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A structural analysis and classification of multivalued fuzzy sets

Análisis estructural y clasificación de los conjuntos difusos multivaluados

Tesis Doctoral

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RESUMEN (en español)

La teoría de conjuntos difusos, tal como la formuló originalmente L. A. Zadeh, es una generalización de la teoría de conjuntos en la que se sustituye la función característica de un conjunto, que asigna a los miembros el valor 1 y a los no miembros 0, por una función más general que asigna a los elementos valores en el intervalo unitario real [0, 1]. Esto hace posibles formas de pertenencia que no son totalmente *verdaderas* ni *falsas*, sino algo *difuso*, borroso o indeterminado situado en algún punto entre los dos valores ideales. El hecho de que estas magnitudes difusas de pertenencia sean de por sí números reales con un valor exacto, no difuso, ha motivado numerosas extensiones del concepto en las que se sustituyen estos números reales por otros objetos matemáticos más sofisticados. En esta memoria, se repasan los resultados más básicos de algunas de estas extensiones obteniéndose, además, resultados matemáticos nuevos que contribuyen a establecer conexiones entre estas teorías, así como con los conjuntos difusos normales de la teoría original.

En particular, se investigan las propiedades de aquellas extensiones de conjuntos difusos en las que la pertenencia viene representada mediante una colección de valores reales posibles, tales como los multiconjuntos difusos y los conjuntos difusos basados en conjuntos, así como el caso menos estudiado de una variante de los multiconjuntos difusos para la que los valores de pertenencia son n-uplas ordenadas en lugar de multiconjuntos. A este tipo de generalizaciones se las designará como “conjuntos difusos multivaluados”. El planteamiento elegido para la búsqueda de conexiones entre estas teorías diversas se basará en establecer varios principios de extensión fundamentales que vinculan conceptos análogos entre diferentes espacios matemáticos. Estos principios de extensión permitirán construir un marco teórico dentro del cual las funciones definidas en el mundo de los conjuntos difusos normales puedan extenderse a espacios difusos más elaborados, recuperando así resultados conocidos en la literatura existente junto con nuevos resultados que juzgamos interesantes para una mejor comprensión de las relaciones existentes entre los diversos conjuntos difusos multivaluados.

La última parte de la memoria está dedicada al estudio de funciones tales como distancias, disimilitudes y divergencias, que proporcionan una medida de cuán diferentes son dos conjuntos difusos, en cualquiera de sus formas. Los principios de extensión presentados en esta memoria conducen a definiciones de estos conceptos para los conjuntos difusos multivaluados, que se comparan con los existentes. La cuantificación de las diferencias entre conjuntos difusos es importante en muchos usos

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prácticos de los conjuntos difusos, por lo que se concluye nuestro estudio con algunos ejemplos de aplicaciones prácticas en los campos del reconocimiento de patrones y de la toma de decisiones, que abren vías de investigación futura para estas técnicas matemáticas.

RESUMEN (en Inglés)

The theory of fuzzy sets, as originally formulated by L. A. Zadeh, is a generalisation of set theory where the characteristic function of a set, which maps members to 1 and non-members to 0, is replaced with a more general function that maps elements into the real unit interval [0, 1]. This allows for membership values that are not quite *true* or *false* but something *fuzzy*, or indeterminate, that sits somewhere between the two ideal values. The fact that these fuzzy membership quantities are themselves real numbers with a precise numeric value has led to a number of extensions of the concept where the real numbers are replaced with more sophisticated mathematical objects. In this dissertation, we sum up the most basic results of some of the more successful of these extensions and arrive at some new mathematical results that help to establish a connection linking these extended theories both among themselves and with the original, or ordinary, fuzzy set theory.

In particular, we will explore the properties of those extended fuzzy sets for which membership can be represented through a collection of possible real values, such as fuzzy multisets and set-based fuzzy sets as well as a less commonly studied case of a variation of fuzzy multisets where membership values are ordered n-tuples rather than multisets. We will refer to extensions of this kind as ‘multivalued fuzzy sets’. Our approach to finding connections between these different theories will be based on establishing some basic extension principles as intuitive mechanisms that map analogous concepts between different mathematical spaces. Such extension principles will allow us to construct a theoretical framework where functions defined in the world of ordinary fuzzy sets can be extended to the more elaborate fuzzy spaces, thereby revisiting well-known results in the existing literature as well as new results that we deem interesting to gain a better understanding of the existing relations among the various multivalued fuzzy sets.

A final chapter is devoted to the study of functions such as distances, dissimilarities and divergences, which provide a measurement of how different two fuzzy sets, in any of their forms, can be. The extension principles introduced by this dissertation lead to definitions of these concepts for multivalued fuzzy sets, which are compared with existing ones. Quantifying the difference between fuzzy sets is important in many practical use cases of fuzzy sets, and we therefore conclude our study with a few examples of some possible practical applications in the fields of pattern recognition and decision-making, which open up the way for further research of these mathematical techniques.

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Agradecimientos

Cuando en el año 2015 decidí embarcarme en un proyecto tan ambicioso como son los estudios de doctorado, era consciente de enfrentarme a un reto que sin duda implicaría sacrificios y renuncias importantes, tanto en un plano profesional como en el estrictamente personal. Y han sido, en efecto, cuatro años de trabajo muy intenso durante los cuales he tenido la inmensa fortuna de sentirme acompañado y apoyado por las personas más cercanas a mí. El principal apoyo, sin duda, ha sido el de Arancha, compañera de aventuras y de sueños, cuya presencia marca y da sentido a todo lo que hago. Junto a Arancha, tengo que agradecer también su ánimo y su apoyo a las las otras dos mujeres más importantes de mi vida, mi hermana, Ana, y mi madre, Marina.

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Resumen

La teoría de conjuntos difusos, tal como la formuló originalmente L. A. Zadeh, es una generalización de la teoría de conjuntos en la que se sustituye la función característica de un conjunto, que asigna a los miembros el valor 1 y a los no miembros 0, por una función más general que asigna a los elementos valores en el intervalo unitario real $[0, 1]$. Esto hace posibles formas de pertenencia que no son totalmente *verdaderas* ni *falsas*, sino algo *difuso*, borroso o indeterminado situado en algún punto entre los dos valores ideales. El hecho de que estas magnitudes difusas de pertenencia sean de por sí números reales con un valor exacto, no difuso, ha motivado numerosas extensiones del concepto en las que se sustituyen estos números reales por otros objetos matemáticos más sofisticados. En esta memoria, se repasan los resultados más básicos de algunas de estas extensiones obteniéndose, además, resultados matemáticos nuevos que contribuyen a establecer conexiones entre estas teorías, así como con los conjuntos difusos normales de la teoría original.

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Abstract

The theory of fuzzy sets, as originally formulated by L. A. Zadeh, is a generalisation of set theory where the characteristic function of a set, which maps members to 1 and non-members to 0, is replaced with a more general function that maps elements into the real unit interval $[0, 1]$. This allows for membership values that are not quite *true* or *false* but something *fuzzy*, or indeterminate, that sits somewhere between the two ideal values. The fact that these fuzzy membership quantities are themselves real numbers with a precise numeric value has led to a number of extensions of the concept where the real numbers are replaced with more sophisticated mathematical objects. In this dissertation, we sum up the most basic results of some of the more successful of these extensions and arrive at some new mathematical results that help to establish a connection linking these extended theories both among themselves and with the original, or ordinary, fuzzy set theory.

In particular, we will explore the properties of those extended fuzzy sets for which membership can be represented through a collection of possible real values, such as fuzzy multisets and set-based fuzzy sets as well as a less commonly studied case of a variation of fuzzy multisets where membership values are ordered n-tuples rather than multisets. We will refer to extensions of this kind as "multivalued fuzzy sets". Our approach to finding connections between these different theories will be based on establishing some basic extension principles as intuitive mechanisms that map analogous concepts between different mathematical spaces. Such extension principles will allow us to construct a theoretical framework where functions defined in the world of ordinary fuzzy sets can be extended to the more elaborate fuzzy spaces, thereby revisiting well-known results in the existing literature as well as new results that we deem interesting to gain a better understanding of the existing relations among the various multivalued fuzzy sets.

A final chapter is devoted to the study of functions such as distances, dissimilarities and divergences, which provide a measurement of how different two fuzzy sets, in any of their forms, can be. The extension principles introduced by this dissertation lead to definitions of these concepts for multivalued fuzzy sets, which are compared with existing ones. Quantifying the difference between fuzzy sets is important in many practical use cases of fuzzy sets, and we therefore conclude our study with a few examples of some possible practical applications in the fields of pattern recognition and decision-making, which open up the way for further research of these mathematical techniques.

1 Introduction

The theory of fuzzy sets was originally proposed by Lotfi A. Zadeh in an article published in 1965 [50]. In that seminal article, Zadeh pointed out the unsuitability of the set-theoretical concept of subsets in cases such as the classification of a group of individuals into two categories "young" and "old". Such assignment into subsets differs, in an essential way, from the classification of individuals into clear-cut groups such as "dead" and "alive". Examples of groupings where membership is semantically subject to gradation are particularly common in the social sciences, where imprecise or idiosyncratic linguistic labels often need to be accounted for. In such cases, a better mathematical model would be one where elements could be labelled as "very young", "rather young" or "somewhat old". The key to this extension of the notion of subset lies in the **characteristic function** of a set. In set theory, given a reference set X, any subset $A \in X$ can be characterised by a function $\chi_A \colon X \to \{0, 1\}$ that maps each element x in X to either 1 or 0, depending on whether x is in A or not. This identification between subsets and characteristic functions can be exploited in order to define more subtle non-Boolean forms of membership. In Zadeh's original formulation, the characteristic function is replaced by a membership function $\mu_A \colon X \to [0, 1]$, where the use of the unit interval makes it possible to define a person as having a membership of, say, 0.3 in the "young" set. This variation on the concept of a set is called a **fuzzy set**. The family of all fuzzy sets over a universe X can formally be identified with the space of functions $[0, 1]^X$ (the family of functions $X \to [0, 1]$), whereas the traditional sets of set theory can be identified with the space of functions $\mathbf{2}^X$ (the functions $X \to \mathbf{2}$, with $\mathbf{2}$ standing for any set of cardinality 2 such as $\{0, 1\}$), and are referred to as **crisp sets** within this broader context of fuzzy set theory.

In the older literature about fuzzy sets, a distinction was made between the concept of a fuzzy set, typically defined by including the reference set (also called the **universe** or **universe of discourse**) and its **membership function**. In the recent literature, however, it has become common to fully identify the notions of "fuzzy set" and "membership function" so that a fuzzy set is defined *to be* a function. In this dissertation, we will follow this approach and define a fuzzy set A as a function rather than as a more elaborate object characterised by a function μ_A. This identification considerably simplifies some of the definitions and formalism.

Once a fuzzy set has been defined to be a function mapping the elements of the universe to the unit interval, operations affecting fuzzy sets need to be defined so that they reduce to the common set-theoretical operations like **complement**, **intersection** and **union** in the crisp set limit case. The first definitions of complement, intersection and union were due to Zadeh [50] and are equivalent to the normal ones for crisp sets when membership is restricted to the crisp values $\{0, 1\}$. Zadeh's original operations are still in common use and are referred to as the "standard" operations, although further generalisations have been defined. Note that while many of the relations that hold for crisp sets are preserved with these and other definitions, some important properties are not fulfilled, like the Law of Contradiction and the Law of Excluded Middle and non-standard operations sacrifice even more properties [20]. Because of this, fuzzy logic systems based on fuzzy sets flout some rules of traditional logic systems.

During the early years of fuzzy set theory, some researchers, including Zadeh himself, made an interesting observation: just as shoehorning elements into a Boolean membership value was a restrictive consequence of the use of crisp sets that could be overcome with the refinement provided by fuzzy sets, the fact that membership was represented by an exact real number such as 0.25 would still fail to qualify as a good characterisation of fuzziness (after all, why not 0.249 or 0.251?). Elaborating on this idea, further "fuzzier" notions of sets were proposed with more sophisticated membership values than those of the unit interval $[0, 1]$. For example, membership values can be given a certain tolerance around a central value by treating them as intervals $[a, b]$. This idea gave rise to the theory of **interval-valued fuzzy sets** [20, 51]. A further refinement consists in **type-2 fuzzy sets**, where the membership values are functions from a subset of the unit interval into the unit interval; i.e. fuzzy sets, in the original sense, over the $[0, 1]$ interval [20, 51]. In their most common form, the type-2 fuzzy sets allow modelling membership as a smooth curve that reaches a maximum around a central value and fades away as the distance to that core membership value increases. Note that the interval-based fuzzy sets include the original fuzzy sets as a particular case (those where the membership intervals are restricted to the degenerate form $[a, a]$) and the type-2 fuzzy sets include the interval-valued fuzzy sets as a particular case where the membership value is given by a boxcar function taking on the values 0 and 1 only. In the context of this diversity of definitions of fuzzy sets, the original concept proposed by Zadeh is usually referred to as **ordinary fuzzy sets** or **type-1 fuzzy sets**.

These generalisations that build on top of one another thus give rise to a beautiful formal picture where the simpler theories fit in as limit cases, and has been the subject of ample attention by researchers over the last fifty-odd years. In fact, many more similar generalisations have been put forward. A very elegant framework for many of the fuzzy set theories was proposed by J. A. Goguen in 1969 in an influential article [13], where **L-fuzzy sets** were defined to be functions from a universe X into an arbitrary set L. Obviously, the ordinary, interval-based and type-2 fuzzy sets are all L-fuzzy sets with the appropriate choice of L. In general, for the L set to be a sensible choice to represent membership grades that can be compared and ordered, Goguen concluded that it should be a completely distributice lattice and, as such, endowed with **meet** and **join** operations that are to be re-interpreted as the fuzzy "intersection" and "union". Not all fuzzy extensions can be identified with a lattice, though. In the more formal theoretical presentations of fuzzy sets, it is common to present each fuzzy set theory as an "algebra" made up of six elements $(L, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$. C. Walker and E. Walker have proved that the operations involved in the ordinary fuzzy sets form the mathematical structure known as a **Kleene algebra** [45]. Other forms of fuzzy sets can be studied by analysing their mathematical properties, and establishing the type of mathematical structure that their algebra matches.

This possibility of crafting fuzzy set extensions by choosing new sets of membership values has been a very fertile ground indeed for researchers, which has led to an abundance of extensions of the original concept. In a recent article, H. Bustince et al. mention more than 20 generalisations of fuzzy sets [6]. Not surprisingly, this ridiculously high number of generalisations has attracted a lot of criticism and scepticism regarding the usefulness of many of these proposals. Besides, many of these generalisations can be shown to be equivalent [19]. In this dissertation, we focus our attention on a particular group of generalisations that share the characteristic of their membership values being composed of several individual values taken together. The first generalisation of this kind was that of **set-valued fuzzy sets**, proposed by I. Grattan-Guinness in 1976 [14]. Two further similar generalisations are **fuzzy multisets**, first put forward by

R. Yager [49] in 1986 and further developed by S. Miyamoto in 2000 [30], and **hesitant fuzzy sets**, proposed by V. Torra in 2010 [44]. We will also consider the **ordered fuzzy multisets**, of a similar nature, that we have introduced in a recent conference paper [36]. These generalisations will be the main focus of our study and we shall refer to all of them collectively as **multivalued fuzzy sets**.

A common problem in practical applications of fuzzy set theory involves quantifying how different two fuzzy sets are. There have been many proposals of functions mapping a pair of fuzzy sets into a real number that represents their difference. Among the existing techniques, we have focused our attention on the **divergence measures** originally defined by S. Montes et al. [34]. In this dissertation, we will extend some of the existing results for divergence measures over ordinary fuzzy sets to the multivalued fuzzy sets and compare our results with some of those found in the existing literature.

The main objective of our research has been to develop a general framework for the multivalued generalisations of fuzzy sets where they appear as particular cases under certain choices of parameters. We have also established a number of extension principles that provide mappings between the different mathematical spaces of the various multivalued extensions. We believe that this approach can lead to a better understanding of the relations among different kinds of extensions of the fuzzy set concept. Finally, by applying these extension principles to the divergence measures for ordinary fuzzy sets, we will describe a family of divergence measures that can be applied to practical uses of fuzzy techniques.

This dissertation is organised as follows. After the introduction in this chapter, Chapter 2 presents the preliminary concepts that the reader should be familiar with before reading the subsequent chapters and spells out the terminological and notational conventions that will be followed throughout the text. Chapter 3 is a study of the relations between sets, multisets and ordered n-tuples as the three basic ways of arranging a collection of elements from a reference set. Chapter 4 sets forth the formalism for L-fuzzy sets that will be used throughout the rest of the dissertation and introduces the set-based and multiset-based types of fuzzy sets based on the relations between sets and multisets established in the preceding chapter. Chapter 5 has a similar structure, this time describing the main properties of ordered fuzzy multisets, which use n-tuples as membership grades, and the relations that exist between this type of fuzzy set with the fuzzy multisets introduced in the previous chapter. In Chapter 6, we approach the relations between the various types of multivalued fuzzy sets and their membership grades from a different angle, by focusing on the difference in information content that underlies the presence v. the absence of repetition and order. We consider this both for the membership grades, in the first half of the chapter, based on the concepts of Chapter 3, and in the three types of multivalued fuzzy sets in the second half. We present some entropy measures that quantify the information loss when repetition or order are ignored. In Chapter 7, we extend the notions of distance, dissimilarity and divergence, which are first introduced in Chapter 2, to the multivalued fuzzy sets that were discussed in chapters 4 and 5 and present some examples of practical applications within the fields of pattern recognition and decision-making. Finally, in the Conclusions we sum up the main results and mention some promising areas for further research.

2 Preliminary concepts

In this chapter, we introduce the basic mathematical concepts that are required as preliminary knowledge for our research, and establish the notational and terminological conventions that we will adhere to throughout this work.

2.1 The characteristic function and the powerset

Before embarking on the details of fuzzy theory, we start off by running through some basic definitions related to sets that will subsequently be used as the starting point for the introduction of fuzzy sets.

The subsets of a set can be characterised by a Boolean-valued function that maps the elements in the set to 1 and the elements not in the set to 0. The formal definition is as follows.

Definition 2.1.1. Given a set U and a subset $A \subset U$, the **characteristic function** (or **indicator function**) of A relative to U is a function $\chi_A\colon U \to \{0,1\}$ defined as:

$$\chi_A(u) := \begin{cases} 1, & \text{if } u \in A \\ 0, & \text{if } u \notin A \end{cases} \tag{2.1}$$

A characteristic function can be defined in this way for any subset of U, and it is trivial to prove that two subsets of U are different if and only if their characteristic functions are different. Thus, subsets can be characterised by such functions, which lends justification to the use of the term "characteristic". This identification between sets and functions is also the basis for some extensions of the concept of a set, like the multisets and the fuzzy sets that we shall discuss later, in which the set $\{0,1\}$ is replaced with larger-cardinality codomains.

From now on, we will usually denote the set $\{0,1\}$ more simply as $\mathbf{2}$, a notational convention justified by the fact that a natural number can be identified with sets using J. Von Neumann's set-theoretical construction of natural numbers [11].

Definition 2.1.2. For any set U, the set containing all the subsets of U is called the **powerset** (also spelled **power set**) of U and is represented by $\mathbf{2}^U$.

The identification between subsets and characteristic functions implies that the set comprising all the subsets of U can be likewise identified with the set of all the functions $U \to \{0,1\}$, which justifies the notation $\mathbf{2}^U$ [1], based on the conventional notation Y^X for the set of all functions $X \to Y$ between any two sets X and Y, which will be used throughout this dissertation. A very common alternative notation for the powerset of U is $\mathcal{P}(U)$, which we will not use here.

In these definitions and results that affect one set U, we will refer to this reference set U as the **universe of discourse**, or simply the **universe**.

2.2 Ordinary fuzzy sets

As the most basic concept in fuzzy set theory, we start this review of preliminary concepts with the definition that extends the traditional notion of a set by modifying the nature of its characteristic function.

Definition 2.2.1. Given a universe X, an **ordinary fuzzy set** over X is a function $A\colon X \to \mathbb{I}$, with $\mathbb{I}$ being the real closed unit interval $[0,1]$.

The image of an element $x \in X$ by a fuzzy set A, $A(x)$, is called the **membership grade** of x by A.

Mainly for notational convenience, it will also be useful to give a name to the family of all the ordinary fuzzy sets over a universe, as follows.

Definition 2.2.2. Given a universe X, the family of all the ordinary fuzzy sets over X is called the **ordinary fuzzy powerset** over X, which is denoted by $\mathbb{I}^X$.

A common alternative notation for the ordinary fuzzy powerset of X is $\mathscr{F}(X)$.

When the image of a fuzzy set is made up of the two values $\{0,1\}$ only, we have a particular case of fuzzy set that can be identified with a set in the traditional sense of set theory, and will be referred to as a **crisp set**. The family of all the crisp sets over X is $\mathbf{2}^X$, a subset of $\mathbb{I}^X$, which can be identified with the powerset in the traditional "crisp" sense of Definition 2.1.2.

The set-theoretical concept of subsets can also be extended to the ordinary fuzzy sets through the following definition:

Definition 2.2.3. Let X be the universe and let A and B be two ordinary fuzzy sets on X. A is said to be a **fuzzy subset** of B, $A \subseteq B$, if the following condition holds:

$$A(x) \leq B(x) \quad \forall x \in X \tag{2.2}$$

This is an order relation, which by itself is called **inclusion** or **subsethood**. When A is a fuzzy subset of B, we can also say that B is a **fuzzy superset** of A, $B \supseteq A$. If $A \subseteq B$ and $A \neq B$, then we can refer to A as a **proper fuzzy subset** of B, $A \subset B$ (or, alternatively, to B as a **proper fuzzy superset** of A, $B \supset A$).

Inclusion thus defined is a Boolean relation that can only be either true or false. In the language of fuzzy set theory, such relations are also called **crisp relations** and it can be argued that they do not fit well within the spirit of fuzzy theory [7]. This criticism has led to various proposals of a fuzzy concept of subsethood [43], a field which goes beyond the scope of this dissertation.

Given any fuzzy set A, it can be turned into a crisp set by replacing its image values in $(0, 1)$ with either 0 or 1 based on a threshold value. This idea leads to the following definition.

Definition 2.2.4. Let X be the universe and A be an ordinary fuzzy set on X. For any $\alpha \in \mathbb{I}$, the α**-cut** of A is the crisp set A_α defined by:

$$A_\alpha := \{x \in X | A(x) \geq \alpha\} \tag{2.3}$$

Similarly, the **strong** α**-cut** of A is the crisp subset $A_\alpha^>$ defined by:

$$A_\alpha^> := \{x \in X | A(x) > \alpha\} \tag{2.4}$$

There are two important special cases of α-cuts. The strong 0-cut $A_0^>$ is called the **support** of A and the 1-cut A_1 is called the **core** of A.

The α-cuts provide a very useful connection between the fuzzy sets and the crisp sets that has led to a good deal of additional definitions and propositions in the fuzzy set literature. An important result is the fact that any fuzzy set A can be expressed in terms of its α-cuts as $A(x) = \sup\{\alpha \mid x \in A_\alpha\}$. Through this representation, many properties of fuzzy sets can be expressed in terms of their α-cuts.

Example 2.2.5. If we have a finite universe $X = \{x_1, x_2, x_3, x_4\}$, an example of an ordinary fuzzy set $A \in \mathbb{I}^X$ is the function $A(x_1) = 0$, $A(x_2) = 0.7$, $A(x_3) = 1$ and $A(x_4) = 0.2$. Its core is the crisp set $Core(A) = \{x_3\}$, its support $Supp(A) = \{x_2, x_3, x_4\}$ and its 0.5-cut is $A_{0.5} = \{x_2, x_3\}$.

The concept of cardinality can be extended to the ordinary fuzzy sets in a natural way.

Definition 2.2.6. Let A be an ordinary fuzzy set over a universe X. The **cardinality** of A is the real number defined as:

$$|A| := \sum_{x \in X} A(x)$$

In order to explore practical uses and results for fuzzy sets as a generalisation of sets, we also need to define the operations that extend the basic operations of classical set theory such as the complement, the intersection and the union.

Definition 2.2.7. Let X be the universe and let A be an ordinary fuzzy set on X. The **standard complement** A^c is defined as the following ordinary fuzzy set over X:

$$A^c(x) := 1 - A(x), \quad x \in X$$

This definition establishes an involution between A and A^c by composing the original fuzzy set with a function $c\colon \mathbb{I} \to \mathbb{I}$ defined as $c(x) := 1 - x$. By choosing alternative definitions for the c function (called a **negation** or **complement function**, which is subject to the conditions that it should be an involution $c(c(x)) = x$ and that $c(0) = 1$ and $c(1) = 0$, so that it preserves the behaviour of the complement for crisp sets), many different complements can be defined, such as the **Sugeno's family of complements**Sugeno and **Yager's family of complements**Yager families described by Klir and Yuan [20, pp. 54-57].

Definition 2.2.8. Let X be the universe and let A and B be two ordinary fuzzy sets on X. The **standard intersection** $A \cap B$ is defined as the following ordinary fuzzy set over X:

$$(A \cap B)(x) := \min(A(x), B(x)), \quad x \in X$$

Definition 2.2.9. Let X be the universe and let A and B be two ordinary fuzzy sets on X. The **standard union** $A \cup B$ is defined as the following ordinary fuzzy set over X:

$$(A \cup B)(x) := \max(A(x), B(x)), \quad x \in X$$

It is straightforward to check that these definitions reduce to the corresponding operations for crisp sets. As was the case with the complement, they are not the only generalisations possible, though. Functions other than min and max can be used as alternative definitions [20, p. 50]. This leads to a variety of possible fuzzy intersections (defined in terms of $\mathbb{I} \times \mathbb{I} \to \mathbb{I}$ functions called **triangular norms**) and possible fuzzy unions (defined in terms of similar functions called **triangular conorms**). The standard ones just happen to be the best-behaved cases of the more general operations. In the next section, we will introduce these operations, as they will be used in the course of this dissertation.

2.3 Aggregation operations

In the previous section, we have briefly mentioned how the max and min operations used in the definition of intersection and union can be generalised to two families of operations called "triangular norms", or **t-norms** for short, and "triangular conorms", or **t-conorms**. In this section we state the conditions that these operations must fulfil in order to respect the expected behaviour of the intersection and union, respectively. We will see that they can be regarded as particular cases of what we will call, following Klir and Yuan, **aggregation operations** [20, p. 88].

Let us define the t-norms first.

Definition 2.3.1. A function $t\colon \mathbb{I}^2 \to \mathbb{I}$ is called a **triangular norm** or **t-norm** if it satisfies the following four conditions for all $a, b, c \in \mathbb{I}$ [20, p. 62]:

1. $t(a, 1) = a$ (boundary condition)
2. $b \leq c \implies t(a, b) \leq t(a, c)$ (monotonicity)
3. $t(a, b) = t(b, a)$ (commutativity)
4. $t(a, t(b, c)) = t(t(a, b), c)$ (associativity)

These four conditions are the minimum requirements for a function to be called a t-norm and hence valid for a working definition of fuzzy intersection.

Three common additional requirements are continuity, subidempotency ($t(a,a) \leq a$) and strict monotonicity ($b < c \implies t(a,b) < t(a,c)$). The min function obviously satisfies all these requirements.

A t-conorm is defined formally in a completely analogous way:

Definition 2.3.2. A function $s\colon \mathbb{I}^2 \to \mathbb{I}$ is a **triangular conorm** or **t-conorm** or **s-norm** if it satisfies the following four conditions for all $a, b, c \in \mathbb{I}$ [20, p. 77]:

1. $s(a, 0) = a$ (boundary condition)
2. $b \leq c \implies s(a,b) \leq s(a,c)$ (monotonicity)
3. $s(a,b) = s(b,a)$ (commutativity)
4. $s(a, s(b,c)) = s(s(a,b), c)$ (associativity)

Three common additional requirements for good mathematical behaviour are continuity, superidempotency ($s(a,a) \geq a$) and strict monotonicity ($b < c \implies s(a,b) < s(a,c)$). The max function obviously satisfies all these requirements.

It can be proved [20, pp. 63, 77] that the min and max functions are the only t-norm and t-conorm that fulfil the more restrictive property of idempotency ($t(a,a) = a$). But if associativity and commutativity are sacrificed, a more general form of operation can be defined in the unit interval $\mathbb{I}$, which we will call "aggregation":

Definition 2.3.3. A function $A\colon \mathbb{I}^n \to \mathbb{I}$ is an **aggregation operation** if it satisfies the following three conditions [20]:

1. $A(0, ..., 0) = 0$ (lower boundary condition)
2. $A(1, ..., 1) = 1$ (upper boundary condition)
3. $u_i \leq v_i (i = 1, ..., n) \implies A(u_1, ..., u_n) \leq A(v_1, ..., v_n)$ (monotonicity)

Two additional axiomatic conditions are usually assumed: symmetry in the arguments, the n-valued equivalent of commutativity ($A(u_1, ..., u_n) = A(u_{\sigma(1)}, ..., u_{\sigma(n)})$ for any index permutation σ) and continuity [20, p. 89]. The monotonicity condition is sometimes required in the stronger form of **strict monotonicity**, where the third condition also has $A(u_1, ..., u_n) < f(v_1, ..., v_n)$ if $u_i \leq v_i$ and there is at least an index j for which the values are different: $u_j < v_j$.

The definition of any t-norm or t-conorm can be extended to n variables without ambiguity thanks to the associativity property and it trivially satisfies the conditions of an aggregation operation, so the family of aggregation operations is more general than the particular cases of t-norms and t-conorms. Continuity is often added as a requirement and we will assume that any t-norm, t-conorm or general aggregation operation is continuous.

We have seen that the max and min operators are the only idempotent t-norm and t-conorm, but it can be proved easily that an aggregation operation is idempotent if and only if it is bounded by the minimum and the maximum [20, p. 89]. An idempotent aggregation operation is called an **averaging operation**. An example of an averaging operation that is neither the max t-norm nor the min t-conorm is the arithmetic mean, which is not associative. Aggregation operations that are associative and commutative are called **norm operations** and include the t-norms and t-conorms as particular cases. A norm operation that is neither a t-norm nor a t-conorm is called an **associative averaging operation** [20].

With these definitions and results, a picture emerges where the aggregation operations form a continuum of related operations with the t-norms at one end, the t-conorms at the other end and the averaging operations in between. A comprehensive account of aggregation can be found in the book by G. Beliakov et al. [3].

2.4 Some generalisations of fuzzy sets

A problem with the original fuzzy sets, the only ones we have introduced so far, is that membership grades are real numbers which are impossible to determine with total accuracy, as I. Grattan-Guninness has agued [14]. Why should somebody be 0.3 tall and not 0.29 or 0.31 tall, for example? In actual use cases, any determination of membership grades is typically an approximation which involves a certain degree of arbitrariness and the use of exact real numbers as membership grades somehow clashes with the very idea of fuzziness.

In order to address this problem, many generalised forms of fuzzy sets have been proposed which extend Definition 2.2.1 through the use of more sophisticated sets of membership grades than the unit interval $\mathbb{I}$. A very simple such refinement, first formally proposed in 1975 by several authors independently [16] (and Zadeh had already mentioned "interval-valued membership" in 1973 [51]), consists in using closed subintervals $[a_l, a_u]$ of the unit interval ($0 \leq a_l \leq a_u \leq 1$) as membership grades. If we denote the set of such closed intervals as $\mathbb{I}^{[2]}$, then the family of these extended fuzzy sets over a universe X can be denoted by $(\mathbb{I}^{[2]})^X$, and they are nowadays referred to as **interval-valued fuzzy sets**. Those interval-valued fuzzy sets such that their membership grades are always degenerate intervals of zero length $[a, a]$ can be put in a one-to-one correspondence with the ordinary fuzzy sets, which justifies the use of the term "extension" for this alternative form of fuzzy sets.

An earlier, more general extension is provided by **type-2 fuzzy sets**, where the membership grade is an ordinary fuzzy set itself. As this is one of the most powerful and successful generalisations of the concept, we present its definition here.

Definition 2.4.1. Given a universe X, a **type-2 fuzzy set** over X is a function $A\colon X \to \mathbb{I}^{\mathbb{I}}$ [20].

In the context of type-2 fuzzy sets, ordinary fuzzy sets are usually referred to as **type-1 fuzzy sets**.

An alternative equivalent definition that is the most common one in the type-2 fuzzy set literature is based on a membership function that maps pairs (x, u), where $x \in X$ and $u \in J_x$ (with J_x being a subset of $\mathbb{I}$, often called the **primary membership**), into $\mathbb{I}$ (the **secondary membership**). In this well-established definition, some restrictions on how J_x is defined are often assumed [28]. Whatever the formal definition, a type-2 fuzzy set can be plotted as a three-dimensional graph and the α-cuts of the type-1 fuzzy sets become α-planes that lead to a **horizontal slice representation**, similar to the α-cut representation for the type-1 fuzzy sets. Other geometrical decompositions that have been found to be useful involve **vertical slices** and **wavy slices** [29]. The projection on the (x, u) plane of those points with a non-zero secondary membership grade determines a 2-dimensional area called the **footprint of uncertainty** (commonly abbreviated as **FOU**; it can also be called the **strong 0-plane** in α-plane terms).

Similarly to the way we have done it for the other cases, we can define the family of all such fuzzy sets.

Definition 2.4.2. The family of all the type-2 fuzzy sets over a universe X is called the **type-2 fuzzy powerset** over X, $(\mathbb{I}^{\mathbb{I}})^X$.

The ordinary fuzzy sets can be identified with the subset of type-2 fuzzy sets where the membership grade is given by a function f satisfying the conditions

$f(a) = 1$ for a single element $a \in X$ and $f(x) = 0$ for any other $x \neq a$. Similarly, the interval-valued fuzzy sets can be identified with those type-2 fuzzy sets where the membership grades are all boxcar functions mapping a subinterval of $\mathbb{I}$ to 1 and the remaining values to 0. Other forms of extended fuzzy sets that we will introduce later can also be interpreted as type-2 fuzzy sets with our general definition (but disregarding the convexity assumptions that are common in the literature on type-2 fuzzy sets).

If we now use type-2 fuzzy sets over $\mathbb{I}$ as membership grades, we can go on to define **type-3 fuzzy sets** in a similar way, and this approach can be extended *ad infinitum* to any arbitrarily large number. Such higher-type fuzzy sets are cumbersome and computationally prohibitive. Because of that, only type-2 fuzzy sets have really been used and investigated to any considerable extent.

But establishing an alternative space of membership grades is not enough for a function to be called a fuzzy set extension. It is also necessary to define the common operations of complement, intersection and union in a way that is compatible with those of the ordinary fuzzy sets. For example, for an interval-valued fuzzy set A such that $A(x) = [a_l, a_u]$ at a point x, the complement can be defined pointwise as $A^c(x) = [1 - a_u, 1 - a_l]$. Similarly, the intersection of two interval-valued fuzzy sets A and B such that $A(x) = [a_l, a_u]$ and $B(x) = [b_l, b_u]$ can be defined as $(A \cap B)(x) = [\min(a_l, b_l), \min(a_u, b_u)]$, and the union as $(A \cup B)(x) = [\max(a_l, b_l), \max(a_u, b_u)]$. These definitions are reasonably intuitive and match the ones for the ordinary fuzzy sets when degenerate intervals of the form $[a, a]$ are considered, which completes the picture of the interval-valued fuzzy sets as an extension fo the ordinary fuzzy sets.

Defining the operations for the type-2 fuzzy set is trickier. Zadeh himself brought up the idea of type-2 fuzzy sets first in 1973 [51] and sketched how to derive the operations from the extension principle that we shall introduce later in this chapter. M. Mizumoto and K. Tanaka later presented complete formal definitions for these operations [32], which we present (with our preferred notation) here.

Definition 2.4.3. Let X be the universe and let $A, B \in (\mathbb{I}^{\mathbb{I}})^X$. The **standard complement** of A, A^c, **intersection** of A and B, $A \cap B$, and **union** of A and B, $A \cup B$, are defined as follows:

$$
\begin{aligned}
A^c(x)(u) &:= A(x)(1-u), \quad x \in X, u \in \mathbb{I} \\
(A \cap B)(x)(u) &:= \sup_{(u_1,u_2) \mid u = \min(u_1,u_2)} \{\min(A(x)(u_1), B(x)(u_2))\}, \quad x \in X, u \in \mathbb{I}
\end{aligned}
$$

$$(A \cup B)(x)(u) := \sup_{(u_1,u_2)|u=\max(u_1,u_2)} \{\min(A(x)(u_1), B(x)(u_2))\}, \quad x \in X, u \in \mathbb{I}$$

It is easy to check that these definitions for the type-2 fuzzy sets lead to the standard definitions for the interval-valued and ordinary fuzzy sets as particular cases, which confirms the suitability of these definitions as an extension of the simpler fuzzy set types.

All these extensions of fuzzy sets are based on a definition that uses a map from the universe X into a set of possible membership grades. This points to a very broad generalisation where the set of membership grades is a parameter in the definition. Such a generalisation was already proposed by J. A. Goguen in the early days of fuzzy set theory [13]. In Goguen's approach, an **L-fuzzy set** is simply a function $L \to X$, where the set L is called the membership grade space. It is obvious that the fuzzy sets we have defined so far fit within this definition as particular cases with the appropriate choice of set L.

In the general form where the domain L can be any set, the definition of L-fuzzy sets would be quite useless, though. For the elements of L to display the expected semantic properties of membership grades, certain conditions on L will be necessary. It is expected that membership grades can be compared and that there must be a maximum membership and a minimum membership that stand for the 0 and 1 values of the characteristic function in the crisp set limit. Goguen posited that L would need to be at least a **complete lattice**, with some additional properties, like distributivity, being desirable. A **lattice** [4] is a partially ordered set where any two elements $l_1, l_2 \in L$ have a greatest lower bound, their **meet** $l_1 \wedge l_2$, and a lowest upper bound, their **join**, $l_1 \vee l_2$. A lattice is said to be **complete** if any subset, and not just any pair, also has a greatest lower bound and a lowest upper bound; a consequence of this being that a complete lattice is also **bounded** with a bottom element $\mathbf{0} := \bigwedge X$ and a top element $\mathbf{1} := \bigvee X$. The $\{0, 1\}$ set with the order $0 \leq 1$ is a complete lattice whose meet and join are the AND and OR logical operations. Similarly, the unit interval $\mathbb{I}$ with the ordinary $\leq$ order also has a complete lattice structure where the meet and join operations are given by the max and min operations. And the same happens with the set of subintervals of $\mathbb{I}$ with a partial order defined by $[a_l, a_u] \leq [b_l, b_u]$ if $a_l \leq b_l$ and $a_u \leq b_u$, which leads to the same definitions for the meet and join as the ones for the intersection and union of interval-valued fuzzy sets. This shows how the meet and join can be used as the definition for the intersection and union of an L-fuzzy set, generalising the role of the minimum and the maximum, provided that L has a lattice structure. If we add a unary operation of **complementation** or **negation**, in this more abstract case denoted by $\neg$, we can specify an L-fuzzy set fully as an algebraic structure made up of six elements $\{L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$, typically called the **algebra of truth values** [16] for the fuzzy set theory that uses L for its membership grades.

But the mathematical properties of the crisp and fuzzy sets get weaker as we proceed along this chain of generalisations. The crisp sets with the normal operations are not only a complete lattice, but a **Boolean algebra**, which satisfies stronger properties. The ordinary fuzzy sets are not a Boolean algebra because they fail to meet the laws of complementation ($x \wedge \neg x = \mathbf{0}$ and $x \vee \neg x = \mathbf{1}$, also called the laws of excluded middle and non-contradiction), and they form a weaker structure known as a **Kleene algebra**. The interval-valued fuzzy sets additionally fail to satisfy Kleene's inequality ($x \wedge \neg x \leq x \vee \neg x$) and form an even weaker structure known as a **De Morgan algebra** [16].

This progressive weakening of the mathematical structure continues with the type-2 fuzzy sets whose algebra is not even a lattice. It is a **bisemilattice** with two partial orders, the **meet order** and the **join order**, which can be expressed in terms of the expressions for the type-2 intersection and union as in Definition 2.4.3 [16, 46]. But when type-2 fuzzy sets are restricted to functions that are concave (using the terminology of calculus; often called "convex" in the fuzzy literature) and have 1 as a supremum, then a De Morgan algebra structure is recovered [16, 46]. This shows the importance of convexity assumptions in the fuzzy theories and should be borne in mind when investigating other extensions of the fuzzy sets, like the **hesitant fuzzy sets** that we will discuss in a later section and which display poor mathematical properties because of their intrinsic non-convexity.

For the general case of not necessarily convex type-2 fuzzy sets, it is interesting to note that a refinement of L-fuzzy sets where the membership grades are bisemilattices, **B-fuzzy sets**, was proposed in 1998 by V. Lazarević et al. [26], but has not garnered much attention. In any case, studying the underlying mathematical structures of fuzzy set theories is a very powerful approach that can shed light on the fundamental similarities and differences among the existing approaches and has been one of the goals of our research.

2.5 Multisets

Before exploring further extensions of fuzzy sets, we are going to introduce yet another extension of the set concept that is formally, albeit not semantically, very similar to the ordinary fuzzy sets. This is the concept of **multisets**, where elements in a set can be repeated.

In a set, elements either belong to it or do not belong to it, but it is not possible to think of an element as belonging to a set twice or three times. There are scenarios, however, where it would be convenient to have set-like mathematical objects like $\{a, b, b\}$ where the element b appears twice. Such mathematical objects are multisets and they can formally be defined through a generalisation of the characteristic function, a form of L-fuzzy set, but with completely different semantics. As in previous sections, we assume that the definitions involve a reference set, the universe, which we will denote by U in the context of multisets.

Definition 2.5.1. Let U be a universe. A **finite multiset** over the universe U is a function $\hat{M}\colon U \to \mathbb{N}$ mapping each element of the universe to a natural number (including 0) [42].

The image of an element $u \in U$ by $\hat{M}$, $\hat{M}(u)$, is called the **multiplicity** of u.

While a distinction is often made between the “multiset” as an arrangement of values and the “multiplicity function” (or “count function”), they can be regarded as formally equivalent, so we will treat them as the same mathematical entity and refer to them with a single-letter notation.

Definition 2.5.2. For a universe U, the family of all the multisets over U is called the **power multiset of** U and can be represented as $\mathbb{N}^U$.

The notion of cardinality can also be extended to the multisets, as follows.

Definition 2.5.3. Let $\hat{M}$ be a multiset over a universe U such that there is a finite set of values $u \in U$ for which $\hat{M}(u) > 0$. The **cardinality** (or **length**) of $\hat{M}$ is the natural number:

$$|\hat{M}| := \sum_{u \in U} \hat{M}(u) \tag{2.5}$$

As we would expect, those multisets with the image set restricted to $\{0, 1\}$ can be identified with the ordinary sets.

Furthermore, by disregarding the actual multiplicity information in a multiset and interpreting it in a Boolean way, as either zero or non-zero, a multiset can be turned into an ordinary set. This is a useful concept that can be defined formally as follows.

Definition 2.5.4. Given a universe U, the **support** of a multiset is a function $Supp\colon \mathbb{N}^U \to 2^U$ defined by the relation:

$$Supp(\hat{M}) := \{u \in U | \hat{M}(u) > 0\} \tag{2.6}$$

Example 2.5.5. Say we have a finite universe $U = \{a, b, c\}$. We can define a multiset $\hat{M} = \langle a, c, c\rangle$ in multiset notation (we will use angular brackets, $\langle\rangle$, for multisets). Using the formal definition above, $\hat{M}$ is actually a function $\hat{M}: U \to \mathbb{N}$ defined as $\hat{M}(a) = 1$, $\hat{M}(b) = 0$ and $\hat{M}(c) = 2$. The cardinality of $\hat{M}$ is $|\hat{M}| = 3$ and its support is the set $Supp(\hat{M}) = \{a, c\}$.

An additional concept that will be found to be useful later is that of the family of those multisets that share a constant cardinality:

Definition 2.5.6. Let U be a universe and let $n \in \mathbb{N}$. The family of all the multisets over U such that their cardinality is n is called the **regular power multiset of length** n over U and is denoted by $\mathbb{N}^U_{=n}$.

And we will also consider multisets of variable cardinality but with a fixed upper bound:

Definition 2.5.7. Let U be a universe and let $n \in \mathbb{N}$. The family of all the multisets over U such that their cardinality is not larger than n is called the **regular power multiset bounded by length** n over U and is denoted by $\mathbb{N}^U_{\leq n}$.

It obviously follows that $\mathbb{N}^U_{\leq n} = \cup^n_{i=1} \mathbb{N}^U_{\leq i}$.

There may be operations between multisets that require that the multisets have the same length. Such operations can be extended to bounded multisets by lengthening the shorter one. This process can be formalised as a function.

Definition 2.5.8. Let U be a universe and let $n \in \mathbb{N}$. A **multiset regularisation** to length n is a function $Reg_n\colon \mathbb{N}^U_{\leq n} \to \mathbb{N}^U_{=n}$ fulfilling the three following conditions:

$$\begin{aligned} &1. \quad Reg_n(\hat{M})(u) \geq \hat{M}(u) \text{ if } \hat{M}(u) > 0 \\ &2. \quad Reg_n(\hat{M})(u) = 0 \text{ if } \hat{M}(u) = 0 \\ &3. \quad |Reg_n(\hat{M})| = n \end{aligned} \tag{2.7}$$

This basically means that the shorter multiset is made longer by repeating values. When the universe U is a totally ordered set, it is common to do this by repeating just the maximum (or the minimum) value in the support, an approach that is referred to as the **optimistic** (**pessimistic**) regularisation.

The operation of complementation cannot be extended trivially to the multisets and we will ignore it. In fact, the absence of a straightforward form of complementation is the main formal difference between the multisets and the fuzzy sets. But the binary operations of intersection and union can be extended to multisets without any difficulty by relying on the same approach as was used for fuzzy sets.

Definition 2.5.9. Let U be the universe and let $\hat{M}$ and $\hat{N}$ be two multisets on U. The **intersection** $\hat{M} \cap \hat{N}$ and the **union** $\hat{M} \cup \hat{N}$ are defined as the following multisets over U:

$$\begin{aligned} (\hat{M} \cap \hat{N})(u) &:= \min(\hat{M}(u), \hat{N}(u)), \quad u \in U \\ (\hat{M} \cup \hat{N})(u) &:= \max(\hat{M}(u), \hat{N}(u)), \quad u \in U \end{aligned}$$

Once the notion of multiset has been defined, it is possible to define extensions of fuzzy sets where multisets over $\mathbb{I}$ are used as membership grades, as we will see in the next section.

2.6 Multivalued fuzzy sets

In the extensions of fuzzy sets that we have introduced so far, the ordinary fuzzy sets are extended by replacing the single-valued membership grade with either an interval around a main value or a function that smoothly reaches a maximum at a central value. Thus, these more general types of fuzzy sets are to be interpreted as modelling a tolerance threshold of uncertainty around the membership value. For the purposes of our discussion, we will refer to them as **smooth fuzzy sets**.

Another generalisation of the ordinary fuzzy sets was proposed by R. R. Yager in 1986 under the name **fuzzy bags**, where the membership values are multisets (sometimes called **bags**) in the unit interval $\mathbb{I}$. The theory of fuzzy bags was further refined by S. Miyamoto, who provided the first definitions of the common operations such as the intersection and union that were consistent with the standard ones for the ordinary fuzzy sets [30]. These generalised fuzzy sets are nowadays usually referred to as **fuzzy multisets**.

A similar approach to fuzzy multisets is used by **hesitant fuzzy sets**, which were proposed by V. Torra in 2010 [44], expanding on an idea first put forward by I. Grattan-Guinness in 1976 [14]. The hesitant theoretical framework broadens the range of the membership functions to encompass any subset of the $[0, 1]$ interval. A multiset-based version was already proposed in Torra's original article, which brings the theory closer to that of fuzzy multisets. The difference lies in the fact that both theories have evolved using different definitions for some of the common operations. In fact, the intersection and union as conventionally defined for fuzzy multisets are not equivalent to the accepted definitions for hesitant fuzzy sets. In a recent paper [39], we introduced a family of definitions for the intersection and union of fuzzy multisets, where the hesitant definitions appear as a particular case. Another possible approach to hesitant fuzzy sets consists in treating them as type-2 fuzzy sets where the membership grade is given by non-continuous, non-convex functions with the image set $\{0, 1\}$. As we mentioned in the previous section, such type-2 fuzzy sets lack the algebraic structure of a lattice and are more cumbersome mathematically than the convex type-2 fuzzy sets.

In terms of their meaning, the peculiarity of using sets or multisets as fuzzy membership values is that they do not represent a tolerance threshold around an ideal precise membership value, as in the smooth fuzzy set generalisations, but rather a collection of distinct possibilities. The real-life use case of fuzzy multisets and hesitant fuzzy sets emerges when the membership value can be determined through several alternative, but related, criteria or mechanisms that yield different values.

In this section we are going to discuss how such situations can, and sometimes should, be modelled through these and other related fuzzy set extensions that also allow for multiple and different membership values. We will refer to these schemes, where for example two values like 0.1 and 0.3 may be valid membership values for an element x while 0.2 is not, as **multivalued fuzzy sets**.

2.6.1 The origin and interpretation of multiple membership values

As we have just mentioned in the preceding section, we have to be particularly careful about the interpretation of the multiple membership values in multivalued fuzzy sets. For example, in a typical hesitant membership value like $\{0.2, 0.3\}$, we might be misled into thinking of the hesitancy as a lack of precise knowledge about the actual value, but that sort of situation would be better modelled through an interval-valued fuzzy set where the membership is $[0.2, 0.3]$. The important thing in a hesitant fuzzy set is that the different membership values do not represent an interval, but distinct possible values, even widely differing ones [40]. In a common intuitive interpretation of the fuzzy multisets and the hesitant fuzzy sets, each membership value is regarded as an independent verdict on membership that one particular "expert" or "decision maker" has produced. In this view, a hesitant membership value of $\{0.1, 0.3\}$ would be regarded as the result of an expert considering that the set has a membership value of 0.1 and a second expert considering that the set has a membership value of 0.3. These "experts" can be human individuals or, more often, simply different criteria or methodologies that produce fuzzy membership values for a common phenomenon. Note that if the experts evaluate different phenomena, then it would make more sense to use separate fuzzy sets.

A problem with the experts' interpretation when we consider the hesitant membership value as simply a subset of $[0, 1]$ is that these experts should be indistinguishable and even the number of experts involved is subject to variation. A more realistic scenario occurs when the number of experts is fixed and it may be possible to link the membership values to the expert that has produced it. In such a situation, the hesitant fuzzy sets are not a good model. On the one hand, it should be possible for a membership value to appear more than once. If both the first and the second expert produce a membership value of 0.2, then the hesitant membership value should be something like $\langle 0.2, 0.2\rangle$, which is a multiset rather than a subset. This example shows the advantage of using fuzzy multisets over hesitant fuzzy sets when repetition is meaningful. But accounting for repetition may not be enough if we consider the experts as being distinguishable. In such a case, which seems sensible for comparison purposes, the membership multisets should be ordered n-tuples, so that $(0.1, 0.2)$ is different from $(0.2, 0.1)$ as a membership grade. We will refer to these ordered vector-like membership grades as **ordered fuzzy multisets**.

From this discussion, we can see that there are four distinct cases in the experts' model.

1. If the experts produce unrelated values for different phenomena, then the values for each expert should be treated as **independent fuzzy sets**, as there is no link between them.

2. If the experts produce related values that account for one phenomenon and there is a fixed number of them producing a membership value for each element in the universe, then the problem can be modelled with **ordered fuzzy multisets**.

3. If the experts produce related values that account for one phenomenon and there is a fixed number of them producing a membership value for each element in the universe and they are indistinguishable (as if in a secret vote; we cannot know which expert produced which value for one element, but we can know that n experts chose the same value) then the problem should be modelled using **fuzzy multisets**.

4. If the experts produce related values that account for one phenomenon and both the association between experts and values and the repetition count for each value are unknown (as in a secret vote where once a value gets a vote, any additional vote for it is irrelevant), then the problem should be modelled using **hesitant fuzzy sets**.

