Universidad de Oviedo
Universidá d'Uviéu
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Estudio Generalizado de Análisis Complejo de la Teoría de Líneas de Transmisión y su Aplicación a Sistemas Electromagnéticos Reales

Pablo Vidal García

Programa de Doctorado en
Tecnologías de la Información y Comunicaciones en Redes Móviles

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Justificación

La calidad técnica e innovación del trabajo realizado por el autor de la tesis están refrendados por diferentes actividades y publicaciones científicas de ámbito internacional y nacional:

PUBLICACIONES
Revistas:
[1] P. Vidal-García, E. Gago-Ribas, "A Logarithmic Version of the Complex Generalized Smith Chart", Progress in Electromagnetic Research Letters, Vol. 68, pp. 53-58, 2017.
(Available online in: http://www.jpier.org/PIERL/pier.php?paper=17022009).

Congresos:
Internacionales:
[1] E. Gago-Ribas, P. Vidal-García, J. Heredia Juesas, "Complex Analysis and Parameterization of the Lossy Transmission Line Theory and its Application to Solve Related Physical Problems", Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA'15), available at IEEE Xplore Online in https://ieeexplore.ieee.org/document/7297091, pp. 141-144, Oct. 2015.
[2] P. Vidal-García, E. Gago-Ribas, "Complex Analysis of the Transmission Line Theory: Anslytical Characterization and Examples of Use", Proceedings of Progress in Electromagnetic Rerearch Symposium 2016, available at IEEE Xplore Online in https://ieeexplore.ieee.org/document/7735278, pp. 3262-3266, Oct. 2016.
[3] P. Vidal-García, E. Gago-Ribas, "The Logarithmic Generalized Smith Chart: Theoretical Analysis", Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA'17), available at IEEE Xplore Online in https: //ieeexplore.ieee.org/document/8065638, pp. 1766-1769, Oct. 2017.
[4] P. Vidal-García, E. Gago-Ribas, "The Logarithmic Generalized Smith Chart: Examples of Use", Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA'17), available at IEEE Xplore Online in https://ieeexplore. ieee.org/document/8065636, pp. 1758-1761, Oct. 2017.
Nacionales:
[1] P. Vidal-García, E. Gago-Ribas, "A Generalized Complex Transmission Line Theory: Characterization and some Examples", XXXI National Symposium of the International Union of Radio

ESTANCIAS EN OTROS GRUPOS DE INVESTIGACIÓN:
Nacionales:
a) Polytechnic University of Valencia
1. (03/26/2018 - 05/07/2018); Introduction to numerical analysis in waveguides; at Instituto de Telecomunicaciones y Aplicaciones Multimedia (iTeAM).
2. (05/25/2018 - 07/04/2018); Numerical algorithms for the resolution of scattering problems in waveguides; at Instituto de Telecomunicaciones y Aplicaciones Multimedia (iTeAM).
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RESUMEN DEL CONTENIDO DE TESIS DOCTORAL

1.- Título de la Tesis	
Español/Otro Idioma: **Estudio Generalizado del Análisis Complejo de la Teoría de Líneas de Transmisión y su Aplicación a Sistemas Electromagnéticos Reales**	Inglés: **Generalized Study of the Complex Analysis of the Transmission Line Theory and its Application to Real Electromagnetic Systems**

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RESUMEN (en español)

La presente Tesis Doctoral se centra en el desarrollo y uso de una novedosa metodología basada en el Análisis Complejo de sistemas guiados. Para tal fin, se han propuesto algunas definiciones basadas en el uso de expresiones complejas para su posterior aplicación al estudio de Líneas de Transmisión (LdT).
En este sentido, la Teoría de Líneas Transmisión (en inglés, *Transmission Line Theory,* TLT) ha tenido que ser reinterpretada desde el punto de vista del Análisis Complejo, y en concreto en el uso de representaciones en planos complejos –en lugar de expresiones complejas– así como transformaciones complejas entre ellos –en lugar de funciones complejas o identidades complejas entre los parámetros bajo estudio. Tanto el cambio de concepto de análisis como la obtención de las interpretaciones físicas asociadas al mismo dan lugar a la Teoría Compleja de Líneas de Transmisión (en inglés, *Complex Transmission Line Theory,* CTLT). Esta CTLT va asociada a una versión de la TLT, la cual puede ser planteada entre (i) los diferentes modos que se propagan, (ii) las características del medio donde se propagan, y (iii) los diferentes dominios de estudio. Puesto que la CTLT está planteada como alternativa a la usual (y particular) TLT, su análisis puede ser dividido en diferentes versiones, todas ellas orientadas a ser analizadas con técnicas en variable compleja.

En concreto, la Teoría de Líneas de Transmisión con Pérdidas (en inglés, *Lossy Transmission Line Theory,* LTLT), ha sido estudiada de forma rigurosa en el Cap. 2 empleando coordenadas generalizadas. Esta LTLT plantea el estudio de (i) ondas planas harmónicas (en inglés, *harmonic plane waves*, HPW) en (ii) medios conductivos y dispersivos reales, analizados en (iii) el dominio de la frecuencia. Esta última clasificación permite el uso del Análisis Complejo en la versión de la CTLT asociada a la LTLT, analizada en el Cap. 4.
En este caso, los parámetros de línea se obtienen integrando los campos en el transverso del sistema de coordenadas generalizadas propuesto, generalizando así la geometría del sistema bajo estudio. A partir de los parámetros de línea, los parámetros básicos y de onda se analizan gráfica y geométricamente en los planos complejos asociados en la llamada caracterización directa de la CTLT.

No obstante, teniendo en cuenta que lo que se desea es un estudio generalizado de la TLT, una versión generalizada de la misma (en inglés, *Generalized Transmision Line Theory*, GTLT-v1) se plantea en el Cap. 3 con el propósito de abarcar el análisis de diferentes modos. Para ello, se considera que los parámetros básicos son los mismos que describen a cada modo, para ser luego “mapeados” en parametrizaciones complejas de los parámetros de línea usados en el consiguiente análisis en la CTLT. Esa forma de caracterizar los parámetros es llamada caracterización inversa (de los parámetros básicos a los parámetros de línea).
A continuación, la GTLT-v1 se particulariza al estudio de HPWs dando lugar a las mismas parametrizaciones que predecía la LTLT, pero facilitando en gran medida el proceso de obtención de éstas.

En la definición de estas caracterizaciones, un “espacio algebraico de parametrizaciones” aparece naturalmente en el análisis. Las transformaciones desde/hasta este “espacio” son las que dan lugar a las caracterizaciones directa/inversa en la CTLT bajo estudio. Los análisis que se llevan a cabo muestran que este “espacio de parametrizaciones” alberga todas aquellas parametrizaciones (vistas como diferentes curvas) que caracterizan la GTLT-v1 estudiada en el contexto de CTLT.
Además, ambas caracterizaciones son necesarias para completar los análisis en función de las pérdidas/frecuencia y a lo largo de la línea, presentados como ejemplos de uso del análisis complejo de LdTs en el Cap. 5.

La metodología sobre la que se sustenta la CTLT es propicia para el análisis de más modos obtenidos a partir de la GTLT-v1 y futuras versiones de la TLT.

RESUMEN (en Inglés)

The present Doctoral Thesis is focused on the development and use of a novel methodology of analysis of EM guided waves based on Complex Analysis. For this purpose, those required analytical resources are expressly defined and rigorously described within the scope of complex numbers applied to the analysis of equivalent Transmission Lines (TL).
Thus, the Transmission Line Theory (TLT) is conveniently reinterpreted so that it is characterized from the Complex Analysis point of view by means of complex graphs –instead of closed-form complex expressions– as well as complex transformations between planes –instead of complex functions or simply complex identities between the parameters in use. Both the change on the mindset this analysis supposes and the obtaining of the related physical interpretations leads to define the Complex Transmission Line Theory (CTLT). This CTLT should be associated to a specifically defined TLT, which may be posed among (i) different mode solutions, existing in (ii) different lossy media, which may be studied in (iii) different domains. Due to the capabilities of the CTLT as alternative analysis of the usual (and particularly defined) TLT, the “General TLT” may be split into different versions oriented to the complex analysis in the associated version of the CTLT.

In particular, the Lossy Transmission Line Theory (LTLT) has been rigorously studied in Chpt. 2 using a generic orthogonal coordinate system. This LTLT is posed for describing the behavior of (i) harmonic plane waves (HPW), which propagate in (ii) conductive and dispersive media, analyzed in (iii) the frequency domain. It is this latter assumption the one which allows for using the complex analysis in the associated CTLT (denoted as CTLT-v1) presented in Chpt. 4.
In this case, the line parameters are obtained integrating the fields on the transverse of the generic coordinate system, so the geometry of the guided system is addressed generalized in the equivalent TL. From these line parameters, the basic and wave parameters are graphically and geometrically analyzed as complex transformations in their respective complex planes in the so called direct characterization of the CTLT.

Nevertheless, keeping the idea of generalizing the study of the TLT, “A Generalized version of the Transmission Line Theory” (denoted as GTLT-v1) is posed in Chpt. 3 in order to analyze (i) different types of waves/modes (not only HPWs) under the same frame, while (ii) considering arbitrary losses in (iii) the frequency domain. This characterization starts considering that the basic parameters are known (they are the same as those for each of the mode solutions) to be then “mapped” into complex parameterizations of line parameters resulting useful for the subsequent analysis in the CTLT. Since the way of analyzing the TL parameters is reverse (from the basic parameters to the line parameters), the corresponding analysis is called the inverse characterization of the CTLT.
Then, the GTLT is particularized to the study of HPWs leading to the same parameterizations of the LTLT and so proving the usefulness of the GTLT for characterizing specific cases, whose analysis are much more straightforward than the those in the LTLT.

As a consequence of these characterizations, an algebraic “space of parameterizations” naturally appears. The transformations from/to this “space” lead to the direct/inverse characterizations of the CTLT. The subsequent analysis show that this “space” contains all the parameterizations (seen as curves) that completely characterize the GTLT-v1 in both directions –direct and inverse– in the context of the CTLT.
In addition, both characterizations are required to finally complete the analysis of HPWs in terms of losses/frequency and along the TL, presented as examples of use of the complex

analysis in Chpt. 5.

The methodology underlying the CTLT is adequate for analyzing more mode solutions characterized by the GTLT-v1 and future versions of the TLT.

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Resumen

La Teoría de Líneas de Transmisión (en inglés, *Transmission Line Theory*) desempeña un papel fundamental tanto en el ámbito académico como en ámbito profesional dentro de la Ingeniría de microondas y radiofrecuencia (RF) para el análisis e interpretación de las características de propagación de ondas electromagnéticas (EM) en sistemas guiados. Aunque el objetivo de la TLT sea caracterizar el comportamiento de ondas ondas equivalentes de tensión y corriente en modelos de Líneas de Transmisión (en inglés, *Transmission Lines*, TLs), son muchos los autores que evitan analizar la TLT de manera rigurosa en medios con pérdidas arbitarias, así como generalizar su estudio a diferentes modos EM en el mismo marco de análisis. Sin duda, evitar el análisis de las pérdidas y particularizar el análisis a una solución simplifica el análisis general, pero ello también limita la descripción y aplicación de la TLT, llevando en todo caso a analizar casos particulares y aproximaciones en lugar de plantear el análisis de forma rigurosa y general desde el principio.

La Teoría Compleja de Líneas de Transmisión (en inglés, *Complex Transmission Line Theory*, CTLT) surge como alternativa al conocido análisis de TLs con el fin de superar las limitaciones de la clásica TLT. La CTLT está basada en el Análisis Complejo, del cual toma las definiciones de números complejos y la representación en planos complejos, que se asocian a los parámetros bajo estudio en la TLT. Esto permite: (i) una representación directa de los parámetros de la TL cuando éstos se caracterizan por medio de las parametrizaciones de interés (p. ej., las parametrizaciones de pérdidas, la asociada a la longitud de la TL, etc.); y (ii) el consiguiente análisis geométrico de las curvas planas bajo estudio. Además, debido a que los parámetros de la TL están relacionados por medio de expresiones y funciones complejas, diferentes transformaciones complejas (*mappins*) entre los planos con las curvas parametrizadas están caracterizadas en la CTLT.

La presente Tesis se centra en el desarrollo de todos los posibles análisis en la CTLT que permitan generalizar el estudio de ondas guiadas analizadas en una versión de la TLT que la acompaña. Para tal fin: (i) la Teoría de Líneas de Transmisión con Pérdidas (en inglés, *Lossy Transmission Line Theory*, LTLT) para el estudio de ondas planas harmónicas (en inglés, *harmonic plane waves*, HPWs) ha sido planteada de forma rigurosa desde el principio. Para ello ha sido necesario obtener las ecuaciones del telegrafista directamente desde las ecuaciones de Maxwell (este proceso se ha denominado "caracterización directa") parametrizando la influencia de las pérdidas y las condiciones de contorno de estructuras que soportan este tipo de soluciones en la TL equivalente; (ii) puesto que este proceso no es eficiente por implicar el desarrollo un equivalente circuital particularizando a un solo tipo de modos (HPWs), se define la Teoría Generalizada de Líneas de Transmisión (en inglés, *Generalized Transmission Line Theory*, GTLT) con el objetivo de aunar todas las posibles soluciones de un sistema guiado estudiado en el dominio de la frecuencia bajo el mismo marco de análisis teórico. En este sentido, no se trata de parametrizar en una TL cada posible modo en una teoría independiente sino utilizar una expresión (compleja) común para los parámetros básicos de la TL equivalente que implique unos prámetros de línea (también complejos) sobre los cuales se pueda mapear cualquier modo que se propague (este proceso es denomiado como "caracterización inversa"). Esta caracterización inversa se ha utilizado para esudiar el caso de HPWs como en la LTLT, con el fin de poder probar su validez para el análisis complejo posterior; y (iii) las caracterizaciones directa e inversa de la LTLT se interpretan en el contexto del análisis comlejo de TLs con pérdidas, dando lugar a su completa caracterización para pasar a establecer la primera versión de la CTLT (CTLT-v1).

Como consecuencia de los análisis realizados, la CTLT, lejos de resultar ser una teoría cerrada para estudiar un tipo particular de modos guiados, se contempla como una metodología de análisis para cualquier tipo de modos dentro de un mismo marco. Con ello, aunque el proósito de la CTLT es el mismo que el de la TLT que la acompaña, el concepto de análisis cambia completamente, y como consecuencia las interpretaciones físicas asociadas y los posibles usos prácticos de ésta última aparecen de forma intuitiva en la CTLT asociada gracias a los análisis de tipo gráfico y geométrico.

Conclusiones

La Teoría Compleja de Líneas de Transmisión (CTLT) ha sido presentada en esta Tesis como una eficaz metodología de análisis alternativa a la descripción en la clásica Teoría de Líneas de Transmisión (TLT), al mismo tiempo que ha sido provista de aquellas definiciones basadas en el análisis complejo útiles para extender la caracterización a cualquier versión que se obtenga la misma. En concreto, se ha ejemplificado el uso de los análisis complejos y de todos aquellos recursos definidos de forma general en la primera versión de la CTLT (CTLT-v1), la cual ha sido planteada para reinterpretar las soluciones en la Teoría de Líneas de Transmisión con Pérdidas (LTLT) y analizarlas en profundidad poniendo especial énfasis a su sentido físico.
Tal y como se dice y se ha presentado en la Tesis, la CTLT no debe quedar relegada a estudiar un tipo concreto de soluciones y por lo tanto aplicada a una TLT en particular, sino que ésta debe poder ser aplicada para estudiar cualquier tipo de sistema EM, tanto propuesto de forma teórica desde el principio como modelado de forma teórica a partir de un sistema EM real. Ésto, sin duda, supone un objetivo ambicioso –sujeto a la capacidad de analizar cada una de las posibles soluciones e interpretarlas físicamente– pero alcanzable en las práctica con las definiciones que en la Tesis se han dado.

La correcta interpretación de la CTLT requiere un "cambio de mentalidad" a la hora de abordar los análisis. Para explicar este nuevo punto de vista, en la Tesis se detallan las definiciones, se recurre a gráficos y esquemas, se presentan ejemplos de uso y se resalta cada una de las partes que implican un cambio en la perspectiva de análisis. Entre los cambios a la hora de abordar el análisis se destaca que:

(i) Las soluciones en forma de ondas de tensión y corriente equivalentes en la Línea de Transmissión (TL) vienen completamente determinadas de forma única en términos de los parámetros de la misma. Por ello, **los parámetros que definen la TL se convierten en el principal objeto de estudio**. Esto trae consigo la importante e intrínseca ventaja de estudiar los límites en la definición de estos parámetros, algo que no se ve directamente en las expresiones matemáticas de las ondas equivalentes.

(ii) Con esta idea de caracterizar los parámetros de la TL en lugar de ofrecer sencillamente las soluciones, se hace necesario un análisis simultáneo de todos estos parámetros con el objetivo de ver su comportamiento en conjunto, previo a ser analizados en su sentido físico. Este hecho introduce inherentemente la idea de observar las transformaciones entre los parámetros involucrados en el análisis. Por ello, la CTLT se centra en el estudio de las transformaciones entre los parametros de la TL.
Cada una de las versiones que surgen de la CTLT se considera completamente caracterizada de forma análitica cuando todas las transformaciones entre los parámetros de la TL están analizadas. Estas transformaciones pueden ser vistas:

(ii.1) analíticamente como transformaciones complejas;

(ii.2) gráficamente entre los planos asociados a cada parámetros; y

(iii.3) geométricamente como curvas planas parametrizadas de igual forma (p. ej. las curvas parametrizadas por la parte real y la parte imaginaria de la impededancia de onda en su plano asociado y las curvas parametrizadas de la misma forma en el plano del coeficiente de reflexión, es decir la Carta de Smith Generalizada)

Para el propósito de definir todas las transfromaciones desde estos tres puntos de vista, es necesario disponer de los dominios o espacios de parámetros donde tales transformaciones operan. Hay dos formas de tratar esta problemática:

a. usando los planos complejos asociados a los parametros que aparecen naturalmente cuando se trabaja con expresiones complejas de los mismos (p. ej. el plano complejo asociado a la impedancia característica), o

b. definiendo de forma adecuada nuevos dominios o espacios de parámetros de forma analítica (con el álgbra asociado), gráfica y geométrica (p. ej. el plano rg que contiene las parametrizaciones asociadas a los parámetros de línea).

(iii) A parte de la definción analítica de las transformaciones entre los planos asociados a los parámetros y su estudio de forma simultánea, las interpretaciones físicas de los análisis quedan en este punto aún pendientes. Por este motivo, **La CTLT se considera únicamente completa cuando se seleccionan las curvas con un significado físico específico en el plano de parametrizaciones (el plano rg) para ser luego transformadas al resto de planos asociaados al resto de los parámetros de la TL**. La selección de estos parámetros atiende a dos motivos fundamentales:

(iii.1) describir los parámetros de la TL físicamente, es decir, estudiar su comportamiento en función de los parámetros físicos del sistema guiado que la TL parametriza (p. ej. las pérdidas de la guía, dimensiones de la misma, etc.), y

(iii.2) describir las soluciones en términos de las variables de estudio: tiempo[1] y longitud en la dirección de propagación; de alguna forma por medio de los parámetros en uso.

En muchas ocasiones, estas caracterizaciones pueden llevarse a cabo de forma simultánea (p. ej. el análisis a lo largo de la TL y términos de las pérdidas o la frecuencia pueden ser tratados a la vez por medio de las parametrizaciones de los ángulos de los parámetros básicos).

(iv) Con el objeto de usar parametrizaciones complejas, en planos complejos, en curvas planas, la adecuadas normalizaciones de los parámetros de la TL han sido escogidos dependiendo del tipo de análisis que se lleve a cabo (p. ej. para el análisis de los parámetros en función de las pérdidas, las normalizaciones se escogen con respecto a caso sin pérdidas). De este modo, **las normalizaciones llevan a "universalizar el comportamiento de los parámetros**, lo que significa agrupar los parámetros en función de las parametrizaciones usadas (anáiticamente esto significa definir clases de equivalencia), representando todos los posibles valores de un parámetro en un mismo punto, y utilizando parametrizaciones para representar las curvas "universales" que describen cada parámetro en el plano.

Bajo estas consideraciones, la CTLT-v1 ha sido planteada de forma satisfactoria en la Tesis. Respecto a esta versión, detacan las siguientes conclusiones:

(i) La LTLT sobre la que se apoya la CTLT-v1 permite obtener los paramétros bajo estudio y cómo estos están conectados. No obstante, se ha apercibido que la manera más eficiente de obtener las relaciones para el análisis complejo de los parámetros de la TL no es utiliando el orden típicamente empleado: (desde los) parámetros de línea → parámetros básicos → (hasta los) parámetros de onda (la llamada "caracterización directa"); incluso para la que se supone la caracterización más sencilla (HPWs, dentro de los modos TEM); ni tampoco resulta ser la de caracterizar los parámetros de forma inversa: (hasta los) parámetros de línea ← parámetros básicos ← parámetros de onda (la llamada "caracterización inversa"). Esta última forma de caracterizar los parámetros, aunque resulta útil para analizar un sólo modo (p. ej. HPW), no resulta eficiente para obtener los parámetros asociados a un conjunto de soluciones.
La manera más eficiente de obtener los parámetros de la TL para su análisis complejo consiste en combinar las caracterizaciones directa e inversa de la siguiente forma: parámetros de línea (generalizados) ← parámetros básicos → parámetros de onda[2]. De esta forma, cada modo es mapeado de forma inversa sobre los parámetros de línea (generalizados), para así proceder con la caracterización directa hasta obtener los parámetros de onda. Así, **no**

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[1] La forma de parametrizar el tiempo es, de forma indirecta,por medio de la frecuencia, caracterizando los parámetros de la TL equivalente en el llamado "análisis a frecuecia variable"

[2] El procedimiento completo sería:

parámetros de línea (generalizados)	←	parámetros básicos (desde funciones potenciales)		
↓				
parámetros de línea (particularizados)	→	parámetors básicos	→	parámetros de onda.

solo los parámetros constitutivos sino también las condiciones de contorno se incluyen como parte de los parámetros de línea (generalizados), y por eso el análisis en términos de las parametrizaciones de las pérdidas cobra espacial relavancia en la Tesis para su posterior uso en nuevas versiones de la CTLT.
Y además, **las parametrizaciones de la CTLT adquieren una importancia aún mayor al poder tratar las variables bajo estudio como parte de éstas** de dos formas diferentes:

(i.1) limitando el análisis a dominios específicos de trabajo, lo cual implica operar en el subespacio que forman los coeficientes que expanden las soluciones en cierta base (p. ej. la expresiones en el dominio de la frecuencia pertenecen al conjunto de coeficientes cuando las funciones exponenciales del tiempo, $e^{j\omega t}$, son seleccionadas para formar un conjunto base); y/o

(i.2) complexificando y/o geometrizando el problema bajo estudio (p. ej. el ángulo de la constante de propgación, φ_γ, determina la variación de los parámetros a lo largo de la línea, por lo que el módulo de este parámetro puede utilizarse para referenciar la longitud de la misma).

Teniendo en cuenta este análisis combinado de caracterizaciones, la primera versión de la Teoría Generalizada de Líneas de Transmissión (GTLT-v1) surge como la teoría más eficiente a la hora de apoyar los análisis complejos en la CTLT. Esta GTLT no trata de resolver las ecuaciones originales sino exclusivamente caracterizar los parámetros básicos que describen las soluciones en función de las pérdidas, condiciones de contorno, etc. En particular las HPWs han sido el objeto de estudio de la GTLT-v1, probando que la caracterización inversa resulta igualmente efectiva a la carcterización directa de este tipo de soluciones en la LTLT. Además, esta caracterización inversa ha demostrado ser especialemente útil a la hora de encontrar parametrizaciones de las pérdidas basadas en el uso de la fase de los parámetros básicos.

(ii) Las curvas en el plano rg (que no es Euclídeo), y en general el "espacio de parametrizaciones" definido y explicado de forma algebraica, gráfica y geométrica, definen por completo aquellas transformaciones de interés en la CTLT:

(ii.1) Las curvas de r y g constantes en el plano rg[3] definen las parametrizaciones de pérdidas en el análisis a frecuencia fija.

(ii.2) Las curvas con módulo (ω_n) constante en el plano rg y las curvas con fase (θ_c) constante en este mismo plano definen las parametrizaciones utilizadas en el análisis en frecuencia variable.

(ii.3) Diferentes conjuntos de hipérbolas en el plano rg definen las parametrizaciones complejas (partes real e imaginaria y módulo y fase) de los parámetros básicos.

Como consecuencia de este análisis, **el plano rg reune todas las parametrizaciones de interés en la CTLT para la descripción completa de los parámetros de la TL a caracterizar**.
Es importante notar que las parametrizaciones complejas son tratadas como una única en lugar de dos divididas en su parte real e imaginaria o su módulo y fase. Esto es así porque siempre existe un parámetro físico en los análisis en la CTLT que relacione estas a priori separadas parametrizaciones (p. ej. las parametrizaiones complejas en el plano del coeficiente de reflexión están relacionadas por medio de la longitud de la TL para equellos análisis a lo largo de su extensión).

Un importante resultado final sobre los análisis de las transformaciones es que **las fases de los parámetros básicos, φ_{Z_0} y φ_γ, determinan por completo el comportamiento de los parámetros de la TL en términos de las pérdidas/frecuencia y a lo largo de**

[3]Puesto que el plano rg es no Euclídeo, estas curvas no se representan como paralelas en el "espacio de parametrizaciones".

la longitud de la misma, respectivamente, siempre que las normalizaciones apropiadas hayan sido escogidas dependiendo del tipo de análisis que se lleve a cabo. Esto es debido a que los ángulos "sobreviven" a las normalizaciones de los parámetros de la TL, que se hacen con respecto a valores reales.

En el mismo sentido, **el coeficiente de reflexión, ρ, es el único parámetro que se mantiene "intacto" tras normalizar el resto**. Por ello, ρ, es el parámetro que describe los análisis en función de las pérdidas, la frecuencia y a lo largo de la línea al mismo tiempo sin re-escalarse. Esto hace a la Carta de Smith Generalizada (el plano ρ parametrizado por las partes real eimaginaria de la impedancia/admitancia) y la Carta de Smith Generalizada inversa (el plano ρ parametrizado por las partes real e imaginaria de la impedancia característica) las herramientas gráficas más útiles a la hora de tratar con la dualidad que supone analizar la TL a lo largo de su longitud y en función de las pérdidas/frecuencia.

> **Importante.** *Los planos que universalmente describen todas las parametriazaciones reunidas y que son verdaderamente útiles para el cometido de la CTLT de caracterizar físicamente la TL son el plano rg y el plano ρ.*

En conclusión, la CTLT tal cual ha sido presentada y ejemplificada en la presente Tesis supone una forma alternativa y novedosa de visualizar la TLT, así como para generalizar a la misma. La metodología de análisis basada en análisis complejos que usa la CTLT resulta verdaderamente intuitiva una vez se ha cambiado la percepción del problema que aquí se explica. Los análisis gráficos apoyan esta nueva idea de análisis a la par que permiten explicar cómo resolver problemas relacionados con líneas.

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Generalized Study of the Complex Analysis of the Transmission Line Theory
and its Application to Real Electromagnetic Systems

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This work has been typeset manually using commands of MiKTEX 2.9; a distribution of TEX/LATEX for Microsoft Windows.
The graphs have been plotted and labeled using AutoCAD 2009.
The curves of the graphs have been generated using MATLAB R2016a.

A Emilio,
quien ha sido y es
mi madre y padre, y hasta mi hermano
en esto.

También
a mi madre, a mi padre, y a mi hermano
en el resto.

"Equations are just the boring part of mathematics.
I attempt to see things in terms of geometry."

— Stephen Hawking
(from *Stephen Hawking: A Biography*, 2005)

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Abstract

The *Transmission Line Theory* (TLT) plays a fundamental role in both academia and the professional background of RF and microwave engineering for the analysis and interpretation of propagative EM waves in guided systems. While the main objective of the TLT is to characterize the propagative behavior of equivalent voltage and current waves in Transmission Lines (TLs), many authors avoid dealing with a TLT that rigorously analyzes those waves in lossy mediums, as well as generalizing the study to different EM modes under the same theoretical framework. Although avoiding losses and studying different solutions one by one simplifies the general analysis, it clearly limits the understanding and applications of the TLT to characterize specific modes which propagate in lossless media, leading to use particular cases and approximations instead of posing the problem rigorously and generalized from the beginning.

The *Complex Transmission Line Theory* (CTLT) arises out as the alternative for the TL analysis with the purpose of overcoming the limitations in the usual TLT. The CTLT is based on the widely-known Complex Analysis, from which it takes the complex number definitions and also advantage of the representation of the parameters that characterize the TL in associated complex planes. This leads to: (i) the intuitive graphical representation of the TL parameters when those are characterized by the parameterizations which are interesting to be rigorously interpreted in the TL analysis (e.g. lossy parameterizations, the length of the TL, etc.); and (ii) the subsequent geometrical analysis of the parameterized plane curves. In addition, since the TL parameters are mutually interconnected, the complex transformations (mappings) between them are defined from complex parameterizations and characterized within the CTLT.

The present Thesis is focused on developing all the possible analysis within the CTLT that lead to generalize the study of guided waves belonging to the underlying TLT. For this purpose: (i) the *Lossy Transmission Line Theory* (LTLT) regarding harmonic plane waves (HPWs) has been posed rigorously from the beginning, which means obtaining the equivalent *telegraher's equations* directly from the original *Maxwell equations* (this process is named as direct characterization) parameterizing the influence of losses and boundary conditions of the structures which support these waves into the line parameters of the equivalent TL; (ii) since the direct characterization of the equivalent TL is not efficient in the sense that it only accepts one type of solutions (HPWs), the *Generalized Transmission Line Theory* (GTLT) is defined with the objective of gathering all the possible solutions which may be complex parameterized in the *frequency domain* under the same theoretical framework. In this sense, it is not about solving each possible mode which propagates in a waveguide but obtaining a expression of the basic parameters of these solutions in terms of constitutive parameters, frequency, etc. in order to be then parameterized into the line parameters (this process is named as inverse characterization). This inverse characterization is particularized to the case considered in the LTLT (involving HPWs); and (iii) both the direct and inverse characterizations regarding the LTLT are interpreted within the context of the *Complex Transmission Line Analysis* (CTLA) of lossy TLs, leading to the their complete characterization and so founding the first version of the CTLT (CTLT-v1).

Far from being a closed theory, the CTLT just as it is introduced in the Thesis represents a methodology of analysis to be expanded for studying different propagative solutions under the same framework. Thus, while the purpose of the CTLT keeps the same as in the original TLT, the concept of analysis completely changes. As a consequence, the physical interpretations and practical uses of the underlying TLT appear intuitively in the associated CTLT thanks to the graphical and geometrical analysis.

List of Acronyms

TL	Transmission Line
TLT	Transmission Line Theory
LTLT	Lossy Transmission Line Theory
GTLT	Generalized Transmission Line Thory
CTLA	Complex Transmission Line Analysis
CTLT	Complex Transmission Line Theory
SST	Signals and Systems Theory
GSST	Generalized Signals and Systems Theory
HPW	Harmonic Plane Wave
BC	Boundary Condition
ffa	Fixed frequency analysis
vfa	Variable frequency analysis
LC	Linear Combination

Notation

Lossy Transmission Line Theory

Variables

Length from the generator	z
Length from the load	l
Time	t
Frequency[4]	ω

Functions

	Time domain	*Frequency domain*
Voltage waves	v	V
Current waves	i	I

Parameters

Line parameters				
Resistance p.u.l.[5]	R			
Conductance p.u.l.	G			
Inductance p.u.l.	L			
Capacitance p.u.l.	C			
Basic parameters				
	Lossy	Lossless	Non dispersive	Low-losses
Characteristic impedance	Z_0	$Z_{0,sp}$	$Z_{0,nd}$	$Z_{0,bp}$
Propagation constant	γ	γ_{sp}	γ_{nd}	γ_{bp}
Attenuation constant	α	α_{sp}	α_{nd}	α_{bp}
Phase constant	β	β_{sp}	β_{nd}	β_{bp}
Wave parameters				
Wave impedance	Z			
Wave admittance	Y			
Reflection coefficient	ρ			

[1]Spectral variable.
[2]Per unit length.

Generalized Transmission Line Theory

Variables

Longitudinal coordinate	z
Time	t
Frequency[6]	ω
Propagation constant[3]	γ
Transversal coordinate	$\mathbf{t} \equiv [\mathrm{t}_1, \mathrm{t}_2]$

Functions

Fields			
	Time domain	*Frequency domain*	*"Propagative" dom.*
Electric (wave)	$\mathcal{E}$	$\boldsymbol{E}$	$\boldsymbol{E_a}$
Magnetic (wave)	$\mathcal{H}$	$\boldsymbol{H}$	$\boldsymbol{H_a}$
Elec. displ. (wave)	$\mathcal{D}$	n.u./n.s.[7]	n.u./n.s.
Mag. flux dens. (wave)	$\mathcal{B}$	n.u./n.s.	n.u./n.s.

Potentials		
	Time domain	*Frequency/"propagative" domain*
Scalar electric	n.u./n.s.	ϕ_e
Scalar magnetic	n.u./n.s.	ϕ_h
Vector electric	n.u./n.s.	$\boldsymbol{A_e} \equiv A_e\hat{z}$
Vector magnetic	n.u./n.s.	$\boldsymbol{A_h} \equiv A_h\hat{z}$

Parameters

Constitutive parameters		
	Time domain	*Frequency/"propagative" domain*
Electric permittiviy	n.u./n.s.	ε_{eq}
Magnetic permeability	n.u./n.s.	μ

Line parameters	
Complex resistance p.u.l.	$\bar{R}$
Complex conductance p.u.l.	$\bar{G}$
Complex inductance p.u.l.	$\bar{L}$
Complex capacitance p.u.l.	$\bar{C}$

Basic parameters	
Characteristic impedance of ξ-mode[8]	$Z_{0,\xi}$
Propagation constant of ξ-mode[5]	γ_ξ

[3]Spectral variable.
[4]Non used/No sense.
[5]The version of the GTLT presented in the Thesis (GTLT-v1) refers to HPWs, so that $\xi \equiv$ HPW.

