{4.24}$$

Finally, from (4.20) again,

$$\|\operatorname{div} \widehat{\boldsymbol{u}}_k\|_{1+s,\Omega_i} \leq \frac{1}{\mu}(\|\operatorname{div} \boldsymbol{u}_k\|_{1+s,\Omega_i} + \|\operatorname{div} \widehat{\boldsymbol{u}}_{k-1}\|_{1+s,\Omega_i}), \qquad i = 1, 2. \tag{4.25}$$

Hence, from inequalities (4.22)–(4.25), the inductive assumption, and (4.19), we derive (4.17) and (4.18) provided $\mu\rho c^2 + 2\nu \neq 0$ in both Ω_1 and Ω_2.

In case that $\mu\rho c^2 + 2\nu$ vanishes in Ω_i, $i = 1$ or 2, arguing as in Remark 2.3 we obtain that $\boldsymbol{u}_1|_{\Omega_i} = \widehat{\boldsymbol{u}}_1|_{\Omega_i} = \boldsymbol{0}$ and, once again, an induction argument allow us to conclude that $\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k = \boldsymbol{0}$ in Ω_i, $k = 1, \ldots, p$. The proof is complete. □

Now, we are in position to establish property P2.

Lemma 4.8. *Property P2 holds true. Moreover, for any* $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \mathcal{E}$*, there exists* $\widetilde{\boldsymbol{u}}_h$*,* $\widetilde{\widehat{\boldsymbol{u}}}_h \in \mathcal{G}_h$ *such that*

$$\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}_h\|_{\mathrm{div},\Omega} + \|\widehat{\boldsymbol{u}} - \widetilde{\widehat{\boldsymbol{u}}}_h\|_{\mathrm{div},\Omega} \leq Ch^s \|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}}.$$

Proof. Let $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \mathcal{E}$. According to Lemma 4.7, we have that $\boldsymbol{u}, \widehat{\boldsymbol{u}} \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$ and $\operatorname{div} \boldsymbol{u}, \operatorname{div} \widehat{\boldsymbol{u}} \in \mathrm{H}^{1+s}(\Omega_1 \cup \Omega_2)$. Let $\Pi_h \boldsymbol{u} \in \mathcal{V}_h$ be the Raviart-Thomas interpolant of $\boldsymbol{u}$. Since $\mathcal{V}_h = \mathcal{G}_h \oplus \mathcal{K}_h$, we decompose $\Pi_h \boldsymbol{u} = \widetilde{\boldsymbol{u}}_h + \check{\boldsymbol{u}}_h$ with $\widetilde{\boldsymbol{u}}_h \in \mathcal{G}_h$ and $\check{\boldsymbol{u}}_h \in \mathcal{K}_h$. The same arguments from the proof of Lemma 4.5 that lead to (4.13) apply in this case and combined with Lemma 4.7 allow us to prove that $\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}_h\|_{\mathrm{div},\Omega} \leq Ch^s \|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}}$. A similar procedure can be used to define $\widetilde{\widehat{\boldsymbol{u}}}_h$ and to prove that $\|\widehat{\boldsymbol{u}} - \widetilde{\widehat{\boldsymbol{u}}}_h\|_{\mathrm{div},\Omega} \leq Ch^s \|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}}$. □

We also have the following auxiliary result when the source terms are in $\mathcal{E}$, which proof follows the arguments in the Case 2 of Lemma 4.7 of [4].

Lemma 4.9. *For* $(\boldsymbol{f}, \boldsymbol{g}) \in \mathcal{E}$*, let* $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) := \boldsymbol{T}(\boldsymbol{f}, \boldsymbol{g})$ *and* $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) := \boldsymbol{T}_h(\boldsymbol{f}, \boldsymbol{g})$*. Then,*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{div},\Omega} + \|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{0,\Omega} \leq Ch^s \|(\boldsymbol{f}, \boldsymbol{g})\|_{\widetilde{\mathcal{V}}}.$$

The above lemmas are the ingredients to prove spectral convergence and to obtain error estimates. Our first result is the following theorem which has been proved in [15] as a consequence of property P1 (cf. Lemma 4.6) and which shows that the proposed method is free of spurious modes.

