Topologies induced by the representation of a betweenness relation as a family of order relations

Hua-Peng Zhang [a,*],
Raúl Pérez-Fernández [b,c] and Bernard De Baets [b]

[a] *School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, China, e-mail: huapengzhang@163.com*

[b] *KERMIT, Department of Data Analysis and Mathematical Modelling, Ghent University, Belgium, e-mail: {raul.perezfernandez,bernard.debaets}@ugent.be*

[c] *Department of Statistics and O.R. and Mathematics Didactics, University of Oviedo, Spain, e-mail: perezfernandez@uniovi.es*

Abstract

Betweenness relations are the mathematical formalization of the geometrical notion of an element being in between other two elements. In this paper, we exploit a well-known result representing a betweenness relation as a family of order relations and analyse the corresponding family of induced (Alexandrov) topologies. In particular, the intersection of this family of topologies is proved to be the anti-discrete topology and a necessary and sufficient condition for the supremum of this family of topologies to be the discrete topology is provided. Interestingly, this condition is proved to hold when dealing with a finite set. We end with a discussion on the relation between the topology induced by an order relation or a metric and the family of topologies induced by the betweenness relation induced by the same order relation or metric.

Key words: Betweenness relation; Topology; Preorder relation; Metric.

1 Introduction

The notion of a betweenness relation, which was first formalized by Pasch [21] and further studied by Huntington [17] and Huntington and Kline [18], de-

* Corresponding author.

scribes when an element is in between two other ones. A betweenness relation can be derived from many mathematical structures. For instance, Blumenthal and Ellis [5], Chvátal [7], and Smiley and Transue [29] characterized betweenness relations induced by (distance) metrics. Düvelmeyer and Wenzel [10], Fishburn [12], Rautenbach and Schäfer [26], and Sholander [27] investigated betweenness relations induced by order relations and related binary relations such as weak order relations and semiorder relations. The particular case of betweenness relations induced by lattices has been studied by Cibulskis [8], Hedlíková and Katriňák [16], Ploščica [25], and Transue [30]. Special attention should be given to the discussion of Padmanabhan [20] on different kinds of betweenness relations induced by the same lattice. Smiley [28] investigated how betweenness relations induced by a real vector space, a metric or a lattice on the same given set relate. In 2013, Bankston [1] introduced the notion of a road system and analysed betweenness relations induced by a road system. It is worth noting that all the works above consider different axiomatizations, resulting in different associated semantics for the notion of a betweenness relation. As will be further explained in the present work, the definition adopted in [22] is here considered since it allows for an intuitive representation as a family of order relations, thus linking the notion of "y being in between x and z" and the notion of "z being farther from x than y".

Quite recently, Pérez-Fernández et al. [23] exploited the geometrical properties of a betweenness relation for defining the concept of a monometric, which is closely related to that of a metric but substitutes the symmetry and the triangle inequality by the compatability with a betweenness relation. This type of function was shown to play an important role in the field of penalty-based data aggregation since it is linked to the construction of many families of penalty-based functions such as medians, centroids, centers and medoids [13]. This field has historically focused on the aggregation of real numbers or real vectors or on the more general setting of the aggregation of elements of a (bounded) partially ordered set [9]. However, the aggregation of many new types of data that do not fit within this historical framework (e.g., rankings [23] and strings [13]) is lately receiving increasing attention, probably due to the incessant amount of data becoming available in this information era. For this very reason, some of the present authors developed a more general framework for penalty-based data aggregation embracing sets equipped with a betweenness relation as interesting sets on which aggregation processes are performed [22].

In the historical setting of real numbers/vectors, continuity has been largely acclaimed as a desirable (and even necessary) property for an aggregation process ensuring that small changes in the objects to be aggregated do not lead to big changes in the result of the aggregation [15]. Unfortunately, unlike in this historical setting, there is still a gap in the study of the notion of continuity of mappings on a set equipped with a betweenness relation. The study of the continuity of penalty-based (aggregation) functions stimulates us to investigate some hitherto unexplored mathematical aspects of a betweenness relation. In particular, since a set equipped with a betweenness relation might not have a natural underlying topology, it is necessary to provide such a structure with a topology generated by the betweenness relation. Admittedly, topologies and betweenness relations have jointly attracted the attention of the scientific community, yet in different directions than in the present paper. Just to name a few relevant works, Bankston studied the notions of gap-freeness [2] and antisymmetry [3] of betweenness relations induced by the topology of Hausdorff continua, Bankston et al. [4] investigated the continuity behavior of betweenness functions, and Bruno et al. [6] analysed betweenness relations in the context of (topological) category theory.

In the present work, we follow a different direction and start by representing a betweenness relation in the sense of [22] as a family of order relations. We analyse the corresponding family of induced (Alexandrov) topologies. More specifically, we pay special attention to the intersection and the supremum of these topologies, which turn out to be the anti-discrete topology for the former and, in case some necessary and sufficient condition is fulfilled, the discrete topology for the latter. The remainder of this paper is organized as follows. We recall some basic notions and results related to topologies, metrics, preorder relations and betweenness relations in Section 2. A representation of a betweenness relation as a family of order relations is presented in Section 3. Section 4 is devoted to the presentation of the main results of the paper, the section being divided into three subsections: Some basic properties of the family of topologies induced by a betweenness relation are studied in Subsection 4.1; the intersection and supremum of these topologies are investigated in Subsection 4.2; the relationship between the family of topologies induced by the betweenness relation induced by an order relation or a metric and the topology directly induced by the same order relation or metric is discussed in Subsection 4.3. A discussion on the use of the (dual) Alexandrov topology given by the lower sets rather than that given by the upper sets is addressed in Section 5. We end with some conclusions and open problems in Section 6.

2 Preliminaries

Throughout this paper, X is always a nonempty set with $|X|$ denoting its cardinality. $\mathcal{P}(X)$ denotes the power set of X, i.e., the set of all subsets of X. In the following, we recall some basic notions and results related to topologies, metrics, preorder relations and betweenness relations.

A subset $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* [11] on X if it is such that $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$ and it is closed under arbitrary unions (if $\{U_i\}_{i\in I} \subseteq \mathcal{T}$, then $\bigcup\limits_{i\in I} U_i \in \mathcal{T}$) and finite intersections ($U \in \mathcal{T}$ and $V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$). Obviously, $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are both topologies on X and are called the *discrete topology* and the *anti-discrete topology* on X, respectively. A topology $\mathcal{T}$ on X is the discrete topology if and only if every singleton $\{x\}$ belongs to $\mathcal{T}$.

A subset $\mathcal{B} \subseteq \mathcal{T}$ is called a *base* for a topology $\mathcal{T}$ on X if every non-empty member of $\mathcal{T}$ can be represented as the union of a subfamily of $\mathcal{B}$. A subset $\mathcal{P} \subseteq \mathcal{T}$ is called a *subbase* for a topology $\mathcal{T}$ on X if the family of all finite intersections of members in $\mathcal{P}$ is a base for $\mathcal{T}$.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then we say that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ or, equivalently, that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$. The discrete topology and the anti-discrete topology are the finest and the coarsest topology on X, respectively.

