On some classes of directionally monotone functions

H. Bustince [a,b,c] R. Mesiar [d,e] A. Kolesárová [f] G. Dimuro [g]
J. Fernandez [a,b,c] I. Diaz [h] S. Montes [i]

[a] *Departamento de Estadistica, Informatica y Matematicas, Universidad Pública de Navarra, Campus Arrosadia, 31006 Pamplona, Spain.*

[b] *Institute of Smart Cities, Universidad Pública de Navarra, Campus Arrosadia, 31006Pamplona, Spain.*

[c] *Laboratory Navarrabiomed, Hospital Complex of Navarre (CHN), Universidad Publica de Navarra, IdiSNA, Irunlarrea 3. 31008, Pamplona, Spain.*

[d] *Department of Mathematics and Descriptive Geometry, Slovak University of Technology, SK-810 05 Bratislava, Slovakia.*

[e] *Palacky University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic.*

[f] *Institute of Information Engineering, Automation and Mathematics, Faculty of Chemical and Food Technology, Slovak University of Technology, Radlinského 9, 812 37 Bratislava 1, Slovakia*

[g] *Centro de Ciência Computacionais, Universidade Federal do Rio Grande, Av. Itália km 08, Campus Carreiros, 96201-900, Rio Grande, Brazil.*

[h] *Department of Computer Science, University of Oviedo, Spain.*

[i] *Department of Statistics and Operational Research and Mathematical Teaching, University of Oviedo, Spain.*

Abstract

In this work we consider some classes of functions with relaxed monotonicity conditions generalizing some other given classes of fusion functions. In particular, DI aggregation functions (called also pre-aggregation functions), DI conjunctors, or DI implications, etc., generalize the standard classes of aggregation functions, conjunctors, or implication functions, respectively. We analyze different properties of these classes of functions and we discuss a construction method in terms of linear combinations of t-norms.

Key words: Directional monotonicity; classes of directionally increasing functions; pre-aggregation functions.

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1 Introduction

Recently, some new classes of functions have been introduced by considering already existing types of functions and relaxing the requirements of monotonicity to allow increasingness or decreasingness along one direction rather than along every direction in some cone [6,10,26]. Additionally, in several recent works the notion of pre-aggregation function has been introduced [18], proving to be a very useful extension of aggregation functions [15] in classification problems [18], specially when some specific pre-aggregation functions built in terms of extensions of the Choquet integral [3] are considered [19,20]. Note that to stress that the standard monotonicity of aggregation functions was relaxed into the directional monotonicity, we will call the class of pre-aggregation functions synonymically as directionally increasing aggregation functions, DI aggregation functions for short.

In the same way as in the case of aggregation functions, every class of functions which is defined in terms of usual monotonicity (apart from possibly other conditions) can be extended just by considering monotonicity along some specific directions. So it is natural to consider a systematic study of such possible extensions which highlights common features for all of them. In particular, because for some functions, even if monotonicity is usually required in the definition, this property is in fact a consequence of the other requirements. For instance, this is the case for continuous t-norms [22] and copulas [23], and for this reason a definition which restricts the direction along which monotonicity is considered does not make full sense in this case.

The goal of this work is to provide a general framework which covers the definition of new classes of functions obtained by restricting the monotonicity of already known functions to some specific directions. In particular, we define and analyze the class of functions obtained by considering monotonicity along some (but not all) directions along which monotonicity was required in the original class. In this way, we are able to generalize the notion of pre-aggregation function, as well as to introduce a natural extension of this concept to encompass other relevant classes of fusion functions, including implication functions, see, for instance, [7,24,25].

In particular, the objectives of this paper are:

- To introduce some class of functions obtained by considering monotonicity along some (but not all) directions along which monotonicity was required

Email addresses: bustince@unavarra.es (H. Bustince), mesiar@math.sk (R. Mesiar), anna.kolesarova@stuba.sk (A. Kolesárová), gracalizdimuro@furg.br (G. Dimuro), fcojavier.fernandez@unavarra.es (J. Fernandez), sirene@uniovi.es (I. Diaz), montes@uniovi.es (S. Montes).

for given classes of fusion functions, such as DI aggregation functions, DI conjunctors, DI implications, etc.

- To analyze some analytical and geometrical properties of these new classes of functions.
- To discuss a specific example of construction of functions in these new classes from well known fusion functions.

The structure of this paper is the following: We start with some preliminary notions and results. In Section 3, we introduce the notion of new classes of functions extending given classes of fusion functions and we consider some algebraical and geometrical properties of them, whereas, in Section 4, we consider a specific example focused on conjunctors. We finish with some conclusions and relevant references.

2 Preliminaries

We start by recalling several well-known concepts which are useful for our subsequent developments. First we recall the notion of fusion function, which has already been used in the previous papers in the literature (see [10,17]) .

In the following, we use any function $F : [0,1]^n \to [0,1]$ to merge values from the unit intervals into values in the same interval. In our context, we call them fusion functions. For some results concerning fusion functions see [10,17].

We denote by $\mathcal{F}_n$ the class of all fusion functions of dimension n. We also denote by $\mathcal{C}\wr_n$ the class of all n-dimensional fusion functions that are constant.

A distinguished class of fusion functions is that of aggregation functions [11,15].

Definition 2.1 *An aggregation function is a fusion function $M : [0,1]^n \to [0,1]$ such that:*

(i) *M is increasing.*
(ii) *$M(0,\dots,0) = 0$ and $M(1,\dots,1) = 1$.*

Among aggregation functions we can also consider some specific types of functions. For instance, triangular norms [16].

Definition 2.2 *A triangular norm (t-norm) is an aggregation function $T : [0,1]^2 \to [0,1]$ such that:*

(i) *T is symmetric.*
(ii) *T is associative.*
(iii) *$T(x,1) = T(1,x) = x$ for every $x \in [0,1]$*

Example 1 *The following are basic examples of t-norms:*

- $T_M(x, y) = \min(x, y)$.
- $T_P(x, y) = xy$.
- $T_L(x, y) = \max(x + y - 1, 0)$.

Note that the chain of inequalities

$$T_L(x, y) \leq T_P(x, y) \leq T_M(x, y)$$

holds for every $x, y \in [0, 1]$.

We denote by $\mathcal{T}$ the class of all t-norms.

T-norms are particular instances of a broader class of aggregation functions called conjunctors, which extend the logical (binary) operator AND to the unit square.

Definition 2.3 *A conjunctor is an aggregation function* $C : [0, 1]^2 \to [0, 1]$ *such that* $C(1, 0) = C(0, 1) = 0$.

Evidently, conjunctors are monotone extensions of the classical Boolean conjunction.

A notion closely related to t-norms is that of overlap functions [2,9], which are also conjunctors.