Theorem 4.1. *Let* $K \subset \mathbb{C}$ *be a compact set such that* $K \cap \mathrm{Sp}(\boldsymbol{T}) = \emptyset$*. Then, there exists* $h_0 > 0$ *such that, for all* $h \leq h_0$*,* $K \cap \mathrm{Sp}(\boldsymbol{T}_h) = \emptyset$*.*

Let $D \subset \mathbb{C}$ be a closed disk centered at μ, such that $D \cap \mathrm{Sp}(\boldsymbol{T}) = \{\mu\}$. Let $\mu_{1h}, \ldots, \mu_{m(h)h}$ be the eigenvalues of $\boldsymbol{T}_h$ contained in D (repeated according to their algebraic multiplicities). Under assumptions P1 and P2, it is proved in [15] that $m(h) = m$ for h small enough and that $\lim_{h \to 0} \mu_{kh} = \mu$ for $k = 1, \ldots, m$.

On the other hand the arguments used in Section 5 of [4] can be readily adapted to our problem, to obtain error estimates. We recall the definition of the gap between two closed subspaces $\mathcal{W}$ and $\mathcal{Y}$ of $\widetilde{\mathcal{V}}$:

$$\widehat{\delta}(\mathcal{W}, \mathcal{Y}) := \max\{\delta(\mathcal{W}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{W})\},$$

with

$$\delta(\mathcal{W}, \mathcal{Y}) := \sup_{\substack{(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{W} \\ \|(\boldsymbol{\phi}, \boldsymbol{\psi})\|_{\widetilde{\mathcal{V}}} = 1}} \left[\inf_{(\widehat{\boldsymbol{\phi}}, \widehat{\boldsymbol{\psi}}) \in \mathcal{Y}} \|(\boldsymbol{\phi}, \boldsymbol{\psi}) - (\widehat{\boldsymbol{\phi}}, \widehat{\boldsymbol{\psi}})\|_{\widetilde{\mathcal{V}}} \right].$$

Let $\mathcal{E}_h$ be the invariant subspace of $\boldsymbol{T}_h$ relative to the eigenvalues $\mu_{1h}, \ldots, \mu_{mh}$ converging to μ. From Lemmas 4.6–4.9, we derive the following results for which we do not include proofs to avoid repeating step by step those of [4, Section 5].

Theorem 4.2. *There exist constants* $h_0 > 0$ *and* $C > 0$ *such that, for all* $h \leq h_0$*,*

$$\widehat{\delta}(\mathcal{E}_h, \mathcal{E}) \leq Ch^s.$$

Theorem 4.3. *There exist constants* $h_0 > 0$ *and* $C > 0$ *such that, for all* $h \leq h_0$*,*

$$\left| \mu - \frac{1}{m} \sum_{k=1}^{m} \mu_{kh} \right| \leq Ch^{2s},$$

$$\left|\frac{1}{\mu} - \frac{1}{m}\sum_{k=1}^{m}\frac{1}{\mu_{kh}}\right| \leq Ch^{2s},$$

$$\max_{k=1,\ldots,m} |\mu - \mu_{kh}| \leq Ch^{2s/p},$$

where p is the ascent of the eigenvalue μ of $\boldsymbol{T}$.

5 Numerical Results

We implemented the proposed method in a MATLAB code. We report in this section the results of some numerical tests, in order to assess its performance. With this end, first we introduce a convenient matrix form of the discrete problem which allows us to use standard eigensolvers. As a by-product, this matrix form also allows us to prove that Problems 3 and 4 are well posed.

Let $\{\phi_j\}_{j=1}^N$ be a nodal basis of $\mathcal{V}_h$. We define the matrices $\mathbf{K}_1 := (\mathbf{K}_{ij}^{(1)})$, $\mathbf{K}_2 := (\mathbf{K}_{ij}^{(2)})$ and $\mathbf{M} := (\mathbf{M}_{ij})$ as follows:

$$\mathbf{K}_{ij}^{(1)} := 2\int_\Omega \nu \operatorname{div}\phi_i \operatorname{div}\phi_j, \quad \mathbf{K}_{ij}^{(2)} := \int_\Omega \rho c^2 \operatorname{div}\phi_i \operatorname{div}\phi_j, \quad \text{and} \quad \mathbf{M}_{ij} := \int_\Omega \rho\phi_i\cdot\phi_j.$$

The matrix form of Problem 3 reads

$$(\lambda_h^2\mathbf{M} + \lambda_h\mathbf{K}_1 + \mathbf{K}_2)\vec{\boldsymbol{u}}_h = \mathbf{0}, \tag{5.1}$$

where we denote by $\vec{\boldsymbol{u}}_h$ the vector of components of $\boldsymbol{u}_h$ in the nodal basis of $\mathcal{V}_h$.

