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Consistency Properties for Fuzzy Choice Functions: An Analysis with the Łukasiewicz T-Norm

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Abstract: We focus on the relationships among some consistency axioms in the framework of fuzzy choice functions. In order to help disclose the role of the t-norm in existing analyses, we start to study the situation that arises when we replace the standard t-norm with other t-norms. Our results allow us to conclude that unless we impose further structure on the domain of application for the choices, the use of the Łukasiewicz t-norm as a replacement for the standard t-norm does not guarantee a better performance.

Keywords: fuzzy choice function; choice function; consistency; fuzzy arrow axiom; triangular norm

1. Introduction

Choice functions are a pre-eminent tool in the abstract theory of decision making. They have been deeply studied from many perspectives. Particular emphasis is put on disclosing their relationship with models like preferences (binary relations) or utility functions. In this interaction, the axioms of consistency draw a bridge between abstract choices and *rational* behavior. Monographs like Aleskerov and Monjardet [1] and Bossert and Suzumura [2] give ample knowledge about these fundamental issues.

These kind of choice functions are termed deterministic because the behavior of the agents is uniquely determined from the menu of alternatives. When they derive from preferences or utilities, these factors also determine the outcomes for any given menu.

However, deterministic choice functions sometimes seem restrictive, since the behavior of the agents is often subject to some randomness. Stochastic choice functions (e.g., Bandyopadhyay, Dasgupta and Pattanaik [3,4], Machina [5]) or correspondences (e.g., Alcantud [6]) replace the deterministic with stochastic or probabilistic behavior. They permit the analysis of their rationality to be reproduced through adapted axioms of consistency.

Related to this, partial degrees of membership seem appealing. Hence, Zadeh's fuzzy sets have provided real applications in many fields (see [7–10] as a short sample). In decision-making models, they permit a more general expression of the choice attitudes of the agents. As a consequence, the notion of the fuzzy choice function was introduced in Economics by Dasgupta and Deb [11]. Georgescu [12] published a monograph on the topic which refers to a successful notion that extends Banerjee's [13] previous concept by the fuzzification of the available domain. It has also been studied by, for example, Alcantud and Díaz [14], Georgescu [15] (i.e., rationality indicators of a fuzzy choice function), Martinetti et al. [16,17], Wang [18] (i.e., congruence conditions of fuzzy choice functions), Wang et al. [19], Wu and Zhao [20], et cetera. Recently, Martinetti et al. [21] established a connection between probabilistic and fuzzy choice functions.

By reference to the standard crisp analysis, the choice functions that can be derived from fuzzy relations by some reasonable mechanism are called rational fuzzy choice functions. Alternatively, their consistency can be verified in terms of rationality axioms. More generally, rationality indicators can be used as a proxy of their closeness to being rational.

It is therefore important to have precise knowledge about the various forms that rationality axioms can adopt. In particular, their relationships with different specifications of the underlying fuzzy concepts, like implications, is a natural field for investigation.

Alcantud and Díaz [14] introduced some consistency axioms and then investigated some relationships among these and other known axioms. All these axiomatic conditions depend upon the choice of the t-norm that defines the logical implications. As a first analysis of this issue as well as the new research programme for sequential application of fuzzy choice functions, this paper [14] concentrates on the case of the standard t-norm. The structure of the feasible domain of choice situations is another important ingredient in the design and consequences of the consistency axioms. However, the most prominent formulations are the subject of a complete analysis in [14].

The current state of matters does not reveal the role of the t-norm in the aforementioned analyses. We begin to study the situation that arises when we use t-norms other than the standard t-norm. In this regard, we performed a partial investigation of the relevant case of the Łukasiewicz t-norm. We conclude that unless we impose further structure on the domain of application for the choices, the application of the Łukasiewicz t-norm to define implications as an alternative to the standard t-norm is of no avail.

This paper is organized as follows. Section 2 provides fundamental concepts and defines the rationality axioms that we need to analyze. Section 3 contains our (negative) results. Section 4 gives a preliminary research summary of a more specialized analysis where the structural restriction on the domain of choice is tightened. We conclude our paper in Section 5.