Example 2.6.1. As an example of a situation where the ordered fuzzy sets can be a better model than the others, let us consider the occasional use of fuzzy sets to represent greyscale images. In such proposals, the pixels in an image are regarded as the elements of the universe and the fuzzy membership value between 0 and 1 represents the greyscale range from black (0) to white (1). We may be tempted to use this model for colour images, and treat the three colour channels as if they were three experts evaluating the fuzzy membership of a pixel. In such a situation, it is clear that the use of hesitant fuzzy sets would result in a colour-blind model, where for example the six colours red, green, blue, yellow, cyan and magenta would all be represented with exactly the same membership value $\{0, 1\}$. Things get only slightly better with the hesitant fuzzy multisets where the red, green and blue colours would be represented as the same pixel value $\langle 0, 0, 1\rangle$ but differentiated from the yellow, cyan and magenta trio, which would be $\langle 0, 1, 1\rangle$. Ordered fuzzy multisets would provide a better model since any two different colours would be represented with two different membership values like, for example, $(1, 0, 0)$ for red and $(0, 1, 0)$ for green.

These three types of multivalued fuzzy sets have received very disparate levels of attention in the fuzzy set research. In fact, there has been very little research so far concerning the ordered fuzzy multisets; they are pretty straightforward if we consider them as a Cartesian product, which may explain the scant interest such mathematical constructions have elicited. Still, it is possible to identify some formal definitions and results related to the multidimensional nature of these objects, which we will explore in this dissertation. A precedent of the ordered fuzzy multisets are the "n-dimensional fuzzy sets", which impose a sorting condition on the coordinates, and which were originally proposed by Y. Shang, X. Yuan and E. S. Lee. [41] in 2010 and were further studied by B. Bedregal et al. [2] in 2011. The concept is not really new, though, and references to similar types of vector-like fuzzy sets go as far back as 1982, when L. Kóczy introduced the concept as **vectorial I-fuzzy sets** [17] and studied applications to image processing [18].

2.6.2 Hesitant fuzzy sets

The hesitant fuzzy sets have been widely studied during the last few years. They were originally introduced by I. Grattan-Guinness in 1976 as **set-based fuzzy sets**, and were revived by V. Torra in 2010, who gave them the name "hesitant" and proposed definitions for their common operations. A comprehensive introduction can be found in a recent book by Z. Xu [48]. The basic definition is as follows [44].

Definition 2.6.2. Let X be the universe. A **hesitant fuzzy set**, or **set-based fuzzy set**, $\tilde{A}$ over X is a function $\tilde{A}\colon X \to 2^{\mathbb{I}}$.

Given an element $x \in X$, $\tilde{A}(x)$ is called its **hesitant element** [48] and the family of all the hesitant fuzzy sets over X, $(\mathbf{2}^{\mathbb{I}})^X$, is called the **hesitant fuzzy powerset** over X.

Let us now see the definitions for the three fundamental operations.

Definition 2.6.3. Let X be the universe and let $\tilde{A}$ and $\tilde{B} \in (\mathbf{2}^{\mathbb{I}})^X$ be two hesitant fuzzy sets. The **hesitant complement**, $\tilde{A}^c$, of $\tilde{A}$, the **hesitant intersection**,

$\tilde{A} \cap \tilde{B}$ and the **hesitant union**, $\tilde{A} \cup \tilde{B}$, of $\tilde{A}$ and $\tilde{B}$ are the hesitant fuzzy sets defined by the following relations:

1. $\tilde{A}^c(x) \quad = \{t | 1 - t \in \tilde{A}(x)\}$
2. $(\tilde{A} \cap \tilde{B})(x) \quad = \{\alpha \in \tilde{A}(x) \cup \tilde{B}(x) | \alpha \leq \min\{\max\{\tilde{A}(x)\}, \max\{\tilde{B}(x)\}\}\}$
3. $(\tilde{A} \cup \tilde{B})(x) \quad = \{\alpha \in \tilde{A}(x) \cup \tilde{B}(x) | \alpha \geq \max\{\min\{\tilde{A}(x)\}, \min\{\tilde{B}(x)\}\}\}$

The hesitant fuzzy sets include the ordinary fuzzy sets as the special case when all the hesitant elements have cardinality 1. In that particular case, the above definitions of complement, intersection and union behave like the ordinary ones [39]. For the purposes of this dissertation, we will only consider **typical hesitant fuzzy sets** [44], where the hesitant elements are always finite sets.

Example 2.6.4. If we have a single-element universe $X = \{x\}$, we can define a typical hesitant fuzzy set $\tilde{A} \in (\mathbf{2}^{\mathbb{I}})^X$ as $\tilde{A}(x) = \{0.1, 0.3\}$. We then say that the crisp set $\{0.1, 0.3\}$ is the hesitant element that represents the membership grade of x. Its complement, according to Definition 2.6.3, is the hesitant fuzzy set given by $\tilde{A}^c(x) = \{0.7, 0.9\}$. If we have a second hesitant fuzzy set $\tilde{B}(x) = \{0.7, 0.8\}$, then the intersection and the union are given by $(\tilde{A} \cap \tilde{B})(x) = \{0.1, 0.3\}$ and $(\tilde{A} \cup \tilde{B})(x) = \{0.7, 0.8\}$, in an obvious generalisation of the ordinary fuzzy definition based on the minimum and the maximum. The outcome of these operations is less intuitive when the ranges of values overlap. For example, with a hesitant fuzzy set $\tilde{C} \in (\mathbf{2}^{\mathbb{I}})^X$ such that $\tilde{A}(x) = \{0.2, 0.4\}$, we have that $(\tilde{A} \cap \tilde{C})(x) = \{0.1, 0.2, 0.3\}$ and $(\tilde{A} \cup \tilde{C})(x) = \{0.2, 0.3, 0.4\}$.

2.6.3 Fuzzy multisets

The fuzzy multisets associate each element in a universe with a multiset in the unit interval $\mathbb{I}$. From now on, we will often refer to the ordinary multisets, as defined in the previous section, as **crisp multisets** in order to tell them apart from the fuzzy multisets.

Definition 2.6.5. Let X be the universe. A **fuzzy multiset** $\hat{A}$ over X is a function $\hat{A} \colon X \to \mathbb{N}^{\mathbb{I}}$. Given an element $x \in X$, $\hat{A}(x)$ is called its **membership multiset** and the family of all the fuzzy multisets over X, $(\mathbb{N}^{\mathbb{I}})^X$, is called the **fuzzy power multiset** over X.

Example 2.6.6. Say we have a single-element universe $X = \{x\}$. We can define a fuzzy multiset $\hat{A}$ as $\hat{A}(x) = \langle 0.1, 0.2, 0.2 \rangle$ in angular-bracket notation. Or in other words, using the formal definition above, the element x is being mapped into a function $\hat{A}(x)\colon \mathbb{I} \to \mathbb{N}$ defined as $\hat{A}(x)(0.1) = 1$, $\hat{A}(x)(0.2) = 2$ and $\hat{A}(x)(t) = 0$ for any $t \notin \{0.1, 0.2\}$.

A fuzzy multiset thus associates each element in the universe with a crisp multiset over the unit interval $\mathbb{I}$. By taking the supports of these membership multisets we can turn the fuzzy multiset into a hesitant fuzzy set. This idea motivates the following definition.

Definition 2.6.7. Let X be the universe and let $\hat{A} \in (\mathbb{N}^{\mathbb{I}})^X$ be a fuzzy multiset. Its **hesitant fuzzy set support** $Supp^h(\hat{A})$ is the hesitant fuzzy set such that, for any element $x \in X$, $(Supp^h(\hat{A}))(x) = Supp(\hat{A}(x))$.

The complement of a fuzzy multiset can be defined in an intuitive and straightforward way as follows:

Definition 2.6.8. Let X be a universe and let $\hat{A} \in (\mathbb{N}^{\mathbb{I}})^X$ be a fuzzy multiset. The **complement** of $\hat{A}$ is the fuzzy multiset $\hat{A}^c$ defined as:

$$(\hat{A}^c(x))(t) := (\hat{A}(x))(1-t), \quad x \in X \quad t \in \mathbb{I} \tag{2.8}$$

The intersection and union, however, are more problematic. The most common definitions in the literature on fuzzy multisets are those due to S. Miyamoto [30], which are based on the idea of arranging the elements of the multisets into ordered sequences and then performing a coordinatewise operation. But for this to be possible, the two operands must have the same cardinality, so we may have to extend one of them by repeating elements when the cardinalities differ. The cardinality was an integer number for the crisp multisets, but in the fuzzy multisets, there will be one cardinality for each element of the universe X, so the cardinality becomes a map, which we can define as follows [39].

Definition 2.6.9. Let X be a universe. Given a function $m\colon X \to \mathbb{N}$, an **m-regular fuzzy multiset** $\hat{A}$ over the universe X is a fuzzy multiset such that, for each element of the universe $x \in X$, $|\hat{A}(x)| = m(x)$. We call m a **cardinality map**. The family of all the m-regular fuzzy multisets over X, $(\mathbb{N}^{\mathbb{I}}_{=m})^X$, is called the **$m$-regular fuzzy power multiset** [39].

We then need to be able to map a multiset into a finite **sequence** of length n ($\in \mathbb{N}$) or **n-tuple**, which for the unit interval $\mathbb{I}$ can be defined as an element of the n-dimensional unit hypercube $\mathbb{I}^n$. Given a membership multiset $M \in \mathbb{N}^{\mathbb{I}}$ with cardinality n, a function mapping it to an n-tuple in $\mathbb{I}^n$ is called an **ordering strategy**, with the family of all such functions being denoted by $\mathcal{OS}(M)$. The number of possible ordering strategies is the number of permutations of n elements with repetition and the two most common sorting strategies are the **ascending sort** $s_\uparrow$ and the **descending sort** $s_\downarrow$, where the elements are sorted in ascending or descending order, respectively. We will use parentheses () for sequences; so we can write, for example, $s_\downarrow(\langle 1, 2, 2\rangle) = (2, 2, 1)$.

It is now possible to define intersection and union by operating on sorted sequences, as follows [39]:

Definition 2.6.10. Let X be a universe and let $m\colon X \to \mathbb{N}$ be a cardinality map. Given two m-regular FM's $\hat{A}$ and $\hat{B}$ and two ordering strategies s_A and s_B, for each element $x \in X$, two new sequences $\mu_{\hat{A}\cap_{(s_A,s_B)}\hat{B}}(x)$ and $\mu_{\hat{A}\cup_{(s_A,s_B)}\hat{B}}(x)$ can be built with the pairwise minima and maxima:

$$(\mu_{\hat{A}\cap_{(s_A,s_B)}\hat{B}}(x))_i := \min\{(s_A(\hat{A}(x)))_i, (s_B(\hat{B}(x)))_i\}, \quad i \in \{1, ..., m(x)\}$$
$$(\mu_{\hat{A}\cup_{(s_A,s_B)}\hat{B}}(x))_i := \max\{(s_A(\hat{A}(x)))_i, (s_B(\hat{B}(x)))_i\}, \quad i \in \{1, ..., m(x)\}$$

The m**-regular** (s_A, s_B)**-ordered intersection** $\hat{A} \cap_{(s_A,s_B)} \hat{B}$ and the m**-regular** (s_A, s_B)**-ordered union** $\hat{A} \cup_{(s_A,s_B)} \hat{B}$ are the two m-regular fuzzy multisets defined by:

$$((\hat{A} \cap_{(s_A,s_B)} \hat{B})(x))(t) = |\{i | 1 \leq i \leq m(x), (\mu_{\hat{A}\cap_{(s_A,s_B)}\hat{B}}(x))_i = t\}|$$
$$((\hat{A} \cup_{(s_A,s_B)} \hat{B})(x))(t) = |\{i | 1 \leq i \leq m(x), (\mu_{\hat{A}\cup_{(s_A,s_B)}\hat{B}}(x))_i = t\}|$$

Miyamoto's intersection and union are the particular case when both ordering strategies are chosen as $s_A = s_B = s_\downarrow$. It can easily be proved that choosing $s_A = s_B = s_\uparrow$ produces the same results for both operations, so sorting in either ascending or descending order is simply a matter of convention. Other ordering strategies, however, lead to different results.

If the two fuzzy multisets $\hat{A}$ and $\hat{B}$ do not share a common cardinality map, then it will be possible to apply the definition above through the additional mechanism of regularisation (see Definition 2.5.8), where the membership multiset with the shorter cardinality for each element $x \in X$ is extended by increasing the multiplicity of either the lowest or the highest value. We omit the details here and will simply assume that definitions can always be extended to the general case by regularising multisets first.

As we have argued in a recent article [39], since we are working with finite sets we can make a definition that is independent of any particular sorting strategy by taking the multiset union of the ordered intersections and unions (2.6.10) resulting from all the combinations of possible sorting strategies (s_A, s_B). This idea leads to the following definitions [39]:

Definition 2.6.11. Let X be a universe and let $\hat{A}, \hat{B} \in (\mathbb{N}^{\mathbb{I}})^X$ be two fuzzy multisets. The **aggregate intersection** and the **aggregate union** of $\hat{A}$ and $\hat{B}$ are the fuzzy multisets $\hat{A} \cap^a \hat{B}$ and $\hat{A} \cup^a \hat{B}$ such that, for any element $x \in X$, $\hat{A} \cap^a \hat{B}(x)$ is the multiset union of the (s_A, s_B)-ordered intersections and $\hat{A} \cup^a \hat{B}(x)$ is the multiset union of the (s_A, s_B)-ordered unions for all the possible pairs of ordering strategies (s_A, s_B):

$$\hat{A} \cap^a \hat{B}(x) := \bigcup_{\substack{s_{\hat{A}} \in \mathcal{OS}(\hat{A}) \\ s_{\hat{B}} \in \mathcal{OS}(\hat{B})}} \hat{A} \cap_{(s_A, s_B)} \hat{B}(x), \quad x \in X \tag{2.9}$$

$$\hat{A} \cup^a \hat{B}(x) := \bigcup_{\substack{s_{\hat{A}} \in \mathcal{OS}(\hat{A}) \\ s_{\hat{B}} \in \mathcal{OS}(\hat{B})}} \hat{A} \cup_{(s_A, s_B)} \hat{B}(x), \quad x \in X \tag{2.10}$$

Example 2.6.12. As in the previous example, we consider a single-element universe $X = \{x\}$ and the fuzzy multiset given by $\hat{A}(x) = \langle 0.1, 0.2, 0.2 \rangle$. The hesitant fuzzy set support of $\hat{A}$ is the hesitant fuzzy set given by $(Supp^h(\hat{A}))(x) = \{0.1, 0.2\}$ and its complement is $\hat{A}^c(x) = \langle 0.8, 0.8, 0.9 \rangle$. If we now have a second fuzzy multiset given by $\hat{B}(x) = \langle 0.1, 0.1, 0.3 \rangle$, Miyamoto's intersection and union are $(\hat{A} \cap_{(s_\downarrow, s_\downarrow)} \hat{B})(x) = \langle 0.1, 0.1, 0.2 \rangle$ and $(\hat{A} \cup_{(s_\downarrow, s_\downarrow)} \hat{B})(x) = \langle 0.1, 0.2, 0.3 \rangle$, whereas the aggregate operations yield different results: $(\hat{A} \cap^a \hat{B})(x) = \langle 0.1, 0.1, 0.1, 0.2 \rangle$ and $(\hat{A} \cup^a \hat{B})(x) = \langle 0.1, 0.2, 0.2, 0.3 \rangle$.

All these forms of intersection and union that we have defined are consistent with the definitions for ordinary fuzzy sets. But this aggregate form of intersection and union is in addition also consistent with the definitions 2.6.3 for the hesitant fuzzy sets, in the sense that both pairs of operations produce the same results when using trivial multisets restricted to count values of 0 and 1. The relations between these concepts will be discussed more in depth in chapters 3 and 4.

2.6.4 Ordered fuzzy multisets

In the experts' model, fuzzy multisets can account for the repetitions in the values produced by the experts, but no distinction is made between, for example, the first expert producing the value 0.1 and the rest producing 0.2, or the second expert being the one who produced 0.1 with the others producing 0.2. This disregard for the order of the experts is a form of information loss, which may not be acceptable. In such situations, we find the need to define a special type of multivalued fuzzy set where the membership values can appear multiple times and are labelled with a coordinate index. We can do this easily by replicating the notion of a fuzzy set over the various dimensions of the unit hypercube $\mathbb{I}^n$ [36].

Definition 2.6.13. Let X be the universe and let n be a natural (non-zero) number. An **n-dimensional ordered fuzzy multiset** $\vec{A}$ over X is a function $\vec{A}\colon X \to \mathbb{I}^n$.

Given an element $x \in X$, $\vec{A}(x)$ is called its **membership sequence** and the family of all the ordered fuzzy multisets over X, $(\mathbb{I}^n)^X$, is called the **n-dimensional ordered fuzzy power multiset** of X.

The restrictions of $\vec{A}$ to the i-th coordinate in the image set, A_i, are by definition ordinary fuzzy sets. These n ordinary fuzzy sets $A_1, ..., A_n$ can be referred to as the **fuzzy coordinates** of $\vec{A}$.

The usual fuzzy set operations such as complement, union and intersection can be carried over to the $\mathbb{I}^n$ space coordinatewise in a straightforward way, as we will see in the definition that follows.

Definition 2.6.14. Let X be a universe, let $n \in \mathbb{N}$ and let $\vec{A}, \vec{B} \in (\mathbb{I}^n)^X$ be two n-dimensional ordered fuzzy multisets over X. The **complement** of $\vec{A}$, $\vec{A}^c$, and the **Cartesian intersection**, $\vec{A} \cap \vec{B}$, and **Cartesian union**, $\vec{A} \cup \vec{B}$, of $\vec{A}$ and $\vec{B}$ are the n-dimensional ordered fuzzy multisets over X given by the following relations:

$$
\begin{aligned}
{A^c}_i(x) &= 1 - A_i(x), \quad i = 1, ..., n, \quad x \in X \\
(A \cap B)_i(x) &= \min\{A_i(x), B_i(x)\}, \quad i = 1, ..., n, \quad x \in X \\
(A \cup B)_i(x) &= \max\{A_i(x), B_i(x)\}, \quad i = 1, ..., n, \quad x \in X
\end{aligned}
$$

As these operations are defined in terms of the ones for the ordinary fuzzy sets, the properties of the latter are replicated in an obvious way. In particular, the Cartesian intersection and union are commutative and associative and the identity element is the ordered fuzzy multiset $\vec{I}\colon X \to \mathbb{I}^n$, defined as the unity membership $\vec{I}(x) = (1, ..., 1)\ \forall x \in X$, for the intersection and the ordered fuzzy multiset $\vec{0}\colon X \to \mathbb{I}^n$, defined as the null membership $\vec{0}(x) = (0, ..., 0)\ \forall x \in X$, for the union.

Example 2.6.15. Let us suppose that we have a single-element universe $X = \{x\}$. A 3-dimensional ordered fuzzy multiset $\vec{A}$ can be defined as $\vec{A}(x) = (0.1, 0.3, 0.3)$. Another 3-dimensional ordered fuzzy multiset $\vec{B}$ would be $\vec{B}(x) = (0.3, 0.1, 0.3)$. Note that in this example the difference in the membership value for $\vec{A}$ and $\vec{B}$ is in the order of the coordinates, a distinction that could not be represented by multisets. The complement of $\vec{A}$ would be given by $\vec{A}^c(x) = (0.9, 0.7, 0.7)$ and the Cartesian intersection and union between $\vec{A}$ and $\vec{B}$ would be $\vec{A} \cap \vec{B} = \{0.1, 0.1, 0.3\}$ and $\vec{A} \cup \vec{B} = \{0.3, 0.3, 0.3\}$.

As the previous example shows, the membership values for the ordered fuzzy multisets are ordered n-tuples, so they carry more information content than the similar fuzzy multisets. Just like the fuzzy multisets extend the hesitant fuzzy sets with multiplicity information, we can think of the ordered fuzzy multisets as extending the fuzzy multisets with the order information.

2.7 Extension principles

In the early days of fuzzy set theory, L. A. Zadeh proposed a technique to convert any function acting on crisp sets into a "fuzzified" version of the function acting on the corresponding fuzzy sets. This technique has traditionally been called **Zadeh's Extension Principle** and, while conceptually very simple, has turned out to be a very powerful tool in the development of the fuzzy theoretical framework. Among other examples, it has played a central role in the original definition of the operations for type-2 fuzzy sets [32, 51] and in the development of the arithmetic of **fuzzy numbers** (fuzzy sets over numeric sets like $\mathbb{R}$) [31]. Here, we will introduce Zadeh's Extension Principle and some other similar principles that we will use in this dissertation. While many of these rules may look like trivial conversion rules, they help to establish a sound formal footing for many intuition-based arguments and can contribute to a clearer understanding of how the concepts that will be the subject of our study are interrelated.

The basic idea of an extension principle is to create a new function based on an existing function between two sets that are conceptually related, one by one, with the domain and codomain sets for the new function. In more formal language, if there is a function $f\colon X \to Y$ mapping elements from a set X into a set Y, and there are two other sets, $\mathscr{E}(X)$ and $\mathscr{E}(Y)$, where $\mathscr{E}$ represents an arbitrary variation of the original set, then an extension principle is a rule that creates a new function $\mathscr{E}(f)\colon \mathscr{E}(X) \to \mathscr{E}(Y)$ through an expression involving the original function f. Using the notation where Y^X stands for the space of functions $X \to Y$, we can define an extension principle as a function mapping functions in Y^X to "extended" functions $\mathscr{E}(Y)^{\mathscr{E}(X)}$.

Definition 2.7.1. Let X and Y be two sets and let $\mathscr{E}(X)$ and $\mathscr{E}(Y)$ be two sets related, in some meaningful way, to X and Y respectively. An **extension principle** is a function $Ext\colon Y^X \to \mathscr{E}(Y)^{\mathscr{E}(X)}$.

In order to handle multivariate functions $f(x_1, ..., x_n)$, alternative forms of the extension principles can be defined where X is replaced by X^n and the extended domain is $(\mathscr{E}(X))^n$. Functions of the form $X^n \to X$ are typically called **n-ary operations** with the English words **unary** and **binary** commonly used instead of "1-ary" and "2-ary", respectively. With this terminology, elements of the reference set X, especially those that play an important role like 0 and 1 in $[0, 1]$, can also be called "0-ary" or **nullary** operations [24, p. 13].

Before introducing Zadeh's Extension Principle, we are going to define a simpler extension principle that we will also use in this dissertation. Given a function $f\colon X \to Y$, the function can be extended trivially to the powersets 2^X and 2^Y, as follows.

Definition 2.7.2 (Powerset Extension Principle)**.** Let X and Y be two sets and let Y^X be the set of all the functions $X \to Y$. Similarly, let $(2^Y)^{(2^X)}$ be the set of all the functions $2^X \to 2^Y$. The **powerset extension** is a function $Ext_s\colon Y^X \to (2^Y)^{(2^X)}$ that maps any function $f\colon X \to Y$ into the function $Ext_s(f)\colon 2^X \to 2^Y$ defined by the expression:

$$Ext_s(f)(A) := \{f(x) | x \in A\}, \quad A \subseteq X \tag{2.11}$$

The image of a function $f\colon X \to Y$ by Ext_s can be denoted more simply by f_s and will be called the **powerset-extended** function.

The Powerset Extension Principle is a relatively trivial technique to extend a function, widely supported in programming languages (for example, the *map* function in Haskell), but it is uncommon to give it an explicit name in the mathematical literature. It is sometimes referred to as the **direct image mapping** [25] or the **united extension** in the field of Interval Analysis [35].

Example 2.7.3. Let us consider as an example the real numbers $\mathbb{R}$ and the trigonometric sine function $\sin\colon \mathbb{R} \to [-1, 1]$. If we now take the family of sets of real numbers; i.e. the powerset of $\mathbb{R}$, $2^{\mathbb{R}}$, we can use the Powerset Extension Principle to build a sine function that operates on sets of real numbers, so that given a set like $\{a, b, c\}$, where a, b and c are three real numbers, then $\sin_s\colon 2^{\mathbb{R}} \to 2^{[-1,1]}$ will map the set $\{a, b, c\} \in 2^{\mathbb{R}}$ to the set $\{\sin(a), \sin(b), \sin(c)\} \in 2^{[-1,1]}$. Note that the cardinality of the image may be less than 3 if any two of the numbers a, b and c have the same sine value (for example, the image by $\sin_s$ of the set $\{0, \pi, 2\pi\}$ is simply $\{0\}$).

Note that those functions $2^A \to 2^B$ that arise from simpler functions $A \to B$ are a strict subset of the whole family of functions $(2^B)^{2^A}$. Adapting the previous example, a function such that $f(\{0, \pi, 2\pi\}) = \{0\}$ and $f(\{0, \pi\}) = \{1\}$ cannot be expressed as a powerset extension.

In many cases, the functions that require an extension to the derived sets are multivariate functions (such as binary functions $X \times X \to Y$) and the extension principle above cannot be used in this form. Because of that, extension principles must also have a more general multivariate version. In the case of the Powerset Extension Principle, we shall define it as follows.

Definition 2.7.4 (Powerset Extension Principle – multivariate form)**.** Let X and Y be two sets and let Y^{X^n} be the set of all the functions $X^n \to Y$ with n being a natural number $n \geq 2$. Similarly, let $(2^Y)^{(2^X)^n}$ be the set of all the functions $(2^X)^n \to 2^Y$. The **multivariate powerset extension** is a function $Ext_s^n\colon Y^{X^n} \to (2^Y)^{(2^X)^n}$ that maps any function $f\colon X^n \to Y$ into the function $Ext_s^n(f)\colon (2^X)^n \to 2^Y$ defined by the expression:

$$Ext_s^n(f)(A_1, ..., A_n) := \{f(x_1, ..., x_n) | x_1 \in A_1, ..., x_n \in A_n\} \tag{2.12}$$

with $A_1, ..., A_n \subseteq X$.

The Powerset Extension Principle turns out to be a particular case of Zadeh's Extension Principle, which is a rule to turn functions $X \to Y$ into functions over the corresponding fuzzy spaces $\mathbb{I}^X \to \mathbb{I}^Y$ [51]. The obvious way of handling this consists in transferring the membership grades from the elements in X to their images in Y. But there is the problem of what to do with functions that are not injective. In that case, the maximum membership grade in the preimage is used.

Definition 2.7.5 (Zadeh's Extension Principle)**.** Let X and Y be two sets and let Y^X be the set of all the functions $X \to Y$. Similarly, let $\mathbb{I}^{Y^{\mathbb{I}^X}}$ be the set of all the functions between their corresponding fuzzy powersets $\mathbb{I}^X \to \mathbb{I}^Y$. **Zadeh's extension** is a function $Ext_z \colon Y^X \to (\mathbb{I}^Y)^{(\mathbb{I}^X)}$ that maps any function $f \colon X \to Y$ into a function between the fuzzy powersets $Ext_z(f) \colon \mathbb{I}^X \to \mathbb{I}^Y$ defined by the expression:

$$Ext_z(f)(A)(y) := \begin{cases} \sup_{x|y=f(x)}\{A(x)\}, & \text{if } \exists x \in X \mid y = f(x) \\ 0, & \text{if } \nexists x \in X \mid y = f(x) \end{cases} \tag{2.13}$$

with $A \in \mathbb{I}^X$ and $y \in \mathbb{I}$.

The image of a function $f \colon X \to Y$ by Ext_z can be denoted more simply by f_z and will be called the **Zadeh-extended** function. Another extension principle that has been widely used in the mathematical framework of fuzzy sets, even though it is rarely mentioned explicitly and nobody seems to have bothered to give it a name before, is the way that an operation on the space of membership grades can be induced on the fuzzy sets in a pointwise manner. This mechanism is independent of the actual choice of membership grades and, in fact, was first identified by J. A. Goguen in his seminal article on L-fuzzy sets, so we will coin the term "Goguen's Extension Principle" for it and define it in the general sense of L-fuzzy sets.

Definition 2.7.6 (Goguen's Extension Principle)**.** Let X be a universe and let L be a partially ordered set that can be used as a membership grade space for X, making L^X the space of L-fuzzy sets over X. And let $f \colon L^n \to L$, with n a natural number, be an n-ary operation on L. The **pointwise extension** or **Goguen's extension** is a function $Ext_g \colon L^{(L^n)} \to (L^X)^{((L^X)^n)}$ that maps any function $f \colon L^n \to L$ into an n-ary operation on the L-fuzzy sets $Ext_g(f) \colon (L^X)^n \to L^X$ defined by pointwise expansion:

$$(Ext_g(f)(A_1, ...A_n))(x) := f(A_1(x), ...A_n(x))) \tag{2.14}$$

with $A_1, ..., A_n \in L^X$ and $x \in L$.

The image of an operation $f\colon L^n \to L$ by Ext_g can be denoted more simply by f_g and will be called the **pointwise-extended** or **Goguen-extended** operation.

Definitions of operations on fuzzy sets through a pointwise expansion include the most common ones and they can be seen as applications of this extension principle. In these cases, the more elegant mathematical formalism often works backwards when compared with the intuitive reasoning. This is what happens with the way t-norms, t-conorms and negation functions induce the operations we call “intersection”, “union” and “complement” on the ordinary fuzzy sets, which is simply Goguen’s extension in action. For example, given a t-norm t on the unit interval $\mathbb{I}$, the Goguen-extended t_g is what we call intersection. Many properties of the operations, such as commutativity and associativity, are preserved when applying Goguen’s extension [13, 16]. This is the reason why exploring the properties of operations on the membership grade space can yield a lot of information that also applies to the extended operations on fuzzy sets.

But not all operations defined on L-fuzzy sets can be expressed as a Goguen extension. For simplicity, let us consider the finite case where the membership grade space has cardinality N $L = \{l_1, ..., l_N\}$ and the universe has cardinality M $X = \{x_1, ...x_M\}$. As this is a finite case, we can count all the possible n-ary operations on L and on L^X. The number of n-ary operations on L is the number of ways one can assign N values into N^n slots, which is $N^{(N^n)}$, whereas the number of n-ary operations on L^X is the number of ways one can assign N^M values into N^{Mn} slots, $(N^M)^{(N^{Mn})}$. In this finite case, we get the result that of all the possible $(N^M)^{(N^{Mn})}$ n-ary operations in L^X only $N^{(N^n)}$ will be Goguen-extended ones. The two numbers become the same when $M = 1$. In this case, when the universe is made up of a single element $X = \{x\}$ (often called a **singleton** universe), there is a one-to-one correspondence between fuzzy sets and membership grades and the Goguen extensions account for all the operations on the fuzzy sets.

2.8 Equivalence relations

We are now going to recall the concept of an **equivalence relation** with the terminology and basic results that we will need in the subsequent chapters.

Definition 2.8.1. Let X be a set. A **binary relation** is a subset of X^2 [24, p.11].

This simple definition gives us a set of ordered pairs of elements; the semantics of equivalence depend on some additional properties.

Definition 2.8.2. Let X be a set. An **equivalence relation** is a binary relation R that satisfies the following properties [24, p.12].

1. $(x, x) \in R, \quad \forall x \in X$ (reflexivity)
2. $(x, y) \in R \Rightarrow (y, x) \in R, \quad \forall x, y \in X$ (symmetry)
3. $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R, \quad \forall x, y, z \in X$ (transitivity)

In this definition we have denoted the equivalence relation as a set R, but from now on we will follow the convention of representing any equivalence relation with a tilde sign $\sim$ placed between the two elements, thus writing $x \sim y$ rather than $(x, y) \in R$.

Definition 2.8.3. Given an equivalence relation $\sim$ on a set X and an element $x \in X$, the **equivalence class** of x, usually denoted by $[x]$, is the set $\{y \in X | x \sim y\}$.

Note that the equivalence class of an element $x \in X$ can never be an empty set since it must at least include x itself because of the reflexivity condition. Furthermore, it follows from the definition of equivalence relation that an element x can only belong to one equivalence class. This means that the equivalence classes induce a partition on the set X. We can give a name to this partition.

Definition 2.8.4. Let $\sim$ be an equivalence relation on a set X. The set of all the equivalence classes $\{[x] | x \in X\}$ is called the **quotient set** of X by the equivalence relation $\sim$ and is denoted by $X/\sim$.

Once we have an equivalence class, we may want to select one of its elements. This operation is an injective function $X/\sim \to X$, which we can give a name to. For this, we will borrow the term "section" from its wide related use in category theory.

Definition 2.8.5. Let $\sim$ be an equivalence relation on a set X. A **section** is an injective function $s \colon X/\sim \to X$ such that $s([x]) \in [x]$.

The image of an equivalence class $c \in X/\sim$ by a section s, $s(c)$, is called a **representative** of the equivalence class. The codomain of a section s is called a **complete set of representatives** for s.

Being an injective function, a section becomes a bijection if its codomain is restricted to its complete set of representatives. Once a particular complete set of representatives is selected, these representatives stand in a one-to-one correspondence with the equivalence classes.

Sometimes there is a choice of section that somehow feels simpler or more natural than others. When that is the case, we will refer to this most natural choice as a **canonical section**, yielding **canonical representatives**.

An important case of equivalence relation that we will be using in this dissertation is the one induced by a surjective function on its domain [24, p. 33].

Definition 2.8.6. Let X and Y be two sets and let $f\colon X \to Y$ be a surjective function. Then the binary relation $\sim_f$ defined by $x \sim_f y$ when $f(x) = f(y)$ is called the **equivalence kernel** for f.

It is straightforward to prove that the relation defined in this way is an equivalence relation. Different names can be found for this type of equivalence relation induced by a function. Here, we are sticking to the terminology used by S. Mac Lane and G. Birkhoff in their well-known textbook *Algebra* [24, p. 33].

The equivalence kernel also induces a bijection between the quotient set $X/\sim_f$ and the codomain Y, an important result that we state in the next proposition.

Proposition 2.8.7. *Let X and Y be two sets and let $f\colon X \to Y$ be a surjective function with an equivalence kernel $\sim_f$. Then the function $\phi_{\sim_f}\colon X/\sim_f \to Y$ defined by the expression $\phi_{\sim_f}(c) = f(s(c))$, where s is any section for $\sim_f$ is a bijection.*

Proof. This proposition is very similar to Theorem 18 in the reference book by Mac Lane and Birkhoff [24, p. 33], but we will prove it here as we have modified the original wording slightly.

Let C and $D \in X/\sim_f$ be two equivalence classes. By definition of the equivalence kernel, $f(s(C))$ will be the same value of Y for any section s and will be different from the corresponding value of Y for D, $f(s(D))$, so ϕ_f is injective.

On the other hand, for any value $y \in Y$, the fact that f is surjective means that there is at least a value $x \in X$ such that $f(x) = y$ and its equivalence class $[x]$ will satisfy $\phi_f([x]) = y$, so ϕ_f is also surjective. □

The relation between f, the section s and the bijection ϕ_f is summed up in Figure 2.1.

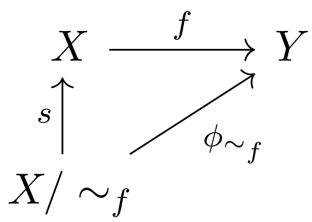

Figure 2.1: Commutation diagram for the $\phi_{\sim_f}$ bijection

2.9 Measuring differences between fuzzy sets

In many applications of both ordinary and fuzzy sets, the need arises to quantify how different (or similar) two sets are. For example, the ordinary set $A = \{1, 2, 3\}$ is different from $B = \{1, 2, 4\}$, but it seems sensible to regard A as being more similar to B than to a set like $C = \{23, 45, 82, 94\}$. Measures of difference between ordinary or fuzzy sets typically take on a form similar to a distance in metric spaces, where a zero value for equality and symmetry appear as two of the axiomatic defining properties and a third one is a variant of the **triangle inequality** that expresses how the difference between two sets is related to the differences involving a third arbitrary set [9]. Borrowing the concept from the theory of metric spaces, we can define a distance between (fuzzy) sets in this way.

Definition 2.9.1. Let U be the universe. A map $d\colon \mathbf{2}^U \times \mathbf{2}^U \to \mathbb{R}^+$ is a **set distance** if the following three conditions for all $A, B, C \in \mathbf{2}^U$ hold:

1. $d(A, B) = 0 \Leftrightarrow A = B$
2. $d(A, B) = d(B, A)$
3. $d(A, B) \leq d(A, C) + d(C, B)$

In this definition, we have used the non-negative real numbers $\mathbb{R}^+$ ($\{x \in \mathbb{R} | x \geq 0\}$) as the codomain to stress the fact that distances can never be negative, although non-negativity follows from the three conditions (by making $A = B$ in 3 and then using 1 and 2 to get $d(A, C) \geq 0$). The first axiomatic condition is sometimes weakened to $d(A, A) = 0$, which would allow distinct sets to have a zero distance. Spaces with such a weak form of distance are usually called **pseudometric spaces** [10].

Coding theory provides a very interesting distance between finite arbitrary sets: the Hamming distance, which is usually defined for strings (sequences) of symbols from an alphabet, but can be generalised to powersets via their characteristic function [9], giving rise to the following definition.

Definition 2.9.2. Let U be a finite universe. The **Hamming distance for sets** is a function $d_H \colon \mathbf{2}^U \times \mathbf{2}^U \to \mathbb{R}^+$ defined in terms of the characteristic function (2.1.1) and the cardinality of a set as:

$$d_H(A, B) := |\{u \in U | \chi_A(u) \neq \chi_B(u)\}| \tag{2.15}$$

It can be proved that the Hamming distance is indeed a distance as it fulfils the conditions in Definition 2.9.1. This distance endows finite powersets with a very simple metric structure that treats the elements of U as labels that can only be compared for equality or difference, and so a set made up of natural numbers like $A = \{1, 2\}$ would be as different from $B = \{3, 4\}$ as it is from $C = \{50, 51\}$, with $d_H(A, B)$ and $d_H(A, C)$ being both 4. Further refinements using the canonical metric structure on the natural numbers would be needed if we wanted numeric differences to contribute to the set distance as well.

It should be noted that the expression (2.15) above is basically the cardinality of the symmetric difference and can be re-arranged by a clever use of arithmetic as $\sum_{u \in U} |\chi_A(u) - \chi_B(u)|$. This hints at a straightforward generalisation to fuzzy sets. We can generalise both the general concept of a distance and the particular case of the Hamming distance, as follows.

Definition 2.9.3. Let X be the universe. A map $d \colon \mathbb{I}^X \times \mathbb{I}^X \to \mathbb{R}^+$ is a **fuzzy set distance** if the following three conditions for all $A, B, C \in \mathbb{I}^X$ hold:

1. $d(A, B) = 0 \Leftrightarrow A = B$
2. $d(A, B) = d(B, A)$
3. $d(A, B) \geq d(A, C) + d(C, B)$

As we see, the axiomatic conditions are exactly the same as for ordinary sets. The Hamming distance can also be defined in an analogous way.

Definition 2.9.4. Let X be a finite universe. The **Hamming distance for fuzzy sets** is a function $d_H \colon \mathbb{I}^X \times \mathbb{I}^X \to \mathbb{R}^+$ defined as:

$$d_H(A, B) = \sum_{x \in X} |A(x) - B(x)| \tag{2.16}$$

Again, the Hamming distance for fuzzy sets is indeed a distance for fuzzy sets as it fulfils the equations in Definition 2.9.3. It is interesting to note that the Hamming distance can also be interpreted as an example of a **taxicab distance** [23] (also called **Manhattan distance**) defined over $\mathbf{2}^{|U|}$ for crisp sets and over $\mathbb{I}^{|X|}$ for fuzzy sets. In an analogous way, we can define a form of Euclidean distance for fuzzy sets, as follows.

Definition 2.9.5. Let X be a finite universe. The **Euclidean distance for fuzzy sets** is a function $d_E \colon \mathbb{I}^X \times \mathbb{I}^X \to \mathbb{R}^+$ defined as:

$$d_E(A, B) = \sqrt{\sum_{x \in X} \left(A(x) - B(x)\right)^2} \tag{2.17}$$

The Hamming and Euclidean distances can be regarded as two particular cases of a more general family of distances parameterised by an exponent value, often called the Minkowski distances [27] (not to be confused with the Minkowski spacetime metric used in relativistic Physics).

Definition 2.9.6. Let X be a finite universe and let $p \geq 1$ be a real number. The **Minkowski distance for fuzzy sets** with exponent p is a function $d_M^p \colon \mathbb{I}^X \times \mathbb{I}^X \to \mathbb{R}^+$ defined as:

$$d_M^p(A, B) = \left(\sum_{x \in X} |A(x) - B(x)|^p\right)^{\frac{1}{p}} \tag{2.18}$$

With this definition, the Hamming and Euclidean distances are the Minkowski distances d_M^1 and d_M^2, respectively. We have excluded values $p < 1$ from the definition as they violate the triangle inequality.

Another common variation of these basic definitions includes a division by the cardinality of the universe X. We will refer to such distances as being "normalised" [27].

Definition 2.9.7. Let X be a finite universe and let $p \geq 1$ be a real number. The **normalised Minkowski distance for fuzzy sets** with exponent p is a function $d^p_{M_n} \colon \mathbb{I}^X \times \mathbb{I}^X \to \mathbb{R}^+$ defined as:

$$d^p_{M_n}(A, B) = \left(\frac{1}{|X|} \sum_{x \in X} |A(x) - B(x)|^p \right)^{\frac{1}{p}} \tag{2.19}$$

In the particular cases when $p = 1$ and $p = 2$, the distances are called the **normalised Hamming distance for fuzzy sets**, d_{H_n}, and **normalised Euclidean distance for fuzzy sets**, d_{E_n}, respectively.

The third axiomatic condition in the definition of a distance, the triangle inequality, is a relatively strong condition and can be replaced with alternative formulations. We can refer to such weaker proposals as **non-metric measures of difference** and we shall mention two of them that have been particularly fruitful in the fuzzy literature: the **dissimilarity measures** and the **divergence measures**.

The dissimilarity measures resort to a third condition based on inclusion by requiring that any two fuzzy sets A and B be more dissimilar between each other than with any other fuzzy set C such that $A \subseteq C \subseteq B$.

Definition 2.9.8. Let X be the universe. A map $D \colon \mathbb{I}^X \times \mathbb{I}^X \to \mathbb{R}^+$ is a **dissimilarity measure** if for all $A, B \in \mathbb{I}^X$ the following three conditions are met:

1. $D(A, B) = 0 \Leftrightarrow A = B$
2. $D(A, B) = D(B, A)$
3. $D(A, B) \geq \max\{D(A, C), D(C, B)\}$, if $A \subseteq C \subseteq B$

There are, nonetheless, examples of dissimilarity measures that fail to behave as intuitive measures of how different two fuzzy sets are [22], so further proposals have been made that modify the third axiom. A successful alternative to distances and dissimilarity measures that is not in general equivalent to either is the **divergence measure** originally proposed by S. Montes et al. [34] for ordinary fuzzy sets, later extended to other types of fuzzy sets and which has played a central role in our research. Since the original introduction of the concept, alternative axiomatic definitions have been proposed [8], but here we will adhere to the original formulation as it most closely resembles the other similar concepts we have defined.

Definition 2.9.9. Let X be the universe. A map $D\colon \mathbb{I}^X \times \mathbb{I}^X \to \mathbb{R}^+$ is a **divergence measure** if for all $A, B \in \mathbb{I}^X$ the following three conditions are met:

1. $D(A, B) = 0 \Leftrightarrow A = B$
2. $D(A, B) = D(B, A)$
3. $D(A, B) \geq \max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\}$, $\forall C \in \mathbb{I}^X$

The new third condition allows for a reasonable intuitive interpretation; basically, the more similar two sets become, the smaller the measure of their difference should be and this can be formalised through the use of a third non-empty set that dilutes the difference between the original sets. As in the distance definition, non-negativity can be deduced from the third and second conditions (in this case, by making C the empty set). It should also be noted that this third condition is actually a compact form of what is often expressed as two separate conditions: the divergence between A and B being an upper bound for the divergence of both the unions and the intersections with any third set C.

It is also worth noting that the third condition depends on how the intersection and union operations are defined, and these operations can be based on a pair of a t-norm and a t-conorm different from the standard ones, so the definition above is actually a family of definitions in the more general case. In the particular case of a singleton universe $X = \{x\}$, A, B and C are fully determined by their image at x, so they become three real numbers a, b and c. In this case, the dissimilarity and divergence measures are equivalent when the standard intersection and union operations are used. To see this, note that the third condition for a dissimilarity measure on a singleton becomes $D(a, b) \geq D(a, c)$ and $D(a, b) \geq D(c, b)$ with $a \leq c \leq b$. And the third condition for the divergence measure can be expressed as two inequalities $D(a, b) \geq D(\max\{a, c\}, \max\{b, c\})$ and $D(a, b) \geq D(\min\{a, c\}, \min\{b, c\})$, which are non-trivial when $a \leq c \leq b$ or $b \leq c \leq a$. And in both cases these conditions become $D(a, b) \geq D(b, c)$, and $D(a, b) \geq D(a, c)$. The apparently two conditions determined by $a \leq c \leq b$ and $b \leq c \leq a$ are a mere change of labels, so it is essentially the same condition as in the dissimilarity measure.

In the general case, however, there are divergence measures that are not dissimilarities and the other way around [22]. In fact, with a more populated universe, the assumption $A \subseteq C \subseteq B$ in the third dissimilarity condition leaves fully free behaviour for any pairs of A and B such that $A(x) > B(x)$ and $A(y) < B(y)$ for two different elements $x, y \in X$, so it is possible to concoct examples of dissimilarities that challenge the intuition of a measure of difference [22]. The third condition in the divergence definition, on the other hand, does not leave such gaps unaccounted for. This is the reason why we do believe that divergence measures can quantify the difference between fuzzy sets more accurately than dissimilarity measures.

A type of divergence measures that are particularly interesting and useful are those where any variation in the membership value for one specific element in the universe results in a variation in the divergence value that is a function of the membership value only. This is referred to as the **local property** and the formal definition is as follows.

Definition 2.9.10. Let X be the universe. A divergence measure $D\colon \mathbb{I}^X \times \mathbb{I}^X \to \mathbb{R}^+$ is a **local divergence measure** if for all $A, B \in \mathbb{I}^X$ and for all $x \in X$ there is a function $h\colon \mathbb{I} \times \mathbb{I} \to \mathbb{R}^+$ such that:

$$D(A,B) - D(A \cup \{x\}, B \cup \{x\}) = h(A(x), B(x)) \tag{2.20}$$

where $\{x\}$ is the fuzzy set such that $\{x\}(x) = 1$ and $\{x\}(x') = 0$ for any other $x' \neq x$.

Local divergence measures can be expressed as a sum of independent contributions for each element in the universe in terms of the function h. This important characterisation, which we are going to express as a representation theorem, imposes some restrictions on the form of the function h, which must behave as if it were a divergence (or dissimilarity) measure on a singleton universe. In order to simplify the local representation theorem that we will introduce later, we now provide a formal definition for these h functions, as follows.

Definition 2.9.11. A function $h\colon \mathbb{I} \to \mathbb{R}^+$ is called a **divergence characteristic function** if it fulfils the following conditions:

1. $h(u,v) = 0 \Leftrightarrow u = v$, $\forall u, v \in \mathbb{I}$
2. $h(u,v) = h(v,u)$, $\forall u, v \in \mathbb{I}$
3. $h(u,w) \geq \max(h(u,v), h(v,w))$, $\forall u, v, w \in \mathbb{I}$ such that $u \leq v \leq w$

Condition 3 implies that h must be bounded, as we now prove.

Proposition 2.9.12. *A divergence characteristic function $h\colon \mathbb{I} \to \mathbb{R}^+$ is bounded, its maximum value being $h(0,1)$.*

Proof. Let $a, b \in \mathbb{I}$ and $0 \leq a \leq b \leq 1$. Using condition $h(u, v) \leq h(u, w)$ with $0 \leq b \leq 1$, it follows that $h(0, b) \leq h(0, 1)$. And using condition $h(v, w) \leq h(u, w)$ with $0 \leq a \leq b$, we have that $h(a, b) \leq h(0, b)$. And so, the combination of both inequalities leads to $h(a, b) \leq h(0, 1)$ for any a and b. □

As we have proved that a divergence characteristic function is always bounded by $h(0, 1)$, we can define a normalised version that restricts the domain to the unit interval.

Definition 2.9.13. A function $h \colon \mathbb{I} \to \mathbb{I}$ is called a **normalised divergence characteristic function** if it fulfils the same properties as a divergence characteristic function with the additional property $h(0, 1) = 1$.

Armed with this definition, we can now state an important theorem [22].

Theorem 2.9.14 (Local Divergence Representation Theorem)**.** *Let X be the universe. A divergence measure $D \colon \mathbb{I}^X \times \mathbb{I}^X \to \mathbb{R}^+$ is a local divergence measure if and only if there is a normalised divergence characteristic function $h \colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ such that:*

$$D(A, B) := \sum_{x \in X} h(A(x), B(x)) \tag{2.21}$$

We omit the proof, which can be found in the referenced article. The Representation Theorem provides a characterisation of the local divergence measures as a sum over the normalised divergence characteristic function values for each element in the universe X. Note that this decomposition can be used as an alternative definition of the concept and this use of h as the building block for the local divergence measures justifies the name we have chosen for such functions. Intuitively, h measures the contribution to the divergence from each particular element of the universe. If the universe X is finite, any local divergence measure can be normalised to values in the unit interval by dividing the sum in expression (2.21) by the cardinality of X.

The Hamming distance d_H between fuzzy sets, which we introduced in Definition 2.9.4, is also a divergence measure and a local one as well, as we will now prove.

Proposition 2.9.15. *Let X be a finite universe. The Hamming distance between fuzzy sets over X is a local divergence measure.*

Proof. The expression (2.16) is a particular case of equation (2.21) with $h(x, y) = |x - y|$, the Euclidean distance on the unit interval, so we just need to prove that this function h is a normalised divergence characteristic function.

If $x, y \in \mathbb{I}$, $|x - y| \in \mathbb{I}$ too, so the domain is $\mathbb{I}$. As for the definition properties that h must fulfil, for any $u, v \in \mathbb{I}$ if $h(u, v) = 0$ then $|u - v| = 0$, which means that $u = v$. Conversely, $h(u, u) = 0$, so the first one of the conditions in Definition 2.9.11 holds. As for symmetry, $h(x, y) = |x - y| = |y - x| = h(y, x)$; and for the third condition, if $u \leq v \leq w$, then $h(u, v) = |u - v| \leq |u - w| = h(u, w)$. Similarly, $h(v, w) = |v - w| \leq |u - w| = h(u, w)$. □

Note that the normalised Hamming distance is also a local divergence measure, with characteristic function $h(x, y) = \frac{1}{|X|}|x - y|$.

As for the Euclidean distance between fuzzy sets of Definition 2.9.5, it is straightforward to prove that it is a divergence measure but not a local one, however, because of the square root. But note that the **squared Euclidean distance** is a local divergence measure with $h(x, y) = (x - y)^2$. Similarly, the Minkowski family of distances are all local divergence measures when the $1/p$ exponent is removed. We will refer to functions such as $f(u) := u^{1/p}$ and $f(u) := (\frac{1}{|X|}u)^{1/p}$ as **dimensionality normalisation functions**. Since such functions are monotonous for $u \in \mathbb{R}^+$, their numeric effect does not change the nature of a pair of fuzzy sets being more or less different than another pair and, consequently, Minkowski distances behave as local divergence measures for all practical purposes.

3 A comparison of finite sets, multisets and n-tuples

As we intend to study extended fuzzy sets where a membership grade can be a collection of various values, it makes sense to start our analysis by discussing the mathematical objects that we will need for such membership grades before introducing new definitions of fuzzy concepts.

Given a reference set U, there are three common approaches to the idea of picking a finite collection of n of its elements. The simplest way consists in taking a subset of U with cardinality n. In this case, the n elements have to be different and there is no order structure. If $a, b \in U$ and $a \neq b$, then $\{a, b\}$ is a subset of two elements. A second approach consists in taking a multiset of U, where repetitions are allowed but there is no order structure. We can have $\langle a, b\rangle$ or $\langle a, a\rangle$ as valid multisets of length 2. In the third common approach, elements of U can be arranged into n-tuples, which are ordered and allow repetition. In this approach (a, b), (b, a) and (a, a) are all valid and different 2-tuples (pairs). This chapter explores how these three concepts are interrelated.