Analogously, the matrix form of Problem 4 reads

$$\begin{pmatrix}\mathbf{K}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{pmatrix}\begin{pmatrix}\vec{\boldsymbol{u}}_h\\ \vec{\widehat{\boldsymbol{u}}}_h\end{pmatrix} = \lambda_h\begin{pmatrix}-\mathbf{K}_1 & -\mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\vec{\boldsymbol{u}}_h\\ \vec{\widehat{\boldsymbol{u}}}_h\end{pmatrix},$$

with $\vec{\widehat{\boldsymbol{u}}}_h$ being the vector of components of $\widehat{\boldsymbol{u}}_h$. However, this problems is not suitable to be solved with standard eigensolvers, since neither the right-hand side nor the left-hand side matrix are Hermitian and positive definite.

Alternatively, for $\lambda_h \neq 0$, let $\mu_h := \frac{1}{\lambda_h}$. Then, problem (5.1) is equivalent to

$$(\mathbf{M} + 2\mu_h\mathbf{K}_1 + \mu_h^2\mathbf{K}_2)\vec{\boldsymbol{u}}_h = \mathbf{0}.$$

Introducing $\vec{\boldsymbol{w}}_h := \mu_h\vec{\boldsymbol{u}}_h$, the problem above is equivalent to

$$\begin{pmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{pmatrix}\begin{pmatrix}\vec{\boldsymbol{u}}_h\\ \vec{\boldsymbol{w}}_h\end{pmatrix} = \mu_h\begin{pmatrix}-\mathbf{K}_1 & -\mathbf{K}_2\\ \mathbf{M} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\vec{\boldsymbol{u}}_h\\ \vec{\boldsymbol{w}}_h\end{pmatrix},$$

which in turn is equivalent to

$$\begin{pmatrix}-\mathbf{K}_1 & -\mathbf{K}_2\\ \mathbf{M} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\vec{\boldsymbol{u}}_h\\ \vec{\boldsymbol{w}}_h\end{pmatrix} = \lambda_h\begin{pmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{pmatrix}\begin{pmatrix}\vec{\boldsymbol{u}}_h\\ \vec{\boldsymbol{w}}_h\end{pmatrix}.$$

Thus, the last problem is equivalent to Problem 3 except for $\lambda_h = 0$ and the matrix in its right-hand side is Hermitian and positive definite. Hence, it is well posed and can be safely solved by standard eigensolvers.

Test 1. We applied it to a 2D rectangular rigid cavity filled with two fluids with different physical parameters as shown in Figure 2. The domain occupied by the fluids are $\Omega_1 := (0, A) \times (0, H)$ and $\Omega_2 := (0, A) \times (H, B)$. For such a simple geometry, it is possible to calculate an analytical solution which will be used to validate our method.

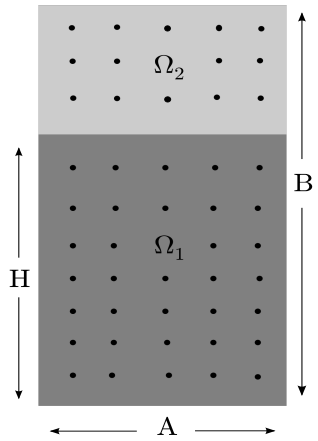

Figure 2: Test 1. Two fluids in a rectangular rigid cavity.