Complex Transmission Line Theory

Variables

Lenght from the load	l
Frequency	ω

Functions

Basic parameter funcs.	
Charact. impedance (func.)	$Z_0 \equiv Z_0(\omega)$
Prop. constant (func.)	$\gamma \equiv \gamma(\omega)$
Reflec. coeff. at the load (func.)	$\rho_L \equiv \rho_L(\omega)$
Wave parameter funcs.	
Wave impedance (func.)	$Z \equiv Z_0(l)$
Wave admittance (func.)	$Y \equiv Y(l)$
Reflec. coeff. along the TL (func.)	$\rho \equiv \rho(l)$

Parameters

Parameterizations		
	Fixed frequency analysis	*Variable frequency anlysis*
Conductor losses	r	r'
Dielectric losses	g	g'
"Dispersivity"	n.u./n.s.	c
Normalized frequency	n.u./n.s.	ω_n
Electrical length	l_e	n.u./n.s.
Normalized length	n.u./n.s.	l_n

Basic parameters				
Normalizations[9]:	Lossless	Non disp.	Modulus	Load
Characteristic impedance	Z_{0n1}	Z_{0n2}	$Z_{0n} \equiv e^{j\varphi_{Z_0}}$	Z_{0nL}
Propagation constant	γ_{n1}	γ_{n2}	$\gamma_n \equiv e^{j\varphi_\gamma}$	n.u./n.s.
Attenuation constant	α_{n1}	α_{n2}	n.u./n.s.	n.u./n.s.
Phase constant	β_{n1}	β_{n2}	n.u./n.s.	n.u./n.s.
Reflec. coeff. at the load	ρ_L	ρ_L	ρ_L	ρ_L

Wave parameters			
Normalizations[6]:	Modulus	Charact. imp. (mod.)	Charact. imp. (lossless)
Wave impedance	Z_n	Z_{n0}	Z_{n1}
Wave admittance	Y_n	Y_{n0}	Y_{n1}
Reflec. coeff. along the TL	ρ	ρ	ρ

[6]In order to have an intuitive explanation of the use of the notation of normalizations in the CTLT see Appendix 4.E.

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Chapter 1

Introduction

1.1 General introduction

The present work is intended to pose, study and use an alternative version of the ***Transmission Line Theory*** (abbreviated as TLT) for the complete characterization of real electromagnetic systems in which EM fields[1] propagate in certain way.
The TLT as it is usually presented, [Mar51, Col90, Poz98], faces this study under specific physical conditions, that is fixing the geometries and constitutive parameters (for example, waves that propagate in multiply connected regions, [BC90], bounded by PECs[2], and filled with lossless materials), as well as several assumptions (for example, zero longitudinal field components for describing TEM modes[3]).
This way to introduce the TLT is quite illustrative (it helps the conception of the problem under study, posed in the context of the TLT just as it is usually presented) but it greatly reduces the analayis to more or less particular cases given by the conditions imposed.

From this point, it is natural –under the author's point of view– questioning whether a generalized study of the TLT is possible in certain way, more or less deep (that is studying more or less cases under the same theory framework), which doubtlessly motivates the analysis presented in this Thesis book (and it also justifies part of the title of the Thesis in which "*Generalized Study*" regarding the "*Transmission Line Theory*" are keywords).
This will not be a study founded in nothing. The analysis developed from this point forward are based of the TLT, combining (i) its well referenced basis (for example those presented in [Col90]), (ii) the particular point of view regarding the TLT introduced in [Gag01], which serves as main framework of the present work, and (iii) the new studies developed throughout the Thesis period, based on the previous basis.
These basis serve to build up some of the pretended generalization on the study, going from particular cases to new general theory, something which is very interesting from the Thesis objectives viewpoint.
This way of proceeding (from particular cases to general theory) clearly has inductive nature, and therefore the general analysis adquire great importance in the context under which the Thesis has been thought.

Getting into the concepts enclosed in the TLT, it may be said that the analysis presented under this Theory suppose a generalization in themselves, as long as the TLT is framed in the appropriate context (otherwise it would be interpreted as a specific "solver", setting aside the generalization it supposes):

[1]ElectroMagnetic fields.
[2]Perfect Electric Conductor.
[3]Transversal ElectroMagnetic modes.

(i) First of all, the TLT has to be seen as the way for transforming the EM fields related through the *Maxwell equations* to equivalent voltage and current waves related by the (equivalent) *telegrapher's equations*, [Poz98].
This transformation is more or less complex depending on the "conditions" under which the study is posed. In particular, the case of characterizing the propagation of TEM modes, which is usually studied to introduce the TLT, reduces the "complixities" to the minimum, and so the *telegrapher's equations* result the easiest to be solved.
In any case, posing the (equivalent) *telegrapher's equations* means reducing the vector EM fields to scalar functions, which is always an important reduction in the "complexity" of the analysis and a crucial step towards the better understanding for the analysis of the underlying physical problem of EM propagation[4].

(ii) Keeping in mind this close relation between a certain EM problem (conditioned in different ways) and the equivalent problem which is posed throrugh the *telegrapher's equations*, it is neccesary to known what can (and thus what can not) solve the TLT. This is commonly omitted when posing the *telegrapher's equations* directly without explaining the origin of the equivalent waves and also the parameters which appear on these equations.
The TLT is intended for (a) ***explaining the physical behavior*** *of* (b) *"almost uniquely solvable"* ***propagative waves***[5].
This general definition of the purpose of the TLT encloses some concepts which are crucial for undersatanding the TLT and how the generalized versions have to be posed: in (a) it is made clear that the physical interpretations regarding the analysis in the TLT are essential. In this way, it is required to think about the "TLT related to what", instead of an "isolated TLT"[6] without a physical problem in the background; and (b) makes reference to an underlying PDE problem[7], [Eva97, Zwi97], in which the spacial dependence of the (individual) solutions representing non-static waves is the direction of propagation. It is also said for these (individual) solutions that they need to be "almost uniquely solvable", which means that there only exists a degree of freedom in the direction of propagation, besides that one in the amplitude of the waves typical when dealing with linear PDEs[8]. The total solutions are the sum of basic solutions, which concretize when the boundary conditions (BCs) are imposed a posteriori along the direction of propagation. However, this solution is given "open", in the sense that multiple solutions based on the combination of multiple individual solutions are possible, so the problem uniqueness is partial (almost verified, or quasi-uniqueness).
The description of both (a) the physical behavior and (b) all the possible solutions are the main issues of every emerging generalized version of TLT.

(iii) The propagative solutions are well defined by means of different types of parameters which have different depth in the context of the TLT. These parameters also characterize the underlying physical problem (for example, there are some parameters that characterize the losses of the medium in which the EM waves propagate).
There are always three types of parameters to deal with: the line parameters, which characterize the equivalent Transmission Line (TL). The TL is the way of representing the *Telegrapher's equtions* by means of a simplified circuit scheme that is defined differentially[9]; the

[4]Here a reference to the understanding of how the EM waves dinamically behave is outlined, which is one of the main problems the TLT has to solve.

[5]Throughout this introductory section, these concepts: the physical interpretation and the solvavbility (denoted with (a) and (b), respectively) are emphasized for founding the subsequent versions of the TLT.

[6]With this consideration it is assumed that there is not a "General Theory" which can afford any EM problem (in fact, this goes against the concept of reducing the original equations faced by the TLT), but some generalized theories which let to explain a especific physical problem in a general way –so feel the difference between "(The) General Theory" and "A Generalied Theory".

[7]Partial Differential Equation problem.

[8]*Maxwell equations* are linear equations, so every equivalent *telegrapher's equtions* also are. An interesting open problem could be affordig non linear PDEs (for example the Navier-Stokes equations, [Zwi97]) through a generalized version of the TLT.

[9]This point of view (the TL seen as an equivalent circuit) supposes one particular among the multiple possible definitions of the concept of TL.
The most general conception of a TL is related with the definition of the TLT main objective: the TL is the physical

basic parameters, which characterize each of the (individual) solutions of the TL, both statically and dynamically (cinematically)[10]; and the wave parameters, which parameterize the way of obtaining the total waves circulating in the TL.
The study of these parameters is completely equivalent to solve the propagative waves in the context of the TLT defined for one specific purpose, for example, and related to the lossy parameters, describe (which entails studying (a) the physical behavior and (b) uniqueness) the solutions in lossy TLs.

And that's all. Every study that supposes any particularization concerning the original *Maxwell equations*, the BCs imposed to solve them, the domain of study, any particular assumption as for example zero longitudinal fields, etc. is a particular case within the most generalized theory which is possible to be described[11].
In any case, the points (i-iii) described above appear common to every particular theory which may be stated, and those possible theories should be presented taking into account these premises, apart from explaining them thoroughly.

Furthermore, it obvious that the issue of generalizing any theory requires the appropriate resources to be well described. In other words, it is worthless to imagine a generalization if it is not possible to be realized (and this means both (a) interpreting the solutions physically and (b) found them according to the TLT ("almost uniqueness") standars).
This suggests explaining a methodology which is capable of describing the proposed TLT as it required to (again, (a) explaining the physical interpretations and (b) contextualizing all the possible solutions).
In this sense, there are few resources like (parameterized) functions to describe all the possible solutions of a posed problem, for which each particularization of the parameters involved in the expressions supposes a particular solution to the problem which is being studied. However, those "spreaded" solutions described by functions lack of (a) the appropriate physical interpretations and (b) the neccesary analysis of the allowed ones (a lot of times based on the physical interpretations, which are met in *Maxwell equations*).

Example 1.1.1. *It is very extended decoupling the EM fields in Maxwell equations by differentiating them. As a result, a pair of parameterized solutions are found based on the coefficients and assumptions on Maxwell equations. However, it is unknown (a) what does each of the solutions physically mean, and (b) whether all the possible waves that follow the expression described by the mathematical solutions are really possible.*
These questions have to be answered by the (right) posed TLT.

These facts make clear that it is needed a new method which, accompanying the underlying expressions that give the general solutions[12], offers the appropriate physical and analytical interpretations.
In these sense, both the supposed method and the original expressions of the solutions would describe appropriately the TLT which is being presented, so they are part of the seeked methodology.
In fact, as it is suggested in the point (iii) itemized above, the characterization of the TL parameters[13] suffices to characterize the solutions completely, since it is well known the role of each parameter in the analytical expression of these solutions.

support for waves to propagate.

[10]The "statical characterization" refers to the study of the behavior of the solutions at any fixed point along the TL, which affects the characterization of the characteristic impedance of the individual solutions, whereas the "dynamical characterization" (actually, "cinematically") refers to the study of the behavior of the solutions along the TL with respect to the time (or any parameter that represents the time variation, e.g. the frequency), which affects the characterization of the propagation constant of the individual solutions.

[11]In fact, this supposed "General Theory" is not possible to be constructed and analyzed in itself and it would be set up by means of different "pieces" which represent more or less particularized theories

[12]It is not only about solving the particular solutions given by the BCs from a general solution, but also interpreting them according to their physical meaning in the problem they model.

[13]The name "TL parameters" refers to the three sets of parameters: line parameters, basic parameters, and wave parameters; in general.

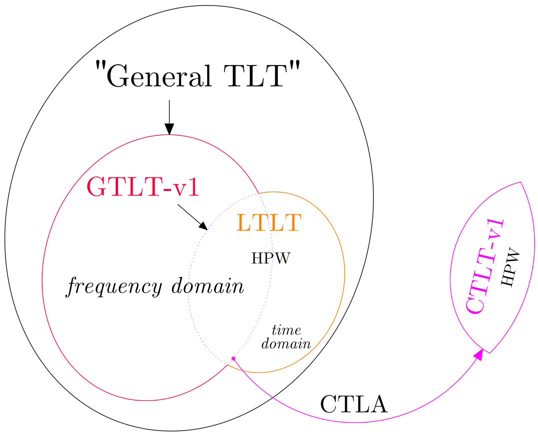


Fig. 1.1: Scheme of different versions within the supposed "General TLT", which gathers every possible TLT, including the GTLT-v1 (a particularization when considering time harmonic regime) and the LTLT (a particularization when HPWs are supposed). The CTLT-v1 refers to the complex analysis (CTLA) of HPWs in the *frequency domain*.

For the issue of characterizing the TLT properly, the ***Complex Transmission Line Analysis*** (CTLA), [Gag01], appears as the method which supports the analysis of the underlying TL equations.
This method of analysis is inspired on some of the resources of Complex Analysis, [BC90]; in particular the graphical representation in complex planes associated to each (complex) parameter under study adquires great importance in the TL analysis.
Assuming the TL parameters are complex is a relatively great particularization[14] from the (supposed and not realizable) "General Analysis of the TLT", although it is perfectly justified, also from the physical and analytical points of view: the use of complex expressions (in the space of complex functions of real variables) comes from considering the solutions associated with a frequency (ω) which characterizes the time harmonic regime. This regime is, in turn, associated with complex exponentials $e^{j\omega t}$, which are eigenfunctions of time derivative operators in the space of complex functions, so analytically their use makes sense for easing the representation of any wave solution. Moreover, from the physical point of view, using time harmonic functions is also justified if supposing "infinite initial conditions"[15], which guarantee the problem is invariant "along the time".
In this sense, the analysis is in certain way particularized (the system is "time invariant"). However, this particularization is overcomed if accompanying the studies in the CTLA with the appropriate spectral analysis, [Her14].
Under the assuption of working in time harmonic regime, the frequency, ω, plays the role of an additional TL parameter. In particular, ω may be seen as a line parameter because its meaning when characterizing the equivalent TL circuit.

The possibility of making the analysis graphical in the context of the CTLA supposes: (i) a huge generalization in the parameterized analysis: each point of the complex planes represent at least one parameterized solution, which supposes the "universalization" of the analysis; (ii) an intuitive way to analyze complex domains of the parameterized solutions, leading to the seeked (a) physical and the subsequent (b) analytical interpretations; and (iii) the possibility of using concepts of both Complex Analysis, for example the complex transformations between planes, and Geometry, [MP77], when studying the plane curves.
These three elements of the CTLA: the "universalization" of the study, the complex and geometrical analysis; are combined to explain (a) the physical interpretations and (b) the analytical results of the TLT. Thus, the CTLA which is focused on explaining a particular TLT, establishes the

[14] This "degree of particularization" is a priori unknown, because not a shadow of knowing more possible particularizations/generalizations for the comparison is found.
[15] These "infinite initial conditions" posed on the "time coordinate" are equivalent to the infinite boundary conditions (IBC) posed on some problems in space, for example on the direction of propagation of those mediums which are object of study when seeing them as a TL, or those problems analyzed in free space.

Complex Transmission Line Theory (CTLT), which emerges to avoid the limitations in the analysis that the underlying (analytical) TLT presents.

The CTLT uses the CTLA as the method of analysis. This method may be extended for being applied to different versions of the TLT. Thus, the CTLT is not strictly a method of analysis but a methodology which uses the CTLA (and thus its resources: "universalizations", different complex analysis, and geometrical interpretations; together with the graphical analysis) for explaining the associated TLT[16].
The definition of (one of) the CTLT and the associated CTLA also motivates this thesis book (and it justifies the mention "*Complex Transmission Line Theory*" in the title). This defintion implies some degree of the seeked "*Generalized Study*".

Finally, the analysis presented in the Thessis are intended to both explaining and being used to characterize "*Real Electromagnetic Systems*". For this purpose, the parameterizations regarding this specific version of the TLT (that one that considers "harmonic plane waves" which propagate in lossy media) are intended to characterize it (in this case, the analysis are parameterized by all the possible sources of losses, frequency, and the TL's length). As a result, each "EM system"[17] which verifies the assumptions in the associated TLT may be rigorously studied by means of these analysis. The possible ***practical uses***[18] are outlined in separated paragraphs for each particular analysis presented in the Thesis.

[16]For this reason, the CTLT is classified in different versions depending on the particular version of the TLT it refers to. For example, the *Lossy Transmission Line Theory* (LTLT) is associated with the first version of the CTLT (CTLT-v1). This is better explained better then when presenting the thesis structure and objectives

[17]The concept of "EM system" may refer to: either a circuit –which could be lumped– if seeing the "EM system" and the equivalent TL –which could be studied at a fixed point– through its line parameters parameters; or a waveguide in which EM fields for different realzations of basic parameters; or even an operator, if seeing the "EM system" in the context of EM Operators and EM Function Theory, [HY02].

[18]This paragraph section: "***practical uses***"; is used throughout the thesis book together with those different sections explained in the structure presented at the end of this introductory chapter.

1.2 Thesis background and previous studies

(Where does everything come from? What really inspires the Thesis? Which analyis are brought to the Thesis?)

In this section those studies which serve as frame for the Thesis, and that clearly inspire not only the work developed throughout the Thesis period but also they settle the basis of the underlying "philosophy of analysis" induced to the Thesis are presented together with the previous studies. On its behalf, these previous studies are "recycled" because either they need to be reinterpretated within the context of the Thesis or they accompany new anaylsis to complete the methodology described in the introduction before.
This section is also intended to introduce the "way of thinking" of the author, and to see how and where the ideas move through.

The section is splitted into: (i) the background of those previous studies which, despite they are not directly related to the Thesis, they result "inspiring" for it; and (ii) the analysis which had been done prior to the beginning of the thesis period (and the author has embraced and analyzed for their reinterpretation and extension).

1.2.1 Thesis background

It is well known that Complex Analysis is a helpful tool for the analysis of different problems in Physics and Engineering, [BC90]. In particular, just an example of use of the complex expressions mentioned in the introduction before, operating in the *frequency domain* makes inherit the approach of using complex exponentials, and thus managing complex values, for example when describing the parameters of the TL.
Nevertheless, related to this point, it is important to understand the difference between the use of complex functions (of real variable), for example complex exponentials, and the study of one specific problem making use of complex variable (which, in general, leads to define complex functions). This latter case is not as common as the first one among the related disciplines, although some specific examples which have been reported, for example those introduced as applications of Complex Analysis in [BC90], are very extended and especially "solvent" when describing physical problems, as well as illustrative of Complex Analysis usefulnesses.
The problems which are described using complex variable or complex parameterizations are of special interest throughout the Thesis (in particular those which refer to EM propagation). Although the work is founded on the basis of complex functions (of real variable), the CTLA induces the use of complex analysis a posteriori.

Aimed for the idea of parameterizing and modelling the solutions of certain EM problems using complex variable, Prof. Emilio Gago-Ribas starts studying the complexification of the space of coordinates for the positioning of the sources which generate Gaussian beams (which are only an approximation of the solution of the wave equation) in free space, which may be also studied in complex coordianates when the complexification is extended to the space of propagation (in fact, these and other complex beams are explained by the analytical deformation of the 2D cylindrical wave solution when the complex extension of the coordinates is applied).
As a result, this complex extension of the a priori real space allows for generalizing different approximations under the same framework. Moreover, those approximations are (a)[19] physically explained and (b) analytically characterized, for example in their allowed regions, by means of complex geometrical parameters, for example complex distances, angles, etc., all of them with clear physical meaning.

[19] The notation "(a)-(b)" used in the introduction is kept here for emphasizing that both (a) the physical interpretations and (b) the analytical characterization are also goals of these previous studies in the background, despite the fact that there the addressed problems are of different (physical) nature from those presented in the Thesis.)

These complex beams are mainly studied for the purpose of analyzing scattering of periodic structures in EM, leading to some interesting contributions in the field, [GG97, GGD97, HF01, GG99, GG00, MGD07, MGD09].

In this point, it is important to understand the difference between, on one hand, complexifying both the solutions and equations which have been initially posed in real variable and, on the other hand, posing the problem in complex coordinates. This latter problem is not solved yet, but it is aimed for being studied under the context of the *Generalized Signals and Systems Theory* (GSST), [Her14].
The GSST is based on the *Signals and Systems Theory* (SST), [OWY97, Lin04], which has been initially posed for providing a general framework for studying the problems different branches in Engineering, such as the Communication Theory, Circuit Theory, Dynamical Systems Analysis, Digital Systems Processing (DSP), and so on.
The GSST is the way to (i) generalize and explain rigoursouly all the resources used in the SST, (ii) propose new mathematical tools which are useful for the description, resolution and parameterization of different physical problems, and (iii) support the existing theories suggesting the way to be reinterpreted from the beginning while evidencing their limitations, for example it is required a GSST on complex variable (CGSST) for analyzing rigorously both the complexifiction of real EM problems and its description and interpretation using complex variable.
Both Prof. Emilio Gago-Ribas and Ph.D. Juan Heredia-Juesas are the main promoters of the GSST for that description, resolution and parameterization of EM problems. In fact, the recent Thesis written by Ph.D. Juan Heredia-Juesas, [Her14], summarizes the previous works in the context of the GSST and provides a detailed and updated analysis of the current version of the GSST.
The most important generalizations regarding the SST achieved in the GSST are:

(i) The generalized representation of the vectorial spaces of functions following algebraic structures equipped with a "solid" definition of an inner product (dot product) which induces, in turn, well defined norms and metrics, leading to Hilbert spaces of functions[20].

(ii) Based on the representation of any function as a linear combination (which is also generalized and generically denoted as **LC**) of a basis set of functions, the thoroughly extended and known concept of "transform", e.g. the Fourier transform of continuous functions with finite norm, is explained as the projection of the function in these set of functions (in case of the Fourier transform the basis are complex exponentials[21].). As a consequence, the inverse tansform, e.g. the inverse Fourier transform, is the expansion of a function by the **LC** of the elements of the basis weighted by the coefficients of the (direct) transfom.
Since the transforms are generically defined in both directions (the projection based on the dot product and the **LC**) they are presented in the GSST as *Generalized Transforms* (GT). From this point, and based on well-known algebraic definitions, it is natural to consider how to change between the different transforms, that is how to change the coefficients which represent a function in different basis sets. The process of changing the basis has been afforded in a generic way leading to the *Generalizd Tranform Changes* (GTC).

(iii) As it is also natural in Algebra, the definition of operators lets to transform functions, which are either in the same space or different space. Linear operators, both invariant and non invariant has been studied in the context of the GSST generalizing how the coefficients change by means of the action of the operator in question, leading to the *Generalized Spectral Analysis* (GSA).

(iv) In order to overcome the limitations of: (i) including functions that are not in the original space of functions, for example the exponentials playing the role of "Fourier basis" in L^2; (ii) managing distributions and generalized functions within the context of the GSST and the defined algebra defined over the original space of functions; and (iii) "homogenize" the

[20]The "completeness" of these spaces for being considered Hilbert spaces is proved by the completeness of the metric defined from the dot product in each possible space of functions: functions of discrete or continuous real variable, in a finite or infinite dimension space, periodic or aperiodic, etc.

[21]This means a dichotomy because the exponentials which act as basis of functions with finite norm (square integrable, indeed) are not in this function space (L^2)

domain and range of operators; a new version of the GSST based on the construction of rigged Hilbert spaces (RHS), [Her14, HGV15, HGV16] is proposed and studied rigorously under the same algebraic basis of the original GSST.

In this way, the basis of a solid theory[22] –the GSST– intended to rigorously explain the problems in EM are establish. However, the studies which would explain the use of complex variable for the application in EM are still undeveloped. This is mainly because unifying an algebraic theory with Complex Analysis sets out a lot of open questions non trivial to be answered, as for example some very basic as how to deal with the **LC** in complex variable (is the Laplace Transform a representative example of this **LC**? What does having an improper integral in a complex domain really mean? And a lot more of related questions).
In this context, it is probably more efficient –and if not, it is also a good practice– having a look at the physical problems which are candidates to be studied under a possible CGSST. Among these problems are, for example, the mentioned scattering problems contextualized in this section, but also the CTLT associated with the study of the TLT using resources typical of Complex Analysis, just as it has been presented in the introductory section.
All these physical problems would serve as examples of use of the supposed CGSST, so notice that the achievement of this latter theory is much more general (because even the physical meaning is lost in favor of be generalized as only an analytical theory) and thus challenging.

Taking into account the idea presented in this summarized background, the focusing on the CTLT may be perfectly understood and justified from the point of view of the GSST. Thus, the most general objective of the GTLT, and in particular the CTLA in the CTLT, even beyond the description of a theory which serves to deal with the TLT rigorously, is to exemplify the use of the GSST and state the fundamentals giving ideas to the potential CGSST.

1.2.2 Previous works

In this section, the previous studies related to CTLA are detailed. These analysis, which are developed in [Gag01], serve as basis for the definition of the first version of the CTLT (CTLT-v1) proposed in this thesis book, which serves, in turn, to analyze rigorously –in this case– the LTLT (that is, leading to a TLT (a) physically right interepreted and (b) analytically well-characterized).

The CTLA *handbook*, [Gag01], has to be read (and the author had to be studied as the main reference in the CTLA) just as it has been written for: it is a reference of the first results regarding the CTLA, and as such it does not provide neither (a) the physical interpretation of the underlying TLT nor (b) the analytical characterizations in the context of Complex Analysis. Thus, this *handbook* does not explain the CTLT as it has been introduced and intended for analyzing the TLT. In this sense, it has been part of the author's work both: (i) completing the studies introduced in the *handbook*, and contextualizing and explaining them within Complex Analysis and the subsequent differential Geometry based on the graphical analysis presented in complex planes; and (ii) formulating the associated CTLT in terms of the CTLA; in this thesis book.
Nevertheless, the analysis presented in the *handbook* are a fundamental part of both the background of the thesis and the analysis presented in the context of the CTLT used "*de jure*".
As a part of the thesis background, the CTLA had been conceived by Prof. Emilio Gago-Ribas as a parallel research line which involves an analysis which is possibly better to be described using complex variable or complexifying the real variables in use. Back then, analyzing the more examples as possible in which Complex Analysis was present would serve to generalize a SST (or GSST) in complex variable (or CGSST), originally named as *Complex Signals Theory* (CST), [Gag09].
Nevertheless, although it could have been the starting point, Prof. Emilio Gago-Ribas realized the

[22] Base an emerging theory on Algebra is, undoubtely, one of the best ways to make this theory well founded and solid.

need of formulating the CTLA met more purposes than uniquely serving as example of use of a possible CST:

(i) From the educational point of view, Prof. Emilio Gago-Ribas appreciated the lack of a general analysis which lets to deal with all the particular TL-related problems under the same methodology. This is maily explained for "the bad" of seeing the TL-realted problems one by one instead of all the problems together seen as a continuation within the same theory.
One illustrative example of this "short-sighted" way on understanding the analysis is the use of the Smith Chart (SC), [Smi39, Smi44], or other related graphical tools, such as the Generalized Smith Chart, [GDG06], despite these graphs are useful tools to analyze the reflection coefficient parameterized by the impedance/admittance for any lossless or lossy TL, respectively. However, under the point of view of the CTLA, these graphical parameterized charts are only one example within the all the possible parameterized complex analysis.

(ii) The approximations regarding losses were not rigorously explained, for example the regions of valifity of the low losses approximation, because it was not obvious establishing an intuitive closed analysis of the approximations and particular cases[23] for obtaining the differences, in which cases they coincide, under which parameterizations these coincidences were produced, etc.
The CTLA was posed to avoid these limitations in the analysis and provide a common framework for the approximations and all the possible real cases at the same time.

(iii) Lossy TLs were not completely analyzed due to the complexity of trying to solve some of the possible TL-related problems implies, often requiring great amount of calculus, or becoming a lot of times impossible only by using algebraic operations. For example, solving the lengths at which the impedance/admittance is purely real (after proving the existence of these solutions) neccesarily requires Complex Analysis techniques.
In this sense, the CTLA had to provide an alternative way to characterize lossy TLs in such a way that the related problems would have become much more easy to be solved.

Under these premises, the CTLA of lossy TLs has been written in the *handbook* in the following terms:

(i) The CTLA starts assuming a TL which obeys the *telegrapher's equations* equations parameterized by (real parameters) R, L, G, and C; being R and G those (lossy) line parameters that model the losses in the TL. These parameters are considered constant coefficients in the *telegrapher's equations* which, written in this form, parameterize TEM modes. Moreover, it is implicitly assumed time harmonic regime for the complex analysis being able to be applied.

(ii) On one hand, with the aim of studying the influence of lossy line parameters, the approparite parameterizations of losses are used to describe the basic parameters both when frequency is fixed and variable in their respective complex planes. For this purpose, the basic parameters are normalized according to the parameterizations used, and so the complex planes which describe them.
Then the graphical analysis directly appears associated with the variation of one of the parameters when keeping fixed the rest. Since the parameterizations regarding losses are presented by pairs, the result of the graphical analysis of basic parameters in terms of losses is having two sets of curves in case the frequency is fixed (parameterized by the source of losses: conductor or dielectric losses) and one set of curves parameterized by losses when frequency is variable (which describes the whole frequency band when losses are fixed).

(iii) On the other hand, it is interesting to see how losses affect the description of wave parameters. For this purpose, the well-known transformations between the wave parameters are studied at any fixed point, normalizing them so that the phase of the characteristic impedance, φ_{Z_0}, is the parameter which describe the influence of losses in these transformations.

[23] An approximation is not a real solution of the wave equation in any case. However, a particular case is an exact solution under certain conditions. Specify these conditions is one of the purposes of the CTLA.

The graphical analysis is strictly addressed in terms of (complex) transformations between planes, due to the simplicity and properties these parameterized transformations present. Among all the possible graphical analysis between the wave parameters which have been described in [Gag01], the one that parameterizes the real and imaginary parts of the wave impedance in this plane to be transformed to the relfection coefficient complex plane is identifyied as the generalized version of the usual SC, called the GSC, [GDG06]. This graphical tool is seen as a particular case of CTLA.

(iv) A series of TL-related questions in the context of the CTLA are left open. For example: How do the wave parameters behave in terms of the explicit influence of lossy parameterizations? How are they along the TL? Which TLs (described in terms of losses) have the same phase of the characteristic impedance (which parameterizes the GSC); among others related to more specific problems, for example, given a loaded lossy TL, solving the lengths –if any– at which the wave impedance/admittance is real, by using the CTLA (along the TL).

The reinterpretation of these analysis in the scope of the CTLA, contextualizing them appropriately, has served to set the thesis objectives as well as propound the first version of the CTLT, giving the consequent ideas to both the future versions of the CTLT and the terms in which the CGSST (or CST) could be posed.

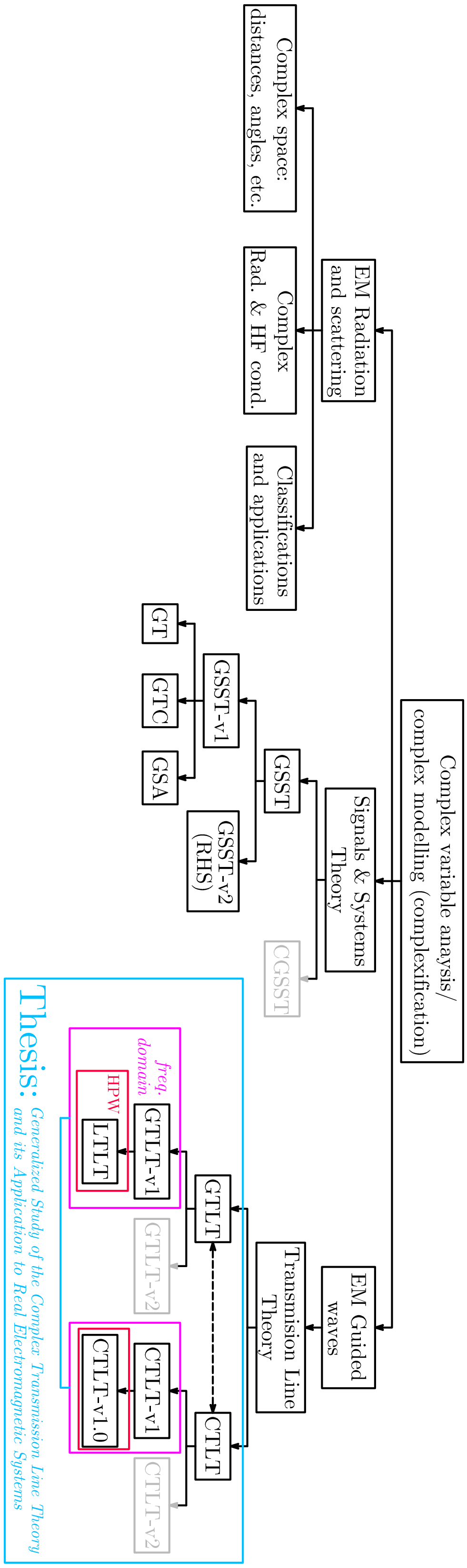


Fig. 1.2: General scheme that summarizes the background in applied complex analysis including the works developed in the Thesis.

1.3 Thesis objectives and contents

(In which goals the Thesis is supported? Which concepts are (and which are not) described in this book?)

Based on the ideas which define the Thesis background and the conclussions which may be deduced from the previous work related to the CTLA, the following reinterpretations are posed to be then analyzed as objectives for the Thesis:

(i) The CTLT is not the same as the CTLA.
While the CTLA refers to the methods within the Complex Analysis which serve to characterize the TLs, the CTLT is the theory –presented as alternative to one particular TLT– which characterizes the TL which physically represents a problem of EM propagative nature under certain conditions. In other words, the CTLA has to be understood as the use of a set of analytical methods typical of Complex Analysis to describe the TL which represents any type of EM system, whereas the CTLT is the physical interpretation of a more or less generalized (but neither particular nor general) TLT from the particular use of the CTLA.

(ii) Both research on the CLTA and on the CTLT are required. A lot of times the developments in both sides –the analysis and the physical interpretations, respectively– come together or, although not all at once, at least they may be identifyed on both parts. One clear example of this correlative behavior happens when characterizing the wave parameters along the TL if the losses on it are fixed, in which the (a priori disregarded) study of the transformations from the reflection coefficient complex plane parameterized by the (fixed) phase of the characteristic impedance is (finally) required. In this way, the study of wave parameters along the TL for physically describe them as a part of the CTLT requires the a priori useless inverse transformation that the GSC, [GDG06], describes.

