A note on “Similarity and dissimilarity measures between fuzzy sets: A formal relational study” and “Additive similarity and dissimilarity measures”

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Abstract

We revisit the relational study between different axiomatic definitions of similarity and dissimilarity of fuzzy sets developed in two previous articles. We observe that every axiomatic definition admits two variants, depending on whether we assume that the measure fulfills the corresponding properties when we restrict ourselves to a fixed finite cardinal universe, or that it complies with them in a general universe. The results and counterexamples presented in those articles concern the implication relations between different axiomatic notions, and they implicitly consider a general finite universe (whose cardinality is not fixed in advance). Most of them fit both variants, but some others need to be adapted, in order to meet the first option. In this short note, we offer the adapted versions of the corresponding results and counterexamples.

Keywords.- Similarity measure, dissimilarity measure, divergence measure, comparison measure, measure of resemblance, measure of similitude

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1. Introduction

Different authors in the 1980’s and 1990’s have proposed different ways of comparing pairs of fuzzy sets, by means of mappings of the form $m : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathbb{R}$ that assign a numerical value to every pair of fuzzy sets. Names such as “similarity” (see [4, 6, 7, 8] among many others),

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"resemblance"([1]) or "similitude" ([1]) were proposed in order to quantify the degree of "equality" between two fuzzy sets. In a dual way, the notions of "dissimilarity" ([1]), "divergence" ([5]) or "distance" ([7, 8]) were independently proposed, in order to quantify the degree of difference or inequality between two fuzzy sets. Each of those axiomatic definitions from the literature establishes a list of properties that the corresponding measure must satisfy.

We have reviewed in [2] the most prominent examples of those axiomatic definitions. In order to offer an organised and clear view, we listed there the whole collection of properties involved in all those definitions and divided it into three separate lists, respectively concerning "general properties", "properties of measures of equality" and "properties of measures of inequality". Then we analysed whether there existed or not a relation of implication between each pair of axiomatic definitions. In [3], we analysed the existence of additional implications in the particular case where the universe was assumed to be finite, and the considered measures were assumed to be "additive" (decomposable as a finite sum, where each term depends on the pair of membership values of the compared fuzzy sets over a single element of the universe).

Now, for simplicity, we will identify any finite universe of cardinal $n \in \mathbb{N}$, with the set of indices $[n] = \{1, \ldots, n\}$. Every property considered in those papers involved the relation between a collection of pairs of fuzzy subsets of the universe and their corresponding numerical images. However, in those works, we have ignored the fact that each of those properties and axiomatic definitions admitted two different versions, one weaker than the other, namely:

1. m is assumed to satisfy the corresponding property for an arbitrary and fixed cardinal $n \in \mathbb{N}$.
2. m is assumed to satisfy the corresponding property for every cardinal $n \in \mathbb{N}$.

This differentiation has relevance on the analysis of formal relationships between different axiomatic definitions, as we will show in this short note.

2. Preliminaries and notation

Let U denote an arbitrary finite universe of cardinal $n \in \mathbb{N}$ and let us identify it with the collection of indices $[n] = \{1, \ldots, n\}$. Let $\wp([n])$ and $\mathcal{F}([n])$ respectively denote the family of crisp and fuzzy subsets of $U = [n]$. Let $\mu_A : [n] \to [0, 1]$ denote the membership function of a fuzzy set $A \in \mathcal{F}([n])$. We will consider the usual min-based intersection and max-based union between fuzzy sets, whose membership functions are defined as follows:

$$\mu_{A\cap B}(i) = \mu_A(i) \wedge \mu_B(i),$$

$$\mu_{A\cup B}(i) = \mu_A(i) \vee \mu_B(i), \forall\, i \in [n],$$

where $\wedge$ and $\vee$ respectively denote the minimum and maximum T-norms.

For the sake of self-containedness of this short note, we will enumerate the whole set of properties considered in [2] and [3]. We will present them in three separate lists. As we did in previous papers, the asterisk will be understood as "stronger than". Some implication relations are straightforward and some others have been proved in [2].

2.1. General properties:

- G1.- $0 \leq m(A, B) \leq 1 \quad \forall A, B \in \mathcal{F}([n])$.

- G1*.- $0 \leq m(A, B) \leq 1 \quad \forall A, B \in \mathcal{F}([n])$ and there exists a pair of fuzzy sets $C, D \in \mathcal{F}([n])$ such that $m(C, D) = 1$.

- G2.- $m(A, B) = m(B, A) \quad \forall A, B \in \mathcal{F}([n])$.