2. Background and Basic Definitions

Zadeh's fuzzy sets are defined as follows:

Definition 1. *A fuzzy set μ on the set X is defined as mapping $\mu : X \to [0,1]$. For each $x \in X$, $\mu(x)$ means the extent to which x belongs to the fuzzy set μ. For this reason μ is called a membership function.*

We say that μ is non-zero if for some $x \in X$, $\mu(x) > 0$.

Let $\mathcal{F}(X)$ denote the set of all fuzzy sets on X. In the fuzzy analysis of choices, the basic notion that we need is the fuzzy choice function as defined below (cf., Definition 5.13 in Georgescu [12]):

Definition 2. *Let $\mathcal{B}$ denote a non-empty set of non-zero fuzzy sets of X, a non-empty set.*

A fuzzy choice function on $(X, \mathcal{B})$ is a mapping $C : \mathcal{B} \to \mathcal{F}(X)$ verifying that $C(S)$ is non-zero and $C(S) \subseteq S$ (i.e., $C(S)(x) \leq S(x)$ for all $x \in X$), when $S \in \mathcal{B}$.

For the purpose of studying the salient characteristics of this idea, we usually assume that the domain of choice functions $\mathcal{B}$ verifies certain structural properties. As explained in [14], two basic options come to mind. The first one is Definition 3 below, and it was suggested in Section 5.2 of [12]. The second one is Definition 4 below, and it was suggested in [16]. Because the latter structure is less restrictive than the former, we concentrated our efforts on the analysis under Definition 4.

Definition 3. *Let C be a fuzzy choice function. Then, it verifies*

- *Condition H1 when the fuzzy sets S and $C(S)$ are always normal, that is, when for every S there is $y \in X$ such that $C(S)(y) = 1$.*

- *Condition H2 when the crisp finite and non-empty subsets of X belong to $\mathcal{B}$, that is, when $\mathcal{B}$ includes $f[X] = \{[x_1, \ldots, x_n] : n \geq 1, x_1, \ldots, x_n \in X\}$.*

Definition 4. *Let C be a fuzzy choice function. We say that it verifies*

- *Condition WH1 if when $S \in \mathcal{B}$, there is $y \in X$ such that $S(y) > 0$ and $C(S)(y) = S(y)$.*
- *Condition WH2 when for each triple $x, y, z \in X$, $\{x\}$, $\{x, y\}$ and $\{x, y, z\}$ belong to $\mathcal{B}$.*

In order to extend the analysis in Alcantud and Díaz [14], and following the suggestion by Humberto Bustince, we considered a different implication to formalize the conditions considered in [14]. However, we are aware of the (usually) good behaviour of the minimum t-norm in this context. Therefore with the objective of preserving the original definition as much as possible, we kept the minimum to define intersection. All in all, the assumptions in this paper are as follows (unless otherwise stated):

(1) We keep the minimum t-norm and t-conorm for the purpose of defining intersection and union of fuzzy sets.

Thus for any S and T fuzzy sets defined on the universe X,

$$(S \cap T)(y) = \min(S(y), T(y)) \text{ for all } y \in X$$

$$(S \cup T)(y) = \max(S(y), T(y)) \text{ for all } y \in X.$$

(2) However we utilize the Łukasiewicz t-norm in order to define implications, which boils down to the following expression:

$$y \to_L x = \min(1, 1 - y + x).$$

Thus for any S and T fuzzy sets on X, the Łukasiewicz t-norm yields a degree of inclusion defined by the expression

$$I(S, T) = \cap_{x \in X} S(x) \to_L T(x) = \min(\min_{x \in X}(1 - S(x) + T(x)), 1).$$

It follows from this expression that for two fuzzy sets S and T defined on a common X, their degree of equality using the Łukasiewicz t-norm is defined as

$$E(S, T) = \cap_{y \in X}(1 - |S(y) - T(y)|).$$

Let us now recap the definitions of some consistency properties of fuzzy choices. The reader is addressed to Alcantud and Díaz [14] for the corresponding motivation and interpretations. For the remaining of this section, C denotes a fuzzy choice function on $(X, \mathcal{B})$.