Definition 2.4 *An overlap function is a continuous aggregation function* $G_O : [0, 1]^2 \to [0, 1]$ *such that:*

(i) G_O *is symmetric.*
(ii) $G_O(x, y) = 0$ *iff* $xy = 0$.
(iii) $G_O(x, y) = 1$ *iff* $xy = 1$.

Every continuous t-norm without zero divisors is an overlap function. But there exist overlap functions which are not t-norms, as, for instance:

$$G_O(x, y) = \min(x^k y, xy^k)$$

for $k > 0$ and $k \neq 1$.

We also recall here some other classes of aggregation functions which will be useful for us in our subsequent developments ([23]).

Definition 2.5 *A fusion function* $S : [0, 1]^2 \to [0, 1]$ *is called a semicopula if it is increasing in each coordinate and* 1 *is its neutral element, i.e.,* $S(x, 1) = S(1, x) = x$ *for all* $x \in [0, 1]$.

Definition 2.6 *A quasi-copula is a semicopula Q which is also a 1-Lipschitz function.*

Definition 2.7 *A copula is a semicopula C which is 2-increasing, i.e.,*

$$C(x,y) + C(x',y') - C(x',y) - C(x,y') \geq 0$$

for all $0 \leq x \leq x' \leq 1, 0 \leq y \leq y' \leq 1$.

We denote by $\mathcal{C}$ the class of all copulas.

Another class of functions which is also of interest for us in this work is provided by implication functions.

Definition 2.8 *An implication function is a fusion function $I : [0,1]^2 \to [0,1]$ such that*

(1) $I(0,0) = I(0,1) = I(1,1) = 1$ *and* $I(1,0) = 0$
(2) I is decreasing in its first variable and increasing in its second variable.

We denote by $\mathcal{I}$ the class of all implication functions.

2.1 Different notions of monotonicity

In all the definitions we have discussed in this section, monotonicity plays a key role. But, from the point of view of applications, monotonicity can be a too strong requirement. For this reason, several researchers have considered weakened versions of monotonicity. In particular, Wilkin et al. [26] introduced the notion of weak monotone function in order to encompass the mode (which is not monotone) in the following way.

Definition 2.9 *([26]) A fusion function $F\colon [0,1]^n \to [0,1]$ is said to be weakly monotone increasing if the inequality*

$$F(x_1 + h, \ldots, x_n + h) \geq F(x_1, \ldots, x_n) \tag{1}$$

holds for every $x_1, \ldots, x_n, h \in [0,1]$ *such that* $x_i + h \leq 1$, $i \in \{1, \ldots, n\}$.

Analogously, F is said to be weakly monotone decreasing if the inequality

$$F(x_1 + h, \ldots, x_n + h) \leq F(x_1, \ldots, x_n) \tag{2}$$

holds for every $x_1, \ldots, x_n, h \in [0,1]$ *such that* $x_i + h \leq 1$, $i \in \{1, \ldots, n\}$.

A function F is said to be weakly monotone if it is weakly monotone increasing or weakly monotone decreasing. Clearly, any increasing (decreasing) fusion function is also weakly monotone increasing (decreasing).

From a geometrical point of view, weak monotone functions are those functions which increase (or decrease) along the fixed ray defined by the vector $(1, \ldots, 1)$ starting at any point of the domain of the function. From a theoretical point of view, there is no reason to restrict the analysis to this specific direction rather than considering more general rays. This idea was developed into the notion of directional monotonicity in [10]. We recall now this definition.

Definition 2.10 ([10]) *Let $\vec{r} = (r_1, \ldots, r_n)$ be a real n-dimensional vector, $\vec{r} \neq \vec{0}$. A fusion function $F\colon [0,1]^n \to [0,1]$ is $\vec{r}$-increasing if for all points $(x_1, \ldots, x_n) \in [0,1]^n$ and for all $c > 0$ such that $(x_1 + cr_1, \ldots, x_n + cr_n) \in [0,1]^n$ we have*

$$F(x_1 + cr_1, \ldots, x_n + cr_n) \geq F(x_1, \ldots, x_n) \ .$$

The notion of $\vec{r}$-decreasing fusion function is defined analogously, just reversing the previous inequality.

In order to follow with our study, we first introduce the following notations. Given a set D of non-null vectors in $\mathbb{R}^n$, we denote by $\mathcal{F}_{n,D}$ the set

$$\mathcal{F}_{n,D} = \{F \in \mathcal{F}_n \quad | \quad F \text{ is } \vec{r} - \text{increasing for some } \vec{r} \in D\}$$

We also define

$$\mathcal{F}_{n,(D)} = \{F \in \mathcal{F}_n \quad | \quad F \text{ is } \vec{r} - \text{ increasing for every } \vec{r} \in D\}$$

Finally, given a fusion function $F \in \mathcal{F}_n$, we denote by R_F the set of all non-null vectors $\vec{r}$ such that F is $\vec{r}$-increasing. Given a set $\mathcal{G} \subseteq \mathcal{F}_n$, we write:

$$R_{\mathcal{G}} = \cap_{G \in \mathcal{G}} R_G$$

With these definitions, the following result is straightforward.

Proposition 2.1 *The following statements hold:*

(1) $\mathcal{F}_{n,(\emptyset)} = \mathcal{F}_n = \mathcal{F}_{n,\emptyset}$;
(2) $\mathcal{F}_{n,(\mathbb{R}^n \setminus \{\vec{0}\})} = \mathcal{C}\wr_n$.

It is worth noticing that every increasing function (in the usual sense) is also $\vec{r}$-increasing for every non-negative real vector $\vec{r}$. However, the class of directionally increasing fusion functions is much wider than that of increasing functions. For instance:

- Fuzzy implication functions (see [1,7]) are $(-1, 1)$-increasing functions. As a consequence, many other functions which are built using implication functions (probably combined with aggregation functions) are also directionally increasing. For instance, some subsethood measures (see [8]) are directionally monotone, since they can be obtained by aggregating appropriate implication functions.

- Particular means are also directionally increasing with respect to some special directions $\vec{r}$, see [3].
- As we have already mentioned, weakly increasing functions are a particular case of directionally increasing functions with respect to the direction $\vec{r} = (1, \ldots, 1)$.

Now we combine the notions of directional increasingness and that of aggregation function.

Definition 2.11 *A directionally increasing (DI) aggregation function is an $\vec{r}$- directionally increasing fusion function $F\colon [0,1]^n \to [0,1]$ with respect to some direction $\vec{r} \in [0,1]^n$ and that satisfies the same boundary conditions as an aggregation function.*

Recall again that the concept of DI aggregation function has already been introduced in the literature with the name of pre-aggregation function, see [18]. Furthermore, also note that the term DI subsethood measure (meaning decreasing and increasing subsethood measure) was introduced in [5].

Every aggregation function is in particular a DI aggregation function. But there exist DI aggregation functions which are not aggregation functions, the mode being one of the most relevant examples. For other properties and construction methods of DI aggregation functions, see also [12].