The intersection $\bigcap\limits_{i\in I} \mathcal{T}_i$ of a family of topologies $\{\mathcal{T}_i\}_{i\in I}$ on X is again a topology, which is the finest topology that is coarser than all the topologies $\{\mathcal{T}_i\}_{i\in I}$. Although the union $\bigcup\limits_{i\in I} \mathcal{T}_i$ of a family of topologies $\{\mathcal{T}_i\}_{i\in I}$ on X might not be a topology, the supremum of these topologies (defined as the coarsest topology on X that is finer than all the topologies $\{\mathcal{T}_i\}_{i\in I}$ and denoted by $\bigvee\limits_{i\in I} \mathcal{T}_i$) is assured to exist. It is clear that $\bigcup\limits_{i\in I} \mathcal{T}_i \subseteq \bigvee\limits_{i\in I} \mathcal{T}_i$, the equality holding if and only if $\bigcup\limits_{i\in I} \mathcal{T}_i$ is a topology on X. In addition, it holds that $\bigcup\limits_{i\in I} \mathcal{T}_i$ is a subbase for $\bigvee\limits_{i\in I} \mathcal{T}_i$.

A subset $U \subseteq X$ is said to be a *neighbourhood* of a point $x \in X$ with respect to a topology $\mathcal{T}$ on X if $x \in U$ and $U \in \mathcal{T}$. A topology $\mathcal{T}$ on X is said to be of type T_0 if, for every pair of distinct points $x, y \in X$, there exists $U \in \mathcal{T}$ containing exactly one of the two points.

A topology $\mathcal{T}$ on X is said to be *Alexandrov* [19] if it is closed under arbitrary intersections, i.e., if $\{U_i\}_{i\in I} \subseteq \mathcal{T}$, then $\bigcap\limits_{i\in I} U_i \in \mathcal{T}$. A topology $\mathcal{T}$ on X is Alexandrov if and only if every $x \in X$ has a smallest neighbourhood. Moreover, the smallest neighbourhood of a point with respect to an Alexandrov topology is precisely the intersection of all the neighbourhoods of that point.

A mapping $d : X \times X \to [0, +\infty[$ is called a *metric* [11] on X if it satisfies the properties of identity of indiscernibles ($d(x, y) = 0$ if and only if $x = y$), symmetry ($d(x, y) = d(y, x)$, for any $x, y \in X$) and the triangle inequality ($d(x, z) \leq d(x, y) + d(y, z)$, for any $x, y, z \in X$). A metric d on a set X induces a topology on X, denoted by $\mathcal{T}_d$, in which all the open balls $\{B(x, r)\}_{x \in X, r > 0}$ constitute a base, where $B(x, r) = \{y \in X \mid d(x, y) < r\}$.

2.2 On preorder relations and induced topologies

A *preorder relation* R on a set X is a binary relation on X satisfying reflexivity and transitivity. An antisymmetric preorder relation is called an *order relation*. An order relation R on X is called a *total order relation* if, for any $x, y \in X$, either xRy or yRx holds. An element $a \in X$ is said to be the *smallest element* with respect to an order relation R on X if, for any $x \in X$, it holds that aRx. It is clear that an order relation does not necessarily have a smallest element, and, if it does, then it is unique.

A subset $U \subseteq X$ is called an *upper set* with respect to a preorder relation R on X if, for any $x, y \in X$, $x \in U$ and xRy imply $y \in U$. For any $x \in X$, the subset $(\uparrow x)_R = \{y \in X \mid xRy\}$ is called the *principal filter* at x with respect to R. It is evident that $(\uparrow x)_R$ is the smallest upper set containing x.

A preorder relation R on a set X induces a topology on X, denoted by $\mathcal{T}_R$, where $\mathcal{T}_R$ is precisely the set of all upper sets with respect to R. The topology $\mathcal{T}_R$ is Alexandrov and, for any $x \in X$, $(\uparrow x)_R$ is the smallest neighbourhood of x with respect to $\mathcal{T}_R$. In addition, $\mathcal{T}_R$ is of type T_0 if and only if R is an order relation. Since $U = \bigcup\limits_{x \in U} (\uparrow x)_R$ for any $U \in \mathcal{T}_R$, it holds that $\{(\uparrow x)_R\}_{x \in X}$ is a base for $\mathcal{T}_R$. Given an Alexandrov topology $\mathcal{T}$ on a set X, it holds that $R_{\mathcal{T}}$ is a preorder relation on X, where $R_{\mathcal{T}}$ is defined as follows: $xR_{\mathcal{T}}y$ if, for any $U \in \mathcal{T}$, $x \in U$ implies $y \in U$. This correspondence between the set of all preorder relations and that of all Alexandrov topologies on a given set is one-to-one. For this very reason, we often refer to the topology $\mathcal{T}_R$ as the Alexandrov topology induced by the preorder relation R.

Note that alternative construction methods for a topology given a (specific type of) preorder relation have been proposed in literature, e.g., the Scott topology and the Lawson topology for directed-complete (partial) order relations (for more details, see, for instance, [14]).

2.3 On betweenness relations

It is important to mention that there exist many different postulates for defining a betweenness relation. In particular, different types of transitivity properties have been proposed by Pitcher and Smiley [24]. Here, we adopt the definition by Pérez-Fernández and De Baets [22], which is a basic betweenness relation in the sense of [1] that additionally satisfies axiom (R7) defined in the same paper. The reason for doing so does not only lie in its applicability in data aggregation, but also in its nice representation as a family of order relations, which will enable us to induce a family of topologies, as will be discussed in Section 4.

Definition 1 *[22] A ternary relation B on a set X is called a betweenness relation if it satisfies the following three properties:*

(i) Symmetry in the end points: for any $x, y, z \in X$,

$$(x, y, z) \in B \iff (z, y, x) \in B\,.$$

(ii) Closure: for any $x, y, z \in X$,

$$((x, y, z) \in B \ \wedge (x, z, y) \in B) \iff y = z\,.$$

(iii) End-point transitivity: for any $o, x, y, z \in X$,

$$((o, x, y) \in B \ \wedge (o, y, z) \in B) \implies (o, x, z) \in B\,.$$

Remark 2 *[22] For any $x, y \in X$, if $(x, y, x) \in B$, then it follows that $x = y$. Equivalently, if $x, y \in X$ with $x \neq y$, then $(x, y, x) \notin B$.*

The simplest and smallest betweenness relation on a set X, denoted by B_0, is the ternary relation $\{(x, y, z) \in X^3 \mid x = y \vee y = z\}$, called the *minimal betweenness relation* on X. One can construct more elaborated betweenness relations from other mathematical structures, such as an ordered set, a metric space and a real vector space.

Example 1 *[1] Let R be an order relation on a set X. The ternary relation $B_R = B_0 \cup \{(x, y, z) \in X^3 \mid xRyRz \vee zRyRx\}$ is a betweenness relation on X.* $\triangleleft$

Example 2 *[1] Let d be a metric on a set X. The ternary relation $B_d = \{(x, y, z) \in X^3 \mid d(x, z) = d(x, y) + d(y, z)\}$ is a betweenness relation on X.* $\triangleleft$

Example 3 *[1] Let X be a subset of a real vector space. The ternary relation $B = \{(x, y, z) \in X^3 \mid (\exists \lambda \in [0, 1])(y = \lambda x + (1 - \lambda)z)\}$ is a betweenness relation on X.* $\triangleleft$

Example 4 *Let X be a set with $|X| > 2$ and $b \in X$ be fixed. The ternary relation $B_b = B_0 \cup \{(x, b, z) \in X^3 \mid x \neq b \wedge x \neq z \wedge b \neq z\}$ is a betweenness* *relation on X. Actually, B_b can be induced by the following metric:*

$$d(x,y) = \begin{cases} 0\,, & \textit{if} \ \ x = y\,, \\ 1\,, & \textit{if} \ \ x \neq y \ \textit{and}\ b \in \{x,y\}\,, \\ 2\,, & \textit{otherwise.} \end{cases}$$

$\triangleleft$

3 Representation of a betweenness relation as a family of order relations

In this section, we show that any betweenness relation can be represented as a family of order relations. The following result was mentioned in [1].