(iii) The transformations explain the parameterized curves in each normalized complex plane. Even when the parameterizations of losses and frequency or the variable that denotes the TL's length parameterize the curve, it is possible to imagine a transformation from a "space/domain of parameterizations/variables" to the complex plane of the parameter to be finally characterized. These transformations are posed, in general, between complex quantities.
If all the parameterized curves are seen as the result of a complex transformation, then these curves may be detransformed using the inverse transformation, provided that the transformation is injective. In turn, thus suggests analyzing the properties of the transformations involved in the CTLA. The appropriate CTLA which lets to describe graphically the parameters as any TLT does analytically, leads to define the subsequent CTLT (supporting the underlying TLT).
On the other hand, if looking at the possible transformations, two types may be considered: those that are directly and naturally related to physical parameters/variables, for example, the transformation which describes a wave parameter in terms of the TL's length; and those transformations which mainly have analytical meaning, for example the transformations between the wave parameters. They can be distinguished because the first type a priori involves real parameterizations, while the second transformations are in general complex. Equivalently, from the GSST point of view, the first transformations describe a complex function of (a priori) real variable, whereas the second ones are complex operators between complex parameters, which may be, in turn, described as complex functions (of complex variable).

(iv) The complex parameterizations (regarding the complex transformations) are typically described separating the real and imaginary parts or the modulus and phase, because they correspond with the usual descriptions of complex numbers, [BC90], athough they can be described by any complex function because sometimes it is interesting to reparameterize[24] them (e.g. the reflection coefficient will be better described as logarithmic reparameterization for describing the impedance/admittance along the TL, [VG17-I]). However, the complex parameterizations have to be understand as only one[25]. In fact, under the context of the CTLT

[24]The concept of reparameterization, [MP77], is clearly bound to the geometrical interpretation of the analysis.
[25]The notations "real-imaginary parts" and "modulus-phase" (instead of "real and imaginary parts" and "moulus

it is expected to have a real physical parameterization which accordingly relate the parts of the complex parameterization.
In any case, the parameterizations are normalized (so they are dimensionless) in such a way that they are able to describe the normalized TL parameters (also dimensionless) by means of an "universal" graphical analysis.

(v) The TL parameters are able to fully desrcribe the solutions in the equivalent TL and thus the EM fields in the system that TL represents.
Their representation in normalized complex planes efficiently substitutes the original analytical expressions, so that it provides all the possible values and the regions of validity at the same time, as well as the required rigorous explanation to particular cases and approximations.

(vi) The CTLA should be able to be re-used for the analysis of different particularizations of the TLT leading to their respective versions of the CTLT. Thus, the definitions have to be as general as the analysis can be. This basically translates to the fact that the parameterizations should be able to describe the couple of equivalent *telegrapher's equations* that models the problem, while the direction of propagation and time are described either as parameter or variable. For example, in the usual *frequency domain* ω parameterizes the time while the TL's length is kept as the variable which describes the direction of propagation.

Based on these premises, the following objectives, which describe the contents of the Thesis, have been proposed and fulfilled:

(i) The Thesis is mainly focused on describing the CTLT referred to the LTLT (the CTLT-v1), using the appropriate CTLA, which includes those analysis presented as previous works: basically (i) the characterization of basic parameters in terms of losses and frequency and (ii) the transformations between the wave parameters when losses are fixed; which are related to direct transformations, so they have to be reinterpreted like this; and those ones new proposed: basically (i) the characterization of the TL (the line parameters) in terms of the basic parameter and (ii) the characterization of basic parameters when the impedance at any fixed point is given; related to the inverse transformation, and thus also interpreted in terms of transformations.

(ii) For being the CTLT-v1 well defined, the LTLT has to be posed and developed. Define the LTLT means define the TLT for (i) harmonic plane waves (HPW), in which (ii) losses come from every possible source: non zero conductivity of materials and dispersivity of them; and (iii) described in both time and space coordinates (see the scheme in Fig. 1.1.
Seen the achievement of the LTLT as objective, it should be advance that: (i) studying HPWs fixes both the BCs and geometries of the domain in which these waves can propagate; (ii) in the abscense of a theoretical model, the constitutive parameters do not depend explicilty on frequency (CTLT-v1.0) so they are not function of time. However, this does not mean that the equivalent TL is not dispersive. In fact, it is, with the exeption of the non dispersive case, which is studied in detail; and (iii) the optimal way to describe the TL parameters for their characterization in terms of losses is in the *frequency domain*, so for being the solutions/parameters desribed in the *time domain*, it is neccesary to analyze the parameters when frequency is variable (a kind of spectral analysis).

(iii) The LTLT can be posed in two ways: as an independent case defined by the conditions exposed before, or as a particular case of a more general TLT. Both ways for obtaining the LTLT are proposed: the first one leads to the "closed" LTLT (the name is kept the same), which is defined particularizing step by step the *Maxwell equations* to be transformed to *telegrapher's equations* to finally obtain the solutions for a equivalent TL; whereas the second one comes from "A Generalized Version of the TLT" (GTLT-v1), which supposes the solutions adjust specific functions, and from these functions the parameters of the equivalent

and phase", respectively) is a direct consequence of considering one complex parameterization (instead of two real parameterizations).

TL are obtained.
These ways of describing the CTLT correspond with the "direct" and "inverse" way of seeing the tranformations in the CTLT, so they are called the direct characterization and the inverse characterization of the CTLT, respectively. Thus, in this case the direct characterization and the invere characterization are denoted as CTLT-v1.0a and CTLT-v1.0b, respectivey.

(iv) Recall that for any version of the CTLT to be completed, the TL parameters have to be fully characterized in terms of the physical parameterizations: losses, frequency and TL's length. Keeping this triple characterization in mind, only by combining the direct and inverse characterizations the CTLT-v1 can be completely studied. As a result, the physical characterization of the lossy TL is presented as example of use of the combination of the direct and inverse transformations. Those examples of use are: (i) the parameters described along the TL, (ii) in terms of losses and analyzed when frequency is variable; and (iii) when they are described both along the TL and in terms of losses/frequency at the same time, presenting this dual behavior when the appropriate parameterizations (based on angles) are chosen.

Once the CTLT-v1 is clearly defined and studied, different types of lossy TL-problems may be analyzed, examplifying some applications of the Theory presented here.

1.4 Thesis structure

(Based on the contents, how is the info organized in this book? What does each chapter include?)

Keeping in mind the global purpose of the Thesis, which is define the CTLT-v1 based on the LTLT (within the GTLT-v1) together with all the needed resources in the CTLA which lead to it and prepare the definition of future versions, the Thesis is organized after this introductory chapter as follows:

Part I

The first part is intended to introduce and describe the LTLT from two perspectives:

In **Chapter II**, the LTLT is presented from *Maxwell equations* being particularized for HPWs, that is, waves whose EM fields are perpendicular to the direction of propagation presenting zero Laplacian in the transverse at the same time.
The analysis is developed from zero in the *time domain*, that is solving the wave equations in full time-space coordinates, and in the *frequency domain*, that is selecting time exponentials (time harmonics) as basis and describing the problem in the spectrum of frequency, [Cle96]; to be then compared.
As a result, it may be shown that the characterization in the *frequency domain* presents better characteristics for the complete description of the parameters in terms of any type of losses.
From this point, the equivalent voltage and current waves are defined by integrating the electric and magnetic fields, and the subsequent integration of *Maxwell equations* while identifying these equivalent waves, which leads to the definition of the line parameters in a generic (orthogonal) coordinate system, together with the equivalent *telegrapher's equations*.
At the end of this chapter, the most important particular cases (from the lossy case, which is going to be studied in the CTLT-v1.0a) are obtained (not efficiently) one by one.
This chapter is based on the usual TLT, [Poz98], but it is enterely presented generalizing: (i) losses (linking the CTLT) in both the *time domain* and the *frequency domain*, and (ii) the coordinate systems; which both suppose new contributions of the author to the Thesis and the state of the art.

In **Chapter III**, a version of the GTLT (GTLT-v1) is introduced. This version is intended to gather all the possible solutions in waveguides under the same theory, generalizing the geometry of the transversal section and the BCs, all of them described in the *frequency domain* for generalizing losses as in the LTLT.
For this purpose, the GTLT-v1 starts supposing that any solution in the waveguide may be described as a linear combination (denoted as **LC**, whatever the form it takes) of a set of orthogonal vectors that are unequivocally defined by the gradient of a scalar potential plus the curl of a vector potential (a scalar product according to this definiton is also proposed).
Using the scalar product besides BCs conciuosly imposed a posteriori, different solutions may be obtained.
Among of them the HPWs that describe the LTLT may be deduced from the general equations. Thus, the LTLT is a particular case of this GTLT-v1 (see this inclusion in the scheme in Fig. 1.1). This particularization is presented as example of obtaining the line parameters by means of the inverse characterization. This inverse characterization lies in identifying the constitutive parameters with the line parameters in the supposed *telegrapher's equation* (the line parameters may be complex, depending on the solutions to parameterize. In the cases of the LTLT are positive real). The resultant line parameters are normalized, but this is actually not critical for the subsequent complex analysis.
This chapter is a original contribution of the author of this thesis book, which is inspired in many viewpoints: (i) the GSST provides a general perspective of the problem; (ii) the way of solving the equations by integration with a scalar product is one of the forms to solve Green functions,

[Sta79], for obtaning the general solution of a EM problem, which is is also used in [Mar51], but particularized to obtain TEM modes, TE modes[26] and TM modes[27] in waveguides with ideal BCs (PECs); and (iii) the inverse characterization proposed in the context of the CTLT.

Part II

This part is intended to develop the CTLT-v1 completely.

In **Chapter IV**, the (basic[28]) transformations between the planes associated to the TL parameters under study are characterized.
The chapter is splitted into two parts: the direct characterization, which mainly supports the analysis in Chpt. II regarding the characterizations of TLs parameterized by losses and frequency; and the inverse characterization, which mainly supports the analysis in Chpt. III obtaining the line parameters from specifications on basic parameters and, these latter parameters from specifications on wave parameters.
In order to understand the graphical analysis presented in the CTLT, the curves in each plane are presented as complex transformations from another plane. For this purpose, the "space of parameterizations (of line parameters)" is algebraically defined. In particular, for the CTLT-v1 this "space" may be reduced to a (non euclidean) plane which is the domain and origin of parameterizations for both the (direct) analysis in terms of losses and the (direct) analysis when frequency is variable; and the plane to (inversely) parameterize the basic parameters.
For each transformation described in the CTLA of the CTLT-v1, the following points are described in detail:

- ***Parameterizations***
- ***Normalizations***
- ***Graphical analysis***
- ***Geometrical analysis***
- ***Physical interpretations***
- ***Practical uses***

The analysis presented in this section are based on the previous works, gathered in [Gag01]. The previous analysis are framed within the direct characterization, but they have to be reinterpreted in terms of transformations, with the exception of the transformations between the wave parameters, which are presented by means of one example.
The rest of analysis which involve the inverse characterization are developed by the author, [GVH15, VG16-I, VG16-II].

In **Chapter V**, the CTLA presented in Chpt. IV (by means of both the direct and inverse characterizations) is used as example to characterize the CTLT regarding HPWs which propagates in lossy media, that is the CTLT-v1.
The TL parameters are characterized:

(i) along the TL when the losses and frequency are fixed (Ex. 01);

(ii) in terms of losses when frequency is fixed, or varying the frequency when losses are fixed, both at a fixed point along the TL in which the load is known (Ex. 02); and

[26]Traversal Electric modes.
[27]Transversal Magnetic modes.
[28]In the sense that they only comprise two complex planes: the transformed and the "transformative"; so they involve transformations of constant parts (e.g. real-imaginary parts) or any modulus-phase.

(iii) parameterizing the losses/frequency and the TL's length by means of the phase of basic parameters in order to be both analysis before (Ex. 01 and Ex. 02) gathered in only one (Ex. 03).

Each example is detailed in:

- ***Normalizations and parameterizations***
- ***Graphical and geometrical analysis***
- ***Physical interpretations***
- ***Practical uses***

In addition, for the purpose of being the graphical analysis properly interpreted (as alternative to the "closed" analytical expressions), all the parameterized complex planes are presented together (or at least grouped in consecutive pages for the right interpretation while they are clearly depicted).
These analysis have been carried out by the author once the direct and inverse characterizations were studied in the associated CTLA. In particular, some partial analysis of Ex. 01 and Ex. 02 have been introduced in [GVH15], and [VG16-I] and [VG16-II], respectively.

Part III

In this part, some **Applications** of the CTLA in different fields: as graphical tools (for example the logarthimic version of the GSC introduced in [VG17-I], and presented in [VG17-II] and [VG17-III]), as theoretical contributions in "complexification" examples, as supportive analysis in numerical problems, and as example for the GSST; are briefly explained.

At the end, both the **General Conclusions** of the CTLA regarding the CTLT version studied in the Thesis and the immediate **Future Lines** taking advantage of these analysis are presented.

Part I

General Transmission Line Theory

Chapter 2

The Lossy Transmission Line Theory

2.1 Introduction

The TLT has been extensively used in RF and microwave engineering to describe the behavior of EM propagative fields by means of equivalent voltage and current waves which propagate in TLs. The reader may refer to some basic bibliography, for example the references [Mar51, Col90], to find several examples of use of the TLT. Some of the examples in which the TLT is helpful could be in modelling waveguides, passive RF and microwave circuits –in general, N-port devices–, propagation in free space, etc.
In addition, the TLT continues being very useful from the educational point of view because it serves to describe the elements of the EM propagation in a very simplified way.
Furthermore, the study of TLs may be combined with the analysis of different networks which simulate pasive and/or active HF circuits by means of some well-known parameters, as for example the scattering parameters (S-parameters), [Poz98]. This possibility also makes the TLT one of the most appropriate ways to characterize microwave components holding propagative waves.

Despite the TLT has demostrable usefulness in many analysis, it is often set aside to analyze lossless TLs or low losses approximations of lossy TLs. This is quite common because: (i) guided waves which propagate in low-lossy mediums hold the major part of applications in the engineering, in which losses are disregarded. The study of EM waves in that "almost lossless" media, or waveguides in practice, is often carried out by means of equivalent TLs in which different rigorousless[1] approximations are imposed over the lossless case; and (ii) these particular cases clearly reduce the complexity of the general analysis.
However, the mentioned simplification when using the lossless case or the usual low losses approximation, [Poz98], not only cast serious doubts on the rigor of the analysis, but also they (i) keep the study of losses widely limited and, as consequence, (ii) a lot of physical interpretations regarding lossy TLs and practical uses of losses are completely missed. And that is not all; limiting the study does not only affect the better understanding of lossy TLs in itself, but also the characterization of many other circuits that present the characteristics of lossy TLs, as for example –and maybe surprisingly– active circuits, [Poz98].
It is for that reason that any methodology which serves to generalize the classical studies to the rigorous analysis of lossy TLs is encouragingly neccesary. If that intended methodology exists, its use would mean avoiding the limitations of particular cases or approximations and taking full

[1] A particular analysis, for example an approximation, is said to be "rigorousless" when neither its validity nor the effects, that is the error it causes, are considered, although it makes sense in practice. On the contrary, the validity of a "rigorous" analysis can not be questionable at all (and this is not a philosofical question). Moreover, a "rigorous" analysis will allow for explaining the suitability of the possible "non rigorous" analysis.

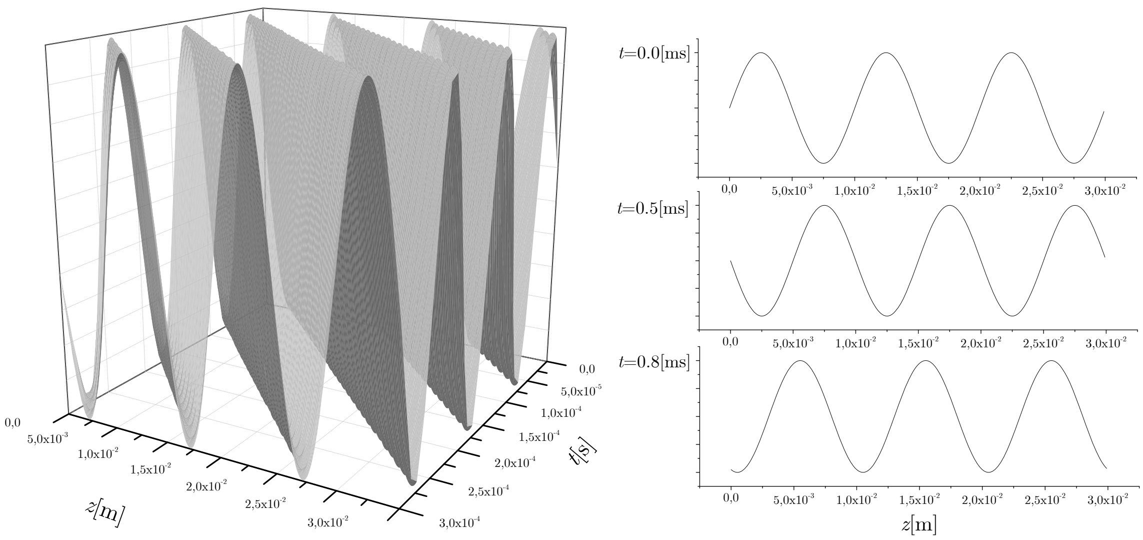


Fig. 2.1: Generic function represented in the $[z, t]$-plane and some time parameterizations of the same function. The graphs "paused" at different times (they are parameterized by time, or time realizations) give an idea of how is the field at different moments and at which velocity the function varies.

advantage of general analysis, too. With no doubt whatsover, the generalization of the TLT, in this case to study lossy TLs thoroughly, is within the purposes of the present Thesis, as it has been previously detached among the general objectives presented in Sect. 1.3 in Chpt. 1.

Of course the TLT has the basis in EM Theory, and so in *Maxwell equations*, [Col90]. *Maxwell equations* are the laws[2] in which every analysis in the present Thesis is based on. In particular, EM plane waves, as the ones used in [Cle96] for expanding any guided waves, are the object of study of the TLT, as it is usually presented.
But the true objective underlying the TLT should be simplifying the study of EM guided waves (they do not have to be plane waves) to equivalent voltage and current waves defined as functions of one variable denoting the coordinate along which the TL extends, besides the time (t) variation. The coordinate z is often used as this "spacial variable" and it coincides with the direction of propagation, whereas time may be seen, for example, as a parameter.
In Fig. 2.1 a representation of a generic function in the $[z, t]$-plane is shown together with different representations of itself parameterized by different time moments. This graphs could represent a wave which is propagating along z.

Once the voltages and currents are defined along the TL, a system of two first order differential equations (ODEs[3]) commonly known as the *telegrapher's equations* relate the variations of voltages and currents in both z and t. The coefficients accompanying the derivatives clearly define the solution of the system of ODEs.
What is more, not only the coefficients in the *telegrapher's equations* serve to define the (mathematical) solution in terms of voltage and current waves in the equivalent TL, but also the "physical nature" of the problem under study. For example, waves propagating in lossless or lossy media correspond with different physical problems, being the coefficients of the first case a particularization of the second one, provided that this latter case is rigorously studied.

[2]Every law, condition, or common/significant equation will be written in *italic* throughout the Thesis; for example *Maxwell equations*

[3]Ordinary differential equations.

When decoupling the *telegrapher's equations*, two second order ODEs of, separately, voltage waves and current waves are obtained. These equations to be solved follow the form of wave equations, [Poz98].

In this chapter, the basis regarding the TLT are reviewed while losses affecting TLs are fully considered. This fact supposes, on one hand, an important generalization if it is compared to the well-known definitions regarding lossless TLs or low losses approximations, [Poz98]. In this way, it is about introducing the LTLT from zero in order to generalize the usual TLT. Nevertheless, on the other hand, this generalization implies an important increase on the complexity of the analysis.
Nevertheless, the basis of the TLT are kept the same, which implies defining and studying the characteristics and behavior of plane waves. In particular "harmonic" plane waves are the object of study, which are defined below in Sect. 2.2 in a generalized orthogonal coordinate system, and analyzed in both *time domain* and *frequency domain*.
Based on the properties of this kind of waves and the facilities the *frequency domain* presents in comparison with the *time domain*, the equivalent voltage and current waves are defined taking the frequency as the variable of study. The particularization of *Maxwell equations* using these equivalent voltage and current waves immediately leads to the equivalent *telegrapher's equations*, which are introduced in Sect. 2.3.
At the end, and overview of several cases/approximations within the LTLT: the general lossy case, the lossless case, the non dispersive, and the low-losses approximations; are described in detail in Sect. 2.4, in order to be then compared and analyzed in a rigorous way.

The LTLT explained in this chapter lets see that it is important to: (i) generalize the TLT to the study of different waves –not only HPWs– which propagate in waveguides, leading to the first version of the GTLT (the GTLT-v1) explained in Chpt. 3; and (ii) define a method of analysis which allows for analyzing the generalizations rigorously. This method is based on Complex Analysis, [BC90], because it results natural when analyzing TLs under time harmonic variation, which is a specific parameterization of t, and it leads to the analysis in *frequency domain*. Thus, the first version of the CTLT (the CTLT-v1) introduced in Chpt. 4 arises out to study the wave solutions included in the GTLT-v1.
Nevertheless, the cited analysis in time harmonic regime is just one possibility to describe the EM propagation by means of TLs. The same characterization may be generalized using concepts of Functional Analysis and Operator Theory, [HP80], as it will be introduced among the **Future Lines**. This latter generalization means an important leap from the usual TLT and it sets the basis of a new version of the GTLT (GTLT-v2).

2.2 The plane waves fundamentals

In this section, the solutions of *Maxwell equations* based on plane waves in homogeneous isotropic media in absence of EM fonts[4] are obtained in both *time domain* and *frequency domain*, generalizing the presence of losses. The most important assumptions which lead to define the TLT based on these plane waves are detached at the same time, in order to later generalize the conditions to impose regarding the GTLT-v1.
In addition, the main aspects regarding the generalization of the TLT for the lossy TL characterization are emphasized in order to introduce the CTLT-v1 as alternative methodology to study the LTLT based on Complex Analysis, which is presented in Chpt. 4 and used in Chpt. 5 for completing the CTLT-v1 characterization.

2.2.1 Plane wave definition

It starts defining the system of reference in generalized coordinates, and the notation to be used to define these plane waves.

Graphically, the system of reference may be represented as follows

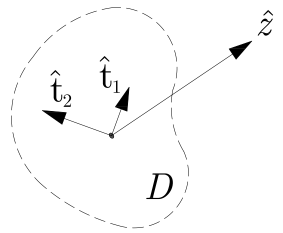


Fig. 2.2: Example of representation of a generic planar cylindrical region. The intrinsec trihedrom $[\hat{t_1}, \hat{t_2}, \hat{z}]$ serves to describe both the direction of propagation (z) and the orientation ($[\hat{t_1}, \hat{t_2}]$) of plane waves, which are in the planar cross section D.

If taking z as the direction of propagation, plane waves are those solutions of *Maxwell equations* which verify the *plane wave condition*:

$$\overbrace{\boxed{\mathcal{E}_z = \mathcal{H}_z \stackrel{\text{def}}{=} 0.}}^{\textit{Plane wave cond.}}$$

This condition means that the field components are in the plane (denoted as D in Fig. 2.2) which is perpendicular to the direction of propagation given by the z-axis, that is, the longitudinal field components are null.
This fact inherentely establishes planar cylindrical regions[5] as the domains in which plane waves are better described. As a result, the *plane wave condition* indirectly geometrizes both the dependence on the position and the orientation of the fields in these planar cylindrical regions, such as it is exemplified in Fig. 2.2.
The generic intrinsec trihedrom $[\hat{t_1}, \hat{t_2}, \hat{z}]$ has been chosen to describe plane waves; both their direction of propagation and the orientation at the same time. The defintion of the coordinates $[t_1, t_2]$ could be both local, that is intrinsecly to the point of study, or universal as a generic orthogonal coordinate system in the space. The selection of one specific orthogonal coordinate system will

[4]Typically, electric charge and current.
[5]A generic planar cylindrical region is a domain defined as invariant along one of the cartesian coordinates. Thus, planar cylindrical regions are geometrically built up by means of the infinite extrusion of planar cross sections (for example D in Fig. 2.2) along the one axis (z-axis in the case of Fig. 2.2).

mainly depend on the boudary conditions (BCs) imposed around D[6].

The dogmatic *Maxwell equations* in homogeneous isotropic media in absence of fonts, [Mar51], particularize for plane waves in planar cylindrical regions as:

Faraday's law

$$\boxed{(\nabla_{\mathrm{t}} + \tfrac{\partial}{\partial z}\hat{z}) \times \boldsymbol{\mathcal{E}}_{\mathrm{t}} = -\tfrac{\partial \mu \boldsymbol{\mathcal{H}}_{\mathrm{t}}}{\partial t};}$$

Ampère's law

$$\boxed{(\nabla_{\mathrm{t}} + \tfrac{\partial}{\partial z}\hat{z}) \times \boldsymbol{\mathcal{H}}_{\mathrm{t}} = \sigma \boldsymbol{\mathcal{E}}_{\mathrm{t}} + \tfrac{\partial \varepsilon \boldsymbol{\mathcal{E}}_{\mathrm{t}}}{\partial t};}$$

Gauss's law (electric field)

$$\boxed{(\nabla_{\mathrm{t}} + \tfrac{\partial}{\partial z}\hat{z}) \cdot \boldsymbol{\mathcal{E}}_{\mathrm{t}} = 0;}$$

Gauss's law (magnetic field)

$$\boxed{(\nabla_{\mathrm{t}} + \tfrac{\partial}{\partial z}\hat{z}) \cdot \boldsymbol{\mathcal{H}}_{\mathrm{t}} = 0.}$$

As it may be seen, the curl and divergence operators in space simplify to the one in the $[\mathrm{t}_1, \mathrm{t}_2]$-plane (where D lives in), besides the vector product and the dot product both with $\hat{z}$ accompanying the directional derivative in z.
Also notice that the current ($\boldsymbol{\mathcal{J}}_{\mathrm{t}}$) is only consequence to the non zero conductivity, and so it should be proportional to the eletric field ($\sigma\boldsymbol{\mathcal{E}}_{\mathrm{t}}$ in the *Ampère's law*). Moreover, the magnetic ($\boldsymbol{\mathcal{B}}_{\mathrm{t}}$) and electric ($\boldsymbol{\mathcal{D}}_{\mathrm{t}}$) displacement fields have been expressed proportionallyl to the electric ($\varepsilon\boldsymbol{\mathcal{E}}_{\mathrm{t}}$ in the *Ampère's law*) and magnetic ($\mu\boldsymbol{\mathcal{H}}_{\mathrm{t}}$ in the *Faraday's law*) fields in *Maxwell equations.*
Once *Maxwell equations* are posed in cylindrical coordinates and the *plane wave condition* has been imposed for the longitudinal field components to vanish, the resultant equations are carefully analyzed to be simplified.

Take in mind the vectors $[\hat{\mathrm{t}}_1, \hat{\mathrm{t}}_2]$ are, indeed, orthogonal. If the fields satisfy the *plane wave condition*, the transversal curl operator ($\nabla_{\mathrm{t}}\times$) applied to the transversal fields also vanishes, and so the electric and magnetic fields are orthogonal. This is proved in Appendix Appendix 2.A in an easy way. As a result, the curl operator applied to plane waves has the form of $\partial/\partial z \times$[7].
In addition, the ortientation of $[\hat{\mathrm{t}}_1, \hat{\mathrm{t}}_2]$ supposes a degree of freedom because *Maxwell equations* are invariant under rotations, which is also proved in Appendix 2.A. This lets to select the vectors $[\hat{\mathrm{t}}_1, \hat{\mathrm{t}}_2]$ being coincident with the electric and magnetic fields, respectively:

$$\boldsymbol{\mathcal{E}}_{\mathrm{t}} \equiv \mathcal{E}_{\mathrm{t}_1} \hat{\mathrm{t}_1}, \tag{2.1}$$

$$\boldsymbol{\mathcal{H}}_{\mathrm{t}} \equiv \mathcal{H}_{\mathrm{t}_2} \hat{\mathrm{t}_2}. \tag{2.2}$$

This assumption does not mean any loss of generality but it simplifies the notation in *Maxwell equations* a lot, which result in the particularized equations:

$$\frac{\partial \mathcal{E}_{\mathrm{t}_1}}{\partial z} = -\frac{\partial \mu \mathcal{H}_{\mathrm{t}_2}}{\partial t}, \tag{2.3}$$

$$-\frac{\partial \mathcal{H}_{\mathrm{t}_2}}{\partial z} = \sigma \mathcal{E}_{\mathrm{t}_1} + \frac{\partial \varepsilon \mathcal{E}_{\mathrm{t}_1}}{\partial t}, \tag{2.4}$$

$$\nabla_{\mathrm{t}} \cdot \mathcal{E}_{\mathrm{t}_1} \hat{\mathrm{t}_1} = 0, \tag{2.5}$$

$$\nabla_{\mathrm{t}} \cdot \mathcal{H}_{\mathrm{t}_2} \hat{\mathrm{t}_2} = 0. \tag{2.6}$$

Notice how the particularized *Faraday and Ampère's laws* in eqs. (2.3) and (2.4) are scalar equations after the components $[\hat{\mathrm{t}}_1, \hat{\mathrm{t}}_2]$ being identified with the fields in eqs. (2.1) and (2.2)[8].

[6]The possibility of generalizing any coordinate system by the local definition in terms of another (known) coordinate system is thanks to leave the imposition of BCs as an (a posteriori) open problem.
[7]The "×" symbol makes reference to the vector product.
[8]This is due to: $\hat{z} \times \hat{\mathrm{t}_1} = \hat{\mathrm{t}_2}$ and $\hat{z} \times \hat{\mathrm{t}_2} = -\hat{\mathrm{t}_1}$.

The field solutions come from decoupling the (general) field functions $\boldsymbol{\mathcal{E}}_{\rm t}$ and $\boldsymbol{\mathcal{H}}_{\rm t}$ from (the general) *Maxwell equations*, to be then particularized with the expressions in eqs. (2.1) and (2.2). For this purpose, the curl on both sides of the *Faraday and Ampère's laws* is taken and, after using the curl-curl vector identity[9] together with *Gauss's laws*, and the particularizations of the fields mentioned, it leads to

$$\Delta_{\rm t}\mathcal{E}_{\rm t_1} + \frac{\partial^2 \mathcal{E}_{\rm t_1}}{\partial z^2} - \frac{\partial \mu\sigma\mathcal{E}_{\rm t_1}}{\partial t} - \frac{\partial \mu \frac{\partial \varepsilon \mathcal{E}_{\rm t_1}}{\partial t}}{\partial t} = 0, \tag{2.7}$$

$$\Delta_{\rm t}\mathcal{H}_{\rm t_2} + \frac{\partial^2 \mathcal{H}_{\rm t_2}}{\partial z^2} - \frac{\partial \mu\sigma\mathcal{H}_{\rm t_2}}{\partial t} - \frac{\partial \mu \frac{\partial \varepsilon \mathcal{H}_{\rm t_2}}{\partial t}}{\partial t} = 0. \tag{2.8}$$

Eqs. (2.7)-(2.8)[10] are of the form scalar wave equation if (i) eliminating the first order term related to the non zero conductivity, and (ii) considering the medium non dispersive so that the constitutive parameters (functions ε, μ, and σ) do not depend explicitly on time. Nevertheless, eqs. (2.7)-(2.8) could me referred as "modified wave equations", which govern the wave solutions in general lossy media.
The solutions to eqs. (2.7)-(2.8) are of extremely different type depending on the BCs imposed. Among of them, there are those which are more straightforard, such as uniform plane waves in free space, for inctance, but most of them are truly hard to condense by one analytical expression, for example, local scattered waves[11]. All the solutions would be object of study of the TLT provided that they present a propagative behavior.

Remark 1. *Despite the large variety of solutions of wave equations –eqs.* (2.7)-(2.8)*– subject to the imposition of BCs, the TLT would be able to generalize the study of all of them under the same methodology. However, this analysis supposes such a huge generalization that the TLT is not expected to solve completely the underlying EM problem, but to describe the meachanisms of propagation of EM waves.*
In the same way, the so called ***Generalized Transmission Line Theory*** *(GTLT) will try to generalize the usually known TLT with respect to the study of: (i) losses; (ii) propagative modes or field solutions (includying BCs); and (iii) parameterizations of time/space.*

	Domain	BCs→modes	Parameterizations
(Usual) TLT	Time & Frequency	Harmonic modes	Lossless
LTLT	Time & Frequency	Harmonic modes	Lossy
GTLT-v1	Frequency	Multi-mode	Lossy
GTLT-v2	Phase velocity	Multi-mode	Lossy

Table 2.1: Scheme of different versions for different (i) parameterizations of losses, (ii) BCs (modes), and (iii) domains of analysis. The LTLT presented here generalizes losses in regions which support "harmonic" modes analyzed in both the *time domain* and the *frequency domain.*

[9] $\nabla \times (\nabla \times \boldsymbol{\mathcal{A}}) = \nabla (\nabla \cdot \boldsymbol{\mathcal{A}}) - \Delta \boldsymbol{\mathcal{A}}$

[10] The hypenation between equations consecutive numbered is used to designate those equations that have been obtained at the same time by substituying one equation into another one, and viceversa, so they are connected in a certain way. An example of these bound equations used throughout the Thesis is the pair of wave equations related to the electric and magnetic fields.

[11] It is supposed that any EM wave (satisfying *Maxwell equations*) presents a plane behavior in certain region in which the appropriate BCs are imposed. In this way, it is about relying each type of possible solutions on the posing of BCs.

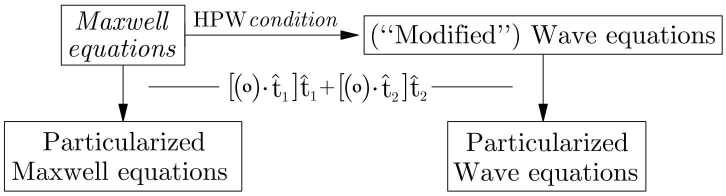


Fig. 2.3: General scheme which guides the procedue to obtain the main equations regarding HPWs which are written in the general coordinate system $[\hat{t_1}, \hat{t_2}]$.