Definition 5. *Let $*$ be a t-norm. We say that C verifies the fuzzy Arrow axiom* **FAA** *when for each $S, T \in \mathcal{B}$ and $x \in X$, the following inequality is true:*

$$I(S, T) * S(x) * C(T)(x) \leq E(S \cap C(T), C(S)).$$

As mentioned above, $*$ will denote the Łukasiewicz t-norm in our study unless otherwise stated. Admittedly, axiom **FAA** may make more sense if we use this t-norm everywhere in this definition. However, our choice does not dissent from existing studies. For example, in the analysis of noteworthy concepts like transitivity, the application of different t-norms has produced remarkable studies [22–24].

Definition 6. *We say that C verifies the fuzzy Chernoff condition* **FCH** *when for each $S, T \in \mathcal{B}$ and $x \in X$, the following inequality is true:*

$$I(S, T) \leq I(S \cap C(T), C(S)) .$$

Definition 7. *We say that C verifies the fuzzy binariness property* **FB** *when for each $S \in \mathcal{B}$,*

$$S(x) * \bigwedge_{y \in X} (S(y) \rightarrow_L C(\{x, y\})(x)) \leq C(S)(x) \,, \textit{ for all } x \in X \,.$$

Definition 8. *We say that C verifies the fuzzy concordance property* **FC** *when for each $S_1, S_2, T \in \mathcal{B}$, the following inequality is true:*

$$E(S_1 \cup S_2, T) \leq I(C(S_1) \cap C(S_2), C(T)) .$$

Definition 9. *We say that C verifies the fuzzy superset property* **FSUP** *when for each $S, T \in \mathcal{B}$ the following inequality is true:*

$$I(S, T) * I(C(T), C(S)) \leq E(C(T), C(S)) .$$

Now we proceed to investigate the relationships among the axioms of consistency that are formulated above.

3. Results under Conditions WH1 and WH2

In [14], the following relationships among consistency properties were established, when both the minimum t-norm (in order to define implications) and both WH1 and WH2 are assumed to hold true:

$$\begin{array}{c} \textbf{FSUP} \\ \not\Uparrow \\ \textbf{FAA} \\ \not\Downarrow \\ \textbf{FC} \end{array} \Longrightarrow \left\{ \begin{array}{l} \textbf{FCH} \\ \\ \textbf{FB} \end{array} \right.$$

Our first purpose is to study these relationships when fuzzy implication and equality are defined through the Łukasiewicz operator in terms of Section 2. By comparison with the aforementioned analysis, our results lead us to conclude that not only the false implications remain false (cf., Counterexamples 3 and 4 below), but also that the implications that hold true under the definitions with the standard t-norm now become false as well (cf., Counterexamples 1 and 2 below). These arguments indicate the lack of robustness of the relationships among axioms that hold under the current assumptions.

Before proceeding with these arguments, we recall that we are bound by assumptions (1) and (2) in Section 2.

Under the assumptions of [14], the fuzzy Arrow axiom is stronger than both the fuzzy Chernoff condition and the fuzzy binariness property. However, under the current assumptions where the minimum t-norm is replaced by Łukasiewicz's for the purpose of defining implications, these consequences no longer hold true:

Example 1. **FAA** *does not imply* **FCH**.

Let $X = \{x, y, z\}$. Let the choice set be $\mathcal{B} = \{X, S, T, \{x, y\}, \{y, z\}, \{x, z\}, \{x\}, \{y\}, \{z\}\}$, with S and T given by Table 1. Let C be the fuzzy choice function given by this table. By inspection, one readily checks that C verifies **FAA**. *Although WH1 and WH2 also hold true, we can observe that C contradicts* **FCH** *because $I(S, T) = 0.7 > 0.6 = I(C(T) \cap S, C(S))$.*

Table 1. The choice function and feasible subsets in Example 1. We omit brackets for simplicity.