3.1 The relation between multisets and sets

We are now going to discuss how multisets are related to sets. Our presentation will be geared to the subsequent generalisation to fuzzy sets. For a comprehensive account of the historical development of the concept of multisets and its mathematical status under the shadow of Cantorian sets, the reader is encouraged to refer to the in-depth article by Wayne D. Blizard [5].

As we know, a multiset contains more information than a set by allowing repeated values. In Chapter 2, we defined the support of a multiset as the set made up of the different elements in the multiset. By applying the support, we can turn a multiset into a set, at the cost of losing the extended multiplicity information. As the support is not injective, there is not an inverse function $\mathbf{2}^U \to \mathbb{N}^U$ that would let us recover the original multiset from its support. But, as with any non-invertible function, we can define the preimage as a function $\mathbf{2}^U \to \mathbf{2}^{(\mathbb{N}^U)}$; i.e. a set-valued function.

Definition 3.1.1. Given a universe U, the **multiset support preimage** of a set is a function $Supp^{-1}\colon \mathbf{2}^U \to \mathbf{2}^{(\mathbb{N}^U)}$ defined by the relation:

$$Supp^{-1}(S) := \{\hat{M} \in \mathbb{N}^U | Supp(\hat{M}) = S\}, \quad S \subseteq U \tag{3.1}$$

The intuitive identification between sets and multisets devoid of their multiplicity information can be formalised through an equivalence relation. As the support is the operation that separates the world of multisets from the world of sets, this equivalence should equate those multisets that share the same support, so its equivalence classes will be the sets that make up the support preimage $Supp^{-1}$. This idea can be expressed through the concept of equivalence kernel, which was introduced in Definition 2.8.6.

Definition 3.1.2. Given a universe U with multisets $\mathbb{N}^U$, the **repetition equivalence** in $\mathbb{N}^U$ is the equivalence kernel of the support function *Supp*.

This equivalence relation is denoted by $\sim_r$, so given two multisets $\hat{M}, \hat{N} \in \mathbb{N}^U$, $\hat{M} \sim_r \hat{N}$ means that $Supp(\hat{M}) = Supp(\hat{N})$.

The simplest way to pick a representative of each repetition-equivalence class consists in selecting the unique multiset where 0 and 1 are the only multiplicity values that appear. This is what we will call the canonical section for the repetition equivalence.

Definition 3.1.3. Given the repetition equivalence $\sim_r$ in the power multiset $\mathbb{N}^U$ over a universe U, the **repetition canonical section** is a section, as in Definition 2.8.5, $s_{rc}\colon \mathbb{N}^U/\sim_r \to \mathbb{N}^U$, such that any equivalence class $c \in \mathbb{N}^U/\sim_r$ is mapped to a multiset that satisfies the following condition:

$$(s_{rc}(c))(u) \in \{0, 1\}, \quad \forall u \in U \tag{3.2}$$

As we mentioned in Chapter 2, a section maps the equivalence classes into a complete set of representatives, so the multisets fulfilling condition (3.2) stand in bijection with the repetition equivalence classes.

Example 3.1.4. The two multisets $\hat{A} = \langle a, c, c\rangle$ and $\hat{B} = \langle a, a, c\rangle$ are repetition-equivalent as they share the same support $\{a, c\}$. The canonical representative of their equivalence class is the multiset $\langle a, c\rangle$, which can be identified with the (single-valued) set $\{a, c\}$.

It is intuitively obvious that there is a bijection between the repetition equivalence classes, or the repetition canonical representatives, and the powerset $\mathbf{2}^U$. This idea is formalised in the next proposition, which follows from Proposition 2.8.7 in the previous chapter as a particular case.

Proposition 3.1.5. *Let U be the universe. There exists a bijection $\phi_r : \mathbb{N}^U/\sim_r \leftrightarrow \mathbf{2}^U$ defined by:*

$$\phi_r = Supp \circ s_{rc} \tag{3.3}$$

We have chosen the repetition canonical section s_{rc} for convenience, but any other section would be equally valid. Thanks to this bijection, we can identify the repetition-equivalent multisets with the sets. This is a mere formalisation of an obvious fact. Note that the inverse bijection ${\phi_r}^{-1}$ can be expressed in terms of the multiset preimage as $[\ ] \circ Supp^{-1}$, where $[\ ]$ is the operation of taking the common equivalence class of any of the multisets in the image by $Supp^{-1}$. This is summed up in Figure 3.1.

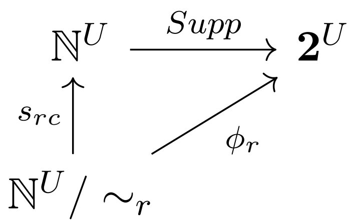

Figure 3.1: Commutation diagram for the repetition equivalence $\phi_{\sim_r}$ bijection.

At the intuition level, the bijection ϕ_r above can be interpreted as if each set $\{a, b\}$ stands for the whole group of repetition-equivalent multisets $\{\langle a, b\rangle, \langle a, a, b\rangle, ...\}$. But, as happens with any equivalence relation, there is also a bijection between the quotient space $\mathbb{N}^U/\sim_r$ and the complete set of representative multisets for any specific section, like the canonical one. This second bijection is the formal counterpart of another intuitive identification where the set $\{a, b\}$ is seen as equivalent to the multiset $\langle a, b\rangle$ with the other multisets with multiplicities larger han 1 being ignored. These two approaches can be used to justify alternative ways of restricting results from the world of multisets to the world of sets, as we will see when working out the extension principles of the next section.

It is also interesting to note that the standard definitions of intersection and union for multisets are compatible with the support function, as we show in the next proposition.

Proposition 3.1.6. *Let U be the universe and let $\hat{A}$ and $\hat{B}$ be two multisets over U. The following relations hold:*

$$\begin{aligned} Supp(\hat{A} \cap \hat{B}) &= Supp(\hat{A}) \cap Supp(\hat{B}) \\ Supp(\hat{A} \cup \hat{B}) &= Supp(\hat{A}) \cup Supp(\hat{B}) \end{aligned} \tag{3.4}$$

where $\cap$ and $\cup$ on the left-hand side refer to the multiset operations and on the right-hand side to the set operations.

Proof. In order to prove the first identity, we note that if an element u is in $Supp(\hat{A} \cap \hat{B})$, then by definition of support $(\hat{A} \cap \hat{B})(u) > 0$, and by definition of the multiset intersection, $\min\{\hat{A}(u), \hat{B}(u)\} > 0$, so $\hat{A}(u) > 0$ and $\hat{B}(u) > 0$, which means that $u \in Supp(\hat{A})$ and $u \in Supp(\hat{B})$. Since u is in both sets, it is also in their intersection: $u \in Supp(\hat{A}) \cap Supp(\hat{B})$

Conversely, if an element u is in $Supp(\hat{A}) \cap Supp(\hat{B})$, then, by definition of support, $\hat{A}(u) > 0$ and $\hat{B}(u) > 0$ and so, if we take the multiset intersection, $(\hat{A} \cap \hat{B})(u) = \min\{\hat{A}(u), \hat{B}(u)\} > 0$ and so u is an element of the support of the intersection $u \in Supp(\hat{A} \cap \hat{B})$.

The proof for the union is completely analogous. □

3.2 The extension principles for multisets

Having set up the basic formal relations between multisets and sets in the previous section, we now turn our attention to how functions operating on multisets can be induced by existing functions that operate on sets and the other way around.

We have already introduced the Powerset Extension Principle in Section 2.7. A completely analogous principle can be established for multisets.

Definition 3.2.1 (Multiset Extension Principle)**.** Let U and V be two sets and let V^U be the set of all the functions $U \to V$. Similarly, let $(\mathbb{N}^V)^{\mathbb{N}^U}$ be the set of all the functions $\mathbb{N}^U \to \mathbb{N}^V$. The **multiset extension** is a function $Ext_m \colon V^U \to (\mathbb{N}^V)^{\mathbb{N}^U}$ that maps any function $f \colon U \to V$ into the function $Ext_m(f) \colon \mathbb{N}^U \to \mathbb{N}^V$ defined by the expression:

$$(Ext_m(f)(\hat{A}))(v) := \sum_{u \in U | f(u) = v} \hat{A}(u), \quad \hat{A} \in \mathbb{N}^U,\, v \in V \tag{3.5}$$

The image of a function $f \colon U \to V$ by Ext_m can be denoted more simply by f_m and will be called the **multiset-extended** function.

Example 3.2.2. Let us revisit the same example that we chose for the Powerset Extension. If we have the real numbers $\mathbb{R}$, the real-valued multisets $\mathbb{N}^{\mathbb{R}}$ and the trigonometric sine function $\sin \colon \mathbb{R} \to [-1, 1]$, we can use the Multiset Extension Principle to build the multiset-extended sine function $\sin_m \colon \mathbb{N}^{\mathbb{R}} \to \mathbb{N}^{[-1,1]}$ which maps any real-valued multiset into a $[-1, 1]$-valued multiset. For example, if we take the multiset $\hat{A} = \langle -\pi, -\pi/2, 0, \pi/2, \pi/2, \pi, \pi \rangle$ then we have $\sin_m(\hat{A}) = \langle -1, 0, 0, 0, 0, 1, 1 \rangle$.

As in the case of the Powerset Extension, those functions $\mathbb{N}^U \to \mathbb{N}^V$ that arise from simpler functions $U \to V$ are a strict subset of the whole family of functions $(\mathbb{N}^V)^{\mathbb{N}^U}$. Adapting the previous example, a function such that $f(\langle 0, \pi, \pi \rangle) = \{0\}$ and $f(\langle 0, \pi \rangle) = \{1\}$ cannot be expresssed as a multiset extension. As with the other extension principles, a multivariate version that takes n arguments can be defined. The important thing to note in the multivariate version is that the multiplicities for each one of the function arguments must be multiplied together in order to account for the different possible combinations, as follows.

Definition 3.2.3 (Multiset Extension Principle – multivariate form)**.** Let U_1, ..., U_n and V be $n{+}1$ sets, with $n > 1$ a natural number, and let $V^{U_1 \times \ldots \times U_n}$ be the set of all the functions $U_1 \times ... \times U_n \to V$. Similarly, let $(\mathbb{N}^V)^{\mathbb{N}^{U_1} \times \ldots \times \mathbb{N}^{U_n}}$ be the set of all the functions $\mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n} \to \mathbb{N}^V$. The **multivariate form of the multiset extension** is a function $Ext_m \colon V^{(U_1 \times \ldots \times U_n)} \to (\mathbb{N}^V)^{(\mathbb{N}^{U_1} \times \ldots \times \mathbb{N}^{U_n})}$ that maps any function $f \colon U_1 \times ... \times U_n \to V$ into the function $Ext_m(f) \colon \mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n} \to \mathbb{N}^V$

defined by the expression:

$$(Ext_m(f)(\hat{A}_1, ..., \hat{A}_n))(v) := \sum_{\substack{(u_1,\ldots,u_n)\in U_1\times\ldots\times U_n \\ |f(u_1,\ldots,u_n)=v}} \left(\prod_{i=1}^{n} \hat{A}_i(u_i) \right), \quad \hat{A}_i \in \mathbb{N}^{U_i}\ ,\ v \in V \tag{3.6}$$

We will only be interested in the particular case with $n = 2$ and with $U_1 = U_2 = V$, which extends binary operators on a reference set.

Example 3.2.4. Let us take multisets over $\mathbb{N}$ and the sum function $+$ on $\mathbb{N}$. By using the Multivariate Multiset Extension Principle, we can build a new sum function $+_m$ that works on pairs of multisets of natural numbers. If we now take two such multisets like $\hat{A} = \langle 1, 2, 2\rangle$ and $\hat{B} = \langle 11, 12\rangle$ and add them together via $+_m$, this results in a multiset containing all the possible sums between elements of $\hat{A}$ and $\hat{B}$:

$$\hat{A} +_m \hat{B} = \langle 1 + 11, 1 + 12, 2 + 11, 2 + 12, 2 + 11, 2 + 12\rangle = \langle 12, 13, 13, 14, 13, 14\rangle$$

By using the formula (3.6), we can check that $(\hat{A} +_m \hat{B})(12) = 1$, $(\hat{A} +_m \hat{B})(13) = 3$, $(\hat{A} +_m \hat{B})(14) = 2$ and $(\hat{A} +_m \hat{B})(n) = 0$ for any n other than 12, 13 and 14. As expected, the formula matches the more intuitive construction used above where all the possible combinations were explicitly enumerated.

We can also define additional extension principles that turn functions that operate on sets into functions that operate on multisets and the other way around. In these cases, we have to deal with the implicit loss of information when going from the world of multisets to the world of sets. For a function operating on powersets, the only sensible thing that can be done consists in a combination of taking the support of the input multiset so that it can be passed to the function f and then picking one of the multisets in the support preimage through a section, typically the canonical one. The next definition encapsulates this idea.

Definition 3.2.5 (Set-to-Multiset Extension Principle)**.** Let U and V be two sets and let $(\mathbf{2}^V)^{(\mathbf{2}^U)}$ be the set of all the functions mapping subsets of U to subsets of V $\mathbf{2}^U \to \mathbf{2}^V$. Similarly, let $(\mathbb{N}^V)^{(\mathbb{N}^U)}$ be the set of all the functions $\mathbb{N}^U \to \mathbb{N}^V$. The **set-to-multiset extension** is a function $Ext_{s\to m}\colon (\mathbf{2}^V)^{(\mathbf{2}^U)} \to (\mathbb{N}^V)^{(\mathbb{N}^U)}$ that maps any function $f\colon \mathbf{2}^U \to \mathbf{2}^V$ into the function $Ext_{s\to m}(f)\colon \mathbb{N}^U \to \mathbb{N}^V$ defined by the expression:

$$(Ext_{s\to m}(f))(\hat{A}) := s_{rc}({\phi_r}^{-1}(f(Supp(\hat{A}))), \quad \hat{A} \in \mathbb{N}^U \tag{3.7}$$

The image of a function $f\colon U \to V$ by $Ext_{s\to m}$ can be denoted more simply by $f_{s\to m}$ and will be called the **set-to-multiset-extended** function.

We can also explore how a set-based function can be induced by an existing multiset-based function. This is the opposite of what we have just done. But in this case, we can come up with two natural approaches. As we mentioned before, sets can be identified with the repetition equivalence classes but also with the canonical complete set of representatives. Depending on which mental picture we stick to, we can establish two alternative extension principles.

Definition 3.2.6 (Canonical Multiset-to-Set Extension Principle)**.** Let U and V be two sets and let $(\mathbb{N}^V)^{(\mathbb{N}^U)}$ be the set of all the functions mapping multisets of U to multisets of V $\mathbb{N}^U \to \mathbb{N}^V$. Similarly, let $(\mathbf{2}^V)^{(\mathbf{2}^U)}$ be the set of all the functions $\mathbf{2}^U \to \mathbf{2}^V$. The **canonical multiset-to-set extension** is a function $Ext_{m\to_c s}\colon (\mathbb{N}^V)^{(\mathbb{N}^U)} \to (\mathbf{2}^V)^{(\mathbf{2}^U)}$ that maps any function $g\colon \mathbb{N}^U \to \mathbb{N}^V$ into the function $Ext_{m\to_c s}(g)\colon \mathbf{2}^U \to \mathbf{2}^V$ defined by the expression:

$$(Ext_{m\to_c s}(g))(A) := Supp(g(s_{rc}(\phi_r^{-1}(A))), \quad A \in \mathbf{2}^U \tag{3.8}$$

This approach depends on the choice of the canonical section. In the second kind of multiset-to-set extension, dependence on a choice of section is avoided by taking a multiset union over all the multisets in the multiset support preimage, as follows.

Definition 3.2.7 (Aggregate Multiset-to-Set Extension Principle)**.** Let U and V be two sets and let $(\mathbb{N}^V)^{(\mathbb{N}^U)}$ be the set of all the functions mapping multisets of U to multisets of V $\mathbb{N}^U \to \mathbb{N}^V$. Similarly, let $(\mathbf{2}^V)^{(\mathbf{2}^U)}$ be the set of all the functions $\mathbf{2}^U \to \mathbf{2}^V$. The **aggregate multiset-to-set extension** is a function $Ext_{m\to_a s}\colon (\mathbb{N}^V)^{(\mathbb{N}^U)} \to (\mathbf{2}^V)^{(\mathbf{2}^U)}$ that maps any function $g\colon \mathbb{N}^U \to \mathbb{N}^V$ into the function $Ext_{m\to_a s}(g)\colon \mathbf{2}^U \to \mathbf{2}^V$ defined by the expression:

$$(Ext_{m\to_a s}(g))(A) := Supp\left(\bigcup_{\hat{M}\in\phi_r^{-1}(A)} g(\hat{M})\right), \quad A \in \mathbf{2}^U \tag{3.9}$$

The two forms of the multiset-to-set extension lead to different results in general, but are equivalent if the original function in $(\mathbb{N}^V)^{(\mathbb{N}^U)}$ can be expressed as a multiset extension. This can be stated as a proposition.

Proposition 3.2.8. *Let U and V be two sets and let g be a function $g\colon \mathbb{N}^U \to \mathbb{N}^V$, such that it admits a decomposition as a multiset extension of a function $f\colon U \to V$; i.e. $(g(\hat{A}))(v) = \sum_{u\in U|f(u)=v} \hat{A}(u)$ with $\hat{A} \in \mathbb{N}^U$. Then the canonical multiset-to-set extension and the aggregate multiset-to-set extension are the same: $Ext_{m\to_c s}(g) = Ext_{m\to_a s}(g)$.*

*In this case, the canonical or aggregate multiset-to-set extension can simply be called **multiset-to-set extension** and is denoted by $Ext_{m\to s}$.*

Proof. Given a set $A \subseteq U$ made up of elements $\{a_1, ..., a_n\}$, we can evaluate its image by $Ext_{m\to_c s}(g)$ according to formula (3.8). Proceeding step by step, we first need to take the repetition equivalence class $\phi_r^{-1}(A)$, which groups together all those multisets whose support is A. We then have to apply the canonical section s_{rc}, which results in a multiset $\langle a_1, ..., a_n\rangle$. The next step consists in applying function g, which gives $g(\langle a_1, ..., a_n\rangle) = \langle f(a_1), ..., f(a_n)\rangle$. And finally, we take the support to turn the multiset into a set $\{f(a_1), ..., f(a_n)\}$, without any repeated values. Note that the result is the same for any section, so it is independent of the choice of the canonical section.

Since the multiset-to-set extensions relative to any fixed section s are the same, the multiset union of all of them, which is the aggregate multiset-to-set extension, will also be the same. □

A consequence of Proposition 3.2.8 is the following corollary.

Corollary 3.2.9. *For any two sets U and V, the powerset extension is the same as the composition of its multiset extension with the multiset-to-set extension.*

$$Ext_s = Ext_{m\to s} \circ Ext_m \tag{3.10}$$

The previous definitions and properties concerning the multiset-to-set extensions can be generalised to the multivariate versions, with all the results being completely analogous, including the latest corollary.

For completeness, we will now go through the equivalent concepts for the more general multivariate case. Let us first introduce the multivariate form of the Canonical Multiset-to-Set Extension Principle.

Definition 3.2.10 (Canonical Multiset-to-Set Extension Principle – multivariate form)**.** Let $U_1, ..., U_n$ and V be $n + 1$ sets, with $n > 1$ a natural number, and let $(\mathbb{N}^V)^{(\mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n})}$ be the set of all the functions $\mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n} \rightarrow \mathbb{N}^V$. Similarly, let $(\mathbf{2}^V)^{(\mathbf{2}^{U_1} \times ... \times \mathbf{2}^{U_n})}$ be the set of all the functions $\mathbf{2}^{U_1} \times ... \times \mathbf{2}^{U_n} \rightarrow \mathbf{2}^V$. The **multivariate form of the canonical multiset-to-set extension** is a function $Ext_{m \rightarrow_c s} \colon (\mathbb{N}^V)^{(\mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n})} \rightarrow (\mathbf{2}^V)^{(\mathbf{2}^{U_1} \times ... \times \mathbf{2}^{U_n})}$ that maps any function $g \colon \mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n} \rightarrow \mathbb{N}^V$ into the function $Ext_{m \rightarrow_c s}(g) \colon \mathbf{2}^{U_1} \times ... \times \mathbf{2}^{U_n} \rightarrow \mathbf{2}^V$ defined by the expression:

$$(Ext_{m \rightarrow_c s}(g))(A_1, ..., A_n) := Supp(g(s_{rc}(\phi_r^{-1}(A_1)), ..., s_{rc}(\phi_r^{-1}(A_n))) \tag{3.11}$$

with $A_1, ..., A_n \in \mathbf{2}^U$.

The Aggregate Multiset-to-Set Extension Principle can likewise be adapted to a multivariate form, as follows.

Definition 3.2.11 (Aggregate Multiset-to-Set Extension Principle – multivariate form)**.** Let $U_1, ..., U_n$ and V be $n+1$ sets, with $n > 1$ a natural number, and let $(\mathbb{N}^V)^{(\mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n})}$ be the set of all the functions $\mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n} \rightarrow \mathbb{N}^V$. Similarly, let $(\mathbf{2}^V)^{(\mathbf{2}^{U_1} \times ... \times \mathbf{2}^{U_n})}$ be the set of all the functions $\mathbf{2}^{U_1} \times ... \times \mathbf{2}^{U_n} \rightarrow \mathbf{2}^V$. The **multivariate form of the aggregate multiset-to-set extension** is a function $Ext_{m \rightarrow_a s} \colon (\mathbb{N}^V)^{(\mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n})} \rightarrow (\mathbf{2}^V)^{(\mathbf{2}^{U_1} \times ... \times \mathbf{2}^{U_n})}$ that maps any function $g \colon \mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n} \rightarrow \mathbb{N}^V$ into the function $Ext_{m \rightarrow_a s}(g) \colon \mathbf{2}^{U_1} \times ... \times \mathbf{2}^{U_n} \rightarrow \mathbf{2}^V$ defined by the expression:

$$(Ext_{m \rightarrow_a s}(g))(A_1, ..., A_n) := Supp\left(\bigcup_{\hat{M}_i \in \phi_r^{-1}(A_i)} g(\hat{M}_1, ..., \hat{M}_n) \right) \tag{3.12}$$

with $A_1, ..., A_n \in \mathbf{2}^U$.

The two forms of the Multiset-to-set extension lead to different results in general, but are equivalent if the original function in $(\mathbb{N}^V)^{(\mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n})}$ can be expressed as a multiset extension. This can be stated as a proposition.

Proposition 3.2.12. *Let $U_1, ..., U_n$ and V be $n+1$ sets, with $n > 1$ a natural number and let g be a function $g \colon \mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_n} \rightarrow \mathbb{N}^V$, such that it admits a decomposition as a multiset extension of a function $f \colon U^n \rightarrow V$; i.e. $g = Ext_m(f)$. Then the canonical multiset-to-set extension and the aggregate multiset-to-set extension are the same: $Ext_{m \rightarrow_c s}(g) = Ext_{m \rightarrow_a s}(g)$.*

*In this case, the canonical or aggregate multiset-to-set extension can simply be called **multiset-to-set extension** and is denoted by $Ext_{m \rightarrow s}$.*

Proof. Given n sets $A_i \subseteq U_i$ with $i = 1, ..., n$, each one made up of m_i elements $\{a_{i1}, ..., a_{im_i}\}$, we can evaluate the image of this n-tuple of sets by $Ext_{m \to_c s}(g)$ according to formula (3.11). Proceeding step by step, we first need to take the n repetition equivalence classes $\phi_r^{-1}(A_i)$, each one of them grouping together all those multisets whose support is A_i. We then have to apply the canonical section s_{rc} to each one of the n equivalence classes, which results in n multisets $\hat{M}_i = \langle a_{i1}, ..., a_{im_i} \rangle$. The next step consists in applying function g, which gives a multiset $\langle f(a_{11}, ..., a_{n1}), ..., f(a_{1m_1}, ..., a_{nm_n}) \rangle$. And finally, we take the support to turn the multiset into a set $\{f(a_{11}, ..., a_{n1}), ..., f(a_{1m_1}, ..., a_{nm_n})\}$, without any repeated values. Note that the result is the same for any section, so it is independent of the choice of the canonical section.

Since the multiset-to-set extensions relative to any fixed section s are the same, the multiset union of all of them, which is the aggregate multiset-to-set extension, will also be the same. □

A consequence of Proposition 3.2.12 is the following corollary.

Corollary 3.2.13. *For any two sets U and V, and in the general multivariate case, the powerset extension is the same as the composition of its multiset extension with the multiset-to-set extension.*

$$Ext_s := Ext_{m \to s} \circ Ext_m \tag{3.13}$$

3.3 The relation between n-tuples and multisets

We now turn our attention to how the n-tuples can be linked to the multisets. As the difference boils down to the presence or absence of an order relation, we can start by establishing an association between an n-tuple and the multiset that results from disregarding the order among the coordinates. As a concept, this is very similar to the way we defined the support of a multiset as the set that results from dropping repetition, so we will use a similar name.

Definition 3.3.1. Given a natural number $n \in \mathbb{N}$ with $n > 1$ and a universe U, the **multiset support** of an n-tuple is a function $Supp_m \colon U^n \to \mathbb{N}^U_{=n}$ defined by the relation:

$$Supp_m(\vec{A})(u) := |\{i \in \{1, ..., n\} | a_i = u\}|, \quad \vec{A} \in U^n, u \in U \tag{3.14}$$

Note that the codomain has been indicated as $\mathbb{N}^U_{=n}$, the n-regular power multiset, rather than $\mathbb{N}^U$, as the function will map any n-tuple into a multiset with cardinality n.

When two n-tuples have the same multiset support, they must obviously comprise the same coordinate values with the same repetition count, so their coordinates are related by a permutation. This characterisation can be expressed as a trivial proposition.

Proposition 3.3.2. *Let $n \in \mathbb{N}$ with $n > 1$ and let U be the universe. Two n-tuples $\vec{A}$ and $\vec{B}$ have the same multiset support, $Supp_m(\vec{A}) = Supp_m(\vec{B})$ if and only if there exists a permutation σ of the coordinate labels (i.e. a bijection $\sigma\colon \{1, ..., n\} \to \{1, ..., n\}$) such that $b_i = a_{\sigma(i)}$ for $i = 1, ..., n$.*

In a complete analogy to the situation with the support for multisets, we can define the n-tuple support preimage as a function $\mathbb{N}^U_{=n} \to \mathbf{2}^{(U^n)}$; i.e. a set-valued function.

Definition 3.3.3. Given a natural number $n \in \mathbb{N}$ with $n > 1$ and a universe U, the **n-tuple support preimage** of an n-regular multiset is a function $Supp_m^{-1}\colon \mathbb{N}^U_{=n} \to \mathbf{2}^{(U^n)}$ defined by the relation:

$$Supp_m^{-1}(\hat{M}) := \{\vec{A} \in U^n | Supp_m(\vec{A}) = \hat{M}\}, \quad \hat{M} \in \mathbb{N}^U_{=n} \tag{3.15}$$

We can now use these concepts in order to define a relation between those n-tuples that share the same multiset support. We express this through the equivalence kernel again.

Definition 3.3.4. Given a universe U and a natural number $n \in \mathbb{N}$ with $n > 1$, the **permutation equivalence** in U^n is the equivalence kernel of the multiset support function $Supp_m$.

This equivalence relation is denoted by $\sim_p$, so given two n-tuples $\vec{A}$, $\vec{B} \in U^n$, $\vec{A} \sim_p \vec{B}$ means that $Supp_m(\vec{A}) = Supp_m(\vec{B})$.

Using Proposition 3.3.2, this equivalence relation can also be stated in terms of a permutation of the coordinate labels; i.e. $\vec{A} \sim_p \vec{B}$ if and only if there is a permutation σ such that $b_i = a_{\sigma(i)}$ for $i = 1, ..., n$.

As in the previous section, the operation of picking a representative from the equivalence class will be referred to as a section, which in this case is an injective function $s\colon U^n/\sim_p \to U^n$.

Depending on the properties of the universe U, it may be possible to single out a section as "canonical". This is what happens when U is a totally ordered set (for example, when U is the real numbers or a real interval). In such a case, we can designate a canonical representative in the quotient space $U^n/\sim_p$ by sorting the values. As we can sort in either ascending or descending order, we will consider both possibilities.

Definition 3.3.5. Let U be a totally ordered universe. For a given dimension $n \in \mathbb{N}$ with $n > 1$, the **(ascending) canonical section** of the permutation equivalence $\sim_p$ is the section $s^{\uparrow}\colon U^n/\sim_p \to U^n$ such that, for any equivalence class $c \in U^n/\sim_p$, the components of $s^{\uparrow}(c)$ are sorted in ascending order: $s^{\uparrow}(c)_1 \leq ... \leq s^{\uparrow}(c)_n$.

Similarly, the **descending canonical section** of $\sim_p$ is the section $s_{\downarrow}$ such that, for any equivalence class $c \in U^n/\sim_p$, $s_{\downarrow}(c)_1 \geq ... \geq s_{\downarrow}(c)_n$.

The image of a permutation equivalence class $c \in U^n/\sim_p$ by its (ascending/descending) canonical section, $s^{\uparrow/\downarrow}$, is called the **(ascending/descending) canonical representative** of c.

As in the general case of Section 2.7, a section maps the equivalence classes into a complete set of representatives, so the sorted n-tuples under a total order stand in bijection to the permutation equivalence classes.

Example 3.3.6. The two triplets $\vec{A} = (1, 2, 2)$ and $\vec{B} = (2, 1, 2)$ are permutation-equivalent as they share the same coordinates. Their ascending canonical representative is $(1, 2, 2)$ and the descending one $(2, 2, 1)$.

As in the case of the repetition equivalence that induces the bijection ϕ_r, we can establish an analogous bijection between the permutation equivalence classes, or the sorted canonical representatives, and the set U^n in the form of a proposition, which can be proved as a particular case of Proposition 2.8.7.

Proposition 3.3.7. *Let U be the universe and let $n \in \mathbb{N}$ with $n > 1$. There exists a bijection $\phi_p\colon U^n/\sim_p \leftrightarrow \mathbb{N}^U_{=n}$ defined by:*

$$\phi_p := Supp_m \circ s \tag{3.16}$$

with s being any section.

This is analogous to what we did for the repetition equivalence. Now it is the permutation-equivalent n-tuples that we can identify with the n-regular multisets. As with the repetition, this is a formalisation of an obvious fact. The inverse bijection ${\phi_p}^{-1}$ can be expressed in terms of the n-tuple support preimage as $[\,] \circ Supp_m^{-1}$, where $[\,]$ is the operation of taking the common equivalence class of any of the n-tuples in the image by $Supp_m^{-1}$. This is shown graphically in Figure 3.2.

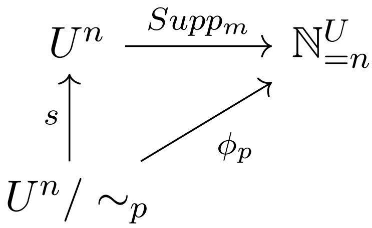

Figure 3.2: Commutation diagram for the permutation equivalence $\phi_{\sim_p}$ bijection.

Just as we did for the repetition equivalence, we can glimpse the intuition behind this new bijection ϕ_p by interpreting it as meaning that each multiset $\langle a, b, b\rangle$ stands for the whole group of permutation-equivalent n-tuples (a, b, b), (b, a, b) and (b, b, a). As with any equivalence relation, there also exists a bijection between the quotient space $U^n/\sim_p$ and the complete set of representative n-tuples for any given section, like a sorted canonical one when there is a total order on the universe U. This second bijection is the formal expression of another intuitive identification where the multiset $\langle a, b, b\rangle$ is seen as equivalent to just the n-tuple (a, b, b) if $a < b$ and we canonically sort in ascending order. In this case, the non-sorted n-tuples are ignored. These two views can be relied on to justify alternative ways of porting operations over n-tuples to the multisets, as we will see in the next section.

3.4 The extension principles for n-tuples

We can now replicate the strategy we followed when defining extension principles for multisets in an earlier section and establish a similar formal framework for the operations on n-tuples. We will first define a mechanism that mimics the Powerset Extension Principle and the Multiset Extension Principle and then study how functions operating on n-tuples can be induced by existing functions operating on multisets and the other way around.

Definition 3.4.1 (Cartesian Extension Principle)**.** Let U and V be two sets and let V^U be the set of all the functions $U \to V$. Similarly, for a natural number $n > 1$ let $(V^n)^{U^n}$ be the set of all the functions $U^n \to V^n$. The **Cartesian extension of dimension** n is a function $Ext_c \colon V^U \to (V^n)^{U^n}$ that maps any function $f \colon U \to V$ into the function $Ext_c(f) \colon U^n \to V^n$ defined by the expression:

$$(Ext_c(f))(a_1, ..., a_n) := (f(a_1), ..., f(a_n)), \quad (a_1, ..., a_n) \in U^n \tag{3.17}$$

The image of a function $f \colon U \to V$ by Ext_c can be denoted more simply by f_c and will be called the **Cartesian-extended** function.

Example 3.4.2. In a variation of the examples that we used for the Powerset Extension and the Multiset Extension, the trigonometric sine function $\sin \colon \mathbb{R} \to [-1, 1]$ can be extended to the space of real triplets $\mathbb{R}^3$ with the Cartesian-extended sine function $\sin_c \colon \mathbb{R}^3 \to [-1, 1]^3$ which maps any triplet of real numbers into a triplet of numbers in the closed interval $[-1, 1]$. For example, for the triplet $\vec{A} = (-\pi, \pi/2, \pi)$ we would have $\sin_c(\vec{A}) = (0, 1, 0)$.

As in the case of the Powerset Extension and Multiset Extension, those functions $U^n \to V^n$ that arise from simpler functions $U \to V$ are a strict subset of the whole family of functions $(V^n)^{(U^n)}$.

The multivariate version of the Cartesian Extension Principle is relevant to our purposes, as it is used in the definitions of binary operations of fuzzy multisets that we will introduce in the next chapter. It can be stated as follows.

Definition 3.4.3 (Cartesian Extension Principle – multivariate form)**.** Let U_1, ..., U_m and V be $m + 1$ sets, with $m > 1$ a natural number, and let $V^{U_1 \times ... \times U_m}$ be the set of all the functions $U_1 \times ... \times U_m \to V$. Similarly, for a natural number $n > 1$, let $(V^n)^{U_1^n \times ... \times U_m^n}$ be the set of all the functions $U_1^n \times ... \times U_m^n \to V^n$. The **multivariate form of the Cartesian extension of dimension** n is a function $Ext_c \colon V^{(U_1 \times ... \times U_m)} \to (V^n)^{U_1^n \times ... \times U_m^n}$ that maps any function $f \colon U_1 \times ... \times U_m \to V$ into the function $Ext_c(f) \colon U_1^n \times ... \times U_m^n \to V^n$ defined by the expression:

$$(Ext_c(f))(\vec{A}_1, ..., \vec{A}_m) := (f(a_{11}, ..., a_{m1}), ..., f(a_{1n}, ..., a_{mn})) \tag{3.18}$$

with $(a_{i1}, ..., a_{in})$ being the coordinates of the i-th n-tuple $\vec{A}_i \in U_i^n$.

We will only be interested in the particular case with $m = 2$ and with $U_1 = U_2 = V$, which extends binary operators on a reference set U.

Example 3.4.4. Let us take the power function with integer exponents $p\colon \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ defined as $p(a, b) := a^b$. If we take its Cartesian extension with dimension 3, we get a power function that relates triplets of bases to triplets of exponents by direct application of formula (3.18) so that, for example, $p_c((2, 3, 4.5), (2, 2, 3)) = (2^2, 3^2, 4.5^3)$.

The previous example helps to clarify the mechanism at work. The interesting case for us will be that of binary operators working on one numeric space, like the min function in $\mathbb{R}$, which can be extended to n-tuples as $min_c((a_1, ..., a_n), (b_1, ..., b_n)) = (\min(a_1, b_1), ..., \min(a_n, b_n))$.

Just as we did with the sets and the multisets, further extension principles can be established that convert functions operating on multisets into functions operating on n-tuples and the other way around, accepting the inevitable loss of information when transitioning from the n-tuples to the multisets. For a function f operating on multisets, there is little that can be done other than taking the multiset support of the input n-tuple so that it can be passed to the function f and then selecting one of the n-tuples in the n-tuple support preimage through a section. This idea is formalised in the next definition.

Definition 3.4.5 (Multiset-to-Cartesian Extension Principle)**.** Let U and V be two sets and let $(V^n)^{(U^n)}$ be the set of all the functions mapping n-tuples of U to n-tuples of V $U^n \to V^n$. Similarly, let $(\mathbb{N}^V_{=n})^{(\mathbb{N}^U_{=n})}$ be the set of all the functions $\mathbb{N}^U_{=n} \to \mathbb{N}^V_{=n}$, restricted to the n-regular multisets. And let $s\colon V^n/\sim_p \to V^n$ be a section defined on the quotient set $V^n/\sim_p$. The **multiset-to-Cartesian extension**, relative to s, is a function $Ext_{m\to_s c}\colon (\mathbb{N}^V_{=n})^{(\mathbb{N}^U_{=n})} \to (V^n)^{(U^n)}$ that maps any function $f\colon \mathbb{N}^U_{=n} \to \mathbb{N}^V_{=n}$ into the function $Ext_{m\to_s c}(f)\colon U^n \to V^n$ defined by the expression:

$$(Ext_{m\to_s c}(f))(\vec{A}) := s({\phi_p}^{-1}(f(Supp_m(\vec{A}))), \quad \vec{A} \in U^n \tag{3.19}$$

The image of a function $f\colon U \to V$ by $Ext_{m\to_s c}$ can be denoted more simply by $f_{m\to_s c}$ and will be called the **multiset-to-Cartesian-extended** function.

We can also study how a multiset-based function can be induced by an existing n-tuple-based function. In this case we can come up with two natural approaches, depending on whether we privilege a particular section in order to pick an n-tuple (typically, a sorted one) or treat all the permutation-equivalent n-tuples on an equal basis. This leads to two techniques that mirror the situation with sets and multisets that we discussed before.

Compared with the multiset-to-set extensions, there is however a significant difference stemming from the fact that an n-tuple-based function requires exactly n arguments. As we will need to resort to the ϕ_p^{-1} function followed by a section in order to turn the multiset into an n-tuple, the Cartesian-to-multiset extensions that we are going to define will have the limitation that the resulting extended function should act on the n-regular multisets rather than on any arbitrary multiset. By settling on a regularisation strategy, it is also possible to extend the domain of the extended functions to the n-bounded multisets, but in order to keep the formal derivations simpler, we will omit the details involving regularisation from the discussion that follows.

With these observations in mind, we can proceed to state the two forms of the Cartesian-to-multiset extension.

Definition 3.4.6 (Sorted Cartesian-to-Multiset Extension Principle)**.** Let U and V be two sets and let $(V^n)^{(U^n)}$ be the set of all the functions mapping n-tuples of U to n-tuples of V $U^n \to V^n$, with a natural number $n > 1$. Similarly, let $(\mathbb{N}^V_{=n})^{(\mathbb{N}^U_{=n})}$ be the set of all the functions $\mathbb{N}^U_{=n} \to \mathbb{N}^V_{=n}$, restricted to n-regular multisets. And let $s\colon U^n/\sim_p \to U^n$ be a section defined on the quotient set $U^n/\sim_p$. The **sorted Cartesian-to-multiset extension**, relative to section s, is a function $Ext_{c\to_s m}\colon (V^n)^{(U^n)} \to (\mathbb{N}^V_{=n})^{(\mathbb{N}^U_{=n})}$ that maps any function $g\colon U^n \to V^n$ into the function $Ext_{c\to_s m}(g)\colon \mathbb{N}^U_{=n} \to \mathbb{N}^V_{=n}$ defined by the expression:

$$(Ext_{c\to_s m}(g))(\hat{A}) := Supp_m(g(s(\phi_p^{-1}(\hat{A})))), \quad \hat{A} \in \mathbb{N}^U_{=n} \tag{3.20}$$

This approach depends on the choice of the section s, the sorting strategy. In the second kind of Cartesian-to-multiset extension, this dependence is avoided by taking a multiset union over those multisets that are multiset supports of the images by the original function of all the n-tuples in the n-tuple support preimage, as follows.

Definition 3.4.7 (Aggregate Cartesian-to-Multiset Extension Principle)**.** Let U and V be two sets and let $(V^n)^{(U^n)}$ be the set of all the functions mapping n-tuples of U to n-tuples of V $U^n \to V^n$, with a natural number $n > 1$. Similarly, let $(\mathbb{N}^V)^{(\mathbb{N}^U_{=n})}$ be the set of all the functions $\mathbb{N}^U_{=n} \to \mathbb{N}^V$. The **aggregate Cartesian-to-multiset extension** is a function $Ext_{c\to_a m}\colon (V^n)^{(U^n)} \to (\mathbb{N}^V)^{(\mathbb{N}^U_{=n})}$ that maps any function $g\colon U^n \to V^n$ into the function $Ext_{c\to_a m}(g)\colon \mathbb{N}^U_{=n} \to \mathbb{N}^V$

defined by the expression:

$$(Ext_{c \to_a m}(g))(\hat{A}) := \bigcup_{\vec{T} \in \phi_p^{-1}(\hat{A})} Supp_m(g(\vec{T})), \quad \hat{A} \in \mathbb{N}^U_{=n} \tag{3.21}$$

Note that in the aggregate form, as opposed to the sorted form, the codomain can no longer be guaranteed to be the n-regular power multiset $\mathbb{N}^V_{=n}$, as the multiset union may increase the cardinality.

These two alternative forms of the Cartesian-to-multiset extension yield different results in general, but are equivalent if the original function in $(V^n)^{(U^n)}$ can be expressed as a Cartesian extension. This can be stated as a proposition.

Proposition 3.4.8. *Let U and V be two sets and let g be a function $g\colon U^n \to V^n$, with a natural number $n > 1$, such that it admits a decomposition as a Cartesian extension of a function $f\colon U \to V$; i.e. $g(u_1, ..., , u_n) = (f(u_1), ..., f(u_n))$ with $(u_1, ..., u_n) \in U^n$. Then the sorted Cartesian-to-multiset extension for any section s defined on the quotient set $U^n/\sim_p$ and the aggregate Cartesian-to-multiset extension are all the same: $Ext_{c\to_a m}(g) = Ext_{c\to_s m}(g)$ for all sections $s\colon U^n/\sim_p \to U^n$.*

*In this case, the sorted or aggregate Cartesian-to-multiset extension can simply be called **Cartesian-to-multiset extension** and is denoted by $Ext_{c\to m}$.*

Proof. Given an n-regular multiset $\hat{A} \in \mathbb{N}^U_{=n}$ made up of elements $\langle a_1, ..., a_n \rangle$, we can evaluate its image by $Ext_{c\to_s m}(g)$ by using equation (3.20). We first need to take the permutation equivalence class $\phi_p^{-1}(\hat{A})$, which groups together all those n-tuples that are permutations of the values in $\hat{A}$ counted as many times as their multiplicity indicates. After that, we have to apply section s, which results in an n-tuple $(a_{\sigma_s(1)}, ..., a_{\sigma_s(n)})$, where σ_s represents the index permutation that characterises the section s. The next step consists in applying function g, which gives $g(a_{\sigma_s(1)}, ..., a_{\sigma_s(n)}) = (f(a_{\sigma_s(1)}), ..., f(a_{\sigma_s(n)}))$. At this stage, it becomes clear that the image by $Ext_{c\to_s m}(g)$ is independent of the section as the next step involves taking the multiset support, which leads to the multiset $\langle f(a_{\sigma_s(1)}), ..., f(a_{\sigma_s(n)}) \rangle$ as the final result regardless of the choice of section. Since all the sorted Cartesian-to-multiset extensions are the same, their multiset union, which is the aggregate Cartesian-to-multiset extension, will also be the same multiset. □

A consequence of Proposition 3.4.8 is the following corollary.

Corollary 3.4.9. *Let U and V be two sets. For any dimension $n > 1$, the multiset extension for n-regular multisets is the same as the composition of its Cartesian extension with the Cartesian-to-multiset extension.*

$$Ext_m = Ext_{c \to m} \circ Ext_c \tag{3.22}$$

But the situation becomes more complicated when tackling the multivariate versions of the above results. Whereas in the multiset-to-set case, the previous corollary would also hold for the multivariate versions, it turns out that this is not true now, as we are going to see.

Let us first introduce the multivariate form of the Sorted Cartesian-to-Multiset Extension Principle.

Definition 3.4.10 (Sorted Cartesian-to-Multiset Extension Principle – multivariate form)**.** Let $U_1, ..., U_m$ and V be $m + 1$ sets, with $m > 1$ a natural number and, for another natural number $n > 1$, let $(V^n)^{U_1^n \times ... \times U_m^n}$ be the set of all the functions $U_1^n \times ... \times U_m^n \to V^n$. Similarly, let $(\mathbb{N}^V_{=n})^{(\mathbb{N}^{U_1}_{=n} \times ... \times \mathbb{N}^{U_m}_{=n})}$ be the set of all the functions $\mathbb{N}^{U_1}_{=n} \times ... \times \mathbb{N}^{U_m}_{=n} \to \mathbb{N}^V_{=n}$. And for each index $i = 1, ..., m$, let $s_i : U_i^n / \sim_p \to U_i^n$ be a section defined on the quotient set $U_i^n / \sim_p$. The **multivariate form of the sorted Cartesian-to-multiset extension**, relative to sections $\{s_i\}_{i=1,...,m}$, is a function $Ext_{c \to_s m} : (V^n)^{U_1^n \times ... \times U_m^n} \to (\mathbb{N}^V_{=n})^{\mathbb{N}^{U_1}_{=n} \times ... \times \mathbb{N}^{U_m}_{=n}}$ that maps any function $g : U_1^n \times ... \times U_m^n \to V^n$ into the function $Ext_{c \to_s m}(g) : \mathbb{N}^{U_1}_{=n} \times ... \times \mathbb{N}^{U_m}_{=n} \to \mathbb{N}^V_{=n}$ defined by the expression:

$$(Ext_{c \to_s m}(g))(\hat{A}_1, ..., \hat{A}_m) := Supp_m(g(s_1(\phi_p^{-1}(\hat{A}_1)), ..., s_n(\phi_p^{-1}(\hat{A}_n)))) \tag{3.23}$$

with $\hat{A}_i \in \mathbb{N}^{U_i}_{=n}$ for $i = 1, ..., m$.

The Aggregate Cartesian-to-Multiset Extension Principle can likewise be adapted to a multivariate form, as follows.

Definition 3.4.11 (Aggregate Cartesian-to-Multiset Extension Principle – multivariate form)**.** Let $U_1, ..., U_m$ and V be $m+1$ sets, with $m > 1$ a natural number and, for another natural number $n > 1$, let $(V^n)^{U_1^n \times ... \times U_m^n}$ be the set of all the functions $U_1^n \times ... \times U_m^n \to V^n$. Similarly, let $(\mathbb{N}^V)^{(\mathbb{N}^{U_1}_{=n} \times ... \times \mathbb{N}^{U_m}_{=n})}$ be the set of all the functions $\mathbb{N}^{U_1}_{=n} \times ... \times \mathbb{N}^{U_m}_{=n} \to \mathbb{N}^V$. The **multivariate form of the aggregate Cartesian-to-multiset extension** is a function $Ext_{c \to_a m} : (V^n)^{U_1^n \times ... \times U_m^n} \to$

$(\mathbb{N}^V)^{\mathbb{N}^{U_1}_{=n} \times \ldots \times \mathbb{N}^{U_n}_{=m}}$ that maps any function $g\colon U_1^n \times ... \times U_m^n \to V^n$ into the function $Ext_{c \to_a m}(g)\colon \mathbb{N}^{U_1}_{=n} \times ... \times \mathbb{N}^{U_m}_{=n} \to \mathbb{N}^V$ defined by the expression:

$$(Ext_{c \to_a m}(g))(\hat{A}_1, ..., \hat{A}_m) = \bigcup_{\{\vec{T}_i \in \phi_p^{-1}(\hat{A}_i)\}_{i=1,\ldots,m}} Supp_m(g(\vec{T}_1, ..., \vec{T}_m)) \tag{3.24}$$

with $\hat{A}_i \in \mathbb{N}^{U_i}_{=n}$ for $i = 1, ..., m$.

So far, it seems as if the extension of functions to n-tuples of values follows exactly the same paradigm as the extensions to multisets and we may feel tempted to think that the multivariate extensions will reproduce analogous results. However, there is a striking difference in store. As it turns out, Proposition 3.4.8 does not hold in the multivariate case. A function $g\colon U_1^n \times ... \times U_m^n \to V^n$ that can be expressed as a Cartesian extension will in general have different sorted Cartesian-to-multiset extensions depending on the choice of section. The aggregate version is therefore also different. The fact that we cannot find a parallel to Proposition 3.4.8 implies that there is no analogous result to Corollary 3.4.9 either. In contrast to the case of the powerset extensions, with the multiset extensions we find that there are different approaches leading to different results in the multivariate case. We might feel that the digression through a Cartesian extension to end up with a multiset is an unnecessary complication, but that is the way operations have traditionally been defined on the fuzzy multisets that we will discuss in the next chapter, so we need to keep these alternative approaches in mind. We can thus conclude our discussion with a different kind of corollary enumerating the three alternative forms of multiset extensions.

Corollary 3.4.12. *Let U_1, ..., U_n and V be $m+1$ sets, with $m \geq 1$ a natural number, and let $V^{U_1 \times \ldots \times U_m}$ be the set of all the functions $U_1 \times ... \times U_m \to V$. Similarly, let $n > 1$ be a natural number and let $(\mathbb{N}^V)^{\mathbb{N}^{U_1}_{=n} \times \ldots \times \mathbb{N}^{U_m}_{=n}}$ be the set of all the functions $\mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_m} \to \mathbb{N}^V$. Three alternative ways in which a function $f\colon U_1 \times ... \times U_m \to V$ can induce a function $\mathbb{N}^{U_1} \times ... \times \mathbb{N}^{U_m} \to \mathbb{N}^V$ are given by the following extensions:*

1. *The **multiset extension** Ext_m (see Definition 3.2.3). This approach results in a multiset with a cardinality that is the product of the cardinalities of the input multisets, so we will also call it the **product multiset extension** to differentiate it from the other two approaches.*

2. *The* ***Cartesian followed by sorted Cartesian-to-multiset extension*** $Ext_{c \to_s m} \circ Ext_c$*, which depends on a choice of sections (see definitions 3.4.3 and 3.4.10). This approach requires that the input multisets be regularised to a common cardinality and the result is a multiset with the same cardinality. We will also call it the* ***sorted multiset extension****.*

3. *The* ***Cartesian followed by aggregate Cartesian-to-multiset extension*** $Ext_{c \to_a m} \circ Ext_c$ *(see definitions 3.4.3 and 3.4.11). This approach also requires that the input multisets be regularised to a common cardinality. We will also call it the* ***aggregate multiset extension****.*

In the case $n = 1$ *and if the function* f *can be expressed as a Cartesian extension, these three extensions are the same, as per Corollary 3.4.9.*

Example 3.4.13. Let us take the unit interval $\mathbb{I}$ as our universe of discourse and the functions $c(x) = 1 - x$, min and max and let us see how we can extend these functions to the power multiset $\mathbb{N}^{\mathbb{I}}$.

In the case of function c, it is a unary operation, so the three extensions in Corollary 3.4.12 give the same result:

$$c_m(\langle a_1, ..., a_n \rangle) = \langle 1 - a_1, ..., 1 - a_n \rangle$$

But the case of the min and max operators is not so simple as these are binary operators, so the three possibilities given in Corollary 3.4.12 are actually different. Let us take two multisets over the unit interval $\mathbb{I}$ $\hat{A} = \langle 0.1, 0.2, 0.2 \rangle$ and $\hat{B} = \langle 0.15, 0.25, 0.25 \rangle$ and we are going to extend the min function to calculate $\min(\hat{A}, \hat{B})$ with the three approaches.

In order to calculate the product multiset extension of min, $\min_m$, we need to take all the combinations of ordered pairs comprising an element of $\hat{A}$ and an element of $\hat{B}$ and take their minimum. As the input multisets have a cardinality of 3, the result will have a cardinality of 9. Starting with $\min(0.1, 0.15)$ and doing all the 9 combinations up to $\min(0.2, 0.25)$, we arrive at the result $\langle 0.1, 0.1, 0.1, 0.15, 0.15, 0.2, 0.2, 0.2, 0.2 \rangle$.