Let $\boldsymbol{u} \in \mathrm{H}_0(\mathrm{div}, \Omega)$ be a solution of Problem 1. Testing it with $\boldsymbol{v} \in \mathcal{D}(\Omega)^2$ we have $\nabla((2\lambda\nu + \rho c^2)\,\mathrm{div}\,\boldsymbol{u}) = -\lambda^2\rho\boldsymbol{u} \in \mathrm{L}^2(\Omega)^2$. Then, $\widehat{p} := -(2\nu\lambda + \rho c^2)\,\mathrm{div}\,\boldsymbol{u} \in \mathrm{H}^1(\Omega)$. Hence, $\widehat{p}_1|_\Gamma = \widehat{p}_2|_\Gamma$. Moreover, $\boldsymbol{u} = -\dfrac{1}{\lambda^2\rho}\nabla\widehat{p}$, which implies that $\dfrac{1}{\rho_1}\dfrac{\partial\widehat{p}_1}{\partial\nu} = \dfrac{1}{\rho_2}\dfrac{\partial\widehat{p}_2}{\partial\nu}$ on Γ. Then, we write problem (2.3)–(2.10), in terms of $\widehat{p}_i$ as follows:

$$
\begin{aligned}
\Delta\widehat{p}_i &= \frac{\lambda^2\rho_i}{\rho_i c_i^2 + 2\nu_i\lambda}\widehat{p}_i && \text{in } \Omega_i, \quad i = 1, 2,\\
\frac{\partial\widehat{p}_i}{\partial\boldsymbol{n}_i} &= 0 && \text{on } \Gamma_i, \quad i = 1, 2,\\
\widehat{p}_1 &= \widehat{p}_2 && \text{on } \Gamma,\\
\frac{1}{\rho_1}\frac{\partial\widehat{p}_1}{\partial\boldsymbol{n}} &= \frac{1}{\rho_2}\frac{\partial\widehat{p}_2}{\partial\boldsymbol{n}} && \text{on } \Gamma.
\end{aligned}
$$

We proceed by separation of variables. Assuming that $\widehat{p}_i(x, y) = X_i(x)Y_i(y)$, we are left with the following problem:

$$
\frac{X_i''(x)}{X_i(x)} + \frac{Y_i''(y)}{Y_i(y)} = \frac{\lambda^2\rho_i}{\rho_i c_i^2 + 2\nu_i\lambda} \quad \text{in } \Omega_i, \tag{5.2}
$$

$$
X_i'(0) = X_i'(A) = 0, \qquad i = 1, 2, \tag{5.3}
$$

$$
Y_1'(0) = Y_2'(B) = 0, \tag{5.4}
$$

$$
\frac{1}{\rho_1}X_1(x)Y_1'(H) = \frac{1}{\rho_2}X_2(x)Y_2'(H), \quad 0 < x < A, \tag{5.5}
$$

$$
X_1(x)Y_1(H) = X_2(x)Y_2(H), \quad 0 < x < A. \tag{5.6}
$$

From (5.2) we have that $X_i(x)''/X_i(x)$ and $Y_i(y)''/Y_i(y)$ are constant. Moreover, from (5.5) and (5.6), it is easy to check that $Y_i(H)$ and $Y_i'(H)$ cannot vanish simultaneously and $X_1(x) = X_2(x)$ (actually, it is derived that $X_1(x) = CX_2(x)$, but the constant C can be chosen equal to one without loss of generality).

From the fact that $X_i(x)''/X_i(x)$ is constant and (5.3), we have that

$$
X_1(x) = X_2(x) = \cos\left(\frac{m\pi x}{A}\right), \qquad m = 0, 1, 2, \ldots.
$$

On the other hand, from the fact that $Y_i(y)''/Y_i(y)$ is also constant and (5.4) we derive

$$
Y_1(y) = C_1\cosh(r_m^{(1)}(\lambda)y) \qquad \text{and} \qquad Y_2(y) = C_2\cosh(r_m^{(2)}(\lambda)(y - B)), \tag{5.7}
$$

where C_1 and C_2 are constants and

$$r_m^{(i)} := \sqrt{\frac{\lambda^2 \rho_i}{\rho_i c_i^2 + 2\nu_i \lambda} + \frac{m^2 \pi^2}{A^2}}, \qquad m = 0, 1, 2, \ldots, \qquad i = 1, 2.$$