Theorem 3 *Consider a betweenness relation B on a set X. For any $x \in X$, we define a binary relation O_x on X as follows:*

$$O_x = \{(y,z) \in X^2 \mid (x,y,z) \in B\}\,.$$

Then $\{O_x\}_{x\in X}$ is a family of order relations on X such that, for any $x,y,z \in X$,

$$(y,z) \in O_x \iff (y,x) \in O_z\,.$$

Proof. It follows from the closure property of B that each O_x is reflexive and antisymmetric. In addition, the end-point transitivity of B implies the transitivity of each O_x. Hence, $\{O_x\}_{x\in X}$ is a family of order relations on X. The symmetry of B in the end points implies $(y,z) \in O_x \iff (y,x) \in O_z$, for any $x,y,z \in X$. $\square$

Remark 4 *(i) Since $(x,y) \in O_x$ for any $x,y \in X$, x is the smallest or greatest element in O_x and $\bigcup\limits_{x\in X} O_x = X^2$.*

(ii) Since every O_x is reflexive, it follows that $\{(x,x) \mid x \in X\} \subseteq \bigcap\limits_{x\in X} O_x$. For any $x,y \in X$ with $x \neq y$, we conclude that $(y,x) \notin O_x$ by Remark 2, which implies $(y,x) \notin \bigcap\limits_{x\in X} O_x$. Hence, it holds that $\bigcap\limits_{x\in X} O_x \subseteq \{(x,x) \mid x \in X\}$, and, thus, $\bigcap\limits_{x\in X} O_x = \{(x,x) \mid x \in X\}$.

Theorem 3 states that we can construct a family of order relations from a given betweenness relation. Conversely, given a family of order relations satisfying an additional condition concerning the symmetry in the end points, we can construct a betweenness relation.

Theorem 5 *Consider a family of order relations $\{O_x\}_{x\in X}$ on a set X such that, for any $x, y, z \in X$,*

$$(y, z) \in O_x \iff (y, x) \in O_z\,.$$

For any $x \in X$, we define a ternary relation B_x on X as follows:

$$B_x = \{(x, y, z) \mid (y, z) \in O_x\}\,.$$

Then $B = \bigcup_{x\in X} B_x$ is a betweenness relation on X.

Proof. It follows from the fact that $(y, z) \in O_x \iff (y, x) \in O_z$ for any $x, y, z \in X$ that B is symmetric in the end points. The reflexivity and anti-symmetry of each O_x imply the closure property of B, and the transitivity of each O_x implies the end-point transitivity of B. $\square$

From Theorems 3 and 5, we obtain the following representation theorem for a betweenness relation via a family of order relations.

Theorem 6 *A ternary relation B is a betweenness relation on a set X if and only if it is induced (as in Theorem 5) by a family of order relations $\{O_x\}_{x\in X}$ on X satisfying that $(y, z) \in O_x \iff (y, x) \in O_z$ for any $x, y, z \in X$.*

4 Main results

4.1 Topologies induced by a betweenness relation

In this subsection, we study some basic properties of the family of topologies induced by a betweenness relation. Note that we fix the semantics "smaller than or equal to" for the family of order relations $\{O_x\}_{x\in X}$. A discussion on the use of the dual semantics "greater than or equal to" is addressed in Section 5.

First, for any $x, y \in X$, we define the subset $(\uparrow y)_x$ of X with respect to a betweenness relation B on X as follows:

$$(\uparrow y)_x = \{z \in X \mid (x, y, z) \in B\}\,.$$

Obviously, $(\uparrow y)_x$ can be equivalently defined via $(\uparrow y)_x = \{z \in X \mid (y, z) \in O_x\}$ and $(\uparrow y)_x$ is precisely the principal filter at y with respect to the order relation O_x, where O_x is defined as in Theorem 3. Some properties of the family $\{(\uparrow y)_x\}_{x,y\in X}$ are listed below.

Proposition 7 *The family $\{(\uparrow y)_x\}_{x,y\in X}$ has the following properties:*

(i) $y \in (\uparrow y)_x$.

(ii) $(\uparrow x)_x = X$.

(iii) $x \notin (\uparrow y)_x$ *if and only if* $x \neq y$.

(iv) $(\uparrow y)_x \cap (\uparrow x)_y = \emptyset$ *if and only if* $x \neq y$.

(v) $\bigcap_{x \in X} (\uparrow y)_x = \{y\}$.

Proof.
(i) Since $(x, y, y) \in B$ for any $x, y \in X$, it holds that $y \in (\uparrow y)_x$.

(ii) Since $(x, x, y) \in B$ for any $x, y \in X$, it holds that $(\uparrow x)_x = X$.

(iii) By (ii), $x = y$ implies $x \in (\uparrow y)_x$. So if $x \notin (\uparrow y)_x$, then $x \neq y$. If $x \neq y$, then $(x, y, x) \notin B$, i.e., $x \notin (\uparrow y)_x$ by Remark 2.

(iv) It follows from (ii) and the closure property of a betweenness relation.

(v) By (iii), $x \notin (\uparrow y)_x$ for any $x \neq y$. Hence, if $x \neq y$, then $x \notin \bigcap_{x \in X} (\uparrow y)_x$, which implies $\bigcap_{x \in X} (\uparrow y)_x \subseteq \{y\}$. On the other hand, $\{y\} \subseteq \bigcap_{x \in X} (\uparrow y)_x$ by (i). We conclude that $\bigcap_{x \in X} (\uparrow y)_x = \{y\}$. $\square$

Consider a betweenness relation B on a set X and the family of order relations $\{O_x\}_{x \in X}$ induced by B as in Theorem 3. A family of topologies $\{\mathcal{T}_x\}_{x \in X}$ is said to be induced by the betweenness relation B, if for any $x \in X$, $\mathcal{T}_x$ is exactly the topology induced by the order relation O_x. For any $x \in X$, $\{(\uparrow y)_x\}_{y \in X}$ is a base for $\mathcal{T}_x$ and $(\uparrow y)_x$ is the smallest neighbourhood of y with respect to $\mathcal{T}_x$ for any $y \in X$.

The following theorem indicates that, for any $x \in X$, $\mathcal{T}_x$ is neither the discrete nor the anti-discrete topology when dealing with a set of cardinality greater than one.