If we now calculate the sorted multiset extension with the ascending canonical section $s^{\uparrow}$, $\min_{c \to_s m}$, we get $\min_{c \to_s m}(\hat{A}, \hat{B}) = Supp_m(\min_c(s^{\uparrow}(\phi_p^{-1}(\hat{A})), s^{\uparrow}(\phi_p^{-1}(\hat{B}))))$, and in this particular example $Supp_m(\min_c((0.1, 0.2, 0.2), (0.15, 0.25, 0.25))) = Supp_m(0.1, 0.2, 0.2) = \langle 0.1, 0.2, 0.2 \rangle$.

Note that the sorted multiset extension preserves the cardinality of the two input multisets, unlike the product multiset extension which multiplies the input cardinalities.

Finally, in order to calculate the aggregate multiset extension of the min function, we need to apply the following formula:

$$\min_{c\rightarrow_a m}(\hat{A},\hat{B}) = \bigcup_{\substack{\vec{S}\in\phi_p^{-1}(\hat{A})\\ \vec{T}\in\phi_p^{-1}(\hat{B})}} Supp_m(\min_c(\vec{S},\vec{T})),$$

which in this particular example expands to a long multiset union over 36 terms $(3!\times 3!)$: $Supp_m(\min_c((0.1,0.2,0.2),(0.15,0.25,0.25)))\cup...\cup Supp_m(\min_c((0.2,0.2,0.1),(0.25,0.25,0.15))) = \langle 0.1,0.2,0.2\rangle \cup ... \cup \langle 0.2,0.2,0.1\rangle$. By going through the 36 terms, we get $= \langle 0.1,0.15,0.2,0.2\rangle$, a result that is different from the previous ones.

This example is based on a real use case of interest: the way multiset-based operations are defined for fuzzy multisets, which will be one of the main topics in the next chapter.

4 Set-based and multiset-based fuzzy sets

In the previous chapter, we have discussed sets, multisets and n-tuples defined over arbitrary reference sets. The discussion led us to the mathematical spaces $\mathbf{2}^U$, $\mathbb{N}^U$ and U^n for an underlying universe U. In this chapter, we use these mathematical structures in the particular case where the universe for such collections of values is a space of fuzzy membership grades, which we denote by L in the general case. In this way, we can define three corresponding forms of fuzzy membership $\mathbf{2}^L$, $\mathbb{N}^L$ and L^n depending on whether we arrange the values as sets, multisets or n-tuples, respectively. In a final step, three types of extended fuzzy sets are defined by mapping a universe X into each one of these three types of membership grades: $\left(\mathbf{2}^L\right)^X$, $\left(\mathbb{N}^L\right)^X$ and $(L^n)^X$. These types of extended fuzzy sets are what we already defined as multivalued fuzzy sets in Chapter 2 and are the main subject of this dissertation.

The key idea when introducing these extended forms of fuzzy sets is that the results from the underlying set of membership grades induce analogous results in the derived fuzzy sets in a pointwise manner, using what we have called Goguen's Extension Principle (Definition 2.7.6) in Chapter 2.

We begin our study by establishing the terminology that we will be using for the underlying space of membership grades.

4.1 The common spaces of membership grades for fuzzy sets

So far, we have been referring to an arbitrary universe U when discussing sets, multisets and n-tuples. As we close in on the fuzzy sets, the universe to consider should be something that works as a membership grade space for fuzzy sets. In the case of crisp sets, membership is represented by a two-valued set $\mathbf{2}$, whereas ordinary fuzzy sets use the unit interval $\mathbb{I}$.

As we mentioned in the chapter on preliminary concepts, membership grades need to support some basic operations. The key idea here is that the membership space $\mathbf{2}$ of crisp sets should have the three operations of logical negation (also called complementation), logical AND (intersection) and logical OR (union). These core operations can be generalised to other membership spaces. Besides, the membership spaces should have at least a partial order and be bounded with a bottom and a top element that generalise the role of 0 and 1, respectively, in the crisp membership space. We will refer to the membership grade spaces as **algebras** when they are endowed with these three operations and, optionally, bottom and top elements.

Definition 4.1.1. The **Boolean algebra** is $(\mathbf{2}, \neg, \wedge, \vee, 0, 1)$, where the membership space $\mathbf{2}$ is endowed with the operations $\neg$, $\wedge$, $\vee$, defined as follows:

$$\begin{array}{ll} 1. & \neg 0 = 1 \ ; \ \neg 1 = 0 \\ 2. & 0 \wedge 0 = 0 \ ; \ 0 \wedge 1 = 0 \ ; \ 1 \wedge 0 = 0 \ ; \ 1 \wedge 1 = 1 \\ 3. & 0 \vee 0 = 0 \ ; \ 0 \vee 1 = 1 \ ; \ 1 \vee 0 = 1 \ ; \ 1 \vee 1 = 1 \end{array} \tag{4.1}$$

For the ordinary fuzzy membership space, we can define a standard algebra based on the general operations of complement, t-norm and t-conorm that we defined in Chapter 2.

Definition 4.1.2. An **ordinary fuzzy algebra** is $(\mathbb{I}, \neg, \wedge, \vee, 0, 1)$, where the unit interval $\mathbb{I}$ is endowed with the three operations $\neg$, $\wedge$, $\vee$, such that $\neg$ is a complement, $\wedge$ is a t-norm and $\vee$ is a t-conorm.

Most often, ordinary fuzzy sets are used with the standard operations.

Definition 4.1.3. The **standard ordinary fuzzy algebra** is an ordinary fuzzy algebra $(\mathbb{I}, c, \min, \max, 0, 1)$, where the unit interval is endowed with the operations of standard complement c, defined as $c(x) := 1 - x$, and min, the minimum, and max, the maximum.

In the case of an L-fuzzy membership space, we can also define an algebra where the operations $\neg$, $\wedge$ and $\vee$ must fulfil certain minimal conditions. It is a bit of a moot point what the absolute minimum to be expected is. J. A. Goguen in his original article on L-fuzzy sets expected them to be bounded lattices, hence the "L" in their name, but it turns out that the algebra for type-2 fuzzy sets is not a lattice but a weaker structure called a De Morgan-Birkhoff system, which is a bounded bisemilattice with a negation operation fulfilling certain properties [46]. This might seem like a good choice for the minimal properties, but we will find situations, particularly when dealing with multisets, where the properties must be even weaker, so we will stick to a very general basic definition and list some additional desirable properties.

Definition 4.1.4. An **L-fuzzy algebra** for a partially ordered set L is a space $(L, \neg, \wedge, \vee)$, where L is endowed with a unary operation $\neg\colon L \to L$ and two binary operations $\wedge\colon L \times L \to L$ and $\vee\colon L \times L \to L$. These operations must fulfil the properties $\neg\neg x = x$ (involution), associativity and commutativity for $\wedge$ and $\vee$, as well as $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg(x \vee y) = \neg x \wedge \neg y$.

In addition to these basic properties, it is usually expected that L should be bounded with a bottom element $\mathbf{0}$ and a top element $\mathbf{1}$ such that $\neg\mathbf{0} = \mathbf{1}$, $\neg\mathbf{1} = \mathbf{0}$ and, for all $x \in L$, $x \wedge \mathbf{0} = \mathbf{0}$, $x \wedge \mathbf{1} = x$, $x \vee \mathbf{0} = x$ and $x \vee \mathbf{1} = \mathbf{1}$. In such cases, we will refer to the L-fuzzy algebra in full as $(L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1})$.

As a partially ordered set, the most basic choice of operations for $\wedge$ and $\vee$ are the meet and join, which are idempotent operations ($x \wedge x = x$, $x \vee x = x$, for any $x \in L$), but non-idempotent operations as generalisations of t-norms and t-conorms on L are also possible.

Just like functions between partially ordered sets that preserve the ordering structure are called **order homomorphisms** and functions between lattices that preserve the meets and joins are called **lattice homomorphisms** [15], or **isomorphisms** when the functions are bijections, it makes sense to use the same terms for functions between L-fuzzy algebras that preserve the mathematical structure given by the three operations $\neg$, $\wedge$ and $\vee$.

Definition 4.1.5. Let $(L_1, \neg_1, \wedge_1, \vee_1)$ and $(L_2, \neg_2, \wedge_2, \vee_2)$ be two L-fuzzy algebras. A function $h\colon L_1 \to L_2$ is called an **L-fuzzy algebra homomorphism** if the following properties are satisfied:

$$
\begin{aligned}
&1. \quad \neg_2(h(u)) = h(\neg_1(u)), \quad u \in L_1 \\
&2. \quad h(u) \wedge_2 h(v) = h(u \wedge_1 v), \quad u, v \in L_1 \\
&3. \quad h(u) \vee_2 h(v) = h(u \vee_1 v), \quad u, v \in L_1
\end{aligned}
\tag{4.2}
$$

When these three properties are satisfied, we will often also say that the L-fuzzy algebra operators **commute with** function h.

Note that as a consequence of properties 2 and 3, if both L_1 and L_2 are bounded with bottom and top elements $\mathbf{0}_1$, $\mathbf{1}_1$, $\mathbf{0}_2$ and $\mathbf{1}_2$, respectively, it also holds that $h(\mathbf{0_1}) = \mathbf{0_2}$ and $h(\mathbf{1_1}) = \mathbf{1_2}$.

It is trivial to prove that an L-fuzzy algebra homomorphism is an order homomorphism and, if L_1 and L_2 are lattices, a lattice homomorphism.

Definition 4.1.6. An L-fuzzy algebra homomorphism that is bijective is called an **L-fuzzy algebra isomorphism** and the two L-fuzzy algebras related by an isomorphism are said to be **isomorphic**.

We now give a formal definition of L-fuzzy sets, a concept we already mentioned in Section 2.4, based on the definition of an L-fuzzy algebra.

Definition 4.1.7. Given a universe X and an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$, the **L-fuzzy powerset** is the set of all functions $X \to L$, denoted by L^X, with the unary and binary operations defined by the Goguen extension of the operations in the L-fuzzy algebra $\neg_g$, $\wedge_g$ and $\vee_g$:

$$
\begin{aligned}
&1. \quad (\neg_g A)(x) := \neg A(x), && A \in L^X, \ x \in X \\
&2. \quad (A \wedge_g B)(x) := A(x) \wedge B(x), && A, B \in L^X, \ x \in X \\
&3. \quad (A \vee_g B)(x) := A(x) \vee B(x), && A, B \in L^X, \ x \in X
\end{aligned}
\tag{4.3}
$$

and, if there are top and bottom elements $\mathbf{0}$ and $\mathbf{1}$ in L, with the two constant functions $\mathbf{0}_g$ and $\mathbf{1}_g$ defined as:

$$
\begin{aligned}
&1. \quad \mathbf{0}_g(A)(x) := \mathbf{0}, \quad A \in L^X, \ x \in X \\
&2. \quad \mathbf{1}_g(A)(x) := \mathbf{1}, \quad A \in L^X, \ x \in X
\end{aligned}
\tag{4.4}
$$

The L-fuzzy powerset is by itself an algebra $\{L^X, \neg_g, \wedge_g, \vee_g, (\mathbf{0}_g, \mathbf{1}_g)\}$. The functions in the L-fuzzy powerset are called **L-fuzzy sets**.

With these basic definitions in place, we can now proceed to build the multivalued versions of membership grades using the techniques described in the previous chapter.

4.2 Set-based L-fuzzy sets

We are now going to discuss the use of finite crisp sets as membership grades for L-fuzzy sets. In the following, we assume that there is a basic L-fuzzy algebra $\{L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$, which we want to extend to the powerset $\mathbf{2}^L$ so that we can use *sets* of values of L, rather than the values of L themselves, as membership grades for new types of fuzzy sets. The strategy here is to extend $\{L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$ to a set-based algebra $\{\mathbf{2}^L, \neg_s, \wedge_s, \vee_s, \mathbf{0}_s, \mathbf{1}_s\}$. For the unary and binary operations, this can be accomplished by means of of the Powerset Extension Principle, which we introduced in Chapter 2 as Definition 2.7.2. As for the bottom and top elements in L, $\mathbf{0}$ and $\mathbf{1}$, they can simply be replaced by the single-element sets containing them (this can be regarded as the Powerset Extension Principle for nullary operations). This idea is summed up in the following definition.

Definition 4.2.1. Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$, the **set-extended L-fuzzy algebra** is $\{\mathbf{2}^L, \neg_s, \wedge_s, \vee_s, \mathbf{0}_s, \mathbf{1}_s\}$, where $\neg_s$, $\wedge_s$ and $\vee_s$ are the set-extended versions of $\neg$, $\wedge$ and $\vee$, respectively, based on the Powerset Extension Principle (Definition 2.7.2), and $\mathbf{0}_s$ and $\mathbf{1}_s$ are the two sets $\{\mathbf{0}\}$ and $\{\mathbf{1}\}$.

The simplest and most usual case for fuzzy sets is the one where the L-fuzzy algebra is the standard ordinary fuzzy algebra. In this case, we get a more specific algebra.

Proposition 4.2.2. *In the particular case of the standard ordinary fuzzy algebra* $\{\mathbb{I}, c, \min, \max, 0, 1\}$*, the set-extended L-fuzzy algebra is* $\{\mathbf{2}^{\mathbb{I}}, c_s, \min_s, \max_s, \{0\}, \{1\}\}$*, where the set-extended operations* c_s*,* $\min_s$ *and* $\max_s$ *fulfil the following relations:*

$$\begin{aligned} &1.\quad c_s(A) = \{x \in X | 1 - x \in A\}, && A \subseteq \mathbb{I} \\ &2.\quad \min_s(A, B) = \{x \in A \cup B | x \leq \min(\sup\{A\}, \sup\{B\})\}, && A, B \subseteq \mathbb{I} \\ &3.\quad \max_s(A, B) = \{x \in A \cup B | x \geq \max(\inf\{A\}, \inf\{B\})\}, && A, B \subseteq \mathbb{I} \end{aligned} \tag{4.5}$$

Proof. The equalities above are derived by direct application of the Powerset Extension Principle. The first one is a straightforward application of the formula (2.11).

For the second and third relations, we need to consider the binary form of the Powerset Extension Principle, a particular case of the multivariate form (Definition 2.7.4). Applying the formula (2.12) when $X = \mathbb{I}$ and $f = \min$, we have:

$$Ext_s(\min)(A, B) := \{\min(a, b) | a \in A, b \in B\}, \quad A, B \subseteq \mathbb{I}$$

The set $\{\min(a, b)|a \in A, b \in B\}$ is obviously a subset of $A \cup B$. Let us suppose the supremum (least upper bound) of A is $M_A \in \mathbb{I}$ and that of B is $M_B \in \mathbb{I}$. Without loss of generality, given the symmetry of the Ext_s function, we can assume that the sets have been named in such a way that $M_A \leq M_B$. Then, if $x \in A$, then there is $m_b \in B$ such that $m_b \leq M_B$ and $x \leq m_B$, so $x = \min(x, m_B)$. And if $x \in B$, we consider two cases: first, if $x \leq M_A$, then there is $m_A \in A$ such that $m_A \leq M_A$ and $x \leq m_A$, so $x = \min(m_A, x)$; secondly, if $M_A < x$ then x cannot be the minimum of any pair with one of the elements in A, so $x \notin \{\min(a, b)|a \in A, b \in B\}$. This proves that $\{\min(a, b)|a \in A, b \in B\} = \{x \in A \cup B | x \leq \min(\sup\{A\}, \sup\{B\})\}$.

The proof for the third equality involving the max operation is completely analogous. □

Definition 4.2.3. The previous set-extended algebra of Proposition 4.2.2 above is called the **hesitant fuzzy algebra** of membership values.

The justification of this name is obvious; the relations (4.5) match the standard definitions for the operations on hesitant fuzzy sets in Definition 2.6.3. Also note that in this dissertation, we will only consider finite sets, so the supremum is the maximum and the infimum is the minimum.

Example 4.2.4. Let us take two sets over the unit interval $A = \{0.2, 0.3, 0.5\}$ and $B = \{0.3, 0.4, 0.6\}$. By using the expressions (4.5), we have that $c_s(A) = \{0.8, 0.7, 0.5\}$ and $c_s(B) = \{0.7, 0.6, 0.4\}$, $\min_s(A, B) = \{0.2, 0.3, 0.4, 0.5\}$ and $\max_s(A, B) = \{0.3, 0.4, 0.5, 0.6\}$.

The set-extended L-fuzzy algebras and, in particular, the hesitant fuzzy algebra can be used to define extended fuzzy sets where the membership grades are made up of crisp sets. As we saw in Chapter 2, Goguen's Extension Principle is the tool that allows us to make this conceptual leap from the world of "membership grades" to the world of "fuzzy sets", with the latter defined as functions that map the elements of a universe X into the space of membership grades and the basic operations being replicated in a pointwise fashion.

Definition 4.2.5. Given a universe X and an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$, the **set-based L-fuzzy powerset** is the set of all functions $X \to \mathbf{2}^L$, denoted by $(\mathbf{2}^L)^X$, with the unary and binary operations defined by the Goguen extension of the operations in the set-extended L-fuzzy algebra $\neg_{gs}$, $\wedge_{gs}$ and $\vee_{gs}$:

$$\begin{aligned}
&1. \quad (\neg_{gs}\tilde{A})(x) := \neg_s \tilde{A}(x), && \tilde{A} \in (\mathbf{2}^L)^X \ , x \in X \\
&2. \quad (\tilde{A} \wedge_{gs} \tilde{B})(x) := \tilde{A}(x) \wedge_s \tilde{B}(x), && \tilde{A}, \tilde{B} \in (\mathbf{2}^L)^X \ , x \in X \\
&3. \quad (\tilde{A} \vee_{gs} \tilde{B})(x) := \tilde{A}(x) \vee_s \tilde{B}(x), && \tilde{A}, \tilde{B} \in (\mathbf{2}^L)^X \ , x \in X
\end{aligned} \tag{4.6}$$

and with the two constant functions $\mathbf{0}_{gs}$ and $\mathbf{1}_{gs}$ defined as:

$$\begin{aligned}
&1. \quad \mathbf{0}_{gs}(\tilde{A})(x) := \{\mathbf{0}\}, \quad \tilde{A} \in (\mathbf{2}^L)^X \ , x \in X \\
&2. \quad \mathbf{1}_{gs}(\tilde{A})(x) := \{\mathbf{1}\}, \quad \tilde{A} \in (\mathbf{2}^L)^X \ , x \in X
\end{aligned} \tag{4.7}$$

With these definitions, the set-based L-fuzzy powerset is by itself an algebra $\{(\mathbf{2}^L)^X, \neg_{gs}, \wedge_{gs}, \vee_{gs}, \mathbf{0}_{gs}, \mathbf{1}_{gs}\}$. The functions in the set-extended L-fuzzy powerset are called **set-based L-fuzzy sets**.

In the context of set-based fuzzy sets, the operations $\wedge_{gs}$ and $\vee_{gs}$ are usually called **intersection** and **union**, respectively, and can also be denoted by the more traditional symbols $\cap$ and $\cup$ when there is no risk of confusion. And the negation, $\neg_{gs}$, can be called **complement** and is often represented as a ^{c} superscript next to the name of the fuzzy set.

Definition 4.2.6. The set-extended L-fuzzy powerset over a universe X for the hesitant fuzzy algebra is called the **hesitant fuzzy powerset**. Its functions are called **hesitant fuzzy sets**.

The previous definition brings back the concept that we already presented in Definition 2.6.2, now as a particular case within a more general approach.

These results show the expressive power of the extension principles as a framework that leads to a simple and novel way of defining hesitant fuzzy sets within a broader context that can also account for other more general spaces of membership grades than the unit interval $\mathbb{I}$ and for operations other than the standard ones.

Example 4.2.7. Let the universe be $X = \{x\}$, comprising one single element. Then two hesitant fuzzy sets over X, $\tilde{A}$ and $\tilde{B}$ are fully specified by the sets that

x is mapped to. For example, $\tilde{A}(x) = \{0.2, 0.3, 0.5\}$ and $\tilde{B}(x) = \{0.3, 0.4, 0.6\}$. These sets, usually called the "hesitant elements", that represent the membership grades are the same as those in the previous Example 4.2.4, so the operations carried out in the hesitant fuzzy algebra are now to be interpreted as the complement, intersection and union of the hesitant fuzzy sets: $\tilde{A}^c(x) = c_s(\tilde{A}(x)) = \{0.8, 0.7, 0.5\}$, $\tilde{B}^c(x) = c_s(\tilde{B}(x)) = \{0.7, 0.6, 0.4\}$, $(\tilde{A} \cap \tilde{B})(x) = \min_s(\tilde{A}(x), \tilde{B}(x)) = \{0.2, 0.3, 0.4, 0.5\}$ and $(\tilde{A} \cup \tilde{B})(x) = \max_s(\tilde{A}(x), \tilde{B}(x)) = \{0.3, 0.4, 0.5, 0.6\}$. If the universe had more elements than x, the three operations would be evaluated at each element in this way, independently of the other elements.

4.3 L-fuzzy multisets

Following a similar approach as in our analysis in the previous section, we are now going to focus our attention on the use of finite multisets as membership grades for L-fuzzy sets. Extending the unary and binary operations from an arbitrary space of membership grades can be done trivially by using any one of the multiset extensions that were enumerated in Corollary 3.4.12.

A trickier issue, where the analogy with the set-based case falters, affects the bottom and top elements in L. If $\mathbf{0}$ is the bottom element in L, we could expect its role to be played by $\langle \mathbf{0} \rangle$. But, since we are dealing with multisets, why not $\langle \mathbf{0}, \mathbf{0} \rangle$ or $\langle \mathbf{0}, \mathbf{0}, \mathbf{0} \rangle$? In fact, if we stick to the definition of the bottom and top elements of L as elements that fulfil the four conditions $x \wedge \mathbf{0} = \mathbf{0}$, $x \vee \mathbf{0} = x$, $x \wedge \mathbf{1} = x$ and $x \vee \mathbf{1} = \mathbf{1}$, a contradiction ensues when multiset-extending the min and max operations on $\mathbb{I}$. A complete treatment of the mathematical structure of such multiset-based extensions is beyond the scope of this dissertation and is an interesting topic for future research, but this preliminary analysis prompts us to conclude that the standard fuzzy algebra, which is a bounded lattice, loses its nature as a bounded lattice when extended to the multisets. For that reason, we will drop any mention of the bottom and top elements from the discussions of these algebras.

It might seem that a way to deal with the unwieldiness of multisets would be to restrict the analysis to those multisets with a fixed cardinality $n \in \mathbb{N}$, which we have called n-regular multisets $\mathbb{N}^L_{=n}$. In that particular case, we could try to find suitable definitions with $\langle 0, \overset{n)}{\ldots}, 0 \rangle$ and $\langle 1, \overset{n)}{\ldots}, 1 \rangle$ as the bottom and top elements, but an additional problem with the regular multisets is that they are not closed by intersections and unions (a simple example: $\langle a,b,b\rangle \cap \langle a,a,b\rangle = \langle a,b\rangle$), so the restriction to regular multisets also comes with its own burden of mathematical weaknesses.

Assuming that we will have to put up with weak mathematical properties, we can nevertheless proceed with a formal narrative similar to the one in the previous section.

Definition 4.3.1. Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$, the **product multiset-extended L-fuzzy algebra** is $\{\mathbb{N}^L, \neg_m, \wedge_m, \vee_m\}$, where $\neg_m$, $\wedge_m$ and $\vee_m$ are the multiset-extended versions of $\neg$, $\wedge$ and $\vee$, respectively, using the Multiset Extension Principle (definitions 3.2.1 and 3.2.3).

Again, we will mostly deal with the particular case of the standard ordinary fuzzy algebra.

Proposition 4.3.2. *In the particular case of the standard ordinary fuzzy algebra* $\{\mathbb{I}, c, \min, \max\}$*, the product multiset-extended L-fuzzy algebra is* $\{\mathbf{2}^{\mathbb{I}}, c_m, \min_m, \max_m\}$*, where the multiset-extended operations* c_m*,* $\min_m$ *and* $\max_m$ *fulfil the following relations for any* $\hat{A}, \hat{B} \in \mathbb{N}^{\mathbb{I}}$ *and* $u \in \mathbb{I}$*:*

1. $c_m(\hat{A})(u) = \hat{A}(1-u)$
2. $(\min_m(\hat{A},\hat{B}))(u) = \hat{A}(u)\hat{B}(u) + \hat{B}(u) \sum_{v\in\mathbb{I}|u<v} \hat{A}(v) + \hat{A}(u) \sum_{v\in\mathbb{I}|u<v} \hat{B}(v)$
3. $(\max_m(\hat{A},\hat{B}))(u) = \hat{A}(u)\hat{B}(u) + \hat{B}(u) \sum_{v\in\mathbb{I}|u>v} \hat{A}(v) + \hat{A}(u) \sum_{v\in\mathbb{I}|u>v} \hat{B}(v)$

We will refer to this algebra as the **standard product fuzzy multiset algebra**.

Proof. The first equality is a straightforward application of the formula (3.5) of the Multiset Extension Principle with one variable:

$$c_m(\hat{A})(u) = \sum_{v\in\mathbb{I}|c(v)=u} \hat{A}(v) = \sum_{v\in\mathbb{I}|1-v=u} \hat{A}(v) = \hat{A}(1-u)$$

For the second relation, we need to use the binary operation version of the Multiset Extension Principle, which is a particular case of the multivariate form (see

Definition 3.2.3). Applying the formula (3.6) when $U = \mathbb{I}$ and $f = \min$, we have:

$$(\min_m(\hat{A},\hat{B}))(u) = \sum_{v,w\in\mathbb{I}|u=\min(v,w)} \hat{A}(v)\hat{B}(w) =$$
$$= \hat{A}(u)\hat{B}(u) + \sum_{v\in\mathbb{I}|u<v} \hat{A}(v)\hat{B}(u) + \sum_{v\in\mathbb{I}|u<v} \hat{A}(u)\hat{B}(v)$$

The proof for the third equality involving max is completely analogous. □

An alternative and more intuitive set of expressions, in angular-bracket notation, for the three operations in Proposition 4.3.2 above and assuming that $\hat{A}$ has a cardinality $n \in \mathbb{N}$ and $\hat{B}$ has a cardinality $m \in \mathbb{N}$, with $\hat{A} = \langle a_1, ..., a_n\rangle$ and $\hat{B} = \langle b_1, ..., b_m\rangle$, is:

$$\begin{aligned} &1. \quad c_{sm}(\hat{A}) = \langle 1 - a_1, ..., 1 - a_n\rangle \\ &2. \quad \min_m(\hat{A},\hat{B}) = \langle \min(a_i, b_j)\rangle_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} \\ &3. \quad \min_m(\hat{A},\hat{B}) = \langle \max(a_i, b_j)\rangle_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} \end{aligned} \tag{4.8}$$

These angular-bracket versions can be worked out by careful expansion of the functional relations for those values $u = a_i$ and $u = b_j$ where the multiplicities are non-zero.

Example 4.3.3. Let us take the multiset $\hat{A} = \langle 0.1, 0.1, 0.5\rangle$ as an example. For the complement, the first relation in Proposition 4.3.2 means that for each $u \in \mathbb{I}$ with $\hat{A}(u) \neq 0$ its multiplicity is transferred to $1 - u$, so we have $c_m(\hat{A}) = \langle 0.9, 0.9, 0.5\rangle$.

In the case of the min and max operations, we can consider an example with two multisets like $\hat{A} = \langle 0.1, 0.1, 0.5\rangle$ and $\hat{B} = \langle 0.2, 0.6, 0.6\rangle$. In this case, we have $\min_m(\hat{A},\hat{B}) = \langle 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.2, 0.5, 0.5\rangle$ and $\max_m(\hat{A},\hat{B}) = \langle 0.2, 0.2, 0.5, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6\rangle$. Note that the product multiset extension of a binary operation yields a result with a cardinality that is the product of the cardinalities of the operands, so the two multisets $\hat{A}$ and $\hat{B}$, each one with cardinality 3, lead to multisets of cardinality 9 when applying $\min_s$ and $\max_s$.

This algebra that results from the direct application of the Multiset Extension Principle is different from the ones that have been introduced in the literature on fuzzy multisets up to now. Two other options that are documented in the related literature, the sorted (Miyamoto) fuzzy multiset algebra and the aggregate fuzzy multiset algebra, naturally arise from the two alternative multiset extension principles in Corollary 3.4.12. We now define these additional extensions.

Definition 4.3.4. Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$, the **$n$-regular sorted multiset-extended L-fuzzy algebra** for a dimension $n > 1$ is $\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}$, where $\neg_{sm_n}$, $\wedge_{sm_n}$ and $\vee_{sm_n}$ are the extended versions of $\neg$, $\wedge$ and $\vee$, respectively, using the sorted multiset extension defined in Corollary 3.4.12.

This family of algebras for specific dimensions n can be combined to support multisets of any cardinality by adding a regularisation strategy (see Definition 2.5.8) into the definition, as follows.

Definition 4.3.5. Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$ and a regularisation strategy involving a family of functions $\{Reg_n \colon \mathbb{N}^L_{\leq n} \to \mathbb{N}^L_{=n}\}_{n\in\mathbb{N}}$, the **sorted multiset-extended L-fuzzy algebra** is $\{\mathbb{N}^L, \neg_{sm}, \wedge_{sm}, \vee_{sm}\}$, where $\neg_{sm}$, $\wedge_{sm}$ and $\vee_{sm}$ are defined in terms of the n-regular sorted multiset-extended L-fuzzy algebras using the maximum cardinalities and the regularisation functions when necessary:

1. $\neg_{sm}\hat{A} := \neg_{sm_{|\hat{A}|}}\hat{A}, \quad \hat{A} \in \mathbb{N}^L$
2. $\hat{A} \wedge_{sm} \hat{B} := Reg_{\max(|\hat{A}|,|\hat{B}|)}(\hat{A}) \wedge_{sm_{\max(|\hat{A}|,|\hat{B}|)}} Reg_{\max(|\hat{A}|,|\hat{B}|)}(\hat{B}), \quad \hat{A}, \hat{B} \in \mathbb{N}^L$
3. $\hat{A} \vee_{sm} \hat{B} := Reg_{\max(|\hat{A}|,|\hat{B}|)}(\hat{A}) \vee_{sm_{\max(|\hat{A}|,|\hat{B}|)}} Reg_{\max(|\hat{A}|,|\hat{B}|)}(\hat{B}), \quad \hat{A}, \hat{B} \in \mathbb{N}^L$

In the case of the standard ordinary fuzzy algebra, we can work out the formulas for the operations.

Proposition 4.3.6. *In the particular case of the standard ordinary fuzzy algebra* $\{\mathbb{I}, c, \min, \max\}$*, the descending sorted multiset-extended L-fuzzy algebra is* $\{\mathbb{N}^{\mathbb{I}}, c_{s_\downarrow m}, \min_{s_\downarrow m}, \max_{s_\downarrow m}\}$*, where the sorted multiset-extended operations* $c_{s_\downarrow m}$*,* $\min_{s_\downarrow m}$ *and* $\max_{s_\downarrow m}$ *relative to the descending sort* $s_\downarrow$ *fulfil the following relations, in angular-bracket notation, for any* $\hat{A} = \langle a_1, ..., a_n\rangle, \hat{B} = \langle b_1, ..., b_n\rangle \in \mathbb{N}^{\mathbb{I}}$*, regularised to a common cardinality* $n > 0$*, and* $u \in \mathbb{I}$*:*

$$
\begin{aligned}
&1.\quad c_{s_\downarrow m}(\hat{A}) = \langle 1 - a_1, ..., 1 - a_n\rangle \\
&2.\quad \min_{s_\downarrow m}(\hat{A}, \hat{B}\} = \langle \min(a_{\sigma_\downarrow(1)}, b_{\sigma_\downarrow(1)}), ..., \min(a_{\sigma_\downarrow(n)}, b_{\sigma_\downarrow(n)})\rangle \\
&3.\quad \max_{s_\downarrow m}(\hat{A}, \hat{B}\} = \langle \max(a_{\sigma_\downarrow(1)}, b_{\sigma_\downarrow(1)}), ..., \max(a_{\sigma_\downarrow(n)}, b_{\sigma_\downarrow(n)})\rangle
\end{aligned}
\tag{4.9}
$$

where $\sigma_\downarrow$ *is the permutation of the coordinates that sorts them in descending order.*

Proof. As the first relation involves a unary operation, it is the same as in the product multiset extension according to Corollary 3.4.9 for one-variable functions, so it is a straight application of the Multiset Extension Principle for one variable (Definition 3.2.1), as in the proof of Proposition 4.3.2 above.

For the second relation involving the binary min operator, we need to apply the sorted multiset extension, which is a composition of the Cartesian extension and the sorted Cartesian-to-multiset extension, so we must use the expressions (3.23) for the multivariate sorted Cartesian-to-multiset extension and (3.18) for the multivariate Cartesian extension:

$$\min_{s_\downarrow m}(\hat{A},\hat{B}) := Ext_{c\to_s m}(Ext_c(\min))(\hat{A},\hat{B}) =$$
$$= Supp_m(Ext_c(\min)(s_\downarrow(\phi_p^{-1}(\hat{A})), s_\downarrow(\phi_p^{-1}(\hat{B})))) =$$
$$= Supp_m((\min(s_\downarrow(\phi_p^{-1}(\hat{A}))_1, s_\downarrow(\phi_p^{-1}(\hat{B}))_1), ..., \min(s_\downarrow(\phi_p^{-1}(\hat{A}))_n, s_\downarrow(\phi_p^{-1}(\hat{B}))_n)) =$$
$$= \langle(\min(s_\downarrow(\phi_p^{-1}(\hat{A}))_1, s_\downarrow(\phi_p^{-1}(\hat{B}))_1), ..., \min(s_\downarrow(\phi_p^{-1}(\hat{A}))_n, s_\downarrow(\phi_p^{-1}(\hat{B}))_n)\rangle$$

The proof for the maximum is completely analogous. □

We will refer to this algebra as **Miyamoto's fuzzy multiset algebra**.

Example 4.3.7. If we take the same multisets as in the previous Example 4.3.3: $\hat{A} = \langle 0.1, 0.1, 0.5\rangle$ and $\hat{B} = \langle 0.2, 0.6, 0.6\rangle$, the complement is given by the first one of the equalities (4.9): $c_{sm}(\hat{A}) = \langle 0.9, 0.9, 0.2\rangle$. This is the same result as with the product algebra, as expected. It is only when we get to the binary operations $\min_{sm}$ and $\max_{sm}$ that we find a different behaviour. Miyamoto's algebra requires that both operands should be regularised and the result preserves the cardinality of the operands, so we now get results of cardinality 3: $\min_{s_\downarrow m}(\hat{A},\hat{B}) = \langle 0.5, 0.1, 0.1\rangle$ and $\max_{s_\downarrow m}(\hat{A},\hat{B}) = \langle 0.6, 0.6, 0.2\rangle$.

And we now turn to the third possibility: the aggregate multiset extension. As in the previous case, the regularisation requirement for the binary operations makes it necessary to introduce the definition in two steps; first, a version for n-regular multisets and then a more general version that uses a regularisation strategy.

Definition 4.3.8. Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$, the ***n*-regular aggregate multiset-extended L-fuzzy algebra** for a dimension $n > 1$ is $\{\mathbb{N}^L_{=n}, \neg_{am_n}, \wedge_{am_n}, \vee_{am_n}\}$, where $\neg_{am_n}$, $\wedge_{am_n}$ and $\vee_{am_n}$ are the multiset-extended versions of $\neg$, $\wedge$ and $\vee$, respectively, using the aggregate multiset extension defined in Corollary 3.4.12.

This is a peculiar L-fuzzy algebra because the binary operators are defined as functions $\mathbb{N}^L_{=n} \times \mathbb{N}^L_{=n} \to \mathbb{N}^L$, as the cardinality may increase. This mathematical weakness is corrected in the next definition, where, as in the sorted case above, a regularisation strategy (see Definition 2.5.8) is added so that a more general algebra supporting multisets of any cardinality can be defined.

Definition 4.3.9. Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$ and a regularisation strategy involving a family of functions $\{Reg_n : \mathbb{N}^L_{\leq n} \to \mathbb{N}^L_{=n}\}_{n \in \mathbb{N}}$, the **aggregate multiset-extended L-fuzzy algebra** is $\{\mathbb{N}^L, \neg_{am}, \wedge_{am}, \vee_{am}\}$, where $\neg_{am}$, $\wedge_{am}$ and $\vee_{am}$ are defined in terms of the n-regular aggregate multiset-extended L-fuzzy algebras using the maximum cardinalities and the regularisation functions when necessary:

1. $\neg_{am}\hat{A} := \neg_{am_M}\hat{A}, \quad \hat{A} \in \mathbb{N}^L, M = |\hat{A}|$
2. $\hat{A} \wedge_{am} \hat{B} := Reg_M(\hat{A}) \wedge_{am_M} Reg_M(\hat{B}), \quad \hat{A}, \hat{B} \in \mathbb{N}^L, M = \max(|\hat{A}|, |\hat{B}|)$
3. $\hat{A} \vee_{am} \hat{B} := Reg_M(\hat{A}) \vee_{am_M} Reg_M(\hat{B}), \quad \hat{A}, \hat{B} \in \mathbb{N}^L, M = \max(|\hat{A}|, |\hat{B}|)$

And the expressions for the particular case of the standard ordinary fuzzy algebra are given by the next proposition.

Proposition 4.3.10. *In the particular case of the standard ordinary fuzzy algebra* $\{\mathbb{I}, c, \min, \max\}$*, the aggregate multiset-extended L-fuzzy algebra is* $\{\mathbb{N}^{\mathbb{I}}, c_{am}, \min_{am}, \max_{am}\}$*, where the aggregate multiset-extended operations* c_{am}*,* $\min_{am}$ *and* $\max_{am}$ *fulfil the following relations, in angular-bracket notation, for any* $\hat{A} = \langle a_1, ..., a_n \rangle, \hat{B} = \langle b_1, ..., b_n \rangle \in \mathbb{N}^{\mathbb{I}}$*, regularised to a common cardinality* $n > 0$*, and* $u \in \mathbb{I}$*:*

$$
\begin{aligned}
&1. \quad c_{am}(\hat{A}) = \langle 1 - a_1, ..., 1 - a_n \rangle \\
&2. \quad \min_{am}(\hat{A}, \hat{B}) = \bigcup_{\substack{(s_1,...,s_n) \in \phi_p^{-1}(\hat{A}) \\ (t_1,...,t_n) \in \phi_p^{-1}(\hat{B})}} \langle \min(s_1, t_1), ..., \min(s_n, t_n) \rangle \\
&3. \quad \max_{am}(\hat{A}, \hat{B}) = \bigcup_{\substack{(s_1,...,s_n) \in \phi_p^{-1}(\hat{A}) \\ (t_1,...,t_n) \in \phi_p^{-1}(\hat{B})}} \langle \max(s_1, t_1), ..., \max(s_n, t_n) \rangle
\end{aligned}
\tag{4.10}
$$

Proof. As in the previous Proposition 4.3.6, the first relation is the same as in the product multiset extension, so it is proved by applying the Multiset Extension Principle, as in Proposition 4.3.2 above.

For the second relation involving the binary min operator, we now need to apply the aggregate multiset extension, which is a composition of the Cartesian extension and the aggregate Cartesian-to-multiset extension, so we must use the expressions (3.24) for the multivariate aggregate Cartesian-to-multiset extension and (3.18) for the multivariate Cartesian extension:

$$\begin{aligned}\min_{am}(\hat{A},\hat{B}) &:= Ext_{c\to am}(Ext_c(\min))(\hat{A},\hat{B}) = \\ &= \bigcup_{\substack{\vec{S}\in\phi_p^{-1}(\hat{A})\\ \vec{T}\in\phi_p^{-1}(\hat{B})}} Supp_m(Ext_c(\min)(\vec{S},\vec{T})) = \\ &= \bigcup_{\substack{\vec{S}\in\phi_p^{-1}(\hat{A})\\ \vec{T}\in\phi_p^{-1}(\hat{B})}} \langle \min(s_1,t_1),...,\min(s_n,t_n)\rangle\end{aligned}$$

The proof for the maximum is completely analogous. □

We will refer to this algebra as the **standard aggregate fuzzy multiset algebra**.

In Section 4.5 below, we will work out simpler equivalent formulas for $\min_{am}$ and $\max_{am}$.

Example 4.3.11. We consider the same multisets as in the previous examples 4.3.3 and 4.3.7 $\hat{A} = \langle 0.1, 0.1, 0.5\rangle$ and $\hat{B} = \langle 0.2, 0.6, 0.6\rangle$. The complement is $c_{am}(\hat{A}) = \langle 0.9, 0.9, 0.2\rangle$, which is the same result as with the other algebras. As for the binary operations, the standard aggregate fuzzy multiset algebra requires, like Miyamoto's, that both operands should be regularised to the same cardinality, but the result can now have a larger cardinality. By using the second and third expressions in the equalities (4.10), we get the results: $\min_{am}(\hat{A},\hat{B}) = \langle 0.1, 0.1, 0.2, 0.5\rangle$ and $\max_{am}(\hat{A},\hat{B}) = \langle 0.2, 0.5, 0.6, 0.6\rangle$.

Armed with the three types of multiset-extended L-fuzzy algebras, we can proceed to define multiset-based fuzzy sets where the elements in these algebras play the role of membership grades. The basic operations are induced via Goguen's extension, just as in the set-based L-fuzzy sets.

Definition 4.3.12. Given a universe X and an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$, the **product multiset-based L-fuzzy powerset** is the set of all functions $X \to \mathbb{N}^L$, denoted by $(\mathbb{N}^L)^X$, with the unary and binary operations defined by Goguen's extension of the operations in the product multiset-extended L-fuzzy algebra $\neg_{gm}$, $\wedge_{gm}$ and $\vee_{gm}$:

$$\begin{array}{lll} 1. & (\neg_{gm}\hat{A})(x) := \neg_m \hat{A}(x), & \hat{A} \in (\mathbb{N}^L)^X \ , \ x \in X \\ 2. & (\hat{A} \wedge_{gm} \hat{B})(x) := \hat{A}(x) \wedge_m \hat{B}(x), & \hat{A}, \hat{B} \in (\mathbb{N}^L)^X \ , \ x \in X \\ 3. & (\hat{A} \vee_{gm} \hat{B})(x) := \hat{A}(x) \vee_m \hat{B}(x), & \hat{A}, \hat{B} \in (\mathbb{N}^L)^X \ , \ x \in X \end{array} \tag{4.11}$$

With these definitions, the product multiset-based L-fuzzy powerset is by itself an algebra $\{(\mathbb{N}^L)^X, \neg_{gm}, \wedge_{gm}, \vee_{gm}\}$. The functions in it are called **product multiset-based L-fuzzy sets** or **L-fuzzy multisets with the product operations**.

Given a universe X together with the standard product fuzzy multiset algebra $\{\mathbb{N}^{\mathbb{I}}, c_m, \min_m, \max_m\}$, its product multiset-based L-fuzzy powerset $\{(\mathbb{N}^{\mathbb{I}})^X, c_{gm}, \min_{gm}, \max_{gm}\}$ is called the **standard product multiset-based fuzzy powerset** and its functions are called **standard product multiset-based fuzzy sets** or **ordinary fuzzy multisets with the standard product operations**.

Example 4.3.13. The simplest case of fuzzy multisets occur when the universe of discourse contains just one element, $X = \{x\}$. In such a case, a fuzzy multiset $\hat{A}$ is determined by its evaluation at x, the multiset $\hat{A}(x)$. For example, if $\hat{A}(x) = \langle 0.1, 0.1, 0.5\rangle$ and $\hat{B}(x) = \langle 0.2, 0.6, 0.6\rangle$, the situation is exactly like the one in Example 4.3.3 and the pointwise operations result in the same numeric values: $(c_{gm}(\hat{A}))(x) = c_m(\hat{A}(x)) = \langle 0.9, 0.9, 0.2\rangle$, $(\min_{gm}(\hat{A}, \hat{B}))(x) = \min_m(\hat{A}(x), \hat{B}(x)) = \langle 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.2, 0.5, 0.5\rangle$ and $(\max_{gm}(\hat{A}, \hat{B}))(x) = \max_m(\hat{A}(x), \hat{B}(x)) = \langle 0.2, 0.2, 0.5, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6\rangle$. When the universe X has additional elements, the three operations are evaluated independently at each element in this way. In the context of fuzzy multisets, it is common to write $\hat{A}^c$, $\hat{A} \cap \hat{B}$ and $\hat{A} \cup \hat{B}$ instead of $c_{gm}(\hat{A})$, $\min_{gm}(\hat{A}, \hat{B})$ and $\max_{gm}(\hat{A}, \hat{B})$, respectively.

And now we are going to state analogous definitions for the sorted operations.

Definition 4.3.14. Given a universe X and an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$, the **sorted multiset-based L-fuzzy powerset**, for a section $s\colon L^n/{\sim_p} \to L^n$, is the set of all functions $X \to \mathbb{N}^L$, denoted by $(\mathbb{N}^L)^X$, with the unary and binary operations defined by the Goguen extension of the operations in the sorted multiset-extended L-fuzzy algebra $\neg_{gsm}$, $\wedge_{gsm}$ and $\vee_{gsm}$:

$$\begin{array}{lll} 1. & (\neg_{gsm}\hat{A})(x) := \neg_{sm}\hat{A}(x), & \hat{A} \in (\mathbb{N}^L)^X \ , x \in X \\ 2. & (\hat{A} \wedge_{gsm} \hat{B})(x) := \hat{A}(x) \wedge_{sm} \hat{B}(x), & \hat{A}, \hat{B} \in (\mathbb{N}^L)^X \ , x \in X \\ 3. & (\hat{A} \vee_{gsm} \hat{B})(x) := \hat{A}(x) \vee_{sm} \hat{B}(x), & \hat{A}, \hat{B} \in (\mathbb{N}^L)^X \ , x \in X \end{array} \tag{4.12}$$

With these definitions, the sorted multiset-based L-fuzzy powerset is by itself an algebra $\{(\mathbb{N}^L)^X, \neg_{gsm}, \wedge_{gsm}, \vee_{gsm}\}$. The functions in it are called **sorted multiset-based L-fuzzy sets** or **L-fuzzy multisets with the sorted operations**.

The sorted multiset-based L-fuzzy powerset over a universe X for Miyamoto's fuzzy multiset algebra is called **Miyamoto's multiset-based L-fuzzy powerset**, $\{(\mathbb{N}^{\mathbb{I}})^X, c_{gs_\downarrow m}, \min_{gs_\downarrow m}, \max_{gs_\downarrow m}\}$, and its functions are called **Miyamoto's multiset-based fuzzy sets** or **ordinary fuzzy multisets with Miyamoto's operations**.

As with the hesitant fuzzy sets in the previous section, in the previous definition we recover the preliminary Definition 2.6.5 for fuzzy multisets, with their usual operations, as a particular case within the general framework that we present in this dissertation.

Example 4.3.15. Let $X = \{x\}$ be a one-element universe and let us consider two fuzzy multisets specified as $\hat{A}(x) = \langle 0.1, 0.1, 0.5\rangle$ and $\hat{B}(x) = \langle 0.2, 0.6, 0.6\rangle$. Since we are now considering the operations in Miyamoto's algebra, the situation is like the one in Example 4.3.7 above and the pointwise operations result in the same numeric values: $\hat{A}^c(x) := (c_{gsm}(\hat{A}))(x) = c_{sm}(\hat{A}(x)) = \langle 0.9, 0.9, 0.2\rangle$, $(\hat{A} \cap \hat{B})(x) := (\min_{gsm}(\hat{A}, \hat{B}))(x) = \min_{sm}(\hat{A}(x), \hat{B}(x)) = \langle 0.1, 0.1, 0.5\rangle$ and $(\hat{A} \cup \hat{B})(x) := (\max_{gsm}(\hat{A}, \hat{B}))(x) = \max_{sm}(\hat{A}(x), \hat{B}(x)) = \langle 0.2, 0.6, 0.6\rangle$.

We now turn our attention to the third type of operations, the aggregate ones, for which we follow the same kind of approach as for the other two types in the previous paragraphs.

Definition 4.3.16. Given a universe X and an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$, the **aggregate multiset-based L-fuzzy powerset** is the set of all functions $X \to \mathbb{N}^L$, denoted by $(\mathbb{N}^L)^X$, with the unary and binary operations defined by the Goguen extension of the operations in the aggregate multiset-extended L-fuzzy algebra $\neg_{gam}$, $\wedge_{gam}$ and $\vee_{gam}$:

$$\begin{array}{lll} 1. & (\neg_{gam}\hat{A})(x) := \neg_{am}\hat{A}(x), & \hat{A} \in (\mathbb{N}^L)^X \ , x \in X \\ 2. & (\hat{A} \wedge_{gam} \hat{B})(x) := \hat{A}(x) \wedge_{am} \hat{B}(x), & \hat{A}, \hat{B} \in (\mathbb{N}^L)^X \ , x \in X \\ 3. & (\hat{A} \vee_{gam} \hat{B})(x) := \hat{A}(x) \vee_{am} \hat{B}(x), & \hat{A}, \hat{B} \in (\mathbb{N}^L)^X \ , x \in X \end{array} \quad (4.13)$$

With these definitions, the aggregate multiset-based L-fuzzy powerset is by itself an algebra $\{(\mathbb{N}^L)^X, \neg_{gam}, \wedge_{gam}, \vee_{gam}\}$. The functions in it are called **aggregate multiset-based L-fuzzy sets** or **L-fuzzy multisets with the aggregate operations**.

The aggregate multiset-based L-fuzzy powerset over a universe X for the standard aggregate fuzzy multiset algebra is called the **standard aggregate multiset-based L-fuzzy powerset**, $\{(\mathbb{N}^{\mathbb{I}})^X, c_{gam}, \min_{gam}, \max_{gam}\}$, and its functions are called **standard aggregate multiset-based fuzzy sets** or **ordinary fuzzy multisets with the standard aggregate operations**.

The name "aggregate" evokes the fact that these operations are based on multiset unions, but also that these operations match the aggregate intersection and union (Definition 2.6.11) of fuzzy multisets that were introduced in a recent article [39], where these operations were derived following a top-down approach starting from the more established definitions due to Miyamoto. Here we are following a bottom-up approach instead, where the algebras are defined prior to their use within fuzzy sets, but the concepts are the same.

Example 4.3.17. As in the previous examples 4.3.13 and 4.3.15, let $X = \{x\}$ be a one-element universe and let us consider the same two fuzzy multisets, specified as $\hat{A}(x) = \langle 0.1, 0.1, 0.5\rangle$ and $\hat{B}(x) = \langle 0.2, 0.6, 0.6\rangle$. With the standard aggregate operations, the situation now resembles Example 4.3.11 above and the pointwise operations lead to the same values: $\hat{A}^c(x) := (c_{gam}(\hat{A})(x) = c_{am}(\hat{A}(x)) = \langle 0.9, 0.9, 0.2\rangle$, $(\hat{A} \cap \hat{B})(x) := (\min_{gam}(\hat{A}, \hat{B}))(x) = \min_{am}(\hat{A}(x), \hat{B}(x)) = \langle 0.1, 0.1, 0.2, 0.5\rangle$ and $(\hat{A} \cup \hat{B})(x) := (\max_{gam}(\hat{A}, \hat{B}))(x) = \max_{am}(\hat{A}(x), \hat{B}(x)) = \langle 0.2, 0.5, 0.6, 0.6\rangle$.

Note that if $\hat{A}$ and $\hat{B}$ had different cardinalities, then an additional regularisation step would be required. Remember that the sorted and aggregate operations must be equipped with a regularisation strategy to be properly defined for multisets of any cardinality.

As shown in the examples above, in the context of L-fuzzy multisets and with any of the algebras, the operations $\wedge_{g(s/a)m}$ and $\vee_{g(s/a)m}$ are usually called **intersection** and **union**, respectively, and can also be denoted by the more conventional symbols $\cap$ and $\cup$ when there is no risk of confusion, as in the examples above. And the negation, $\neg_{gm}$, can be called **complement** and can be represented as a ^{c} superscript next to the name of the fuzzy set. When it is necessary to identify the operations in use, we may refer to them as the **product/Miyamoto/aggregate intersection**, etc.