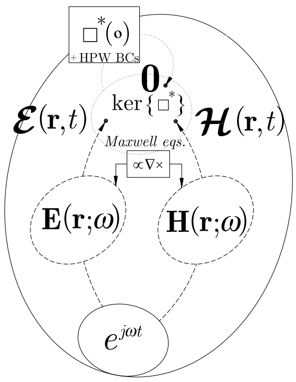


Fig. 2.4: Function space in which the "modified" d'Alembert operator ($\Box^*$) representing the wave equations, and the curl operators ($\propto \nabla\times$) representing *Maxwell equations* act over the functions which represent the electric and magnetic fields. These fields may be obtained by the expansion of harmonics $e^{j\omega t}$ weighted by the functions $E(\bar{\boldsymbol{r}};\omega)$ and $H(\bar{\boldsymbol{r}};\omega)$ studied in the *frequency domain.*

The usual TLT is posed for (i) lossless mediums in which (ii) "harmonic" plane waves, which are introduced in next section, propagate. Moreover, the TLT may be referred in (iii) either $[z,t]$-coordinates (*time domain*) when solving the PDEs[12] (2.7)-(2.8), or parameterizing the time variation when supposing time harmonic dependence (*frequency domain*) for fields to be solved from an equivalent complex Helmholtz-type ODE.

Here the LTLT, that is, the TLT in which losses are fully taken into account, is posed for HPWs (for example, those associated with the infinite boundary conditions (IBCs) or uniform plane waves) in both *time domain* and *frequency domain.* The advantages of parameterizing the losses in each domain are detached, being the *frequency domain* the most appropriate domain (which is represented as a basis in the space represented in Fig. 2.4) to analyze losses.

2.2.2 Plane waves in *time domain*

Here the general solution of eqs. (2.7)-(2.8) is fully obtained in $[z,t]$-cordinates. For this purpose, the "harmonicity" of plane waves is imposed.
In addition, the medium is supposed to be non dispersive in order to simplify the PDEs in eqs. (2.7)-(2.8).
An example when IBCs are imposed gives a particular solution of harmonic plane waves in *time domain*: the uniform plane waves.

[12]Partial differential equations

HPWs are those solutions to wave equation which verify:

Harmonic plane wave cond.

$$\boxed{\Delta_{\mathrm{t}}\mathcal{E}_{\mathrm{t}_1} = \Delta_{\mathrm{t}}\mathcal{H}_{\mathrm{t}_2} = 0.}$$

This condition tells that both $\mathcal{E}_{\mathrm{t}_1}$ and $\mathcal{H}_{\mathrm{t}_2}$ are harmonic functions in the cross section.

Additionally, if the mediums are non dispersive, the constitutive parameters do not depend on time, so they are constant for time derivatives. As a result, the particularized *Faraday and Ampère's laws* in eqs. (2.3) and (2.4) result in

$$\frac{\partial \mathcal{E}_{\mathrm{t}_1}}{\partial z} = -\mu \frac{\partial \mathcal{H}_{\mathrm{t}_2}}{\partial t}, \tag{2.9}$$

$$-\frac{\partial \mathcal{H}_{\mathrm{t}_2}}{\partial z} = \sigma \mathcal{E}_{\mathrm{t}_1} + \varepsilon \frac{\partial \mathcal{E}_{\mathrm{t}_1}}{\partial t}, \tag{2.10}$$

whereas the *Gauss's laws* in eqs. (2.5) and (2.6) remain the same.
On the other hand, the so called modified wave equations in eqs. (2.7)-(2.8) reduce to

$$\frac{\partial^2 \mathcal{E}_{\mathrm{t}_1}}{\partial z^2} - \mu\sigma \frac{\partial \mathcal{E}_{\mathrm{t}_1}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathcal{E}_{\mathrm{t}_1}}{\partial t^2} = 0, \tag{2.11}$$

$$\frac{\partial^2 \mathcal{H}_{\mathrm{t}_2}}{\partial z^2} - \mu\sigma \frac{\partial \mathcal{H}_{\mathrm{t}_2}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathcal{H}_{\mathrm{t}_2}}{\partial t^2} = 0. \tag{2.12}$$

The general (individual) solutions[13] of the PDEs (2.11)-(2.12) are

$$\mathcal{E}^{\pm}_{\mathrm{t}_1} \cong f_e(\mathrm{t}_1, \mathrm{t}_2) e^{\mp\alpha z} \sin(\omega t \mp \beta z + \varphi_e), \tag{2.13}$$

$$\mathcal{H}^{\pm}_{\mathrm{t}_2} \cong f_h(\mathrm{t}_1, \mathrm{t}_2) e^{\mp\alpha z} \sin(\omega t \mp \beta z + \varphi_h), \tag{2.14}$$

$$\omega \geq 0$$

$$\alpha^2 = \tfrac{\mu\varepsilon\omega^2}{2} \left[-1 + \sqrt{1 + \left(\tfrac{\sigma}{\omega\varepsilon}\right)^2}\right] \geq 0,$$

$$\beta^2 = \tfrac{\mu\varepsilon\omega^2}{2} \left[1 + \sqrt{1 + \left(\tfrac{\sigma}{\omega\varepsilon}\right)^2}\right] > 0,$$

in which $f_e(\mathrm{t}_1, \mathrm{t}_2)$ and $f_h(\mathrm{t}_1, \mathrm{t}_2)$ are harmonic functions in the cross-section.

The general solutions in eqs. (2.13) and (2.14) are plane waves which propagate along the z-axis at the same velocity $v_p = \omega/\beta \leq c[\mathrm{m \cdot s^{-1}}]$, c the speed of light, whereas they both attenuate exponentially at the same rate set by α.
The parameters α and β are the well-known attenuation constant and phase constant, respectively. They determine the characteristics of propagation along the z-axis.
Notice that the dependence of the conductivity σ, which is the only source of losses considered in *time domain*, is present in both parameters α and β.
Also notice the dependence of frequency (ω) in both parameters α and β, which are not linear functions of frequency as long as losses due to conductivity are present. Nevertheless, the function $f(\alpha, \beta) = \pm\sqrt{(\alpha^2 - \beta^2)}$ presents a linear behavior with frequency, even though the conductor losses exist.
All these particular behaviors regarding the attenuation and phase constants due to conductivity

[13]There are two possible individual solutions to wave equations regarding the electric and magenetic fields. Each solution is denoted by the superscript "+" or "−", which respectively refer to the electric or the magnetic field which propagates in the $\hat{z}$ direction (z increasing) or the $-\hat{z}$ (z decreasing) direction. The total solution is the linear combination (**LC**) of both waves. The coefficients of the **LC** are found a posteriori when the BCs are imposed in the z direction. Thus, the total solution (the total wave) distinguishes from the individual solutions (or basic solutions), and so the parameters which relate them. When solving the wave equations, the individual solutions are the ones interesting to obtain.

losses, and also the changes with frequency, are object of study latter in Chpt. 4 when frequency is parameterized (*frequency domain*).

If the solutions in eqs. (2.13) and (2.14) are expected to model an unique EM wave which propagates at $\omega/\beta[\mathrm{m}\cdot\mathrm{s}^{-1}]$ and it is attenuated e at a point which is $1/\alpha[\mathrm{m}]$ away from each z, then ω, α, and β have to be the same for $\mathcal{E}_{\mathrm{t}_1}$ and $\mathcal{H}_{\mathrm{t}_2}$. These correspondences between the parameters of the electric and the magnetic fields are due to the physics of the problem: α and β are unique because they depend on the constitutive parameters and frequency which depends, in turn, on the speed of the EM wave. As a consequence, there is only one value of ω for which the EM field solutions in eqs. (2.13) and (2.14) present the same parameters α, β, and v_p, which clearly determines the kinematics of the wave.

Remark 2. *The constitutive parameters of the medium together with the kinematics (phase speed) of the electric and magnetic fields regarding the same EM wave make the frequency have to be equal for both fields. Thus, it seems to be a logic practice arranging the fields in frequency, something which is done in the frequency domain.*
*Contrarily, it is possible to see the constitutive parameters and frequency as the elements for building up equivalence classes which arrange the EM waves. For example, the phase speed v_p fix an equivalence relation between the constitutive parameters to build up the partitions of the v_p-equivalence classes, that is, to see the EM waves in the "velocity domain". This arrangement is useful for studying the waves that propagate in lossy mediums which are invariant along the z-axis, and it constitutes the basis to develop an alternative version of the GTLT in future analysis (see **Future Lines**).*

Furthermore, if the EM fields fulfill the particularized *Faraday's law* in eq. (2.9), then

$$f_e(\mathrm{t}_1, \mathrm{t}_2) \propto f_h(\mathrm{t}_1, \mathrm{t}_2) \equiv f(\mathrm{t}_1, \mathrm{t}_2), \tag{2.15}$$

and so the general solutions are

$$\mathcal{E}_{\mathrm{t}_1}^{\pm} \cong f(\mathrm{t}_1, \mathrm{t}_2) e^{\mp\alpha z} \sin(\omega t \mp \beta z + \varphi_e), \tag{2.16}$$

$$\mathcal{H}_{\mathrm{t}_2}^{\pm} \cong f(\mathrm{t}_1, \mathrm{t}_2) e^{\mp\alpha z} \sin(\omega t \mp \beta z + \varphi_h),. \tag{2.17}$$

Notice that, if the fields belong to the same EM wave, they only differ in some constant of proportionality and their phases φ_e and φ_h, but they present the same variation in the cross section. Thanks to the particularized *Maxwell equations*, it is possible to obtain the constant of proportionality and the phase difference.
With this purpose in mind, eqs. (2.16) and (2.17) are substituyed in the particularized *Faraday and Ampère's laws* in eqs. (2.9) and (2.10) and, after operating (see the Appendix 2.B), it leads to the relation of proportionality between the EM fields (in eq. (2.B.11) in Appendix 2.B)

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \left(\frac{\sqrt{-1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}}{2\sqrt{2}} \sin(\Delta\varphi) + \sqrt{\frac{-1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}{8} \sin^2(\Delta\varphi) + \cos^2(\Delta\phi)} \right), \tag{2.18}$$

in which $\Delta\varphi = \varphi_e - \varphi_h$ is the phase difference between the electric ans magnetic fields. The parameter η is the same for both individual solutions (the ones defines in each direction $+z$ or $-z$) in eqs. (2.16) and (2.17).

This parameter η is known as the impedance of the medium, or simply the impedance. It relates the electric and magnetic fields regarding harmonic plane waves expressed in *time domain*.
Notice again the influence of the conductivity σ and the phase difference $\Delta\varphi$ in the expression of the impedance.
Also notice the dependence of η with ω, which makes the impedance be non constant if seeing this as a function of frequency, as long as neither the conductivity nor the phase difference are zero. The fact that either the zero conductivity or the zero phase difference make the characteristic impedance non frequency dependent suggests that there is a relation between the presence of conductivity and

the phase difference.
In Appendix 2.B this relation is solved leading to (eq. (2.B.12) in Appendix 2.B)

$$\Delta\varphi = \pm\tan^{-1}\left(\frac{\alpha}{\beta}\right) = \pm\tan^{-1}\left(\sqrt{\frac{-1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}{1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}}\right) \in \left[-\frac{\pi}{4},\frac{\pi}{4}\right). \tag{2.19}$$

The sign of $\Delta\varphi$ is not related to each of the individual solutions in eqs. (2.16) and (2.17), but it makes an extension of the usually known $\tan^{-1}(\circ)$ function by reflecting it. Both individual electric and magnetic fields have the same $\Delta\varphi$.

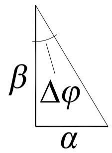


Fig. 2.5: Geometrical construction which represents the relations between the parameters α, β, and $\Delta\varphi$ regarding the analysis of wave solutions achieved in *time domain*.

Notice the geometrical interpretation of the relation between $\Delta\varphi$, and α and β, which is detached in Fig. 2.5: the complementary angle to that one that α and β form (β vs. α) is $\Delta\varphi$. This fact gives to the analysis of parameters clear geometrical meaning.

Remark 3. *The study of the parameters which determine the characteristics of EM waves in terms of both the presence of losses and frequency, requires a methodology which facilitates these analysis and gives at the same time physical interpretations regarding the behavior of these parameters. Since the parameters are related by means of trigonometric identities, every analysis has geometrical meaning. This indicates Geometry to be one possible strong method of analysis.*

In summary, solving the PDEs in $[z,t]$-coordinates when the *harmonic plane wave condition* and the non dispersivity are imposed, that is solving eqs. (2.11)-(2.12), leads to the parameterized solution of EM fields in eqs. (2.16) and (2.17) in *time domain*.
On one side, the parameters α and β and, on the other side, η and $\Delta\varphi$, completely determine the general solution of EM waves in *time domain* when losses due conductivity are studied rigorously.

The particular solution to this problem comes when imposing the appropriate BCs[14]. The following example serves to illustrate what appropriate BCs means when looking for a particular solution of the general one.

Example 2.2.1. *Imagine the deleted plane, [BC90],* $D=(x,y)\equiv(\rho\cos(\varphi),\rho\sin(\varphi))\in\mathbb{R}^2\backslash(0,0), \rho\in(0,+\infty), \varphi\in[-\pi,\pi)$*, in which the harmonic plane wave condition is imposed, so*

$$\Delta_t\mathcal{E}_\rho = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\mathcal{E}_\rho}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\mathcal{E}_\rho}{\partial\varphi^2}=0, \tag{2.20}$$

$$\Delta_t\mathcal{H}_\varphi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\mathcal{H}_\varphi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\mathcal{H}_\varphi}{\partial\varphi^2}=0, \tag{2.21}$$

having been chosen, $[\hat{t}_1\equiv\hat{\rho}, \hat{t}_2\equiv\hat{\varphi}]$*, and* $\mathcal{E}_{t_1}\equiv\mathcal{E}_\rho$ *and* $\mathcal{H}_{t_2}\equiv\mathcal{H}_\varphi$*, because of the symmetries the operator* Δ_t *has in the domain* D*.*

[14]Here the term "appropriate" is quite ambiguous because it depends on the solution (within a space of functions) which is looking for. In the case of looking for a example which represents the behavior of harmonic plane waves, appropriate BCs would be, for instance, those imposed on the general solution in eqs. (2.16) and (2.17), which serve to obtain the amplitude function $f(\mathrm{t}_1,\mathrm{t}_2)$ of the fields $\mathcal{E}_{t_1}$ and $\mathcal{H}_{t_2}$. In Example 2.2.1 the IBC lets to obtain uniform plane waves (which are a particular solution of harmonic plane waves) in a specified domain.

Eqs. (2.20)-(2.21) *have not an unique solution. For example, both* $f_1(\rho) = \log(\rho)$ *and* $f_2(\rho) = C \in \mathbb{R}$ *are solutions to eqs.* (2.20)-(2.21)*. However, if IBC*[15] *are imposed, only* $\mathcal{E}_\rho = C$ *and* $\mathcal{H}_\varphi = \eta C$*, with* η *as defined in eq.* (2.18)*, are possible. In this way, the fields are uniform in D in the sense of they are constant in D. Thus, the resultant waves when IBCs are imposed are called uniform plane waves.*

Remark 4. *Uniform plane waves are a type of harmonic plane waves verifying the Laplace equation in the not bounded or infinite (IBC) transversal section which is perpenticular to the direction of propagation.*

Uniform plane wave cond.

$$\boxed{\begin{array}{c}\Delta_{\mathrm{t}}\mathcal{E}_{\mathrm{t}_1} = \Delta_{\mathrm{t}}\mathcal{H}_{\mathrm{t}_2} = 0, \\ +\mathrm{IBC}.\end{array}}$$

By means of the example Example 2.2.1 before, it has been seen that each particular HPW solution in the cross section does not determine the propagative behavior or relation between the electric and magnetic fields at all. This fact makes the study of these characteristics: propagation and relation between the fields; generalized, which is exactly for what the any version of the TLT is intended to.
Nevertheless, it has been seen that the study of the parameters regarding this proposed TLT clearly becomes much more complex when the most basic losses (those due to conductivity) are taken into account. In this case, only the first order term $-(\sigma\mu)\partial(\cdot)/\partial t$ would produce losses in those solutions in *time domain*.
Furthermore, notice that the frequency ω, which is a time scaling factor, explicitly appears in each parameter α, β, η, and $\Delta\varphi$, which are also written in terms of constitutive parameters. This, together with the fact that losses essentially come with time derivatives, suggests "parameterizing" ω and, consequently, transforming time. Therefore, so far the analysis in *time domain* are proposed in the *frequency domain*, with the corresponding advantages that come with it.

2.2.3 Plane waves in the *frequency domain*

In this section, the general solutions to eqs. (2.7)-(2.8) are obtained parameterized by frequency. In this way, the total solution would be expanded by means of the Inverse Fourier Transform (IFT) of the (time) Fourier Transform (t-FT) using the complex basis[16] $\mathrm{e}(t;\omega) \equiv e^{j\omega t}$ (see this basis in the scheme in Fig. 2.4). In this way, the proposed study is focused on the harmonic parameterized by ω, so talking in terms of *time harmonic regime* and the use of complex functions parameterized by ω.

Time harmonic regime transformations

Funcs.	$\mathcal{E}_{\mathrm{t}_1}(\bar{r},t) \overset{t-\mathrm{FT}}{\longrightarrow} E_{\mathrm{t}_1}(\bar{r};\omega)$ $\mathcal{H}_{\mathrm{t}_1}(\bar{r},t) \overset{t-\mathrm{FT}}{\longrightarrow} H_{\mathrm{t}_2}(\bar{r};\omega)$
Ops.	$\partial(\cdot)/\partial t) \overset{t-\mathrm{FT}}{\longrightarrow} j\omega(\cdot)$ $\partial^2(\cdot)/\partial t^2 \overset{t-\mathrm{FT}}{\longrightarrow} -\omega^2(\cdot)$

Notice that time derivatives regarding any complex time exponential lead to the same exponential weigthed by the complex factor $j\omega$, that is complex exponentials are eigenfunctions of time deriva-

[15] Infinite boundary conditions ensure that the field is zero or constant when the position vector $\bar{r} \to \infty$ if either the field fonts (which are not in the domain of study) are finite or "infinite functions" (in the sense of distributions), respectively.

[16] The notation $\mathrm{e}(t;\omega)$ ($e^{j\omega t}$ in Fig. 2.4) is a particular case of the generic notation $\mathrm{e}(\tau;\mu)$ presented in [Her14], which refers the element 'e' parameterized by μ in the basis set of functions of variable τ. The use of this notation is quite recommendable to study parameterized functions, for example those in the TLT, but especially when designating the basis in the space of functions in which the fields are repressented.

tives. This fact makes (i) solving the wave equations in the *frequency domain* much more easy than in *time domain* and (ii) the analysis in terms of losses generalizes to dispersive cases, that is, the analysis when complex constitutive parameters may be frequency dependent.
The same example of uniform plane waves (harmonic plane wave solutions + IBC) may be addressed in the *frequency domain*.

Again the *harmonic plane wave condition* is imposed.
By considering the *time harmonic regime* transformations above, the particularized *Faraday and Ampère's laws* in the *frequency domain* follow the form

$$\frac{\partial E_{\mathrm{t}_1}}{\partial z} = -j\omega\mu H_{\mathrm{t}_2}, \tag{2.22}$$

$$-\frac{\partial H_{\mathrm{t}_2}}{\partial z} = (\sigma + j\omega\varepsilon)E_{\mathrm{t}_1}, \tag{2.23}$$

whereas the *Gauss's laws* in the *frequency domain* are witten as

$$\nabla_{\mathrm{t}} E_{\mathrm{t}_1}\hat{\mathrm{t}_1} = 0, \tag{2.24}$$

$$\nabla_{\mathrm{t}} H_{\mathrm{t}_2}\hat{\mathrm{t}_1} = 0, \tag{2.25}$$

if the mediums are suposed to be homogeneous with no fonts, although they can be dispersive if $\varepsilon \equiv \varepsilon(\omega) \in \mathbb{C}$, $\mu \equiv \mu(\omega) \in \mathbb{C}$, and even $\sigma \equiv \sigma(\omega) \in \mathbb{C}$, that is, the constitutive parameters are complex functions of frequency (parameterized by frequency for each harmonic).

For its part, if the wave equations in eqs. (2.7)-(2.8) are transformed to the *frequency domain* by using the *time harmonic regime* transformations above, and the *harmonic wave condition* is imposed, they result in

$$\frac{\partial^2 E_{\mathrm{t}_1}}{\partial z^2} + k^2 E_{\mathrm{t}_1} = 0 \tag{2.26}$$

$$\frac{\partial^2 H_{\mathrm{t}_2}}{\partial z^2} + k^2 H_{\mathrm{t}_2} = 0 \tag{2.27}$$

$k \equiv k(\omega) = \omega\sqrt{\mu\varepsilon_{eq}} \in \mathbb{C}$, $\varepsilon_{eq} \equiv \varepsilon_{eq}(\omega) = \left(\varepsilon(\omega) - j\frac{\sigma(\omega)}{\omega}\right) \in \mathbb{C}$. Notice that eqs. (2.26)-(2.27) are Helmholtz-type with coefficients in $\mathbb{C}$. Thus, the complexity of solving the fields from these equations has descreased significantly in comparison with the original eqs. (2.7)-(2.8).

The complex parameters in the transformed equations are briefly characterized from the physical point of view: the complex constitutive parameters ε, μ, and σ, have the real part positive but the imaginay part negative, so they are written as: $\varepsilon = \varepsilon' - j\varepsilon''$, $\varepsilon' \geq \varepsilon_0$, $\varepsilon'' \geq 0$; $\mu = \mu' - j\mu''$, $\mu' \geq \mu_0$, $\mu'' \geq 0$; and $\sigma = \sigma' - j\sigma''$, $\sigma' \geq 0$, $\sigma'' \geq 0$; then the parameters ε_{eq} and k defined from the complex constitutive parameters are written as (see the mathematical analysis in Appendix 2.C): $\varepsilon_{eq} = \varepsilon' - \frac{\sigma''}{\omega} - j\left(\varepsilon'' + \frac{\sigma'}{\omega}\right)$, so $\varepsilon_{eq} = \varepsilon'_{eq} - j\varepsilon''_{eq}$, $\varepsilon''_{eq} \geq 0$ (while $\varepsilon'_{eq} \in \mathbb{R}$); and $k = k' + jk''$, $k' = \sqrt{\frac{\omega^2}{2}\left(\mu'\varepsilon'_{eq} - \mu''\varepsilon''_{eq} + \sqrt{(\mu'^2 + \mu''^2)(\varepsilon'^2_{eq} + \varepsilon''^2_{eq})}\right)} \geq \omega\sqrt{\mu_0\varepsilon_0}$ (eq. (2.C.16) in Appendix 2.C), $k'' = \sqrt{\frac{\omega^2}{2}\left(-\mu'\varepsilon'_{eq} + \mu''\varepsilon''_{eq} + \sqrt{(\mu'^2 + \mu''^2)(\varepsilon'^2_{eq} + \varepsilon''^2_{eq})}\right)} \leq 0$ (eq. (2.C.17) in Appendix 2.C).

A graphical analysis of these parameters based on complex transformations between themselves is carried out in [Rie98]. This analysis lets to see how these parameters vary in terms of physically realizable parameterizations of the constitutive parameters, and it is useful when analyzing the parameterizations regarding the TLT from a physical point of view.
The parameter k is called the complex wavenumber. It depends on frequency and the constitutive parameters, which depend, in turn, on frequency. It determines the solutions of eqs. (2.26)-(2.27).

The general solutions regarding eqs. (2.26)-(2.27) are

$$E_{t_1} \cong f_e(t_1, t_2)e^{\mp jkz}, \tag{2.28}$$
$$H_{t_1} \cong f_h(t_1, t_2)e^{\mp jkz}, \tag{2.29}$$
$$jk = j\omega\sqrt{\mu\varepsilon_{eq}},$$

$f_e(t_1, t_2)$ and $f_h(t_1, t_2)$ are complex harmonic functions in the cross section.

Notice that eqs. (2.28) and (2.29) are fully equivalent to those eqs. (2.13) and (2.14) once the first ones are tranformed to the *time domain* provided that

$$\gamma \equiv \gamma(\omega) = \alpha + j\beta \equiv jk \in \mathbb{C} \tag{2.30}$$

has been defined.
The parameter γ is known as the propagation constant. It determines the propagative behavior of EM waves in lossy media. Thus, eqs. (2.28) and (2.29) may be compactly written in terms of γ as

$$E_{t_1} \cong f_e(t_1, t_2)e^{\mp\gamma z} = f_e(t_1, t_2)e^{\mp\alpha z}e^{\mp j\beta z}, \tag{2.31}$$
$$H_{t_1} \cong f_e(t_1, t_2)e^{\mp\gamma z} = f_h(t_1, t_2)e^{\mp\alpha z}e^{\mp j\beta z}, \tag{2.32}$$
$$\gamma = \alpha + j\beta \equiv jk,$$
$$\alpha = -k'' = \sqrt{\tfrac{\omega^2}{2}\left(-\mu'\varepsilon'_{eq} + \mu''\varepsilon''_{eq} + \sqrt{(\mu'^2 + \mu''^2)(\varepsilon'^2_{eq} + \varepsilon''^2_{eq})}\right)} \geq 0,$$
$$\beta = k' = \sqrt{\tfrac{\omega^2}{2}\left(\mu'\varepsilon'_{eq} - \mu''\varepsilon''_{eq} + \sqrt{(\mu'^2 + \mu''^2)(\varepsilon'^2_{eq} + \varepsilon''^2_{eq})}\right)} \geq \omega\sqrt{\mu_0\varepsilon_0}.$$

The parameters α and β defined above –the complex attenuation and phase constants– generalize the ones presented in *time domain.* In the *frequency domain*, not only the losses due to the presence of non zero conductivity but also the losses due to dispersivity are taken into account.
Moreover, the congruency (symbolized by "≅") regarding the field solutions in eqs. (2.31)-(2.32) have to be understood in "complex sense", that is, a function which is the result of multiplying eqs. (2.31) or (2.32) by any complex scalar factor is also a solution.
Even so, the physical properties regarding the attenuation and velocity of waves are the same as in *time domain.*
The greatest usefulness regarding the analysis in the *frequency domain* is in having both real parameters α and β in only one complex parameter γ, which, in turn, generalizes losses due to conductivity and dispersivity.

Just like the solutions in *time domain*, the parameters α, β, and of course the parameterized frequency ω, are the same for the electric and magnetic fields regarding the same EM wave. Likewise, the amplitude functions should be (complex) proportional if the field is required to verify both particularized *Faraday and Ampère's laws* in eqs. (2.22) and (2.23). In this way,

$$f(t_1, t_2) \equiv f_e(t_1, t_2) \propto f_h(t_1, t_2) \equiv \eta f_h(t_1, t_2), \tag{2.33}$$

, in which $\eta \in \mathbb{C}$ plays the role of the complex impedance of the medium, so the fields result in

$$E_{t_1} \cong f(t_1, t_2)e^{\mp\alpha z}e^{\mp j\beta z}, \text{ and} \tag{2.34}$$
$$H_{t_1} \cong \eta f(t_1, t_2)e^{\mp\alpha z}e^{\mp j\beta z}. \tag{2.35}$$

If substituying the field eqs. (2.34) and (2.35) in either the particularized *Faraday's law* in eq. (2.22) or the particularized *Ampère's law* in eq. (2.23), it gets to the same expression of the complex impedance

$$\eta \equiv \eta(\omega) = \pm\frac{j\omega\mu}{\gamma} = \pm\frac{\omega\mu}{k} \equiv \pm\frac{\gamma}{j\omega\varepsilon_{eq}} = \pm\frac{k}{\omega\varepsilon_{eq}} = \pm\sqrt{\frac{\mu}{\varepsilon_{eq}}} \in \mathbb{C}. \tag{2.36}$$

Notice that the expression of the impedance in the *frequency domain* is much more simple than the expression of the impedance in eq. (2.18) expressed in *time domain.* In addition, the phase difference between the electric and magnetic fields, $\Delta\varphi$, is included in the phase of the complex impedance, φ_η, so

$$\Delta\varphi \equiv \varphi_\eta = -\tfrac{1}{2}\tan^{-1}\left(\tfrac{\mu''}{\mu'}\right) + \tfrac{1}{2}\tan^{-1}\left(\tfrac{\varepsilon''_{eq}}{\varepsilon'_{eq}}\right), \tag{2.37}$$

which not only depends on conductor losses but also on the losses due to dispersivity.

Studying the parameters in the *frequency domain* lets analyizing them geometretrically because of the inmmedate representation of complex parameters on complex planes. This is just as it has been shown in analysis presented in [Rie98].
When relating the phases of the propagation constant, φ_γ, and the characteristic impedance, φ_η, it leads to the following condition regarding HPW in *frequency domain*

Harmonic plane wave phase cond.

$$\boxed{\varphi_\gamma + \varphi_\eta \leq \tfrac{\pi}{2}.}$$

Notice that this condiction generalizes the geometrical interpretation between α, β, and $\Delta\varphi$ which has been presented in Fig. 2.5 in the *time domain* analysis.

Remark 5. *The analysis in the frequency domain, which leads to the complex analysis of the parameters that determine the characteristics of EM propagation of each harmonic (parameterized by ω), noticeably reduce the complexity of the expressions in time domain.*
In addition, since these complex parameters have direct representation in complex planes, the geometrical analysis become natural when doing the appropriate complex analysis. These complex analysis are expected to leave the required physical interpretations regarding the behavior of the parameters in terms of losses (the constitutitve parameters).

The same Example 2.2.1 solved in *time domain* may be addressed here in the *frequency domain* but taking into accout that uniform plane waves in the cross section are complex constant.
This analysis is different from complexifying the coordinates $[\mathrm{t}_1, \mathrm{t}_2]$ in the cross section under the same complex variable, and impose the *harmonic plane condition* in complex variable on the resultant complex domain. In fact this method of complexifying the coordinates is a representative example of that for which the complex analysis of EM is aimed for (introducedd as objective for previous works in Sect. 1.2 in Chpt. 1 and inherited by CTLA).
This analysis leads to have complex derivatives and thus the Laplacian operator becomes "universal" regardless the geometry of the problem included the BCs, just as it is presented in Appendix 2.D.

In any case, the dependence in the cross section does not determine the characteristics of propagation of harmonic plane waves. This significative result may be used to study the wave solutions in different structures under the same theory. Thus, equivalent voltage and current waves would generalize the propagative fields.
The origin of these equivalent waves and the particularization of main equations to the ones that govern them: the *telegrapher's equations*; is studied in next section. The definition of these waves and the transformations of the underlying equations that govern them are the bases of the TLT. Moreover, any generalization from the usual TLT should do the same. In particular, the GTLT-v1 introduced in Chpt. 3 defines equivalent waves from a priori non-harmonic non-plane waves and particularizes the *Maxwell equations* to obtain the equivalent *telegrapher's equations.* Of course, this generalized version is able to explain the particular solutions such as HPWs, but the process is inverse: from a general case to those particular cases which are of interest.

2.3 Equivalent waves. Telegrapher's equations.

The usual Transmission Line Theory (TLT) bases on the study and parameterization of HPW defined in Sect. 2.2. As it has been seen, these waves behave in such a way that the propagation is completely independent of the variation in the cross section. This fact makes that different waves which propagate in different domains (physical structures) may be studied under the same theory, simplifying the original EM problem.
Keeping these ideas in mind, the TLT means a generalization in itself, in the sense that it tries to gruop all the possible scenarii in which the solutions to the particularized wave equations in eqs. (2.7)-(2.8) are HPWs, that is, those solutions regarding the particularized wave equations in eqs. (2.11)-(2.12) expressed in the *time domain*, or those regarding the Helmholtz equations in eqs. (2.26)-(2.27) in the *frequency domain*. These generalizations are, without a doubt, useful when analyzing the propagation of HPWs in different media by simply parameterizing the physical properties into the same circuit scheme.

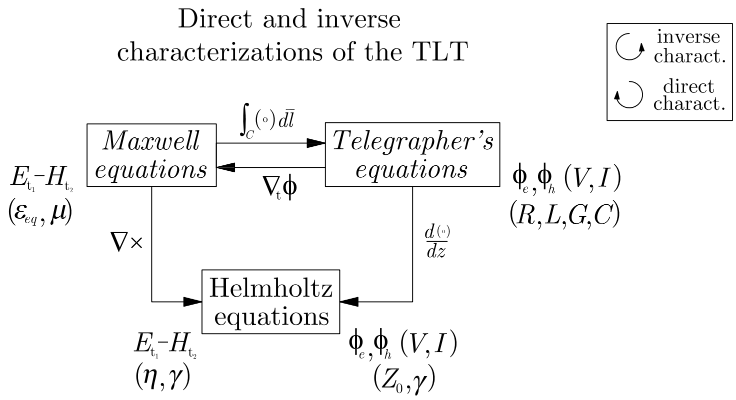


Fig. 2.6: Scheme that summarizes both procedures to obtain the Helmholtz equations corresponding to the direct and the inverse characterizationn of the TLT.

The objective in this section is to introduce the generalization of defining the equivalent voltage and current waves (the functions V, I in Fig. 2.6, which are equivalent to ϕ_e and ϕ_h, respectively, if seeing these in a more general frameworks such as the one in which GTLT-v1 introduced in Chpt. 3 is posed), to deal with them in the LTLT[17]. As a result of posing this generalization, the parameterization of the problem may be done by means of circuital components (the line parameters in Fig. 2.6). These circuital elements are the parameters in the *telegrapher's equations*, which appear replacing *Maxwell equations* (*Maxwell equations* $\rightarrow$ *Telegrapher's equations* in Fig. 2.6). The *telegrapher's equations*, which differentially relate the voltage and currents by means of the circuital elements, define two networks based on lumped components per unit lenght (p.u.l.). These networks physically parameterize the medium (losses included) and they define the equivalent TL. Then, the *telegrapher's equations* are solved by decoupling the voltage and current waves (*Telegrapher's equations* $\rightarrow$ *Helmholtz equations* in Fig. 2.6). The parameters which define the voltages and currents in the TL are congruent with those that parameterize the EM waves, in such a way that these latter ones can be restored from the equivalent waves and circuital parameters, and so the original solutions of *Maxwell equations*.
This procedure to finally obtain the solutions on an equivalent TL is called the direct characterization of the TLT (represented clockwise in Fig. 2.6). This is the natural way to define the TLT. It is also the most accurate because the equivalence between the medium and the representative TL is found in the physical meaning of the problem.