Theorem 8 *Let X be a set with $|X| > 1$ and $\{\mathcal{T}_x\}_{x \in X}$ be the family of topologies induced by a betweenness relation B on X. Then $\{\emptyset, X\} \subset \mathcal{T}_x \subset \mathcal{P}(X)$ for any $x \in X$.*

Proof. Obviously, $\{\emptyset, X\} \subseteq \mathcal{T}_x \subseteq \mathcal{P}(X)$. It remains to prove that $\mathcal{P}(X) \neq \mathcal{T}_x \neq \{\emptyset, X\}$. For any $x \in X$, it holds that $\{x\} \in \mathcal{P}(X)$, but, as we will prove next, $\{x\} \notin \mathcal{T}_x$. Suppose $\{x\} \in \mathcal{T}_x$. On the one hand, $\{x\}$ is the smallest neighbourhood of x with respect to $\mathcal{T}_x$. On the other hand, $(\uparrow x)_x = X$ is also the smallest neighbourhood of x with respect to $\mathcal{T}_x$. Hence, it holds that $\{x\} = X$, which contradicts $|X| > 1$. This shows that $\mathcal{P}(X) \neq \mathcal{T}_x$. Next, consider any $y \in X \setminus \{x\}$. It follows from $y \in (\uparrow y)_x$ and $x \notin (\uparrow y)_x$ that $\emptyset \neq (\uparrow y)_x \neq X$. Note that $(\uparrow y)_x \in \mathcal{T}_x$. We conclude that $\mathcal{T}_x \neq \{\emptyset, X\}$. $\square$

Remark 9 *It is evident that, if* $|X| = 1$*, then* $\{\emptyset, X\} = \mathcal{T}_x = \mathcal{P}(X)$*.*

From Theorem 8, the following interesting question arises: *Is it possible that* $\bigcap_{x\in X} \mathcal{T}_x = \{\emptyset, X\}$ *and* $\bigvee_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$ *if* $|X| > 1$*?* The main purpose of the next section is to address this question.

4.2 Intersection and supremum of the induced topologies

In this subsection, we investigate the intersection and supremum of the family of topologies induced by a betweenness relation.

The intersection of the family of topologies $\{\mathcal{T}_x\}_{x\in X}$ induced by a betweenness relation B on a set X is necessarily the anti-discrete topology on X.

Theorem 10 *Let* $\{\mathcal{T}_x\}_{x\in X}$ *be the family of topologies induced by a betweenness relation* B *on a set* X*. Then* $\bigcap_{x\in X} \mathcal{T}_x = \{\emptyset, X\}$*.*

Proof. It is obvious that $\{\emptyset, X\} \subseteq \bigcap_{x\in X} \mathcal{T}_x$. Now consider any $A \in \bigcap_{x\in X} \mathcal{T}_x$ with $A \neq \emptyset$, and choose any $x_0 \in A$. On the one hand, it holds that $A \in \mathcal{T}_{x_0}$, which implies that A is a neighbourhood of x_0 with respect to $\mathcal{T}_{x_0}$. On the other hand, $(\uparrow x_0)_{x_0} = X$ is the smallest neighbourhood of x_0 with respect to $\mathcal{T}_{x_0}$. Therefore, $A = X$. $\square$

For the supremum of these topologies, we first consider the minimal betweenness relation.

Theorem 11 *Let* $\{\mathcal{T}_x\}_{x\in X}$ *be the family of topologies induced by the minimal betweenness relation* B_0 *on a set* X*. Then* $\bigcup_{x\in X} \mathcal{T}_x = \bigvee_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$*.*

Proof. Obviously, $\bigcup_{x\in X} \mathcal{T}_x = \bigvee_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$ if $|X| = 1$. We therefore assume that $|X| > 1$. Since $\bigcup_{x\in X} \mathcal{T}_x \subseteq \bigvee_{x\in X} \mathcal{T}_x \subseteq \mathcal{P}(X)$, it suffices to show that $\mathcal{P}(X) \subseteq \bigcup_{x\in X} \mathcal{T}_x$. For any $A \in \mathcal{P}(X)$, we distinguish two possible cases:

(i) If $A = \emptyset$ or $A = X$, then it is evident that $A \in \bigcup_{x\in X} \mathcal{T}_x$.

(ii) If $\emptyset \subset A \subset X$, choose any $x_0 \in X \setminus A$. It follows from the definition of B_0 that $(\uparrow y)_{x_0} = \{y\}$ for any $y \in A$. Note that $(\uparrow y)_{x_0} \in \mathcal{T}_{x_0}$. Hence, $A = \bigcup_{y\in A} \{y\} = \bigcup_{y\in A} (\uparrow y)_{x_0} \in \mathcal{T}_{x_0}$. We conclude that $A \in \bigcup_{x\in X} \mathcal{T}_x$. $\square$

The following question now arises: *Are the equalities* $\bigcup_{x\in X} \mathcal{T}_x = \bigvee_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$

still valid for any betweenness relation? The following example shows that these equalities do not hold in general.

Example 5 *Consider an infinite set X and the betweenness relation B_b defined in Example 4. Then $\mathcal{T}_b = \{X\} \cup \{A \subseteq X \mid b \notin A\}$ and, for any $x \in X$ with $x \neq b$, it holds that $\mathcal{T}_x = \{X, X \setminus \{x\}\} \cup \{A \subseteq X \mid x, b \notin A\}$. Hence,*

$$\bigvee_{x \in X} \mathcal{T}_x = \{A \subseteq X \mid b \notin A\} \cup \{X \setminus B \mid B \text{ is a finite subset of } X \setminus \{b\}\} .$$

Since $\{b\} \notin \bigvee\limits_{x \in X} \mathcal{T}_x$, $\bigvee\limits_{x \in X} \mathcal{T}_x$ is not the discrete topology. In addition, if $a, c \in X$ and $b \neq c \neq a \neq b$, then $X \setminus \{a, c\} \in \bigvee\limits_{x \in X} \mathcal{T}_x$, but it is not difficult to see that $X \setminus \{a, c\} \notin \bigcup\limits_{x \in X} \mathcal{T}_x$. This implies that $\bigvee\limits_{x \in X} \mathcal{T}_x \neq \bigcup\limits_{x \in X} \mathcal{T}_x$. Therefore, $\bigcup\limits_{x \in X} \mathcal{T}_x \subset \bigvee\limits_{x \in X} \mathcal{T}_x \subset \mathcal{P}(X)$. ◁

Although the two equalities might not hold in general, the second equality $\bigvee\limits_{x \in X} \mathcal{T}_x = \mathcal{P}(X)$ is valid in many cases, as we shall see next. To that end, we consider three different axioms for a betweenness relation and study the relationships among them. Later on, we will show that either of the first two axioms is a sufficient condition for the equality $\bigvee\limits_{x \in X} \mathcal{T}_x = \mathcal{P}(X)$ to hold (Theorems 20 and 21), whereas the third one is a necessary and sufficient condition (Theorem 18). The three axioms read as follows:

(A1) For any $a, b, c \in X$ with $a \neq b \neq c$ such that $(a, b, c) \in B$, it holds that $(a, b, d) \in B$ and $(c, b, d) \in B$ imply $d = b$.
(A2) For any $b \in X$, there exists $a \in X$ such that $(a, b, c) \in B$ implies $c = b$.
(A3) For any $b \in X$, there exists a finite subset $\{a_i\}_{i=1}^n$ of X such that, if $(a_i, b, c) \in B$ for all $i \in \{1, \ldots, n\}$, then $c = b$.

Obviously, the minimal betweenness relation B_0 satisfies all three axioms. The next two propositions indicate that any betweenness relation induced by an order relation or a real vector space satisfies (A1). A non-minimal betweenness relation satisfying (A2) is given in Example 6, which is a slight modification of Example 4.