Remark 1 *Note that formally, with respect to the null vector $\vec{r} = (0, \ldots, 0)$, any fusion function is $\vec{r}$-directionally increasing (this is an empty condition). On the other hand, for any direction $\vec{r}$, if a fusion function is $\vec{r}$-directionally increasing, then it is also $\alpha\vec{r}$-directionally increasing for any $\alpha > 0$, see [10]. Thus, concerning the directions, it is equivalent to write that a fusion function is directionally increasing for some $\vec{r}$ from $[0,1]^n$, from $[0,1]^n \setminus \{(0, \ldots, 0)\}$, from $[0, \infty[^n$ or from $[0, \infty[^n \setminus \{(0, \ldots, 0)\}$. For the sake of simplicity, we will use in such cases only $[0,1]^n$.*

3 Classes of DI fusion functions

In the same way as DI aggregation functions were introduced from aggregation functions by considering directional monotonicity instead of monotonicity, it is possible to define DI t-norms, DI overlaps, and, in general, a class of DI functions for every class of functions in whose definition monotonicity enters. In this section, we start such an analysis, providing a general framework for all these definitions. In particular, we propose to consider a system $\mathcal{G}$ of fusion functions characterized by some properties p not related to monotonicity forming a set P, and, possibly, by some monotonicity conditions. Then P and $R_{\mathcal{G}}$ completely characterize $\mathcal{G}$.

Now, observe that, if for any non-null vector $\vec{r}$, we denote :

$\mathcal{G}_{\vec{r}} = \{F \in \mathcal{F}_n \quad | \quad F$ satisfies all the properties in P and F is $\vec{r}-$increasing$\}$

then

$$\mathcal{G} = \cap_{\vec{r} \in R_{\mathcal{G}}} \mathcal{G}_{\vec{r}}$$

Example 2 *Consider the class of two-dimensional copulae, $\mathcal{C}$. In this case, the set of properties not related to monotonicity is*

$$P = \{0 \textit{ annihilator }, 1 \textit{ neutral element }, \textit{ supermodularity}\}$$

and

$$R_{\mathcal{C}} = [0, \infty[^2 \setminus \{(0,0)\}$$

Then, for any $\vec{r} \in R_{\mathcal{C}}$, we have that $\mathcal{C}_{\vec{r}} = \mathcal{C}$. But, furthermore,

$$\mathcal{C}_{(-1,0)} = \emptyset$$

and

$$\mathcal{C}_{(-1,1)} = \{T_L\}$$

where $T_L(x, y) = \max(0, x + y - 1)$ is the smallest copula.

Now take $\alpha \in [0, 1]$, and define a function $C_\alpha : [0, 1]^2 \to [0, 1]$ as

$$C_\alpha(x, y) = \max(0, \min(x + \alpha(y - 1), y + \alpha(x - 1))) \ .$$

Then C_α is a copula and R_{C_α} is a cone bounded by the vectors $c(-\alpha, 1)$ and $c(1, -\alpha)$, with $c > 0$. Observe that $C_0 = T_M$ and then R_{C_0} is the first quadrant (without the null vector), and $C_1 = T_L$ and R_{C_1} is the half plane above the axis of the second and the fourth quadrant.

In order to avoid inconsistencies, it is important that the set P actually comprises all boundary conditions. This is due to the fact that some conditions can be deduced from other ones if all the directions in $R_{\mathcal{G}}$ are considered, but this is not any more the case if less conditions are considered. For instance, take, e.g., the class $\mathcal{I}$ of all implication functions. Then, $R_{\mathcal{I}}$ is the cone corresponding to the second quadrant (without its vertex, which corresponds to the null vector). But regarding the definition of P, several possibilities may be considered, e.g.,

$$P_1 = \{I(1, 0) = 0, I(0, 0) = I(0, 1) = I(1, 1) = 1\}$$

or

$$P_2 = \{I(1,0) = 0, I(0,0) = I(1,1) = 1\} .$$

Note that from the $(0,1)$-monotonicity of implications, also in case P_2 the equality $I(0,1) = 1$ follows. But we intend that DI implication functions (i.e., DI fusion functions which satisfy the same boundary constraints as implication functions but which are just $\vec{r}$-increasing for some direction $\vec{r}$ in the second quadrant) related to P_1 should always satisfy $F(0,1) = 1$, which is not true for DI implication functions related to P_2, as this is much broader class, which contains, e.g., the function F given by $F(0,0) = F(1,1) = 1$ and $F(x,y) = 0$ elsewhere, which is a particular instance of bi-implication. So in order to define DI implication functions, we should take:

$$P_{\mathcal{I}} = \{F(1,0) = 0 \text{ and } F(x,y) = 1 \text{ if } x = 0 \text{ or } y = 1\} . \tag{3}$$

Analogously, if we consider the class of t-norms, $\mathcal{T}$, we should have

$$\begin{aligned} P_{\mathcal{T}} = \{ & F(x,y) = \min(x,y) \text{ whenever } \{x,y\} \cap \{0,1\} \neq \emptyset; \\ & F \text{ is symmetric and associative}\} \end{aligned}$$

Taking into account this discussion, the key definition from which the rest of the work originates is the following one.

Definition 3.1 *Let $\mathcal{G}$ be a set of n-dimensional fusion functions characterized, possibly by some monotonicity and by some properties p not related to monotonicity forming a set P which also contains all the boundary conditions fulfilled by functions in $\mathcal{G}$; that is, if for all $F \in \mathcal{G}$ and some argument $\mathbf{x}$ it holds that $F(\mathbf{x}) = f(\mathbf{x})$ with some fixed function f, then this boundary condition should be included in P.*

Then, DI-$\mathcal{G}$ is the class of all n dimensional fusion functions such that $G \in$ DI-$\mathcal{G}$ if and only if:

(i) G satisfies the same boundary conditions and properties not related to ordering as the functions in $\mathcal{G}$.
(ii) G is $\vec{r}$-increasing for some $\vec{r} \in R_{\mathcal{G}}$.

Note that a fusion function G such that $G \in DI - \mathcal{G}$ but $G \notin \mathcal{G}$ will be called a proper $DI - \mathcal{G}$ function.

Example 3 *(1) The first example for this definition, which is also the motivation for the present work, is provided by DI aggregation functions. In this case, we take $\mathcal{G}$ as the set of aggregation functions, $P = \{G(0,\ldots,0) = 0$ and $G(1,\ldots,1) = 1\}$ and $R_{\mathcal{G}} = \{\vec{r} \in [0,1]^n\}$.*

(2) Analogously, we can obtain new examples if we consider specific examples of aggregation functions. For instance, DI overlap functions are defined

taking $\mathcal{G}$ as the set of overlap functions (see [9]) and $R_{\mathcal{G}} = \{\vec{r} \in [0,1]^2\}$. Analogously, DI semicopulae can be defined taking $\mathcal{G}$ as the set of semicopulae and $R_{\mathcal{G}} = \{\vec{r} \in [0,1]^2\}$. In general, given any specific class of aggregation function, we can build the corresponding class of DI fusion functions by keeping the same boundary conditions and relaxing the directional increasingness with respect to all vectors $\vec{r} \in R_{\mathcal{G}}$ into an $\vec{r}$-increasingness with respect to some (at least one but arbitrary) vector from $R_{\mathcal{G}}$.