4.4 The relation between set-based L-fuzzy sets and L-fuzzy multisets

The relations linking sets and multisets that we discussed in the previous chapter can be transposed to the corresponding fuzzy sets using Goguen's extension. We will begin by defining an extended concept of support that provides a link between set-based and multiset-based L-fuzzy sets.

Definition 4.4.1. Let X be the universe and let L be a membership grade space over which a set-based L-fuzzy powerset $(\mathbf{2}^L)^X$ and a multiset-based L-fuzzy powerset $(\mathbb{N}^L)^X$ are defined. The **fuzzy-set-based support** $Supp^f\colon (\mathbb{N}^L)^X \to (\mathbf{2}^L)^X$ is the Goguen extension (see Definition 2.7.6) of the support function $Supp\colon \mathbb{N}^L \to \mathbf{2}^L$ (see Definition 2.5.4):

$$Supp^f := Ext_g(Supp)$$

When the membership grade space is the unit interval $\mathbb{I}$, we will refer to the fuzzy-set-based support as the **hesitant support**.

Proposition 4.4.2. *$Supp^f$ can be expressed in terms of how it acts on each element $x \in X$, as follows:*

$$(Supp^f(\hat{A}))(x) = Supp(\hat{A}(x)), \quad \hat{A} \in (\mathbb{N}^L)^X, x \in X \tag{4.14}$$

Proof. A trivial substitution, as Goguen's extension is simply a pointwise expansion. □

This support at the fuzzy multiset level replicates the properties of the support for multisets, so the concepts are very similar. Just as we did with the plain multiset version, we can define the preimage as a function $(\mathbf{2}^L)^X \to \mathbf{2}^{((\mathbb{N}^L)^X)}$.

Definition 4.4.3. Let X be the universe and let L be a membership grade space over which a set-based L-fuzzy powerset $(\mathbf{2}^L)^X$ and a multiset-based L-fuzzy powerset $(\mathbb{N}^L)^X$ are defined. The **fuzzy multiset support preimage** of a set-based L-fuzzy set is a function $Supp^{f\ -1}\colon (\mathbf{2}^L)^X \to \mathbf{2}^{((\mathbb{N}^L)^X)}$ defined by the relation:

$$Supp^{f\ -1}(\tilde{A}) := \{\hat{T} \in (\mathbb{N}^L)^X | Supp^f(\hat{T}) = \tilde{A}\}, \quad \tilde{A} \in (\mathbf{2}^L)^X$$

The identification between sets and repetition-equivalent multisets that we brought up in the previous chapter can also be replicated in the corresponding types of fuzzy sets. For this, we will once more resort to the concept of equivalence kernel of Definition 2.8.6.

Definition 4.4.4. Let X be the universe and let L be a membership grade space over which a multiset-based L-fuzzy powerset $(\mathbb{N}^L)^X$ is defined. The **repetition equivalence** in $(\mathbb{N}^L)^X$ is the equivalence kernel of the fuzzy-set-based support function $Supp^f$.

This equivalence relation is denoted by $\sim_{fr}$, so given two fuzzy multisets $\hat{A}$, $\hat{B} \in (\mathbb{N}^L)^X$, $\hat{A} \sim_{fr} \hat{B}$ means that $Supp^f(\hat{A}) = Supp^f(\hat{B})$.

Example 4.4.5. Let us consider a two-element universe $X = \{x, y\}$ and the ordinary membership grade space $\mathbb{I}$. If $\hat{A}$ and $\hat{B}$ are two fuzzy multisets such that $\hat{A}(x) = \langle 0.2, 0.3, 0.3\rangle$, $\hat{A}(y) = \langle 0.7, 0.7, 0.8\rangle$, $\hat{B}(x) = \langle 0.2, 0.2, 0.3\rangle$ and $\hat{B}(y) = \langle 0.7, 0.8, 0.8\rangle$, the respective multisets evaluated at x and y only differ in the multiplicities, so their hesitant supports are the same: $(Supp^f(\hat{A}))(x) = (Supp^f(\hat{B}))(x) = \{0.2, 0.3\}$ and $(Supp^f(\hat{A}))(y) = (Supp^f(\hat{B}))(y) = \{0.7, 0.8\}$. As $Supp^f(\hat{A}) = Supp^f(\hat{B})$, we can say that $\hat{A}$ and $\hat{B}$ are repetition-equivalent, $\hat{A} \sim_{fr} \hat{B}$.

The simplest way to pick a representative of each repetition-equivalence class consists in selecting the unique set-based L-fuzzy set that maps elements into multisets with the multiplicity values of 0 and 1 only. This is what we will call the canonical section for the repetition equivalence.

Definition 4.4.6. Given the repetition equivalence $\sim_{fr}$ in the multiset-based L-fuzzy powerset $(\mathbb{N}^L)^X$ over a universe X with membership grades in L, the **repetition canonical section** is a section, as in Definition 2.8.5, $s_{frc}\colon (\mathbb{N}^L)^X/{\sim_{fr}} \to (\mathbb{N}^L)^X$, such that equivalence classes are mapped to L-fuzzy multisets that satisfy the following condition:

$$((s_{frc}(c))(x))(u) \in \{0, 1\} \quad \forall x \in X, \forall u \in L, c \in (\mathbb{N}^L)^X/{\sim_{fr}} \tag{4.15}$$

As we mentioned in Section 2.8, a section maps the equivalence classes into a complete set of representatives, so the L-fuzzy multisets fulfilling condition (4.15) stand in bijection with the repetition equivalence classes.

Example 4.4.7. If we consider the two fuzzy multisets of the previous Example 4.4.5 again, their canonical representative would be the fuzzy multiset $\hat{C}$ defined by $\hat{C}(x) = \langle 0.2, 0.3 \rangle$ and $\hat{C}(y) = \langle 0.7, 0.8 \rangle$. Such a fuzzy multiset can be identified with a hesitant fuzzy set $\tilde{C}$ defined by $\tilde{C}(x) = \{0.2, 0.3\}$ and $\tilde{C}(y) = \{0.7, 0.8\}$. This identification between the repetition canonical fuzzy multisets and the hesitant fuzzy sets will be formalised in the next proposition.

As we have hinted in the previous example, it is intuitively obvious that there is a bijection between the repetition equivalence classes, or the repetition canonical representatives, and the set-based L-fuzzy powerset $(\mathbf{2}^L)^X$. This idea can be stated as a result that follows as a particular case of Proposition 2.8.7.

Proposition 4.4.8. *Let X be the universe and let L be a membership grade space over which a set-based L-fuzzy powerset $(\mathbf{2}^L)^X$ and a multiset-based L-fuzzy powerset $(\mathbb{N}^L)^X$ are defined. There exists a bijection $\phi_{fr} \colon (\mathbb{N}^L)^X/{\sim_{fr}} \leftrightarrow (\mathbf{2}^L)^X$ defined by:*

$$\phi_{fr} = Supp^f \circ s_{frc} \tag{4.16}$$

We have chosen the repetition canonical section s_{frc} for convenience, but any other section would be equally valid.

Thanks to this bijection, we can identify the repetition-equivalent L-fuzzy multisets with the set-based L-fuzzy sets. This is completely analogous to what we did with the repetition equivalence class for multisets in the previous chapter and is shown in the form of a commutation diagram in Figure 4.1. The ϕ_{fr} bijection makes it possible to identify the set-based L-fuzzy sets (or the typical hesitant fuzzy sets, in their most usual incarnation) with a subset or partition of the L-fuzzy multisets. But this identification will be quite pointless if the basic operations of complementation, intersection and union are not compatible with the bijection. It is when the operations are compatible, in the sense that the bijection commutes with the corresponding version of the operation for each type of mathematical objects, that the identification between the mathematical theories is complete. As we have presented three alternative algebras that provide the operations for the L-fuzzy multisets, we are going to study the situation with each one of them.

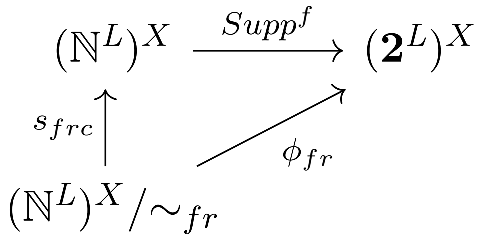

Figure 4.1: Commutation diagram for the fuzzy repetition equivalence ϕ_{fr} bijection.

By careful inspection of the examples 4.3.3 and 4.3.11 above, we can see that the results of the product and aggregate multiset-extended operations $\min_{(a)m}$ and $\max_{(a)m}$ are multisets whose supports are the same in both cases and also match those that appear in the set-extended operations $\min_s$ and $\max_s$ in Example 4.2.4, something that does not happen with Miyamoto's operations in Example 4.3.7. This is just a hint that points to a general result that we will prove: the product and aggregate multiset-extended operations, but not the sorted ones, are compatible with the set-extended operations relative to the repetition equivalence. This is related to the fact that, when the product or the aggregate multiset-extended operations are used, the support of the multisets commutes with these operations or, in other words, that the support is an L-fuzzy algebra homomorphism in the terms of Definition 4.1.5. This is a result that can be stated and proved in the general case, as we do in the next proposition.

Proposition 4.4.9. *Given an L-fuzzy algebra* $\{L, \neg, \wedge, \vee\}$*, the support function Supp, as in Definition 2.5.4, mapping its product multiset-extended L-fuzzy algebra* $\{\mathbb{N}^L, \neg_m, \wedge_m, \vee_m\}$ *to its set-extended L-fuzzy algebra* $\{\mathbf{2}^L, \neg_s, \wedge_s, \vee_s\}$ *is an L-fuzzy algebra homomorphism.*

Proof. By definition, the support being an L-fuzzy algebra homomorphism means that the following relations, as a particular case of equations (4.2) for any multisets $\hat{A}, \hat{B} \in \mathbb{N}^L$, hold:

$$\begin{aligned} &1. \quad Supp(\neg_m \hat{A}) = \neg_s Supp(\hat{A}) \\ &2. \quad Supp(\hat{A} \wedge_m \hat{B}) = Supp(\hat{A}) \wedge_s Supp(\hat{B}) \\ &3. \quad Supp(\hat{A} \vee_m \hat{B}) = Supp(\hat{A}) \vee_s Supp(\hat{B}) \end{aligned} \qquad (4.17)$$

For the first relation, let $x \in L$ be an element of the universe such that $x \in$

$Supp(\neg_m \hat{A})$. By definition of support, this is equivalent to $(\neg_m \hat{A}))(x) > 0$, and by definition of the multiset extension, this is also equivalent to $\sum_{y \in L | \neg y = x} \hat{A}(x) > 0$ but as $\neg$ is an involution and consequently a bijection which is its own inverse, there is only one term in the sum and this is simply $\hat{A}(\neg x) > 0$. And, by definition of support again, this is equivalent to $\neg x \in Supp(\hat{A})$, which, by definition of powerset extension, is equivalent to $x \in \neg_s Supp(\hat{A})$ and, therefore, the two sides of the first equation in (4.17) are equivalent.

For the second relation, let $x \in L$ be an element of the universe such that $x \in Supp(\hat{A} \wedge_m \hat{B})$. By definition of support, this is equivalent to $(\hat{A} \wedge_m \hat{B})(x) > 0$, and by the Multiset Extension Principle 3.2.3 in its multivariate form, this is also equivalent to $\sum_{y,z \in L | y \wedge z = x} \hat{A}(y)\hat{B}(z) > 0$. This means that at least one of the terms in the sum is non-zero, so it is equivalent to $\exists y, z \in L$ such that $y \wedge z = x$, $\hat{A}(y) > 0$ and $\hat{B}(z) > 0$ and, by definition of support again, this is equivalent to $\exists y, z \in L$ such that $y \wedge z = x$, $y \in Supp(\hat{A})$ and $z \in Supp(\hat{B})$. And, by definition of the powerset extension, this is equivalent to $x \in Supp(\hat{A}) \wedge_s Supp(\hat{B})$, which proves the second equation in (4.17).

The proof of the third equation in (4.17) is completely analogous, with $\vee$ replacing $\wedge$. □

A similar proposition can be stated for the aggregate multiset extensions.

Proposition 4.4.10. *Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$, the support function Supp, mapping its aggregate multiset-extended L-fuzzy algebra $\{\mathbb{N}^L, \neg_{am}, \wedge_{am}, \vee_{am}\}$ to its set-extended L-fuzzy algebra $\{\mathbf{2}^L, \neg_s, \wedge_s, \vee_s\}$ is an L-fuzzy algebra homomorphism.*

Proof. By definition, the support being an L-fuzzy algebra homomorphism means that the following relations, as a particular case of equations (4.2), hold:

$$\begin{aligned}
&1. \quad Supp(\neg_{am} \hat{A}) = \neg_s Supp(\hat{A}) \\
&2. \quad Supp(\hat{A} \wedge_{am} \hat{B}) = Supp(\hat{A}) \wedge_s Supp(\hat{B}) \\
&3. \quad Supp(\hat{A} \vee_{am} \hat{B}) = Supp(\hat{A}) \vee_s Supp(\hat{B})
\end{aligned} \tag{4.18}$$

Since the aggregate multiset extension is the same as the product multiset extension for unary operators, the first relation is proved in the same way as in Proposition 4.4.9 above.

For the second relation, let $x \in L$ be an element of the universe such that $x \in Supp(\hat{A} \wedge_{am} \hat{B})$. By definition of support, this is equivalent to $(\hat{A} \wedge_{am} \hat{B})(x) > 0$. The aggregate multiset extension results from taking a multiset union of all the possible n-tuples built with the elements of $\hat{A}$ and $\hat{B}$, so the fact that that union has non-zero multiplicity for x means that there is at least a term in the union with non-zero multiplicity. In terms of permutations, if $\hat{A} = \langle a_1, ..., a_n \rangle$ and $\hat{B} = \langle b_1, ..., b_n \rangle$, then $\hat{A} \wedge_{am} \hat{B} = \cup_{\sigma,\rho \in S_n} \langle a_{\sigma(1)} \wedge b_{\rho(1)}, ..., a_{\sigma(n)} \wedge b_{\rho(n)} \rangle$ (where S_n is the set of all possible permutations of n elements), and this is equivalent to the existence of a pair of indices $0 \leq i, j \leq n$ for which $a_i \wedge b_j = x$. And, by definition of the powerset extension, this is equivalent to $x \in Supp(\hat{A}) \wedge_s Supp(\hat{B})$, which proves the second equation in (4.18).

The proof of the third equation in (4.18) is completely analogous, with $\vee$ replacing $\wedge$. □

It is important to emphasise that propositions 4.4.9 and 4.4.10 do not have an equivalent for the sorted multiset extension.

Example 4.4.11. As a very simple counterexample that illustrates that the sorted operations do not lead to a homomorphism of L-fuzzy algebras, consider the two 3-regular multisets $\hat{A} = \langle 0.1, 0.2, 0.3 \rangle$ and $\hat{B} = \langle 0.15, 0.2, 0.25 \rangle$ with Miyamoto's algebra. Their minimum is $\min_{sm}(\hat{A}, \hat{B}) = \langle 0.1, 0.2, 0.25 \rangle$ and their maximum is $\max_{sm}(\hat{A}, \hat{B}) = \langle 0.15, 0.2, 0.3 \rangle$ and if we take the supports, we get $Supp(\min_{sm}(\hat{A}, \hat{B})) = \{0.1, 0.2, 0.25\}$ and $Supp(\max_{sm}(\hat{A}, \hat{B})) = \{0.15, 0.2, 0.3\}$, which are different from $\min_s(Supp(\hat{A}), Supp(\hat{B})) = \min_s(\{0.1, 0.2, 0.3\}, \{0.15, 0.2, 0.25\})$ $= \{0.1, 0.15, 0.2, 0.25\}$ and $\max_s(Supp(\hat{A}), Supp(\hat{B})) = \max_s(\{0.1, 0.2, 0.3\}, \{0.15, 0.2, 0.25\}) = \{0.15, 0.2, 0.25, 0.3\}$, respectively.

An immediate implication of propositions 4.4.9 and 4.4.10 is that the product and aggregate multiset-extended operations preserve the integrity of the repetition equivalence classes, as the two following propositions state.

Proposition 4.4.12. *Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$ with a product multiset-extended L-fuzzy algebra $\{\mathbb{N}^L, \neg_m, \wedge_m, \vee_m\}$ and aggregate multiset-extended L-fuzzy algebras $\{\mathbb{N}^L, \neg_{am}, \wedge_{am}, \vee_{am}\}$, for any two multisets $\hat{A}, \hat{B} \in \mathbb{N}^L$, if $\hat{A} \sim_r \hat{B}$ then $\neg_m \hat{A} \sim_r \neg_m \hat{B}$ and $\neg_{am} \hat{A} \sim_r \neg_{am} \hat{B}$.*

Proof. By definition of repetition equivalence, $\hat{A} \sim_r \hat{B}$ is the same as $Supp(\hat{A}) = Supp(\hat{B})$, which is equivalent to $\neg_s Supp(\hat{A}) = \neg_s Supp(\hat{B})$. And, by the first equality in the relations (4.17), $Supp(\neg_m \hat{A}) = Supp(\neg_m \hat{B})$, which, by definition of repetition equivalence, is the same as $\neg_m \hat{A} \sim_r \neg_m \hat{B}$. Note that as all these relations are equivalent, the inverse implication is also true.

Using the aggregate multiset extensions and the same reasoning, this time based on the relations (4.18), leads to $\neg_{am} \hat{A} \sim_r \neg_{am} \hat{B}$. □

Because of these properties, we say that the product and aggregate multiset-extended negations **are compatible with** the repetition equivalence relation.

Proposition 4.4.13. *Given an L-fuzzy algebra* $\{L, \neg, \wedge, \vee\}$ *with a product multiset-extended L-fuzzy algebra* $\{\mathbb{N}^L, \neg_m, \wedge_m, \vee_m\}$ *and aggregate multiset-extended L-fuzzy algebras* $\{\mathbb{N}^L, \neg_{am}, \wedge_{am}, \vee_{am}\}$*, for any four multisets* $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in \mathbb{N}^L$*, if* $\hat{A} \sim_r \hat{C}$ *and* $\hat{B} \sim_r \hat{D}$*, then* $\hat{A} \wedge_m \hat{B} \sim_r \hat{C} \wedge_m \hat{D}$ *and* $\hat{A} \wedge_{am} \hat{B} \sim_r \hat{C} \wedge_{am} \hat{D}$ *and* $\hat{A} \vee_m \hat{B} \sim_r \hat{C} \vee_m \hat{D}$ *and* $\hat{A} \vee_{am} \hat{B} \sim_r \hat{C} \vee_{am} \hat{D}$*.*

Because of these properties, we say that the product and aggregate multiset-extended t-norm and t-conorm ***are compatible with*** *the repetition equivalence relation.*

Proof. By definition of repetition equivalence, $\hat{A} \sim_r \hat{C}$ and $\hat{B} \sim_r \hat{D}$ are the same as $Supp(\hat{A}) = Supp(\hat{C})$ and $Supp(\hat{B}) = Supp(\hat{D})$, respectively, which entails that $Supp(\hat{A}) \wedge_s Supp(\hat{B}) = Supp(\hat{C}) \wedge_s Supp(\hat{D})$. And, by the second equality (4.17) $Supp(\hat{A} \wedge_m \hat{B}) = Supp(\hat{C} \wedge_m \hat{D})$, which, by definition of repetition equivalence, is the same as $\hat{A} \wedge_m \hat{B} \sim_r \hat{C} \wedge_m \hat{D}$.

Using the aggregate multiset extensions and the same reasoning, this time based on the relations (4.18), leads to $\hat{A} \wedge_{am} \hat{B} \sim_r \hat{C} \wedge_{am} \hat{D}$.

The proofs for $\vee_m$ and $\vee_{am}$ are completely analogous. □

Propositions 4.4.12 and 4.4.13 make it possible to extend the three operations $\neg$, $\wedge$ and $\vee$ to the repetition quotient space in a trivial way, as follows.

Definition 4.4.14. Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$ with a product multiset-extended L-fuzzy algebra $\{\mathbb{N}^L, \neg_m, \wedge_m, \vee_m\}$ (or, alternatively, an aggregate multiset-extended L-fuzzy algebra $\{\mathbb{N}^L, \neg_{am}, \wedge_{am}, \vee_{am}\}$), the operations $\neg_{\sim_r}$, $\wedge_{\sim_r}$ and

$\vee_{\sim_r}$ can be defined on the quotient space $\mathbb{N}^L/{\sim_r}$ in the following way:

$$\begin{array}{ll} 1. & \neg_{\sim_r} c = [\neg_m s(c)], \quad c \in \mathbb{N}^L/{\sim_r} \\ 2. & c \wedge_{\sim_r} d = [s(c) \wedge_m s(d)], \quad c, d \in \mathbb{N}^L/{\sim_r} \\ 3. & c \vee_{\sim_r} d = [s(c) \vee_m s(d)], \quad c, d \in \mathbb{N}^L/{\sim_r} \end{array} \tag{4.19}$$

where $s\colon \mathbb{N}^L/{\sim_r} \to \mathbb{N}^L$ is any section and $[\ ]\colon \mathbb{N}^L \to \mathbb{N}^L/{\sim_r}$ is the operation of selecting the equivalence class of a multiset.

These operations on the quotient space are well-defined thanks to propositions 4.4.12 and 4.4.13, which make the choice of section s irrelevant. They can be defined equivalently using the aggregate operators $\neg_{am}$, $\wedge_{am}$ and $\vee_{am}$ with a suitable regularisation strategy.

The repetition quotient space is now equipped with these three operations that make it an L-fuzzy algebra. Thanks to these operations, the bijection ϕ_r can now be interpreted as an L-fuzzy algebra isomorphism, an important result that we sum up in the proposition that follows.

Proposition 4.4.15. *Given an L-fuzzy algebra* $\{L, \neg, \wedge, \vee\}$ *with a product multiset-extended L-fuzzy algebra* $\{\mathbb{N}^L, \neg_m, \wedge_m, \vee_m\}$ *and an aggregate multiset-extended L-fuzzy algebra* $\{\mathbb{N}^L, \neg_{am}, \wedge_{am}, \vee_{am}\}$ *and a set-extended L-fuzzy algebra* $\{\mathbf{2}^L, \neg_s, \wedge_s, \vee_s\}$ *and a repetition equivalence quotient space* $\{\mathbb{N}^L/{\sim_r}, \neg_{\sim_r}, \wedge_{\sim_r}, \vee_{\sim_r}\}$*, the bijection* $\phi_r\colon \mathbb{N}^L/{\sim_r} \to \mathbf{2}^L$*, as in Proposition 3.1.5, defined as* $\phi_r := Supp \circ s$*, with* $s\colon \mathbb{N}^L/{\sim_r} \to \mathbb{N}^L$ *being any section, is an L-fuzzy algebra isomorphism.*

Proof. The function ϕ_r is a bijection according to Proposition 3.1.5. We simply have to prove that the operations fulfil the following conditions of an L-algebra homomorphism:

$$\begin{array}{ll} 1. & \phi_r(\neg_{\sim_r} c) = \neg_s \phi_r(c), \quad c \in \mathbb{N}^L/{\sim_r} \\ 2. & \phi_r(c \wedge_{\sim_r} d) = \phi_r(c) \wedge_s \phi_r(d), \quad c, d \in \mathbb{N}^L/{\sim_r} \\ 3. & \phi_r(c \vee_{\sim_r} d) = \phi_r(c) \vee_s \phi_r(d), \quad c, d \in \mathbb{N}^L/{\sim_r} \end{array} \tag{4.20}$$

Using the definitions of ϕ_r and the operations $\neg_{\sim_r}$, $\wedge_{\sim_r}$ and $\vee_{\sim_r}$, the three equations adopt the following form:

$$\begin{array}{ll} 1. & Supp(s([\neg_m s(c)])) = \neg_s Supp(s(c)) \\ 2. & Supp(s([s(c) \wedge_m s(d)])) = Supp(s(c)) \wedge_s Supp(s(d)) \\ 3. & Supp(s([s(c) \vee_m s(d)])) = Supp(s(c)) \vee_s Supp(s(d)) \end{array}$$

On the left-hand side, the operations $s([\,])$ have no effect as the results will be passed to *Supp*, so the equalities can be simplified as:

1. $Supp(\neg_m s(c)) = \neg_s Supp(s(c))$
2. $Supp(s(c) \wedge_m s(d)) = Supp(s(c)) \wedge_s Supp(s(d))$
3. $Supp(s(c) \vee_m s(d)) = Supp(s(c)) \vee_s Supp(s(d))$

And these three equalities are true as *Supp* is an L-algebra homomorphism according to Proposition 4.4.9 and match equations (4.17) when the product operations are used. If the aggregate operations are used instead, then the above equations tally with equations (4.18) under Proposition 4.4.10. □

The previous propositions and definitions for L-fuzzy algebras can be adapted to L-fuzzy multisets. In the paragraphs that follow, we present the fuzzy set versions without proof as they can all be derived as Goguen extensions.

Proposition 4.4.16. *Given a universe X and an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$, the fuzzy-set-based support function $Supp^f$, as in Definition 4.4.1, mapping its product multiset-based L-fuzzy powerset $\{(\mathbb{N}^L)^X, \neg_{gm}, \wedge_{gm}, \vee_{gm}\}$ to its set-based L-fuzzy powerset $\{(\mathbf{2}^L)^X, \neg_{gs}, \wedge_{gs}, \vee_{gs}\}$ is an L-fuzzy algebra homomorphism. That is, the following relations hold:*

$$\begin{aligned}
&1. \quad Supp^f(\neg_{gm}\hat{A}) = \neg_{gs} Supp^f(\hat{A}), \quad \hat{A} \in (\mathbb{N}^L)^X \\
&2. \quad Supp^f(\hat{A} \wedge_{gm} \hat{B}) = Supp^f(\hat{A}) \wedge_{gs} Supp^f(\hat{B}), \quad \hat{A}, \hat{B} \in (\mathbb{N}^L)^X \\
&3. \quad Supp^f(\hat{A} \vee_{gm} \hat{B}) = Supp^f(\hat{A}) \vee_{gs} Supp^f(\hat{B}), \quad \hat{A}, \hat{B} \in (\mathbb{N}^L)^X
\end{aligned} \tag{4.21}$$

Proposition 4.4.17. *Given a universe X and an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$, the fuzzy-set-based support function $Supp^f$, as in Definition 4.4.1, mapping its aggregate multiset-based L-fuzzy powerset $\{(\mathbb{N}^L)^X, \neg_{gam}, \wedge_{gam}, \vee_{gam}\}$ to its set-based L-fuzzy powerset $\{(\mathbf{2}^L)^X, \neg_{gs}, \wedge_{gs}, \vee_{gs}\}$ is an L-fuzzy algebra homomorphism. That is, the following relations hold:*

$$\begin{aligned}
&1. \quad Supp^f(\neg_{gam}\hat{A}) = \neg_{gs} Supp^f(\hat{A}), \quad \hat{A} \in (\mathbb{N}^L)^X \\
&2. \quad Supp^f(\hat{A} \wedge_{gam} \hat{B}) = Supp^f(\hat{A}) \wedge_{gs} Supp^f(\hat{B}), \quad \hat{A}, \hat{B} \in (\mathbb{N}^L)^X \\
&3. \quad Supp^f(\hat{A} \vee_{gam} \hat{B}) = Supp^f(\hat{A}) \vee_{gs} Supp^f(\hat{B}), \quad \hat{A}, \hat{B} \in (\mathbb{N}^L)^X
\end{aligned} \tag{4.22}$$

Proposition 4.4.18. *Given a universe X, an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$ with a set-based L-fuzzy powerset $\{(\mathbf{2}^L)^X, \neg_{gs}, \wedge_{gs}, \vee_{gs}\}$, a product multiset-based L-fuzzy powerset $\{(\mathbb{N}^L)^X, \neg_{gm}, \wedge_{gm}, \vee_{gm}\}$ and an aggregate multiset-based L-fuzzy powerset $\{(\mathbb{N}^L)^X, \neg_{gam}, \wedge_{gam}, \vee_{gam}\}$:*

- *For any two L-fuzzy multisets* $\hat{A}, \hat{B} \in (\mathbb{N}^L)^X$*, if* $\hat{A} \sim_{fr} \hat{B}$ *then* $\neg_{gm}\hat{A} \sim_{fr} \neg_{gm}\hat{B}$ *and* $\neg_{gam}\hat{A} \sim_{fr} \neg_{gam}\hat{B}$*.*
- *For any four L-fuzzy multisets* $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in (\mathbb{N}^L)^X$*, if* $\hat{A} \sim_{fr} \hat{C}$ *and* $\hat{B} \sim_{fr} \hat{D}$*, then* $\hat{A} \wedge_{gm} \hat{B} \sim_{fr} \hat{C} \wedge_{gm} \hat{D}$ *and* $\hat{A} \wedge_{gam} \hat{B} \sim_{fr} \hat{C} \wedge_{gam} \hat{D}$ *and* $\hat{A} \vee_{gm} \hat{B} \sim_{fr} \hat{C} \vee_{gm} \hat{D}$ *and* $\hat{A} \vee_{gam} \hat{B} \sim_{fr} \hat{C} \vee_{gam} \hat{D}$*.*

The previous Proposition 4.4.18 makes it possible to extend the three operations $\neg$, $\wedge$ and $\vee$ to the fuzzy repetition quotient space in a trivial way, as follows.

Definition 4.4.19. Given a universe X and an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$ with a product multiset-based L-fuzzy powerset $\{(\mathbb{N}^L)^X, \neg_{gm}, \wedge_{gm}, \vee_{gm}\}$ (or, alternatively, an aggregate multiset-based L-fuzzy powerset $\{\mathbb{N}^L)^X, \neg_{gam}, \wedge_{gam}, \vee_{gam}\}$), the operations $\neg_{\sim_{fr}}$, $\wedge_{\sim_{fr}}$ and $\vee_{\sim_{fr}}$ can be defined on the quotient space $(\mathbb{N}^L)^X/{\sim_{fr}}$ in the following way:

$$\begin{aligned}
&1. \quad \neg_{\sim_{fr}} c = [\neg_{gm} s((c))], \quad c \in (\mathbb{N}^L)^X/{\sim_{fr}} \\
&2. \quad c \wedge_{\sim_{fr}} d = [s(c) \wedge_{gm} s(d)], \quad c, d \in (\mathbb{N}^L)^X/{\sim_{fr}} \\
&3. \quad c \vee_{\sim_{fr}} d = [s(c) \vee_{gm} s(d)], \quad c, d \in (\mathbb{N}^L)^X/{\sim_{fr}}
\end{aligned} \tag{4.23}$$

where $s\colon (\mathbb{N}^L)^X/{\sim_{fr}} \to (\mathbb{N}^L)^X$ is any section and $[\,]\colon (\mathbb{N}^L)^X \to (\mathbb{N}^L)^X/{\sim_{fr}}$ is the operation of selecting the equivalence class of a fuzzy multiset.

The fuzzy repetition quotient space is now equipped with these three operations that make it an L-fuzzy algebra. Thanks to these operations, the bijection ϕ_{fr} can now be interpreted as an L-fuzzy algebra isomorphism.

Proposition 4.4.20. *Given a universe* X*, an L-fuzzy algebra of membership grades* $\{L, \neg, \wedge, \vee\}$ *with a set-based L-fuzzy powerset* $\{(\mathbf{2}^L)^X, \neg_{gs}, \wedge_{gs}, \vee_{gs}\}$*, a product multiset-based L-fuzzy powerset* $\{(\mathbb{N}^L)^X, \neg_{gm}, \wedge_{gm}, \vee_{gm}\}$*, an aggregate multiset-based L-fuzzy powerset* $\{(\mathbb{N}^L)^X, \neg_{gam}, \wedge_{gam}, \vee_{gam}\}$ *and a fuzzy repetition equivalence quotient space* $\{(\mathbb{N}^L)^X/{\sim_{fr}}, \neg_{\sim_{fr}}, \wedge_{\sim_{fr}}, \vee_{\sim_{fr}}\}$*, then the bijection* $\phi_{fr}\colon (\mathbb{N}^L)^X/{\sim_{fr}} \to (\mathbf{2}^L)^X$ *as in Proposition 4.4.8, defined as* $\phi_{fr} := Supp^f \circ s$*, with* $s\colon (\mathbb{N}^L)^X/{\sim_{fr}} \to (\mathbb{N}^L)^X$ *being any section, is an L-fuzzy algebra isomorphism.*

We have therefore found that there is a partition (or a subset) of the L-fuzzy multisets that can be put in a bijection with the set-based L-fuzzy sets and, furthermore, that when either the product or the aggregate multiset-extended operations are used, then this bijection is an isomorphism that preserves the mathematical structure determined by the three basic operations of fuzzy sets. As a result of this, when using the unit interval and the standard operations, the hesitant fuzzy sets can be regarded as a particular case of the ordinary fuzzy multisets with the standard product or aggregate operations. This is not the case, however, with the more popular Miyamoto's operations, as the basic operations do not commute with the support, as we showed in Example 4.4.11 above, and such an isomorphism is not possible.

4.5 An explicit formula for the aggregate intersection and union

In this last section, we address a weakness of the definition for the aggregate binary operations on multisets, which we have discussed in a forthcoming article [37]. As these operations are based on a multiset union over all the possible permutation of the n-tuples, actual calculations can involve a huge amount of terms and its computational cost may be prohibitive, both in the general form (4.10) and in the standard aggregate fuzzy multiset algebra (4.10). In fact, the time complexity of any algorithm that uses the formula in the definitions is $\mathcal{O}(n!)$, since permutations of n elements are involved.

We are going to show that less computationally intensive formulas, displaying time complexity $\mathcal{O}(n)$, can be derived. We will do this for the particular case of the standard aggregate fuzzy multiset algebra, so we will be working on the unit interval $\mathbb{I}$ and our goal is to derive two explicit formulas for $\min_{am}$ and $\max_{am}$.

Before introducing these formulas, we need the concept of α-cut cardinalities for multisets. This is related to the α-cuts defined by S. Miyamoto for fuzzy multisets, but the formalism is simpler when we work on the membership grade algebras. We will define two versions of α-cut cardinality, one that counts the elements strictly greater than the cut value α and another one that counts the elements strictly less than α. For brevity, we group the two versions under one definition.

Definition 4.5.1. Let $\mathbb{N}^{\mathbb{I}}$ be the power multiset over the unit interval. For a multiset $\hat{A} \in \mathbb{N}^{\mathbb{I}}$ and a real number $\alpha \in [0, 1]$, the **strict upper (lower) α-cut cardinality** of $\hat{A}$ is the sum of the multiplicities of those values that are strictly greater (less) than α:

$$|\hat{A}|_{>\alpha} := \sum_{t\in\mathbb{I},t>\alpha} \hat{A}(t) \tag{4.24}$$

$$|\hat{A}|_{<\alpha} := \sum_{t\in\mathbb{I},t<\alpha} \hat{A}(t) \tag{4.25}$$

We need these definitions of the strong α-cuts for the next proposition:

Proposition 4.5.2. *In the standard aggregate fuzzy multiset algebra* $\{\mathbb{N}^{\mathbb{I}}, c_{am}, \min_{am}, \max_{am}\}$*, the* $\min_{am}$ *and* $\max_{am}$ *binary operations can be expressed explicitly in terms of the multiplicities of the operands as follows:*

$$\begin{aligned}
1.\ (\min_{am}(\hat{A},\hat{B}))(t) &= \min(\hat{A}(t),|\hat{B}|_{>t}) + \min(\hat{B}(t),|\hat{A}|_{>t}) + \\
&+ \max(0, \min(\hat{A}(t) - |\hat{B}|_{>t}, \hat{B}(t) - |\hat{A}]_{>t})), \quad t\in[0,1] \\
2.\ (\max_{am}(\hat{A},\hat{B}))(t) &= \min(\hat{A}(t),|\hat{B}|_{<t}) + \min(\hat{B}(t),|\hat{A}|_{<t}) + \\
&+ \max(0, \min(\hat{A}(t) - |\hat{B}|_{<t}, \hat{B}(t) - |\hat{A}]_{<t})), \quad t\in[0,1]
\end{aligned} \tag{4.26}$$

Proof. We will only prove the first expression, as the second one is completely analogous.

In order to prove the equality, we are going to analyse the different possible cases for the membership grade $t \in \mathbb{I}$.

Case 1. Let t be such that both $\hat{A}(t) = 0$ and $\hat{B}(t) = 0$. As t does not appear in either fuzzy multiset, the result of $(\min_{am}(\hat{A},\hat{B}))(t)$, according to the union-based formula in the equalities (4.10), must be 0. And the above formula yields 0 as a result too.

Case 2. Let t be such that $\hat{A}(t) > 0$ and $\hat{B}(t) = 0$. Now t appears in one of the multisets, and the result of $(\min_{am}(\hat{A},\hat{B}))(t)$ according to the formula (4.10), and taking into account Definition 2.5.9 for the multiset union, will be the maximum number of possible pairings between each one of the $\hat{A}(t)$ occurrences of t in the n-tuples of the permutation equivalence class for $\hat{A}$ and those values in $\hat{B}$ that are greater than t. That number is obviously $\min(\hat{A}(t), |\hat{B}|_{>t})$. As we have $\hat{B}(t) = 0$, the other two terms evaluate to zero and the equality holds in this case too.

Case 3. Let t be such that $\hat{A}(t) = 0$ and $\hat{B}(t) > 0$. This is the same as Case 2, with the roles of the two operands swapped, so the aggregate operation $\min_{am}$ will be given by the second term $\min(\hat{B}(t), |\hat{A}|_{>t})$, with the remaining terms being zero.

Case 4. Let t be such that $\hat{A}(t) > 0$ and $\hat{B}(t) > 0$. This is the non-trivial case where the three terms contribute to the result. As the known $\min_{am}$ formula in the equalities (4.10) is based on taking the maximum possible multiplicity among all the permutations of n-tuples, we have to identify the most favourable situations. The $\hat{A}(t)$ occurrences of t in the n-tuples of the permutation equivalence class for $\hat{A}$ will make it into the aggregate result if they can be paired with values in the n-tuples of the permutation equivalence class for $\hat{B}$ that are greater than t. And the maximum number of such pairings is obviously $|\hat{B}|_{>t}$. We have thus accounted for $\min(\hat{A}(t), |\hat{B}|_{>t})$ contributions. Now if $\hat{A}(t) \leq |\hat{B}|_{>t}$ we have exhausted the t values in $\hat{A}$, but if that is not the case there will be a positive number $\hat{A}(t) - |\hat{B}|_{>t}$ of occurrences of t that can still be paired with the t values in $\hat{B}$ in the final step. Before that, we repeat the same reasoning for the $\hat{B}(t)$ occurrences of t in the n-tuples of the permutation equivalence class for $\hat{B}$ and, again, a maximum of $|\hat{A}|_{>t}$ will find their way into the aggregate result, independently of the ones in $\hat{A}$, which results in the $\min(\hat{B}(t), |\hat{A}]_{>t})$ contribution. Finally, if both $\hat{A}(t) > |\hat{B}|_{>t}$ and $\hat{B}(t) > |\hat{A}|_{>t}$, there remains a number of t values, the minimum of the two subtractions, that have not been paired with any greater values but which can, in the most favourable sequence combination, be paired with each other, leading to the third term in the equality. □

We conclude this analysis with an observation that is relevant when calculating the aggregate multiset minimum (intersection) and maximum (union) and that helps to understand why the difference between the aggregate and the sorted operations is, in practice, much less important than would appear from the mathematical formalism of this chapter.

We have already mentioned how the aggregate operations do not preserve the cardinality of the operands, unlike the sorted operations. For example, if we have two multisets $\hat{A} = \langle 0.1, 0.4\rangle$ and $\hat{B} = \langle 0.2, 0.4\rangle$, their aggregate minimum is $\min_{am}(\hat{A}, \hat{B}) = \langle 0.1, 0.2, 0.4\rangle$. But it can be proved that such an increase in cardinality only occurs when the involved multisets have overlapping ranges of values, but not when the ranges do not overlap. In fact, in this latter case, the aggregate operations yield the same results as Miyamoto's sorted operations. In the more usual cases when working with fuzzy multisets, it would be normal to have membership multisets like $\langle 0.1, 0.2, 0.2\rangle$ or $\langle 0.8, 0.8, 0.9\rangle$ but a membership multiset like $\langle 0.1, 0.5, 0.9\rangle$ would be hard to justify. When overlapping ranges of values are rare, the operational differences between the sorted and the aggregate operations fade in importance.

This equivalence between the sorted and the aggregate operations when ranges do not overlap is formalised in the following proposition.

Proposition 4.5.3. *Let $\hat{A}$ and $\hat{B}$ be two n-regular multisets over the unit interval $\mathbb{I}$. If $\hat{A}$ and $\hat{B}$ span ranges of values that do not overlap; i.e.* $\max Supp(\hat{A}) \leq \min Supp(\hat{B})$ *or* $\max Supp(\hat{B}) \leq \min Supp(\hat{A})$*, then the result of* $\min_{sm}(\hat{A}, \hat{B})$ *and of* $\max_{sm}(\hat{A}, \hat{B})$ *is independent of the choice of section.*

Proof. Let us assume, without loss of generality, that $\hat{A}$ is the multiset with the lower values, $\max Supp(\hat{A}) \leq \min Supp(\hat{B})$. Then for any pair of index permutations σ and ρ, if $\hat{A} = \langle a_1, ..., a_n\rangle$ and $\hat{B} = \langle b_1, ..., b_n\rangle$ we get the n-tuples $(a_{\sigma(1)}, ..., a_{\sigma(n)})$ and $(b_{\rho(1)}, ..., b_{\rho(n)})$. As none of the a_i values are greater than any of the b_j values for any pair of indices i, j, we have $\min_{sm}(\hat{A}, \hat{B}) = \hat{A}$ and $\max_{sm}(\hat{A}, \hat{B}) = \hat{B}$ regardless of how the elements are sorted. □

If the operations are independent of how the elements are sorted, then the corresponding aggregate operations, being based on a multiset union of all the possible sorted arrangements, will obviously produce the same result.

Corollary 4.5.4. *Let $\hat{A}$ and $\hat{B}$ be two n-regular multisets over the unit interval $\mathbb{I}$. If $\hat{A}$ and $\hat{B}$ span ranges of values that do not overlap, then* $\min_{am}(\hat{A}, \hat{B}) = \min_{sm}(\hat{A}, \hat{B})$ *and* $\max_{am}(\hat{A}, \hat{B}) = \max_{sm}(\hat{A}, \hat{B})$*, with the choice of section being irrelevant for the sorted operations.*

In an ordinary fuzzy multiset, if the condition of non-overlapping ranges holds for any element, then the independence of the sorting strategy applies to the whole fuzzy multiset. The following corollary expresses this idea.

Corollary 4.5.5. *Let X be the universe and let $\hat{A}$ and $\hat{B} \in (\mathbb{N}^{\mathbb{I}})^X$ be two fuzzy multisets such that they span ranges of values that do not overlap for any element, i.e. either* $\max Supp(\hat{A}(x)) \leq \min Supp(\hat{B}(x))$ *or* $\max Supp(\hat{B}(x)) \leq \min Supp(\hat{A}(x))$ *for all $x \in X$, then the aggregate intersection is the same as Miyamoto's intersection and the aggregate union is the same as Miyamoto's union.*

We include two pseudocode listings showing the algorithms that compute the aggregate intersection and union for a fixed element $x \in X$. It is assumed that there is a data structure representing a multiset together with operations for element insertion and look-up (like, for example, the *std::multiset* class in the C++ standard library) and a similar data structure for sets (like a *std::set* in C++). As the algorithm involves an iteration over the elements of the multisets, its time complexity is $\mathcal{O}(n)$ in terms of the size of the input multisets.

Algorithm 1 Function that calculates the aggregate intersection

```
function INTERSECTION(a, b)                ▷ Input arguments: two multisets a and b
    result ← create_multiset()             ▷ Empty initialisation of a new multiset
    a_max ← multiset_max(a)
    b_min ← multiset_min(b)
    if a_max ≤ b_min then
        result ← a
    else
        a_min ← multiset_min(a)
        b_max ← multiset_max(b)
        if b_max ≤ a_min then
            result ← b
        else
            a_support ← multiset_support(a)
            b_support ← multiset_support(b)
            if a_max ≠ b_max then
                if a_max < b_max then      ▷ Ignore values > min(a_max, b_max)
                    b_support ← multiset_lower_bound(b_support, a_max)
                else
                    a_support ← multiset_lower_bound(a_support, b_max)
                end if
            end if
            support ← set_union(a_support, b_support)
            for all t ∈ support do
                a_upper_bound ← multiset_strict_upper_bound(a, t)
                a_up_length ← multiset_length(a_upper_bound)      ▷ This is [Â]_{>t}
                b_upper_bound ← multiset_strict_upper_bound(b, t)
                b_up_length ← multiset_length(b_upper_bound)      ▷ This is [B̂]_{>t}
                t_a_count ← multiset_element_count(a, t)
                t_b_count ← multiset_element_count(b, t)
                t_count ← min(t_a_count, b_up_length) + min(t_b_count, a_up_length)
                if t_a_count ≥ b_up_length AND t_b_count ≥ a_up_length then
                    t_count += min(t_a_count − b_up_length, t_b_count − a_up_length)
                end if
                multiset_insert(result, t, t_count)               ▷ Inserts t t_count times
            end for
        end if
    end if
    return result
end function
```

Algorithm 2 Function that calculates the aggregate union

```
function Union(a, b)                                  ▷ Input arguments: two multisets a and b
    result ← create_multiset()                        ▷ Empty initialisation of a new multiset
    a_max ← multiset_max(a)
    b_min ← multiset_min(b)
    if a_max ≤ b_min then
        result ← b
    else
        a_min ← multiset_min(a)
        b_max ← multiset_max(b)
        if b_max ≤ a_min then
            result ← a
        else
            a_support ← multiset_support(a)
            b_support ← multiset_support(b)
            if a_min ≠ b_min then
                if a_min < b_min then                 ▷ Ignore values < max(a_min, b_min)
                    a_support ← multiset_upper_bound(a_support, b_min)
                else
                    b_support ← multiset_upper_bound(b_support, a_min)
                end if
            end if
            support ← set_union(a_support, b_support)
            for all t ∈ support do
                a_lower_bound ← multiset_strict_lower_bound(a, t)
                a_low_length ← multiset_length(a_lower_bound)      ▷ This is [Â]_{<t}
                b_lower_bound ← multiset_strict_lower_bound(b, t)
                b_low_length ← multiset_length(b_lower_bound)      ▷ This is [B̂]_{<t}
                t_a_count ← multiset_element_count(a, t)
                t_b_count ← multiset_element_count(b, t)
                t_count ← min(t_a_count, b_low_length) + min(t_b_count, a_low_length)
                if t_a_count ≥ b_low_length AND t_b_count ≥ a_low_length then
                    t_count += min(t_a_count−b_low_length, t_b_count−a_low_length)
                end if
                multiset_insert(result, t, t_count)        ▷ Inserts t t_count times
            end for
        end if
    end if
    return result
end function
```

5 Ordered fuzzy multisets

In this chapter, we continue our exploration of multivalued fuzzy sets with the use of n-tuples as fuzzy membership grades. We shall follow the same approach as we did in the previous chapter with the fuzzy sets based on sets and multisets. We assume that there is a certain space L that can be used to represent fuzzy membership, which will be the unit interval $\mathbb{I}$ in the ordinary case. Our aim now is to study the use of an n-dimensional space L^n as a new space of membership grades. By using the Cartesian Extension Principle (see Definition 3.4.1), we can generalise the operations $\neg$, $\wedge$ and $\vee$ to L^n. After that, Goguen's Extension Principle allows us to induce the operations on the n-tuple-based fuzzy sets, which we will call **ordered fuzzy multisets**, as in a yet unpublished article currently under review [38].

5.1 The n-dimensional L-fuzzy membership grades

Just as we did when we dealt with the set and multiset-based membership grades, we assume that there is a basic L-fuzzy algebra $\{L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$, which we want to extend to an n-dimensional Cartesian product L^n, where n is a fixed non-zero natural number (we will usually assume that $n > 1$, as $n = 1$ would be a trivial case $L^1 = L$). This way, we will be able to use *n-tuples* of values of L as membership grades for a new type of fuzzy sets.

The mechanism we will need in order to extend the operations from L to L^n is the Cartesian Extension Principle, whereby the L-fuzzy algebra $\{L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$ can be extended to an n-dimensional algebra where the operations are generalised to L^n. The following definition encapsulates this idea.

Definition 5.1.1. Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$ and a non-zero natural number $n \in \mathbb{N}$, the **Cartesian-extended n-dimensional L-fuzzy algebra** is $\{L^n, \neg_c, \wedge_c, \vee_c, \mathbf{0}_c, \mathbf{1}_c\}$, where $\neg_c$, $\wedge_c$ and $\vee_c$ are the Cartesian-extended versions of $\neg$, $\wedge$ and $\vee$, respectively, based on the Cartesian Extension Principle (see Definition 3.4.1), and $\mathbf{0}_c$ and $\mathbf{1}_c$ are the two n-tuples $\mathbf{0}_c := (\mathbf{0}, \overset{n)}{\ldots}, \mathbf{0})$ and $\mathbf{1}_c := (\mathbf{1}, \overset{n)}{\ldots}, \mathbf{1})$.

The typical situation occurs when the L-fuzzy algebra is the standard ordinary fuzzy algebra, which leads to a more specific algebra. In this case, we arrive at the next proposition through a straightforward application of the Cartesian Extension Principle.

Proposition 5.1.2. *In the particular case of the standard ordinary fuzzy algebra* $\{\mathbb{I}, c, \min, \max, 0, 1\}$*, the Cartesian-extended n-dimensional L-fuzzy algebra is* $\{\mathbb{I}^n, c_n, \min_c, \max_c, \vec{0}, \vec{1}\}$*, where* $\vec{0} = (0, \overset{n)}{\ldots}, 0)$*,* $\vec{1} = (1, \overset{n)}{\ldots}, 1)$ *and the Cartesian-extended operations* c_c*,* $\min_c$ *and* $\max_c$ *fulfil the following relations for any n-tuples* $\vec{A}, \vec{B} \in \mathbb{I}^n$ *with coordinates* $\vec{A} = (a_1, ..., a_n)$ *and* $\vec{B} = (b_1, ..., b_n)$*:*

$$\begin{aligned} &1. \quad c_c(\vec{A}) = (1 - a_1, ..., 1 - a_n) \\ &2. \quad \min_c(\vec{A}, \vec{B}) = (\min(a_1, b_1), ..., \min(a_n, b_n)) \\ &3. \quad \max_c(\vec{A}, \vec{B}) = (\max(a_1, b_1), ..., \max(a_n, b_n)) \end{aligned} \tag{5.1}$$

The previous Cartesian-extended algebra of Proposition 5.1.2 above is called the **standard ordinary Cartesian n-dimensional fuzzy algebra** of membership values.

Example 5.1.3. Let us take two triplets over the unit interval $\vec{A} = (0.2, 0.3, 0.5)$ and $\vec{B} = (0.3, 0.1, 0.7)$. By using the expressions (5.1), we have that $c_c(\vec{A}) = (0.8, 0.7, 0.5)$ and $c_c(\vec{B}) = (0.7, 0.9, 0.3)$, $\min_c(\vec{A}, \vec{B}) = (0.2, 0.1, 0.5)$ and $\max_c(\vec{A}, \vec{B}) = (0.3, 0.3, 0.7)$.

The Cartesian-extended n-dimensional L-fuzzy algebras and, in particular, the standard ordinary Cartesian n-dimensional fuzzy algebra can be used to define extended fuzzy sets where the membership grades are made up of n-tuples. As in the previous chapter, we will rely on Goguen's Extension Principle (see Definition 2.7.6) to extend the operations on the Cartesian fuzzy algebras to functions mapping those algebras into the elements of a universe X.