Proposition 12 *Let B be the betweenness relation induced by an order relation R on a set X. Then B satisfies axiom (A1).*

Proof. For any $a, b, c \in X$ with $a \neq b \neq c$ and $(a, b, c) \in B$, it follows from $a \neq b \neq c$ that $aRbRc$ or $cRbRa$. Without loss of generality, we can assume that $aRbRc$. Suppose that $(a, b, d) \in B$ and $(c, b, d) \in B$. We also assume that $d \neq b$.

It follows from $a \neq b \neq d$ and $(a, b, d) \in B$ that $aRbRd$ or $dRbRa$. Similarly, we have $cRbRd$ or $dRbRc$. We distinguish four possible cases:

(i) $aRbRd$ and $cRbRd$. Recall that $aRbRc$. It holds that bRc and cRb imply $b = c$, a contradiction.
(ii) $aRbRd$ and $dRbRc$. It holds that bRd and dRb imply $d = b$, a contradiction.
(iii) $dRbRa$ and $cRbRd$. It holds that dRb and bRd imply $d = b$, a contradiction.
(iv) $dRbRa$ and $dRbRc$. Recall that $aRbRc$. It holds that aRb and bRa imply $a = b$, a contradiction.

Therefore, $(a, b, d) \in B$ and $(c, b, d) \in B$ imply $d = b$. $\square$

Proposition 13 *The betweenness relation B induced on a subset X of a real vector space satisfies axiom (A1).*

Proof. For any $a, b, c \in X$ such that $a \neq b \neq c$ and $(a, b, c) \in B$, there exists $\alpha \in]0, 1[$ such that $b = \alpha a + (1 - \alpha)c$. Suppose that $(a, b, d) \in B$ and $(c, b, d) \in B$. Then there exist $\beta, \gamma \in [0, 1[$ such that $b = \beta a + (1 - \beta)d$ and $b = \gamma c + (1 - \gamma)d$. We distinguish two possible cases:

(i) If $\beta = 0$ or $\gamma = 0$, then it is obvious that $d = b$.
(ii) If $\beta \neq 0$ and $\gamma \neq 0$, then $\beta, \gamma \in]0, 1[$. Multiplying both sides of the three equalities above by the appropriate factor, we obtain what follows:

$$\begin{cases} \beta\gamma b = \alpha\beta\gamma a + (1 - \alpha)\beta\gamma c \\ \alpha\gamma b = \alpha\beta\gamma a + \alpha(1 - \beta)\gamma d \\ (1 - \alpha)\beta b = (1 - \alpha)\beta\gamma c + (1 - \alpha)\beta(1 - \gamma)d. \end{cases}$$

Hence, it holds that

$$\alpha\gamma b + (1 - \alpha)\beta b = \beta\gamma b + (\alpha(1 - \beta)\gamma d + (1 - \alpha)\beta(1 - \gamma)d)\,,$$

which implies

$$(\alpha(1 - \beta)\gamma + (1 - \alpha)\beta(1 - \gamma))d = (\alpha(1 - \beta)\gamma + (1 - \alpha)\beta(1 - \gamma))b\,.$$

Note that $(\alpha(1 - \beta)\gamma + (1 - \alpha)\beta(1 - \gamma)) > 0$ since $\alpha, \beta, \gamma \in]0, 1[$. Thus, we conclude that $d = b$. $\square$

Example 6 *Let X be an infinite set, S be a finite subset of X with $|S| > 3$ and $b \in S$ be fixed. Define $B = B_0 \cup \{(x, b, z) \in X^3 \mid (x, z \in S \setminus \{b\}) \wedge (x \neq z)\}$. It is easy to verify that B is a betweenness relation on X. For any $y \in X$, choose an $x \in X \setminus (S \cup \{y\})$. It holds that $(x, y, z) \in B$ implies $z = y$. Thus, we conclude that B satisfies (A2).* $\triangleleft$

The following theorem characterizes the three axioms in terms of principal filters.

Theorem 14 *Let B be a betweenness relation on a set X.*

(i) B satisfies (A1) if and only if for any $a, b, c \in X$ with $a \neq b \neq c$ and $(a, b, c) \in B$, $(\uparrow b)_a \cap (\uparrow b)_c = \{b\}$.
(ii) B satisfies (A2) if and only if for any $b \in X$, there exists $a \in X$ such that $(\uparrow b)_a = \{b\}$.
(iii) B satisfies (A3) if and only if for any $b \in X$, there exists a finite subset $\{a_i \mid i \in \{1, 2, \ldots, n\}\}$ of X such that $\bigcap_{i=1}^{n} (\uparrow b)_{a_i} = \{b\}$.

Proof. We only prove (i). The proofs of (ii) and (iii) are quite similar to that of (i).

Suppose that B satisfies (A1). Consider $a, b, c \in X$ such that $a \neq b \neq c$ and $(a, b, c) \in B$. By definition, it holds that $d \in (\uparrow b)_a \cap (\uparrow b)_c$ is equivalent to $(a, b, d) \in B$ and $(c, b, d) \in B$. Then $d = b$ by (A1), which implies $(\uparrow b)_a \cap (\uparrow b)_c \subseteq \{b\}$. Proposition 7(i) implies the opposite inclusion $\{b\} \subseteq (\uparrow b)_a \cap (\uparrow b)_c$. We conclude that $(\uparrow b)_a \cap (\uparrow b)_c = \{b\}$.

Conversely, for any $a, b, c \in X$ with $a \neq b \neq c$ and $(a, b, c) \in B$, if $(a, b, d) \in B$ and $(c, b, d) \in B$, i.e., $d \in (\uparrow b)_a \cap (\uparrow b)_c$, then it follows from $(\uparrow b)_a \cap (\uparrow b)_c = \{b\}$ that $d = b$. So B satisfies axiom (A1). $\square$

From Theorem 14(iii) and Proposition 7(v), it follows that any betweenness relation on a finite set necessarily satisfies axiom (A3).

Proposition 15 *If X is a finite set, then any betweenness relation B on X satisfies (A3).*

Now, we study the relationships among the three axioms on a betweenness relation.

Theorem 16 *Let B be a betweenness relation on a set X.*

(i) Axiom (A1) implies axiom (A3).
(ii) Axiom (A2) implies axiom (A3).

Proof. (i) Assume that B satisfies (A1). For any $b \in X$, we distinguish two possible cases:

(a) If there exist $a, c \in X$ such that $a \neq b \neq c$ and $(a, b, c) \in B$, then $(\uparrow b)_a \cap (\uparrow b)_c = \{b\}$ by Theorem 14(i).
(b) Otherwise, for any $a \in X$ with $a \neq b$, it is easy to verify that $(\uparrow b)_a =$

$\{b\}$. We conclude that B satisfies (A3) by Theorem 14(iii).