- G3.- Let $\rho : [n] \to [n]$ be a permutation on $[n]$. For each $A \in \mathcal{F}([n])$ denote A^ρ the fuzzy set whose membership function is defined from μ_A as follows: $\mu_{A^\rho}(x) = \mu_A(\rho(x))$. Then $m(A, B) = m(A^\rho, B^\rho)$.

- G3*.- There exists $h : [0, 1] \times [0, 1] \to \mathbb{R}$ such that $m(A, B) = \sum_{i\in[n]} h(\mu_A(i), \mu_B(i))$.

- G4.- There exists a mapping $f : \mathcal{F}([n]) \times \mathcal{F}([n]) \times \mathcal{F}([n]) \to \mathbb{R}$ such that

$$m(A, B) = f(A \cap B, A \setminus B, B \setminus A).$$

- G4*.- There exists a function $F_m : \mathbb{R}^3 \to \mathbb{R}$ and a fuzzy measure[1]
  $M : \mathcal{F}([n]) \to \mathbb{R}$ such that, for all $A, B \in \mathcal{F}([n])$, $m(A, B)$ can be expressed as follows:

$$m(A, B) = F_m(M(A \cap B), M(A \setminus B), M(B \setminus A)).$$

- G5: If $A \cap B = \emptyset$, $A' \cap B' = \emptyset$, $m(A, \emptyset) \leq m(A', \emptyset)$ and $m(B, \emptyset) \leq m(B', \emptyset)$, then $m(A, B) \leq m(A', B')$.

[1]Several formal definitions of this notion can be found in the literature. Here, we will refer to a monotone increasing set-function satisfying the restriction $M(\emptyset) = 0$.

2.2. Properties for measures of "equality"

- S1.- $\forall A, B, C \in \mathcal{F}([n])$, if $A \subseteq B \subseteq C$ and $\max_{i\in[n]} \mu_A(i) = \max_{i\in[n]} \mu_B(i)$ then $s(A, C) \leq s(A, B)$.

- S1*.- $\forall A, B, C \in \mathcal{F}([n])$,
$$A \subseteq B \subseteq C \Rightarrow s(A, C) \leq s(A, B)$$

- S1*-var.-$\forall A, B, C \in \mathcal{F}([n])$,
$$A \subsetneq B \subsetneq C \Rightarrow s(A, C) < s(A, B)$$

- S2.- $\forall A, B, C \in \mathcal{F}([n])$,
$$A \subseteq B \subseteq C \Rightarrow s(A, C) \leq s(B, C).$$

- S2*.- If $A, B, C \in \mathcal{F}([n])$ satisfy:
  - $\mu_A(i_0) < \mu_B(i_0) \leq \mu_C(i_0)$, for some $i_0 \in [n]$, and
  - $\mu_B(i) = \mu_A(i),\ \forall\, i \in [n], i \neq i_0$,

  then $s(A, C) < s(B, C)$.

- S3.- $\forall D \in \wp(U),\ s(D, D^c) = 0$.

- S3*.- $s(D, D^c) = 0 \Leftrightarrow D \in \wp(U)$.

- S4.- $s(C, C) = \max_{A,B\in\mathcal{F}([n])} s(A, B)\ \forall C \in \mathcal{F}([n])$.

- S4*.- $C = D \Leftrightarrow s(C, D) = \max_{A,B\in\mathcal{F}([n])} s(A, B)$.

- S5.- $s(A, B) = 0 \Rightarrow A \cap B = \emptyset$.

- S5*.- Consider $A, B \in \mathcal{F}([n])$ and an arbitrary $i_0 \in [n]$. Define C and D such that:
  - $\mu_C(i) = \mu_A(i)$ and $\mu_D(i) = \mu_B(i)$, $\forall\, i \neq i_0$
  - $\mu_C(i_0) = \mu_A(i_0) + \alpha$ and $\mu_D(i_0) = \mu_B(i_0) + \alpha$, where $0 \leq \alpha \leq 1 - \max\{\mu_A(i_0), \mu_B(i_0)\}$

  Then:

– If $\max_{i\in[n]} \mu_{C\cap D}(i) = \max_{i\in[n]} \mu_{A\cap B}(i)$ then $s(C,D) = s(A,B)$.

– If $\max_{i\in[n]} \mu_{C\cap D}(x) > \max_{i\in[n]} \mu_{A\cap B}(i)$ then $s(C,D) > s(A,B)$.