Definition 5.1.4. Given a universe X, an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$ and a non-zero natural number $n \in \mathbb{N}$, the **(Cartesian) n-dimensional L-fuzzy powerset** is the set of all functions $X \to L^n$, denoted by

$(L^n)^X$, with the unary and binary operations defined by Goguen's extension of the operations in the Cartesian-extended n-dimensional L-fuzzy algebra $\neg_{gc}$, $\wedge_{gc}$ and $\vee_{gc}$:

$$\begin{array}{lll} 1. & (\neg_{gc}\vec{A})(x) := \neg_c \vec{A}(x), & \vec{A} \in (L^n)^X \ , x \in X \\ 2. & (\vec{A} \wedge_{gc} \vec{B})(x) := \vec{A}(x) \wedge_c \vec{B}(x), & \vec{A}, \vec{B} \in (L^n)^X \ , x \in X \\ 3. & (\vec{A} \vee_{gc} \vec{B})(x) := \vec{A}(x) \vee_c \vec{B}(x), & \vec{A}, \vec{B} \in (L^n)^X \ , x \in X \end{array} \tag{5.2}$$

and with the two constant functions $\mathbf{0}_{gc}$ and $\mathbf{1}_{gc}$ defined as:

$$\begin{array}{lll} 1. & \mathbf{0}_{gc}(\vec{A})(x) := \mathbf{0}_c = (\mathbf{0}, \overset{n)}{\ldots}, \mathbf{0}), & \vec{A} \in (L^n)^X \ , x \in X \\ 2. & \mathbf{1}_{gc}(\vec{A})(x) := \mathbf{1}_c = (\mathbf{1}, \overset{n)}{\ldots}, \mathbf{1}), & \vec{A} \in (L^n)^X \ , x \in X \end{array} \tag{5.3}$$

With these definitions, the n-dimensional L-fuzzy powerset is by itself an algebra $\{(L^n)^X, \neg_{gc}, \wedge_{gc}, \vee_{gc}, \mathbf{0}_{gc}, \mathbf{1}_{gc}\}$. The functions in the n-dimensional L-fuzzy powerset are called **(Cartesian) n-dimensional L-fuzzy sets** or simply **ordered L-fuzzy multisets**.

Being a function with a multidimensional codomain, an ordered L-fuzzy multiset can be decomposed into one-dimensional functions.

Definition 5.1.5. Given an n-dimensional ordered L-fuzzy multiset $\vec{A} \in (L^n)^X$ over a universe X and with an underlying L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$, the n single-valued functions $A_i \colon X \to L$ defined by the following expression:

$$A_i(x) := \pi_i(\vec{A}(x)), \quad x \in X, i = 1, ..., n$$

are called the **coordinates** of the ordered L-fuzzy multiset, with π_i being the projection function $\pi_i \colon L^n \to L$ defined as $\pi_i(a_1, ..., a_n) := a_i$ for any $(a_1, ..., a_n) \in L^n$ and for each index $i = 1, ..., n$.

It is trivial to prove that each coordinate is by itself an L-fuzzy set over X with the underlying L-fuzzy algebra.

As with the other generalisations of fuzzy sets, the extended operations $\wedge_{gc}$ and $\vee_{gc}$ are usually called **intersection** and **union**, respectively, and can be denoted by the more traditional symbols $\cap$ and $\cup$ when there is no risk of confusion. The negation, $\neg_{gc}$, can be called **complement** and is often represented as a ^{c} superscript next to the name of the fuzzy set.

The n-dimensional L-fuzzy powerset over a universe X for the standard ordinary fuzzy algebra is called the **standard ordinary (Cartesian) n-dimensional fuzzy powerset**. Its functions are called **standard ordinary (Cartesian) n-dimensional fuzzy sets** or simply **ordered fuzzy multisets with the standard operations**.

Example 5.1.6. Let the universe be $X = \{x\}$, comprising one single element. Then two 3-dimensional ordered fuzzy multisets over X, $\vec{A}$ and $\vec{B}$ are fully specified by the triplets that x is mapped to. For example, $\vec{A}(x) = (0.2, 0.3, 0.5)$ and $\vec{B}(x) = (0.3, 0.1, 0.7)$. These triplets that stand for the membership grades are the same as those in the previous Example 5.1.3, so the operations carried out in the standard ordinary 3-dimensional fuzzy algebra are now to be interpreted as the complement, intersection and union of the ordered fuzzy multisets: $\vec{A}^c(x) = c_c(\vec{A}(x)) = (0.8, 0.7, 0.5)$, $\vec{B}^c(x) = c_c(\vec{B}(x)) = (0.7, 0.9, 0.3)$, $(\vec{A} \cap \vec{B})(x) = \min_c(\vec{A}(x), \vec{B}(x)) = (0.2, 0.1, 0.5)$ and $(\vec{A} \cup \vec{B})(x) = \max_c(\vec{A}(x), \vec{B}(x)) = (0.3, 0.3, 0.7\}$. As with any other type of fuzzy set, if the universe had more elements than x, the three operations would be evaluated at each element in this way, independently of the other elements.

This derivation of the ordered fuzzy multisets parallels those for the set-based and multiset-based L-fuzzy sets. The key difference that this new concept introduces is the fact that elements in an n-tuple are ordered, so (a, a, b) is different from (a, b, a) unlike the case with multisets, which ignore order, or sets, which do not account for either order or repetition. As we mentioned in Chapter 2, taking both order and repetition into account makes sense in the common scenario where the multiple membership values in a multivalued fuzzy set come from a number n of distinguishable experts or criteria, each one of them producing a membership value for each element in the universe. Sacrificing order or repetition in such a scenario may result in a flawed mathematical model.

5.2 The relation between L-fuzzy multisets and ordered L-fuzzy multisets

Ordered L-fuzzy multisets differ from L-fuzzy multisets in the use of n-tuples rather than multisets, so the relations between these two types of collections of values discussed in Chapter 3 can be adapted to the corresponding fuzzy sets by means of Goguen's extension. This is analogous to what we did with set-based

L-fuzzy sets and L-fuzzy multisets in the previous chapter, so the derivation is similar. We begin with an extension of the concept of multiset support (see Definition 3.3.1) from the n-tuples to the ordered L-fuzzy multisets.

Definition 5.2.1. Let X be the universe, let $n \in \mathbb{N}$ be a non-zero natural number and let L be a membership grade space over which a multiset-based L-fuzzy powerset $(\mathbb{N}^L)^X$ and an n-dimensional L-fuzzy powerset $(L^n)^X$ are defined. The **fuzzy-multiset-based support** $Supp^f_m \colon (L^n)^X \to (\mathbb{N}^L)^X$ is Goguen's extension of the multiset support function $Supp_m \colon L^n \to \mathbb{N}^L$:

$$Supp^f_m := Ext_g(Supp_m)$$

By application of Goguen's extension, $Supp^f_m$ has a pointwise expression, which we can state as a trivial proposition.

Proposition 5.2.2. *$Supp^f_m$ can be expressed in terms of how it acts on each element $x \in X$, as follows:*

$$(Supp^f_m(\vec{A}))(x) = Supp_m(\vec{A}(x)), \quad \vec{A} \in (L^n)^X, x \in X \tag{5.4}$$

This new type of support in the realm of the ordered fuzzy multisets has similar properties to the multiset support for n-tuples. Just as we did with the other versions of the support concept, we can define the preimage as a function $(\mathbb{N}^L)^X \to \mathbf{2}^{((L^n)^X)}$.

Definition 5.2.3. Let X be the universe, let n be a non-zero natural number and let L be a membership grade space over which a multiset-based L-fuzzy powerset $(\mathbb{N}^L)^X$ and an n-dimensional L-fuzzy powerset $(L^n)^X$ are defined. The **ordered fuzzy multiset support preimage** of a multiset-based L-fuzzy set is a function $Supp^{f\,-1}_m \colon (\mathbb{N}^L)^X \to \mathbf{2}^{((L^n)^X)}$ defined by the relation:

$$Supp^{f\,-1}_m(\hat{A}) := \{\vec{T} \in (L^n)^X | Supp^f_m(\vec{T}) = \hat{A}\}, \quad \hat{A} \in (\mathbb{N}^L)^X$$

As in the previous chapters, here we can also introduce an equivalence relation to extend to these types of fuzzy sets the identification between multisets and permutation-equivalent n-tuples of Chapter 3. We will use the concept of equivalence kernel of Definition 2.8.6 again.

Definition 5.2.4. Let X be the universe, let n be a non-zero natural number and let L be a membership grade space over which a multiset-based L-fuzzy powerset

$(\mathbb{N}^L)^X$ and an n-dimensional L-fuzzy powerset $(L^n)^X$ are defined. The **permutation equivalence** in $(L^n)^X$ is the equivalence kernel of the fuzzy-multiset-based support function $Supp^f_m$.

This equivalence relation is denoted by $\sim_{fp}$, so given two ordered fuzzy multisets $\vec{A}$, $\vec{B} \in (L^n)^X$, $\vec{A} \sim_{fp} \vec{B}$ means that $Supp^f_m(\vec{A}) = Supp^f_m(\vec{B})$.

Example 5.2.5. Let us consider a two-element universe $X = \{x, y\}$ and the ordinary membership grade space $\mathbb{I}$. If $\vec{A}$ and $\vec{B}$ are two ordered fuzzy multisets such that $\vec{A}(x) = (0.3, 0.1, 0.2)$, $\vec{A}(y) = (0.7, 0.7, 0.8)$, $\vec{B}(x) = (0.2, 0.1, 0.3)$ and $\vec{B}(y) = (0.7, 0.8, 0.7)$, the respective n-tuples evaluated at x and y only differ in the ordering of the coordinates, so their fuzzy-multiset-based supports are the same: $(Supp^f_m(\vec{A}))(x) = (Supp^f_m(\vec{B}))(x) = \langle 0.1, 0.2, 0.3\rangle$ and $(Supp^f_m(\vec{A}))(y) = (Supp^f_m(\vec{B}))(y) = \langle 0.7, 0.7, 0.8\rangle$. As $Supp^f_m(\vec{A}) = Supp^f_m(\vec{B})$, we can say that $\vec{A}$ and $\vec{B}$ are permutation-equivalent, $\vec{A} \sim_{fp} \vec{B}$.

Choosing a representative for each permutation-equivalence class involves the use of a sorting strategy to single out a particular n-tuple for each element in the universe. In the usual case when the space of membership grades L is totally ordered, we can identify as "canonical" the strategy that consists in picking the sorted n-tuples, in either ascending or descending order, just as we did with the repetition equivalence classes for n-tuples in Chapter 3.

Definition 5.2.6. Given the permutation equivalence $\sim_{fp}$ in the ordered L-fuzzy power multiset $(L^n)^X$ over a universe X with membership grades in L, the **(ascending) canonical section** of the permutation equivalence $\sim_{fp}$ is the section $s^{\uparrow}_f \colon (L^n)^X/\sim_{fp} \to (L^n)^X$ such that, for any equivalence class $c \in (L^n)^X/\sim_{fp}$ and for any element $x \in X$, the components of $s^{\uparrow}_f(c)(x)$ are sorted in ascending order: $(s^{\uparrow}_f(c)(x))_1 \leq ... \leq (s^{\uparrow}_f(c)(x))_n$.

Similarly, the **descending canonical section** of $\sim_{fp}$ is the section $s_{f\downarrow}$ such that, for any equivalence class $c \in (L^n)^X/\sim_{fp}$ and for any element $x \in X$, $(s_{f\downarrow}(c)(x))_1 \geq ... \geq (s_{f\downarrow}(c)(x))_n$.

The image of a permutation equivalence class $c \in (L^n)^X/\sim_{fp}$ by its (ascending/descending) canonical section, $s^{\uparrow/\downarrow}_f$, is an ordered L-fuzzy multiset called the **(ascending/descending) canonical representative** of c.

As we mentioned in Chapter 2, a section maps the equivalence classes into a complete set of representatives, so the ordered L-fuzzy multisets with sorted membership n-tuples stand in bijection with the permutation equivalence classes.

The complete set of ascending or descending canonical representatives, where the coordinates are always sorted, is by itself an important family of fuzzy sets, which has been studied in the existing literature. This type of fuzzy set was first proposed, with the name of "n-dimensional fuzzy sets" by Y. Shang et al. [41] in 2010. Further work by B. Bedregal et al. [2] in 2011 contributed to the development of the concept, but the idea has not gained much traction among the fuzzy-set community. The problem with the restriction to sorted coordinates is that it bypasses the notion of order completely and, in the end, is equivalent to a multiset-based approach, as we will soon elaborate on. The restriction to sorted coordinates is, however, attractive from a theoretical viewpoint because it cuts the full daunting L^n down to a subset that can be given a bounded lattice structure. This shows an inevitable problem with multivalued fuzzy sets: the more flexible they become, by supporting first repetition and then order, the weaker their mathematical properties are. That is why we believe that a thorough analysis of their properties and their limitations is useful.

Example 5.2.7. If we consider the first of the two ordered fuzzy multisets in the previous Example 5.2.5, their ascending canonical representative would be the ordered fuzzy multiset $\vec{C}$ defined by $\vec{C}(x) = (0.1, 0.2, 0.3)$ and $\vec{C}(y) = (0.7, 0.7, 0.8)$. This ordered fuzzy multiset can be identified with the fuzzy multiset $\hat{C}$ defined by $\hat{C}(x) = \langle 0.1, 0.2, 0.3\rangle$ and $\hat{C}(y) = \langle 0.7, 0.7, 0.8\rangle$.

As the previous example shows, it is intuitively obvious that there must be a bijection between the permutation equivalence classes, or their canonical representatives, and the L-fuzzy power multiset $(\mathbb{N}^L)^X$. This idea can be stated as a new proposition, which is a particular case of Proposition 2.8.7.

Proposition 5.2.8. *Let X be the universe, let $n \in \mathbb{N}$ be a non-zero natural number and let L be a membership grade space over which a multiset-based L-fuzzy powerset $(\mathbb{N}^L)^X$ and an n-dimensional L-fuzzy powerset $(L^n)^X$ are defined. There exists a bijection $\phi_{fp}\colon (L^n)^X/{\sim_{fp}} \leftrightarrow (\mathbb{N}^L)^X$ defined by:*

$$\phi_{fp} = Supp^f_m \circ s_{fp} \tag{5.5}$$

where $s_{fp}\colon (L^n)^X/{\sim_{fp}} \to (L^n)^X$ is any section .

Thanks to this bijection, we can identify the permutation-equivalent ordered L-fuzzy multisets with the L-fuzzy multisets. As we did with other equivalence classes in the preceding chapters, this is summed up as a commutation diagram in Figure 5.1.

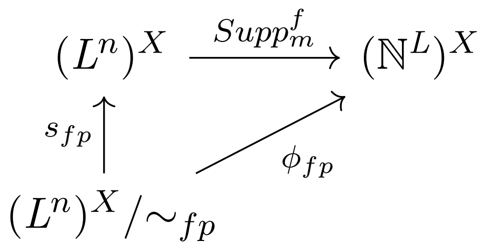

Figure 5.1: Commutation diagram for the fuzzy permutation equivalence ϕ_{fp} bijection.

But for this identification embodied by the ϕ_{fp} bijection to be effective as an L-algebra isomorphism, it is also necessary that the basic operations of complementation, intersection and union be compatible with the bijection.

In the previous chapter, we introduced three different algebras that provide the basic operations for the L-fuzzy multisets. Among the three possibilities, the product and aggregate operations were revealed to be more interesting when we were comparing the L-fuzzy multisets with the set-based L-fuzzy sets as those operations led to an isomorphism between L-fuzzy algebras. But this time we are comparing the L-fuzzy multisets with the ordered L-fuzzy multisets, and it turns out that it is the n-regular sorted multiset-extended L-fuzzy algebra (see Definition 4.3.4) the only one that can lead to an isomorphism. We will devote the rest of this chapter to proving that there is such an isomorphism.

But before turning our attention to the sorted operations, it is worth clarifying why the product and aggregate operations are not even candidates for the good mathematical behaviour described by an isomorphism. The key observation here is the fact that we have defined the ordered fuzzy multisets and their operations as relative to a fixed dimension n. If, for example, we work with pairs, then the generalised operations of negation, minimum and maximum can only yield pairs. There is absolutely no way we can get a triplet as the result of a binary operation between pairs with the Cartesian algebra we have described. We could extend the definitions to the family of n-tuples of any dimension by taking the infinite, countable union of all the L^n for every dimension $n \in \mathbb{N}$ and allow operations between n-tuples of different dimensions through an intermediate regularisation step, but the operations would still yield results within the dimension of the (regularised) operands. Because of this, there is no way that the Cartesian binary operations $\wedge_{gc}$ and $\vee_{gc}$ can be paired with either the product $(\wedge_{gm}, \vee_{gm})$ or the aggregate $(\wedge_{gam}, \vee_{gam})$ operations in such a way that they commute with the fuzzy-multiset-based support. If we were to accept the possibility of ditching the Cartesian operations and study other options for the n-tuples, then it would be possible to construct bespoke algebras replicating the behaviour of the product or aggregate operations with the fuzzy multisets on the permutation-equivalence classes. But then n-tuples would be used only nominally as little more than a notational quirk since the operations would obey the logic of unordered multisets.

So, having established that the only candidate for an L-fuzzy algebra isomorphism between the quotient space by the permutation equivalence relation and the L-fuzzy multisets is the n-regular sorted multiset-extended L-fuzzy algebra, we will follow the same steps as we did in the previous chapter in the similar case of Section 4.4.

Proposition 5.2.9. *Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$ and a non-zero natural number $n \in \mathbb{N}$, the multiset support function $Supp_m$, as in Definition 3.3.1, mapping its Cartesian-extended n-dimensional L-fuzzy algebra $\{L^n, \neg_c, \wedge_c, \vee_c\}$ to its n-regular sorted multiset-extended L-fuzzy algebra $\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}$ is an L-fuzzy algebra homomorphism.*

Proof. By definition, the multiset support being an L-fuzzy algebra homomorphism means that the following relations, as a particular case of equations (4.2) for any n-tuples $\vec{A}, \vec{B} \in L^n$, hold:

$$\begin{aligned} &1. \quad Supp_m(\neg_c \vec{A}) = \neg_{sm_n} Supp_m(\vec{A}) \\ &2. \quad Supp_m(\vec{A} \wedge_c \vec{B}) = Supp_m(\vec{A}) \wedge_{sm_n} Supp_m(\vec{B}) \\ &3. \quad Supp_m(\vec{A} \vee_c \vec{B}) = Supp_m(\vec{A}) \vee_{sm_n} Supp_m(\vec{B}) \end{aligned} \tag{5.6}$$

For the first relation, if the n-tuple $\vec{A}$ has coordinates $(a_1, ..., a_n)$, the left-hand side expression can be expanded as follows:

$$Supp_m(\neg_c(a_1, ..., a_n)) = Supp_m((\neg a_1, ..., \neg a_n)) = \langle \neg a_1, ..., \neg a_n \rangle \tag{5.7}$$

And the right-hand side expression is:

$$\neg_{sm_n} Supp_m((a_1, ..., a_n)) = \neg_{sm_n} \langle a_1, ..., a_n \rangle = \langle \neg a_1, ..., \neg a_n \rangle \tag{5.8}$$

As both sides of the first equality result in the same multiset in (5.7) and (5.8), the first relation is proved.

For the second relation, if we name the coordinates of $\vec{B}$ as $(b_1, ..., b_n)$, then the left-hand side can be expanded as follows:

$$\begin{aligned} Supp_m((a_1, ..., a_n)\wedge_c(b_1, ..., b_n)) &= Supp_m((a_1 \wedge b_1, ..., a_n \wedge b_n)) = \\ &= \langle a_1 \wedge b_1, ..., a_n \wedge b_n \rangle \end{aligned} \tag{5.9}$$

And the right-hand side expression is:

$$\begin{aligned} Supp_m((a_1, ..., a_n))\wedge_{sm_n} Supp_m((b_1, ..., b_n)) = \\ = \langle a_1, ..., a_n \rangle \wedge_{sm_n} \langle b_1, ..., b_n \rangle = \langle a_1 \wedge b_1, ..., a_n \wedge b_n \rangle \end{aligned} \tag{5.10}$$

Again, the two sides of the equality result in the same multiset in (5.9) and (5.10), so the second relation is also proved. The proof of the third relation is completely analogous, with $\vee$ replacing $\wedge$. □

An immediate implication of Proposition 5.2.9 is that the Cartesian-extended operations preserve the integrity of the permutation equivalence classes, as the following proposition states.

Proposition 5.2.10. *Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$ with a Cartesian-extended n-dimensional L-fuzzy algebra $\{L^n, \neg_c, \wedge_c, \vee_c\}$ for a non-zero dimension $n \in \mathbb{N}$, for any two n-tuples $\vec{A}, \vec{B} \in L^n$, if $\vec{A} \sim_p \vec{B}$ then $\neg_c\vec{A} \sim_p \neg_c\vec{B}$.*

Proof. By definition of permutation equivalence, $\vec{A} \sim_p \vec{B}$ is the same as $Supp_m(\vec{A})$ $= Supp_m(\vec{B})$, which is equivalent to $\neg_{sm_n} Supp_m(\vec{A}) = \neg_{sm_n} Supp_m(\vec{B})$. And, by the first equality in (5.6), $Supp_m(\neg_c\vec{A}) = Supp_m(\neg_c\vec{B})$, which, by definition of permutation equivalence, is the same as $\neg_c\vec{A} \sim_p \neg_c\vec{B}$. Note that as all these relations are equivalent, the inverse implication is also true. □

Because of these properties, we say that the Cartesian-extended negations **are compatible with** the permutation equivalence relation.

Proposition 5.2.11. *Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$ with a Cartesian-extended n-dimensional L-fuzzy algebra $\{L^n, \neg_c, \wedge_c, \vee_c\}$ for a non-zero dimension $n \in \mathbb{N}$, for any four n-tuples $\vec{A}, \vec{B}, \vec{C}, \vec{D} \in L^n$, if $\vec{A} \sim_p \vec{C}$ and $\vec{B} \sim_p \vec{D}$, then $\vec{A}\wedge_c\vec{B} \sim_p \vec{C}\wedge_c\vec{D}$ and $\vec{A}\vee_c\vec{B} \sim_p \vec{C}\vee_c\vec{D}$.*

Proof. By definition of permutation equivalence, $\vec{A} \sim_p \vec{C}$ and $\vec{B} \sim_p \vec{D}$ are the same as $Supp_m(\vec{A}) = Supp_m(\vec{C})$ and $Supp_m(\vec{B}) = Supp_m(\vec{D})$, respectively, which entails that $Supp_m(\vec{A}) \wedge_{sm_n} Supp_m(\vec{B}) = Supp_m(\vec{C}) \wedge_{sm_n} Supp_m(\vec{D})$. And, by the second equality in (5.6) above, $Supp_m(\vec{A} \wedge_c \vec{B}) = Supp_m(\vec{C} \wedge_c \vec{D})$, which, by definition of permutation equivalence, is the same as $\vec{A} \wedge_c \vec{B} \sim_p \vec{C} \wedge_c \vec{D}$.

The proof for $\vee_c$ is completely analogous. □

Because of these properties, we say that the Cartesian-extended meet and join **are compatible with** the permutation equivalence relation.

Propositions 5.2.10 and 5.2.11 make it possible to extend the three operations $\neg$, $\wedge$ and $\vee$ to the permutation quotient space in a trivial way, as follows.

Definition 5.2.12. Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$ with a Cartesian-extended n-dimensional L-fuzzy algebra $\{L^n, \neg_c, \wedge_c, \vee_c\}$ for a non-zero dimension $n \in \mathbb{N}$, the operations $\neg_{\sim_p}$, $\wedge_{\sim_p}$ and $\vee_{\sim_p}$ can be defined on the quotient space $L^n/{\sim_p}$ in the following way:

$$\begin{array}{ll} 1. & \neg_{\sim_p} c = [\neg_c s(c)], \quad c \in L^n/{\sim_p} \\ 2. & c \wedge_{\sim_p} d = [s(c) \wedge_c s(d)], \quad c, d \in L^n/{\sim_p} \\ 3. & c \vee_{\sim_p} d = [s(c) \vee_c s(d)], \quad c, d \in L^n/{\sim_p} \end{array} \tag{5.11}$$

where $s\colon L^n/{\sim_p} \to L^n$ is any section and $[\ ]\colon L^n \to L^n/{\sim_p}$ is the operation of selecting the permutation equivalence class of an n-tuple.

These operations on the quotient space are well-defined thanks to propositions 5.2.10 and 5.2.11, which make the choice of section s irrelevant.

The permutation quotient space is now equipped with these three operations that make it an L-fuzzy algebra. Thanks to these operations, the bijection ϕ_p can now be interpreted as an L-fuzzy algebra isomorphism, an important result that we sum up in the proposition that follows.

Proposition 5.2.13. *Given an L-fuzzy algebra $\{L, \neg, \wedge, \vee\}$ with a Cartesian-extended n-dimensional L-fuzzy algebra $\{L^n, \neg_c, \wedge_c, \vee_c\}$ for a non-zero dimension $n \in \mathbb{N}$, an n-regular sorted multiset-extended L-fuzzy algebra $\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}$ relative to a section $s^*\colon L^n/{\sim_p} \to L^n$ and a permutation equivalence quotient space $\{L^n/{\sim_p}, \neg_{\sim_p}, \wedge_{\sim_p}, \vee_{\sim_p}\}$, the bijection $\phi_p\colon L^n/{\sim_p} \to \mathbb{N}^L$ as in Proposition 3.3.7, defined as $\phi_p := Supp_m \circ s$, with $s\colon L^n/{\sim_p} \to L^n$ being any section, is an L-fuzzy algebra isomorphism.*

Proof. The function ϕ_p is a bijection according to Proposition 3.3.7. We simply have to prove that the operations fulfil the following conditions of an L-algebra homomorphism:

$$\begin{array}{ll} 1. & \phi_p(\neg_{\sim_p} c) = \neg_{sm}\phi_p(c) \quad c \in L^n/{\sim_p} \\ 2. & \phi_p(c \wedge_{\sim_p} d) = \phi_p(c) \wedge_{sm} \phi_p(d) \quad c, d \in L^n/{\sim_p} \\ 3. & \phi_p(c \vee_{\sim_p} d) = \phi_p(c) \vee_{sm} \phi_p(d) \quad c, d \in L^n/{\sim_p} \end{array} \tag{5.12}$$

Using the definitions of ϕ_p and the operations $\neg_{\sim_p}$, $\wedge_{\sim_p}$ and $\vee_{\sim_p}$, the three equations adopt the following form:

$$\begin{array}{ll} 1. & Supp_m(s([\neg_c s(c)])) = \neg_{sm_n} Supp_m(s(c)) \\ 2. & Supp_m(s([s(c) \wedge_c s(d)])) = Supp_m(s(c)) \wedge_{sm_n} Supp_m(s(d)) \\ 3. & Supp_m(s([s(c) \vee_c s(d)])) = Supp_m(s(c)) \vee_{sm_n} Supp_m(s(d)) \end{array}$$

On the left-hand side, the operations $s([\;])$ have no effect as the results will be passed to $Supp_m$, so the equalities can be simplified as:

$$\begin{array}{ll} 1. & Supp_m(\neg_c s(c)]) = \neg_{sm_n} Supp_m(s(c)) \\ 2. & Supp_m(s(c) \wedge_c s(d)) = Supp_m(s(c)) \wedge_{sm_n} Supp(s(d)) \\ 3. & Supp_m(s(c) \vee_c s(d)) = Supp_m(s(c)) \vee_{sm_n} Supp(s(d)) \end{array}$$

And these three equalities are true as $Supp_m$ is an L-algebra homomorphism according to Proposition 5.2.9 and match equations (5.6). □

The previous propositions and definitions for L-fuzzy algebras can be adapted to ordered L-fuzzy multisets. As in the similar derivations in the previous chapter, we present the fuzzy set versions without proof as they are all Goguen extensions.

Proposition 5.2.14. *Given a universe X, a dimension $n \in \mathbb{N}$ and an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$, the fuzzy-multiset-based support function $Supp_m^f$, as in Definition 5.2.1, mapping its n-dimensional L-fuzzy powerset $\{(L^n)^X, \neg_{gc}, \wedge_{gc}, \vee_{gc}\}$ to its n-regular sorted multiset-based L-fuzzy powerset $\{(\mathbb{N}^L_{=n})^X, \neg_{gsm_n}, \wedge_{gsm_n}, \vee_{gsm_n}\}$ is an L-fuzzy algebra homomorphism. That is, the following relations hold:*

$$\begin{array}{ll} 1. & Supp_m^f(\neg_{gc}\vec{A}) = \neg_{gsm_n} Supp_m^f(\vec{A}), \quad \vec{A} \in (L^n)^X \\ 2. & Supp_m^f(\vec{A} \wedge_{gc} \vec{B}) = Supp_m^f(\vec{A}) \wedge_{gsm_n} Supp_m^f(\vec{B}), \quad \vec{A}, \vec{B} \in (L^n)^X \\ 3. & Supp_m^f(\vec{A} \vee_{gc} \vec{B}) = Supp_m^f(\vec{A}) \vee_{gsm_n} Supp_m^f(\vec{B}), \quad \vec{A}, \vec{B} \in (L^n)^X \end{array} \tag{5.13}$$

Proposition 5.2.15. *Given a universe X, a dimension $n \in \mathbb{N}$, an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$ with an n-regular sorted multiset-based L-fuzzy powerset $\{(\mathbb{N}^L_{=n})^X, \neg_{gsm_n}, \wedge_{gsm_n}, \vee_{gsm_n}\}$ and an n-dimensional L-fuzzy powerset $\{(L^n)^X, \neg_{gc}, \wedge_{gc}, \vee_{gc}\}$:*

- *For any two ordered L-fuzzy multisets $\vec{A}, \vec{B} \in (L^n)^X$, if $\vec{A} \sim_{fp} \vec{B}$ then $\neg_{gc}\vec{A} \sim_{fp} \neg_{gc}\vec{B}$.*
- *For any four ordered L-fuzzy multisets $\vec{A}, \vec{B}, \vec{C}, \vec{D} \in (L^n)^X$, if $\vec{A} \sim_{fp} \vec{C}$ and $\vec{B} \sim_{fp} \vec{D}$, then $\vec{A} \wedge_{gc} \vec{B} \sim_{fp} \vec{C} \wedge_{gc} \vec{D}$ and $\vec{A} \vee_{gc} \vec{B} \sim_{fp} \vec{C} \vee_{gc} \vec{D}$.*

Proposition 5.2.15 makes it possible to extend the three operations $\neg$, $\wedge$ and $\vee$ to the fuzzy permutation quotient space in a trivial way, as follows.

Definition 5.2.16. Given a universe X, a dimension $n \in \mathbb{N}$ and an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$ with an n-dimensional L-fuzzy powerset $\{(L^n)^X, \neg_{gc}, \wedge_{gc}, \vee_{gc}\}$, the operations $\neg_{\sim_{fp}}$, $\wedge_{\sim_{fp}}$ and $\vee_{\sim_{fp}}$ can be defined on the quotient space $(L^n)^X/{\sim_{fp}}$ in the following way:

$$\begin{aligned} &1. \quad \neg_{\sim_{fp}} c = [\neg_{gc} s((c))], \quad c \in (L^n)^X/{\sim_{fp}} \\ &2. \quad c \wedge_{\sim_{fp}} d = [s(c) \wedge_{gc} s(d)], \quad c, d \in (L^n)^X/{\sim_{fp}} \\ &3. \quad c \vee_{\sim_{fp}} d = [s(c) \vee_{gc} s(d)], \quad c, d \in (L^n)^X/{\sim_{fp}} \end{aligned} \tag{5.14}$$

where $s\colon (L^n)^X/{\sim_{fp}} \to (L^n)^X$ is any section and $[\,]\colon (L^n)^X \to (L^n)^X/{\sim_{fp}}$ is the operation of selecting the equivalence class of an ordered L-fuzzy multiset.

The fuzzy permutation quotient space is now equipped with these three operations that make it an L-fuzzy algebra. Thanks to these operations, the bijection ϕ_{fp} can now be interpreted as an L-fuzzy algebra isomorphism.

Proposition 5.2.17. *Given a universe X, a dimension $n \in \mathbb{N}$, an L-fuzzy algebra of membership grades $\{L, \neg, \wedge, \vee\}$ with an n-regular sorted multiset-based L-fuzzy powerset $\{(\mathbb{N}^L_{=n})^X, \neg_{gsm_n}, \wedge_{gsm_n}, \vee_{gsm_n}\}$, an n-dimensional L-fuzzy powerset $\{(L^n)^X, \neg_{gc}, \wedge_{gc}, \vee_{gc}\}$ and the quotient space for a fuzzy permutation equivalence $\{(L^n)^X/{\sim_{fp}}, \neg_{\sim_{fp}}, \wedge_{\sim_{fp}}, \vee_{\sim_{fp}}\}$, the bijection $\phi_{fp}\colon (L^n)^X/{\sim_{fp}} \to (\mathbb{N}^L)^X$ as in Proposition 5.2.8, defined as $\phi_{fp} := Supp^f_m \circ s$, with $s\colon (L^n)^X/{\sim_{fp}} \to (L^n)^X$ being any section, is an L-fuzzy algebra isomorphism.*

This final proposition encapsulates the idea that when the ordered L-fuzzy multisets are deprived of the notion of order through the permutation equivalence relation, the behaviour in their equivalence classes (or, equivalently, in a complete set of sorted canonical representatives like Shang's n-dimensional fuzzy sets) is the same as the behaviour of the L-fuzzy multisets with the sorted operations. This is not really surprising if we take into account how the sorted operations are defined via a Cartesian extension. The isomorphism we have established appears as a natural consequence of the way the sorted extensions are built.

But regardless of how the ordered L-fuzzy multisets may generalise other more restricted fuzzy theories, we believe that there are many use cases where both repetition and order are relevant, and in such cases the ordered L-fuzzy multisets are an interesting tool by themselves that can help to model fuzzy decision-making problems successfully. In the following chapters, we will continue exploring their properties and applications.

6 The entropy of repetition and order

We have mentioned how restrictions that disregard order and repetition are intrinsically linked to a loss of information. If a triplet (3-tuple) like (a, b, a) is deprived of order by taking its multiset support, we get a multiset $\langle a, a, b\rangle$ where the original order of the three elements is lost. And if we go a step further and take its support to rid it of repetition, we end up with the set $\{a, b\}$, where the fact that the element a originally appeared twice is lost. These differences in information content among the sets, the multisets and the n-tuples are obviously carried over to the different types of multivalued fuzzy sets.

In this chapter, we will provide formulas that quantify the information loss incurred when order and repetition are disregarded. We will follow the traditional method, well founded in statistical mechanics and information theory, of counting the number of states in a system under a set of rules that become one single state under a different set of rules (like the microscopic and the macroscopic states in statistical mechanics) and then taking the logarithm of that count, which we will call the **entropy** associated with that state, as a measure of information content. As in information theory, we will use binary logarithms, which prompt a natural interpretation of the entropy as the number of bits required to codify the information loss in the less granular system.

An important remark about terminology is in order here. The entropy that we study in this chapter affects the information content associated with repetition and order of values in multisets and n-tuples. This is different from the concept of entropy as a measure of fuzziness, where crisp sets have zero entropy, which has been amply studied in the literature on fuzzy sets and which is not being addressed in this dissertation.

6.1 The entropy of repetition

Let us begin by considering the regular power multisets of length $n \in \mathbb{N}$, as in Definition 2.5.6. With the cardinality fixed, we can count the total number of multisets in the support preimage. This is the result of the following proposition.

Proposition 6.1.1. *Given a finite universe U and a subset $A \subseteq U$ and a natural number $m \geq |A|$ (with $|A|$ being the cardinality of A), the number N of different multisets $\hat{A}_i$ with cardinality m ($\hat{A}_i \in \mathbb{N}^U_{=m}$) and such that $Supp(\hat{A}_i) = A$ is:*

$$N = \binom{m-1}{m-|A|} \tag{6.1}$$

Proof. As A has a finite cardinality $|A|$, its elements can be denoted by $\{a_1, ..., a_{|A|}\}$. As the multisets $\hat{A}_i$ have A as their support, they necessarily have the form:

$$\hat{A}_i(u) = \begin{cases} 1 + r_j & \text{if } u = a_j \in A \\ 0 & \text{if } u \notin A \end{cases} \qquad u \in U$$

where the $\{r_j\}$, with $j = 1, ..., |A|$, are non-negative integers. The cardinality of the multiset $\hat{A}_i$ can then be expressed as:

$$|\hat{A}_i| = \sum_{j=1}^{|A|} (1 + r_j) = |A| + \sum_{j=1}^{|A|} r_j$$

And since this multiset cardinality equals m, we have:

$$|A_i| = m \Leftrightarrow |A| + \sum_{j=1}^{|A|} r_j = m \Leftrightarrow \sum_{j=1}^{|A|} r_j = m - |A|$$

Finding the number of combinations of $\{r_j\}_{j=1,\ldots,|A|}$ values such that $\sum_{j=1}^{|A|} r_j = m - |A|$ is a form of the "stars and bars" problem [12], where $m - |A|$ undistinguishable balls must be placed into $|A|$ distinguishable boxes, so the total number is given by:

$$N = \binom{m-1}{m-|A|}$$

□

Note that this number is the cardinality of the equivalence class for the repetition equivalence $\sim_r$ (see Definition 3.1.2).

In addition, if a fixed cardinality cannot be assumed, we may want to consider those multisets bounded by a certain length, leading to the next similar proposition for the total number of multisets in the support preimage.

Proposition 6.1.2. *Given a finite universe U and a subset $A \subseteq U$ and a natural number $m \geq |A|$, the number N of different multisets $\hat{A}_i$ with cardinality $|\hat{A}_i| \leq m$ ($\hat{A}_i \in \mathbb{N}^U_{\leq m}$) and such that $Supp(\hat{A}_i) = A$ is:*

$$N = \binom{m}{|A|} \tag{6.2}$$

Proof. According to the previous Proposition 6.1.1, for a fixed cardinality k, the number of multisets $\hat{A}_i$ in the support preimage is:

$$N_k = \binom{k-1}{k-|A|}$$

As we are now counting all the multisets with cardinalities k_i such that $|A| \leq k_i \leq m$, we must sum all the N_k values for the $1+m-|A|$ possible cardinalities:

$$N = \sum_{k=|A|}^{m} N_k = \sum_{k=|A|}^{m} \binom{k-1}{k-|A|} = \sum_{k'=0}^{m-|A|} \binom{k'+|A|-1}{k'}$$

Using some well-known properties of the binomial coefficients, it is possible to conclude that:

$$N = \sum_{k'=0}^{m-|A|} \binom{k'+|A|-1}{k'} = \binom{m}{m-|A|} = \binom{m}{|A|}$$

and (6.2) is satisfied. □

Propositions 6.1.1 and 6.1.2 give the number of possible multisets that share the same support under different restrictions on the valid cardinalities. We can now turn these two numbers of multisets that share a common support into two matching definitions of entropy, a measure of the amount of information that gets lost when repetition is discarded. First, if we have a set A with cardinality $|A|$ that may come from an underlying multiset with fixed cardinality $m \geq |A|$, its repetition entropy derived from Proposition 6.1.1 will be given by the following definition.

Definition 6.1.3. Given a finite universe U, a subset $A \subseteq U$ and a natural number $m \geq |A|$, the m**-fixed-cardinality repetition entropy** of A is a function $S^r_{=m} \colon \mathbf{2}^U \to \mathbb{R}^+$ given by:

$$S^r_{=m}(A) := \log_2 \binom{m-1}{m-|A|}, \quad A \subseteq U \tag{6.3}$$

When the set A may come from an underlying multiset with bounded, but not fixed, cardinalities we can define a second form of repetition entropy based on Proposition 6.1.2:

Definition 6.1.4. Given a finite universe U and a subset $A \subseteq U$ and a natural number $m \geq |A|$, the m**-bounded-cardinality repetition entropy** of A is a function $S^r_{\leq m} \colon \mathbf{2}^U \to \mathbb{R}^+$ given by:

$$S^r_{\leq m}(A) := \log_2 \binom{m}{|A|}, \quad A \subseteq U \tag{6.4}$$

Example 6.1.5. In a universe $U = \{a, b, c\}$, let us suppose there is a set $A = \{a, b\}$. Its 3-fixed-cardinality repetition entropy will be $S_{(r,=3)} = \log_2 \binom{2}{1} = \log_2 2 = 1$ and its 3-bounded-cardinality repetition entropy is $S_{(r\leq 3)} = \log_2 \binom{3}{2} = \log_2 3 = 1.58$. The idea here is that if we know that the set A originally comes from a multiset of cardinality 3, there are two possibilities $\langle a, a, b\rangle$ or $\langle a, b, b\rangle$ so we would need one additional bit of information to specify the original multiset. But if we know that the original multiset cardinality was *at most* 3, the possibilities also include $\langle a, b\rangle$, so we will need more than one bit of information to account for the three possibilities, and hence the 1.58 entropy value.

6.2 The entropy of order

In a similar way, we can count the number of n-tuples that produce the same multiset via the operation of taking their multiset supports. Given a multiset of cardinality n, we want to address the question of how many n-tuples there are in the multiset support preimage. This question is answered by the next proposition.

Proposition 6.2.1. *Given a universe U and a finite multiset $\hat{A}$ over U of cardinality n ($n \in \mathbb{N}$ with $n > 1$) with a support Supp($\hat{A}$) with cardinality $k \leq n$ whose elements can be denoted by $\{a_1, ..., a_k\}$, the number N of n-tuples that have $\hat{A}$ as their multiset support is:*

$$N = \binom{n}{\hat{A}(a_1), ..., \hat{A}(a_k)} = \frac{n!}{\prod_{i=1}^{k} \hat{A}(a_i)!} \tag{6.5}$$

Proof. This is a simple combinatorics exercise of the permutations of the k elements $\{a_1, ..., a_k\}$, with repetition counts $\{r_i\}_{i=1,...,k}$ times into $n = \sum_{i=1}^{k} r_i$ slots. □

By taking the logarithm of this number, we can define a form of entropy associated to order, as follows.

Definition 6.2.2. Given a universe U and a finite multiset $\hat{A}$ over U of cardinality $n > 1$, and using the same notation as in Proposition 6.2.1 above, the **ordering entropy** of $\hat{A}$ is a function $S_o\colon \mathbb{N}^U \to \mathbb{R}^+$ defined by:

$$S_o(\hat{A}) := \log_2 \frac{n!}{\prod_{i=1}^{k} \hat{A}(a_i)!} \qquad \hat{A} \subseteq \mathbb{N}^U \tag{6.6}$$

Example 6.2.3. Using the same universe $U = \{a, b, c\}$ as in the previous Example 6.1.5, the triplets $\vec{A} = (a, b, c)$ and $\vec{B} = (b, a, c)$ are different as elements of U^3, but they have the same multiset support $\hat{M} = \langle a, b, c \rangle$. The ordering entropy for this multiset is:

$$S_o(\hat{M}) = \log_2 \frac{3!}{\prod_{i=1}^{3} 1!} = \log_2 6 = 2.58$$

This basically means that if the multiset support is known, then a minimum of 3 additional bits of information are required in order to codify the particular permutation of the original triplet.

6.3 The entropy of the hesitant support of fuzzy multisets

By following the well-trodden path of translating concepts from the membership grades to the various styles of fuzzy sets in a pointwise way, we can replicate the results of the previous two subsections to the set-based (hesitant sets) and multiset-based (fuzzy multisets) forms of multivalued fuzzy sets. We will do this for the repetition entropy first. Again, we introduce two forms that account for the fixed and bounded cardinality cases.

As in the non-fuzzy case, we start off by counting the number of fuzzy multisets that lead to a particular hesitant set as their common hesitant support.

Proposition 6.3.1. *Given a finite universe X, a typical hesitant fuzzy set $\tilde{A} \in (\mathbf{2}^{\mathbb{I}})^X$ and a natural number $m \geq \sup_{x\in X}\{|\tilde{A}(x)|\}$ (i.e. m is an upper bound on the maximum cardinality of all the hesitant elements in $\tilde{A}$), the number N of different fuzzy multisets $\hat{A}_i \in (\mathbb{N}^{\mathbb{I}})^X$ that have $\tilde{A}$ as their hesitant support and such that their cardinality $|\hat{A}_i(x)| = m$ for all the elements $x \in X$ is:*

$$N = \prod_{x\in X|\tilde{A}(x)\neq\emptyset} \binom{m-1}{m-|\tilde{A}(x)|} \tag{6.7}$$

Proof. For each element x in the universe X, the hesitant element $\tilde{A}(x)$ is a set. As the universe is finite, there will be a finite number of elements $n \leq |X|$ for which $\tilde{A}(x) \neq \emptyset$. If we denote these elements by $\{x_i\}_{i=1,\dots,n}$, for each hesitant element $\tilde{A}(x_i)$ the number of multisets in its support preimage that have a fixed cardinality m is given by Proposition 6.1.1:

$$N_i = \binom{m-1}{m-|\tilde{A}(x_i)|}$$

And in order to count all the possibilities for the different x_i, we must take the product of the values above, which leads to the expected result for N. □

Proposition 6.3.2. *Given a finite universe X, a typical hesitant fuzzy set $\tilde{A} \in (\mathbf{2}^{\mathbb{I}})^X$ and a natural number $m \geq \sup_{x\in X}\{|\tilde{A}(x)|\}$, the number N of different fuzzy multisets $\hat{A}_i \in (\mathbb{N}^{\mathbb{I}})^X$ that have $\tilde{A}$ as their hesitant support and such that* *their cardinality $|\hat{A}_i(x)| \leq m$ for all the elements $x \in X$ is:*

$$N = \prod_{x \in X | \tilde{A}(x) \neq \emptyset} \binom{m}{|\tilde{A}(x)|} \tag{6.8}$$

Proof. Using the same notational conventions, for each hesitant element $\tilde{A}(x_i)$ the number of multisets in its support preimage that have a cardinality bounded by m is given by Proposition 6.1.2:

$$N_i = \binom{m}{|\tilde{A}(x_i)|}$$

And, again, we must take the product for all the indices i such that $\tilde{A}(x_i)$ is not empty in order to count all the possibilities, which leads to the expected result for N. □

Just as we did in the non-fuzzy case we can now use these results as the basis for the two following definitions of entropy. First, the one for fixed cardinality is based on taking the binary logarithm of the expression (6.7).

Definition 6.3.3. Given a finite universe X, a typical hesitant fuzzy set $\tilde{A} \in (\mathbf{2}^{\mathbb{I}})^X$ and a natural number $m \geq \sup_{x \in X}\{|\tilde{A}(x)|\}$, the **$m$-fixed-cardinality repetition entropy** of $\tilde{A}$ is a function $S^{fr}_{=m} \colon (\mathbf{2}^{\mathbb{I}})^X \to \mathbb{R}^+$ given by:

$$\begin{aligned} S^{fr}_{=m}(\tilde{A}) := \log_2 \prod_{x \in X | \tilde{A}(x) \neq \emptyset} \binom{m-1}{m - |\tilde{A}(x)|} = \\ = \sum_{x \in X | \tilde{A}(x) \neq \emptyset} \log_2 \binom{m-1}{m - |\tilde{A}(x)|} \end{aligned} \tag{6.9}$$

When the typical hesitant fuzzy set $\tilde{A}$ may have originated from an underlying fuzzy multiset with bounded, but not fixed, cardinalities we can define a second form of repetition entropy based on the expression (6.8).

Definition 6.3.4. Given a finite universe X, a typical hesitant fuzzy set $\tilde{A} \in (\mathbf{2}^{\mathbb{I}})^X$ and a natural number $m \geq \sup_{x \in X}\{|\tilde{A}(x)|\}$, the **$m$-bounded-cardinality**

repetition entropy of $\tilde{A}$ is a function $S^{fr}_{\leq m}\colon (\mathbf{2}^{\mathbb{I}})^X \to \mathbb{R}^+$ given by:

$$\begin{aligned} S^{fr}_{\leq m}(\tilde{A}) &:= \log_2 \prod_{x\in X|\tilde{A}(x)\neq\emptyset} \binom{m}{|\tilde{A}(x)|} = \\ &= \sum_{x\in X|\tilde{A}(x)\neq\emptyset} \log_2 \binom{m}{|\tilde{A}(x)|} \end{aligned} \tag{6.10}$$

Note that the second term in these definitions shows that the entropies thus defined yield the same value as the sum of the analogous non-fuzzy entropies of expressions (6.2) and (6.3) over the elements x with non-empty hesitant elements in $\tilde{A}$. It is possible to use that sum as the definition for the fuzzy versions, but we have preferred to follow the longer path of counting the possibilities in order to make the intuition behind the definitions more obvious.

Example 6.3.5. In a two-element universe $X = \{a, b\}$, let us suppose we have a hesitant fuzzy set $\tilde{A}$ defined by $\tilde{A}(a) = \{0.1, 0.2\}$ and $\tilde{A}(b) = \{0.5, 0.6\}$. Its 3-fixed-cardinality repetition entropy is $S^f_{r,=3}(\tilde{A}) = \log_2 \binom{2}{1} + \log_2 \binom{2}{1} = \log_2 2 + \log_2 2 = 2$ and its 3-bounded-cardinality repetition entropy is $S^f_{r\leq 3} = \log_2 \binom{3}{2} + \log_2 \binom{3}{2} = \log_2 3 + \log_2 3 = 3.16$. The idea here is that if we know that the hesitant elements (i.e. the membership grades) of $\tilde{A}$ have originally come from the values given by three undistinguishable experts, there are four possibilities resulting from the combination of either $\langle 0.1, 0.2, 0.2\rangle$ or $\langle 0.1, 0.1, 0.2\rangle$ for a and $\langle 0.5, 0.6, 0.6\rangle$ or $\langle 0.5, 0.5, 0.6\rangle$ for b. so we would need an additional two bits of information to specify the original membership grades. But if we know that there were *at most* three experts, the possibilities would also include $\langle 0.1, 0.2\rangle$ for a and $\langle 0.5, 0.6\rangle$ for b, where only two experts produced a membership value, so we will need more than three bits of information to account for a total of nine possibilities, and hence the 3.16 entropy value.

6.4 The entropy of the fuzzy multiset support of ordered fuzzy multisets

We can now do the same for the ordering entropy as we did for the repetition entropy in the previous section. Now, instead of dealing with multisets and n-tuples, we have to deal with multiset-based membership grades and n-tuple-based membership grades; i.e. with fuzzy multisets and ordered fuzzy multisets. The question now is how many ordered fuzzy multisets have the same fuzzy multiset as their fuzzy multiset support. This is answered by the following proposition, based on Proposition 6.2.1.

Proposition 6.4.1. *Given a finite universe X and a regular fuzzy multiset $\hat{A} \in (\mathbb{N}^{\mathbb{I}})^X$ of length n, for each $x \in X$, the crisp multiset $\hat{A}(x)$ has a support made up by different values that we can denote by $\{a_1(x), \dots a_{k(x)}(x)\}$, where $k(x)$ is the number of different values and is in general a function of x. Then the number N of different ordered fuzzy multisets $\vec{A}_i \in (\mathbb{I}^n)^X$ that have $\hat{A}$ as their fuzzy multiset support is:*

$$N = \prod_{x \in X} \binom{n}{\hat{A}(x_i)(a_1(x_i)), \dots, \hat{A}(x_i)(a_{k(x_i)}(x_i))} \tag{6.11}$$

Proof. As the universe X is finite, we can denote its elements by $\{x_i\}_{i=1,\dots,|X|}$. For each element x_i, the membership grade $\hat{A}(x_i)$ is a crisp multiset and we can apply expression (6.5) to it in order to get the number N_i of n-tuples in the multiset support preimage:

$$N_i = \binom{n}{\hat{A}(x_i)(a_1(x_i)), \dots, \hat{A}(x_i)(a_{k(x_i)}(x_i))}$$

And in order to count all the possibilities for the different x_i, we simply take the product of the values above, which leads to the expected result $N = \prod_{i=1}^{n} N_i = \prod_{i=1}^{n} \binom{n}{\hat{A}(x_i)(a_1(x_i)), \dots, \hat{A}(x_i)(a_{k(x_i)}(x_i))}$. □

Finally, we can turn this result into a definition of entropy by taking the binary logarithm.

Definition 6.4.2. Given a finite universe X and a regular fuzzy multiset $\hat{A} \in (\mathbb{N}^{\mathbb{I}})^X$ of length n, the **ordering entropy** of $\hat{A}$ is a function $S_o^f : (\mathbb{N}^{\mathbb{I}})^X \to \mathbb{R}^+$ given by:

$$\begin{aligned} S_o^f(\hat{A}) &:= \log_2 \prod_{x \in X} \binom{n}{\hat{A}(x_i)(a_1(x_i)), \dots, \hat{A}(x_i)(a_{k(x_i)}(x_i))} = \\ &= \sum_{x \in X} \log_2 \binom{n}{\hat{A}(x_i)(a_1(x_i)), \dots, \hat{A}(x_i)(a_{k(x_i)}(x_i))} \end{aligned} \tag{6.12}$$

with the same notation as in Proposition 6.4.1 above.