Example 4 *Another interesting example to be discussed is that of DI implication function, that we have already started to discuss. In this case, we take as $\mathcal{G}$ the set of implication functions, with $P_{\mathcal{I}}$ defined as above, and we get $R_{\mathcal{G}} = [-1, 0] \times [0, 1]$. For instance, we have that:*

$$I(x,y) = \begin{cases} 1 & \textit{if } x = y = 1 \textit{ or } x = 0; \\ 0 & \textit{if } (x,y) = (1,0); \\ A(x,y) & \textit{in other cases,} \end{cases}$$

where $A : [0,1]^2 \to [0,1]$ is any $(0,1)$-increasing function, is an example of DI implication. In particular, we can take as A any aggregation function, and in this way, the resulting function is a proper DI implication function.

Remark 2 *According to the philosophy behind Definition 3.1, note that, for speaking of a class of DI fusion functions for a given class $\mathcal{G}$ of functions, it is necessary that monotonicity is a property defining the original class of functions. This means a proper class of DI fusion functions exists for every class of functions.*

Example 5 *(1) Let us consider the class of proper DI copulae. These functions are defined in terms of appropriate boundary conditions, supermodularity or, equivalently, 2-increasingness property. Note that supermodularity (2-increasingness) together with the boundary conditions implies the increasingness and 1-Lipschitzianity of copulae. So it does not exist a set of directions $R_{\mathcal{G}}$ for characterizing these functions. Even if these functions are $\vec{r}$-increasing for every $\vec{r} \in [0,1]^2$, these properties follow from the constraints imposed in the definition so it is not necessary to impose directions.*

(2) An analogous situation holds for the class of continuous t-norms. Actually, these functions are usually defined requiring monotonicity. But, from an original result by Mostert and Shields [22], see also [16], any continuous fusion function which is associative, and for which 1 is a neutral element and 0 is idempotent must be a continuous t-norm, so it must be, in particular, monotone. So, according with our definition, no proper DI continuous t-norms exist. But since monotonicity does not follow from the other conditions in the definition of a t-norm which is not continuous,

it actually makes sense to speak of the class of general DI t-norms, see more examples in [13].

(3) *From duality, if a proper class of DI fusion functions of a class $\mathcal{G}$ exists, it also exists the proper class of DI fusion functions of the dual class of $\mathcal{G}$. In particular, this means that proper DI t-conorms exist but this is not the case for the class of continuous t-conorms (which is obtained by duality from the class of continuous t-norms). See more examples in [14].*

Example 6 *Consider the class $\mathcal{W}$ of weighted arithmetic means. In this case*

$$P_{\mathcal{W}} = \{idempotency; additivity\}$$

and

$$R_{\mathcal{W}} = [0, \infty[^n \setminus \{\vec{0}\}$$

If F belongs to the corresponding class of DI weighted arithmetic means, note that additivity implies that

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i$$

for some vector $(w_1, \ldots, w_n)$. And the idempotency implies that $\sum_{i=1}^{n} w_i = 1$.

Now, in order to have a fusion function, if we consider an input which takes value 1 at position i and 0 in any other position, then

$$F(0, \ldots, 0, 1, 0, \ldots, 0) = w_i$$

so $(w_1, \ldots, w_n) \in [0, 1]^n$, thus F is in fact a weighted arithmetic mean, and there does not exist a proper class of DI weighted arithmetic means.

But consider now the class $\mathcal{H}$ of OWA operators. Then

$$P_{\mathcal{H}} = \{idempotency; comonotone\ additivity; symmetry\}$$

and $R_{\mathcal{H}} = R_{\mathcal{W}}$, as above. In this case, if we consider a function in the corresponding class of DI OWA operators, the comonotone additivity and symmetry imply that

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_{\sigma(i)}$$

where σ is a permutation of $\{1, \ldots, n\}$ such that $\sigma(1) \geq \sigma(2) \geq \cdots \geq \sigma(n)$. Furthermore, idempotency implies that $\sum_{i=1}^{n} w_i = 1$. To be a fusion function, consider $\mathbf{x}_i$ to be the input with 1 in the first i-th positions and 0 in all the other ones. Then:

$$F(\mathbf{x}_i) = w_1 + \cdots + w_i \in [0, 1]$$

and in fact, this is also a sufficient condition. This means that, for $n = 2$, proper DI OWA operators do not exist, but for instance, for $n = 3$, we can *consider $w_1 = w_3 = 1$ and $w_2 = -1$, and then*

$$F(x_1, x_2, x_3) = \max(x_1, x_2, x_3) - median(x_1, x_2, x_3) + \min(x_1, x_2, x_3)$$

Obviously, if $x_1+c, x_2+c, x_3+c \in [0,1]$ for some positive c, then $F(x_1+c, x_2+c, x_3+c) = F(x_1, x_2, x_3) + c \geq F(x_1, x_2, x_3)$, i.e., F is weakly increasing and thus a proper DI OWA operator. Note that this function F is often considered in algebra as a prototypical example of a proper Post-associative ternary function, i.e., for any $x_1, \ldots, x_5 \in [0,1]$ it satisfies $F(F(x_1, x_2, x_3), x_4, x_5) = F(x_1, F(x_2, x_3, x_4), x_5) = F(x_1, x_2, F(x_3, x_4, x_5))$.

In the following we start analyzing some properties of the class of DI fusion functions of a given class $\mathcal{G}$. First of all, we consider the relationship between $\mathcal{G}$ and its corresponding class DI-$\mathcal{G}$. The following result is straightforward.

Proposition 3.1 *For every set $\mathcal{G}$ of fusion functions as in Definition 3.1, it holds that $\mathcal{G} \subseteq$ DI-$\mathcal{G}$.*

Proposition 3.2 *Let $\mathcal{G}$ and DI-$\mathcal{G}$ be as in Definition 3.1. Assume that all vectors in $R_{\mathcal{G}}$ are linearly dependent. Then, $\mathcal{G} =$ DI-$\mathcal{G}$.*

Proof. From the results in [10], it follows that every $\vec{r}$-increasing function is also $\alpha\vec{r}$-increasing for every $\alpha > 0$. In particular, this means that $\mathcal{G}_{\vec{r}} = \mathcal{G}_{\alpha\vec{r}}$ for every $\alpha > 0$, and the result follows. □

Remark 3 *In particular, Prop. 3.2 means that $\mathcal{G} =$DI-$\mathcal{G}$ either if $R_{\mathcal{G}} = \emptyset$ or $R_{\mathcal{G}}$ is a one-dimensional subset of vectors from $\mathbb{R}^n$.*

Example 7 *Take $\mathcal{G}$ as the class of weakly increasing DI aggregation functions. Then we can take $\vec{u} = (1, \ldots, 1)$ and we are in the setting of Prop. 3.2, so $\mathcal{G} =$ DI-$\mathcal{G}$.*

Remark 4 *Observe that the converse of Prop. 3.2 does not hold. Just consider the case $\mathcal{G} = \mathcal{C}$.*

The following result is straightforward.