Since $Y_i(H)$ and $Y_i'(H)$ cannot vanish simultaneously, (5.5) and (5.6) lead to

$$\frac{1}{\rho_1} Y_1'(H) = \frac{1}{\rho_2} Y_2'(H) \qquad \text{and} \qquad Y_1(H) = Y_2(H),$$

respectively. Thus, substituting (5.7) into these equation yields the following linear system for the coefficients C_1 and C_2:

$$\begin{aligned} C_1 \cosh(r_m^{(1)}(\lambda) H) &= C_2 \cosh(r_m^{(2)}(\lambda)(H - B)), \\ \frac{C_1 r_m^{(1)}(\lambda)}{\rho_1} \sinh(r_m^{(1)}(\lambda) H) &= \frac{C_2 r_m^{(2)}(\lambda)}{\rho_2} \sinh(r_m^{(2)}(\lambda)(H - B)). \end{aligned}$$

For this system to have non trivial solutions, its determinant must vanish, which yields the following non linear equation in λ for $m = 0, 1, 2, \ldots$ whose roots are the eigenvalues of Problem 1:

$$\begin{aligned} f_m(\lambda) := \frac{r_m^{(1)}(\lambda)}{\rho_1} \sinh(r_m^{(1)}(\lambda) H) \cosh(r_m^{(2)}(\lambda)(H - B)) \\ - \frac{r_m^{(2)}(\lambda)}{\rho_2} \sinh(r_m^{(2)}(\lambda)(H - B)) \cosh(r_m^{(1)}(\lambda) H) = 0. \end{aligned}$$

We have computed some roots of the above equation and used these roots as exact eigenvalues to compare those obtained with the method proposed in this paper. For the geometrical parameters, we have taken $A = 1\,\mathrm{m}$, $B = 2\,\mathrm{m}$ and $H = 1.25\,\mathrm{m}$.

We have used physical parameters of water and air for the density and acoustic speed of the fluids in Ω_1 and Ω_2, respectively: $c_1 = 1430$ m/s, $\rho_1 = 1000$ kg/m^3, $c_2 = 340$ m/s and $\rho_2 = 1$ kg/m^3. We have used uniform meshes as those shown in Figure 3. The refinement parameter N refers to the number of elements per width of the rectangle.

Figure 3: Test 1. Meshes for $N = 4$ (left) and $N = 8$ (right).

In presence of dissipation ($\nu \neq 0$), the eigenvalues λ are complex numbers $\lambda = \eta + i\omega$, with $\eta < 0$ being the decay rate and ω the vibration frequency. In absence of dissipation ($\nu = 0$), the eigenvalues λ are purely imaginary ($\eta = 0$). The same holds for the computed eigenvalues λ_h.

In our first test, we neglected the viscosity damping effects by taking $\nu_1 = \nu_2 = 0$. In this case, the eigenvalues λ are actually purely imaginary Accurate values of the zeros of $f_m(\lambda)$ have been obtained with the MATLAB command `fminsearch` applied to $|f_m(\lambda)|$.

Table 1 shows the four smallest eigenvalues computed with the proposed method on successively refined meshes. Accurate values obtained with the MATLAB command `fminsearch` applied to $|f_m(\lambda)|$ are also reported on the last line of the table as exact eigenvalues.

m	1	0	1	0
$N = 8$	$1066.07\,i$	$1418.42\,i$	$1784.37\,i$	$1796.61\,i$
$N = 16$	$1067.78\,i$	$1422.52\,i$	$1781.49\,i$	$1797.09\,i$
$N = 32$	$1068.21\,i$	$1423.54\,i$	$1780.73\,i$	$1797.21\,i$
$N = 64$	$1068.33\,i$	$1423.79\,i$	$1780.55\,i$	$1797.23\,i$
Order	2.00	2.00	1.99	2.00
Exact	$1068.36\,i$	$1423.87\,i$	$1780.49\,i$	$1797.24\,i$

Table 1: Test 1. Computed and exact eigenvalues for dissipative fluids in a rigid cavity.

As predicted by the theory, these eigenvalues are purely imaginary. The high accuracy of the computed eigenvalues can be observed from Table 1 even for the coarsest mesh. We have used a least squares fitting to estimate the convergence rate for each eigenvalue, which are also reported in Table 1. A clear order $\mathcal{O}(h^2)$ can be seen in all cases.