- S6.- Consider $A, B \in \mathcal{F}([n])$ and an arbitrary $x_0 \in U$. Define C and D such that:

  – $\mu_C(i) = \mu_A(i)$ and $\mu_D(i) = \mu_B(i)$, $\forall\, i \neq i_0$

  – $\mu_C(i_0) = \mu_A(i_0) + \alpha$ and $\mu_D(i_0) = \mu_B(i) + \alpha$.

  Then, $s(A,C) = s(B,D)$.

- S7.- If $A, B, C, D \in \mathcal{F}([n])$ satisfy the following restrictions: $A \cap B \subseteq C \cap D$, $A \setminus B \supseteq C \setminus D$, and $B \setminus A \supseteq D \setminus C$ then $s(A,B) \leq s(C,D)$.

2.3. Properties for measures of "inequality" or "difference"

- D1.- If $A, B, C \in \mathcal{F}([n])$ satisfy the restrictions $A \subseteq B \subseteq C$, then:

  (a) $d(A,C) \geq d(A,B)$ and

  (b) $d(A,C) \geq d(B,C)$.

- D2.- $d(D, D^c) = \max_{A,B\in\mathcal{F}([n])} d(A,B)$ $\forall D \in \wp(U)$.

- D3.- $A = B \Rightarrow d(A,B) = 0$.

- D3*.- $A = B \Leftrightarrow d(A,B) = 0$.

- D4.- $\forall A, B, C \in \mathcal{F}([n])$, $d(A \cap C, B \cap C) \leq d(A,B)$.

- D5.- $\forall A, B, C \in \mathcal{F}([n])$, $d(A \cup C, B \cup C) \leq d(A,B)$.

- D6.- If $A, B, C, D \in \mathcal{F}([n])$ satisfy the following restrictions: $A \cap B \subseteq C \cap D$, $A \setminus B \supseteq C \setminus D$, and $B \setminus A \supseteq D \setminus C$ then $d(A,B) \geq d(C,D)$.

- D7.- $d(A,B) = d(A \setminus B, B \setminus A)$.

Properties S7, D6 and D7 involve the notion of "difference" between fuzzy sets. There is not a unique universally accepted difference operator. The following definition has been considered in [2]:

$$A \setminus B = A \cap B^c,$$

i.e.

$$\mu_{A\setminus B}(i) = \mu_A(i) \wedge (1 - \mu_B(i)),\ \forall\, i = 1, \ldots, n.$$

3. Two types of statements and their influence on the study of formal relations between different axiomatic definitions

As we have mentioned in the Introduction, each of the properties listed in Subsections 2.1, 2.2 and 2.3 admits the following variants:

(a) We select and fix an arbitrary $n \in \mathbb{N}$ and assume that $m : \mathcal{F}([n]) \times \mathcal{F}([n]) \to \mathbb{R}$ satisfies the corresponding property for this particular n.

(b) We assume that m is defined over the union $\cup_{n\in\mathbb{N}}\mathcal{F}([n]) \times \mathcal{F}([n])$ and its restriction to every Cartesian product $\mathcal{F}([n]) \times \mathcal{F}([n])$ satisfies the corresponding property, for every $n \in \mathbb{N}$.

We will use respectively the nomenclature G$i^{(n)}$, S$i^{(n)}$ and D$i^{(n)}$ in the first case (Case (a)) and the nomenclature Gi, Si and Di in Case (b). Every notion of similarity, dissimilarity, divergence, etc. reviewed in [2] and [3] corresponds to a collection of properties selected from Subsections 2.1, 2.2 and 2.3. Thus, we can also consider two different variants for each of those notions. I.e., given a particular notion, N, we can differentiate between the weaker version, $N^{(n)}$, and the stronger one, N. In [2] and [3] we did not mention this difference, but we implicitly considered the stronger version of each notion, i.e., Case (b). In those papers, we studied whether implications between the different notions (i.e., implications of the type $N \Rightarrow N'$) were satisfied or not. We have established mathematical results or counterexamples respectively proving or disproving those implications.

In most of the cases, slight modifications of our proofs would prove implications of the type $N^{(n)} \Rightarrow N'^{(n)}$ for an arbitrary $n \in \mathbb{N}$. With respect to the counterexamples, we can consider three variants of them:

A.- An example of a measure m satisfying a notion N (i.e., satisfying $N^{(n)}$ for every $n \in \mathbb{N}$), but not satisfying another notion N' (i.e., not satisfying $N'^{(n)}$ for some $n \in \mathbb{N}$).