	S	*T*	*C* (*S*)	*C* (*T*)	*C* (*x*,*y*)	*C* (*x*,*z*)	*C* (*y*,*z*)	*C* (*X*)
x	0.9	0.6	0.9	0.6	1	1	0	1
y	0.9	0.9	0.5	0.9	0.5	0	1	0.5
z	0.5	0.6	0.5	0.5	0	0.5	0.5	0.5

Example 2. **FAA** *does not imply* **FB**.

Let $X = \{x, y\}$ *and let* $\mathcal{B} = \{X, S, \{x\}, \{y\}\}$, *with* S *given by Table 2. The fuzzy choice function* C *given by this table verifies* **FAA**. *Although WH1 and WH2 also hold true, this fuzzy choice function contradicts* **FB** *because*

$$S(x) *_L \bigwedge_{y \in X}(S(y) \to_L C(\{x,y\})(x)) \quad = 0.9 *_L ((0.9 \to_L 1) \wedge (0.7 \to_L 0.7)) = 0.9 *_L 1 = 0.9 \not\leq 0.8 = C(S)(x).$$

Table 2. The choice function and feasible subsets in Example 2.

	S	*C* (*S*)	*C* (*X*)
x	0.9	0.8	1
y	0.7	0.7	0.7

Now we proceed to check that the implications that failed to hold true in the context of [14] still remain false in our setting:

Example 3. *In order to confirm that* **FAA** *does not imply* **FSUP** *(when WH1and WH2 and the Łukasiewicz t-norm are considered), it suffices to check the counterexample provided in [14] for the minimum t-norm. For the sake of completeness, we reproduce it through Table 3. In that example, the set* X *contains three elements,* $X = \{x, y, z\}$. *Using the Łukasiewicz t-norm, the set* $\mathcal{B} = \{X, T, S, \{x, y\}, \{x, z\}, \{y, z\}, \{x\}, \{y\}, \{z\}\}$ *verifies* **FAA** *but it contradicts* **FSUP** *because* $1 = I(C(T), C(S)) = I(S, T)$ *and* $E(C(S), C(T)) = 1 - |0.9 - 0.7| = 0.8$ *but*

$$I(S, T) *_L I(C(T), C(S)) = 1 \not\leq 0.8 = E(C(S), C(T)).$$

Table 3. The choice function and feasible subsets in Example 3. We omit brackets for simplicity.

	T	*S*	*C* (*T*)	*C* (*S*)	*C* (*x*,*y*)	*C* (*x*,*z*)	*C* (*y*,*z*)	*C* (*X*)
x	0.9	0.9	0.7	0.9	1	0.7	0	0.7
y	0.7	0.6	0.6	0.6	0.6	0	0.6	0.6
z	0.7	0.7	0.7	0.7	0	1	1	1

In order to prove that axiom **FC** is not implied by **FAA** in the current setting, we can also benefit from the corresponding counterexample given in [14]:

Example 4. **FAA** *does not imply* **FC**.

For the sake of completeness, we reproduce the choice setting provided in [14] for the minimum t-norm in Table 4. In that example we work with the set $X = \{x, y, z\}$.

Using the Łukasiewicz t-norm, the collection of non-zero fuzzy sets of X *consisting of* $\mathcal{B} = \{X, T, S_1, S_2, \{x, y\}, \{y, z\}, \{x, z\}, \{x\}, \{y\}, \{z\}\}$ *satisfies the fuzzy Arrow axiom but it contradicts axiom* **FC** *because* $1 = I(C(T), C(S)) = I(S, T)$ *and* $E(C(S), C(T)) = 1 - |0.9 - 0.7| = 0.8$, *but*

$$I(S, T) *_L I(C(T), C(S)) = 1 \not\leq 0.8 = E(C(S), C(T)).$$

Table 4. The choice function and feasible subsets in Example 4. We omit brackets for simplicity.