The alternative to this definition consists in assuming the equivalence between the parameters

[17]The difference between the TLT and the LTLT will not be explicitly detached anymore. The (lossless) TLT is a particular case of the LTLT, which allows for studying lossy TLs rigorously.

which characterize the propagative solutions of *Maxwell equations* obtained when decoupling in Helmholtz equations, and the parameters which characterize the equivalent *telegrapher's equations*. For this purpose, the equivalent waves ϕ_e and ϕ_h (playing the roll of V and I) are asummed to verify the (general) *telegrapher's equations* (in which the line parameters may be even complex), while they represent (by differentiation) the electric and magnetic field solutions (*Maxwell equations*←*telegrapher's equations* in Fig. 2.6). Then the obtained field Helmholtz equations are solved in terms of the constitutive parameters (Helmholtz equations←*Maxwell equations* in Fig. 2.6), and the parameters which identify the field solutions are mapped onto the parameters which identify the solutions of *telegrapher's equations* through the line parameters.
This procedure to define the solutions of the TL is called the inverse characterization of the TLT (represented counterclockwise in Fig. 2.6). This way to proceed is much more generalized but less specific than the direct characterization, so that it is the method in a generalized version of the TLT (then presented as GTLT-v1 when operating in the *frequency domain*).

Furthermore, each form of obtaining the solutions in the TLT is in the same sense of analyzing the parameters of the TL in Chpt. 4.

Notice that it is assumed working in the *frequecy domain* because the facilities in the analysis it presents in comparison with the *time domain*, just as it has been explained in Sect. 2.2. This assumption does not suppose any loss of generality in the study but a useful simplification in the expression of the parameters and a great generalization regarding the inclusion of losses.

This section is intended to explain and develop the direct characterization of HPW, and so obtaining the equivalent voltage and current waves which define the *telegrapher's equations* and the circuital parameters in the *frequency domain*.
Notice that this analysis could be repeated for non harmonic plane waves leading to a new direct characterization. However, this proccess is not efficient in the objective of parameterizing the solutions. Thus, a generalized version of the TLT based on the inverse characterization is introduced in Chpt. 3. In this generalized version of the TLT the same solutions based on HPWs may be obtained, which is exactly what it is done in Chpt. 3, allowing the comparison between the inverse and the direct characterization presented here.

2.3.1 Direct characterization of harmonic plane waves

As it is previously mentioned, the direct characterization of HPWs deals with the transformation of *Maxwell equations* to obtain the equivalent *telegrapher's equations*, to be these ones decoupled obtaining the equivalent Helmholtz equations.

Direct characterization of the TLT

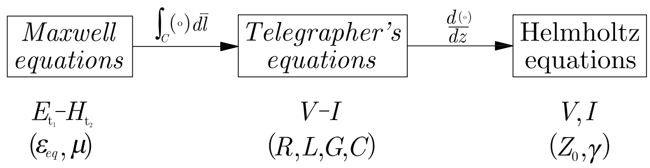


Fig. 2.7: Scheme that summarizes the transformations regarding the direct characterization of the LTLT.

For this purpose, the analysis starts obtaining the equivalent voltage and current waves by means of definite integrals of the electric and magnetic fields, respectively. Then, from these wave definitions, the equivalent *telegrapher's equations* are obtained by integrating *Maxwell equations*, at the same time that the line parameters of the equivalent TL are deduced emphasizing their physical interpretation. Finally, the equivalent voltage and current waves are solved by decoupling the *telegrapher's equations*. Each solution is parameterized by the basic parameters, which are expressed in terms of line parameters. The sum of the solutions leads to the total voltage and currrent waves together with the definition of wave parameters which relate them.

Equivalent waves

In this section, the equivalent voltage and current waves are obtained by using properties inherited from HPWs. These solutions are described in a general coordinate system, taking advantage of the general notation which has been previously introduced in Sect. 2.2. The properties of harmonic functions lets the integration being definite and unique, and so the expression of equivalent waves results simplified.
This study leads to define the characteristics of the contours bounding the cross section, as well as the BCs imposed on them.
The equivalences will be helpful when next defining the equivalent *telegrapher's equations* from *Maxwell equations*.

Let's start supposing a random planar region D like the one in Fig. 2.2, but now delimited by the contour C, and the cylindrical volume[18] $\mathcal{V}$ sorrounding D, which is obtained by the extrusion of D along the z-axis.

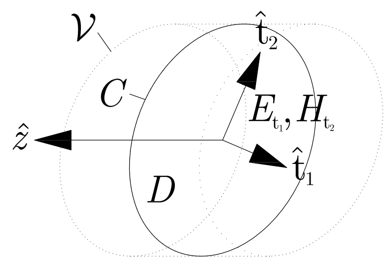


Fig. 2.8: Cylindrical volumen $\mathcal{V}$ obtained by the extrusion of the domain D delimited by C where the harmonic plane wave propagates.

Suppose that HPWs exist in $\mathcal{V}$, so eqs. (2.26)-(2.27) verify there. Recall that the *harmonic plane wave condition* written in the *frequency domain* ensures

$$\Delta_{\mathrm{t}} E_{\mathrm{t}_1} = 0, \tag{2.38}$$
$$\Delta_{\mathrm{t}} H_{\mathrm{t}_2} = 0, \tag{2.39}$$

in D, so the fields are harmonic there.
The fact that both E_{t_1} and H_{t_2} are harmonic is useful to generalize the field expressions by means of equivalent voltage and current waves.

First of all, it is interesting to describe the appropritе BCs over C for the HPWs to live in D (and so in $\mathcal{V}$). For this purpose, the properties of harmonic functions are consciuously used.

On one hand, the fact that the field functions are harmonic in D guarantees that the integration of E_{t_1} and H_{t_2} along any path in $\bar{D} \equiv D \cup C$ does not depend on the shape of the path itself. This

[18]Remember that a "cylindrical volume" does not refer uniquely a cylinder, but a geometry invariant along the z-axis.

is demostrated by using lemmas of harmonic functions, [BC90][19].
In addition, there exists primitive functions, [BC90], which are denoted by $\phi_e \equiv \phi_e(\mathrm{t}_1, \mathrm{t}_2)$[20] and $\phi_h \equiv \phi_h(\mathrm{t}_1, \mathrm{t}_2)$ associated with E_{t_1} and H_{t_2}, respectively, such that

$$\int_{P_{1-2}} E_{\mathrm{t}_1} dl = \phi_e(\mathrm{t}_{1,2}, \mathrm{t}_{2,2}) - \phi_e(\mathrm{t}_{1,1}, \mathrm{t}_{2,1}), \tag{2.40}$$

$$\int_{P_{1-2}} H_{\mathrm{t}_2} dl = \phi_h(\mathrm{t}_{1,2}, \mathrm{t}_{2,2}) - \phi_h(\mathrm{t}_{1,1}, \mathrm{t}_{2,1}). \tag{2.41}$$

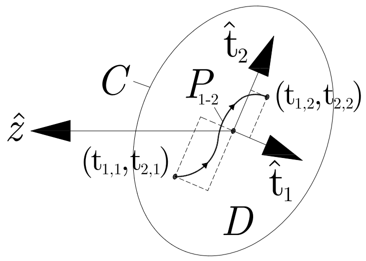


Fig. 2.9: A random path P_{1-2} between the points $(\mathrm{t}_{1,1}, \mathrm{t}_{2,1})$ and $(\mathrm{t}_{1,2}, \mathrm{t}_{2,2})$ in the cross section D.

The points $(\mathrm{t}_{1,1}, \mathrm{t}_{2,1})$ and $(\mathrm{t}_{1,2}, \mathrm{t}_{2,2})$ are, respectively, the initial and final points of the path P_{1-2}, as depicted in Fig. 2.9. If the path is a parametizable simple curve, [MP77], with general equation

$$l(\mathrm{t}_1, \mathrm{t}_2) = 0, \tag{2.42}$$

then the integrals in eqs. (2.40) and (2.41) may be generalized to

$$\int_{P_{1,2}} E_{\mathrm{t}_1} dl \equiv \int_{\mathrm{t}_{1,1}}^{\mathrm{t}_{1,2}} (\nabla\phi_e \nabla l)\, h_1 d\mathrm{t}_1 \equiv \phi_e(\mathrm{t}_{1,2}, \mathrm{t}_2) - \phi_e(\mathrm{t}_{1,1}, \mathrm{t}_2) \ \ \forall \mathrm{t}_2, \tag{2.43}$$

$$\int_{P_{1,2}} H_{\mathrm{t}_2} dl \equiv \int_{\mathrm{t}_{2,1}}^{\mathrm{t}_{2,2}} (\nabla\phi_h \nabla l)\, h_2 d\mathrm{t}_2 \equiv \phi_h(\mathrm{t}_1, \mathrm{t}_{2,2}) - \phi_h(\mathrm{t}_1, \mathrm{t}_{2,1}) \ \ \forall \mathrm{t}_1, \tag{2.44}$$

in which h_1 and h_2 are the scale factors of the coordinates t_1 and t_2, respectively.
Notice that the equivalences in eqs. (2.43) and (2.44) are possible thanks to the definition of the electric and magnetic fields in the $\hat{\mathrm{t}_1}$ and $\hat{\mathrm{t}_2}$ components, respectively, and also because the ortogonality between the coordinates $[\mathrm{t}_1, \mathrm{t}_2]$. This reduces the integrals to simple ones.
If the integrals do not depend on the shape of the path but these ones may be reduced to integrals in only one coordinate as in eqs. (2.43) and (2.44), it means that the primitive functions ϕ_e and ϕ_h are constant along t_2 and t_1, respectively. Congruently, E_{t_1} and H_{t_2} are orthogonal to these curve levels of ϕ_e and ϕ_h, respectively.

On the other hand, the harmonicity ensures that fields are at least class $\mathcal{C}^2$ in D, and class $\mathcal{C}^1$ in $\bar{D}$. For the primitive functions ϕ_e and ϕ_h not to be constant on D, the contour C can not be simple connected. This is direct consequence of the mean value theorem, [Bro96], and it is explained in Appendix 2.E in detail. Thus, C is mandatorily multiply connected.

[19] It is possible to use the generalization of harmonic complex functions in order to get the lemmas. For this purpose, the domain D should be complexified, just as it is presented in Appendix 2.D, instead of particularizing the lemmas of complex functions to real bivariate functions.
[20] The dependence on z is omitted in the definitions regarding D. This does not affect the equations based on integrals of fields (field integral equations, FIEs) expressed in D because the dependece on z is separable and it can be eliminated from both sides of the related equations.

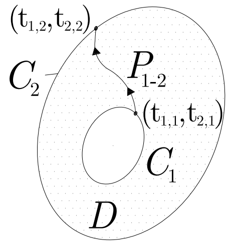
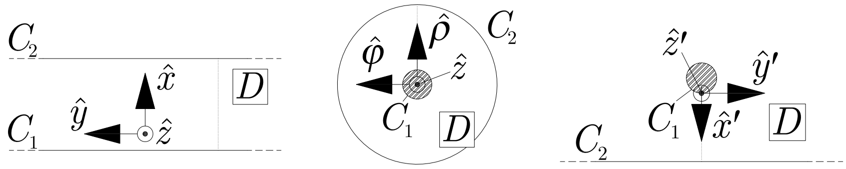


Fig. 2.10: A generic representation of the multiply connected cross section in which a random path P_{1-2} goes from C_1 to C_2.



(a) Infinite parallel plates (b) Coaxial (c) Wire and infinite plate

Fig. 2.11: Some examples of multiply connected sections in which C_1 and C_2 are separated. They all support harmonic plane waves, and thus unique equivalent voltage and current waves may be defined in these structures.

In Fig. 2.10 a generic multiply connected cross section is represented. There the path P_{1-2} represents a random curve which goes between the separated contours C_1 and C_2.
Some examples of multiply connected regions are depicted in Fig. 2.11. For each case, the coordinates $[\mathrm{t}_1, \mathrm{t}_2]$ have been particularized to any well-known orthogonal coordinate system which adjusts to the geometry of the domain. This may be done by defining either an universal coordinate system as in Fig. 2.11a or Fig. 2.11b, or a local one as in Fig. 2.11c[21].
Multiply connected regions could be extended to more than two separated closed contours. Also notice that any contour which is infinitely extended is also a closed contour[22], for example, in the infinite parallel plates in Fig. 2.11a.
In any case, the primitive functions ϕ_e and ϕ_h should be constant wherever the coordinates t_1 and t_2 are constant, respectively, and the coordinate system is mandatorily othogonal for the definitions above to verify.

Remark 6. *If the fields are constant along one of the transveral coorditates (as the primitives are), the geometry of the domain D has to be invariant along this coordinate. This is inverse consequence of imposing IBCs in that coordinate, which makes any field constant on it.*
Some geometries parameterizable with orthogonal coordinate systems in which one of them varies in all the range are the parallel plates or the coaxial, for example.

Both primitive functions ϕ_e and ϕ_h have clear physical meaning. They are the electric and magnetic potential funtions, respectively.
The electric potential function, ϕ_e, which is the antiderivative of the electric field, is the voltage function, it has volts ([V]) as units, and the difference of this function evaluated in different points on the cross section is the electric potential difference. Since ϕ_e is constant along the contours described by the equation

$$C\,(\equiv C_e) : \mathrm{t}_1 = a, \tag{2.45}$$

[21]In the case of describing a geometry in a coordinate system which does not adjust to it, which means that the multiply connected contours are non constant in the selected coordinate system, the BCs to impose become non constant functions. This increases the difficulty of solving the fields in the cross section, although this is not strictly required for defining equivalent voltage and current waves.

[22]The idea of considering infinitely large contours as closed contours is topologically accepted.

being a a real constant in the range of t_1, $\phi_e(C_2)-\phi_e(C_1)$ is also constant if C_1 and C_2 are written in terms of eq. (2.45). As a consequence, the definite integral in eq. (2.43) is constant if the path P_{1-2} connects the contours C_1 and C_2:

$$V_0 = \int_{P_{1-2}} E_{\mathrm{t}_1} dl = \phi_e(\mathrm{t}_{1,2}, \mathrm{t}_2) - \phi_e(\mathrm{t}_{1,1}, \mathrm{t}_2),\ \mathrm{t}_{1,2} \in C_2 \text{ and } \mathrm{t}_{1,1} \in C_1,\ \forall \mathrm{t}_2. \tag{2.46}$$

The magnetic potential function, ϕ_h, which is the antiderivative of the magnetic field, is the current function, it has amps ([A]) as units, and the difference of this function evaluated in different points on the cross section is the magnetic potential difference. Since ϕ_h is constant along the contours described by the equation

$$C\,(\equiv C_h) : \mathrm{t}_1 = b, \tag{2.47}$$

in which b is a real constant in the range of t_2, $\phi_h(C_2) - \phi_e(C_1)$ is also constant if C_1 and C_2 are written in terms of eq. (2.47). As consequence, the definite integral in eq. (2.44) is constant if the path P_{1-2} connects these contours:

$$I_0 = \int_{P_{1-2}} H_{\mathrm{t}_2} dl = \phi_h(\mathrm{t}_1, \mathrm{t}_{2,2}) - \phi_h(\mathrm{t}_1, \mathrm{t}_{2,1}),\ \mathrm{t}_{2,2} \in C_2 \text{ and } \mathrm{t}_{2,1} \in C_1,\ \forall \mathrm{t}_1. \tag{2.48}$$

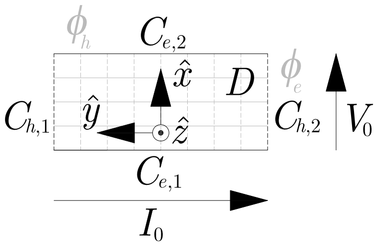


Fig. 2.12: Example of structure based on finite parallel plates which supports both ϕ_e and ϕ_h potentials whose curve levels are represented in horizontal continuous and vertical dashed lines, respectively. The potential difference between the contours on the extremes is V_0 and I_0 for the electric and magnetic potentials, respectively.

Although it is usual to define ϕ_e in structures which are totally symmetric respect to the $\hat{\mathrm{t}_2}$ direction (in the sense of they are invariant along this direction), for example the parallel plates along $\hat{y}$ in Fig. 2.11a, or the coaxial along $\hat{\varphi}$ in Fig. 2.11b, in which imposing $\phi_e(C_2) - \phi_e(C_1) = V_0$, the theory which is presented here may be generalized to non symmetric structures, for example, finite parallel plates in Fig. 2.12.
The first approach in which ϕ_e is constant along a contour is the most common case because contours in which the electric potential is constant are physically realizable. They are the so called Perfect Electric Conductors (PEC) in comparison with Perfect Magnetic Conductors (PMC), which are more difficult to synthesize.
However, the first case in which the contours are PECs also invariant along the t_2 direction makes ϕ_h constant, also because the contours are not multiply connected in this direction. As a result, the magnetic field can not be obtained by differentiating the primitive ϕ_h, and I_0 is not the result of the integration in eq. (2.48). Thus, these "invariant cases" can not be addressed using strictly the direct characterization. Nevertheless, they can be studied if a equivalent structure delimited by PMCs is defined in such a way that the EM solution in the cross section does not vary from the original problem.

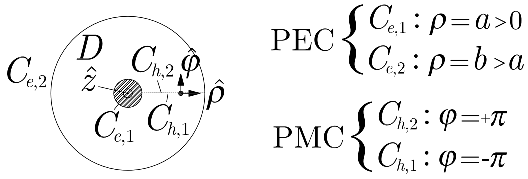


Fig. 2.13: Example of equivalent coaxial structure in which PMCs are placed at $\varphi = -\pi$ and $\varphi = +\pi$. These placements do no vary the form of the fields in the cross section of a coaxial geometry but lets to define I_0 from the magnetic potential ϕ_h.

In Fig. 2.13 an equivalent structure with PMCs placed in the interior of a coaxial geometry lets analyzing this case by using both potential function ϕ_e and ϕ_h without changing the solution regarding the original problem. Another example of an equivalence between an infinite (invariant) structure and a structure bounded by PMCs is that established between infinite parallel plates and finite parallel plates depicted in Fig. 2.12.

With these definitions of V_0 and I_0, it is possible to define the voltage and current waves and the parameter which relates them.

Remark 7. *The TLT as it is usually known does not analyze the electrostatical and magnetostatical problem for every possible structure which supports plane waves. The fact that the EM fields come from potential functions (primitives) is used to "universalize" the solutions making use of voltage and electric current waves. This connects with the idea of having the propagative behavior of EM fields independent from the form that they present in the cross section.*
In addition, since the fields are harmonic, the equivalent waves are defined by constants in the cross section obtained by means of definite integrals of the electric and magnetic fields. However, these integrals are not strictly neccesary, so they could be non definite without changing the solutions, precisely because the independence between the propagative behavior and the variation of the fields/equivalent waves in the cross section.

The amplitudes are written as factors of the exponential terms inherited from the fields in eqs. (2.28)-(2.29), which determine the variation along the z-axis:

$$V^{\pm}(z) = V_0 e^{\mp jkz} = V_0^{\pm} e^{\mp \gamma z}, \tag{2.49}$$

$$I^{\pm}(z) = I_0 e^{\mp jkz} = I_0^{\pm} e^{\mp \gamma z}. \tag{2.50}$$

These are the equivalent voltage and current waves. They have V_0 and I_0 as amplitudes, respectively, regardless the structure under study that they parameterize.
The relation between V_0 and I_0 is given by:

$$Z_0 = \frac{V^{\pm}(z)}{I^{\pm}(z)} = \frac{V_0^{\pm}}{I_0^{\pm}}, \tag{2.51}$$

which is known as the characteristic impedance, and it fixes the true dependence on physical and constitutive parameters.

Remark 8. *The TLT focuses on the study of the characteristic impedance instead of the equivalent waves in themselves. The characteristic impedance parameterizes the dependence of the equivalent waves in the cross section, both dimensionally and in its constitutive parameters.*

Once the equivalent waves have been obtained and the procedure that leads to define them has been described, the equivalent *telegrapher's equations* may be rigorously obtained from *Maxwell equations* in order to see the equivalent TL problem.

Equivalent ***telegrapher's equations***

With the equivalent voltage and current waves defined by integrating the field solutions regarding HPWs, the *telegrapher's equations* are obtained by repeating this integration of those fields in *Maxwell equations*.
The procedure of transforming *Maxwell equations* lets defining the line parameters of an equivalent TL. The physical definitions of these line parameters are generalized in the sense that they are written as functions of constitutive and dimensional parameters associated with any geometry in the cross section.
The transformations and definitions here lead to important physical interpretations of the equivalent TL seen as circuit. These phsysical interpretations are then analyzed in Chpt. 4 in detail. Moreover, they let to define the basic parameters as functions of the line parameters once the equivalent *telegrapher's equations* are decoupled leading to *Helmholtz equations*.

For the purpose of obtaining the equivalent *telegrapher's equations*, let's start tranforming the *Ampère's law*[23]. For this purpose, consider the generic cylindrical volume extruded dz from the cross section D, which is multiply connected:

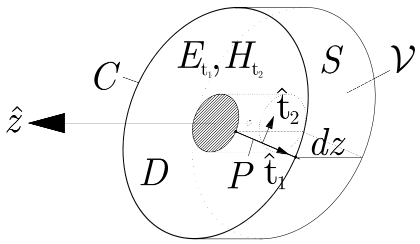


Fig. 2.14: Representation[24]of the volumen extruded dz from the cross section surrounded by the surface $S \equiv C \cup dz$ whose normal vector is t_1. Moreover, the path P connects the separated contours in the cross section (P goes from $(t_{1,1}, t_2)$ to $(t_{1,2}, t_2)$), and it is useful for the definition of the conductance and capacitance (or complex capacitance).

Integrating the particularized *Ampère's law* in eq. (2.23) in D and using the Stokes' theorem p.u.l.[25] in the right hand side (r.h.s.):

$$\begin{aligned} \int_{t_1=t_{1,1}}^{t_{1,2}} \left(\int_{\langle t_2 \rangle} \frac{\partial \left(\hat{z} \times H_{t_2} \hat{t_2} \right)}{\partial z} h_2 dt_2 \hat{t_1} \right) h_1 dt_1 = \\ \int_{t_1=t_{1,1}}^{t_{1,2}} \left(\int_{\langle t_2 \rangle} H_{t_2} \hat{t_2} h_2 dt_2 \hat{t_2} \right) h_1 dt_1 = \\ = j\omega\varepsilon_{eq} \int_{t_1=t_{1,1}}^{t_{1,2}} \left(\int_{\langle t_2 \rangle} E_{t_1} \hat{t_1} h_1 dt_1 \hat{t_1} \right) h_1 dt_1 \end{aligned} \tag{2.54}$$

[23]This law is much more intuitive to transform because the integration uncovers the electric effects: conductance and capacitance; which appears connected with electric potentials. This potentials are –as it has been detached in the previous section– more intuitive to understand because of the classic correspondence with the electric potentials in electrostatic, in comparision with the "magnetic potentials" (currents) in magnetostatic.

[24]This figure is only a scheme, not a geometry in itself. In this case, the contours in C represent curves with t_1 constant and so t_2 varies in the whole range where it is defined ($\langle t_2 \rangle$).

[25]The Stokes' theorem applied to the surface $S' \equiv \langle t_2 \rangle \times dz$ having $\hat{t_1}$ as normal vector:

$$\iint_S \nabla \times \boldsymbol{A} dS \hat{t_1} = \oint_{C \equiv \bar{S}} \boldsymbol{A} d\bar{l}; \tag{2.52}$$

reduces to the Stokes' theorem p.u.l. as

$$\iint_{\langle t_2 \rangle \times dz} \frac{\partial \left(\hat{z} \times \partial A_{t_2} \hat{t_2} \right)}{\partial z} h_2 dt_2 \hat{t_1} = \oint_{t_2} A_{t_2} \hat{t_2} h_2 dt_2 \hat{t_2}, \tag{2.53}$$

because the vector $\boldsymbol{A} \equiv A_{t_2} t_2$ is orthogonal to P.

The equation before has been organized in such a way that the Stokes' theorem p.u.l. can be applied to the magnetic field in the r.h.s. The integration along t_2 is in the complete range of t_2 but it is not closed becuase the definition of the contour p.u.l. In Appendix 2.F the transformation of both sides of eq. (2.54) to the equivalent waves has been carried out. The result in eq. (2.F.29) which relates $I(z)$ and $V(z)$ is brought here:

$$I(z) = j\omega\varepsilon_{eq}\frac{L_{t_2}}{H_{1,2}}V(z), \tag{2.55}$$

in which L_{t_2}[26] is the measure of the range of t_2, and $H_{1,2}$[27] is the definite integral of the scale factor h_1/h_2 in the range of t_1.
Notice that the coefficient $\varepsilon_{eq}L_{t_2}/H_{1,2}$ plays the role of "complex capacitance" p.u.l. in the sense that it relates the current and the trasnformation of the derivative of the voltage in the *frequency domain*. The "complex capacitance" refers to a capacitance with a parallel condutance. If dividing $\varepsilon_{eq}L_{t_2}/H_{1,2}$ in the real and imaginary parts, the capacitance p.u.l., C, and conductance p.u.l., G, are obtained separately:

$$\begin{cases} C = \varepsilon'_{eq}\frac{L_{t_2}}{H_{1,2}} & \geq \varepsilon_0 \\ G = \omega\varepsilon''_{eq}\frac{L_{t_2}}{H_{1,2}} & \geq 0 \end{cases}{}^{28}. \tag{2.56}$$

As it has been remarked, the integration in eq. (2.54) has been done in the cross section D and thus the circuital parameters C and G are expressed p.u.l. Nevertheless, the integration may be extended to the volume $\mathcal{V}$ in Fig. 2.14 in the following form:

$$\oiint_S \frac{\partial\left(\hat{z}\times H_{t_2}\hat{t_2}\right)}{\partial z}d\bar{S} = j\omega\varepsilon_{eq}\oiint_S E_{t_1}\hat{t_1}d\bar{S}, \tag{2.57}$$

in which S is the surface sorroundig $\mathcal{V}$, that is the clausure of $\mathcal{V}$ ($S \equiv \bar{\mathcal{V}}$). The r.h.s. in eq. (2.57) integrates dz times the r.h.s. in eq. (2.54). That is because the field E_{t_1} does not vary along dz and it repits dz times in the $\hat{z}$ direction, where the lateral surface has $\hat{t_1}$ as normal vector, so the total voltage is $V(z)dz$.
The l.h.s. in (2.57) equals the application of the Stokes' theorem continuously along dz. The adjacent line integrals cancel so it only remains the first integral minus last one, that is $I(z) - I(z+dz) \equiv -dI(z)$.
As a result, if considering the circuital parameters in eq. (2.56), eq. (2.57) is equivalent to

$$\frac{dI(z)}{dz} = -\left(G + j\omega C\right)V(z), \tag{2.58}$$

which is of the form of one of the *telegrapher's equations*.

Now let's transforming the *Faraday's law*. Consider the same cylindrical volume as in Fig. 2.14 but now the path covers the whole range of t_1 ($\langle t_1 \rangle$) and it also goes from $t_{2,1}$ to $t_{2,2}$ in the cross section:

[26] Do not confuse L_{t_2} with any self-inductance.
[27] Do not confuse $H_{1,2}$ with any magnetic field.
[28] The parameter ε'_{eq} may be lower than ε_0 if σ'' is not zero. However, this case is not usual because the conductivity is real in practice.

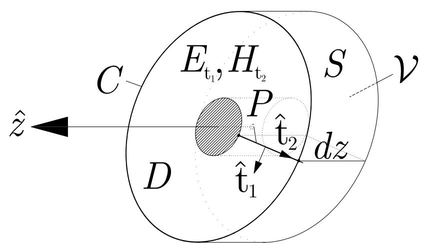


Fig. 2.15: Representation[29]of the volumen extruded dz from the cross section surrounded by the surface $S \equiv C \cup dz$ whose normal vector is t_2. Moreover, the path P connects the separated contours in the cross section (P goes from $(t_1, t_{2,1})$ to $(t_1, t_{2,2})$), and it is useful for the definition of the resistance and inductance (or complex inductance).

Integrating the particularized *Faraday's law* in eq. (2.22) in D, and using the Stokes' theorem p.u.l.[30] in the r.h.s.:

$$\begin{aligned} \int_{t_2=t_{2,1}}^{t_{2,2}} \left(\int_{\langle t_1 \rangle} \frac{\partial \left(\hat{z} \times E_{t_1} \hat{t_1} \right)}{\partial z} h_1 dt_1 \left(-\hat{t_2} \right) \right) h_2 dt_2 = \\ = \int_{t_2=t_{2,1}}^{t_{2,2}} \left(\int_{\langle t_1 \rangle} E_{t_1} \hat{t_1} h_1 dt_1 \hat{t_1} \right) h_2 dt_2 = \\ = -j\omega\mu \int_{t_2=t_{2,1}}^{t_{2,2}} \left(\int_{\langle t_1 \rangle} H_{t_2} \hat{t_2} h_1 dt_1 \left(-\hat{t_2} \right) \right) h_2 dt_2 \end{aligned} \tag{2.61}$$

This equation is reciprocally equivalent to the one defined in eq. (2.54). It has been organized in such a way that the Stokes' thorem p.u.l. can be applied to the electric field in the l.h.s. The integration along t_1 is in the whole range of t_1 but the path it is not closed. The procedure of transforming both sides of eq. (2.61) is completely equivalent of the one transforming eq. (2.54), which has been presented in Appendix 2.F. The result in eq. (2.F.30), which relates $V(z)$ and $I(z)$, is brought here:

$$V(z) = j\omega\mu \frac{H_{1,2}}{L_{t_2}} I(z), \tag{2.62}$$

in which L_{t_2} and $H_{1,2}$ are defiend as for eq. (2.54).
Notice that the coefficient $\mu H_{1,2}/L_{t_2}$ plays the role of "complex inductance" p.u.l. in the sense that it relates the voltage and the trasnformation of the time derivative of the current in the *frequency domain.* The "complex inductance" refers to a self-inductance with a series resistance. If splitting the term $\mu H_{1,2}/L_{t_2}$ into its real and imaginary parts, the self-inductance p.u.l., L, and the resistance p.u.l., R, are separately obtained:

$$\begin{cases} L = \mu' \frac{H_{1,2}}{L_{t_2}} & \geq 0 \\ R = \omega\mu'' \frac{H_{1,2}}{L_{t_2}} & \geq 0 \end{cases}. \tag{2.63}$$

Since the integration in eq. (2.61) has been done in the cross section D, the circuital parameters L and R are expressed p.u.l. Nevertheless, the integration may be extended to the surface S

[29]This figure is only a scheme, not a geometry in itself. In this case, the contours in C represent curves with t_2 constant and so t_1 varies in the whole range where it is defined ($\langle t_1 \rangle$).

[30]The Stokes' theorem applied to the surface $\langle t_1 \rangle \times dz$ having $\hat{t_2}$ as normal vector:

$$\iint_S \nabla \times \boldsymbol{B} dS \hat{t_2} = \oint_{C \equiv \bar{S}} \boldsymbol{B} \bar{dl}; \tag{2.59}$$

reduces to the Stokes' theorem p.u.l. as

$$\iint_{\langle t_1 \rangle} \frac{\partial \left(\hat{z} \times \partial B_{t_1} \hat{t_b} \right)}{\partial z} h_1 dt_1 \hat{t_2} = \oint_{t_1} B_{t_1} \hat{t_1} h_1 dt_1 \hat{t_1}, \tag{2.60}$$

because the vector $\boldsymbol{B} \equiv B_{t_1} \hat{t_1}$ is orthogonal to P.

sorrounding the volume $\mathcal{V}$ (the clausure of $\mathcal{V}$ or $\bar{\mathcal{V}}$) in Fig. 2.15 in the following form:

$$\oiint_S \frac{\partial \left(\hat{z} \times E_{t_1} \hat{t_1}\right)}{\partial z} d\bar{S} = -j\omega\mu \oiint_S H_{t_2} \hat{t_2} d\bar{S}. \tag{2.64}$$

The r.h.s. in eq. (2.64) integrates dz times the r.h.s. in eq. (2.61). That is because the field H_{t_2} does not vary along dz and it repits dz times in the $\hat{z}$ direction, where the internal section has $\hat{t_2}$ as normal vector, so $I(z)dz$.
The l.h.s. in (2.64) equals the application of the Stokes' theorem continuously along dz. The adjacent line integrals cancel so they only remains the first minus last ones, that is $V(z) - V(z + dz) \equiv -dV(z)$.
As a result, if considering the circuital parameters in eq. (2.63), eq. (2.64) is equivalent to

$$\frac{dV(z)}{dz} = -\left(R + j\omega L\right) I(z), \tag{2.65}$$

which is of the form of the other *telegrapher's equation*.

Once the equivalent *telegrapher's equations* in eqs. (2.58) and (2.65) have been obtained, the equivalent voltage and current waves may be decoupled in the equivalent Helmholtz equations and solved in terms of line parameters.
The general solutions to the *telegrapher's equations* and the general expression of basic parameters are next obtained.
Next sections are intended to particularize the parameters to those cases that will be analyzed in Chpt. 4. These cases may be obtained by particularizing the general solutions and parameters, which avoids to rewrite and solve the *telegrapher's equations* repeatedly.

Equivalent Helmholtz equations

In this section the equivalent voltage and current waves are solved by decoupling the equivalent *telegrapher's equations*, which have been obtained in the previous part when tranforming the particularized *Maxwell equations* by using the physical definition of line parameters, that is the direct characterization of the equivalent TL.
The procedure which is going to be followed to obtain the wave solutions of *telegrapher's equations* is well-known in the literature, [Poz98], so the method which guides the analysis is only briefly summarized. Nevertheless, the parameters which are going to be particularized in the following epigraphs are outlined to be then analyzed in Chpt. 4

The decoupled Helmholtz equations written in terms of the line parameters are obtained by differentiating eq. (2.65) in z (see this in the scheme in Fig. 2.6) and substituying eq. (2.58) on it, and viceversa, leading to

$$\frac{d^2V^{\pm}(z)}{dz^2} - \left(R + j\omega L\right)\left(G + j\omega C\right) V^{\pm}(z) = 0, \text{ and} \tag{2.66}$$

$$\frac{d^2I^{\pm}(z)}{dz^2} - \left(R + j\omega L\right)\left(G + j\omega C\right) I^{\pm}(z) = 0, \tag{2.67}$$

respectively. The solutions to eqs. (2.66) and (2.67) are of the form of the equivalent voltage and current waves in eqs. (2.49) and (2.50), respectively.