B.- An example of a measure m satisfying a notion $N^{(n)}$ for some $n \in \mathbb{N}$, but not satisfying $N'^{(n)}$ for the same n.

C.- An example of a measure m satisfying a notion N (i.e., satisfying $N^{(n)}$ for every $n \in \mathbb{N}$), but not satisfying $N'^{(n)}$ for any $n \geq 2$.

In [2] and [3] we did not take into account the existence of those three variants, and we implicitly considered Case A in all our examples. Most of them also apply in Case C, but a few of them do not.

In the next sections, we will analyse, case by case, whether we can adapt our previous results and counterexamples, in order to provide alternative versions of them.

4. Analysis of of implications and counterexamples included in [2]

All the mathematical results proving implications of the type $N \Rightarrow N'$ included in [2] can be easily adapted to the case $N^{(n)} \Rightarrow N'^{(n)}$, by means of slightly adapting the corresponding proofs. On the other hand, all the counterexamples correspond to Case A in the above classification (the one mentioned at the end of Section 3. Most of them fit the third version (Case C), but some others need adjustment. In this section, we will focus on those that need adjustment:

Example 8. In this example from [2] a universe of cardinal $n = 1$ is considered and a mapping $d : \mathcal{F}([1]) \times \mathcal{F}([1]) \to \mathbb{R}$ is defined as follows:

$$d(\emptyset, [1]) = d([1], \emptyset) = 5,\ d(A, B) = 0, \text{otherwise}.$$

It illustrates a trivial situation where d satisfies Properties G2, D1, D2 and D3, but it does not satisfy G1*, because G1$^{*(n)}$ is not fulfilled for $n = 1$. It can be easily adapted in order to show a situation where properties G2, D1, D2 and D3 are satisfied, but G1$^{*(n)}$ is not, for any $n \in \mathbb{N}$. Consider the mapping $d : \cup_{n\in\mathbb{N}}\mathcal{F}([n]) \times \mathcal{F}([n]) \to \mathbb{R}$ defined as follows. Given an arbitrary $n \in \mathbb{N}$, and an arbitrary pair of fuzzy subsets $A, B \in \mathcal{F}([n])$,

$$d(A,B) := \begin{cases} 5 & \text{if } (A,B) = (\emptyset, [n]) \text{ or } (A,B) = ([n], \emptyset) \\ 0 & \text{otherwise.} \end{cases}$$

It does not satisfy property G1$^{*(n)}$, for any $n \in \mathbb{N}$, as $d(\emptyset, [n]) = d([n], \emptyset)$ is strictly greater than 1 for every $n \in \mathbb{N}$.

Example 10. The mapping $d : \cup_{n\in\mathbb{N}}\mathcal{F}([n]) \times \mathcal{F}([n]) \to \mathbb{R}$ considered in this example is defined as follows:

$$d(A, B) = \frac{\#\{x \,:\, \mu_A(i) \neq \mu_B(i)\}^{\frac{1}{\#(A\cap B)+1}}}{n},$$

where $\#C = \sum_{i\in[n]} \mu_C(i)$, $\forall\, C \in \mathcal{F}([n])$, denotes the cardinal of a fuzzy set C. It satisfies Properties G1*, G2, D1, D2 and D3* and therefore it fulfills [[2], Definition 13] (for every $n \in \mathbb{N}$). However it does not satisfy [[2], Definition 11], as it does not meet the equality $d(A, B) = d(A \setminus B, B \setminus A)$, for any $n \geq 2$. In [2], we explicitly check that d does not satisfy the above equality for $n = 4$. In order to do so we select for the pair of (crisp) sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, and observe that $d(A, B) = \sqrt[3]{2}/4 \neq 2/4 = d(A \setminus B, B \setminus A)$. We can consider instead the pair of (crisp) sets $A = \{1, 2\}$ and $B = \{2, 3\}$ in order to prove that $d(A, B) \neq d(A \setminus B, B \setminus A)$, for any $n \geq 3$.

We need an additional example in order to check that the implication Definition 13 $\Rightarrow$ Definition 11 does not either hold for $n = 2$ (i.e., to show that Definition $13^{(2)} \not\Rightarrow$ Definition $11^{(2)}$). Let us consider $d : \mathcal{F}([2]) \times \mathcal{F}([2]) \to \mathbb{R}$ defined as follows:

$$d(A, B) = \begin{cases} 0 & \text{if } A = B \\ 1 & \text{if } B = A^c, A \in \wp([2]) \\ 0.5 & \text{otherwise.} \end{cases}$$

It satisfies Properties G1*, G2, D1, D2 and D3*, but it does not satisfy the equality $d(A, B) = d(A \setminus B, B \setminus A)$, for every pair of subsets $A, B \in \mathcal{F}([2])$. In particular, if $A = \{1\}$ and $B = [2] = \{1, 2\}$ (crisp sets) then $d(A, B) = 0.5$ and $d(A \setminus B, B \setminus A) = 1$.