	T	S_1	S_2	$C(X)$	$C(T)$	$C(S_1)$	$C(S_2)$	$C(x,y)$	$C(x,z)$	$C(y,z)$
x	0.9	0.9	0.6	0.5	0.5	0.9	0.6	0.6	0.5	0
y	0.9	0.6	0.9	0.5	0.5	0.6	0.9	1	0	0.5
z	0.5	0.5	0.5	1	0.5	0.5	0.5	0	1	1

4. Results under H1 and H2 Conditions

If one selects the minimum t-norm in order to define implication and equality between two fuzzy sets, some properties that are not true under WH1 and WH2 do hold under the more stringent requirements H1 and H2. For example, under the assumption that both the minimum t-norm (in order to define implications) and conditions H1 and H2 hold true, Alcantud and Díaz [14] establish the following relationships:

$$\textbf{FAA} \Rightarrow \begin{cases} \textbf{FCH} \\ \textbf{FSUP} \\ \textbf{FC} \\ \textbf{FB} \end{cases}$$

We observe that H1 and H2 provide a much richer environment for the existence of implications among consistency properties, in comparison with the milder WH1 and WH2.

In light of the results obtained in the previous section, conditions WH1 and WH2 are clearly insufficient to provide relationships when we consider the implication defined by the Łukasiewicz t-norm. The question that naturally arises is if we can obtain positive results when we reinforce WH1 and WH2 to H1 and H2.

The first implication that we study under H1 and H2 is whether the fuzzy Arrow axiom guarantees the fuzzy Chernoff condition in the context of the Łukasiewicz implicator. However, by contrast with the case of the standard implicator explained above, the result that we obtain is negative:

Example 5. *When the Łukasiewicz t-norm is considered, and even when we assume that H1 and H2 hold true,* **FAA** *does not imply* **FCH**. *As a counterexample, we fix $X = \{x, y, z\}$. Let the choice set be $\mathcal{B} = \{X, T, S, \{x, y\}, \{y, z\}, \{x, z\}\}$, with T and S given by Table 5. Using the Łukasiewicz t-norm, one can check that $\mathcal{B}$ satisfies* **FAA** *but it contradicts* **FCH** *because $I(S, T) = 0.8$ and $I(S \cap C(T), C(S)) = 0.75$, but*

$$I(S,T) = 0.8 \not\leq I(S \cap C(T), C(S)) = 0.75\,.$$

Table 5. The choice function and feasible subsets in Example 5. We omit brackets for simplicity.

	T	S	$C(T)$	$C(S)$	$C(x,y)$	$C(x,z)$	$C(y,z)$	$C(X)$
x	1	0.7	1	0.7	1	1	0	1
y	0.8	1	0.75	1	0.75	0	1	0.75
z	0.75	0.75	0.75	0.5	0	0.75	0.5	0.75

However, the next two propositions prove that fuzzy binariness and the fuzzy superset property follow from the fuzzy Arrow axiom.

Proposition 1. *The fuzzy Arrow axiom* **FAA** *implies the fuzzy binariness property* **FB**.

Proof. Fix C, a fuzzy choice function on $\mathcal{B}$. Assume that C verifes **FAA** but it contradicts **FB**. Then, there exist a subset $S \in \mathcal{B}$ and $x \in X$ for which the following holds true:

$$S(x) * \bigwedge_{y\in X} (S(y) \to_L C(\{x,y\})(x)) > C(S)(x)\,.$$

Since H1 and H2 hold, there must exist $y \in X$ with the property $S(y) = 1 = C(S)(y)$. This element cannot be x since the previous inequality implies $C(S)(x) < 1$. Therefore $x \neq y$ and we can derive

$$S(x) * (S(y) \to_L C(\{x,y\})(x)) \geq S(x) * \bigwedge_{y\in X} (S(y) \to_L C(\{x,y\})(x)) > C(S)(x)\,.$$

Since $S(y) = 1$, then $S(y) \to_L C(\{x,y\})(x) = 1 \to_L C(\{x,y\})(x) = C(\{x,y\})(x)$. In addition,

$$S(x) * (S(y) \to_L C(\{x,y\})(x)) = S(x) * C(\{x,y\})(x) > C(S)(x)\,.$$

From this, we conclude $S(x) > 1 - (C(\{x,y\})(x) - C(S)(x))$.