Secondly, test we have used the same physical parameters as above for both fluids, but considering now non vanishing viscosities. In order to make the dissipation effects more visible, we have used unrealistically large viscosity values: $\nu_1 = 9\,\mathrm{N/m}s^2$ and $\nu_2 = 1\,\mathrm{N/m}s^2$. We have repeated the scheme used above. We report in Table 2 the computed and 'exact' eigenvalues and the estimated convergence rates, which are in accordance with the theory once again. Notice that now all λ have negative real parts (decay rate) as predicted by the theory.

m	1	0	1	0
$N = 8$	$-9.83127 + 1066.03\,i$	$-17.38526 + 1418.31\,i$	$-27.54238 + 1784.16\,i$	$-0.04746 + 1796.61\,i$
$N = 16$	$-9.86298 + 1067.74\,i$	$-17.48513 + 1422.41\,i$	$-27.45337 + 1781.27\,i$	$-0.04875 + 1797.08\,i$
$N = 32$	$-9.87090 + 1068.17\,i$	$-17.50995 + 1423.43\,i$	$-27.43029 + 1780.53\,i$	$-0.04908 + 1797.20\,i$
$N = 64$	$-9.87288 + 1068.38\,i$	$-17.51614 + 1423.78\,i$	$-27.42447 + 1780.34\,i$	$-0.04916 + 1797.23\,i$
Order	2.00	2.00	1.99	2.00
Exact	$-9.87354 + 1068.32\,i$	$-17.51817 + 1423.76\,i$	$-27.42236 + 1780.27\,i$	$-0.04919 + 1797.24\,i$

Table 2: Test 1. Computed and exact eigenvalues for dissipative fluids in a rigid cavity.

It can be seen from Table 2 that even in the coarsest mesh the vibration frequencies are computed with at least three correct significant digits. In turn, the decay rates are computed with at least two correct significant digits, in spite of the fact that they are much smaller than the vibration frequencies. Moreover, a least square fitting of the computed decay rates shows that they also converge quadratically. As a consequence, the decay rates computed with the finest mesh attain three or four significant correct digits.

Finally, Figure 4 show the real and imaginary parts of the computed pressure as defined in (2.2) for the smallest eigenvalue.

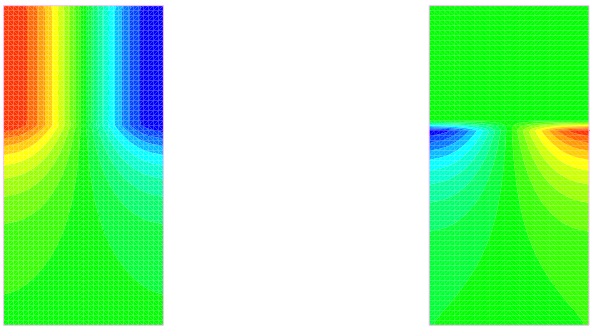

Figure 4: Test 1. Real (left) and imaginary (right) parts of the computed pressure for the first eigenvalue.

Test 2. As a second test, we have applied our code to solve a problem with a curved interface. In spite of the fact that such a problem does not lie in our theoretical framework, we will report experimental results which will allow us to asses the performance of the method in this case.

The whole domain Ω is the same as in the previous experiment, but with a curved interface Γ. We report in Table 3 the computed eigenvalues. In this case, there is no analytical solution available. Therefore, we have obtained more accurate approximations of the exact eigenvalues by means of a least square fitting. These more accurate values are reported on the last line of Table 3 as 'Exact' We also report in this table the estimated order of convergence, which once more is clearly quadratic.