Example 11. This example considers a universe of cardinal $n = 2$. It illustrates the fact that [[2], Definition 14] does not either imply [[2], Definition 12] or [[2], Definition 13] for the particular case $n = 4$, because Property D3$^{(2)}$ is not satisfied. An alternative example would be needed in order to prove that [[2], Definition 14] does not imply either Definition $12^{(n)}$ or Definition $13^{(n)}$, for any $n \neq 2$. Let us consider $d : \cup_{n \in \mathbb{N}} \mathcal{F}([n]) \times \mathcal{F}([n]) \to \mathbb{R}$ defined as follows:

$$d(A, B) = \#(A \setminus B) \vee \#(B \setminus A), \ \forall A, B \in \wp([n]), \ \forall n \in \mathbb{N}$$

and

$$d(A, B) = d(C_A, C_B), \ \forall A, B \in \mathcal{F}(U),$$

where C_A denotes the nearest crisp set of A, (i.e. $C_A = \{i \in [n] : \mu_A(i) \geq 0.5\}$). This mapping satisfies all the properties included in [[2], Definition 14] for every $n \in \mathbb{N}$ (Properties G2, D3, D4 and D5), but it does not satisfy Property D2 for any $n \geq 2$ ($d(D, D^c)$ does not attain its maximum value for any $D \in \wp([n])$ satisfying the inequalities $0 < \#D < n$). Therefore d does not satisfy either Definition $12^{(n)}$ or Definition $13^{(n)}$ for any $n \neq 2$.

Example 12. This example considers a universe of cardinal $n = 1$, but it can be easily adapted to any finite universe. Let us consider an arbitrary $n \in \mathbb{N}$, and let the fuzzy measure M be defined as $M(A) = \mu_A(1), \ \forall A \in \mathcal{F}(U)$. Let us consider the mapping $F : \mathbb{R}^3 \to \mathbb{R}$ defined as $F(u, v, w) = \frac{v+w}{2}$. The mapping $d : \cup_{n \in \mathbb{N}} \mathcal{F}([n]) \times \mathcal{F}([n]) \to \mathbb{R}$ defined as $d(A, B) = F(M(A \cap B), M(A \setminus B), M(B \setminus A)) = \frac{\mu_{A \setminus B}(1) + \mu_{B \setminus A}(1)}{2}$ satisfies all the properties included in [[2], Definition 11] (It satisfies Properties G1 and D3, and furthermore the mapping F does not depend on the first argument, and is increasing wrt the second and the third arguments). However, it does not satisfy Axiom D1, for any $n \in \mathbb{N}$. If we consider a triple of nested sets $A \subseteq B \subseteq C$ satisfying $\mu_A(1) = 0.2$, $\mu_B(1) = 0.8$, $\mu_C(1) = 0.9$ and $\mu_A(i) \leq \mu_B(i) \leq \mu_C(i)$, for every $i \geq 2$, we observe that $d(A, C) = 0.45 < d(A, B) = 0.5$, if we consider the set-difference $A \setminus B = A \cap B^c$.

5. Analysis of implications and counterexamples included in [3]

We analyse in [3] the existence of additional implications between the different axiomatic definitions, under the additional assumption G3*. The same as in [2], all the results included there correspond to the situation (b) described in Section 3, and all the counterexamples correspond to Case A. Notwithstanding, most of them can be easily adapted respectively to situations (a) and C. In this section we will focus on those results and examples that need an additional adjustment, in order to adapt them to those situations.