7 Divergence measures for multivalued fuzzy sets

In this chapter, we are going to investigate some techniques that make it possible to measure how different two multivalued fuzzy sets are. We will do this by extending the notions of distance, dissimilarity and divergence to these types of objects. We start by generalising the concepts, as were introduced in chapter 2, to L-fuzzy sets.

7.1 Distances, dissimilarities and divergences in L-fuzzy sets

In Section 2.9, we presented the well-known ideas of distance (Definition 2.9.1), dissimilarity measure (Definition 2.9.8) and divergence measure (Definition 2.9.9) for ordinary fuzzy sets. Those definitions do not depend on the particular nature of the unit interval and, consequently, can be extended to any space of membership grades without any difficulty. We introduce the generalised definitions now.

Definition 7.1.1. Let X be the universe and let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra (see Definition 4.1.4) with a corresponding L-fuzzy powerset L^X (See Definition 4.1.7). A map $d\colon L^X \times L^X \to \mathbb{R}^+$ is an **L-fuzzy set distance** if the following three conditions, for all $A, B, C \in L^X$, are met:

$$1_{\text{dist}}.\quad d(A,B) = 0 \Leftrightarrow A = B$$
$$2_{\text{dist}}.\quad d(A,B) = d(B,A)$$
$$3_{\text{dist}}.\quad d(A,B) \geq d(A,C) + d(C,B)$$

The definition of distance is very simple, as it does not depend on the operations defined on L at all. On the other hand, the definitions of dissimilarity and divergence, which we now turn our attention to, do depend on how the operations are defined.

We have seen that the definition of dissimilarity measure for ordinary fuzzy sets is based on the concept of inclusion (see Definition 2.9.8), so in order to generalise it we will need to establish the meaning of $A \subseteq B$ when A and B are L-fuzzy sets. This is straightforward when there is a lattice structure on L, as in the case of the unit interval $\mathbb{I}$. In such a situation, the properties $A \subseteq B \Leftrightarrow A \cap B = A$ and $A \subseteq B \Leftrightarrow A \cup B = B$ can be used as a definition for the inclusion order relation and, in fact, the standard Definition 2.2.3 of inclusion for ordinary fuzzy sets can be derived in this way. Basically, in lattice-theory terms we are following the algebraic approach where we define the meet and join operations first and then induce an order relation from them [15]. If L has a bisemilattice structure, the situation is more complicated as the order relations induced by the two properties are different. We begin with the definitions for this general case.

Definition 7.1.2. Let X be the universe and let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra with a corresponding L-fuzzy powerset $\{L^X, \neg_g, \wedge_g, \vee_g\}$. An L-fuzzy set $A \in L^X$ is said to be a **meet-subset** of an L-fuzzy set $B \in L^X$, $A \subseteq_\wedge B$, if the following condition holds:

$$A \wedge_g B = A$$

Definition 7.1.3. Let X be the universe and let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra with a corresponding L-fuzzy powerset $\{L^X, \neg_g, \wedge_g, \vee_g\}$. An L-fuzzy set $A \in L^X$ is said to be a **join-subset** of an L-fuzzy set $B \in L^X$, $A \subseteq_\vee B$, if the following condition holds:

$$A \vee_g B = B$$

Definition 7.1.4. Let X be the universe and let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra with a corresponding L-fuzzy powerset $\{L^X, \neg_g, \wedge_g, \vee_g\}$. An L-fuzzy set $A \in L^X$ is said to be a **subset** of B, $A \subseteq B$, if it is both a meet-subset and a join-subset of B.

The distinction between meet and join-subsets vanishes when the membership space is a lattice. This is an important result, so we sum it up in a proposition.

Proposition 7.1.5. *Let X be the universe and let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra. If this L-fuzzy algebra is a lattice, then for any two L-fuzzy sets $A, B \in L^X$, $A \subseteq_\wedge B$ if and only if $A \subseteq_\vee B$.*

Proof. If L is a lattice, then it is both a meet-semilattice and a join-semilattice and the order relations induced by the expressions $x \wedge y = x$ and $x \vee y = y$ are the same [15]. By taking Goguen's extension of the lattice relations, the result is extended pointwise to the L-fuzzy sets. □

Now that we have defined inclusion for L-fuzzy sets, we have all the building blocks in place to define dissimilarity measures.

Definition 7.1.6. Let X be the universe and let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra with an associated L-fuzzy powerset L^X where an order relation $\subseteq$ has been defined. A map $D_{\text{iss}}\colon L^X \times L^X \to \mathbb{R}^+$ is a **dissimilarity measure** (or **dissimilarity**, for short), for the order $\subseteq$, if for all $A, B \in L^X$ the following three conditions are met:

1_{diss}. $D_{\text{iss}}(A, B) = 0 \Leftrightarrow A = B$

2_{diss}. $D_{\text{iss}}(A, B) = D_{\text{iss}}(B, A)$

3_{diss}. $D_{\text{iss}}(A, B) \geq \max(D_{\text{iss}}(A, C), D_{\text{iss}}(C, B))$, if $A \subseteq C \subseteq B$

For non-lattice membership spaces, we can refer to a **meet-dissimilarity** or a **join-dissimilarity** when the order relation is $\subseteq_\wedge$ or $\subseteq_\vee$, respectively. For lattice membership spaces, the natural $\subseteq$ relation associated with the meet and join operations is usually assumed. And when the operations $\wedge$ and $\vee$ on L are a general t-norm and a t-conorm, not a meet and a join, other definitions of inclusion may be possible. Besides, we have already seen how, with families of multivalued fuzzy sets, the properties for the L-fuzzy algebra may get even weaker than a bounded bisemilattice. In such situations, for example with L-fuzzy multisets, any definition of inclusion may display additional weaknesses. This is yet another reason why we find the definition of divergence measure, which follows, more expressive as it is based on the more fundamental concepts of intersection and union, which must always be defined for any kind of L-fuzzy powerset.

Definition 7.1.7. Let X be the universe and let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra with a corresponding L-fuzzy powerset $\{L^X, \neg_g, \wedge_g, \vee_g\}$. A map $D\colon L^X \times L^X \to \mathbb{R}^+$ is a **divergence measure** (or simply **divergence**) if for all $A, B \in L^X$ the following three conditions are met:

1_{div}. $D(A, B) = 0 \Leftrightarrow A = B$

2_{div}. $D(A, B) = D(B, A)$

3_{div}. $D(A, B) \geq \max(D(A \wedge_g C, B \wedge_g C), D(A \vee_g C, B \vee_g C)),\ \forall C \in L^X$

As we mentioned in Section 2.9, the first axiomatic condition for a divergence measure can be weakened to the pseudometric condition $D(A, A) = 0$, which would allow for distinct L-fuzzy sets to have zero divergence. We will refer to functions that satisfy the less strict form of the first axiom, together with the other two axioms, as **weak divergence measures**.

Divergence measures and dissimilarities are equivalent in some cases. The simplest such case occurs when L is totally ordered and so is the order induced on the L-fuzzy powerset L^X. This typically happens when the universe X is a singleton, a result we already mentioned within the context of ordinary fuzzy sets in Chapter 2, and which we can sum up in a proposition.

Proposition 7.1.8. *Let $X = \{x\}$ be a universe and let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra which is totally ordered. A function $D\colon L^X \times L^X \to \mathbb{R}^+$ is a divergence measure if and only if it is a dissimilarity measure, with inclusion being defined through Definition 7.1.4.*

Proof. As X is a singleton, the total order on L is induced on L^X as $A \subseteq B$ when $A(x) \leq B(x)$ or, equivalently, through Definition 7.1.4 using Goguen's pointwise extension. As the first two conditions in the definitions of dissimilarity and divergence are exactly the same, the only difference between the two concepts is in the third condition. We are going to prove that the two forms of the third condition are equivalent in this case.

For the divergence measure, Condition 3_{div} can be expanded as two conditions: $D(A \wedge_g C, B \wedge_g C) \leq D(A, B)$ and $D(A \vee_g C, B \vee_g C) \leq D(A, B)$, for any C. We assume $A \subseteq B$ without loss of generality. As L^X has a total order, there are three ordering possibilities for C: $C \subseteq A \subseteq B$, $A \subseteq C \subseteq B$ and $A \subseteq B \subseteq C$. But when $C \subseteq A \subseteq B$, we have $A \wedge_g C = B \wedge_g C = C$, which leads to $D(A \wedge_g C, B \wedge_g C) = D(C, C) = 0$ and, similarly, $A \vee_g C = A$ and $B \vee_g C = B$, which lead to $D(A \vee_g C, B \vee_g C) = D(A, B)$, so Condition 3_{div} is always fulfilled. Similarly, when $A \subseteq B \subseteq C$, we have $A \vee_g C = C$ and $B \vee_g C = C$, which leads to $D(A \vee_g C, B \vee_g C) = D(C, C) = 0$ and, similarly, $A \wedge_g C = A$ and $B \wedge_g C = B$, which lead to $D(A \wedge_g C, B \wedge_g C) = D(A, B)$, so Condition 3_{div} is always fulfilled too. This means that the only non-trivial case of the divergence condition occurs when $A \subseteq C \subseteq B$ and in that case $D(A \wedge_g C, B \wedge_g C) = D(A, C)$ and $D(A \vee_g C, B \vee_g C) = D(C, B)$, so $\max(D(A \wedge_g C, B \wedge_g C), D(A \vee_g C, B \vee_g C)) = \max(D(A, C), D(C, B))$ and we arrive at Condition 3_{diss} for a dissimilarity measure through a chain of equivalences provided that $A \subseteq C \subseteq B$, so the two forms of the condition are equivalent. □

The simplest example of an L-fuzzy powerset where this proposition is applicable is the ordinary fuzzy sets over a singleton universe $X = \{x\}$. The relation between divergences and dissimilarities has been studied in the recent literature [22], where counterexamples of dissimilarities that are not divergences and the other way around have been identified. We should note that dissimilarities that are not divergences involve the use of t-norms other than the standard ones and further research, which goes beyond the scope of our present work, is needed to determine the exact circumstances under which such situations arise.

7.2 Distances, dissimilarities and divergences for multivalued fuzzy sets

By simply choosing the appropriate L-fuzzy algebra in the definitions of distance, dissimilarity and divergence, the definitions in the previous section can be particularised to the various types of multivalued fuzzy sets. Definition 7.1.1 for distance is the only one that is independent of how the basic operations of intersection and union are defined, whereas Definition 7.1.7 for the divergence measure explicitly uses the intersection and union, so for the L-fuzzy multisets there will be three kinds of divergence depending on the choice of operations, and similarly for the definitions of dissimilarity, where intersection and union play a role through Definition 7.1.4 for inclusion.

In the following sections, we will focus our attention on the divergence measures, so we will frequently refer to Definition 7.1.7 but with L^X replaced by any one of $(\mathbf{2}^L)^X$ (set-based fuzzy sets), $(\mathbb{N}^L)^X$ (fuzzy multisets) and $(L^n)^X$ (ordered fuzzy multisets).

7.3 Local divergence measures in L-fuzzy sets

Among the various types of measures of difference for ordinary fuzzy sets, those divergence measures that fulfil the local property 2.9.10 display particularly good behaviour and include common cases like the Hamming distance for fuzzy sets [34]. We will now tackle the issue of extending the concept of a local divergence measure to the L-fuzzy sets first and then to the particular case of multivalued fuzzy sets. For this, we need to revisit the preliminary concepts introduced in

Chapter 2. While Definition 2.9.10 for ordinary fuzzy sets could be used as the starting point for a more general definition, we will base the extension on the Representation Theorem 2.9.14, as the derived formalism becomes simpler in this way. In order to accomplish this, we start with a general definition of the divergence characteristic function on any arbitrary space of membership grades.

Definition 7.3.1. Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra. A function $h\colon L \times L \to \mathbb{R}^+$ is called an **L-fuzzy divergence characteristic function** if it fulfils the following conditions:

1. $h(u, v) = 0 \Leftrightarrow u = v$, $\forall u, v \in L$
2. $h(u, v) = h(v, u)$, $\forall u, v \in L$
3. $h(u, v) \geq \max(h(u \wedge w, v \wedge w), h(u \vee w, v \vee w))$, $\forall u, v, w \in L$

If the first condition is replaced with $h(u, u) = 0$, we say that h is an **L-fuzzy weak divergence characteristic function**.

This is like the definition of a divergence measure, but on the space of membership grades. When the space L is totally ordered and $\wedge$ and $\vee$ are the meet (minimum) and join (maximum) operations, then the third condition can be expressed as in a dissimilarity.

Proposition 7.3.2. *Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra where $\wedge$ and $\vee$ induce a total order. For an L-fuzzy (weak) divergence characteristic function $h\colon L \times L \to \mathbb{R}^+$, Condition 3 in Definition 7.3.1 above is equivalent the following condition:*

3′. $h(u, w) \geq \max(h(u, v), h(v, w))$, *$\forall u, v, w \in L$ such that $u \leq v \leq w$*

Proof. If L is a totally ordered set, then it is a lattice where its meet and join operations must be the minimum and maximum, respectively [4, p. 7]) and the condition $a \leq b$ is equivalent to $a \wedge b = \min(a, b) = a$ and $a \vee b = \max(a, b) = b$. We can prove that Condition 3' is equivalent to Condition 3 in Definition 7.3.1 using the same reasoning as in the proof for Proposition 7.1.8. Basically, we assume, without loss of generality, that the labels u and v are chosen so that $u \leq v$. If $w \leq u \leq v$, then $h(u \wedge w, v \wedge w) = h(w, w) = 0$ and $h(u \vee w, v \vee w) = h(u, v)$. And if $u \leq v \leq w$, then $h(u \wedge w, v \wedge w) = h(u, v)$ and $h(u \vee w, v \vee w) = h(w, w) = 0$. Thus, Condition 3 in Definition 7.3.1 is always satisfied when w lies outside the range of values between u and v. This means that the only non-trivial case of Condition 3 is $u \leq w \leq v$ and in that case $h(u \wedge w, v \wedge w) = h(u, w)$ and $h(u \vee w, v \vee w) =$

$h(w,v)$, so $\max(h(u \wedge w, v \wedge w), h(u \vee w, v \vee w)) = \max(h(u,w), h(w,v))$, so we arrive at the new Condition 3' through a chain of equivalences when $u \leq w \leq v$ is assumed. □

A trivial result is that an L-fuzzy divergence characteristic function defines an L-fuzzy divergence measure on an L-fuzzy powerset over a singleton universe $X = \{x\}$ by simply equating $D(A,B) := h(A(x), B(x))$. When the universe comprises more than one element, we will see that an L-fuzzy divergence measure can be constructed as a sum over the h values.

If an L-fuzzy divergence characteristic function has an upper bound, then a normalised version can be defined.

Definition 7.3.3. Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra. A function $h \colon L \times L \to \mathbb{I}$ is called a **normalised L-fuzzy (weak) divergence characteristic function** if it is an L-fuzzy (weak) divergence characteristic function and $h(u,v) \leq 1$ for any $u, v \in L$.

In particular, if L is a bounded lattice, then there exist both a bottom element $\mathbf{0}_L$ ($\mathbf{0}_L \leq a$ for any $a \in L$) and a top element $\mathbf{1}_L$ ($a \leq \mathbf{1}_L$ for any $a \in L$) and for any L-fuzzy divergence characteristic function condition 3' in Proposition 7.3.2 implies that $0 \leq h(u,v) \leq h(\mathbf{0}_L, \mathbf{1}_L)$, so $h'(u,v) := \frac{h(u,v)}{h(\mathbf{0}_L, \mathbf{1}_L)}$ is a normalised L-fuzzy divergence measure.

In order to complete our extension of the concept of divergence measure to the L-fuzzy sets, we will now qualify as "local" those L-fuzzy divergence measures that are constructed as a sum of L-fuzzy divergence characteristic functions. We will do this in two steps; first defining local divergence as an independent concept and then proving that it is always a form of divergence.

Definition 7.3.4. Let X be the universe and let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra with a corresponding L-fuzzy powerset $\{L^X, \neg_g, \wedge_g, \vee_g\}$. A function $D \colon L^X \times L^X \to \mathbb{R}^+$ is a **local L-fuzzy (weak) divergence measure** (or simply a **local divergence**, when the context is clear) if for each element $x \in X$, there is a normalised L-fuzzy (weak) divergence characteristic function $h_x \colon L \times L \to \mathbb{R}^+$ such that for any $A, B \in L^X$:

$$D(A,B) = \sum_{x \in X} h_x(A(x), B(x)) \tag{7.1}$$

Intuitively, each h_x measures the contribution to the divergence of each particular element of the universe.

The local divergence is often defined with the same characteristic function h for all the elements $x \in X$, but a very common scenario where a different h_x is needed for each element occurs when the relative relevance of membership for each element $x \in X$ is adjusted through the use of multiplicative factors or weights α_x affecting each element. So, given a base divergence characteristic function $h(u, v)$, an adjusted divergence characteristic function can be defined on a per element basis as $h_x(u, v) := \alpha_x h(u, v)$. We will refer to such functions as **weighted divergence characteristic functions**.

We now need to prove that any local divergence measure as in Definition 7.3.4 is indeed a divergence measure, which justifies the appropriateness of the chosen name.

Proposition 7.3.5. *Let X be the universe and let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra with a corresponding L-fuzzy powerset $\{L^X, \neg_g, \wedge_g, \vee_g\}$. A local L-fuzzy (weak) divergence measure $D\colon L^X \times L^X \to \mathbb{R}^+$ is an L-fuzzy (weak) divergence measure.*

Proof. As D is a local L-fuzzy divergence measure, for each element $x \in X$, there exists an L-fuzzy divergence characteristic function $h_x\colon L \times L \to \mathbb{R}^+$ such that $D(A, B) = \sum_{x \in X} h(A(x), B(x))$. We are going to prove that the conditions for an L-fuzzy divergence measure are all fulfilled.

First, if $A = B$, we have $D(A, A) = \sum_{x \in X} h_x(A(x), A(x))$, and this is 0 because of the first condition in Definition 7.3.1. Similarly, if $D(A, B) = 0$, then $\sum_{x \in X} h_x(A(x), B(x)) = 0$ and, as h_x can only take non-negative values, this means that $h_x(A(x), B(x)) = 0$ for any $x \in X$ and from the first condition in Definition 7.3.1 again, it follows that $A(x) = B(x)$ for any $x \in X$; i.e. $A = B$. So, the first condition of a divergence measure is proved. For a local weak divergence, the second part is not true and only the condition for a weak divergence is satisfied.

As a consequence of the symmetry of h_x, $D(A, B) = \sum_{x \in X} h_x(A(x), B(x)) = \sum_{x \in X} h_x(B(x), A(x)) = D(B, A)$, so symmetry also holds for D.

For the third condition, we have $D(A \wedge_g C, B \wedge_g C) = \sum_{x \in X} h_x((A \wedge_g C)(x), (B \wedge_g C)(x)) = \sum_{x \in X} h(A(x) \wedge C(x), B(x) \wedge C(x))$. And, as a result of the third condition in Definition 7.3.1, $\sum_{x \in X} h(A(x) \wedge C(x), B(x) \wedge C(x)) \leq$

$\sum_{x \in X} h(A(x), B(x)) = D(A, B)$. So, $D(A \wedge_g C, B \wedge_g C) \leq D(A, B)$. The additional condition $D(A \vee_g C, B \vee_g C) \leq D(A, B)$ is proved in the same way, with $\vee$ replacing $\wedge$. □

The expression (7.1) of a local divergence is often modified through the addition of a dimensionality normalisation operation, a division by $|X|$ and sometimes a dimensionality exponent $1/p$, of the kind that we mentioned at the end of the preliminary Section 2.9, when local divergence measures for ordinary fuzzy sets were discussed. Note that dividing by $|X|$ preserves the local property of the divergence measure while an exponent $1/p \neq 1$ makes the whole expression non-local, but remains a divergence measure with similar properties.

7.4 Local divergence measures in multivalued fuzzy sets

An important consequence of the results in Section 7.3 is that local divergence measures for the three types of multivalued fuzzy sets can be constructed by establishing normalised L-fuzzy divergence characteristic functions on the relevant membership grade spaces. The question we are going to address now is how we can build such characteristic functions for the multivalued membership grades out of well-known divergence characteristic functions for single-valued membership grades. For example, given the Minkowski divergence characteristic function $h(u, v) := |u - v|^p$ $(p \geq 1)$ on the unit interval $\mathbb{I}$, does this function induce matching set-based, multiset-based and n-tuple-based characteristic functions? As we will see, this is indeed what happens and we are going to present the formal procedures that will allow us to translate any local divergence measure for single-valued fuzzy sets into the corresponding local divergence measure for multivalued fuzzy sets.

The n-tuples provide the most straightforward case for this, as it is possible to take a Cartesian multivariate extension of the divergence characteristic function over a single-valued space of membership grades, such as $\mathbb{I}$, and then perform an aggregation operation on the resulting coordinates in order to get a real number as a final result. Let us turn this idea into a definition.

Definition 7.4.1. Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra, let n be a non-zero natural number, let $h \colon L \times L \to \mathbb{R}^+$ be a normalised L-fuzzy divergence characteristic function and let $A\colon \mathbb{I}^n \to \mathbb{I}$ be a strictly monotone aggregation operation. The **A-aggregated n-dimensional Cartesian-extended L-fuzzy divergence characteristic function** of h is a function $h^A_{c_n} : L^n \times L^n \to \mathbb{I}$ defined as:

$$h^A_{c_n}(\vec{U},\vec{V}) := A(h(u_1,v_1),...,h(u_n,v_n)) \tag{7.2}$$

with $\vec{U} = (u_1,...,u_n)$, $\vec{V} = (v_1,...,v_n) \in L^n$.

We now need to prove that the function $h^A_{c_n}$ defined in this way is a normalised divergence characteristic function.

Proposition 7.4.2. *Under the same conditions as in Definition 7.4.1 above, an A-aggregated n-dimensional Cartesian-extended L-fuzzy divergence characteristic function is a normalised L-fuzzy divergence characteristic function on the Cartesian-extended n-dimensional L-fuzzy algebra $\{L^n, \neg_c, \wedge_c, \vee_c\}$ (see Definition 5.1.1).*

Proof. First, $h^A_{c_n}(\vec{U},\vec{U}) = A(h(u_1,u_1),...,h(u_n,u_n)) = A(0,...,0) = 0$. Conversely, if $A(h(u_1,v_1),...,h(u_n,v_n)) = 0$, as we have imposed that A should be strictly monotone, then $h(u_1,v_1) = ... = h(u_n,v_n) = 0$ and, by definition of a divergence characteristic function, $u_i = v_i$ with $i = 1,...,n$ and, therefore, $\vec{U} = \vec{V}$. Note that the condition of strict monotonicity on A is not needed for the weak divergence version of this proposition.

Symmetry follows from the symmetry of h: $h^A_{c_n}(\vec{U},\vec{V}) = A(h(u_1,v_1),...,h(u_n, v_n)) = A(h(v_1,u_1),...,h(v_n,u_n)) = h^A_{c_n}(\vec{V},\vec{U})$.

As for the third condition for a divergence characteristic function, it can be split into two conditions. The first one is the inequality for the Cartesian-extended $\wedge$ operation: $h^A_{c_n}(\vec{U}\wedge_c\vec{W},\vec{V}\wedge_c\vec{W}) \leq h^A_{c_n}(\vec{U},\vec{V})$, for any $\vec{W}$, which is proved by substitution: $h^A_{c_n}(\vec{U}\wedge_c\vec{W},\vec{V}\wedge_c\vec{W}) = h^A_{c_n}((u_1\wedge w_1,...,u_n\wedge w_n),(v_1\wedge w_1,...,v_n\wedge w_n)) = A(h(u_1\wedge w_1,v_1\wedge w_1),...,h(u_n\wedge w_n,v_n\wedge w_n)) \leq A(h(u_1,v_1),...,h(u_n,v_n)) = h^A_{c_n}(\vec{U},\vec{V})$, where we have used the monotonicity property of A together with the fact that $h(u_i\wedge w_i,v_i\wedge w_i) \leq h(u_i,v_i)$ as h is a divergence characteristic function. The second condition is the inequality for the Cartesian-extended $\vee$ operation: $h^A_{c_n}(\vec{U}\vee_c\vec{W},\vec{V}\vee_c\vec{W}) \leq h^A_{c_n}(\vec{U},\vec{V})$, for any $\vec{W}$, which is also proved by substitution: $h^A_{c_n}(\vec{U}\vee_c\vec{W},\vec{V}\vee_c\vec{W}) = h^A_{c_n}((u_1\vee w_1,...,u_n\vee w_n),(v_1\vee w_1,...,v_n\vee w_n)) = A(h(u_1\vee w_1,v_1\vee w_1),...,h(u_n\vee w_n,v_n\vee w_n)) \leq A(h(u_1,v_1),...,h(u_n,v_n)) = h^A_{c_n}(\vec{U},\vec{V})$.

Finally, from the second and third condition in Definition 2.3.3, it follows that if $h(u_i, v_i) \leq 1$ for all $i = 1, ..., n$, then $A(h(u_1, v_1), ..., h(u_n, v_n)) \leq A(1, ..., 1) = 1$, so h being normalised means that $h^A_{c_n}$ is also normalised. □

This proposition that we have just proved means that any A-aggregated n-dimensional Cartesian-extended L-fuzzy divergence characteristic function determines a local L-fuzzy divergence measure over the ordered L-fuzzy multisets $(L^n)^X$.

Example 7.4.3. One of the simplest cases of a local divergence measure for the ordinary fuzzy sets is the Hamming distance (see Definition 2.9.4). Its divergence characteristic function is $h_H(u, v) := |u - v|$. Let us suppose that we are working with the space of triplets of real numbers in the unit interval as fuzzy membership grades. In this particular case, $L = \mathbb{I}$ and $n = 3$ and, by using Definition 7.4.1, the 3-dimensional Cartesian-extended ordinary fuzzy divergence characteristic function for the Hamming distance is a function $h^A_{Hc} \colon \mathbb{I}^3 \times \mathbb{I}^3 \to \mathbb{I}$. If we choose the arithmetic mean as the aggregation operation, then this function can be expressed as $h^A_{Hc}((u_1, u_2, u_3), (v_1, v_2, v_3)) = \frac{1}{3}(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|)$.

Let us take as an example a two-valued universe $X = \{x, y\}$ and two 3-dimensional ordered fuzzy multisets $\vec{A}$ and $\vec{B}$ defined as $\vec{A}(x) = (0.1, 0.2, 0.3)$, $\vec{A}(y) = (0.4, 0.2, 0.2)$, $\vec{B}(x) = (0.7, 0.8, 0.9)$ and $\vec{B}(y) = (0.3, 0.4, 0.5)$. The 3-dimensional Cartesian-extended ordinary fuzzy divergence characteristic function of the Hamming distance h^A_{Hc} that we worked out in the previous paragraph characterises a local divergence measure on $(\mathbb{I}^3)^X$ through Definition 7.3.4 as $D^S_{Hc}(\vec{A}, \vec{B}) = h^A_{Hc}(\vec{A}(x), \vec{B}(x)) + h^A_{Hc}(\vec{A}(y), \vec{B}(y))$. By substituting the values of $\vec{A}$ and $\vec{B}$, the Cartesian-extended Hamming divergence between these two ordered fuzzy multisets is:

$$
\begin{aligned}
D_{Hc}(\vec{A}, \vec{B}) = h^A_{Hc}(\vec{A}(x), \vec{B}(x)) + h^A_{Hc}(\vec{A}(y), \vec{B}(y)) = \\
= \frac{1}{3}(|0.1 - 0.7| + |0.2 - 0.8| + |0.3 - 0.9|) + \\
+ \frac{1}{3}(|0.4 - 0.3| + |0.2 - 0.4| + |0.2 - 0.5|) = \\
= \frac{1}{3}(0.9) + \frac{1}{3}(0.6) = 0.8
\end{aligned}
$$

We now turn our attention to the L-fuzzy multisets; i.e. the functions in $(\mathbb{N}^L)^X$. In order to define a local L-fuzzy divergence measure through Definition

7.4.1, we need a divergence characteristic function $h_m\colon \mathbb{N}^L \times \mathbb{N}^L \to \mathbb{I}$ and the question is how such a function can be induced by a simpler characteristic function in the L space of membership grades $h\colon L \times L \to \mathbb{I}$.

Corollary 3.4.12 gives us three possible courses of action to extend functions defined on L to new functions on $\mathbb{N}^L$. By taking one of those extension principles and rounding it off with an aggregation operation, we can get a multiset-based divergence characteristic function. An apparent alternative approach consists in combining the Cartesian-extended divergence characteristic function h^A_{cn}, which we have defined before, with a Cartesian-to-multiset extension, but we will see that both approaches lead to the same extended function.

If we rely on the three options of Corollary 3.4.12, the simplest one of these recipes is the multiset extension, which would allow us to build a multiset of h-based differences that can be aggregated in a final step. But such an approach cannot be valid as it would violate the rule that $h(u, u) = 0$. To see why, consider the multiset $\hat{M} = \langle 0.1, 0.2\rangle$ and the real-valued Hamming characteristic function $h(u, v) := |u - v|$. The multiset-extended version of h for M would yield $h^*_m(\hat{M}, \hat{M}) = \langle 0, 0, 0.1, 0.1\rangle$, with the 0 values coming from $|0.1 - 0.1|$ and $|0.2 - 0.2|$, and 0.1 coming from $|0.1 - 0.2|$ and $|0.2 - 0.1|$. While we can impose an aggregation operation that turns $\langle 0, 0, 0.1, 0.1\rangle$ into 0, like the minimum, these values could also come from two different multisets like $\hat{M}_1 = \langle 0.1, 0.2\rangle$ and $\hat{M}_2 = \langle 0.1, 0.1\rangle$, and the image by the divergence characteristic function should then be non-zero as the two input multisets are different. As it turns out, the aggregate multiset extension is problematic in the same way as it will also yield positive values when combining a non-trivial multiset with itself. We therefore conclude that those two types of multiset extension are not viable for our goal.

The previous observation suggests that only the sorted multiset extension looks likely to be compatible with the rule that $h(u, v) = 0 \Leftrightarrow u = v$. And this is indeed the case when working with regular multisets of a fixed cardinality n, although not when the cardinality is not fixed, as we will discuss later. The second axiomatic condition of symmetry does not present any trouble, and will follow directly from the symmetry of the multivariate multiset extensions. The third axiomatic condition, on the other hand, is also problematic when the cardinality can vary freely, but not when it is fixed. This is reminiscent of the problem with the definition of top and bottom elements, where we were able to circumvent the issues derived from differences in cardinality by restricting the multisets to a fixed cardinality n. We will later comment on the problems with the first and the third axioms in the general case. Our conclusion is that a divergence characteristic function based on aggregating the sorted multiset extension for a divergence characteristic function h on the underlying space of membership grades L can be defined only if it is restricted to the n-regular multisets $\mathbb{N}^L_{=n}$.

The sorted multiset extension that we are going to define depends on the choice of an ordering strategy, which we introduced in Chapter 2 and corresponds to a composition of functions $s \circ \phi_p^{-1}$ in the formalism of Chapter 3. Here we will express the ordering strategy more succinctly as a function $s\colon \mathbb{N}^L_{=n} \to L^n$ that sorts the values in the multiset into an n-tuple, typically in either ascending or descending order when L is totally ordered.

But building a valid divergence characteristic function on $\mathbb{N}^L_{=n}$ out of a characteristic function on L will also require an additional consistency property for the ordering strategy, which we define now.

Definition 7.4.4. Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra, let n be a non-zero natural number and let s be an ordering strategy that determines an n-regular sorted multiset-extended L-fuzzy algebra $\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}$ (see Definition 4.3.4). The ordering strategy s is said to be **compatible** with the L-fuzzy algebra if the following relations hold:

1. $s(\hat{A} \wedge_{sm_n} \hat{B}) = s(\hat{A}) \wedge_{c_n} s(\hat{B}), \qquad \forall \hat{A}, \hat{B} \in \mathbb{N}^L_{=n}$
2. $s(\hat{A} \vee_{sm_n} \hat{B}) = s(\hat{A}) \vee_{c_n} s(\hat{B}), \qquad \forall \hat{A}, \hat{B} \in \mathbb{N}^L_{=n}$

Note that if we expand the left-hand sides, for example $s(\hat{A} \wedge_{sm_n} \hat{B})$, using the definition of the sorted multiset-extended operations we get $s(Supp_m(s(\hat{A}) \wedge_{c_n} s(\hat{B})))$, so this basically means that $s \circ Supp_m$ is the identity when applied to the result of a Cartesian-extended operation on operands that were sorted by s; or, in other words, that the sorted state of the n-tuples is preserved by the Cartesian-extended operations. This is always true when the operations are t-norms and t-conorms and the ordering strategy is an ascending or descending sort, but not necessarily so in other cases.

We can now state the following definition.

Definition 7.4.5. Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra, let n be a non-zero natural number, let $h\colon L \to \mathbb{R}^+$ be a normalised L-fuzzy divergence characteristic function and let $A\colon L^n \to \mathbb{I}$ be a strictly monotone aggregation operation on L.

Furthermore, let s be the ordering strategy that determines an n-regular sorted multiset-extended L-fuzzy algebra $\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}$ and that is compatible with the L-fuzzy algebra (see Definition 7.4.4).

Then the **A-aggregated n-regular sorted multiset-extended L-fuzzy divergence characteristic function** of h, with respect to the ordering strategy s, is a function $h^A_{sm_n}: \mathbb{N}^L_{=n} \times \mathbb{N}^L_{=n} \to \mathbb{I}$ defined as:

$$h^A_{sm_n}(\hat{A}, \hat{B}) := A(h_{sm_n}(\hat{A}, \hat{B})) \tag{7.3}$$

where $\hat{A}, \hat{B} \in \mathbb{N}^L_{=n}$, and $h_{sm_n}: \mathbb{N}^L_{=n} \times \mathbb{N}^L_{=n} \to \mathbb{N}^L_{=n}$ is the sorted-multiset extension of h: $h_{sm_n}(\hat{A}, \hat{B}) := Supp_m(h_{c_n}(s(\hat{A}), s(\hat{B})))$. Note that the aggregation operation A can be applied to a multiset without any ambiguity as it is a symmetric operation.

Thanks to the fact that s is compatible with the L-fuzzy algebra, $h^A_{sm_n}$ can equivalently be expressed in terms of the A-aggregated n-dimensional Cartesian-extended L-fuzzy divergence characteristic function:

$$h^A_{sm_n}(\hat{A}, \hat{B}) = h^A_{c_n}(s(\hat{A}), s(\hat{B})) \tag{7.4}$$

This alternative expression for $h^A_{sm_n}$ follows from the definition of h_{sm_n}, together with the compatibility conditions of Definition 7.4.4 and the expression (7.2) that defines $h^A_{c_n}$.

We now need to prove that the function $h^A_{sm_n}$ defined in this way is a divergence characteristic function.

Proposition 7.4.6. *Under the same conditions as in Definition 7.4.5, an A-aggregated n-regular sorted multiset-extended L-fuzzy divergence characteristic function using a compatible ordering strategy $s: \mathbb{N}^L_{=n} \to L^n$ is an L-fuzzy divergence characteristic function on the n-regular sorted multiset-extended L-fuzzy algebra $\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}$ with the same ordering strategy s.*

Proof. First, $h^A_{sm_n}(\hat{A}, \hat{A}) = A(h_{sm_n}(\hat{A}, \hat{A})) = A(0, ..., 0) = 0$. Conversely, if $h^A_{sm_n}(\hat{A}, \hat{B}) = 0$ and if we use the notation $s(\hat{A}) = (a_{\sigma(1)}, ..., a_{\sigma(n)})$ and $s(\hat{B}) = (b_{\rho(1)}, ..., b_{\rho(n)})$, with σ and ρ being the index permutations induced by s on $\hat{A}$ and $\hat{B}$ respectively, then $A(\langle h(a_{\sigma(1)}, b_{\rho(1)}), ..., h(a_{\sigma(n)}, b_{\rho(n)})\rangle) = 0$, which, if A is strictly monotone, means that $h(a_{\sigma(i)}, b_{\rho(i)}) = 0$ for all $i = 1, ..., n$ and, since h is a divergence characteristic function $a_{\sigma(i)} = b_{\rho(i)}$ for all $i = 1, ..., n$, so the two multisets $\hat{A}$ and $\hat{B}$ are the same.

Symmetry can be proved through substitution: $h^A_{sm_n}(\hat{A},\hat{B}) := A(h_{sm_n}(\hat{A},\hat{B})) = A(h_{sm_n}(\hat{B},\hat{A})) = h^A_{sm_n}(\hat{B},\hat{A})$.

The third condition can be expanded as two conditions, as we always do in proofs. We begin with the one for the $\wedge$ operation: $h^A_{sm_n}(\hat{A} \wedge_{sm_n} \hat{C}, \hat{B} \wedge_{sm_n} \hat{C}) \leq h^A_{sm_n}(\hat{A},\hat{B})$, for any $\hat{C}$, where $\wedge_{sm_n}$ is the sorted multiset-extended t-norm operation. This inequality is proved through careful substitution. Using the definitions of $h^A_{sm_n}$ and of the $\wedge_{sm_n}$ operation, we have $h^A_{sm_n}(\hat{A}\wedge_{sm_n}\hat{C}, \hat{B}\wedge_{sm_n}\hat{C})$ $= A(h_{sm_n}(\hat{A}\wedge_{sm_n}\hat{C}, \hat{B}\wedge_{sm_n}\hat{C})) = A(h_{sm_n}(Supp_m(s(\hat{A})\wedge_{c_n}s(\hat{C})), Supp_m(s(\hat{B})\wedge_{c_n} s(\hat{C})))) = A(Supp_m(h_{c_n}(s(Supp_m(s(\hat{A}) \wedge_{c_n} s(\hat{C}))), s(Supp_m(s(\hat{B}) \wedge_{c_n} s(\hat{C}))))))$. We now use the requirement that the ordering strategy s should be compatible with the L-fuzzy algebra: $A(Supp_m(h_{c_n}(s(Supp_m(s(\hat{A})\wedge_{c_n} s(\hat{C}))), s(Supp_m(s(\hat{B}) \wedge_{c_n} s(\hat{C})))))) = A(Supp_m(h_{c_n}(s(\hat{A}) \wedge_{c_n} s(\hat{C}), s(\hat{B}) \wedge_{c_n} s(\hat{C}))))$. And, finally, as we proved in Proposition 7.4.2 that h_{c_n} is a divergence characteristic function, it follows that $A(Supp_m(h_{c_n}(s(\hat{A}) \wedge_{c_n} s(\hat{C}), s(\hat{B}) \wedge_{c_n} s(\hat{C})))) \leq A(Supp_m(h_{c_n}(s(\hat{A}), s(\hat{B})))) = A(h_{sm_n}(\hat{A},\hat{B})) = h^A_{sm_n}(\hat{A},\hat{B})$.

The condition for the $\vee$ operation is $h^A_{sm_n}(\hat{A}\vee_{sm_n}\hat{C}, \hat{B}\vee_{sm_n}\hat{C}) \leq h^A_{sm_n}(\hat{A},\hat{B})$, which is is proved through essentially the same sequence of substitutions, replacing $\wedge$ with $\vee$. □

Note that if we exclude the condition that A should be strictly monotone, then Proposition 7.4.6 would still be valid if $h^A_{sm_n}$ is downgraded to a weak divergence characteristic function.

At this point, we have managed to build a divergence characteristic function $h^A_{sm_n}$ on $\mathbb{N}^L_{=n}$ and consequently, through Definition 7.3.4, a local divergence on $(\mathbb{N}^L_{=n})^X$ with the sorted multiset operations.

By combining $h^A_{sm_n}$ with a regularisation strategy, we can extend this characteristic function to any cardinality. But there is an important problem with this idea. Basically, as we already hinted before, the resulting extended function does not fulfil the defining properties of a divergence characteristic function on the $\mathbb{N}^L$ space. In fact, the first axiomatic condition, $h(u,v) = 0 \Leftrightarrow u = v$, does not hold with multisets of different cardinality. Consider, for example, two multisets like $\hat{A}_1 = \langle 0.1, 0.2, 0.2\rangle$ and $\hat{A}_2 = \langle 0.1, 0.2\rangle$, which would need a regularisation step to turn the second one into, for example, $Reg_3(\hat{A}_2) = \langle 0.1, 0.2, 0.2\rangle$. But then we would have $h(\hat{A}_1,\hat{A}_2) := h^A_{sm_3}(\hat{A}_1, Reg_3(\hat{A}_2)) = 0$ with $\hat{A}_1 \neq \hat{A}_2$. Despite these difficulties, the problem can be avoided if we work with regular multisets that share a common cardinality only, as in the previous definitions, or if we content ourselves with a weak divergence characteristic function. In the latter case, the first condition is simply $h(u,u) = 0$, which we will prove to be valid for any sorted multiset extension of an L-valued divergence characteristic function h.

The third axiomatic condition is also problematic with non-regular multisets. To understand why, let us suppose that we have two multisets $\hat{A} = \langle 0, 0.5\rangle$ and $\hat{B} = \langle 0.1\rangle$. Using the optimistic regularisation strategy and an ascending sort, the sorted-multiset extension of a divergence characteristic function h would be $h^*_{sm}(\hat{A}, \hat{B}) = \langle h(0, 0.1), h(0.5, 0.1)\rangle$, where h^*_{sm} maps a pair of multisets into a multiset of real values, the images by h of the pairwise combinations of elements. In the specific case where we use the Hamming divergence characteristic function, we would have $h^*_{sm}(\hat{A}, \hat{B}) = \langle 0.1, 0.4\rangle$. If we then use the arithmetic mean as the final aggregation step to produce a divergence characteristic function, the end result is $h_{sm}(\hat{A}, \hat{B}) = 0.25$. Having opted for the most obvious parameter choices (optimistic regularisation, ascending sort, Hamming divergence characteristic function and arithmetic mean aggregation), we would expect this definition to fulfil the axiom but, unfortunately, this is not the case. Consider a third multiset $\hat{C} = \langle 1.0, 1.0, 1.0\rangle$. Then the standard t-norms with $\hat{C}$, using the sorted multiset-extended algebra, are $\hat{A} \wedge_{sm} \hat{C} = \langle 0, 0.5, 0.5\rangle$ and $\hat{B} \wedge_{sm} \hat{C} = \langle 0.1, 0.1, 0.1\rangle$ and the would-be divergence characteristic function h_{sm} yields the result $h_{sm}(\hat{A} \wedge_{sm} \hat{C}, \hat{B} \wedge_{sm} \hat{C}) = (0.1 + 0.4 + 0.4)/3 = 0.3$. If we choose $\hat{C}' = \langle 0.2, 0.2, 0.2\rangle$ instead, then $\hat{A} \wedge_{sm} \hat{C}' = \langle 0, 0.2, 0.2\rangle$ and $\hat{B} \wedge_{sm} \hat{C}' = \langle 0.1, 0.1, 0.1\rangle$, leading to $h_{sm}(\hat{A} \wedge_{sm} \hat{C}', \hat{B} \wedge_{sm} \hat{C}') = (0.1 + 0.1 + 0.1)/3 = 0.1$. So, we find that the t-norm operation with an arbitrary multiset $\hat{C}$ can both increase and decrease the value of the h_{sm} function, so it cannot be a divergence characteristic function. This can only happen when the multiset operands $\hat{A}$ and $\hat{B}$ are combined with a multiset $\hat{C}$ that has larger cardinality than both.

The conclusion that can be drawn from the preceding reasoning is that a form of divergence characteristic function for general multisets can be defined if the axiomatic conditions are weakened in two ways. First, only the weak form of the first axiom, $h(u,u) = 0$, will be required. And, secondly, the arbitrary third multiset $\hat{C}$ in the third axiom will be required not to be longer than either one of the operands $\hat{A}$ and $\hat{B}$. We now state this definition formally.

Definition 7.4.7. Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra with an associated sorted multiset-extended L-fuzzy algebra $\{\mathbb{N}^L, \neg_{sm}, \wedge_{sm}, \vee_{sm}\}$ based on an ordering strategy s. An **L-fuzzy multiset-divergence characteristic function** is a function $h\colon \mathbb{N}^L \times \mathbb{N}^L \to \mathbb{R}^+$ that fulfils the following conditions:

1. $h(\hat{A}, \hat{A}) = 0$
2. $h(\hat{A}, \hat{B}) = h(\hat{B}, \hat{A})$
3. $h(\hat{A}, \hat{B}) \geq \max(h(\hat{A} \wedge_{sm} \hat{C}, \hat{B} \wedge_{sm} \hat{C}), h(\hat{A} \vee_{sm} \hat{C}, \hat{B} \vee_{sm} \hat{C}))$

for all $\hat{A}$, $\hat{B}$, $\hat{C} \in \mathbb{N}^L$ such that $|\hat{C}| \leq \max(|\hat{A}|, |\hat{B}|)$. Additionally, h is said to be **normalised** if $h(\hat{A}, \hat{B}) \leq 1$ for all $\hat{A}, \hat{B} \in \mathbb{N}^L$.

Based on this definition, we can go on to define an **L-fuzzy local multiset-divergence measure** on an L-fuzzy power multiset $(\mathbb{N}^L)^X$ as a function $D\colon (\mathbb{N}^L)^X \times (\mathbb{N}^L)^X \to \mathbb{R}^+$ such that $D(\hat{A}, \hat{B}) := \sum_{x \in X} h(\hat{A}(x), \hat{B}(x))$.

A similar concept of **multiset-divergence** with the weakened axioms can be defined and it can then be proved that any local multiset-divergence is a multiset-divergence. For the sake of brevity, we will omit the details, which would involve a repetition of Proposition 7.3.5 and its proof with minor obvious changes.

We are now going to establish a proposition that will enable us to build an L-fuzzy multiset-divergence characteristic function out of a family of divergence characteristic functions on $\mathbb{N}^L_{=n}$. But, before that, we need a consistency condition on the regularisation strategies (see Definition 2.5.8).

Definition 7.4.8. Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra and let $\{s_n\}_{n \in \mathbb{N}}$ be the family of ordering strategies that determines the family of n-regular sorted multiset-extended L-fuzzy algebras $\{\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}\}_{n \in \mathbb{N}}$. We will say that a family of multiset regularisation functions $\{Reg_n\colon \mathbb{N}^L_{\leq n} \to \mathbb{N}^L_{=n}\}_{n \in \mathbb{N}}$ is **compatible** with the L-fuzzy algebras if the following relations, for $\kappa_1, \kappa_2 \in \mathbb{N}$ with $\kappa_1 \leq \kappa_2$ and for all $\hat{A}, \hat{B} \in \mathbb{N}^L_{\leq \kappa_1}$, hold:

1. $Reg_{\kappa_2}(Reg_{\kappa_1}(\hat{A}) \wedge_{sm_{\kappa_1}} Reg_{\kappa_1}(\hat{B})) = Reg_{\kappa_2}(\hat{A}) \wedge_{sm_{\kappa_2}} Reg_{\kappa_2}(\hat{B})$
2. $Reg_{\kappa_2}(Reg_{\kappa_1}(\hat{A}) \vee_{sm_{\kappa_1}} Reg_{\kappa_1}(\hat{B})) = Reg_{\kappa_2}(\hat{A}) \vee_{sm_{\kappa_2}} Reg_{\kappa_2}(\hat{B})$

In the ordinary case when $L = \mathbb{I}$, it is easy to check that the optimistic and pessimistic regularisation strategies are compatible with the standard ordinary fuzzy algebra and its associated sorted multiset-extended L-fuzzy algebra.

With this consistency check in place, we can now state the proposition.

Proposition 7.4.9. *Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra, let $\{Reg_n \colon \mathbb{N}^L_{\leq n} \to \mathbb{N}^L_{=n}\}_{n\in\mathbb{N}}$ be a family of regularisation strategies compatible with the L-fuzzy algebras (see Definition 7.4.8) and let $\{h_n\}_{n\in\mathbb{N}}$ be a family of L-fuzzy divergence characteristic functions for the n-regular sorted multiset-extended L-fuzzy algebras $\{\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}\}_{n\in\mathbb{N}}$.*

Under these conditions, the function $h_r \colon \mathbb{N}^L \times \mathbb{N}^L \to \mathbb{R}^+$, defined as follows:

$$h_r(\hat{A}, \hat{B}) := h_\kappa(Reg_\kappa(\hat{A}), Reg_\kappa(\hat{B})), \quad \hat{A}, \hat{B} \in \mathbb{N}^L \tag{7.5}$$

and where $\kappa := \max(|\hat{A}|, |\hat{B}|)$, is an L-fuzzy multiset-divergence characteristic function.

Proof. The first condition of a weak divergence characteristic function is obviously satisfied, as $h_r(\hat{A}, \hat{A}) = h_{|\hat{A}|}(Reg_{|\hat{A}|}(|\hat{A}|), Reg_{|\hat{A}|}(|\hat{A}|)) = h_{|\hat{A}|}(\hat{A}, \hat{A})$, which is 0, as $h_{|\hat{A}|}$ is a divergence characteristic function.

Symmetry is also obviously satisfied, as $h_r(\hat{A}, \hat{B}) = h_\kappa(Reg_\kappa(\hat{A}), Reg_\kappa(\hat{B})) = h_\kappa(Reg_\kappa(\hat{B}), Reg_\kappa(\hat{A})) = h_r(\hat{B}, \hat{A})$.

The third condition is proved, as usual, in two parts. We are going to check the inequality for the $\wedge$ operation first: $h_r(\hat{A} \wedge_{sm} \hat{C}, \hat{B} \wedge_{sm} \hat{C}) \leq h_r(\hat{A}, \hat{B})$ for any multiset $\hat{C}$ such that $|\hat{C}| \leq \max(|\hat{A}|, |\hat{B}|)$, which can be done by careful substitution. Starting from the left-hand side, we have $h_r(\hat{A} \wedge_{sm} \hat{C}, \hat{B} \wedge_{sm} \hat{C}) = h_{\kappa'}(Reg_{\kappa'}(\hat{A} \wedge_{sm} \hat{C}), Reg_{\kappa'}(\hat{B} \wedge_{sm} \hat{C}))$, where $\kappa' := \max(|\hat{A} \wedge_{sm} \hat{C}|, |\hat{B} \wedge_{sm} \hat{C}|)$. To simplify the cumbersome long expression, we first calculate the parts involving $\wedge_{sm}$. $\hat{A} \wedge_{sm} \hat{C} = Reg_{\kappa''}(\hat{A}) \wedge_{sm_{\kappa''}} Reg_{\kappa''}(\hat{C})$, with $\kappa'' := \max(|\hat{A}|, |\hat{C}|)$. And, similarly, $\hat{B} \wedge_{sm} \hat{C} = Reg_{\kappa'''}(\hat{B}) \wedge_{sm_{\kappa'''}} Reg_{\kappa'''}(\hat{C})$, with $\kappa''' := \max(|\hat{B}|, |\hat{C}|)$. And so, combining the two expressions we get $h_r(\hat{A} \wedge_{sm} \hat{C}, \hat{B} \wedge_{sm} \hat{C}) = h_{\kappa'}(Reg_{\kappa'}(Reg_{\kappa''}(\hat{A}) \wedge_{sm_{\kappa''}} Reg_{\kappa''}(\hat{C})), Reg_{\kappa'}(Reg_{\kappa'''}(\hat{B}) \wedge_{sm_{\kappa'''}} Reg_{\kappa'''}(\hat{C})))$. Now the regularisation value κ' must be $\kappa' = \max(\kappa'', \kappa''')$, as it is is the maximum of the two cardinalities. And because the regularisation is compatible with the operations, the latter expression can be simplified as $h_{\kappa'}(Reg_{\kappa'}(\hat{A}) \wedge_{sm_{\kappa'}} Reg_{\kappa'}(\hat{C}), Reg_{\kappa'}(\hat{B}) \wedge_{sm_{\kappa'}} Reg_{\kappa'}(\hat{C}))$, with $\kappa' = \max(|\hat{A}|, |\hat{B}|, |\hat{C}|)$. And using the axiomatic condition 3 for the divergence characteristic functions $h_{\kappa'}$, we get to the following inequality: $h_{\kappa'}(Reg_{\kappa'}(\hat{A}) \wedge_{sm_{\kappa'}} Reg_{\kappa'}(\hat{C}), Reg_{\kappa'}(\hat{B}) \wedge_{sm_{\kappa'}} Reg_{\kappa'}(\hat{C})) \leq h_{\kappa'}(Reg_{\kappa'}(\hat{A}), Reg_{\kappa'}(\hat{B}))$. We now use the axiomatic condition that $|\hat{C}| \leq \max(|\hat{A}|, |\hat{B}|)$, which means that $\kappa' = \max(|\hat{A}|, |\hat{B}|, |\hat{C}|)$ is the same as $\kappa := \max(|\hat{A}|, |\hat{B}|)$. And so we have the equality $h_{\kappa'}(Reg_{\kappa'}(\hat{A}), Reg_{\kappa'}(\hat{B})) = h_\kappa(Reg_\kappa(\hat{A}), Reg_\kappa(\hat{B})) = h_r(\hat{A}, \hat{B})$, which completes the proof. □

In the previous proof, we have had to use the restriction on the cardinality of $\hat{C}$ to equate $\max(|\hat{A}|, |\hat{B}|, |\hat{C}|)$ with $\max(|\hat{A}|, |\hat{B}|)$. If we wanted to allow any cardinality for $\hat{C}$, then we would need an additional consistency condition stating that for any two natural numbers $\kappa_1 \leq \kappa_2$, then $h_{\kappa_2}(Reg_{\kappa_2}(\hat{A}), Reg_{\kappa_2}(\hat{B})) \leq h_{\kappa_1}(Reg_{\kappa_1}(\hat{A}), Reg_{\kappa_1}(\hat{B}))$. The problem with such a compatibility condition is that it is violated in the typical cases, such as the A-aggregated sorted multiset-extended L-fuzzy divergence characteristic functions of Definition 7.4.5 over $\mathbb{I}$ with the optimistic or pessimistic regularisation strategies. This is the reason why we have chosen to weaken the conditions for the definition of divergence on multisets rather than impose an additional consistency condition and try to find uncanny forms of regularisation and divergence characteristic functions.