Now, if we focus on S and $\{x,y\}$, **FAA** entails

$$I(\{x,y\}, S) * \{x,y\}(y) * C(S)(y) \leq E(\{x,y\} \cap C(S), C(\{x,y\}))\,.$$

Since $I(\{x,y\}, S) = S(x)$, $\{x,y\}(y) = 1$, $C(S)(y) = 1$ and

$$E(\{x,y\} \cap C(S), C(\{x,y\})) \leq 1 - |C(S)(x) - C(\{x,y\})(x)|\,,$$

the previous inequality implies

$$S(x) \leq 1 - |C(S)(x) - C(\{x,y\})(x)|\,.$$

However, $S(x) > 1 - (C(\{x,y\})(x) - C(S)(x))$, which yields the final contradiction. □

Proposition 2. *The fuzzy Arrow Axiom* **FAA** *implies the fuzzy superset property* **FSUP**.

Proof. Fix C, a fuzzy choice function on $\mathcal{B}$. Assume that C verifies **FAA** but it contradicts **FSUP**. There must be subsets $S, T \in \mathcal{B}$ with the property

$$I(S,T) * I(C(T), C(S)) > E(C(S), C(T))\,.$$

Since $E(C(S), C(T)) = \min(I(C(T), C(S)), I(C(S), C(T)))$ and $I(C(T), C(S)) \geq I(S,T) * I(C(T), C(S)) > E(C(S), C(T))$. We must have $E(C(S), C(T)) = I(C(S), C(T))$ and

$$I(S,T) * I(C(T), C(S)) > I(C(S), C(T))\,,$$

whereas $1 > I(C(S), C(T))$. Now for every $\epsilon > 0$ sufficiently small, there is $z \in X$ that verifies $1 - C(S)(z) + C(T)(z) \leq I(C(S), C(T)) + \epsilon < 1$, and this yields $C(T)(z) < C(S)(z) \leq S(z)$. From this we derive

$$\begin{aligned} E(C(T) \cap S, C(S)) &\leq 1 - |S \cap C(T)(z) - C(S)(z)| \\ &= 1 - |C(T)(z) - C(S)(z)| \\ &\leq 1 - C(S)(z) + C(T)(z) \\ &\leq I(C(S), C(T)) + \epsilon\,. \end{aligned}$$

Therefore, $E(C(T) \cap S, C(S)) \leq I(C(S), C(T))$. Now, since we assume **FAA** but S, T do not satisfy **FSUP**, we get

$$I(S,T) * S(x) * C(T)(x) < I(S,T) * I(C(T),C(S)), \quad \forall x \in X.$$

Equivalently,

$$I(S,T) + S(x) + C(T)(x) - 2 < I(S,T) + I(C(T),C(S)) - 1.$$

Therefore,

$$S(x) + C(T)(x) < I(C(T),C(S)) + 1, \quad \forall x \in X.$$

Now $I(C(T),C(S)) = \inf_{x \in X} \min\{1 - C(T)(x) + C(S)(x), 1\}$. Then, for each element x from X it holds that $S(x) + C(T)(x) < 1 - C(T)(x) + C(S)(x) + 1$, or equivalently,

$$S(x) + 2C(T)(x) < C(S)(x) + 2, \quad \forall x \in X.$$

However, since H2 holds true, there is $y \in X$ with the property $T(y) = C(T)(y) = 1$. For this element we get $S(y) + 2 \cdot 1 < C(S)(y) + 2$, or equivalently, $S(y) < C(S)(y)$. A contradiction. □

Finally, the following counterexample proves that **FAA** does not imply **FC**:

Example 6. *When the Łukasiewicz t-norm is considered, and even when we assume that H1 and H2 are true,* **FAA** *does not imply* **FC**. *As a counterexample, we fix* $X = \{x, y, z\}$. *As a choice set we take* $\mathcal{B} = \{X, S, T, \{x,y\}, \{y,z\}, \{x,z\}\}$ *with S and T given by Table 6. Using the Łukasiewicz t-norm, one readily checks that* $\mathcal{B}$ *satisfies* **FAA**. *However it does not satisfy* **FC** *because* $E(S \cup S, T) = E(S,T) = 0.9$ *and* $I(C(S), C(T)) = 0.8$, *but*

$$E(S \cup S, T) = 0.9 \not\leq 0.8 = I(C(S), C(T)).$$

Table 6. The choice function and feasible subsets in Example 6. We omit brackets for simplicity.