Theorem 2, part 12. In [3], we prove that any mapping $s : \cup_{n\in\mathbb{N}}\mathcal{F}([n]) \times \mathcal{F}([n]) \to \mathbb{R}$ fulfilling property G3* satisfies S1 if and only if it satisfies S1*. Now we want to prove a stronger statement: Under the assumption $\mathrm{G3}^{(n)}$, $\mathrm{S1}^{(n)}$ and $\mathrm{S1}^{*(n)}$ are also equivalent, for every $n \geq 2$. Taking into account that $\mathrm{S1}^{*(n)}$ straightforwardly implies $\mathrm{S1}^{(n)}$, we just need to prove the reverse implication, $\mathrm{S1}^{(n)} \Rightarrow \mathrm{S1}^{*(n)}$, for every $n \in \mathbb{N}$. Let us fix an arbitrary $n \geq 2$. According to property $\mathrm{G3}^{*(n)}$, $s : \mathcal{F}([n]) \times \mathcal{F}([n]) \to \mathbb{R}$ is assumed to be decomposed as a sum in terms of a generating function h, as follows:

$$s(A, B) = \sum_{i\in[n]} h(\mu_A(i), \mu_B(i)),\ \forall\, A, B \in \mathcal{F}([n]).$$

In order to prove that s satisfies $\mathrm{S1}^{*(n)}$, and according to Theorem 2, part 1 in [3], we just need to prove that h satisfies, for every triple $a, b, c, \in [0, 1]$ with $a \leq b \leq c$ then $h(a, b) \geq h(a, c)$. In order to do so, let us consider the triple of fuzzy subsets $A, B, C \in \mathcal{F}([n])$ defined as follows: $\mu_A(1) = a$, $\mu_B(1) = b$, $\mu_C(1) = c$, and $\mu_A(i) = \mu_B(i) = \mu_C(i) = c$, $\forall\, i > 1$. According to $\mathrm{S1}^{(n)}$, $h(a, b) + (n - 1)h(c, c) \geq h(a, c) + (n - 1)h(c, c)$, which implies that $h(a, b) \geq h(a, c)$. $\square$

Theorem 2, part 13. We have proved in [3] that Properties G3* and S5* are mutually exclusive. Now we want to prove a stronger statement saying that that Properties $\mathrm{G3}^{*(n)}$ and $\mathrm{S5}^{*(n)}$ are incompatible for every $n \in \mathbb{N}$. We will prove it by *reductio ab absurdum*, following a similar course of reasoning as there. Let us fix an arbitrary $n \in \mathbb{N}$ and let us consider the pair of tuples (A, B, C, D) and (A', B', C', D') of fuzzy subsets of $[n]$ respectively defined as follows:

$$\begin{array}{llll} \mu_A(1) = 0.1 & \mu_B(1) = 0.2 & \mu_C(1) = 0.4 & \mu_D(1) = 0.5 \\ \mu_{A'}(1) = 0.1 & \mu_{B'}(1) = 0.2 & \mu_{C'}(1) = 0.4 & \mu_{D'}(1) = 0.5 \end{array}$$

$$\mu_A(i) = \mu_B(i) = \mu_C(i) = \mu_D(i) = 0.5 \text{ and}$$

$$\mu_{A'}(i) = \mu_{B'}(i) = \mu_{C'}(i) = \mu_{D'}(i) = 0.1, \forall\, i \geq 2.$$

Then, according to S5* we get the following expressions:

$$h(0.1, 0.2) + (n-1)h(0.5, 0.5) = h(0.4, 0.5) + (n-1)h(0.5, 0.5) \text{ and}$$

$$h(0.1, 0.2) + (n-1)h(0.1, 0.1) < h(0.4, 0.5) + (n-1)h(0.1, 0.1),$$

leading us to a contradiction, as the mapping h cannot simultaneously satisfy the equality $h(0.1, 0.2) = h(0.4, 0.5)$ and the strict inequality $h(0.1, 0.2) < h(0.4, 0.5)$. $\square$

Example 9. This example illustrates the fact that measures of similitude do not necessarily satisfy Property G2 (symmetry), as they do not necessarily satisfy it for the particular case $n = 1$. The same measure of similitude considered there illustrates the fact that there exists a mapping $s : \cup_{n \in \mathbb{N}} \mathcal{F}([n] \times \mathcal{F}([n]) \to \mathbb{R}$ satisfying the properties of measures of similitude for every $n \in \mathbb{N}$ that do not necessarily satisfy Property G2$^{(n)}$ for any $n \in \mathbb{N}$. In order to check it, instead of restricting to the case $n = 1$, we can consider an arbitrary $n \in \mathbb{N}$ and the pair of fuzzy subsets of $[n]$, A, B, defined as follows:

$\mu_A(1) = 0.3$, $\mu_B(1) = 0.7$, $\mu_A(i) = 0$ and $\mu_B(i) = 1$, $\forall\, i \geq 2$.

We observe that $s(A, B) = 0.45 \neq 0.65 = s(B, A)$, and therefore we conclude that s does not satisfy the property of symmetry G2$^{(n)}$.

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