With a suitable choice of compatible ordering and regularisation strategies, the $h^A_{sm_n}$ functions of Definition 7.4.5 fulfil the properties of a family h_n as in Proposition 7.4.9, which leads to the following corollary and definition.

Corollary 7.4.10. *Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra, let $h\colon L \times L \to \mathbb{I}$ be a normalised L-fuzzy divergence characteristic function and let $A\colon L^n \to \mathbb{I}$ be an aggregation operation on L.*

Furthermore, let s_n be a family of ordering strategies that determines the family of n-regular sorted multiset-extended L-fuzzy algebras $\{\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}\}_{n\in\mathbb{N}}$ such that s_n is compatible with L for all values $n \in \mathbb{N}$. And let $\{Reg_n\colon \mathbb{N}^L_{\leq n} \to \mathbb{N}^L_{=n}\}_{n\in\mathbb{N}}$ be a family of regularisation strategies compatible with the L-fuzzy algebras.

Under these conditions, the function $h^A_{rsm}\colon \mathbb{N}^L \times \mathbb{N}^L \to \mathbb{I}$, defined as follows:

$$h^A_{rsm}(\hat{M}, \hat{N}) := h^A_{sm_\kappa}(Reg_\kappa(\hat{M}), Reg_\kappa(\hat{N})), \quad \hat{M}, \hat{N} \in \mathbb{N}^L \tag{7.6}$$

and where $\kappa := \max(|\hat{M}|, |\hat{N}|)$ and $h^A_{sm_n}$ is the A-aggregated sorted multiset-extended L-fuzzy divergence characteristic function of h (See Definition 7.4.5), is a normalised L-fuzzy multiset-divergence characteristic function on $\mathbb{N}^L$.

Definition 7.4.11. Under the same conditions as in Corollary 7.4.10, the function h^A_{rsm} will be called the ***A*-aggregated sorted multiset-extended L-fuzzy multiset-divergence characteristic function** for h.

With these L-fuzzy multiset-divergence characteristic functions that have been defined, it is now possible to build multiset-divergence measures by summing the values of the characteristic functions over all the elements in the universe, as we show in the next example.

Example 7.4.12. Let us consider, once again, the Hamming distance for ordinary fuzzy sets, which is a distance, a dissimilarity measure and a local divergence measure with characteristic function $h_H(u,v) := |u-v|$. This characteristic function induces a local divergence measure for the ordinary fuzzy multisets through the extended characteristic function in Definition 7.4.5. We will choose the optimistic regularisation and the ascending sort as the strategies that turn multisets into n-tuples and the arithmetic mean for the aggregation step.

Let us take the example of the two-valued universe $X = \{x, y\}$ and two fuzzy multisets $\hat{A}$, $\hat{B} \in (\mathbb{N}^L)^X$ defined as $\hat{A}(x) = \langle 0.1, 0.2\rangle$, $\hat{A}(y) = \langle 0.6, 0.8\rangle$, $\hat{B}(x) = \langle 0.2, 0.3, 0.3\rangle$ and $\hat{B}(y) = \langle 0.8\rangle$. As the multisets display various cardinality values, the tool to compare $\hat{A}$ and $\hat{B}$ will be the L-fuzzy multiset local multiset-divergence measure expressed in terms of the characteristic function according to Corollary 7.4.10 as:

$$\begin{aligned} D^A_{sm}(\hat{A}, \hat{B}) = h^A_{sm_{\kappa_x}}(Reg_{\kappa_x}(\hat{A}(x)), Reg_{\kappa_x}(\hat{B}(x)))+ \\ +h^A_{sm_{\kappa_y}}(Reg_{\kappa_y}(\hat{A}(y)), Reg_{\kappa_y}(\hat{B}(y))) \end{aligned} \tag{7.7}$$

In order to evaluate the two terms in the addition, we have to calculate the expression in Definition 7.4.5 for each one of them. But we need the fuzzy multisets to be regularised to a common cardinality. Note that we can do this for each element in the universe separately as the characteristic functions can be different for each element. Let us begin with the term for element x. The first step consists in calculating the maximum cardinality for the regularisation step:

$$\kappa_x := \max(|\hat{A}(x)|, |\hat{B}(x)|) = \max(2, 3) = 3 \tag{7.8}$$

We now need to regularise the two multisets to cardinality 3 using the optimistic regularisation, which repeats the maximum value as many times as needed:

$$\begin{aligned} Reg_3(\hat{A}(x)) = Reg_3(\langle 0.1, 0.2\rangle) = \langle 0.1, 0.2, 0.2\rangle \\ Reg_3(\hat{B}(x)) = Reg_3(\langle 0.2, 0.3, 0.3\rangle) = \langle 0.2, 0.3, 0.3\rangle \end{aligned} \tag{7.9}$$

And now we must sort them into triplets, using the ascending sort:

$$\begin{aligned} \vec{A}(x)_{rs} := s^{\uparrow}(Reg_3(\hat{A}(x))) = (0.1, 0.2, 0.2) \\ \vec{B}(x)_{rs} := s^{\uparrow}(Reg_3(\hat{B}(x))) = (0.2, 0.3, 0.3) \end{aligned} \tag{7.10}$$

And we now use the arithmetic-mean-aggregated 3-dimensional Cartesian-extended fuzzy divergence characteristic function h^A_{Hc} for the Hamming distance, as in the previous Example 7.4.3:

$$\begin{aligned} h^A_{sm_{\kappa_x}}(Reg_{\kappa_x}(\hat{A}(x)), Reg_{\kappa_x}(\hat{B}(x))) &= h^A_c(\vec{A}(x)_{rs}, \vec{B}(x)_{rs}) = \\ &= h^A_c((0.1, 0.2, 0.2), (0.2, 0.3, 0.3)) = \\ &= \frac{1}{3}(|0.1 - 0.2| + |0.2 - 0.3| + |0.2 - 0.3|) = 0.1 \end{aligned} \tag{7.11}$$

The second term in the expression (7.7) is solved in the same way:

$$\begin{aligned} h^A_{sm_{\kappa_y}}(Reg_{\kappa_y}(\hat{A}(y)), Reg_{\kappa_y}(\hat{B}(y))) &= h^A_c(\vec{A}(y)_{rs}, \vec{B}(y)_{rs}) = \\ &= h^A_c((0.6, 0.8), (0.8, 0.8)) = \\ &= \frac{1}{2}(|0.6 - 0.8| + |0.8 - 0.8|) = 0.1 \end{aligned} \tag{7.12}$$

And so, the final solution for the local multiset-divergence measure for fuzzy multisets induced by the Hamming distance is:

$$D^A_{sm}(\hat{A}, \hat{B}) = 0.1 + 0.1 = 0.2 \tag{7.13}$$

We have referred to D as a "multiset-divergence" because of the need for regularisation, which, as we have argued above, weakens the mathematical properties of such functions. If all the multisets involved had the same cardinality, say 3, then the problem would be defined on the 3-regular fuzzy multisets and the function $D^S_{sm_3}$, without the regularisation step, would fully comply with the conditions of a local divergence measure.

Having already analysed the ordered fuzzy multisets and the fuzzy multisets, we finally consider the set-valued L-fuzzy sets; i.e. the functions in $(\mathbf{2}^L)^X$. Again, local L-fuzzy divergence measures can be defined for the set-valued L-fuzzy sets through Definition 7.4.1 by using an appropriate divergence characteristic function where the domain is made up of pairs of sets: $h\colon \mathbf{2}^L \times \mathbf{2}^L \to \mathbb{I}$. Our goal now is to figure out how this type of function can be built from a divergence characteristic function in the single-valued space of membership grades $h\colon L \times L \to \mathbb{I}$. As in the multiset case, we will find that the basic definition of divergence must be refined with some additional restrictions to account for this particular case.

The most obvious idea in order to extend a function $L \times L \to \mathbb{I}$ to a new function $\mathbf{2}^L \times \mathbf{2}^L \to \mathbb{I}$ would consist in using the multivariate version of the Powerset Extension Principle (see Definition 2.7.2). In that way, a set of differences could be built with the extended function to be aggregated in the final step. But such an approach cannot be valid as it would violate the rule that $h(u, u) = 0$ (consider, for example, $h(S, S)$ with $S = \{0.1, 0.2\}$). In fact, the situation is the same as the one with the L-fuzzy multisets that we discussed earlier. As in that case, it turns out that the only extension mechanism that we have found to be compatible with the condition $h(u, v) = 0 \Leftrightarrow u = v$ is the one based on extending the sets to n-tuples first. This is essentially the same mechanism as for the L-fuzzy multisets, with the addition of an initial purely formal step where the input sets are reinterpreted as multisets. Interestingly, the first axiom can be restored to its strong form when dealing with sets, as the problems with the repeated values of multisets fade away.

The second axiom is not problematic at all as all the formulas we will use are implicitly symmetric.

As in the case of the multisets, satisfying the third axiom involves more difficulty and we will need to impose some additional consistency conditions. First, as we will be reinterpreting the input sets as multisets, we need to keep the same restriction on the cardinality of the third arbitrary set. But this is not the only obstacle. An additional problem is caused by the fact that the set-extended L-fuzzy algebra is not isomorphic to the sorted multiset-extended algebra that we have used for the multiset-based divergence. A manifestation of this lack of an isomorphism is the fact that the set-extended operations $\wedge_s$ and $\vee_s$ can result in sets with a higher cardinality than the original operands. in fact, this circumstance makes the inequality of the third axiom, in its general form, impossible with the most common parameter choices. As an example, consider the two sets $A = \{0, 0.2\}$ and $B = \{0.1, 0.5\}$ with the ordinary hesitant algebra. Reinterpreting the sets as multisets, we can evaluate their difference using the divergence characteristic function h^{A}_{rsm} (see Definition 7.4.11) based on the Hamming distance with the ascending sort and the optimistic regularisation. For A and B, this characteristic function yields the result $h(A, B) = (|0 - 0.1| + |0.2 - 0.5|)/2 = 0.2$. Now if we introduce a third set $C = \{0.44, 0.49\}$ the hesitant algebra operations produce $A \wedge_s C = \{0, 0.2\}$ and $B \wedge_s C = \{0.1, 0.44, 0.49\}$. If we now apply the same characteristic function h to these sets, we get $h(A \wedge_s C, B \wedge_s C) = (|0-0.1|+|0.2-0.44|+|0.2-0.49|)/3 = 0.21$, which is greater than $h(A, B)$. If we choose $C' = \{0.42, 0.45\}$ instead, then we get $h(A \wedge_s C', B \wedge_s C') = 0.19$, less than $h(A, B)$. This shows that the hesitant intersection of A and B with an arbitrary set C can both increase and decrease the difference between the two sets, even when $|C| = |A| = |B|$. This counter-example shows that additional conditions on the nature of C are needed in the third axiom for the kind of extension that we propose to be acceptable. The next definition shows the restrictions on C that will be needed to guarantee a behaviour as similar as possible to that of the single-valued divergence characteristic functions.

Definition 7.4.13. Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra with an associated set-extended L-fuzzy algebra $\{\mathbb{N}^L, \neg_s, \wedge_s, \vee_s\}$. An **L-fuzzy set-divergence characteristic function** is a function $h\colon \mathbf{2}^L \times \mathbf{2}^L \to \mathbb{R}^+$ that fulfils the following conditions:

1. $h(A, B) = 0 \Leftrightarrow A = B$
2. $h(A, B) = h(B, A)$
3. $h(A, B) \geq \max(h(A \wedge_s C, B \wedge_s C), h(A \vee_s C, B \vee_s C))$

for all A, $B \in \mathbf{2}^L$ and for any $C \in \mathbf{2}^L$ satisfying the following cardinality-related conditions:

1. $|C| \leq \min(A|, |B|)$
2. $|A \wedge_s C| = |A \vee_s C| = |A|$
3. $|B \wedge_s C| = |B \vee_s C| = |B|$

Additionally, h is said to be **normalised** if $h(A, B) \leq 1$ for all $A, B \in \mathbf{2}^L$.

In the ordinary case when $L = \mathbb{I}$, the restrictions on the arbitrary set C mean that it cannot be longer than either operand and that it cannot span a range of values that overlaps with the ranges of the operands (so that there is no difference between the sorted and aggregate operations involving C; see Proposition 4.5.3).

Using this definition, it is straightforward to define an **L-fuzzy local set-divergence measure** on a set-based L-fuzzy powerset $(\mathbf{2}^L)^X$ as a function $D\colon (\mathbf{2}^L)^X \times (\mathbf{2}^L)^X \to \mathbb{R}^+$ such that $D(\tilde{A}, \tilde{B}) := \sum_{x \in X} h(\tilde{A}(x), \tilde{B}(x))$.

As with the multiset-divergence, it is also possible to define a similar concept of **set-divergence** with the weakened third axiom and then prove that any local set-divergence is a set-divergence. We omit the details, which again involve minor obvious changes with respect to the general result.

As for the reinterpretation of a set as a multiset that we mentioned before, it can be formalised through the use of a **set-to-multiset function** $m\colon \mathbf{2}^L \to \mathbb{N}^L$ mapping a set to the repetition-canonical multiset with the same elements. This corresponds to a composition of functions $s_{rc} \circ \phi_r^{-1}$ in the formalism of Chapter 3. In terms of notation, this is simply the conversion from $\{a, b, c, ...\}$ (an element of $\mathbf{2}^L$) to $\langle a, b, c, ...\rangle$ (an element of $\mathbb{N}^L$).

There is an additional group of consistency conditions that will be needed and which affect the behaviour of the regularisation strategy. Up to this point, we have defined regularisation as an operation that simply lengthens a multiset by repeating elements and have defined a compatibility condition with the sorted multiset-extended L-fuzzy algebra. For our next result, we will need to prescribe a stronger form of compatibility that the regularisation functions must meet.

Definition 7.4.14. Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra with an associated set-extended L-fuzzy algebra $\{\mathbf{2}^L, \neg_s, \wedge_s, \vee_s\}$, let $\{s_n\}_{n\in\mathbb{N}}$ be the family of ordering strategies that determines the family of n-regular sorted multiset-extended L-fuzzy algebras $\{\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}\}_{n\in\mathbb{N}}$ and let $m\colon \mathbf{2}^L \to \mathbb{N}^L$ be the set-to-multiset function that maps a set to its corresponding repetition-canonical multiset. We say that a family of multiset regularisation functions $\{Reg_n\colon \mathbb{N}^L_{\leq n} \to \mathbb{N}^L_{=n}\}_{n\in\mathbb{N}}$ is **strongly compatible** with the L-fuzzy algebras if the following conditions apply:

1. It is compatible with the L-fuzzy algebras (See Definition 7.4.8).

2. The regularisation of a set $A \in \mathbf{2}^L$ to a size $n > |A|$ via its repetition-canonical multiset $m(A) \in \mathbb{N}^L_{=n}$ repeats just one value; i.e. there is only one element in $Reg_n(m(A))$ with a multiplicity larger than 1. In the typical case when L is a lattice, this is the minimum or the maximum.

3. For the ordering strategy s used in the sorted multiset-extended L-fuzzy algebra, the regularisation of a repetition-canonical multiset of cardinality m to a larger cardinality n will always have the repeated elements at the same positions, regardless of the elements that make up the set. In the typical case when L is a lattice, those positions correspond to either the initial or the final $n - m$ indices.

4. For A, B and C such that $|A \wedge_s C| = |A \vee_s C| = |A|$, $|B \wedge_s C| = |B \vee_s C| = |B|$, if $|C| \leq |A| < |B|$, the repeated element in $Reg_{|B|}(m(A \wedge_s C))$ will be $a_\lambda \wedge c_\mu$ and the repeated element in $Reg_{|B|}(m(A \vee_s C))$ will be $a_\lambda \vee c_\mu$, with a_λ being the repeated element for $Reg_{|B|}(m(A))$ and c_μ being the repeated element for $Reg_{|B|}(m(C))$ And the indices of the repeated elements in $Reg_{|B|}(m(A \wedge_s C))$ will correspond to $1 + |B| - |A|$ elements in $|B \wedge_s C|$ of the form $(b_{\tau_i} \wedge_s c_\mu)_{i=1,\ldots,1+|B|-|A|}$, and similarly for $Reg_{|B|}(m(A \vee_s C))$.

These conditions of strong compatibility may appear daunting, but it can be checked by substitution that the optimistic and pessimistic regularisation strategies on a laticce L comply with them, so they encapsulate the good mathematical behaviour that is expected from the interaction between regularisation and ordering strategies and will be needed for the proof of a proposition that will follow shortly.

We can now define the L-fuzzy set-divergence characteristic function that is induced by a simpler L-fuzzy divergence characteristic function on the underlying space of membership grades.

Definition 7.4.15. Let $\{L, \neg, \wedge, \vee\}$ be an L-fuzzy algebra, let $h\colon L \times L \to \mathbb{I}$ be a normalised L-fuzzy divergence characteristic function and let $A\colon \mathbb{I}^n \to \mathbb{I}$ be a strictly monotone aggregation operation.

Furthermore, let s_n be a family of ordering strategies that determines the family of n-regular sorted multiset-extended L-fuzzy algebras $\{\{\mathbb{N}^L_{=n}, \neg_{sm_n}, \wedge_{sm_n}, \vee_{sm_n}\}\}_{n\in\mathbb{N}}$ such that s_n is compatible with L for all values $n \in \mathbb{N}$. And let $\{Reg_n\colon \mathbb{N}^L_{\leq n} \to \mathbb{N}^L_{=n}\}_{n\in\mathbb{N}}$ be a family of regularisation strategies strongly compatible with the L-fuzzy algebras.

Then the A**-aggregated sorted set-extended L-fuzzy set-divergence characteristic function** of h, with respect to the given ordering and regularisation strategies, is a function $h^A_{ss}\colon \mathbb{N}^L \times \mathbb{N}^L \to \mathbb{I}$ defined as:

$$h^A_{ss}(A, B) := h^A_{rsm}(m(A), m(B)) \ , \quad A, B \subseteq L \tag{7.14}$$

where $m\colon \mathbf{2}^L \to \mathbb{N}^L$ is the set-to-multiset function that turns a set into its corresponding repetition-canonical multiset and h^A_{rsm} is the A-aggregated sorted multiset-extended L-fuzzy multiset-divergence characteristic function for h (see Definition 7.4.11).

We now need to prove that the function h^A_{ss} defined in this way is a set-divergence characteristic function.

Proposition 7.4.16. *Under the same conditions as in Definition 7.4.15, an A-aggregated sorted set-extended L-fuzzy set-divergence characteristic function is an L-fuzzy divergence characteristic function on the set-extended L-fuzzy algebra* $\{\mathbf{2}^L, \neg_s, \wedge_s, \vee_s\}$.

Proof. For the first axiomatic condition, $h^A_{ss}(A, A) = h^A_{rsm}(m(A), m(A)) = 0$, as h^A_{rsm} is a multiset-divergence characteristic function on $\mathbb{N}^L$. Conversely, if $h^A_{ss}(A, B) = 0$, then $h^A_{rsm}(m(A), m(B)) = 0$ and, by definition of h^A_{rsm}, we have $h^A_{sm_\kappa}(Reg_\kappa(m(A)), Reg_\kappa(m(B))) = 0$. with $\kappa := \max(|A|, |B|)$. As $h^A_{sm_n}$ is a divergence characteristic function, $Reg_\kappa(m(A)) = Reg_\kappa(m(B))$. But A and B being sets rather than multisets, their regularisations can only be equal if $A = B$.

Symmetry is a consequence of the symmetry of h^A_{rsm} : $h^A_{ss}(A, B) = h^A_{rsm}(m(A), m(B)) = h^A_{rsm}(m(B), m(A)) = h^A_{ss}(B, A)$.

The third axiomatic condition can be split into two conditions, as usual. We begin with $h^A_{ss}(A \wedge_s C, B \wedge_s C) \leq h^A_{ss}(A, B)$. Let us call the cardinalities of the three sets $m := |A|$, $n := |B|$ and $p := |C|$. Without loss of generality, we can assume that the sets have been named in such a way that $m \leq n$. Besides, as a result of the restrictions imposed by the definition of a set-divergence characteristic function, $|A \wedge_s C| = |A| = m$ and $|B \wedge_s C| = |B| = n$. Furthermore, we will assume that the indices are initially ordered according to the ordering strategy s. With these notational conventions, we have $A \wedge_s C = \{a_i \wedge c_k\}^{i=1,\dots,m}_{k=1,\dots,p}$, where there will be $(m-1)p$ repeated values among the $a_i \wedge c_k$ combinations so that the cardinality stays at the value m. Similarly, $B \wedge_s C = \{b_j \wedge c_k\}^{j=1,\dots,n}_{k=1,\dots,p}$, with $(n-1)p$ repeated values so that the cardinality stays at the value n in this case. Using the definition of h^A_{ss}, we have $h^A_{ss}(A \wedge_s C, B \wedge_s C) = h^A_{rsm}(\langle a_i \wedge c_k \rangle_{i;k}, \langle b_j \wedge c_k \rangle_{j;k})$, with any repeated values $a_i \wedge c_k$ or $b_j \wedge c_k$ only appearing once in each multiset. And now using the definition of h^A_{rsm}, we have $h^A_{rsm}(\langle a_i \wedge c_k \rangle_{i;k}, \langle b_j \wedge c_k \rangle_{j;k}) = h^A_{sm_n}(Reg_n(\langle a_i \wedge c_k \rangle_{i;k}), Reg_n(\langle b_j \wedge c_k \rangle_{j;k}))$, where the multisets are regularised to the larger cardinality n. As $B \wedge_s C$ has $n - m$ more values than $A \wedge_s C$, the effect of regularisation will be: $h^A_{sm_n}(Reg_n(\langle a_i \wedge c_k \rangle_{i;k}), Reg_n(\langle b_j \wedge c_k \rangle_{j;k})) = h^A_{sm_n}(\langle a_1 \wedge c_1, ..., a_\lambda \wedge c_\mu, \overset{1+n-m}{\cdots}, a_\lambda \wedge c_\mu, ..., a_m \wedge c_p\rangle, \langle b_1 \wedge c_1, ..., b_n \wedge c_p\rangle)$, where $a_\lambda \wedge c_\mu$ is the value repeated by the regularisation operation, which will have a multiplicity of $1 + n - m$, while all the other values have a multiplicity of 1. And now from the definition of $h^A_{sm_n}$, it follows that $h^A_{sm_n}(\langle a_1 \wedge c_1, ..., a_\lambda \wedge c_\mu, \overset{1+n-m}{\cdots}, a_\lambda \wedge c_\mu, ..., a_m \wedge c_p\rangle, \langle b_1 \wedge c_1, ..., b_n \wedge c_p\rangle) = h^A_{c_n}(s(\langle a_1 \wedge c_1, ..., a_\lambda \wedge c_\mu, \overset{1+n-m}{\cdots}, a_\lambda \wedge c_\mu, ..., a_m \wedge c_p\rangle), s(\langle b_1 \wedge c_1, ..., b_n \wedge c_p\rangle))$. Since the ordering strategy s is compatible with L and the indices have been assigned with the order given by it, the effect of s is simply to reinterpret the multisets as n-tuples: $h^A_{c_n}(s(\langle a_1 \wedge c_1, ..., a_\lambda \wedge c_\mu, \overset{1+n-m}{\cdots}, a_\lambda \wedge c_\mu, ..., a_m \wedge c_p\rangle), s(\langle b_1 \wedge c_1, ..., b_n \wedge c_p\rangle)) = h^A_{c_n}((a_1 \wedge c_1, ..., a_\lambda \wedge c_\mu, \overset{1+n-m}{\cdots}, a_\lambda \wedge c_\mu, ..., a_m \wedge c_p), (b_1 \wedge c_1, ..., b_n \wedge c_p))$. And substituting the definition of $h^A_{c_n}$, it follows that $h^A_{c_n}((a_1 \wedge c_1, ..., a_\lambda \wedge c_\mu, \overset{1+n-m}{\cdots}, a_\lambda \wedge c_\mu, ..., a_m \wedge c_p), (b_1 \wedge c_1, ..., b_n \wedge c_p)) = A(h(a_1 \wedge c_1, b_1 \wedge c_1), ..., h(a_\lambda \wedge c_\mu, b_{\tau_1} \wedge c_\mu), ..., h(a_\lambda \wedge c_\mu, b_{\tau_{1+n-m}} \wedge c_\mu), ..., h(a_m \wedge c_p, b_n \wedge c_p))$. And, by virtue of h being a divergence characteristic function and A being an aggregation operation, we arrive at the inequality $A(h(a_1 \wedge c_1, b_1 \wedge c_1)..., h(a_\lambda \wedge c_\mu, b_{\tau_1} \wedge c_\mu), ..., h(a_\lambda \wedge c_\mu, b_{\tau_{1+n-m}} \wedge c_\mu), ..., h(a_m \wedge c_p, b_n \wedge c_p)) \leq A(h(a_1, b_1)..., h(a_\lambda, b_{\tau_1}), ..., h(a_\lambda, b_{\tau_{1+n-m}}, ..., h(a_m, b_n))$. This is where the need for the cardinality conditions that we have imposed on these characteristic functions and for the strong compatibility for regularisation functions becomes apparent. Because of the nature of regularisation and the set-extended operations, without those conditions the number of terms and the indices might not match the ones in the operation on A and B and it would not be possible to arrive at any inequality. It is thanks to those conditions that it is now possible to identify the terms in the preceding expression with those that result from the expansion of $h^A_{ss}(A, B)$, getting the final result $h^A_{ss}(A \wedge_s C, B \wedge_s C) \leq A(h(a_1, b_1), ..., h(a_\lambda, b_{\tau_1}), ..., h(a_\lambda, b_{\tau_{1+n-m}}), ..., h(a_m, b_n)) = h^A_{c_n}((a_1, ..., a_\lambda, ..., a_\lambda, ..., a_m), (b_1, ..., b_n)) = h^A_{sm_n}(\langle a_1, ..., a_\lambda, ..., a_\lambda, ..., a_m\rangle, \langle b_1, ..., b_n\rangle) = h^A_{sm_n}(Reg_n(\langle a_1, ..., a_m\rangle), Reg_n(\langle b_1, ..., b_n\rangle)) = h^A_{rsm}(m(A), m(B)) = h^A_{ss}(A, B)$, which completes the proof.

The proof for the other inequality $h^A_{ss}(A \vee_s C, B \vee_s C) \leq h^A_{ss}(A, B)$ is similar and depends on the same assumptions about C and the regularisation functions.

□

Thanks to Proposition 7.4.16, any A-aggregated sorted set-extended L-fuzzy set-divergence characteristic function h^A_{ss} determines a local L-fuzzy set-divergence measure over the set-based L-fuzzy sets $(\mathbf{2}^L)^X$.

Example 7.4.17. If we take the unit interval $\mathbb{I}$ as the basic space of membership grades and the Minkowski divergence characteristic function $h^p_M(u, v) := |u - v|^p$, we can use Definition 7.4.15 to extend the Minkowski-style local divergence measures to the corresponding versions for hesitant fuzzy sets.

Given a finite universe X and two hesitant fuzzy sets $\tilde{A}, \tilde{B} \in (\mathbf{2}^{\mathbb{I}})^X$, in order to work out an expression for their divergence, as induced by h^p_M, we will have to first regularise their cardinalities to their maximum κ and then sort them, for example in descending order. If we represent their sorted values, evaluated at an element $x \in X$, as $\tilde{A}^{\sigma(j)}(x)$ and $\tilde{B}^{\sigma(j)}(x)$, with σ standing for the index permutation that sorts the values in descending order, then for an arbitrary aggregation operation S, the local divergence measure of the two hesitant fuzzy sets will be given by the following expression:

$$D^p_M(\tilde{A}, \tilde{B}) = \sum_{x \in X} S^{\kappa}_{j=1}(|\tilde{A}^{\sigma(j)}(x) - \tilde{B}^{\sigma(j)}(x)|^p)$$

The Minkowski-style local divergence measures are usually combined with a dimensionality normalisation operation that takes the p-th root of the result. This restores the right dimension for the result as a length at the cost of sacrificing the local property. The non-local normalised version of the Minkowski divergence has the following form:

$$D^p_{M_n}(\tilde{A}, \tilde{B}) = \left(\sum_{x \in X} S^{\kappa}_{j=1}(|\tilde{A}^{\sigma(j)}(x) - \tilde{B}^{\sigma(j)}(x)|^p) \right)^{\frac{1}{p}}$$

This explicit form of a local divergence measure for two hesitant fuzzy sets, which we have worked out using the extension principle of Definition 7.4.15, was first proposed by V. Kobza et al. [21] (with an additional $1/|X|$ normalisation factor), where the aggregation operation S was also required to be a t-conorm. While that restriction is not necessary from a formal viewpoint, the semantics of t-conorms (as generalisations of the union) are often regarded as more useful for divergence measures. Testing how useful these techniques are and comparing different parameter choices is still an open field of research.

7.5 A practical application: pattern recognition

In this section, we will study an interesting application of divergence measures between multivalued types of fuzzy sets in the field of pattern recognition. We will adapt an example that has been discussed by I. Montes et al. [33] for Atanassov intuitionistic fuzzy sets, a type of fuzzy extension that is similar to interval-valued fuzzy sets.

Following Montes's approach, we consider that there is a finite universe $X = \{x_1, ..., x_n\}$ (the attributes used for patterns) and will regard a finite collection of

m multivalued fuzzy sets $\{A_1, ..., A_m\}$ over X as the reference patterns that other fuzzy sets must be compared with. The idea is to classify any possible fuzzy set (an image) as being closest to one particular pattern. So, given a multivalued fuzzy set A, the pattern it resembles will be indicated by the index $1 \le j \le m$ such that $D(A, A_j) = \min_{i=1,...,m}\{D(A, A_i\}$ where D is one of the measures of difference that we have studied. This setting provides an effective testing ground both for different types of multivalued fuzzy sets and for the different types of measures of difference.

In the following examples, we will perform two simple tests of the standard ordinary ordered fuzzy multisets with the Hamming-extended local divergence measure of Example 7.4.3.

Example 7.5.1. Adapting the example provided by Montes [33], let us consider a universe $X = \{x_1, x_2, x_3\}$. We can construct ordered fuzzy multisets of dimension 2 over X with the standard operations and the ordinary membership space $\mathbb{I}^2$ and we will regard the following three ordered fuzzy multisets as the patterns used for classification:

$$\begin{aligned}
\vec{A}_1(x_1) &= (0.1, 0.9); \quad \vec{A}_1(x_2) = (0.5, 0.6); \quad \vec{A}_1(x_3) = (0.1, 0.1)\\
\vec{A}_2(x_1) &= (0.5, 0.5); \quad \vec{A}_2(x_2) = (0.7, 0.7); \quad \vec{A}_2(x_3) = (0.0, 0.2)\\
\vec{A}_3(x_1) &= (0.7, 0.8); \quad \vec{A}_3(x_2) = (0.1, 0.2); \quad \vec{A}_3(x_3) = (0.4, 0.6)
\end{aligned}$$

Let us now assume that there is a sample given by the following ordered fuzzy multiset $\vec{B}$:

$$\vec{B}(x_1) = (0.4, 0.6); \quad \vec{B}(x_2) = (0.6, 0.8); \quad \vec{B}(x_3) = (0.0, 0.2)$$

We consider the two-dimensional Hamming-based divergence characteristic function with the arithmetic mean as the aggregation operation, as in Example 7.4.3:

$$h^A_{Hc}(u_1, u_2, v_1, v_2) := \frac{1}{2}(|u_1 - v_1| + |u_2 - v_2|) \tag{7.15}$$

And the Hamming-based local divergence measure for two-dimensional ordered fuzzy multisets over the universe X is then defined as:

$$D_{Hc}(\vec{A}, \vec{B}) := \sum_{x \in X} h^A_{Hc}(\vec{A}(x), \vec{B}(x)) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{2} |(\vec{A}(x_i))_j - (\vec{B}(x_i))_j| \tag{7.16}$$

Let us now calculate the Hamming-based local divergence measures (using the arithmetic mean as the aggregation operation) between $\vec{B}$ and each one of the $\vec{A}_i$ patterns:

$$D_{Hc}(\vec{A}_1, \vec{B}) = \frac{1}{2}(0.6 + 0.3 + 0.2) = 0.55$$
$$D_{Hc}(\vec{A}_2, \vec{B}) = 0.2$$
$$D_{Hc}(\vec{A}_3, \vec{B}) = 1.2$$

As $D_{Hc}(\vec{A}_2, \vec{B})$ yields the minimum divergence value, we conclude that $\vec{B}$ should be classified in the $\vec{A}_2$ pattern.

Example 7.5.2. The second example is also adapted from Montes's article [33]. Following Montes, we now consider a universe with six elements $X = \{x_1, ..., x_6\}$, where each x_i represents a mineral and we have five reference patterns of typical hybrid elements defined by the two-dimensional ordered fuzzy multisets $\vec{A}_1, ..., \vec{A}_5$. In addition, there is a mineral sample $\vec{B}$, which needs to be to classified under one of the reference patterns. The six ordered fuzzy multisets for the five patterns and the sample to classify are displayed in Table 7.1.

	x_1	x_2	x_3	x_4	x_5	x_6
$\vec{A}_1(x_i)$	$(0.739, 0.875)$	$(0.033, 0.182)$	$(0.188, 0.374)$	$(0.492, 0.642)$	$(0.020, 0.372)$	$(0.739, 0.875)$
$\vec{A}_2(x_i)$	$(0.124, 0.335)$	$(0.030, 0.175)$	$(0.048, 0.200)$	$(0.136, 0.352)$	$(0.019, 0.177)$	$(0.393, 0.347)$
$\vec{A}_3(x_i)$	$(0.449, 0.613)$	$(0.662, 0.702)$	$(1.000, 1.000)$	$(1.000, 1.000)$	$(1.000, 1.000)$	$(1.000, 1.000)$
$\vec{A}_4(x_i)$	$(0.280, 0.285)$	$(0.521, 0.632)$	$(0.470, 0.577)$	$(0.295, 0.342)$	$(0.188, 0.194)$	$(0.735, 0.882)$
$\vec{A}_5(x_i)$	$(0.326, 0.548)$	$(1.000, 1.000)$	$(0.182, 0.275)$	$(0.156, 0.235)$	$(0.049, 0.104)$	$(0.675, 0.737)$
$\vec{B}(x_i)$	$(0.629, 0.697)$	$(0.524, 0.644)$	$(0.210, 0.311)$	$(0.218, 0.247)$	$(0.069, 0.124)$	$(0.658, 0.744)$

Table 7.1: Six kinds of materials represented by ordered fuzzy multisets.

We have mentioned how the divergence characteristic functions can depend on the element $x \in X$. A situation in which this is common occurs when the measured difference for each element of the universe must be given a variable relative importance. A common setting in pattern recognition involves attributes whose relative importance is not uniform. This can be modelled through a local divergence measure where the divergence characteristic function for each element differs by a multiplicative constant, the weight of the attribute. In that way, once a base characteristic function $h(u_1, ..., u_n)$ has been defined, for each element $x \in X$ we have an x-dependent refinement $h_x(u_1, ..., u_n) := \alpha_x h(u_1, ..., u_n)$, with $\alpha_x \in \mathbb{R}$ and typically normalised to the unit interval. These manipulations show the expressive power of the local divergence measures as they only depend on the membership grades in a pointwise way, so building local divergences is usually simpler and much more computationally efficient than using non-local divergences [33]. In this particular example, we will give the six basic minerals the weights $\{\alpha_i\}_{i=1,\ldots,6} = \{1/4, 1/4, 1/8, 1/8, 1/8, 1/8\}$. We consider the same two-dimensional Hamming-based divergence characteristic function with the standard operations and the underlying ordinary membership as in the previous Example 7.5.1, but modified by the specific weight α_i for each element. This gives rise to a family of six characteristic functions h_i with $i = 1, ..., 6$.

$$h^A_{Hci}(u_1, u_2, v_1, v_2) := \frac{\alpha_i}{2}(|u_1 - v_1| + |u_2 - v_2|) \tag{7.17}$$

And the weighted Hamming-based local divergence measure for two-dimensional ordered fuzzy multisets over the universe X is then defined as:

$$D_{Hc}(\vec{A}, \vec{B}) = \sum_{i=1}^{6} \frac{\alpha_i}{2} \left(\sum_{j=1}^{2} |(\vec{A}(x_i))_j - (\vec{B}(x_i))_j| \right) \tag{7.18}$$

We can now calculate the local divergences between $\vec{B}$ and each one of the reference patterns $\vec{A}_i$, getting the following results.

$$D_{Hc}(\vec{A}_1, \vec{B}) = \frac{1}{16}(2(|0.739 - 0.629| + |0.875 - 0.697|) + (2(|0.033 - 0.524|+$$
$$+|0.182 - 0.644|)) + |0.188 - 0.210| + |0.374 - 0.311| + |0.492 - 0.218|+$$
$$+|0.642 - 0.247| + |0.020 - 0.069| + |0.372 - 0.124| + |0.739 - 0.658|+$$
$$+|0.875 - 0.744| = 0.234$$
$$D_{Hc}(\vec{A}_2, \vec{B}) = 0.305$$
$$D_{Hc}(\vec{A}_3, \vec{B}) = 0.396$$
$$D_{Hc}(\vec{A}_4, \vec{B}) = 0.164$$
$$D_{Hc}(\vec{A}_5, \vec{B}) = 0.173$$

From the divergence values above, it follows that $\vec{B}$ should be classified in the $\vec{A}_4$ pattern as this is the one that minimises the divergence value.

7.6 A practical application: decision-making

In this second use case for multivalued fuzzy sets, we are going to present a simple adaptation to two-dimensional ordered fuzzy multisets of an example introduced by I. Montes et al. [33], The basic assumption is like the one in the previous section. Montes's example is also a modification of a case originally studied by Z. Xu [47] in the context of Atanassov intuitionistic fuzzy sets. The situation here is that there is a universe comprising n attributes $C = \{c_1, ..., c_n\}$ and a possible decision is a fuzzy set over C. While Xu and Montes used Atanassov intuitionistic fuzzy sets to represent the possible decisions, we will use two-dimensional ordered fuzzy multisets in our treatment, translating the membership and non-membership values used in these sets into two membership values that represent two criteria matching what were originally the lower and upper bounds of a reasonable membership interval. As in the similar example that we worked out in the previous section, there is a relevant difference in semantics between the two values used in an Atanassov intuitionistic fuzzy set and the two values in a two-dimensional ordered fuzzy multiset, but this is fine as an example to showcase how calculations can be carried out. Thus, in the approach presented here, the decision to make consists in picking one among a given set of m alternatives $\{\vec{A}_1, ..., \vec{A}_m\}$.

Once the description of the problem has been laid out, we need a mechanism to measure the difference in a pair of alternatives. Here come into play the local divergence measures again. As in Example 7.5.2 above, we will use a basic two-dimensional Hamming-based divergence characteristic function as in Equation (7.19) above, and then define n weighted divergence characteristic functions $h_i((u_1, u_2), (v_1, v_2)) := \alpha_i h((u_1, u_2), (v_1, v_2))$ with weights $\{\alpha_i\}_{i=1,...,n}$ that assign different relative relevances to each attribute c_i. The resulting weighted Hamming-based local divergence measure for two-dimensional ordered fuzzy multisets over the universe of attributes C is defined by the following expression:

$$D_{Hc}(\vec{A}, \vec{B}) = \frac{1}{2} \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{2} |(\vec{A}(c_i))_j - (\vec{B}(c_i))_j| \right) \tag{7.19}$$

Following Xu's idea, the solution to the decision-making problem involves identifying the intersection and the union (in the sense of ordered fuzzy multisets, in our case) of the existing alternatives with an "optimal" and "least optimal" decision, respectively, beyond the available alternatives. This leads to two additional definitions:

$$\vec{A}^- := \bigcap_{i=1}^{n} \vec{A}_i \qquad (7.20) \qquad\qquad \vec{A}^+ := \bigcup_{i=1}^{n} \vec{A}_i \qquad (7.21)$$

As Xu and Montes have argued, there are various ways in which an optimal alternative can be selected. One possibility consists in picking the alternative $\vec{A}_i$ that minimises $D_{Hc}(\vec{A}_i, \vec{A}^+)$; i.e. selecting the available alternative that is closest to the optimal decision. Another possibility consists in picking the $\vec{A}_i$ that maximises $D_{Hc}(\vec{A}_i, \vec{A}^-)$; i.e. the one that lies farthest from the least optimal decision. A more general possibility combining both approaches as limit cases can be specified by means of a function $\psi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that is non-increasing in the first argument and non-decreasing in the second one. With such a function, we can pick the alternative $\vec{A}_i$ as the one that maximises the value of $a_i := \psi(D_{Hc}(\vec{A}_i, \vec{A}^+), D_{Hc}(\vec{A}_i, \vec{A}^-))$. Using this approach, we can model different decision policies through the ψ function. In particular, $\psi_1(x, y) = 1/x$ is the decision that selects the alternative that minimises $D_{Hc}(\vec{A}_i, \vec{A}^+)$, whereas $\psi_2(x, y) = y$ selects the alternative that maximises $D_{Hc}(\vec{A}_i, \vec{A}^-)$ and $\psi_3(x, y) = 1/x + y$ is yet another decision policy that averages the effects of being close to the optimal decision and far from the least optimal one.

After this general introduction to the problem, let us now adapt the original example brought up by Xu and Montes using two-dimensional ordered fuzzy multisets. The problem involves a city where a library is going to be built and there are five alternative proposals for the air-conditioning system, which differ in how they meet the requirements under three attributes c_1, the economic factor, c_2, the functional factor, and c_3, the operational factor, that make up the universe C. The five alternatives are represented by the two-dimensional ordered fuzzy multisets $\{\vec{A}_i\}_{i=1,\ldots,5}$ in $(\mathbb{I}^2)^C$ with the Cartesian-extended standard operations. Adapting the values used by Montes, the five alternatives and the optimal and least optimal decisions are represented by the two-dimensional ordered fuzzy multisets displayed in Table 7.2.

We can now calculate the Hamming-based local divergence measures using Equation (7.19). Following Montes, we will use the weights $\alpha = \{0.3, 0.5, 0.2\}$ for the local divergence. In Table 7.3, we summarise the divergence measures between each one of the alternatives and the optimal and least optimal decisions.

Finally, in Table 7.4, we display the values for the three decision policies we mentioned before: ψ_1, ψ_2 and ψ_3. As each one of these ψ_i reaches its maximum value for $\vec{A}_5$, we conclude that the fifth proposal for the air-conditioning system is the best alternative according to the three policies.

	c_1	c_2	c_3
$\vec{A}_1(c_i)$	$(0.2, 0.6)$	$(0.7, 0.9)$	$(0.6, 0.7)$
$\vec{A}_2(c_i)$	$(0.4, 0.8)$	$(0.5, 0.8)$	$(0.8, 0.9)$
$\vec{A}_3(c_i)$	$(0.5, 0.6)$	$(0.6, 0.8)$	$(0.9, 1.0)$
$\vec{A}_4(c_i)$	$(0.3, 0.5)$	$(0.8, 0.9)$	$(0.7, 0.8)$
$\vec{A}_5(c_i)$	$(0.8, 0.8)$	$(0.7, 1.0)$	$(0.1, 0.4)$
$\vec{A}^-(c_i)$	$(0.2, 0.5)$	$(0.5, 0.8)$	$(0.1, 0.4)$
$\vec{A}^+(c_i)$	$(0.8, 0.8)$	$(0.8, 1.0)$	$(0.9, 1.0)$

Table 7.2: The five alternative options for the air-conditioning system represented by ordered fuzzy multisets, together with the associated optimal and least optimal decisions.

D_{Hc}	$\vec{A}^-$	$\vec{A}^+$
$\vec{A}_1$	0.170	0.230
$\vec{A}_2$	0.195	0.205
$\vec{A}_3$	0.225	0.175
$\vec{A}_4$	0.215	0.185
$\vec{A}_5$	0.235	0.165

Table 7.3: The divergence measures between the five available alternatives and the optimal and least optimal decisions.

	ψ_1	ψ_2	ψ_3
$\vec{A}_1$	4.348	0.170	4.518
$\vec{A}_2$	4.878	0.195	5.073
$\vec{A}_3$	5.714	0.225	5.939
$\vec{A}_4$	5.405	0.215	5.620
$\vec{A}_5$	6.061	0.235	6.296

Table 7.4: The decision values for each alternative according to the three ψ decision policies we have considered.

8 Conclusiones

A lo largo de nuestra investigación de los distintos tipos de conjuntos difusos multivaluados, hemos prestado especial atención a la identificación de aquellos aspectos clave que distinguen a los diversos planteamientos, al tiempo que hemos tratado de dilucidar las partes que tales teorías presentan en común. Un resultado importante es el hecho de que los multiconjuntos difusos admiten, junto a las operaciones tradicionales debidas a S. Miyamoto, otras formas de operaciones básicas como la intersección y unión agregadas, las cuales hemos presentado, demostrando que estas operaciones agregadas son un caso general, basado en multiconjuntos, de las operaciones para los conjuntos difusos *hesitant*.

Nuestro estudio de los conjuntos difusos basados en conjuntos y en multiconjuntos se ha visto complementado por un análisis detallado de lo que hemos denominado multiconjuntos difusos ordenados, que tienen n-uplas como grados de pertenencia. Aun cuando el concepto no es nuevo, lo hemos vinculado a los planteamientos basados en multiconjuntos y conjuntos de una manera original al investigar lo que ocurre cuando se pasa por alto el orden en las n-uplas, lo que da lugar a multiconjuntos, o cuando se pasa por alto la repetición, lo que hace que los multiconjuntos degeneren a la forma de conjuntos. El mecanismo que hemos empleado para pasar por alto el orden y la repetición ha consistido en definir relaciones de equivalencia que igualan bien los multiconjuntos con el mismo soporte o bien aquellas n-uplas que comparten las mismas coordenadas bajo distintas permutaciones. Estas relaciones de equivalencia nos han permitido definir a los conjuntos como una partición de los multiconjuntos y a los multiconjuntos como una partición de las n-uplas, de tal modo que hemos podido pasar a estudiar cómo se interrelacionan las operaciones sobre estos tipos diversos de grados de pertenencia difusa. El planteamiento que hemos seguido ha constado de dos fases, analizando primero los conjuntos, multiconjuntos y n-uplas como objetos matemáticos fundamentales para después trasladar los resultados al caso en que estas cantidades se utilizan como grados de pertenencia para cada tipo correspondiente de conjunto difuso multivaluado.

Para extender las operaciones desde los espacios matemáticos más sencillos a los espacios compuestos formados por agrupaciones de objetos y a los conjuntos difusos, hemos logrado identificar reglas comúnmente aplicadas como principios de extensión que pueden considerarse como un conjunto de herramientas que nos permiten trasladar los conceptos desde los conjuntos difusos normales hasta los tipos multivaluados, así como a través de los distintos tipos de conjuntos difusos multivaluados. Hemos estudiado cómo estas transformaciones conllevan una diferencia en contenido de información, que hemos cuantificado en forma de varias medidas de entropía.

Finalmente, se ha utilizado el planteamiento basado en principios de extensión para aportar definiciones nuevas para los conceptos de distancia, disimilitud y divergencia en los conjuntos difusos multivaluados. Para llevar esto a cabo, hemos presentado en primer lugar nuevas definiciones para conjuntos L-difusos, que después hemos generalizado mediante los principios de extensión enunciados inicialmente. Esto ha conducido a lo que creemos que es una formalización interesante de estos conceptos en un marco muy general, que puede contribuir a la experimentación de aplicaciones prácticas de estas técnicas difusas. En concreto, se ha mostrado el uso de medidas de divergencia para los multiconjuntos difusos ordenados mediante dos ejemplos de casos prácticos de uso en los campos del reconocimiento de patrones y de la toma de decisiones. Creemos que el marco teórico general y las definiciones proporcionadas por nuestro trabajo abren vías interesantes para la investigación futura de estas técnicas y para una mejor comprensión tanto de los beneficios como de las limitaciones de las técnicas difusas basadas en valores múltiples de pertenencia.

9 Conclusions

In our research of the different types of multivalued fuzzy sets, we have paid special attention to identifying the key aspects that differentiate the various approaches while also seeking to clarify the parts that they have in common. An important result that we have obtained is the fact that fuzzy multisets can support, alongside the traditional operations due to S. Miyamoto, other forms of basic operations like the aggregate intersection and union, which we have introduced. We have proved that these aggregate operations are a general multiset-based case of the operations for hesitant fuzzy sets.

Our study of set-based and multiset-based fuzzy sets has been complemented with an in-depth analysis of what we have called ordered fuzzy multisets, where membership grades are n-tuples. While the concept is not new, we have linked it to the approaches based on multisets and sets in an original way by investigating what happens when order in n-tuples is ignored, which gives rise to multisets, or when repetition is ignored, which makes multisets degrade into sets. The mechanism we have used to disregard order and repetition has consisted in defining equivalence relations that equalise either those multisets with the same support or those n-tuples that share the same coordinates under different permutations. These equivalence relations have allowed us to define sets as a partition of multisets and multisets as a partition of n-tuples and then go on to study how operations on these different types of fuzzy membership grades are interrelated. We have followed a two-step approach where we first study the sets, multisets and n-tuples as basic mathematical objects and then transpose the results to the use of these quantities as membership grades for the corresponding types of multivalued fuzzy sets.

In order to extend operations from the simpler mathematical spaces to the compound spaces made up of collections of objects and fuzzy sets, we have managed to identify some commonly used rules as extension principles that can be seen as a toolbox of mechanisms that allow us to translate the concepts from the ordinary fuzzy sets to the multivalued types and also across the different types of multivalued fuzzy sets. We have also studied how these transformations imply a difference in information content, which we have quantified in the form of several entropy measures.

Finally, the approach based on extension principles has been used to provide new definitions for the concepts of distance, dissimilarity and divergence in multivalued fuzzy sets. In order to do this, we have first introduced new definitions for L-fuzzy sets, which we have then generalised by means of the extension principles laid out in the initial chapters. This has led to what we believe is an elegant conceptualisation of these ideas in a very general setting, which can help to test practical applications of these fuzzy techniques. In particular, the use of divergence measures for ordered fuzzy multisets has been shown through a couple of examples of practical use cases in the fields of pattern recognition and decision-making. We believe that the general framework and definitions provided by our work opens up interesting paths for future research of these approaches and for a better understanding of both the benefits and the limitations of fuzzy techniques based on multiple membership values.

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