	S	*T*	*C* (*S*)	*C* (*T*)	*C* (*x*,*y*)	*C* (*x*,*z*)	*C* (*y*,*z*)	*C* (*X*)
x	1	0.9	1	0.8	0.9	0.9	0	0.9
y	0.9	1	0.9	1	1	0	0.9	0.9
z	0.9	0.9	0.9	0.9	0	1	1	1

The following figure summarizes the connections that hold and do not hold when H1 and H2 are imposed in the conditions of our analysis:

$$\begin{array}{c} \mathbf{FCH} \\ \not\Uparrow \\ \mathbf{FAA} \\ \not\Downarrow \\ \mathbf{FC} \end{array} \Longrightarrow \left\{ \begin{array}{l} \mathbf{FB} \\ \\ \mathbf{FSUP} \end{array} \right.$$

Again, we observe that the relationships among axioms that hold true under the assumptions in this section are sensible to the choice of the t-norm that defines implications.

5. Conclusions

The analysis of the consistency of choices is an important challenge that aspires to establish the limits of rational behavior from a theoretical point of view. The meaning of consistency is subtly complex, hence it is subject to debate and loaded with nuances. Just like apparently direct concepts (e.g., transitivity of binary relations) adopt different forms in the fuzzy theories, consistency axioms for choice functions abound in this setting even if inspired by a common rationale from the abstract theory of choice in the crisp environment. Therefore one of the sources of disparity is the basic principle that the axiom intends to convey (e.g., expansion or contraction consistency). Another one that befits the fuzzy domain is the selection of the t-norm from which implications, unions, and intersections are derived. Our theoretical contribution goes in the direction of exploring the variations of some known principles when the t-norm is changed. Particularly, we investigate what relationships apply so that the researcher can be aware of the implications of adopting a given principle.

There are of course some antecedents of research in similar lines. Particularly, Alcantud and Díaz [14] established a number of relationships among some consistency properties of fuzzy choice functions in a concrete setting. The results in this paper suffice to conclude that when we use the Łukasiewicz t-norms instead of the standard t-norm to define logical implications, those axioms are independent from each other. Therefore, the hypothesis that we can advance is that the role of the t-norm is crucial in the analysis of consistency axioms. Our conclusion is that the implications among consistency axioms that were proven in [14] are not robust in a loose sense.

In the future we intend to complete this investigation with a full inspection of the case in Section 4. By doing so we will check whether our conjecture is true. In order to fully reveal the role of the t-norm in the analysis of the rationality axioms defined for fuzzy choice functions, other cases should be looked into carefully. Particularly, the following possibilities have been suggested by a perceptive referee and deserve special attention.

(a) In this study we resorted to the Łukasiewicz operator in the definition of implication only. However intersections are still defined through the minimum operator. Replacing it with the Łukasiewicz operator would produce a completely new set of consistency conditions, and axioms **FAA**, **FCH**, . . . would acquire a different meaning. We believe that the analysis of the relationships among these renewed axioms will probably agree with the verdict that the implications among consistency axioms are not robust.
(b) In any of these logical frameworks, one can also wonder whether suitable combinations of consistency axioms ensure the verification of other axioms.

Nevertheless, a full inspection of these two lines of research should be the purpose of a separate investigation.

In a different line of research, we shall also investigate the framework of the sequential application of fuzzy choice functions with the Łukasiewicz and other t-norms. The inspiration for this approach lies in the research programme posed by [14] as well as previous research in the crisp case like García-Sanz and Alcantud [25] and the references therein.

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