(ii) It follows directly from Theorem 14(ii) and Theorem 14(iii). $\square$

Remark 17 *(i) Axiom (A1) does not imply axiom (A2). Actually, it is not difficult to prove that the betweenness relation induced by a total order relation on a set X with $|X| > 2$ and the betweenness relation induced by a convex subset X of a real vector space with $|X| > 2$ do not satisfy axiom (A2). From Propositions 12 and 13, we know that both betweenness relations satisfy axiom (A1).*

(ii) Axiom (A2) does not imply axiom (A1). Actually, let B be the betweenness relation defined in Example 6, $a, c \in S \setminus \{b\}$. If $a \neq c$ (otherwise $(a, b, c) \notin B$), then $(a, b, c) \in B$ and $(\uparrow b)_a \cap (\uparrow b)_c = S \setminus \{a, c\}$ since $(\uparrow b)_a = S \setminus \{a\}$ and $(\uparrow b)_c = S \setminus \{c\}$. It follows from $|S| > 3$ that $\{b\} \subset S \setminus \{a, c\}$, which implies $(\uparrow b)_a \cap (\uparrow b)_c \neq \{b\}$. Hence, B does not satisfy axiom (A1), but B satisfies axiom (A2).

(iii) Axiom (A3) neither implies axiom (A1) nor axiom (A2), which follows from the example below.

Example 7 *Let X be an infinite set that can be written as the union of a pairwise disjoint family of finite subsets $\{S_i\}_{i\in I}$ with $|S_i| > 3$ for all i and fix $b \in X$. Define $B = B_0 \cup \{(x, b, z) \in X^3 \mid (\exists i \in I)(x, z \in S_i \setminus \{b\} \wedge \ x \neq z)\}$. It is easy to verify that B is a betweenness relation on X.*

For any $x \in X$, we can show that there exists a finite subset $\{a_i \mid i \in \{1, 2, \ldots, n\}\}$ of X such that $\bigcap\limits_{i=1}^{n} (\uparrow x)_{a_i} = \{x\}$. In fact, if $x \neq b$, choose any $y \in X \setminus \{x\}$, then $(\uparrow x)_y = \{x\}$. If $x = b$, then $\bigcap\limits_{y\in S_i} (\uparrow x)_y = \{x\}$ for any $i \in I$. Therefore, B satisfies (A3).

However, similarly to Remark 17(ii), B does not satisfy (A1). It follows from the construction of X that $|(\uparrow b)_a| > 2$ for any $a \in X$, which implies $(\uparrow b)_a \neq \{b\}$ for any $a \in X$. Therefore, B does not satisfy (A2). $\triangleleft$

The following theorem shows that axiom (A3) is a necessary and sufficient condition for the equality $\bigvee\limits_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$ to hold.

Theorem 18 *Let $\{\mathcal{T}_x\}_{x\in X}$ be the family of topologies induced by a betweenness relation B on a set X. Then $\bigvee\limits_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$ if and only if B satisfies axiom (A3).*

Proof. *Necessity:* Assume that $\bigvee\limits_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$. Then $\{b\} \in \bigvee\limits_{x\in X} \mathcal{T}_x$ for any $b \in X$. Additionally, since $\bigcup\limits_{x\in X} \mathcal{T}_x$ is a subbase for $\bigvee\limits_{x\in X} \mathcal{T}_x$, there exists a family of subsets $\{U_j\}_{j\in J}$ of X such that $\{b\} = \bigcup\limits_{j\in J} U_j$, where each U_j is the intersection of a finite number of members in $\bigcup_{x\in X}\mathcal{T}_x$. Hence, there exist $j_0 \in J$ such that $\{b\} = U_{j_0}$ and a finite family $\{V_i \mid i \in \{1, 2, \ldots, n\}\}$ of members in $\bigcup_{x\in X}\mathcal{T}_x$ such that $U_{j_0} = \bigcap_{i=1}^{n} V_i$. Hence, $\{b\} = \bigcap_{i=1}^{n} V_i$.

Since $V_i \in \bigcup_{x\in X}\mathcal{T}_x$ for any $i \in \{1, 2, \ldots, n\}$, there exists $a_i \in X$ such that $V_i \in \mathcal{T}_{a_i}$, which implies that V_i is a neighbourhood of b with respect to $\mathcal{T}_{a_i}$. Note that $(\uparrow b)_{a_i}$ is the smallest neighbourhood of b with respect to $\mathcal{T}_{a_i}$, which implies $b \in (\uparrow b)_{a_i} \subseteq V_i$. It follows from $\{b\} = \bigcap_{i=1}^{n} V_i$ that $\bigcap_{i=1}^{n}(\uparrow b)_{a_i} = \{b\}$. Therefore, B satisfies (A3) by Theorem 14(iii).

Sufficiency: If B satisfies (A3), then it follows from Theorem 14(iii) that for any $b \in X$, there exists a finite subset $\{a_i \mid i \in \{1, 2, \ldots, n\}\}$ of X such that $\bigcap_{i=1}^{n}(\uparrow b)_{a_i} = \{b\}$. Note that $(\uparrow b)_{a_i} \in \mathcal{T}_{a_i} \subseteq \bigvee_{x\in X}\mathcal{T}_x$ for any $i \in \{1, 2, \ldots, n\}$, which implies $\{b\} \in \bigvee_{x\in X}\mathcal{T}_x$. We conclude that $\bigvee_{x\in X}\mathcal{T}_x = \mathcal{P}(X)$ since the result holds for any $b \in X$. $\square$

Since any betweenness relation on a finite set necessarily satisfies (A3), the following corollary is immediate.

Corollary 19 *Let $\{\mathcal{T}_x\}_{x\in X}$ be the family of topologies induced by a betweenness relation B on a set X. If X is finite, then $\bigvee_{x\in X}\mathcal{T}_x = \mathcal{P}(X)$.*

Note that both axioms (A1) and (A2) imply axiom (A3), but the converse is not true. Hence, either axiom (A1) or axiom (A2) is a sufficient (but not necessary) condition for the equality $\bigvee_{x\in X}\mathcal{T}_x = \mathcal{P}(X)$ to hold, as indicated in the following two theorems.

Theorem 20 *Let $\{\mathcal{T}_x\}_{x\in X}$ be the family of topologies induced by a betweenness relation B on a set X. If B satisfies axiom (A1), then $\bigvee_{x\in X}\mathcal{T}_x = \mathcal{P}(X)$.*

Theorem 21 *Let $\{\mathcal{T}_x\}_{x\in X}$ be the family of topologies induced by a betweenness relation B on a set X. If B satisfies axiom (A2), then $\bigvee_{x\in X}\mathcal{T}_x = \mathcal{P}(X)$.*

Finally, from the perspective of topology, we propose another necessary and sufficient condition for the equality $\bigvee_{x\in X}\mathcal{T}_x = \mathcal{P}(X)$ to hold.

Theorem 22 *Let $\{\mathcal{T}_x\}_{x\in X}$ be the family of topologies induced by a betweenness relation B on a set X. Then $\bigvee_{x\in X}\mathcal{T}_x = \mathcal{P}(X)$ if and only if $\bigvee_{x\in X}\mathcal{T}_x$ is Alexandrov.*

Proof. *Necessity:* It is evident that $\bigvee_{x\in X} \mathcal{T}_x$ is Alexandrov if $\bigvee_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$.