ω_h^i	ω_h^1	ω_h^2	ω_h^3	ω_h^4
$N = 8$	$-10.40320 + 1096.60\,i$	$-10.45430 + 1099.28\,i$	$-10.46725 + 1099.96\,i$	$-10.47051 + 1100.13\,i$
$N = 16$	$-13.49718 + 1249.28\,i$	$-13.42914 + 1246.13\,i$	$-13.41172 + 1245.32\,i$	$-13.40737 + 1245.12\,i$
$N = 32$	$-26.06756 + 1735.74\,i$	$-25.88822 + 1729.76\,i$	$-25.84076 + 1728.18\,i$	$-25.82875 + 1727.78\,i$
$N = 64$	$-0.05457 + 1969.38\,i$	$-0.05782 + 1974.72\,i$	$-0.05890 + 1976.45\,i$	$-0.05918 + 1976.93\,i$
Order	1.98	1.97	1.93	1.67
'Exact'	$-10.47163 + 1100.19\,i$	$-13.40581 + 1245.04\,i$	$-25.82423 + 1727.63\,i$	$-0.05934 + 1977.19\,i$

Table 3: Test 2. Computed and 'Exact' eigenvalues for dissipative fluids in a rigid cavity with a curved interface.

Finally, Figure 5 show the real and imaginary parts of the computed pressure as defined in (2.2) for the smallest eigenvalue.The curved interface can be clearly appreciated in the imaginary part of the pressure.

References

[1] R. Ardito, C. Comi, A. Corigliano, and A. Frangi, *Solid damping in micro electro mechanical systems*, Meccanica, **43** (2008), 419–428.

[2] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numerica, **15** (2006), 1–155.

[3] A. Bermúdez, R. Durán, M.A. Muschietti, R. Rodríguez, and J. Solomin, *Finite element vibration analysis of fluid-solid systems without spurious modes*, SIAM J. Numer. Anal., **32** (1995) 1280–1295.

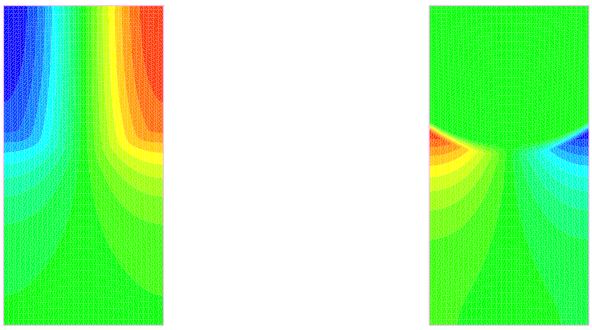

Figure 5: Test 2. Real (left) and imaginary (right) parts of the pressure for the first eigenvalue

[4] A. Bermúdez, R. Durán, R. Rodríguez, and J. Solomin, *Finite element analysis of a quadratic eigenvalue problem arising in dissipative acoustics*, SIAM J. Numer. Anal., **38** (2000) 267–291.

[5] A. Bermúdez, P. Gamallo, L. Hervella-Nieto, R. Rodríguez, and D. Santamarina, *Fluid-structure acoustic interaction*, in *Computational Acoustics of Noise Propagation in Fluids. Finite and Boundary Element Methods.* S. Marburg, B. Nolte, eds., Springer, 2008, pp. 253–286.

[6] A. Bermúdez and R. Rodríguez, *Numerical computation of elastoacoustic vibrations with interface damping*, in *Équations aux Dérivées Partielles et Applications*, Gauthier-Villars, Paris, 1998, pp. 165–187.

[7] A. Bermúdez and R. Rodríguez, *Modeling and numerical solution of elastoacoustic vibrations with interface damping*, Internat. J. Numer. Methods Engng., **46** (1999) 1763–1779.

[8] A. Bermúdez, R. Rodríguez, and D. Santamarina, *Two discretization schemes for a time-domain dissipative acoustics problem*, Math. Models Methods Appl. Sci., **16** (2006) 1559–1598.

[9] M. Blumenfeld, *The regularity of interface-problems on corner-regions*, in *Singularities and constructive methods for their treatment (Oberwolfach, 1983)*, Lecture Notes in Math., vol. 1121, Springer, Berlin, 1985, pp. 38–54.

[10] D. Boffi, *Fortin operator and discrete compactness for edge elements*, Numer. Math. **87** (2000) 229-246.

[11] D. Boffi, *Finite element approximation of eigenvalue problems*, Acta Numerica **19** (2010) 1–120.

[12] A. Buffa, *Remarks on the discretization of some non-coercive operator with applications to heterogeneous Maxwell equations.* SIAM J. Numer. Anal., **43** (2005) 1–18.