The parameter which identifies with the propagation constant of the equivalent waves is written in terms of the line parameters as

$$\gamma = \alpha + j\beta = \sqrt{\left(R + j\omega L\right)\left(G + j\omega C\right)} \in \mathbb{C}, \tag{2.68}$$

in which

$$\alpha = \frac{1}{\sqrt{2}}\sqrt{RG - \omega^2 LC + \sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}} \in \mathbb{R}, \text{ and} \tag{2.69}$$

$$\beta = \frac{1}{\sqrt{2}}\sqrt{-RG + \omega^2 LC + \sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}} \in \mathbb{R}. \tag{2.70}$$

In order to obtain the relation between the voltage and current waves in eq. (2.51), eqs. (2.58) and (2.65) are divided, separating the magnitudes and integrating the sides, leading to

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega G}} \in \mathbb{C}, \tag{2.71}$$

which is the expression of the characteristic impedance in terms of line parameters.

The line parameters in both the propagation constant in eq. (2.68) and the characteristic impedance in eq. (2.71) may be frequency dependent, and so these latter basic parameters are, besides their explicit dependence on the noted ω.

The total voltage in the TL is expanded by adding the individual solutions in eq. (2.49), while the total current is expanded by substracting the individual solutions in (2.50). Thus,

$$\begin{aligned} V(z) =& V^+(z) + V^-(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \equiv \\ \equiv& V_0^+ e^{-\gamma z} + \rho_0 V_0^+ e^{\gamma z} = V_0^+ e^{-\gamma z}\left(1 + \rho_0 e^{2\gamma z}\right),^{31} \end{aligned} \tag{2.72}$$

$$\begin{aligned} I(z) =& I^+(z) - I^-(z) = I_0^+ e^{-\gamma z} - I_0^- e^{\gamma z} \equiv \\ \equiv& I_0^+ e^{-\gamma z} - \rho_0 I_0^+ e^{\gamma z} = I_0^- e^{-\gamma z}\left(1 - \rho_0 e^{2\gamma z}\right), \end{aligned} \tag{2.73}$$

in which

$$\rho_0 \equiv \frac{V_0^-}{V_0^+} = \frac{I_0^-}{I_0^+} \in \mathbb{C}, \tag{2.74}$$

is the so called reflection coefficient.
Notice that the definition of ρ_0 is the same as the one obtained when relating the reflected and incident voltage or current waves in $z = 0$. Then, it is possible to define the reflection coefficient at any point of the TL from ρ_0 as

$$\rho(z) = \rho_0 e^{2\gamma z} \in \mathbb{C}, \tag{2.75}$$

so that the total voltage and current waves may be rewritten as

$$V(z) = V_0^+ e^{-\gamma z}(1 + \rho(z)), \tag{2.76}$$

$$I(z) = I_0^+ e^{-\gamma z}(1 + \rho(z)). \tag{2.77}$$

Relating these total voltage and current waves and taking into account the definition of the characteristic impedance in eq. (2.51), the impedance of the total wave, or wave impedance, is defined as

$$Z(z) \equiv \frac{V(z)}{I(z)} = Z_0 \frac{1+\rho(z)}{1-\rho(z)} = Z_0 \frac{1+\rho_0 e^{2\gamma z}}{1-\rho_0 e^{2\gamma z}} \in \mathbb{C}, \tag{2.78}$$

while the admittance of the total wave, or wave admittance, is the inverse quotient of the wave impedance:

$$Y(z) \equiv \frac{I(z)}{V(z)} \equiv \frac{1}{Z(z)} = Z_0 \frac{1-\rho(z)}{1+\rho(z)} \in \mathbb{C}. \tag{2.79}$$

[31] Notice the difference in the notation between the total solution, $V(z)$, and the individual solutions, $V^{\pm}(z)$.

Conversely, the reflection coefficient may be rewritten as function of $Z(z)$ or $Y(z)$, if it is solved from eq. (2.78) or eq. (2.79):

$$\rho(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} = \frac{1 - Z_0 Y(z)}{1 + Z_0 Y(z)} \in \mathbb{C}. \tag{2.80}$$

For defining the total waves, it is usual to reference the waves at the load ($Z_L \in \mathbb{C}$). Then, the voltage and current waves are defined towards the beginning of the TL from the load (at l from the load). With this reference, the voltage and current waves are

$$V^{\pm}(l) = V_L^{\pm} e^{\pm\gamma l}, \text{ and} \tag{2.81}$$

$$I^{\pm}(l) = I_L^{\pm} e^{\pm\gamma l}. \tag{2.82}$$

They are related by means of the relection coefficient

$$\rho(l) = \frac{V^-(l)}{V^+(l)} = \frac{I^-(l)}{I^+(l)} = \rho_L e^{-2\gamma l} \in \mathbb{C}, \tag{2.83}$$

for which $\rho_L \equiv \rho(l = 0)$ is the reflection coefficient at the load, which is defined from the impedance at the load as

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0} \in \mathbb{C}. \tag{2.84}$$

The wave impedance along the TL from the load is then written as

$$Z(l) = Z_0 \frac{1 + \rho(l)}{1 - \rho(l)} = Z_0 \frac{Z_L \cosh(\gamma l) + Z_0 \sinh(\gamma l)}{Z_L \sinh(\gamma l) + Z_0 \cosh(\gamma l)} \in \mathbb{C}, \tag{2.85}$$

whereas the wave admittance is

$$Y(l) \equiv \frac{1}{Z(l)} = \frac{1}{Z_0} \frac{1 - \rho(l)}{1 + \rho(l)} = \frac{1}{Z_0} \frac{Z_L \sinh(\gamma l) + Z_0 \cosh(\gamma l)}{Z_L \cosh(\gamma l) + Z_0 \sinh(\gamma l)} \in \mathbb{C}. \tag{2.86}$$

It is possible to obtain the expression of $\rho(l)$ in terms of $Z(l)$ or $Y(l)$ as:

$$\rho(l) = \frac{Z(l) - Z_0}{Z(l) + Z_0} = \frac{1 - Z_0 Y(l)}{1 + Z_0 Y(l)} \in \mathbb{C}. \tag{2.87}$$

The conversions which have been introduced in this section: (i) the equivalent voltage and current waves from harmonic EM waves, (ii) the equivalent *telegrapher's equations* from *Maxwell equations*, and (iii) the (equivalent) basic parameters defined from (equivalent) line parameters, and the wave parameters defined from the basic parameters; lets to study particular cases based on different definitions of line parameters and particularizations or approximations imposed on the basic and wave parameters.

Since the (equivalent) line parameters determine the parameters of the individual and total wave solutions, their definition p.u.l. in an equivalent circuital based on lumped components schematizes the equivalences presented in this section and defines de equivalent TL.

Two basic examples of TLs based on the definition of line parameters: the lossy TL and the lossless TL; an additional example based on the conditions imposed on the propagation constant: the non dispersive TL; and an approximation in the basic parameters: the low-losses approximation; are next presented. The circuit p.u.l. defines de equivalent TL in each case. Then, the basic parameters and total wave parameters are obtained particularizing the general solutions presented in this direct characterization.

In the following section, the physical meaning of each parameter under study is emphasized in first instance. Nevertheless, the analysis in terms of losses and frequency, and the subsequent physical interpretations, are obtained by means of Complex Analysis in Chpt. 4.

2.4 Transmission Line particular cases

In this section, four different TL cases or approximations are studied by means of the definition of their respective line parameters, basic parameters, and wave parameters, following the scheme presented as the direct characteriation (see Fig. 2.7).

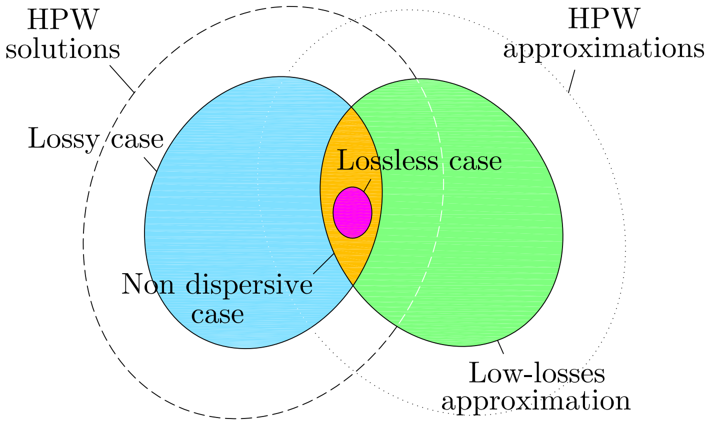


Fig. 2.16: Scheme that represents the sets of harmonic plane wave (HPW) solutions and approximations. The non dispersive cases are the HPW solutions with coincide with their respective approximations.

These cases are: the lossy case, the lossless case, the non dispersive case and the low-losses approximation.
Notice that the low-losses approximation is not a solution in the rigorous sense of the analysis presented in this chapter, as it is reflected in Fig. 2.16, but it will be useful for the analysis in Chpt. 4, in which may be rigorously interpreted.

2.4.1 Lossy Transmission Lines

In this section, the solution of lossy TLs in which the line parameters are not frequency dependent is obtained. This case corresponds with the physical description of harmonic plane waves which propagates in mediums whose constitutive parameters are specifically defined as functions of frequency.
This non frequency dependent lossy case is a particular case of the general lossy case presented in the section before, at the same time that it generalizes some particular cases, for example the lossless case presented in next section.
Because of the generalization this case supposes, the parameters are studied in deep in Chpt. 4 by means of Complex Analysis.

Line parameters

The line parameters regarding this lossy case are required to be non frequency dependent. This means that the constitutive parameters present the following specific behavior with frequency:

$$\begin{cases} \varepsilon' \geq \varepsilon_0 \\ \varepsilon'' \cong \frac{\varepsilon_0''}{\omega} \geq 0,\, \varepsilon_0'' \in \mathbb{R} \end{cases}, \begin{cases} \mu' \geq \mu_0 \\ \mu'' \cong \frac{\mu_0''}{\omega} \geq 0,\, \mu_0'' \in \mathbb{R} \end{cases}, \begin{cases} \sigma' \geq 0 \\ \sigma'' \cong \sigma_0''\omega,\, \sigma_0'' \leq \varepsilon',\, \sigma_0'' \in \mathbb{R} \end{cases}, \tag{2.88}$$

so that the line parameters

$$\begin{cases} C = (\varepsilon' - \sigma_0'')\,\frac{L_{t_2}}{H_{1,2}} \\ G = (\varepsilon_0'' + \sigma')\,\frac{L_{t_2}}{H_{1,2}} \end{cases} , \begin{cases} L = \mu'\,\frac{H_{1,2}}{L_{t_2}} \\ R = \mu_0''\,\frac{H_{1,2}}{L_{t_2}} \end{cases} , \tag{2.89}$$

and thus R, L, G, $C \geq 0$ are constant on frequency.

The equivalent circuits p.u.l. are:

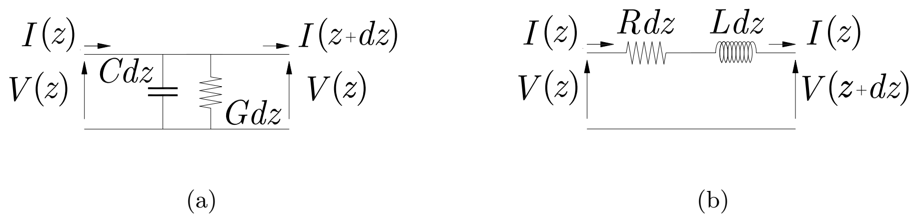


Fig. 2.17: Circuit schemes of line parameters defined p.u.l. in the equivalent TL. The analysis of these circuits leads to the equivalent *telegrapher's equations* which govern the general lossy case.

The analysis of the circuital schemes in Figs. 2.17a and 2.17b define the *telegrapher's equations* in eqs. (2.58) and (2.65), respectively.

Basic parameters

Since the non frequency dependent basic parameters of this case are a particularization of the general lossy case, the characterisstic impedance, Z_0, and the propagation constant, γ, expression are those in eqs. (2.71) and (2.68), respectively, and also the attenuation constant, α, and the phase constant, β, in eqs. (2.69) and (2.70), respectively.
The basic parameters determine the individual solutions, which are of the form of eqs. (2.49) and (2.50) for the voltage and current waves, respectively.

Notice that the basic parameters depend explicitly on frequency and thus, the equivalent waves will present certain degree of dispersivity.
The study of basic pameters in terms of line parametes and frequency regarding the direct characterization of the basic parameters is done in Chpt. 4 while emphasizing the physical interpretation of losses and their behavior when frequency varies.

Wave parameters

The wave parameters $\rho(z)$, $Z(z)$, and $Y(z)$ keep the form of eqs. (2.75), (2.78), and (2.79).
If the wave parameters are defined along l from the load: $\rho(l)$, $Z(l)$, and $Y(l)$; they are of the form of eqs. (2.83), (2.85), and (2.86).
These wave parameters determine the total solutions in the equivalent TL, which are of the form of eqs. (2.72) and (2.73) for the total voltage and current waves, respectively.

Since the wave parameters depend on the basic parameters, which depend, in turn, on frequency, the first ones also present different behavior when frequency varies.
The study of wave parameters in terms of both losses and frequency is carried out in Chpt. 4 while outlining the physical interpretations underlying the analysis.

2.4.2 Lossless Transmission Lines

In this section, the lossless mediums are parameterized into lossless TLs whose line parameters are not frequency dependent. This case also corresponds with the physical description of harmonic plane waves which propagate in lossless mediums whose constitutive parameters are not frequency dependent.
This lossless case is just a particular case of the lossy case presented before. Thus, it will be useful for the normalizations regarding the Complex Analysis when analyzing the basic and wave parameters in terms of losses in Chpt. 4.

Line parameters

In this case, the specifications of a lossless medium are imposed over the constitutive parameters:

$$\begin{cases} \varepsilon' \geq \varepsilon_0 \\ \varepsilon'' = 0 \end{cases}, \begin{cases} \mu' \geq \mu_0 \\ \mu'' = 0 \end{cases}, \begin{cases} \sigma' = 0 \\ \sigma'' = 0 \end{cases}, \tag{2.90}$$

so that the line parameters are:

$$\begin{cases} R = 0 \\ L = \mu' \frac{H_{1,2}}{L_{t_2}} \geq 0 \end{cases}, \begin{cases} G = 0 \\ C = \varepsilon' \frac{L_{t_2}}{H_{1,2}} \geq 0 \end{cases}. \tag{2.91}$$

By means of the line parameters, it can be seen that this case is a particular case of the lossy TL.

The equivalent circuits p.u.l. are:

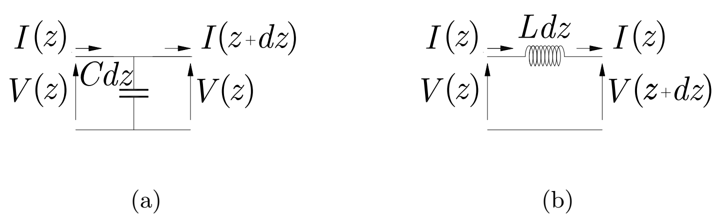


Fig. 2.18: Circuit schemes of line parameters defined p.u.l. in the equivalent lossless TL. The analysis of these circuits leads to the equivalent *telegrapher's equations* which govern the particular lossless case.

If analyzing the circuits in Fig. 2.18, the equivalent *telegrapher's equations* regardig the lossless case are obtained:

$$\frac{I(z)}{dz} = -j\omega CV(z), \tag{2.92}$$

$$\frac{V(z)}{dz} = -j\omega LI(z), \tag{2.93}$$

which is are the form of those in eqs. (2.58) and (2.65) when making $G = R = 0$.

Basic parameters

The basic parameters regarding the lossless case are denoted by $Z_{0,sp}$[32] and γ_{sp}, for the characteristic impedance and the propagation constant, respectively.
Since the lossless case is a particular case of both the general lossy case and the lossy case presented in Sect. 2.4.1, the lossless basic parameters are a particular case of those ones in these lossy cases when $G = R = 0$. The resultant lossless basic parameters are the same as the ones obtained when dividing or decoupling eqs. (2.92) and (2.93) to get the characteristic impedance and the propagation constant, respectively.

The lossless characteristic impedance and propagation constant result in:

$$Z_{0,sp} = \sqrt{\frac{L}{C}} \in \mathbb{R}, \tag{2.94}$$

and

$$\gamma_{sp} \equiv j\beta_{sp} = j\omega\sqrt{LC} \in \mathbb{I}, \tag{2.95}$$

respectively.
In this case, $\alpha_{sp} = 0$, and the characteristic impedance is pure resistive (real) so it presents 0-phase ($\varphi_{Z_{0,sp}} = 0$). Remember that α and φ_{Z_0} are the parameters which quantify lossless, so they are congruently zero in this lossless case.
When decoupling eqs. (2.92)-(2.93) the resultant voltage and current waves regarding the lossless TL are:

$$V^{\pm}(z) = V_0^{\pm} e^{\mp j\beta_{sp} z} \equiv V_L^{\pm} e^{\pm j\beta_{sp} l}, \tag{2.96}$$

$$I^{\pm}(z) = I_0^{\pm} e^{\mp j\beta_{sp} z} \equiv I_L^{\pm} e^{\pm j\beta_{sp} l}. \tag{2.97}$$

Notice that $Z_{0,sp}$ does not depend on frequency. Thus, this parameter is appropriate to normalize the lossy characteristic impedance when frequency varies. However, the propagation constant is a linear function of frequency, which means that, although lossless TLs are not dispersive (each harmonic propagates at the same v_p), this parameter is not appropriate for normalizing the general lossy case when the studies are frequency variable.

Wave parameters

The wave parameters $\rho(z)$, $Z(z)$, and $Y(z)$ in eqs. (2.75), (2.78), and (2.79) particularize when taking into account the basic parameters defined for this lossless case as:

$$\rho_{sp}(z) = \rho_0 e^{2j\beta z} \in \mathbb{C}, \tag{2.98}$$

$$Z_{sp}(z) = Z_{0,sp} \frac{1 + \rho_{sp}(z)}{1 - \rho_{sp}(z)} \in \mathbb{C}, \text{ and} \tag{2.99}$$

$$Y_{sp}(z) = \frac{1}{Z_{0,sp}} \frac{1 - \rho_{sp}(z)}{1 + \rho_{sp}(z)} \in \mathbb{C}. \tag{2.100}$$

If the wave parameters for this lossless case are defined from the load ($Z_L \in \mathbb{C}$), they result in:

$$\rho_{sp}(l) = \rho_{L,sp} e^{-2j\beta_{sp} l} \in \mathbb{C}, \tag{2.101}$$

in which

$$\rho_{L,sp} = \frac{Z_L - Z_{0,sp}}{Z_L + Z_{0,sp}}, \tag{2.102}$$

[32] The subindex "$_{sp}$" referes to the lossless cases. It comes from the translation of the term "lossless" to Spanish: "sin pérdidas".

$$Z_{sp}(l) = Z_{0,sp}\frac{1+\rho_{sp}(l)}{1-\rho_{sp}(l)} = Z_{0,sp}\frac{Z_L\cos(\beta_{sp}l) + jZ_{0,sp}\sin(\beta_{sp}l)}{jZ_L\sin(\beta_{sp}l) + Z_{0,sp}\cos(\beta_{sp}l)} \in \mathbb{C}, \text{ and} \tag{2.103}$$

$$Y_{sp}(l) = \frac{1}{Z_{0,sp}}\frac{1-\rho_{sp}(l)}{1+\rho_{sp}(l)} = \frac{jZ_L\sin(\beta_{sp}l) + Z_{0,sp}\cos(\beta_{sp}l)}{Z_L\cos(\beta_{sp}l) + jZ_{0,sp}\sin(\beta_{sp}l)} \in \mathbb{C}. \tag{2.104}$$

Since the characteristic impedance of this lossless case does not depend on frequency and the propagation constant is linear function of frequency, the appropriate length normalization with respect to the wavelength, which is inversely proportional to ω, "univeralizes"[33] the behavior of wave parameters along the TL.

2.4.3 Non dispersive Transmission Lines

Here the non dispersive TLs are defined by means of imposing that the phase velocity is constant, which translates to the line parameters, which are, in turn, non frequency dependent. In this way, this case corresponds with the physical description of harmonic plane waves which propagate in lossy or lossless mediums in which the phase velocity is constant.
This non dispersive case is a particular case of the lossy case; and the lossless case is, in turn, a particular case of the non dispersive case described here. This case will be useful in future analysis in which the phase velocity is required to be constant, for example the ones related with the analysis briefly introduced among the **Future Lines**.

Line Parameters

In this case, the specification regarding non dispersive mediums is imposed on the phase velocity. This phase speed has to be constant:

$$v_p = v_0 \in \mathbb{R} \to \beta \cong \omega\beta_0,\ \beta_0 \geq \sqrt{\mu_0\varepsilon_0} \in \mathbb{R}. \tag{2.105}$$

Since the non dispersive case is a particular case of the lossy case, it is possible to examine β written in terms of line parameters in eq. (2.70), and impose the condition which makes it a linear function of frequency. This condition is well-known, [Poz98],

$$\frac{G}{C} = \frac{R}{L}, \tag{2.106}$$

which, in terms of the constitutive parameters defined in eq. (2.88) in the lossy case, means

$$\frac{\varepsilon_0'' + \sigma'}{\varepsilon' - \sigma_0''} = \frac{\mu_0''}{\mu'}. \tag{2.107}$$

In this case, the equivalent circuits may be defined as in Fig. 2.17 but now only by means of three values if solving one of them from eq. (2.106). The same occurs in the parameterization of the equivalent *telegrapher's equations*.

[33]The term "universal" is frequently used throughout the Thesis to designate those behaviors that may be gathered within the same complex expression. This translates in different "operations" depending on the type of analysis it is doing. For example, from the geometrical point of view, "to universalize" means "to reparameterize" a curve regarding the analysis in complex planes.

Basic parameters

Since this case is a particular case of the lossy case, the basic parameters are the same but particularized with the condition in eq. (2.106)

The characteristic impedance regarding this non dispersive case, $Z_{0,nd}$[34] is

$$Z_{0,nd} = \sqrt{\frac{L}{C}} \equiv Z_{0,sp} \in \mathbb{R}. \tag{2.108}$$

The propagation constant results in

$$\begin{aligned} \gamma_{nd} = \alpha_{nd} + j\beta_{nd} =& G\sqrt{\frac{L}{C}} + j\omega\sqrt{LC} = R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC} \equiv \\ \equiv& G Z_{0,sp} + j\beta_{sp} \equiv \frac{R}{Z_{0,sp}} + j\beta_{sp} \in \mathbb{C}. \end{aligned} \tag{2.109}$$

Notice that losses only appears in α_{nd}, whereas the phase of the characteristic impedance is zero as in the lossless case. Moreover, the non dispersive case presents the same lossless phase constant. As a result, α_{nd} and $Z_{0,nd}$ does not depend on frequency, while β_{nd} is a linear function of frequency.

The voltage and current waves regarding this non dispersive case are of the form of of eqs. (2.49) and (2.50), respectively, but particulaizing the basic parameters.

Wave parameters

The wave parameters $\rho_{nd}(z)$, $Z_{nd}(z)$, and $Y_{nd}(z)$ keep the form of eqs. (2.75), (2.78), and (2.79), but particularized with the basic parameters defined above.
If the wave parameters are defined along l from the load: $\rho_{nd}(l)$, $Z_{nd}(l)$, and $Y_{nd}(l)$; they are of the form of eqs. (2.83), (2.85), and (2.86), also particularizing the basic parameters for this non dispersive case. Nevertheless, notice that the reflection coefficient at the load, $\rho_{L,nd}$ is the same as in the lossless case presented in eq. (2.102).
These wave parameters determine the total solutions in the equivalent TL, which are of the form of eqs. (2.72) and (2.73) for the total voltage and current waves, respectively, but particularizng the basic parameters with the ones presented for this non dispersive case.

2.4.4 Low-losses approximation for Transmission Lines

In this section, the low-losses approximation is introduced and compared to the rest of solutions, specially to the non-dispersive case. This case is not a physical solution among the harmonic plane waves, but a mathematical approximation obtained from the lossy case, which is frequently used in the literature, [Poz98, Col01].
This case will be helpful when normalizing the basic parameters for the variable frequency analysis in Chpt. 4.

[34] The same Spanish notation on the subindex of this non dispersive case is followed here. In this case "$_{nd}$" refers to "no dispersivo", which means "non dispersive".

Line parameters

In this case, the line parameters regarding the lossy case in eqs. (2.68) and (2.71) are required to verify

$$\begin{cases} R \ll \omega L \\ G \ll \omega C \end{cases}, \tag{2.110}$$

which means that the constitutive parameters fulfill

$$\begin{cases} \mu_0'' & \ll \omega\mu' \\ \varepsilon_0'' + \sigma' & \ll \omega\left(\varepsilon' - \sigma_0''\right) \end{cases}. \tag{2.111}$$

Notice that both the line parameters in eq. (2.106) and the constitutive parameters in eq. (2.107) fulfill the conditions in eqs. (2.110) and (2.111), respectively, if ω is high enough [35]. Thus, the non dispersive case is within the low-losses approximations.

Basic parameters

The Taylor series of the propagation constant and the characteristic impedance in the general lossy case in eqs. (2.68) and (2.71) are

$$Z_0 \approx \sqrt{\frac{L}{C}}\left[1 + j\frac{1}{2}\left(\frac{G}{\omega C} - \frac{R}{\omega L}\right) + \dots\right] \text{ and} \tag{2.114}$$

$$\gamma \approx \frac{1}{2}\left(G\sqrt{\frac{L}{C}} + R\sqrt{\frac{C}{L}}\right) + j\omega\sqrt{LC}\left(1 - \frac{1}{4}\frac{G}{\omega C}\frac{R}{\omega L}\right) + \dots, \tag{2.115}$$

respectively.

The characteristic impedance of the low-losses approximation, $Z_{0,bp}$ [36], is built by taking the first term of the expansion of Z_0 in eq. (2.114), so

$$Z_{0,bp} = \sqrt{\frac{L}{C}} \equiv Z_{0,sp}. \tag{2.116}$$

The propagation constant of the low-losses approximation, γ_{bp}, is built by taking the first real and imaginary terms of the Taylor series of γ in eq. (2.115), which results in

$$\gamma_{bp} = \frac{1}{2}\left(G\sqrt{\frac{L}{C}} + R\sqrt{\frac{C}{L}}\right) + j\omega\sqrt{LC} \equiv \alpha_{bp} + j\beta_{sp}, \tag{2.117}$$

in which

$$\alpha_{bp} = \frac{1}{2}\left(G\sqrt{\frac{L}{C}}\right) \equiv \frac{1}{2}\left(GZ_{0,sp} + \frac{R}{Z_{0,sp}}\right). \tag{2.118}$$

[35] This condition has to been understood in asymptotic sense (from asymptotic analysis). That is, a generic function $f(\omega)$ is asymptotically equivalent to $g(\omega)$ as $\omega \to \infty$, which is written as

$$f(\omega) \sim g(\omega) \quad (\text{as } \omega \to \infty), \tag{2.112}$$

if

$$\lim_{\omega\to\infty} \frac{f(\omega)}{g(\omega)} = 1. \tag{2.113}$$

[36] In this case, the subindex "$_{bp}$" refers to "bajas pérdidas", which is the Spanish term which means "low losses".

Notice that the parameters in this low-losses approximation are the same that in the lossless case excepto to $\alpha \equiv \alpha_{bp}$ which quantifies losses in the equivalent TL. This parameter does not depend on frequency as long as the line parameters are not functions of frequency, so it is appropriate for the normalizations when analyzing the propagation constant in frequency, as it is used in the analysis in Chpt. 4.
Also notice that, when the non dispersive condition in eq. (2.106) verifies,

$$\alpha_{bp} \underset{R/L=G/C}{\equiv} \alpha_{nd}, \tag{2.119}$$

and so the basic parameters are all the same as in the non dispersive case. In fact, only if the non dispersive condition is fulfilled, the low-losses approximation coincides with real harmonic plane waves solutions.

The voltage and current waves regarding this non dispersive case are of the form of of eqs. (2.49) and (2.50), respectively, but particulaizing the basic parameters.

Wave parameters

The wave parameters $\rho_{bp}(z)$, $Z_{bp}(z)$, and $Y_{bp}(z)$ keep the form of eqs. (2.75), (2.78), and (2.79), but particularized with the basic parameters defined above.
If the wave parameters are defined along l from the load: $\rho_{bp}(l)$, $Z_{bp}(l)$, and $Y_{bp}(l)$; they are of the form of eqs. (2.83), (2.85), and (2.86), also particularizing the basic parameters for this low-losses approximation. Nevertheless, notice that the reflection coefficient at the load, $\rho_{L,bp}$ is the same as in the lossless case presented in eq. (2.102).
These wave parameters determine the total solutions in the equivalent TL, which are of the form of eqs. (2.72) and (2.73) for the total voltage and current waves, respectively, but particularizng the basic parameters with the ones presented for this low-losses approximation.

2.4.5 Summary of Transmission Line particular cases

	Line parameters	Basic parameters	Wave parameters
Lossy case	R, L, G, C	$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$, $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$	$Z(z)$ $(Z(l))$ $Y(z)$ $(Y(l))$ $\rho(z)$ $(\rho(l))$
Lossless case $(R=0, G=0)$	L, C	$Z_{0,sp} = \sqrt{\frac{L}{C}}$, $\gamma_{sp} = j\omega\sqrt{LC}$	$Z_{sp}(z)$ $(Z_{sp}(l))$ $Y_{sp}(z)$ $(Y_{sp}(l))$ $\rho_{sp}(z)$ $(\rho_{sp}(l))$
Non dispersive case $(R/L = G/C)$	R, L, (G), C	$Z_{0,nd} = \sqrt{\frac{L}{C}} \equiv Z_{0,sp}$, $\gamma_{nd} = G\sqrt{\frac{L}{C}} + j\omega\sqrt{LC} =$ $= R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC} \equiv$ $\equiv GZ_{0,sp} + \gamma_{sp} \equiv \frac{R}{Z_{0,sp}} + \gamma_{sp}$	$Z_{nd}(z)$ $(Z_{nd}(l))$ $Y_{nd}(z)$ $(Y_{nd}(l))$ $\rho_{nd}(z)$ $(\rho_{nd}(l))$
Low-losses approximation $(R \ll \omega L, G \ll \omega C)$	R, L, G, C	$Z_{0,bp} = \sqrt{\frac{L}{C}} \equiv Z_{0,sp}$, $\gamma_{bp} = \frac{1}{2}\left(G\sqrt{\frac{L}{C}} + R\sqrt{\frac{C}{L}}\right) +$ $+j\omega\sqrt{LC} \equiv$ $\equiv \frac{1}{2}\left(GZ_{0,sp} + \frac{R}{Z_{0,sp}}\right) + j\beta_{sp}$	$Z_{bp}(z)$ $(Z_{bp}(l))$ $Y_{bp}(z)$ $(Y_{bp}(l))$ $\rho_{bp}(z)$ $(\rho_{bp}(l))$

Table 2.2: Summary of the parameters regarding the TL particular cases: the lossy case, the lossless case, and the non dispersive case; and approximations: the low-losses approximation.

2.5 Conclusions

In this chapter, the LTLT regarding HPW has been presented with the goal of parameterize lossy mediums in which this type of waves propagate in equivalent TLs. For this purpose, the equivalent waves are obtained by integrating the fields taking advantage of the properties of harmonic functions to generalize the integration which is, in addition, definite.
This characterization (named as direct characterization) lets to define the line parameters from the constitutive parametrs and the geometry of the structure which supports HPWs. This direct characterization has been carried out in a generalized way in the sense that: (i) all the possible sources of losses are considered, and (ii) the coordinates which describe the problem are taken in a generalized coordinate system; so that the LTLT presented in this chapter is analytically "complete" for the purpose of studying lossy TLs in wich these types of waves propagate.

The resultant individual equivalent waves (that waves that propagate in each direction) are completely characterized by the basic parameters and, in turn, the total waves are characterized by the wave parameters. These parameters have been obtained for a general lossy case by simply using the LTLT. However, there are still open questions which are non easy to answer only by inspecting the resultant expressions of this analysis, as for example answering which are the allowed values of the basic and wave parameters for the lossy case and also the particular cases and approximations. It is also important to detach the geometrical meaning these pararemeters show, for example when adding the angles of the basic parameters, they not exceed $\pi/2$. In fact, angles and modulus of basic parameters are directly connected with physical behaviors of the equivalent waves, for example the angle of the characteristic impedance is the phase shifting between the voltage and current waves. From these physical interpretations, it is possible to pose more questions that the analytical expressions does not directly answer. For example, related to the phase shifting, are all the displacements possible?

As it is showing in this section, the LTLT developed in this chapter is not "complete" when looking beyond the analytical expressions because it lacks of large amount of interpretations that are very useful in the analysis of the TLs. In fact, it is introducing the physical aspects (in this case the losses, but also the influence of TL's length to the parameters and the equivalent waves) when the analytical expressions show the most important weaknesses.
All these facts suggest studying the inlfuence of losses from another ("complete") perspective that complements the analytical results, so it is based in the LTLT posed here.
Since the expressions that generalize the presence of losses in TLs are addressed in *frrequency domain* are complex, the complex analysis of the TL parameters turns to be the method to overcome the limitations in the analysis presented in this chapter. The results of this type of analysis together with the physical interpretations will lead to the CTLT regarding the LTLT, that is, the CTLT-v1.