Sufficiency: Suppose that $\bigvee_{x\in X} \mathcal{T}_x$ is Alexandrov. On the one hand, for any $x, y \in X$, it holds that $(\uparrow y)_x \in \mathcal{T}_x \subseteq \bigvee_{x\in X} \mathcal{T}_x$. On the other hand, from Proposition 7(v), we know that $\bigcap_{x\in X} (\uparrow y)_x = \{y\}$. Since $\bigvee_{x\in X} \mathcal{T}_x$ is Alexandrov, it holds that $\{y\} \in \bigvee_{x\in X} \mathcal{T}_x$ for any $y \in X$. Therefore, $\bigvee_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$. $\square$

Remark 23 *Note that $\{\mathcal{T}_x\}_{x\in X}$ is a family of type T_0 and Alexandrov topologies on X. Therefore, Theorem 22 and Example 5 indicate that the supremum of a family of type T_0 and Alexandrov topologies is not necessarily Alexandrov.*

4.3 Relationship between the topology induced by an order relation or a metric and the family of topologies induced by the betweenness relation induced by this same order relation or metric

On the one hand, any order relation or metric can induce a topology and a betweenness relation. On the other hand, any betweenness relation can induce a family of topologies. Thus, it might be interesting to investigate the relationship between the topology induced by an order relation or a metric and the family of topologies induced by the betweenness relation induced by this same order relation or metric.

We have the following two results in the case of an order relation.

Theorem 24 *Let B_R be the betweenness relation induced by an order relation R on a set X, $\mathcal{T}_R$ be the topology induced by R and $\{\mathcal{T}_x\}_{x\in X}$ be the family of topologies induced by B_R. The following results hold:*

(i) $\mathcal{T}_R \subseteq \bigvee_{x\in X} \mathcal{T}_x$.
(ii) $\mathcal{T}_R = \bigvee_{x\in X} \mathcal{T}_x$ *if and only if* $R = \{(x, x) \mid x \in X\}$.

Proof. (i) It follows from Proposition 12 and Theorem 20 that $\bigvee_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$, which implies $\mathcal{T}_R \subseteq \bigvee_{x\in X} \mathcal{T}_x$.

(ii) Since $\bigvee_{x\in X} \mathcal{T}_x = \mathcal{P}(X)$, we may equivalently prove that $\mathcal{T}_R = \mathcal{P}(X)$ if and only if $R = \{(x, x) \mid x \in X\}$.

If $R = \{(x, x) \mid x \in X\}$, then every singleton is an upper set, which implies $\{x\} \in \mathcal{T}_R$ for all $x \in X$. Hence, $\mathcal{T}_R = \mathcal{P}(X)$.

Conversely, if $\mathcal{T}_R = \mathcal{P}(X)$, then it follows from the one-to-one correspondence (mentioned in Subsection 2.2) between the set of all preorder relations and that of all Alexandrov topologies on X that $R = \{(x, x) \mid x \in X\}$. $\square$

Theorem 25 *Let B_R be the betweenness relation induced by an order relation R on a set X, $\mathcal{T}_R$ be the topology induced by R and $\{\mathcal{T}_x\}_{x\in X}$ be the family of topologies induced by B_R. Then, for any $a \in X$, $\mathcal{T}_a = \mathcal{T}_R$ if and only if a is the smallest element with respect to R.*

Proof. *Necessity:* Since the correspondence between the set of all preorder relations and that of all Alexandrov topologies on X is one-to-one, it follows from $\mathcal{T}_a = \mathcal{T}_R$ that $O_a = R$. So a is the smallest element with respect to R by Remark 4.

Sufficiency: Assume that a is the smallest element with respect to R. It is equivalent to show that $O_a = R$. If $(b, c) \in O_a$, then we have by the definition of O_a that $(a, b, c) \in B_R$. We distinguish four possible cases: (i) $a = b$ or (ii) $b = c$ or (iii) $aRbRc$ or (iv) $cRbRa$:

(i) $a = b$ implies $(b, c) \in R$ since a is the smallest element in R.
(ii) $b = c$ trivially implies $(b, c) \in R$ since R is reflexive.
(iii) It is evident that $aRbRc$ implies $(b, c) \in R$.
(iv) Since a is the smallest element in R and R is antisymmetric, $cRbRa$ implies that $a = b = c$. Thus, due to the reflexivity of R, it holds that $(b, c) \in R$.

Hence, $O_a \subseteq R$.

Conversely, if $(b, c) \in R$, then $aRbRc$ since a is the smallest element in R, which implies $(a, b, c) \in B_R$, and, also, $(b, c) \in O_a$. Therefore, it holds that $R \subseteq O_a$, and, thus, $O_a = R$. $\square$

Remark 26 *From Theorem 25, we conclude that, if R has no smallest element, then $\mathcal{T}_a \neq \mathcal{T}_R$ for any $a \in X$.*

In the upcoming two examples, we show that similar results do not hold in the case of a metric.

Example 8 *Let X be an infinite set with a fixed element b, d be the metric defined in Example 4 and B_d be the betweenness relation induced by d. For any $x \in X$, $B(x, 1) = \{y \in X \mid d(x, y) < 1\} = \{x\}$, which implies $\{x\} \in \mathcal{T}_d$. Hence, $\mathcal{T}_d = \mathcal{P}(X)$. On the other hand, it follows from Example 5 that $\bigvee\limits_{x\in X} \mathcal{T}_x \subset \mathcal{P}(X)$. Therefore, $\bigvee\limits_{x\in X} \mathcal{T}_x \subset \mathcal{T}_d$.* ◁

Example 9 *Let $X =]-\infty, +\infty[$ be the real line and d be the metric on X defined as $d(x, y) = |x - y|$, for any $x, y \in X$. It is easy to verify that the* *betweenness relation B_d induced by d is exactly the one induced by the usual total order relation $\leq$ on X. For any $x, y \in X$, it holds that*

$$(\uparrow y)_x = \begin{cases} [y, +\infty[\,, & \textit{if } \; y > x\,, \\ X\,, & \textit{if } \; y = x\,, \\ ]-\infty, y]\,, & \textit{if } \; y < x\,. \end{cases}$$

For any $x \in X$, we may consider $y = |x| + 1 > x$, thus resulting in $[y, +\infty[\in \mathcal{T}_x$. However, $[y, +\infty[\notin \mathcal{T}_d$. Therefore, $\mathcal{T}_x \nsubseteq \mathcal{T}_d$. It is also true that $\mathcal{T}_d \nsubseteq \mathcal{T}_x$ since $]-y, y[\in \mathcal{T}_d$, but $]-y, y[\notin \mathcal{T}_x$. $\triangleleft$

Remark 27 *In Example 9, we know from Proposition 12 and Theorem 20 that $\bigvee_{x \in X} \mathcal{T}_x = \mathcal{P}(X)$. However, it is well known that $\mathcal{T}_d \subset \mathcal{P}(X)$. We thus conclude that $\mathcal{T}_d \subset \bigvee_{x \in X} \mathcal{T}_x$.*

5 Discussion on the use of lower sets

Note that the set of all lower sets with respect to a preorder relation R is also an Alexandrov topology, and coincides with the Alexandrov topology induced by the preorder relation obtained by transposing R. Thus, a natural question arises: *Are the main results on the intersection and supremum of the induced topologies in Subsection 4.2 still valid in such case?* The following theorem indicates that, if we consider the topology of the lower sets, then the intersection is still anti-discrete, but the supremum is now necessarily discrete for any betweenness relation (the betweenness relation does not need to satisfy any additional condition such as (A3)).