Proposition 3.3 *Let $\mathcal{G}$ be a set as in Definition 3.1. Then:*

(1) $\mathcal{G} = \bigcap_{\vec{r} \in R_{\mathcal{G}}} \mathcal{G}_{\vec{r}}$.
(2) DI-$\mathcal{G} = \bigcup_{\vec{r} \in R_{\mathcal{G}}} \mathcal{G}_{\vec{r}}$.

Taking into account this approach, it is possible to define a topology on the set DI-$\mathcal{G}$ as follows.

Proposition 3.4 *The family $\{\cup_{\vec{r} \in \mathcal{A}} \mathcal{G}_{\vec{r}} \mid \mathcal{A} \subseteq R_{\mathcal{G}}\}$ (where $\cup_{\vec{r} \in \emptyset} \mathcal{G}_{\vec{r}} = \emptyset$ by* *definition) is a topology over the set DI-$\mathcal{G}$.*

Proof.

The fact that both $\emptyset$ and DI-$\mathcal{G}$ belong to the family is straightforward. If $A_1 = \cup_{\vec{r}\in\mathcal{A}_1}\mathcal{G}_{\vec{r}}$ and $A_2 = \cup_{\vec{r}\in\mathcal{A}_2}\mathcal{G}_{\vec{r}}$ are two sets in the family, then:

$$A_1 \cap A_2 = \cup_{\vec{r}\in\mathcal{A}_1\cup\mathcal{A}_2}\mathcal{G}_{\vec{r}} \,.$$

Finally, the fact that the union of sets in the family is again a set in the family is straightforward from our construction. □

Regarding the preservation of classes of DI fusion functions by proper operators, we first have the following result.

Proposition 3.5 *Let $\mathcal{G}$ be a set as in Definition 3.1 and let $\vec{r} \in R_{\mathcal{G}}$. Let $F : [0,1]^n \to [0,1]$ be an increasing function. Then, for every $G_1, \dots, G_n \in \mathcal{G}_{\vec{r}}$, if the function*

$$F \circ (G_1, \dots, G_n)(x_1, \dots, x_n) = F(G_1(x_1, \dots, x_n), \dots, G_n(x_1, \dots, x_n))$$

satisfies the same boundary conditions required to functions in $\mathcal{G}$, it is also a fusion function in $\mathcal{G}_{\vec{r}}$.

Proof.

Since $G_1, \dots, G_n \in \mathcal{G}_{\vec{r}}$, it follows that, for every $(x_1, \dots, x_n) \in [0,1]^n$ and $c > 0$ such that $x_1 + cr_1, \dots, x_n + cr_n \in [0,1]$, the inequality

$$G_i(x_1 + cr_1, \dots, x_n + cr_n) \geq G_i(x_1, \dots, x_n)$$

holds for any $i \in \{1, \dots, n\}$. Then, from the monotonicity of F, we have that

$$F \circ (G_1, \dots, G_n)(x_1 + cr_1, \dots, x_n + cr_n) \geq F \circ (G_1, \dots, G_n)(x_1, \dots, x_n)$$

so the result follows. □

Note that if the increasing fusion function F in Proposition 3.5 is idempotent (thus it is also an aggregation function), it automatically satisfies all other constraints considered in Proposition 3.5.

Corollary 3.6 *Let $\mathcal{G}$ be the set of DI aggregation functions and let $\vec{r} \in R_{\mathcal{G}}$. Let $F : [0,1]^n \to [0,1]$ be an aggregation function which preserves the boundary conditions of DI aggregation functions. Then, for every $G_1, \dots, G_n \in \mathcal{G}_{\vec{r}}$, the function $F \circ (G_1, \dots, G_n)$ is also a fusion function in $\mathcal{G}_{\vec{r}}$.*

Proof.

It follows from the fact that every aggregation function is in particular weakly increasing. □

As a consequence of these results, we can state the following result.

Proposition 3.7 *Let $\mathcal{G}$ be a set as in Definition 3.1, where the boundary conditions are given by the value of some function on some subset of* $[0,1]^n$*. Then, for every $\vec{r} \in R_{\mathcal{G}}$, the class $\mathcal{G}_{\vec{r}}$ is a lattice with the operations joint and meet defined in terms of the pointwise maximum and minimum of functions, respectively.*

Proof.

It follows from the fact that the maximum and the minimum are aggregation functions which are idempotent and will hence preserve the value of the function which defines the boundary condition. □

3.1 Classes of DI fusion functions and duality

Recall that we say that two classes $\mathcal{G}_1$ and $\mathcal{G}_2$ are dual to each other if $G_1 \in \mathcal{G}_1$ if and only if there exists a function $G_2 \in \mathcal{G}_2$ such that:

$$G_1(x_1, \ldots, x_n) = 1 - G_2(1 - x_1, \ldots, 1 - x_n)$$

for every $x_1, \ldots, x_n \in [0,1]$.

Then, for dual classes we have the following result.

Proposition 3.8 *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be dual to each other. Then, DI-$\mathcal{G}_1$ and DI-$\mathcal{G}_2$ are also dual to each other.*

Proof. First of all note that, since $\mathcal{G}_1$ and $\mathcal{G}_2$ are dual to each other, the boundary conditions for defining $\mathcal{G}_1$ and $\mathcal{G}_2$ are appropriately transformed by duality. So we only need to worry about monotonocity. But, from Corollary 2 in [10], we know that a fusion function F is $\vec{r}$-increasing if and only if its dual is also $\vec{r}$-increasing. As a consequence, $R_{\mathcal{G}_1} = R_{\mathcal{G}_2}$ and the result follows. □

Regarding this analysis of duality, recall that in [21] the authors proposed to consider the so-called extended Boolean (EB) functions, i.e., fusion functions $F : [0,1]^2 \to [0,1]$ such that $F(0,0)$, $F(0,1)$, $F(1,0)$, $F(1,1) \in \{0,1\}$. The authors denoted by Φ the class of all involutive monotone functions in $[0,1]$ (i.e., monotone functions $\varphi : [0,1] \to [0,1]$ such that $\varphi(\varphi(x)) = x$ for every $x \in [0,1]$) and by $\Psi = \Phi^3 \backslash \{id, id, id\}$, where $id : [0,1] \to [0,1]$ denotes the identity function $id(x) = x$. Then, the authors introduced the following definition of duality.