Chapter 2

Appendices

Appendix 2.A

Appendix 2.A The curl operator applied to a vector $\boldsymbol{\mathcal{A}}_{\mathrm{t}}$ which represents a plane wave in some generic cylindrical coordinate system denoted by $[\mathrm{t}_1, \mathrm{t}_2, z]$,

$$\begin{aligned} \nabla \times \boldsymbol{\mathcal{A}}_{\mathrm{t}} = \left(\nabla_{\mathrm{t}} + \frac{\partial}{\partial z}\hat{z}\right) \times \boldsymbol{\mathcal{A}}_{\mathrm{t}} = \frac{1}{h_1 h_2}\begin{vmatrix} h_1\hat{\mathrm{t}_1} & h_2\hat{\mathrm{t}_2} & \hat{z} \\ \frac{\partial}{\partial \mathrm{t}_1} & \frac{\partial}{\partial \mathrm{t}_2} & \frac{\partial}{\partial z} \\ h_1\mathcal{A}_{\mathrm{t}_1} & h_2\mathcal{A}_{\mathrm{t}_2} & 0 \end{vmatrix} = \\ = \frac{\partial \mathcal{A}_{\mathrm{t}_1}}{\partial z}\hat{\mathrm{t}_2} - \frac{\partial \mathcal{A}_{\mathrm{t}_2}}{\partial z}\hat{\mathrm{t}_1} + \frac{1}{h_1 h_2}\left(\frac{\partial h_2\mathcal{A}_{\mathrm{t}_2}}{\partial \mathrm{t}_1} - \frac{\partial h_2\mathcal{A}_{\mathrm{t}_1}}{\partial \mathrm{t}_2}\right)\hat{z} = \\ = \frac{\partial}{\partial z}\hat{z} \times \boldsymbol{\mathcal{A}}_{\mathrm{t}} + \frac{1}{h_1 h_2}\left(\frac{\partial h_2\mathcal{A}_{\mathrm{t}_2}}{\partial \mathrm{t}_1} - \frac{\partial h_2\mathcal{A}_{\mathrm{t}_1}}{\partial \mathrm{t}_2}\right)\hat{z} = \frac{\partial}{\partial z}\hat{z} \times \boldsymbol{\mathcal{A}}_{\mathrm{t}} + \nabla_{\mathrm{t}} \times \boldsymbol{\mathcal{A}}_{\mathrm{t}}, \end{aligned} \tag{2.A.1}$$

in which h_1 and h_2 are the scale factors of the cordinates t_1 and t_2, respectively, which are taken non z dependent.
Furthermore, *Maxwell equations* relate two vectors belonging to the same plane, $\boldsymbol{\mathcal{A}}_{\mathrm{t}}$ and $\boldsymbol{\mathcal{B}}_{\mathrm{t}}$, under the form

$$\nabla \times \boldsymbol{\mathcal{A}}_{\mathrm{t}} = \boldsymbol{\mathcal{B}}_{\mathrm{t}}. \tag{2.A.2}$$

Thus, identifying this equation with the curl of plane waves identity in eq. (2.A.1), it leads to

$$\begin{cases} \frac{\partial}{\partial z}\hat{z} \times \boldsymbol{\mathcal{A}}_{\mathrm{t}} = \boldsymbol{\mathcal{B}}_{\mathrm{t}} \\ \nabla_{\mathrm{t}} \times \boldsymbol{\mathcal{A}}_{\mathrm{t}} = 0 \end{cases}, \tag{2.A.3}$$

and so the transversal curl operator $(\nabla_{\mathrm{t}}\times)$ is null if it is applied to plane waves.

Moreover, if expanding the vectors $\boldsymbol{\mathcal{A}}_{\mathrm{t}}$ and $\boldsymbol{\mathcal{B}}_{\mathrm{t}}$ in the first identity in eq. (2.A.3), and identifying the components between themselves, it gets to

$$\begin{cases} \frac{\partial A_{\mathrm{t}_1}}{\partial z} = \mathcal{B}_{\mathrm{t}_2} \\ -\frac{\partial A_{\mathrm{t}_2}}{\partial z} = \mathcal{B}_{\mathrm{t}_1} \end{cases}. \tag{2.A.4}$$

Now, scalar multiplying the vectors $\boldsymbol{\mathcal{A}}_{\mathrm{t}}$ and $\boldsymbol{\mathcal{B}}_{\mathrm{t}}$, and using the identities in eq. (2.A.4) as

$$\boldsymbol{\mathcal{A}}_{\mathrm{t}} \cdot \boldsymbol{\mathcal{B}}_{\mathrm{t}} = \mathcal{A}_{\mathrm{t}_1}\mathcal{B}_{\mathrm{t}_1} + \mathcal{A}_{\mathrm{t}_2}\mathcal{B}_{\mathrm{t}_2} = -\mathcal{A}_{\mathrm{t}_1}\frac{\partial \mathcal{A}_{\mathrm{t}_2}}{\partial z} + \mathcal{A}_{\mathrm{t}_2}\frac{\partial \mathcal{A}_{\mathrm{t}_1}}{\partial z}. \tag{2.A.5}$$

Since $\boldsymbol{\mathcal{A}}_{\mathrm{t}}$ and $\boldsymbol{\mathcal{B}}_{\mathrm{t}}$ are written using a generic coordinate system, it is possible to chose that one which makes $\boldsymbol{\mathcal{A}}_{\mathrm{t}} \equiv \mathcal{A}_{\mathrm{t}_1}\hat{\mathrm{t}_1}$, and so $\mathcal{A}_{\mathrm{t}_2} = 0$. Thus, the scalar product in eq. (2.A.5) equals zero, making $\boldsymbol{\mathcal{A}}_{\mathrm{t}}$ and $\boldsymbol{\mathcal{B}}_{\mathrm{t}}$ orthogonal.

Finally, *Maxwell equations* of the form of eq. (2.A.2) are proved to verify that they are invariant regarding rotations, that is, if rotating the field $\boldsymbol{\mathcal{A}}_{\mathrm{t}}$, then the field $\boldsymbol{\mathcal{B}}_{\mathrm{t}}$ appears rotated the same angle.
Here this property is proved for plane waves but the same result may be easily generalized to non plane waves.

Appendix 2.B

If the fields $\mathcal{E}_{t_1}$, $\mathcal{H}_{t_2}$ in eqs. (2.16)-(2.17) are brought to the particularized *Faraday and Ampère's laws* in eqs. (2.9) and (2.10), the following system of equations is obtained:

$$\begin{cases} \pm\eta\alpha\sin(\omega t \mp \beta z + \phi_e) \pm \eta\beta\cos(\omega t \mp \beta z + \varphi_e) = \mu\omega\cos(\omega t \mp \beta z + \phi_h) \\ \pm\alpha\sin(\omega t \mp \beta z + \phi_h) \pm \beta\cos(\omega t \mp \beta z + \varphi_h) = \eta\sigma\sin(\omega t \mp \beta z + \phi_e) + \\ \qquad + \eta\varepsilon\omega\cos(\omega t \mp \beta z + \phi_e) \end{cases}, \quad (2.B.6)$$

in which η is defined as

$$\mathcal{E}_{t_1} = \eta\mathcal{H}_{t_2}. \quad (2.B.7)$$

Since the system in eq. (2.B.6) verifies for each $[z, t]$, if taking for example

$$\omega t \mp \beta z + \phi_e = 0, \quad (2.B.8)$$

the same reduces to

$$\begin{cases} \eta\beta = \pm\,\mu\omega\cos(\Delta\varphi) \\ \eta\varepsilon\omega = \mp\,\alpha\sin(\Delta\varphi) \pm \beta\cos(\Delta\varphi) \end{cases}, \quad (2.B.9)$$

in which $\Delta\varphi = \varphi_e - \varphi_h$, the phase difference between the electric and magnetic fields. Solving η as a function of α and $\Delta\varphi$,

$$\eta = \pm\frac{\alpha\sin(\Delta\varphi) + \sqrt{\alpha^2\sin^2(\Delta\varphi) + 4\omega^2\varepsilon\mu\cos^2(\Delta\varphi)}}{2\varepsilon\omega}. \quad (2.B.10)$$

Writing α in terms of constitutive parameters and substituying it in eq. (2.B.10) leaves

$$\eta = \pm\sqrt{\tfrac{\mu}{\varepsilon}}\left(\frac{\sqrt{-1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}}{2\sqrt{2}}\sin(\Delta\varphi) + \sqrt{\frac{-1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}{8}\sin^2(\Delta\varphi) + \cos^2(\Delta\varphi)}\right), \quad (2.B.11)$$

which is the expression of the characteristic impedance solved in *time domain* when the conductor losses are fully taken into account.

On the other hand, if dividing the equations of system (2.B.9), it is possible to solve $\Delta\phi$ as

$$\Delta\varphi = \pm\tan^{-1}\left(\frac{\alpha}{\beta}\right) = \pm\tan^{-1}\left(\sqrt{\frac{-1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}{1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}}\right), \quad (2.B.12)$$

which is the expression of the phase difference solved in *time domain* when conductor losses are rigorously considered.

Appendix 2.C

From the complex permeability $\mu = \mu' - j\mu''$, and the complex equivalent permitivity $\varepsilon_{eq} = \varepsilon'_{eq} - j\varepsilon''_{eq}$, the complex wavenumber $k = k' + jk''$ is defined as follows

$$k = \omega\sqrt{(\mu' - j\mu'')\left(\varepsilon'_{eq} - j\varepsilon''_{eq}\right)} = \omega\sqrt{\left(\mu'\varepsilon'_{eq} - \mu''\varepsilon''_{eq}\right) - j\left(\mu'\varepsilon''_{eq} + \mu''\varepsilon'_{eq}\right)}. \tag{2.C.13}$$

Now, taking the square of k,

$$k^2 = \left(k'^2 - k''^2\right) + j2k'k'', \tag{2.C.14}$$

and identifying the real and imaginary parts with those which result when squaring k in eq. (2.C.13), it lead to a system of equations

$$\begin{cases} k'^2 - k''^2 = \omega^2\left(\mu'\varepsilon'_{eq} - \mu''\varepsilon''_{eq}\right) \\ k'k'' = \omega^2\left(\mu'\varepsilon''_{eq} + \mu''\varepsilon'_{eq}\right) \end{cases}. \tag{2.C.15}$$

Solving k' and k'' from the system above:

$$k' = \sqrt{\frac{\omega^2}{2}\left(\mu'\varepsilon'_{eq} - \mu''\varepsilon''_{eq} + \sqrt{(\mu'^2 + \mu''^2)(\varepsilon'^2_{eq} + \varepsilon''^2_{eq})}\right)} \geq \omega\sqrt{\mu_0\varepsilon_0}, \tag{2.C.16}$$

$$k'' = \sqrt{\frac{\omega^2}{2}\left(-\mu'\varepsilon'_{eq} + \mu''\varepsilon''_{eq} + \sqrt{(\mu'^2 + \mu''^2)(\varepsilon'^2_{eq} + \varepsilon''^2_{eq})}\right)} \leq 0. \tag{2.C.17}$$

The limits for k' and k'' have been obtained by means of the analysis which has been presented in [Rie98][37]. They correspond with the lossless case for which the mediums are non dispersive and the conductivity is zero.

[37] The anaysis presented in this work has to be extended to the case in which $\varepsilon'_{eq} < 0$ (due to the presence of σ''). However, this case does not affect the limits of k because the terms $\pm\left(\mu'\varepsilon'_{eq} - \mu''\varepsilon''_{eq}\right)$ in eqs. (2.C.16) and (2.C.17) are in $]-\infty, \infty[$ for any lossy case.

Appendix 2.D

The complexification of transversal coordinates $[\mathrm{t}_1, \mathrm{t}_2]$ in the cross section means an important simplification in both the complexity of the problem to be solved, and notation of functions and operators to be applied.
This transformation is specially useful when dealing with harmonic plane fields, as it is here proved.

The following change is proposed:

$$\boxed{[\mathrm{t}_1, \mathrm{t}_2] \in D \subseteq \mathbb{R}^2 \rightarrow \tilde{\mathrm{t}} \in D^* \subseteq \mathbb{C}.}$$

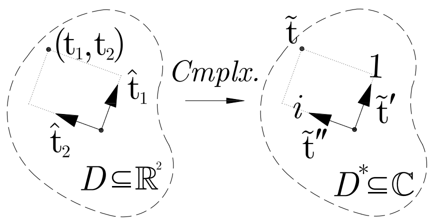


Fig. 2.19: Transformation from the original cross section $D \in \mathbb{R}^2$ to the complex domain $D^* \in \mathbb{C}$ by means of the complexification of the coordinates $[\mathrm{t}_1, \mathrm{t}_2]$ in only one complex coordinate $\tilde{\mathrm{t}}$.

The complex variable $\tilde{\mathrm{t}}$ parameterizes the cross section just like it is represented in Fig. 2.19.

On one hand, if the EM wave to be parameterized in D^* belong to a plane wave, the electric and magnetic fields are orthogonal. This has been proved in Appendix 2.A. In fact, the magnetic field is rotated $\pi/2$ from the electric field (and scaled by η). Thus, in the *frequency domain*, the fields E_{t_1} and H_{t_2} of eqs. (2.34) and (2.35) are possible to be gathered in only one complex function F:

$$F = E_{\mathrm{t}_1} + iH_{\mathrm{t}_2} = f(\mathrm{t}_1, \mathrm{t}_2)(1 + i\eta)e^{\mp\gamma z} \in \mathbb{C}. \qquad (2.D.18)$$

Regarding the function F in eq. (2.D.18), it is important to detach that the imaginary units i and h are used in a different manner. While i represents a rotation in D^*, j refers to the basis parameterized by ω in the *frequency domain*, so both can not be simplified together.

On the other hand, the function F is said to be complex harmonic in D^*, [BC90], because the functions E_{t_1} and H_{t_2} are harmonic in D. This immediately means that F is analytical in D^* if supposing this region is open, [BC90], so the inherent properties of complex analytical functions may be applied to F.

Appendix 2.E

Imagine C is a contour in which the primitive $\phi_e(\mathrm{t}_1, \mathrm{t}_2)$ is constant, that is, C is given by the equation $\mathrm{t}_1 = c$, because of the definition of ϕ_e in Sect. 2.3.

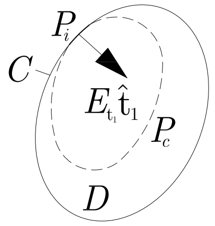


Fig. 2.20: Simply connected contour C enclosing a domain D where the random closed path P is chosen to demostrate that E_{t_1} is constant.

Imagine C is simply connected, as for example the one depicted in Fig. (2.20). A randomly chosen closed path P_c which interects C in more than a point ($P_i \equiv Pc \cap C$ denotes intersection, which is an open curve) makes zero the integration of E_{t_1} over there. That is only because E_{t_1} is harmonic. Additionally, the integration in the part of P_c which intersects C is constant:

$$\int_{P_c \cap C} E_{\mathrm{t}_1} dl = a, \, \forall P_c. \tag{2.E.19}$$

That means that

$$\int_{P_c - P_i} E_{\mathrm{t}_1} dl = -a, \, \forall P_c. \tag{2.E.20}$$

Since P_c is randomly chosen, then E_{t_1} has to be constant. This contradicts the fact that E_{t_1} varies with t_1.

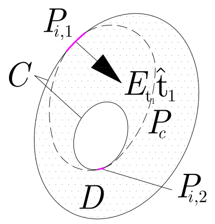


Fig. 2.21: Multiply connected contour C enclosing a deleted domain D. The parts of the closed path P which intersect C are denoted by $P_{i,1}$ and $P_{i,2}$. There the function E_{t_1} is constant different from part to part.

As a result, C has to be multiply connected and ϕ_e has to be different in each separated contour belonging to C, as it is represented in Fig. (2.21).

Appendix 2.F

The transformation of the particularized *Ampère's law* in eq. (2.23) to one of the *telegrapher's equations* is here studied. This is equivalent to transform its integral version in eq. (2.54) to finally obtain eq. (2.58).

Firstly, the integration of the $\hat{\mathrm{t}_1}$-field components in the *Ampére's law* in the whole range of t_2,

$$\int_{\langle \mathrm{t}_2 \rangle} \frac{\partial \left(\hat{z} \times H_{\mathrm{t}_2} \hat{\mathrm{t}_2}\right)}{\partial z} h_2 d\mathrm{t}_2 \hat{\mathrm{t}_1} = j\omega\varepsilon_{eq} \int_{\langle \mathrm{t}_2 \rangle} E_{\mathrm{t}_1} \hat{\mathrm{t}_1} h_2 d\mathrm{t}_2 \hat{\mathrm{t}_1}, \tag{2.F.21}$$

may be transformed if using the Stokes' theorem p.u.l., leading to

$$\int_{\langle \mathrm{t}_2 \rangle} H_{\mathrm{t}_2} \hat{\mathrm{t}_2} h_2 d\mathrm{t}_2 \hat{\mathrm{t}_2} = j\omega\varepsilon_{eq} \int_{\langle \mathrm{t}_2 \rangle} E_{\mathrm{t}_1} \hat{\mathrm{t}_1} h_2 d\mathrm{t}_2 \hat{\mathrm{t}_1}. \tag{2.F.22}$$

The scale factor h_2 does not depend on t_2. This is proved as follows:
The scale factor h_2 is defined as:

$$h_2 = \left| \frac{\partial \boldsymbol{r}}{\partial \mathrm{t}_2} \right|, \tag{2.F.23}$$

$\boldsymbol{r} \equiv \boldsymbol{r}\,(\mathrm{t}_1, \mathrm{t}_2)$ is the vector position in the generic coordinate system. Geometrically, the definition of h_2 corresponds with the modulus of the tangent vector of curve defined in the whole range of t_2. These curves (C_h) follow the equation

$$C_h : \mathrm{t}_1 = a, \tag{2.F.24}$$

with $a \in \mathbb{R}$. Thus,

$$\frac{\partial h_2}{\partial \mathrm{t}_2} = 0, \tag{2.F.25}$$

in which $\partial h_2/\partial \mathrm{t}_2$ is geometrically the same as the modulus of the normal vector in the curves described by eq. (2.F.24), because the orthogonality between t_1 and t_2.
Eq. (2.F.25) means that h_2 does not depend on t_2.

Then, the factor h_2 is solved in the r.h.s. of eq. (2.F.22). Moreover, E_{t_1} in the r.h.s. is constant along t_2 so it leaves the integral. This leads to

$$\frac{1}{h_2} \int_{\langle \mathrm{t}_2 \rangle} H_{\mathrm{t}_2} \hat{\mathrm{t}_2} h_2 d\mathrm{t}_2 \hat{\mathrm{t}_2} = j\omega\varepsilon_{eq} E_{\mathrm{t}_1} \hat{\mathrm{t}_1} \hat{\mathrm{t}_1} \int_{\langle \mathrm{t}_2 \rangle} d\mathrm{t}_2 = j\omega\varepsilon_{eq} E_{\mathrm{t}_1} \hat{\mathrm{t}_1} \hat{\mathrm{t}_1} L_{\mathrm{t}_2}, \tag{2.F.26}$$

in which L_{t_2} is the measure or arc length of the range of t_2.
Notice that the integral in the l.h.s. is that one defined in eq. (2.48), so

$$\frac{1}{h_2} I(z) = j\omega\varepsilon_{eq} E_{\mathrm{t}_1} \hat{\mathrm{t}_1} \hat{\mathrm{t}_1} L_{\mathrm{t}_2}. \tag{2.F.27}$$

Now, integrating both sides of eq. (2.F.27) in the path along t_1 defined in the interval $(\mathrm{t}_{1,1}, \mathrm{t}_{1,2})$,

$$\left(\int_{\mathrm{t}_1 = \mathrm{t}_{1,1}}^{\mathrm{t}_{1,2}} \frac{h_1}{h_2} d\mathrm{t}_1 \right) I(z) = j\omega\varepsilon_{eq} L_{\mathrm{t}_2} \left(\int_{\mathrm{t}_1 = \mathrm{t}_{1,1}}^{\mathrm{t}_{1,2}} E_{\mathrm{t}_1} \hat{\mathrm{t}_1} h_1 d\mathrm{t}_1 \hat{\mathrm{t}_1} \right). \tag{2.F.28}$$

The integral in the r.h.s. identifies with that one in eq. (2.46). Moreover, if denoting the definite integral in the l.h.s as $H_{1,2}$, and solving $I(z)$, it leads to,

$$I(z) = j\omega\varepsilon_{eq} \frac{L_{\mathrm{t}_2}}{H_{1,2}} V(z). \tag{2.F.29}$$

The transformation of the *Faraday's law* is completely equivalent when taking the same assumptions to transform eq. (2.61). It leads to

$$V(z) = j\omega\mu \frac{H_{1,2}}{L_{\mathrm{t}_2}} I(z), \tag{2.F.30}$$

with $H_{1,2}$ and L_{t_2} as defined above.

Chapter 3

A Generalized Version of the Transmission Line Theory

3.1 Introduction and fundamentals

In this chapter the GTLT for the parameterization and description in the *frequency domain* of all the possible modes which propagate in lossy media (GTLT-v1) is introduced and explained.
The purpose with this generalized Theory introduction is describing the modes based on HPWs in an easy way from the general equations once these ones are particularized by imposing the *harmonic plane wave condition*, at the same time that the inverse characterization introduced in Sect. 2.3 in Chpt. 2 is explained in comparison with the direct characterization described in the LTLT detailed in that chapter.
Furthermore, as a generalization in itself, this Theory lets to parameterize more modes by means of equivalent TLs describing equivalent *telegrapher's equations*. However, those possible solutions of general equations beyond HPWs are not obtained here for the sake of focusing the analysis presented in the Thesis (both under the TLT and the CTLT points of view) on these type solutions (HPWs). Neverthelss, since this version of the GTLT is posed from the general case point of view to be then particularized to specific cases, the inverse characterization for HPWs presented here could be repeated imposing different conditions leading to parameterize different modes and wave solutions.

In certain way, the generalized analysis presented by means of this chapter is induced from the particular case studied in the LTLT in which HPWs propagate. Nevertheless, while the process in the LTLT goes from imposing the *harmonic plane wave condition* in the field equations for obtaining the solutions, and from them the equivalent equations that govern the equivalent TL (and this direct characterization could be repeated for different *conditions*), here the process is reverse making all the analysis generalized in terms of the final expressions, structures, relations between fields, etc. In particular, the following characteristics of the LTLT developed in the previous chapter are taken generalized as assumptions for being the analysis presented in this chapter able to be parameterized using equivalent TLs:

(i) The modes which are of interest are propagative. This means working in "propagative harmonic regime"; a concept which has to be explained.
For the purpose of understafing this specific behavior of waves, recall the definition of "time harmonic regime" under the GSST or the Function Theory point of view: a basis $e^{j\omega t}$ is selected in the space of functions which describes the fields solutions to the EM problem (see this in the scheme in Fig. 2.4). Operating with the coefficients in this basis set of time exponentials parameterized by ω reduces the dimension of the original problem while converting

ω as the parameter in the so called *frequency domain*, which is the domain of operation including these coefficients. The time exponentials are separable from the coefficients because the problem under study is time invariant, which is direct consequence of imposing IBC on the time coordinate.
Taking the idea of representing the solutions by means of coefficients accompanying a basis, it is imposed for the soutions to be propagative, besides operating in the space of coefficients in the *frequency domain* (which is assumed in the version of the GTLT). This assumption, which is natural for the analysis of propagative waves in which the analysis of this thesis book is focused on, supposes chosing the basis set[1] $\mathrm{e}(z;\gamma) \equiv e^{\mp\gamma z}$ for describing propagative waves. With respect to this set it should be detached that: (i) within the set of propagative exponentials there are two subsets of solutions distinguished by the signs "$\mp$", which make reference to the propagative solutions in the $\hat{z}$ direction and the $-\hat{z}$ direction; (ii) the parameter γ, which plays the role of spectral variable, is complex ($\gamma \in \mathbb{C}$) for generalizing the propagative solutions to those which attenuate at the same time they propagate; and (iii) because the division in (i) and the complex nature of γ in (ii), the real and imaginary parts of γ, α and β, respectively, both are greater or equal than zero. When $\alpha = 0$ the mode in question is not attenuating, so it should propagate in lossless media, whereas when $\beta = 0$ the mode is called evanescent, [Mar51]. It may be foreseen that these behaviors are, separately, not physically realizable. Thus, if $\alpha = \beta = 0$, it may be assumed that the EM is operating in DC-regime, which does not describe a propagative behavior of solutions, although it is a parameterization physically realizable.
On the other hand, since the propagative exponentials are separated from the dependence on the cross section (as well as the time variation), the EM system in which the modes propagate is invariant along the direction of propagation, which lets to define physically the structures which support these propagative modes.

(ii) The structures in which the propagative modes exist are defined by the imposition of invariance related to the use of complex propagative exponentials. These structures are called uniform waveguides (uniform WG), [Mar51].

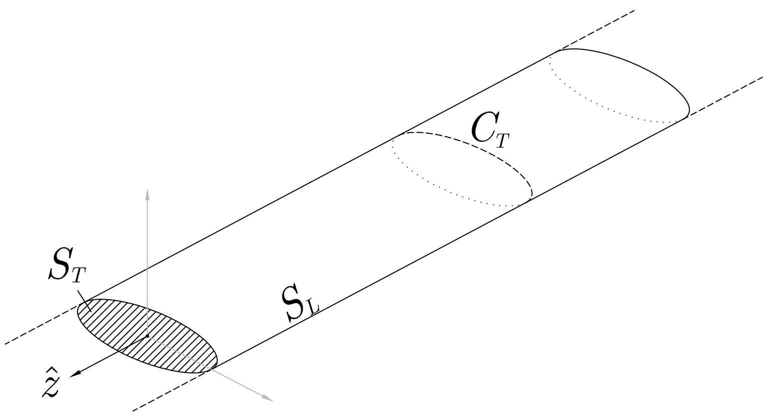


Fig. 3.1: An uniform waveguide infinite along the direction of propagation, $\hat{z}$. The section of the waveguide is denoted as S_T while its contour is C_T. The extrusion of C_T along the $\hat{z}$-direction lead to the lateral surface, S_L.

Although the uniform WGs generalize the direction of propagation, which means that any unitary vector may describe the direction along which the modes propagate and thus the uniform WG is defined invariant, the direction $\hat{z}$ is generally chosen for describing a (cylindrical) uniform WG, as the one represented in Fig. 3.1. In this way, the subsequent equations are particularized to this structure in which z is the direction of propagation and the coordinates $[\mathrm{t}_1, \mathrm{t}_2]$ act as the generic coordinate system in the cross section S_T. This selection does not

[1]The notation presented in [Her14] which has been used to denote the set of time exponentials ($\mathrm{e}(t;\omega)$) is reused here to describe the set of "propagative exponentials" ($\mathrm{e}(z;\gamma)$).

suppose a restriction to the generalization but an important reduction in the expressions of fields and operators (for example $\nabla \times \hat{z} = \mathbf{0}$, because $\hat{z}$ is contant in S_T).
It should be said that S_T may represent an unbounded domain, such as the free space, but, in any case, the constitutive parameters are required to be the same from cross section to cross section, so invariant along the z-coordinate.

(iii) If the electric and magnetic fields propagate with the same $\mp\gamma$ along the z-axis, then there exists a relation between the electric and magnetic fields given by *Maxwell equations* in such a way that for the ith mode solution projected to the S_T (that is the transversal field components $\boldsymbol{E}_{\mathrm{t}}$ and $\boldsymbol{H}_{\mathrm{t}}$), it verifies:

$$\boldsymbol{E}_{\mathrm{t},i} = Z_{0,i}\mathrm{rot}^{-1}\left[\boldsymbol{H}_{\mathrm{t},i}\right], \text{ or} \tag{3.1}$$

$$\boldsymbol{H}_{\mathrm{t},i} = \frac{\mathrm{rot}\left[\boldsymbol{E}_{\mathrm{t},i}\right]}{Z_{0,i}}, \tag{3.2}$$

in which rot[$\circ$] is an operator which rotates the field components in the space, $\mathrm{rot}^{-1}[\circ]$ causes the same rotation but inversely oriented, and $Z_{0,i}$ would play the role of the characteristic impedance in the equivalent TL.
This assumption is forced for the electric and magnetic fields be analyzed by means of an equivalent TL.

There are some open questions that the previous itemized premises leave open.
On one hand, selecting γ as the sprectral variable for describing a set of solutions by means of the set of "propagative harmonics" $e^{\mp\gamma z}$ is clearly an "overstimation" in the sense that not all the possible values of γ refer to real solutions in the waveguide once the BCs and constitutive parameters are imposed. However, a priori every γ whose α, $\beta \geq 0$ is physically possible, something which is a problem when expanding the a particular set field solutions based on γ.
The **LC** that defines the expansions in terms of γ, that is $\mathbf{LC}_\gamma$, is of the form of inverse Laplace Transform (LT), inverse Z-transform (ZT), etc., depending on the space of fuctions in which the solutions are found, [Her14], which, in turn, depends on BCs and constitutive parameters of the WG. In any case, the spectral variable is complex which means that: (i) it is neccesary to study the region of convergence (ROC) in which the expansion converge and (ii) define which form adquires that expansion, for example, if the expansion is an integral, in which path (represented by a curve in the γ-plane) the integration is defined.
The issue of (i) determining the ROC is not critical because the physical restriction on the direction os propagation (making $\alpha > 0$ and $\beta > 0$ in practice) produces that any definite integral converges to a finite function (square integrable/summable, for instance). However (ii) determining the form of the $\mathbf{LC}_\gamma$ requires more specific analysis. The most intuitive way to define $\mathbf{LC}_\gamma$ is using paths based on the complex analysis of γ in terms of losses and BCs, besides the frequency. In this way, the integration along these parameterized path makes the integrals be completely defined. The example presented below helps understanding this idea:

Example 3.1.1. *Imagine the field functions $\boldsymbol{E}$ and $\boldsymbol{H}$ are obtained continuously expanding "propagative harmonics" on the domain of γ in such a way that the $\mathbf{LC}_\gamma$ is of the form of Laplace tranform represented by the operator $\mathbf{LT}^{-1}$, which is a $\mathbf{LC}$, [Her14], so:*

$$\boldsymbol{E}(\mathrm{t}, z) \equiv \mathbf{LT}^{-1}\left\{\boldsymbol{E}_{\boldsymbol{a}}^{\pm}(\mathrm{t}, \gamma)\right\}, \text{ and}$$
$$\boldsymbol{H}(\mathrm{t}, z) \equiv \mathbf{LT}^{-1}\left\{\boldsymbol{H}_{\boldsymbol{a}}^{\pm}(\mathrm{t}, \gamma)\right\},$$

in which $\boldsymbol{E}_{\boldsymbol{a}}(\mathrm{t}, \gamma)$ and $\boldsymbol{H}_{\boldsymbol{a}}(\mathrm{t}, \gamma)$ play the role of coefficients.
If the BCs and the constitutive parameters make the path of integration in the complex plane of γ is P_ς (regularly) parameterized by ς, then the $\mathbf{LT}^{-1}$ and the fields may be written as:

$$\boldsymbol{E}(\mathrm{t}, z) \equiv \mathbf{LT}^{-1}\left\{\boldsymbol{E}_{\boldsymbol{a}}^{\pm}(\mathrm{t}, \gamma)\right\} = \int_{P_\varsigma} \boldsymbol{E}_{\boldsymbol{a}}^{\pm}(\mathrm{t}, \gamma) e^{\mp\gamma(\varsigma) z} d\varsigma, \text{ and}$$

$$\boldsymbol{H}(\mathrm{t}, z) \equiv \mathbf{LT}^{-1}\left\{\boldsymbol{H}_{\boldsymbol{a}}^{\pm}(\mathrm{t}, \gamma)\right\} = \int_{P_\varsigma} \boldsymbol{H}_{\boldsymbol{a}}^{\pm}(\mathrm{t}, \gamma) e^{\mp\gamma(\varsigma) z} d\varsigma.$$

Moreover, if the parameteric expression of the path P_ς is written in general form, and α is solved as a function of β ($\alpha(\beta)$), then the $\mathbf{LT}^{-1}$ *may be written as an inverse Fourier Transform (*$\mathbf{FT}^{-1}$ *in the range in which β varies (denoted as $\langle\beta\rangle$))*

$$\boldsymbol{E}(\mathrm{t}, z) \equiv \int_{\langle\beta\rangle} \left(\boldsymbol{E}^{\pm}_{\boldsymbol{a}}(\mathrm{t}, \beta) e^{\mp\alpha(\beta)z}\right) e^{-j\beta(z)} d\beta \equiv \mathbf{FT}^{-1}\left\{\boldsymbol{E}^{\pm}_{\boldsymbol{a}}(\mathrm{t}, \beta) e^{\mp\alpha(\beta)z}\right\}, \text{ and}$$

$$\boldsymbol{H}(\mathrm{t}, z) \equiv \int_{\langle\beta\rangle} \left(\boldsymbol{H}^{\pm}_{\boldsymbol{a}}(\mathrm{t}, \beta) e^{\mp\alpha(\beta)z}\right) e^{-j\beta(z)} d\beta \equiv \mathbf{FT}^{-1}\left\{\boldsymbol{H}^{\pm}_{\boldsymbol{a}}(\mathrm{t}, \beta) e^{\mp\alpha(\beta)z}\right\}.$$

By means of this example it can be seen that the characterization of γ becomes essential for the analysis.
Notice that this requirement is slightly and subtly different to expand the fields by means of time harmonics because in that analysis ω is always in $\mathbb{R}$, whereas γ as complex spectral variable has to be characterized in terms of losses, the BCs and the frequency in itself. Thus, in this characterization of γ the BCs have to be parameterized at the same level as the losses or the frequency (taking it as fixed parameter, so in the fixed frequency analysis). The process of parameterizing the BCs as losses is consequence of the inverse characterization when looking for different solutions regarding *Maxwell equtions*.

Keeping these ideas in mind, in this chapter the propagative solutions of *Maxwell equations* in uniform WGs are obtained in a generalized form in Sect. 3.2. Over these equations it is possible to impose different conditions for obtaining different solutions. This would be, for example, the case of HPWs presented in Sect. 2.2 in Chpt. 2, in which introducing the *HPW conditions* (*plane wave cond.* plus *harmonic wave cond.*) leads to solve these waves from the wave equations. However, the solutions here are left open and written as a $\mathbf{LC}_\gamma$ of the fields. Then, the domain of is reduced to the so called *propagative domain*, which is no more than the domain of the coefficients which results when the propagative basis $e^{\mp\gamma z}$ introduced above are chosen.
Then, some potential functions which describe the fields by differentiation (see the scheme presented in Fig. 2.6 in Sect. 2.3 in Chpt. 2) are supposed to fully describe the equivalent voltage and current waves in generalized *telegrapher's equations*. A well-defined scalar product based on these potentials lets to solve the *Maxwell equations* posed using the $\mathbf{LC}_\gamma$ (posed in integral/summable form).
The parameters which characterize the solutions of *Maxwell equations* are identifyed with the parameters which define the solutions in the supposed *telegrapher's equations* by inverse characterization. This procedure is briefly described in Sect. 3.3.
The analysis presented in this chapter are finally guided and particularized in Sect. 3.4 to obtain the HPW solutions and parameterize them in order to be compared with the solutions and parameters obtained by direct characterization in Chpt. 2.