Theorem 28 *Consider a betweenness relation B on a set X and the family of order relations $\{O_x\}_{x \in X}$ induced by B. For any $x \in X$, $\mathcal{T}_x$ denotes the topology of all lower sets with respect to O_x. Then $\bigcap_{x \in X} \mathcal{T}_x = \{\emptyset, X\}$ and $\bigvee_{x \in X} \mathcal{T}_x = \mathcal{P}(X)$.*

Proof. It is obvious that $\{\emptyset, X\} \subseteq \bigcap_{x \in X} \mathcal{T}_x$. Now consider any $A \in \bigcap_{x \in X} \mathcal{T}_x$ with $A \notin \{\emptyset, X\}$, and choose any $x_0 \in X \setminus A$. Since $A \in \mathcal{T}_{x_0}$ and $\{(\downarrow x)_{x_0}\}_{x \in X}$ is a base for $\mathcal{T}_{x_0}$, there exists $\{x_i\}_{i \in I} \subseteq X$ such that $A = \bigcup_{i \in I} (\downarrow x_i)_{x_0}$, where $(\downarrow x)_{x_0} = \{y \in X \mid (x_0, y, x) \in B\}$. Note that $x_0 \in (\downarrow x)_{x_0}$ for any $x \in X$. Hence $x_0 \in A$, which contradicts the fact that $x_0 \in X \setminus A$. Therefore, $\bigcap_{x \in X} \mathcal{T}_x = \{\emptyset, X\}$.

For the supremum, it is equivalent to prove that $\{x_0\} \in \bigvee_{x \in X} \mathcal{T}_x$ for any $x_0 \in X$.

This follows from $(\downarrow x_0)_{x_0} = \{x_0\}$ and $(\downarrow x_0)_{x_0} \in \mathcal{T}_{x_0} \subseteq \bigvee_{x\in X} \mathcal{T}_x$. $\square$

6 Conclusions and open problems

A betweenness relation can be represented as a family of order relations, and, thus, induce a family of topologies. In this paper, the intersection of this family of topologies has been proven to be the anti-discrete topology, whereas a necessary and sufficient condition for the supremum of this family of topologies to be the discrete topology has been given. In particular, this condition holds in case the considered set is finite. Future work concerns the definition of alternative construction methods for a topology given a betweenness relation. For instance, we highlight the use of alternative topologies to the Alexandrov topology, such as the Scott topology and the Lawson topology in case we are dealing with a family of directed-complete (partial) order relations (see Subsection 2.2), or even the construction of a topology that avoids the intermediate step of representing the betweenness relation as a family of order relations. Also, the construction of a fuzzy topology given a fuzzy betweenness relation could be an interesting future study subject. We end by noting that, in the field of data aggregation, continuity is strongly acclaimed as a desirable property assuring an aggregation process to be well behaving. The study here addressed entails a first step towards the study of the continuity of aggregation processes on sets equipped with a betweenness relation (such as sets of rankings or sets of strings [22]).

Acknowledgements

Hua-Peng Zhang is supported by National Natural Science Foundation of China (grant No. 11571006), the scholarship of Jiangsu Overseas Visiting Scholar Program and NUPTSF (grant No. NY218059). Raúl Pérez-Fernández acknowledges the support of the Research Foundation of Flanders (FWO17/PDO/160) and the Spanish MINECO (TIN2017-87600-P).

References

[1] P. Bankston, *Road systems and betweenness.* Bull. Math. Sci. **3** (2013), 389–408.

[2] P. Bankston, *When Hausdorff continua have no gaps.* Topology Proceedings **44** (2014), 177–188.

[3] P. Bankston, *The antisymmetry betweenness axiom and Hausdorff continua.* Topology Proceedings **45** (2015), 189–215.

[4] P. Bankston, A. McCluskey, R.J. Smith, *Semicontinuity of betweenness functions.* Topology and its Applications **246** (2018), 22–47.

[5] L.M. Blumenthal, D.O. Ellis, *Notes on lattices.* Duke Math. J. **16** (1949), 585–590.

[6] J. Bruno, A. McCluskey, P. Szeptycki, *Betweenness relations in a categorical setting.* Results in Mathematics **72** (2017), 649–664.

[7] V. Chvátal, *Sylvester–Gallai theorem and metric betweenness.* Discrete Comput. Geom. **31** (2004), 175–195.

[8] J.M. Cibulskis, *A characterization of the lattice orderings on a set which induce a given betweenness.* J. London Math. Soc. **s2-1** (1969), 480–482.

[9] D. Dubois, H. Prade, *On the use of aggregation operations in information fusion processes.* Fuzzy Sets and Systems **142** (2004) 143–161.

[10] N. Düvelmeyer, W. Wenzel, *A characterization of ordered sets and lattices via betweenness relations.* Results in Mathematics **46** (2004), 237–250.

[11] R. Engelking, *General Topology.* Heldermann Verlag, Berlin, 1989.

[12] P.C. Fishburn, *Betweenness, orders and interval graphs.* J. Pure Appl. Algebra **1** (1971), 159–178.

[13] M. Gagolewski, *Data Fusion. Theory, Methods and Applications.* Institute of Computer Science, Polish Academy of Sciences, Warsaw, 2015.

[14] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, *Continuous Lattices and Domains.* Cambridge University Press, Cambridge, 2003.

[15] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, *Aggregation Functions.* Cambridge University Press, Cambridge, 2009.

[16] J. Hedlíková, T. Katriñák, *On a characterization of lattices by the betweenness relation –on a problem of M. Kolibiar.* Algebra Universalis **28** (1991), 389–400.

[17] E.V. Huntington, *A new set of postulates for betweenness, with proof of complete independence.* Trans. Amer. Math. Soc. **26** (1924), 257–282.

[18] E.V. Huntington, J.R. Kline, *Sets of independent postulates for betweenness.* Trans. Amer. Math. Soc. **18** (1917), 301–325.

[19] P.T. Johnstone, *Stone Spaces.* Cambridge University Press, Cambridge, 1982.

[20] R. Padmanabhan, *On some ternary relations in lattices.* Colloquium Mathematicum **15** (1966), 195–198.

[21] M. Pasch, *Vorlesungen über neuere Geometrie, vol. 23.*, Teubner, Leipzig, Berlin, 1882.

[22] R. Pérez-Fernández, B. De Baets, *On the role of monometrics in penalty-based data aggregation.* IEEE Transactions on Fuzzy Systems, in press, DOI: 10.1109/TFUZZ.2018.2880716

[23] R. Pérez-Fernández, M. Rademaker, B. De Baets, *Monometrics and their role in the rationalisation of ranking rules.* Information Fusion **34** (2017), 16–27.

[24] E. Pitcher, M.F. Smiley, *Transitivities of betweenness.* Trans. Amer. Math. Soc. **52** (1942), 95–114.

[25] M. Ploščica, *On a characterization of distributive lattices by the betweenness relation.* Algebra Universalis **35** (1996), 249–255.

[26] D. Rautenbach, P.M. Schäfer, *Strict betweennesses induced by posets as well as by graphs.* Order **28** (2011), 89–97.

[27] M. Sholander, *Trees, lattices, order, and betweenness.* Proc. Amer. Math. Soc. **3** (1952), 369–381.

[28] M.F. Smiley, *A comparison of algebraic, metric, and lattice betweenness.* Bull. Amer. Math. Soc. **49** (1943), 246–252.

[29] M.F. Smiley, W.R. Transue, *Applications of transitivities of betweenness in lattice theory.* Bull. Amer. Math. Soc. **49** (1943), 280–287.

[30] W.R. Transue, *Remarks on transitivities of betweenness.* Bull. Amer. Math. Soc. **50** (1944), 108–109.