Definition 3.2 *[21] Let $D = (\varphi_1, \varphi_2, \varphi_3) \in \Psi$. Then, the mapping*

$$\mathcal{D}^D : \mathcal{F}_2 \to \mathcal{F}_2$$

defined, for each $F \in \mathcal{F}_2$ by $\mathcal{D}^D(F) = F^D$, with

$$F^D(x, y) = \varphi_1(F(\varphi_2(x), \varphi_3(y))$$

is called a D-duality.

One advantage of this definition is that it allows to relate well-known classes of fusion functions. For instance, the class of implication functions $\mathcal{I}$ is D-dual of the class of aggregation functions $M : [0,1]^2 \to [0,1]$ such that $M(1,0) = M(0,1) = 1$ for any $D = (id, \varphi_2, id) \in \Psi$ with $\varphi_2 \neq id$ (see [25] for a detailed analysis of this relation).

A particular class of D-dualities is provided by basic D-dualities, which are those obtained with $D \in \{id, N_s\}^3 \backslash \{id, id, id\}$ (with $N_s(x) = 1 - x$). It is possible to define new classes of DI fusion functions by using this D-duality. In particular, from the results in [21], we can state the following.

Proposition 3.9 *Let DI-$\mathcal{G}$ be a class obtained from a given class of two-dimensional EB functions $\mathcal{G}$. Assume that for every $F \in$ DI-$\mathcal{G}$ and for every increasing bijection $\theta : [0,1] \to [0,1]$, it follows that $\theta^{-1}(F(\theta(x), \theta(y))) \in$ DI-$\mathcal{G}$. Then, if DI-$\mathcal{G}$ is autodual with respect to the negation $N(x) = 1 - x$, then it is also autodual with respect to any other strong negation.*

Remark 5 *Observe that, in general, it is not true that if the class of fusion functions is D-dual to some other class, then the corresponding classes of DI fusion functions are also D-dual.*

3.2 The problem of convexity

If the class $\mathcal{G}$ is convex, this does not mean that DI-$\mathcal{G}$ is also convex. This is due to the fact that a linear combination of functions which are $\vec{r}$-increasing for different vectors $\vec{r}$ need not be directionally increasing. For instance, if we consider the functions $A, B : [0,1]^2 \to [0,1]$ given by

$$A(x, y) = \min(1, x + \max(0, x - 4y))$$

and

$$B(x, y) = \min(1, y + \max(0, y - 4x))$$

so $A(x, y) = B(y, x)$. Let $C : [0,1]^2 \to [0,1]$ be defined as

$$C(x, y) = \frac{A(x, y) + B(x, y)}{2}$$

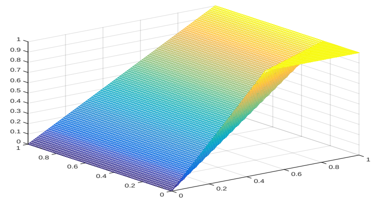

Fig. 1. Graphical depiction of function $A(x, y) = \min(1, x + \max(0, x - 4y))$.

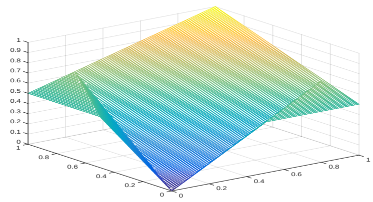

Fig. 2. Graphical depiction of function $C(x, y) = \frac{A(x,y)+B(x,y)}{2}$.

then C is not $\vec{r}$-increasing whatever $\vec{r}$ is, even if A is $(1, 0)$-increasing and B is $(0, 1)$-increasing (see Fig. 1 and 2). Then C is not a DI aggregation function even if both A and B are coordinate-wise monotone DI aggregation functions.

However, the following result holds.

Proposition 3.10 *Let $\mathcal{G}$ be a convex class. Then, for every $\vec{r} \in R_{\mathcal{G}}$, the set $\mathcal{G}_{\vec{r}}$ is also convex.*

Proof. Since the class $\mathcal{G}$ is convex, it follows that the boundary conditions which define the functions (which are independent of the monotonicity conditions, by assumption) in this class are preserved by convex combinations of functions (observe that a convex combination can be seen as a weighted arithmetic mean playing the role of function F in Proposition 3.5). So consider $G_1, G_2 \in \mathcal{G}_{\vec{r}}$ and take $\lambda \in [0, 1]$. The function:

$$\lambda G_1 + (1 - \lambda)G_2$$

verifies the same boundary conditions as G_1 and G_2, as we have already said. Since a convex combination of fusion functions which are $\vec{r}$-increasing for the same vector $\vec{r}$ is also $\vec{r}$-increasing. The result follows. □

However, the converse of this result is also true.

Proposition 3.11 *Let the class $\mathcal{G}$ be such that, for every $\vec{r} \in R_{\mathcal{G}}$, the set $\mathcal{G}_{\vec{r}}$ is convex. Then the class $\mathcal{G}$ is also convex.*

Proof.

Just notice that the intersection of convex sets is again a convex set. □

4 A specific example: Construction of DI conjunctors

In this section we analyze some specific examples related to our previous theory. Specifically, we are interested in building DI conjunctors starting from appropriate linear combinations of t-norms. We start with a result that will be of interest for our study.

Proposition 4.1 *Each symmetric DI aggregation function F is weakly monotone.*

Proof.

Let $\vec{r} \in [0,1]^n$ be a vector such that F is $\vec{r}$-increasing and let $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ be a permutation. From the symmetry of F, it follows that F is also $\vec{r_\sigma} = (r_{\sigma(1)},\ldots,r_{\sigma(n)})$ increasing. From [10], we know that F is also increasing along the direction

$$\frac{1}{n!}\sum_{\sigma} \vec{r_\sigma}$$

where the summation extends for every permutation of $\{1,\ldots,n\}$. But

$$\frac{1}{n!}\sum_{\sigma} \vec{r_\sigma} = (t,\ldots,t)$$

where $t = \frac{1}{n}\sum_{i=1}^{n} r_i$, so the result follows. □

As a consequence of this result, it does not exist any symmetric function F satisfying the boundary conditions from a class of DI aggregation functions unless F is also weakly increasing.

Now let us consider the following specific example.