[13] S. Caorsi, P. Fernandes, and M. Raffetto, *On the convergence of Galerkin finite element approximations of electromagnetic eigenproblems.* SIAM J. Numer. Anal., **38** (2005) 580–607.

[14] S-H. Chou, T-M. Huang, W-Q. Huang, and W-W. Lin, *Efficient Arnoldi-type algorithms for rational eigenvalue problems arising in fluid-solid systems*, J. Comp. Phys., **230** (2011), 2189–2206.

[15] J. Descloux, N. Nassif, and J. Rappaz, *On spectral approximation. Part 1: The problem of convergence.* RAIRO Anal. Numér., **12** (1978) 97–112.

[16] J. Descloux, N. Nassif, and J. Rappaz, *On spectral approximation. Part 2: Error estimates for the Galerkin method.* RAIRO Anal. Numér., **12** (1978) 113–119.

[17] C. Engstöm, S. Giani, and L. Grubišić, *Efficient and reliable hp-FEM estimates for quadratic eigenvalue problems and photonic crystal applications*, Comp. Math. Appl. **72** (2016), 952–973.

[18] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numerica, **11** (2002), 237–339.

[19] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1966.

[20] L.E. Kinsler, A.R, Frey, A.B. Coppens, and J.V. Sanders, *Fundamentals of Acoustics*, John Wiley & Sons, 2000.

[21] M.G. Krein and H. Langer, *On some mathematical principles in the linear theory of damped oscillations of continua I.* Integ. Eqns. Oper. Theory, **1** (1978) 364–399.

[22] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, Pergamon Press, 1982.

[23] W. Larbi, J.-F. Deŭ, and R. Ohayon, *A new finite element formulation for internal acoustic problems with dissipative walls*, Internat. J. Numer. Methods Engrg. **68** (2006) 381–399.

[24] K. Lemrabet, *An interface problem in a domain of* $\boldsymbol{R}^3$, J. Math. Appl., **63** (1978) 549–562.

[25] K. Lemrabet, *Régularité de la solution d'un problème de transmission*, J. Math, Pures Appl. (9), **56** (1977) 253-267.

[26] A. Målqvist and D. Peterseim, *Computation of eigenvalues by numerical upscaling*, Numer. Math., **130** (2015), 337–361.

[27] A. Målqvist and D. Peterseim, *Generalized finite element methods for quadratic eigenvalue problems*, M^2AN, Math. Model. Numer. Anal., **51** (2017), 147–163.

[28] S. Meddahi, D. Mora, and R. Rodríguez, *Finite element spectral analysis for the mixed formulation of the elasticity equations*, SIAM J. Numer. Anal., **51** (2013) 1041–1063.

[29] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford Clarendon Press, 2003.

[30] P. Monk and L. Demkowicz, *Discrete compactness and the approximation of Maxwell's equations in* R^3, Math. Comp., **70** (2001), 234, 507-523.

[31] P.M. Morse, *Vibration and Sound*, Acoustical Society of America through the American Institute of Physics, 1995.

[32] R. Ohayon and C. Soize, *Structural Acoustic and Vibration. Mechanical Models Variational Formulations and Discretization*, Academic Press, New York, 1998.

[33] D. Petersein, *Composite finite elements for elliptic problems*, Math. Comp., **83** (2014) 2657–2674.

[34] M. Petzoldt, *Regularity and error estimates for elliptic problems and discontinuous coefficients*, Ph.D. Thesis, Berlin, Freie Univ. (2001).

[35] M. Petzoldt, *Regularity result for the Laplace interface problems in two dimensions*, Z. Anal. Anw., **20** (2001) 431–455.

[36] M. Petzoldt, *A posteriori error estimates for elliptic equations with discontinuous coefficients*, Adv. Comp. Math., **16** (2002), 47–75.

[37] A.D. Pierce, *Acoustics: An Introduction to its Physical Principles and Applications*, Acoustical Society of America through the American Institute of Physics, 1994.

[38] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators* Academic Press, New York, 1978.

[39] F. Tisseur and K. Meerbergen, *The quadratic eigenvalue problem*, SIAM Rev., **43** (2001) 235–286.