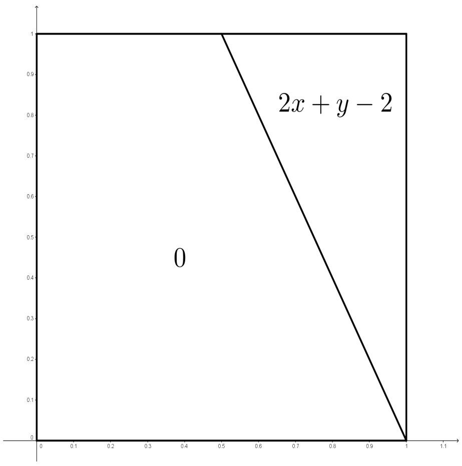

Fig. 3. Representation of C

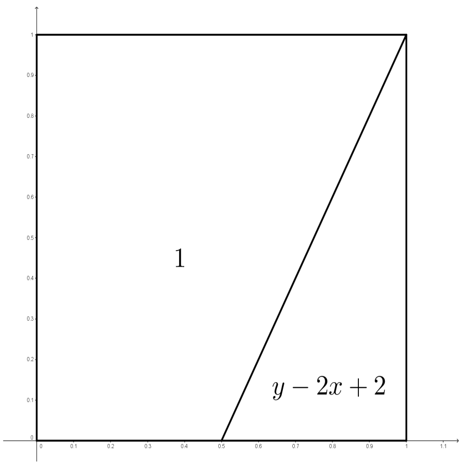

Fig. 4. Representation of I

Let $C : [0,1]^2 \to [0,1]$ be the function:

$$C(x,y) = \max(2x + y - 2, 0)$$

which is a conjunctor (see Fig. 3).

Then the function

$$I(x,y) = 1 - C(x, 1-y)$$

is an implication function (see Fig. 4).

Now let us consider the function $C_1(x,y) = 2xy - C(x,y) = 2xy - \max(2x + y - 2, 0)$ (see Fig. 5)

This function is a DI conjunctor, since it is directionally increasing with respect to $\vec{r} = (r_1, r_2) \neq (0,0)$ and $2r_1 \leq r_2$. We can build a DI implication related to this DI conjunctor in the following way:

$$I_1(x,y) = 1 - C_1(x, 1-y)$$

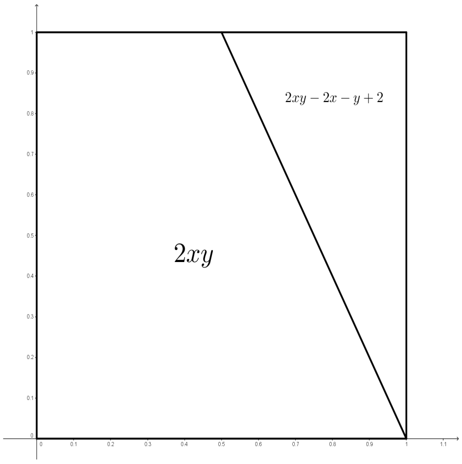

Fig. 5. Representation of C_1.

Observe that I_1 extends the standard Boolean implication, so it fulfills the same boundary conditions as implication functions (Eq. 3). However, this function does not satisfy the monotonicity properties of implication functions, so it is a proper DI implication function.

Now, let us consider the three examples of t-norms $T_M(x, y)$, $T_P(x, y)$ and $T_L(x, y)$, already introduced in Example 1. We are going to study the following families of linear combinations. Given $k \geq 0$, we define:

(1)
$$G^k_{M,P} = (k+1)T_M - kT_P$$

(2)
$$G^k_{M,L} = (k+1)T_M - kT_L$$

(3)
$$G^k_{P,L} = (k+1)T_P - kT_L$$

First of all, note that $G^k_{M,P}, G^k_{M,L}$ and $G^k_{P,L}$ are t-norms only if $k = 0$.

Furthermore, we have the following result.

Proposition 4.2 *(1) The function $G^k_{M,P}$ is a fusion function if and only if $k \in [0, 1]$.*
(2) The function $G^k_{M,L}$ is a fusion function if and only if $k \in [0, 1]$.
(3) The function $G^k_{P,L}$ is a fusion function if and only if $k \in [0, 3]$.

Proof.

(1) We start considering the function

$$G^k_{M,P}(x, y) = (k+1)T_M - kT_P \; .$$

Since the minimum is the greatest of all the t-norms, the inequality

$$G^k_{M,P}(x,y) \geq 0$$

holds for every $k \geq 0$ and for every $x, y \in [0,1]$. Besides, take $k > 1$. In this setting, taking $x = y = \frac{k+1}{2k}$ it follows that

$$G^k_{M,P}(x,y) = (k+1)x - kx^2 = \frac{(k+1)^2}{2k} - \frac{(k+1)^2}{4k} = \frac{(k+1)^2}{4k} > 1$$

In fact, an analogous inequality holds taking $x \in]\frac{1}{k}, 1[$.

Now let us take $k = 1$. Without loss of generality we can assume $x \leq y$. Then:

$$G^1_{M,P}(x,y) = 2\min(x,y) - xy = x(2-y) \leq x(2-x)\ .$$

But the function $f(x) = x(2-x)$ attains its maximum at $x = 1$ and $f(1) = 1$, so it follows that

$$G^1_{M,P}(x,y) \leq 1$$

for every $x, y \in [0,1]$. Since the function $G^k_{M,P}$ is increasing as a function of k, it follows that $G^k_{M,P}$ is a fusion function if and only if $k \in [0,1]$, as we intended to prove.

(2) Let us consider now the function $G^k_{M,L}$ (see Fig. 6 and 7). Since the minimum is the greatest of all the t-norms, the inequality

$$G^k_{M,L}(x,y) \geq 0$$

holds for every $k \geq 0$ and for every $x, y \in [0,1]$. If $k > 1$, take $x \in [\frac{1}{k+1}, 0.5]$. Then

$$G^k_{M,L}(x,x) = (k+1)x > 1,$$

so $G^k_{M,L}$ is not a fusion function in this case. Now, it is straightforward that for $k = 1$ it holds that

$$G^1_{M,L}(x,y) \leq 1.$$

So, as the function $G^k_{M,L}$ is increasing in the parameter k, it follows that it is a fusion function if and only if $k \in [0,1]$, as we intended to prove.

(3) Finally, let us consider

$$G^k_{P,L}(x,y) = (k+1)T_P(x,y) - kT_L(x,y).$$

Observe that, if $k > 3$, it follows that

$$G^k_{P,L}(\frac{1}{2}, \frac{1}{2}) = \frac{k+1}{4} \geq 1$$

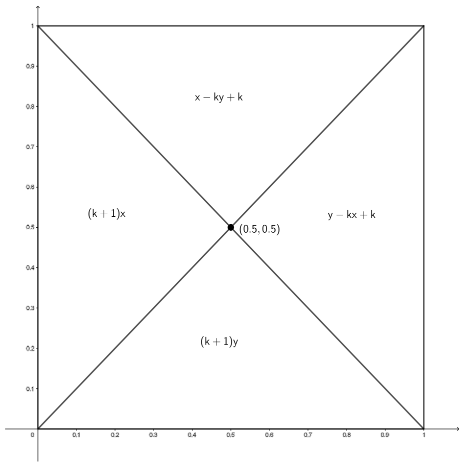

Fig. 6. Representation of $G^k_{M,L}$

so $G^k_{P,L}$ is not a fusion function. Now, if we take $k = 3$, we can distinguish two possibilities:

- $x + y \leq 1$. In this case, we have that

$$G^3_{P,L}(x, y) = 4xy \leq 4x(1 - x) \leq 1 \ .$$

- $x + y > 1$, i.e., $x + y = 1 + a$ for some $a \in]0, 1]$. Note that then

$$xy \leq \left(\frac{1+a}{2}\right)^2 \ .$$

Let us consider the function $F(x, y) = 4xy - 3(x + y - 1)$ and thus $F(x, y) \leq (1+a)^2 - 3a = 1 - a + a^2 \leq 1$.Then, this function is decreasing in x (in y) for $x \leq \frac{3}{4}$ ($y \leq \frac{3}{4}$) and increasing otherwise in $[0, 1]$. But, for $x, y \in [0, 1]$ such that $x + y - 1 = 0$, it follows that, as we have seen before

$$F(x, y) \leq 1.$$

Furthermore, $F(1, 1) = 1$. So it follows that $G^3_{P,L}(x, y) \leq 1$.

Taking into account that $G^k_{P,L}(x, y)$ is increasing in the parameter k, and since $T_L(x, y) \leq T_P(x, y)$ for every $x, y \in [0, 1]$, it follows that it is a fusion function if and only if $k \in [0, 3]$ and the result follows □

In order to analyze if these functions are proper DI conjunctors (i.e., DI conjunctors which are not conjunctors), recall that a symmetric DI aggregation function is always weakly increasing (so it is also a symmetric DI copula). Since conjunctors are particular instances of aggregation functions, we can state the following result.

Proposition 4.3 *(1) The function $G^k_{M,P}$ is a proper DI conjunctor if and only if $k \in]0, 1]$.*

(2) The function $G^k_{M,L}$ is a proper DI conjunctor if and only if $k \in]0, 1]$.

(3) The function $G^k_{P,L}$ is a proper DI conjunctor if and only if $k \in]0, 1]$.

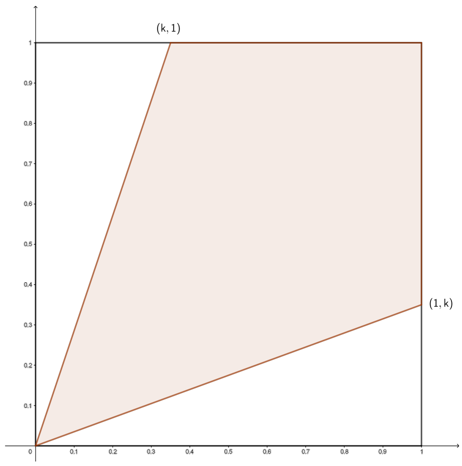

Fig. 7. Cone $\mathcal{R}_G$ for $G^k_{M,L}$

Proof.

First of all, it is clear that these functions satisfy all the boundary conditions demanded to conjunctors. Regarding the directional monotonicity of these functions, observe that, for $k \in [0,1]$:

$$G^k_{M,P}(x,y) = \begin{cases} x[k(1-y)+1] & \text{if } x \leq y \\ y[k(1-x)+1] & \text{otherwise.} \end{cases}$$

$$G^k_{M,L}(x,y) = \begin{cases} x - k(y-1) & \text{if } 1-y < x \leq y \\ (k+1)x & \text{if } x \leq \min(y,,1-y) \\ y - k(x-1) & \text{if } x > \max(y, 1-y) \\ (k+1)y & \text{otherwise.} \end{cases}$$

And, for $k \in [0,3]$,

$$G^k_{P,L}(x,y) = \begin{cases} (k+1)xy - k(x+y-1) & \text{if } x+y \geq 1 \\ (k+1)xy & \text{otherwise.} \end{cases}$$

In this way, it is easy to see that none of the three functions is monotone since none of them is $(1,0)$-increasing in its whole domain. So they are not conjunctors. In order to prove that they are DI conjunctors, as we have said, it is enough to check whether they are weakly increasing.

(1) Let us see that $G^k_{M,P}$ is $(1,1)$-increasing.

$$G^k_{M,P}(x+c, y+c) = (k+1)\min(x+c, y+c) - k(x+c)(y+c)$$
$$= (k+1)\min(x,y) + (k+1)c - kxy - kc(x+y+c) \geq G^k_{M,P}(x,y)$$

as $(k+1) - k(x+y+c) \geq 0$ due to the fact that $x, y \in [0, 1-c]$ (from the hypothesis on c).

(2)

$$\begin{aligned} & G^k_{M,L}(x+c, y+c) - G^k_{M,L}(x,y) \\ = & (k+1)\min(x+c, y+c) - k(\max(x+y+2c-1, 0) - k\min(x,y) \\ + & k(\max(x+y+2c-1, 0) \\ \geq & (k+1)\min(x,y) + (k+1)c - k(\max(x+y+2c-1,0) > 0 \end{aligned}$$

(3) It follows from a calculation analogous to the previous one.

□

Remark 6 *Note that, in the three cases, the resulting functions satisfy all the properties demanded to overlap functions, except monotonicity. So they provide examples of a proper DI overlap functions.*

Remark 7 *Observe that in general, if we consider arbitrary t-norms T_1 and T_2, $T_1 \geq T_2$, it is not true that the function $G^k_{T_1,T_2}$ is a DI conjunctor for some $k \neq 0$. To see it, just take T_1 as*

$$T_1(x,y) = \begin{cases} 0.5 + 0.5T_L(2(x-0.5), 2(y-0.5)) & \textit{if } x, y \in [0.5, 1] \\ \min(x,y) & \textit{otherwise,} \end{cases}$$

and $T_2(x,y) = T_L(x,y)$. Then $(k+1)T_1(x,y) - kT_2(x,y)$ is weakly increasing if and only if $k = 0$.

5 Conclusions

Given a class of fusion functions defined in terms of some monotonicity conditions and possibly some other conditions independent of the monotonicity, we have introduced the notion of class of DI fusion functions of the given class by imposing monotonicity along some (but not all) of the original monotonicity directions. We have studied some analytical and geometrical properties of these classes and we have considered some specific examples.

As a general claim, we can mention that in several applications, some monotonicity constraints for applied fusion functions are superfluously strong and our proposal enables to relax them, still preserving the other properties important for the considered application. This is the case, in classification problems [20], where the relaxation of monotonicity condition leads to a very relevant improvement in the performance of the algorithms.

In future works we intend to analyze some specific cases of these classes, focusing on the possible applications in the fields such as classification, image processing or decision making. Note also that recently the cone-monotonicity was introduced by Beliakov et al. [4]. For a fixed cone C of possible directions, an interesting topic for the further study and applications can be the research in C-aggregation functions (i.e., fusion functions satisfying boundary conditions of aggregation functions which are increasing in any direction from the cone C), C-conjunctors, C-implications, etc.

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