3.2 Guided waves in uniform waveguides

Referencing a generic orthogonal coordinate system in the S_T of the the uniform WG depicted in Fig. 3.1,

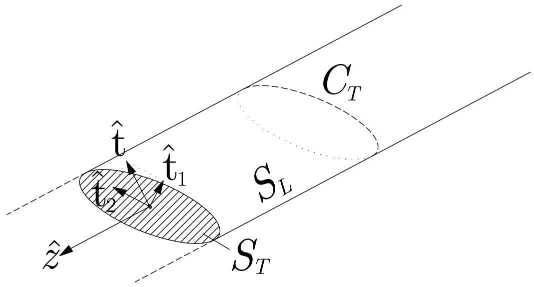


Fig. 3.2: Generic coordinate system used for describing the WG geometry in which $\hat{z}$ points to the longituinal direction and $\hat{\mathrm{t}} \equiv [\hat{\mathrm{t}_1}, \hat{\mathrm{t}_2}]$ is the unitary vector in the crosst section S_T.

The fields in in the cylindrical coordinate system established in the waveguide depicted in Fig. 3.2 are written as:

$$\mathbf{E} = \mathbf{E}_{\mathrm{t}} + E_z \hat{z}, \tag{3.3}$$
$$\mathbf{H} = \mathbf{H}_{\mathrm{t}} + H_z \hat{z}; \tag{3.4}$$

and the differential operators in this frame of reference may be written as:

$$\nabla \equiv \nabla_t + \frac{\partial [\circ]}{\partial z} \hat{z},$$
$$\Delta \equiv \Delta_t + \frac{\partial^2 [\circ]}{\partial z^2};$$

so the two main Maxwell equations: the *Faraday's law* and the *Ampère's law*; are splitted into four equations separating the transversal and longitudinal components of each field, [Mar51]:

$$\nabla_{\mathrm{t}} \times E_z \hat{z} + \frac{\partial [\hat{z} \times \mathbf{E}_{\mathrm{t}}]}{\partial z} = -j\omega\mu \mathbf{H}_{\mathrm{t}}, \tag{3.5}$$
$$\nabla_{\mathrm{t}} \times \mathbf{E}_{\mathrm{t}} = -j\omega\mu H_z \hat{z}, \tag{3.6}$$
$$\nabla_{\mathrm{t}} \times H_z \hat{z} + \frac{\partial [\hat{z} \times \mathbf{H}_{\mathrm{t}}]}{\partial z} = j\omega\varepsilon_{eq} \mathbf{E}_{\mathrm{t}}, \text{ and} \tag{3.7}$$
$$\nabla_{\mathrm{t}} \times \mathbf{H}_{\mathrm{t}} = j\omega\varepsilon_{eq} E_z \hat{z}. \tag{3.8}$$

Eqs. (3.5)-(3.8) are written in the *frequency domain* as this analysis is supposed to operate in, so ω is another parameter to take into account, and thus ε_{eq} and μ are complex parameters.
The fields $\mathbf{E}_{\mathrm{t}}$ and $\mathbf{H}_{\mathrm{t}}$ are complex fuctions which depend on the spacial coordinates $[\mathbf{t}, z] \in \bar{S_T} \times \hat{z}$. The real and imaginary parts of these fields are supposed to be at least $\mathcal{C}^2(S_T \times z)$ and $\mathcal{C}^1(\bar{S_T} \times z)$.

The longitudinal components E_z and H_z are solved from eqs. (3.8) and (3.6) to be replaced in eqs. (3.5) and (3.7), respectively. Assuming that there is no volume density charge in the waveguide, the solved equations are:

$$\frac{\partial \mathbf{E}_{\mathrm{t}}}{\partial z} = \frac{1}{j\omega\varepsilon_{eq}} \left[k^2 + \Delta_t\right] \mathbf{H}_{\mathrm{t}} \times \hat{z} \text{ and} \tag{3.9}$$
$$\frac{\partial \mathbf{H}_{\mathrm{t}}}{\partial z} = \frac{1}{j\omega\mu} \left[k^2 + \Delta_t\right] \hat{z} \times \mathbf{E}_{\mathrm{t}}. \tag{3.10}$$

Notice that eqs. (3.9) and (3.10) relate the transversal components of the electric and magentic fields, so once these equations are solved, the longitudinal components E_z and H_z would be obtained by eqs. (3.8) and (3.6).
Also notice that eqs. (3.9) and (3.10) are of the form of *telegrapher's equations*, [Mar51], despite they relate vector instead of scalar functions which also depend on the coordinates in the cross sesction. Nevertheless, they relate first derivatives of a magnitude along the direction of propagation with a scaling of another magnitude provided that the eigenvalues of the operator $[k^2 + \Delta_\mathrm{t}]$ were found, just as the *telegrapher's equations* do.

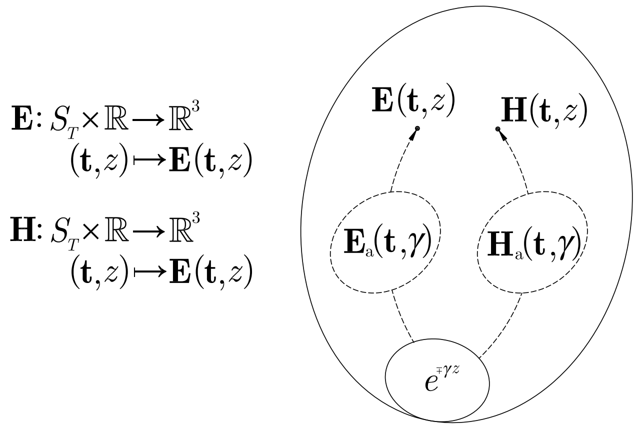


Fig. 3.3: Representation of the solutions $\boldsymbol{E} \equiv \boldsymbol{E}^{\pm}(\mathbf{t}, z)$ and $\boldsymbol{H} \equiv \boldsymbol{H}^{\pm}(\mathbf{t}, z)$ in the *frequency domain*, which are expanded by means of a $\mathbf{LC}_\gamma$ of complex propagative exponentials $e^{\pm\gamma z}$ weighted by coefficients $\boldsymbol{E_a}(\mathbf{t}, \gamma)$ and $\boldsymbol{H_a}(\mathbf{t}, \gamma)$, respectively.

Now, taking advantage of the fact that the electric and magnetic fields in a waveguide can be expanded by means of complex propagative exponentials as

$$\boldsymbol{E}^{\pm}(\mathbf{t}, z) \equiv \mathbf{LC}_\gamma \left\{ \boldsymbol{E}_{\boldsymbol{a}}^{\pm}(\mathbf{t}, \gamma) e^{\mp\gamma z} \right\}, \tag{3.11}$$

$$\boldsymbol{H}^{\pm}(\mathbf{t}, z) \equiv \mathbf{LC}_\gamma \left\{ \boldsymbol{H}_{\boldsymbol{a}}^{\pm}(\mathbf{t}, \gamma) e^{\mp\gamma z} \right\}; \tag{3.12}$$

it is possible to restrict the analysis to the coefficients $\boldsymbol{E_a}(\mathbf{t}, \gamma)$ and $\boldsymbol{H_a}(\mathbf{t}, \gamma)$, depicted in Fig. 3.3, which means working in the "*propagative domain*" with the "propagative harmonics" $\boldsymbol{E_a}(\mathbf{t}; \gamma) \equiv \boldsymbol{E_a}(\mathbf{t})$ and $\boldsymbol{H_a}(\mathbf{t}; \gamma) \equiv \boldsymbol{H_a}(\mathbf{t})$. In this way, γ transforms to a parameter, equivalently to what happens with ω when operating in the *frequency domain* with time harmonics.
When γ is seen as a parameter of functions $\boldsymbol{E_a}(\mathbf{t}, \gamma)$ and $\boldsymbol{H_a}(\mathbf{t}, \gamma)$, it can be proved that they form a 2-dim subspace (see Appendix 3.A).
In addition, the derivatives in z transform as:

$$[\partial(\circ)^{\pm}]/\partial z \overset{z-\mathrm{LT}}{\longrightarrow} \mp\gamma \cdot (\circ)^{\pm};$$

so eqs. (3.9) and (3.10) are equivalent to:

$$\boldsymbol{E_a} = \frac{1}{\mp\gamma} \frac{1}{j\omega\varepsilon_{eq}} \left[k^2 + \Delta_t\right] \boldsymbol{H_a} \times \hat{z} \text{ and} \tag{3.13}$$

$$\boldsymbol{H_a} = \frac{1}{\mp\gamma} \frac{1}{j\omega\mu} \left[k^2 + \Delta_t\right] \hat{z} \times \boldsymbol{E_a}, \tag{3.14}$$

respectively, when operating in the "*propagative domain*".

Analyzing eqs. (3.13) and (3.14), and comparing them with (3.1) and (3.2), the eigenvalues of the Helmholtz operator $\left[k^2 + \Delta_t\right]$ are those that define the characteristic impedance. Integrating $\mathbf{E}_t$

and $\mathbf{H}_t$ by means of an orthogonal set of functions that form a basis lets to obtain these eigenvalues. Rememeber that the space of functions in which $\boldsymbol{E_a}$ and $\boldsymbol{H_a}$ are in needs to be at least twice differentiable in S_T and the functions there must have finite energy. A Hilbert space equiped with a scalar product in which their functions verify these two properties is a Sobolev space, $W^{2,2}$. Subspaces of the domain and/or range of the Helmholtz operator are the sets of mode solutions.

With these ideas in mind, equations (3.13) and (3.14) are rewriten when considering the **LC**s

$$\begin{aligned} \boldsymbol{E_a^{\pm}} &= \mathbf{LC}_\xi \left[V^{\pm}(\xi)\mathbf{e}(t_1,t_2;\xi) \right] \text{ and} \\ \boldsymbol{H_a^{\pm}} &= \mathbf{LC}_\xi \left[I^{\pm}(\xi)\mathbf{h}(t_1,t_2;\xi) \right] \end{aligned}$$

as

$$\begin{aligned} \mathbf{LC}_\xi \left[V^{\pm}(\xi)\mathbf{e}(t_1,t_2;\xi) \right] &= \\ &= \frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}} k^2 \mathbf{LC}_{\xi'} \left[I^{\pm}(\xi')\mathbf{h}(t_1,t_2;\xi') \times \hat{z} \right] + \\ &+ \frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}} \mathbf{LC}_{\xi'} \left[I^{\pm}(\xi')\Delta_t\mathbf{h}(t_1,t_2;\xi') \times \hat{z} \right] \text{ and} \end{aligned} \tag{3.15}$$

$$\begin{aligned} \mathbf{LC}_\xi \left[I^{\pm}(\xi)\mathbf{h}(t_1,t_2;\xi) \right] &= \\ &= \frac{1}{\mp\gamma}\frac{1}{j\omega\mu} k^2 \mathbf{LC}_{\xi'} \left[V^{\pm}(\xi')\hat{z} \times \mathbf{e}(t_1,t_2;\xi') \right] + \\ &+ \frac{1}{\mp\gamma}\frac{1}{j\omega\mu} \mathbf{LC}_{\xi'} \left[V^{\pm}(\xi')\Delta_t\hat{z} \times \mathbf{e}(t_1,t_2;\xi') \right], \end{aligned} \tag{3.16}$$

taking advantage of the linearity of **LC** operators.
In this case, the "spectral variable" ξ (or ξ') is useful to describe the order of modes, and it is directly related to γ because it acts as the cutoff wavenumber, just as it has been presented in Appendix 3.A. Depending on the belonging of ξ (ξ') to different field of numbers, the **LC** will be accordingly defined. The vector functions **e** and **h** are the basis functions which are required to be orthogonal with recpect to the scalar product defined in the space in which the solutions are.

Notice that in eqs. (3.15) and (3.16), the conditions

$$\begin{aligned} \mathbf{e}(t_1,t_2;\xi) &= \mathbf{h}(t_1,t_2;\xi) \times \hat{z} \text{ and} \\ \mathbf{h}(t_1,t_2;\xi) &= \hat{z} \times \mathbf{e}(t_1,t_2;\xi) \end{aligned}$$

have to be fulfilled for the fields **E** and **H** to propagate in the same TL belonging to the same wave, so eqs. (3.15) and (3.16) turn into

$$\begin{aligned} \mathrm{LC}_\xi \left[V^{\pm}(\xi)\mathbf{e}(t_1,t_2;\xi) \right] &= \\ &= \frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}} k^2 \mathrm{LC}_{\xi'} \left[I^{\pm}(\xi')\mathbf{e}(t_1,t_2;\xi') \right] + \\ &+ \frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}} \mathrm{LC}_{\xi'} \left[I^{\pm}(\xi')\Delta_t\mathbf{e}(t_1,t_2;\xi') \right] \text{ and} \end{aligned} \tag{3.17}$$

$$\begin{aligned} \mathbf{LC}_\xi \left[I^{\pm}(\xi)\mathbf{h}(t_1,t_2;\xi) \right] &= \\ &= \frac{1}{\mp\gamma}\frac{1}{j\omega\mu} k^2 \mathrm{LC}_{\xi'} \left[V^{\pm}(\xi')\mathbf{h}(t_1,t_2;\xi') \right] + \\ &+ \frac{1}{\mp\gamma}\frac{1}{j\omega\mu} \mathbf{LC}_{\xi'} \left[V^{\pm}(\xi')\Delta_t\mathbf{h}(t_1,t_2;\xi') \right]. \end{aligned} \tag{3.18}$$

Now the orthogonality between the $\mathbf{e}(t_1,t_2;\xi)$ and $\mathbf{h}(t_1,t_2;\xi)$ functions in eqs. (3.17) and (3.18) is going to be used.

Scalar multiplying both sides of eqs. (3.17) and (3.18) by $\mathbf{e}(t_1,t_2;\xi_0)$ and $\mathbf{h}(t_1,t_2;\xi_0)$, respectively, leads to

$$\begin{aligned}\mathrm{LC}_{\xi}\left[V^{\pm}(\xi)\left\langle\mathbf{e}(t_1,t_2;\xi_0),\mathbf{e}(t_1,t_2;\xi)\right\rangle\right]=&\\ =\frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}k^2\mathrm{LC}_{\xi'}\left[I^{\pm}(\xi')\left\langle\mathbf{e}(t_1,t_2;\xi_0),\mathbf{e}(t_1,t_2;\xi')\right\rangle\right]+&\\ +\frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}\mathrm{LC}_{\xi'}\left[I^{\pm}(\xi')\left\langle\mathbf{e}(t_1,t_2;\xi_0),\Delta_t\mathbf{e}(t_1,t_2;\xi')\right\rangle\right]\ \text{and}&\end{aligned}\tag{3.19}$$

$$\begin{aligned}\mathrm{LC}_{\xi}\left[I^{\pm}(\xi)\left\langle\mathbf{h}(t_1,t_2;\xi_0),\mathbf{h}(t_1,t_2;\xi)\right\rangle\right]=&\\ =\frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}k^2\mathrm{LC}_{\xi'}\left[V^{\pm}(\xi')\left\langle\mathbf{h}(t_1,t_2;\xi_0),\mathbf{h}(t_1,t_2;\xi')\right\rangle\right]+&\\ +\frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}\mathrm{LC}_{\xi'}\left[V^{\pm}(\xi')\left\langle\mathbf{h}(t_1,t_2;\xi_0),\Delta_t\mathbf{h}(t_1,t_2;\xi')\right\rangle\right],&\end{aligned}\tag{3.20}$$

in which

$$\left\langle\mathbf{e}(t_1,t_2;\xi_0),\mathbf{e}(t_1,t_2;\xi)\right\rangle=\delta(\xi-\xi_0),\ \text{and also}$$
$$\left\langle\mathbf{h}(t_1,t_2;\xi_0),\mathbf{h}(t_1,t_2;\xi)\right\rangle=\delta(\xi-\xi_0),$$

being $\delta(\xi-\xi_0)$ the Dirac delta distribution, [Her14], so the $\mathbf{LC}_{\xi}$ in which $\delta(\xi-\xi_0)$ appears has to be understood in distribution sense, [Her14], acting the δ as the kernel. Thus, eqs. (3.19) and (3.20) particularize to

$$\begin{aligned}V^{\pm}(\xi_0)=&\frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}k^2I^{\pm}(\xi_0)+\\ +&\frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}\mathrm{LC}_{\xi'}\left[I^{\pm}(\xi')\left\langle\mathbf{e}(t_1,t_2;\xi_0),\mathrm{LC}_{\xi''}\left[-k_{c,e}^2(\xi';\xi'')\mathbf{e}(t_1,t_2;\xi'')\right]\right\rangle\right]=\\ =&\frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}k^2I^{\pm}(\xi_0)+\frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}\mathrm{LC}_{\xi'}\left[-k_{c,e}^2(\xi';\xi_0)I^{\pm}(\xi')\right]\ \text{and}\end{aligned}\tag{3.21}$$

$$\begin{aligned}I^{\pm}(\xi_0)=&\frac{1}{\mp\gamma}\frac{1}{j\omega\mu}k^2V^{\pm}(\xi_0)+\\ +&\frac{1}{\mp\gamma}\frac{1}{j\omega\mu}\mathrm{LC}_{\xi'}\left[V^{\pm}(\xi')\left\langle\mathbf{h}(t_1,t_2;\xi_0),\mathrm{LC}_{\xi''}\left[-k_{c,h}^2(\xi';\xi'')\mathbf{h}(t_1,t_2;\xi'')\right]\right\rangle\right]=\\ =&\frac{1}{\mp\gamma}\frac{1}{j\omega\mu}k^2V^{\pm}(\xi_0)+\frac{1}{\mp\gamma}\frac{1}{j\omega\mu}\mathrm{LC}_{\xi'}\left[-k_{c,h}^2(\xi';\xi_0)V^{\pm}(\xi')\right],\end{aligned}\tag{3.22}$$

in which the functions $-k_{c,e}^2(\xi';\xi'')$ and $-k_{c,h}^2(\xi';\xi'')$ in eqs. (3.21) and (3.22) are the transfer functions of the operator $\Delta_t\left[\circ\right]$ between the ξ'^{th} mode and the ξ''^{th} mode.
The sets of orthogonal functions $\{\mathbf{e}(t_1,t_2;\xi)\}_{\xi}$ and $\{\mathbf{h}(t_1,t_2;\xi)\}_{\xi}$, which are complete with respect to the defined scalar product,together with the transfer functions $-k_{c,e}^2(\xi';\xi'')$ and $-k_{c,h}^2(\xi';\xi'')$, completely define the mode solutions.

Appendix 3.B shows how to define the scalar product between two-variable vector functions using scalar and vector potentials. In this way, notice that the analysis of the solutions of eqs. (3.21) and (3.22) reduces to find the potentials which are orthogonal on the S_T domain. This analysis leads to establish different BCs on C_T and/or particularizing the operators acting over these functions on S_T. For example, the operator Δ_{t} which is applied to the fields leading to the null vector lets to define HPWs.

Next subsections are intended to describe the HPWs from scalar potentials, which is one solution among all the possible TEM modes, particularizing the equations above (refer to the scheme in Fig. 3.5 below to see a "map" of all the possible solutions indicating where the HPWs are).
Nevertheless, this analysis could be particularized imposing different conditions on these equations leading to different solutions. In this way, the GTLT-v1 is finally established. Its scope covers all

the possible solutions that result from the particularization of the general equations presented in this section.

3.2.1 TEM modes in uniform waveguides

The most trivial set of solution for eqs. (3.17)-(3.18) comes from imposing

TEM modes cond.

$$\boxed{\begin{array}{c}\Delta_t \mathbf{e} = \Delta_t \left(-\nabla_t \phi_e + \nabla_t \times A_e \hat{z}\right) \overset{2}{=} -\nabla_t \Delta_t \phi_e + \nabla_t \left(\Delta_t A_e\right) \times \hat{z} = \mathbf{0} \text{ and} \\ \Delta_t \mathbf{h} = \Delta_t \left(-\nabla_t \phi_h + \nabla_t \times A_h \hat{z}\right) = -\nabla_t \Delta_t \phi_h + \nabla_t \left(\Delta_t A_h\right) \times \hat{z} = \mathbf{0}.\end{array}}$$

The dependence of the potentials on the transversal coordinates has been omitted for the sake of simplicity. Thus, in the following analysis only the spectral variable ξ is indicated.

Replacing the dot products in eqs. (3.19) and (3.20) (in which the second addend is set to zero in order to verify the *TEM modes cond.*) with the definition introduced in Appendix 3.B based on potentials, it gets to:

$$\begin{aligned}
\mathbf{LC}_\xi \Bigg[-V^{\pm}(\xi) \iint_{S_T} \phi_e(\xi_0) \Delta_t \phi_e(\xi) dS + V^{\pm}(\xi) \oint_{C_T} \phi_e(\xi_0) \nabla_t \phi_e(\xi) \cdot \hat{n} dl \, + \\
+ V^{\pm}(\xi) \iint_{S_T} \left(\nabla_t \phi_e(\xi_0) \times \nabla_t A_e^*(\xi)\right) dS \hat{z} - \\
- V^{\pm}(\xi) \iint_{S_T} \left(\nabla_t A_e(\xi_0) \times \nabla_t \phi_e^*(\xi)\right) dS \hat{z} - \\
- V^{\pm}(\xi) \iint_{S_T} A_e(\xi_0) \Delta_t A_e^*(\xi) dS + V^{\pm}(\xi) \oint_{C_T} A_e(\xi_0) \nabla_t A_e^*(\xi) \cdot \hat{n} dl \Bigg] = \\
= \frac{1}{\mp\gamma} \frac{1}{j\omega\varepsilon_{eq}} k^2 \mathrm{LC}_{\xi'} \Bigg[-I^{\pm}(\xi') \iint_{S_T} \phi_e(\xi_0) \Delta_t \phi_e(\xi') dS + \\
I^{\pm}(\xi') \oint_{C_T} \phi_e(\xi_0) \nabla_t \phi_e(\xi') \cdot \hat{n} dl + \\
+ I^{\pm}(\xi) \iint_{S_T} \left(\nabla_t \phi_e(\xi_0) \times \nabla_t A_e^*(\xi')\right) dS \hat{z} - \\
- I^{\pm}(\xi') \iint_{S_T} \left(\nabla_t A_e(\xi_0) \times \nabla_t \phi_e^*(\xi')\right) dS \hat{z} - \\
- I^{\pm}(\xi') \iint_{S_T} A_e(\xi_0) \Delta_t A_e^*(\xi') dS + I^{\pm}(\xi') \oint_{C_T} A_e(\xi_0) \nabla_t A_e^*(\xi') \cdot \hat{n} dl \Bigg] \text{ and}
\end{aligned} \tag{3.23}$$

[2] $\Delta_t \left(\nabla_t \times A_e \hat{z}\right) = \nabla_t \times \left[\Delta_t \left(A_e \hat{z}\right)\right] = \nabla_t \times \left(\hat{z} \Delta_t A_e\right) = \nabla_t \left(\Delta_t A_e\right) \times \hat{z}$

$$
\begin{aligned}
\mathrm{LC}_\xi \Bigg[-I^\pm(\xi) \iint_{S_T} \phi_h(\xi_0)\Delta_t\phi_h(\xi)dS + I^\pm(\xi) \oint_{C_T} \phi_h(\xi_0)\nabla_t\phi_h(\xi)\cdot\hat{n}dl + \\
+I^\pm(\xi) \iint_{S_T} \left(\nabla_t\phi_h(\xi_0)\times\nabla_t A_h^*(\xi)\right) dS\hat{z} - \\
-I^\pm(\xi) \iint_{S_T} \left(\nabla_t A_h(\xi_0)\times\nabla_t \phi_h^*(\xi)\right) dS\hat{z} - \\
-I^\pm(\xi) \iint_{S_T} A_h(\xi_0)\Delta_t A_h^*(\xi)dS + I^\pm(\xi) \oint_{C_T} A_h(\xi_0)\nabla_t A_h^*(\xi)\cdot\hat{n}dl \Bigg] = \\
= \frac{1}{\mp\gamma}\frac{1}{j\omega\mu}k^2\mathrm{LC}_{\xi'} \Bigg[-V^\pm(\xi') \iint_{S_T} \phi_h(\xi_0)\Delta_t\phi_h(\xi')dS + \\
V^\pm(\xi') \oint_{C_T} \phi_h(\xi_0)\nabla_t\phi_h(\xi')\cdot\hat{n}dl + \\
+V^\pm(\xi) \iint_{S_T} \left(\nabla_t\phi_h(\xi_0)\times\nabla_t A_h^*(\xi')\right) dS\hat{z} - \\
-V^\pm(\xi') \iint_{S_T} \left(\nabla_t A_h(\xi_0)\times\nabla_t \phi_h^*(\xi')\right) dS\hat{z} - \\
- V^\pm(\xi') \iint_{S_T} A_h(\xi_0)\Delta_t A_h^*(\xi')dS + V^\pm(\xi') \oint_{C_T} A_h(\xi_0)\nabla_t A_h^*(\xi')\cdot\hat{n}dl \Bigg].
\end{aligned}
\tag{3.24}
$$

Equivantely,

$$
\begin{aligned}
\mathrm{LC}_\xi \Bigg[-V^\pm(\xi) \iint_{S_T} \phi_e(\xi_0)\Delta_t\phi_e(\xi)dS + V^\pm(\xi) \oint_{C_T} \phi_e(\xi_0)\nabla_t\phi_e(\xi)\cdot\hat{n}dl + \\
+V^\pm(\xi) \oint_{C_T} \phi_e(\xi_0)\nabla_t A_e^*(\xi)\cdot(\hat{z}\times\hat{n})dl + \\
+V^\pm(\xi) \oint_{C_T} \phi_e^*(\xi)\nabla_t A_e(\xi_0)\cdot(\hat{z}\times\hat{n})dl - \\
-V^\pm(\xi) \iint_{S_T} A_e(\xi_0)\Delta_t A_e^*(\xi)dS + V^\pm(\xi) \oint_{C_T} A_e(\xi_0)\nabla_t A_e^*(\xi)\cdot\hat{n}dl \Bigg] = \\
= \frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}k^2\mathrm{LC}_{\xi'} \Bigg[-I^\pm(\xi') \iint_{S_T} \phi_e(\xi_0)\Delta_t\phi_e(\xi')dS + \\
I^\pm(\xi') \oint_{C_T} \phi_e(\xi_0)\nabla_t\phi_e(\xi')\cdot\hat{n}dl + \\
+I^\pm(\xi') \oint_{C_T} \phi_e(\xi_0)\nabla_t A_e^*(\xi')\cdot(\hat{z}\times\hat{n})dl + \\
+I^\pm(\xi') \oint_{C_T} \phi_e^*(\xi')\nabla_t A_e(\xi_0)\cdot(\hat{z}\times\hat{n})dl - \\
- I^\pm(\xi') \iint_{S_T} A_e(\xi_0)\Delta_t A_e^*(\xi')dS + I^\pm(\xi') \oint_{C_T} A_e(\xi_0)\nabla_t A_e^*(\xi')\cdot\hat{n}dl \Bigg] \text{ and}
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
\mathrm{LC}_\xi \Big[-I^\pm(\xi) \iint_{S_T} \phi_h(\xi_0)\Delta_t\phi_h(\xi)dS + I^\pm(\xi) \oint_{C_T} \phi_h(\xi_0)\nabla_t\phi_h(\xi)\cdot\hat{n}dl\, + \\
+ I^\pm(\xi) \oint_{C_T} \phi_h(\xi_0)\nabla_t A_h^*(\xi)\cdot(\hat{z}\times\hat{n})dl + \\
+ I^\pm(\xi) \oint_{C_T} \phi_h^*(\xi)\nabla_t A_h(\xi_0)\cdot(\hat{z}\times\hat{n})dl - \\
- I^\pm(\xi) \iint_{S_T} A_h(\xi_0)\Delta_t A_h^*(\xi)dS + I^\pm(\xi) \oint_{C_T} A_h(\xi_0)\nabla_t A_h^*(\xi)\cdot\hat{n}dl \Big] = \\
= \frac{1}{\mp\gamma}\frac{1}{j\omega\mu}k^2\mathrm{LC}_{\xi'} \Big[-V^\pm(\xi') \iint_{S_T} \phi_h(\xi_0)\Delta_t\phi_h(\xi')dS + \\
V^\pm(\xi') \oint_{C_T} \phi_h(\xi_0)\nabla_t\phi_h(\xi')\cdot\hat{n}dl + \\
+ V^\pm(\xi') \oint_{C_T} \phi_h(\xi_0)\nabla_t A_h^*(\xi')\cdot(\hat{z}\times\hat{n})dl + \\
+ V^\pm(\xi') \oint_{C_T} \phi_h^*(\xi')\nabla_t A_h(\xi_0)\cdot(\hat{z}\times\hat{n})dl - \\
- V^\pm(\xi') \iint_{S_T} A_h(\xi_0)\Delta_t A_h^*(\xi')dS + V^\pm(\xi') \oint_{C_T} A_h(\xi_0)\nabla_t A_h^*(\xi')\cdot\hat{n}dl \Big].
\end{aligned} \tag{3.26}
$$

Let's now particularize the equations above to obtain the so called HPWs based on harmonic scalar potentials, which are the same type of waves of those presented and used in Chpt. 2 for developing the LTLT.

Harmonic plane waves from Harmonic scalar potentials in uniform waveguides

First and most intuitive solutions (because of its physical meaning directly related to voltages and currents) comes from imposing:

HPW cond.

$A_e = A_h = 0$ on $\bar{S_T}$,	(no vector potential sources everywhere)
$\Delta_t\phi_e = \Delta_t\phi_h = 0$ on S_T,	(Harmonic Scalar Potential sources)

being $\bar{S_T} \equiv S_T \cup C_T$ the closed domain and S_T the cross section region.

Using the *HPW cond.* in eqs. (3.23)-(3.24) (or equivalenty in eqs. (3.25)-(3.26)) leads to:

$$
\begin{aligned}
\mathrm{LC}_\xi \left[V^\pm(\xi) \oint_{C_T} \phi_e(\xi_0)\nabla_t\phi_e(\xi)\cdot\hat{n}dl \right] = \\
= \frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}k^2\mathrm{LC}_{\xi'} \left[I^\pm(\xi') \oint_{C_T} \phi_e(\xi_0)\nabla_t\phi_e(\xi')\cdot\hat{n}dl \right] \text{ and}
\end{aligned} \tag{3.27}
$$

$$
\begin{aligned}
\mathrm{LC}_\xi \left[I^\pm(\xi) \oint_{C_T} \phi_h(\xi_0)\nabla_t\phi_h(\xi)\cdot\hat{n}dl \right] = \\
= \frac{1}{\mp\gamma}\frac{1}{j\omega\mu}k^2\mathrm{LC}_{\xi'} \left[V^\pm(\xi') \oint_{C_T} \phi_h(\xi_0)\nabla_t\phi_h(\xi')\cdot\hat{n}dl \right].
\end{aligned} \tag{3.28}
$$

Since ϕ_e and ϕ_h are both harmonic on S_T they are $\mathcal{C}^2$ differentiable. It also means that the potentials verify the Cauchy-Riemann equations, [BC90], on S_T and therefore they are analytic on

S_T.
The analiticity of ϕ_e and ϕ_h guarantees the derivatives are omnidirectional, and because they are $\mathcal{C}^2$, continuous. It means that [3]:

$$\left.\frac{\partial\phi_e(t_1,t_2)}{\partial n}\right|_{C_T(t)} = \lim_{(t_1,t_2)\to C_T(t)} \frac{\partial\phi_e(t_1,t_2)}{\partial n} \equiv \lim_{(t_1,t_2)\to C_T(t)} \nabla_t\phi_e(t_1,t_2)\cdot\hat{n}, \text{ and}$$

$$\left.\frac{\partial\phi_h(t_1,t_2)}{\partial n}\right|_{C_T(t)} = \lim_{(t_1,t_2)\to C_T(t)} \frac{\partial\phi_h(t_1,t_2)}{\partial n} \equiv \lim_{(t_1,t_2)\to C_T(t)} \nabla_t\phi_h(t_1,t_2)\cdot\hat{n},$$

in which $C_T(t)$ is a (piecewise) regular parameterization of the boundary curve C_T, which may be any (closed) curve and thus

$$\boxed{\begin{array}{c}\nabla_t\phi_e = c_{e,1}\in\mathbb{R} \text{ on } C_T, \text{ and}\\ \nabla_t\phi_h = c_{h,1}\in\mathbb{R} \text{ on } C_T,\end{array}}$$

because $\Delta_t\phi_e = \nabla_t\left(\nabla_t\phi_e\right) = 0$ as $\Delta_t\phi_h = \nabla_t\left(\nabla_t\phi_h\right)$.

With this idea in mind, the conditions

$$\boxed{\begin{array}{c}\phi_e = c_{e,2}\in\mathbb{R} \text{ on } C_T, \text{ and}\\ \phi_h = c_{h,2}\in\mathbb{R} \text{ on } C_T,\end{array}}$$

have to be fulfilled. That is because the derivatives are continuous on S_T and the potentials are analytic, so if

$$d\phi_e(n,t) = \nabla_{\rm t}\phi_e(n,t)\cdot\mathbf{dr} = \left(\frac{\partial\phi_e(n,{\rm t})}{\partial n}\hat{n} + \frac{1}{h_{\rm t}}\frac{\partial\phi_e(n,{\rm t})}{\partial{\rm t}}\hat{\rm t}\right)\cdot\mathbf{dr},$$

on C_T, and the limit

$$\lim_{\Delta t\to 0}\phi_e(n,{\rm t}+\Delta{\rm t}) - \phi_e(n,{\rm t})|_{C_T} = d\phi_e(n,{\rm t})|_{C_T} = \nabla_t\phi_e(n,{\rm t})|_{C_T}\cdot dl\hat{\rm t} = \frac{\partial\phi_e(n,{\rm t})}{\partial n}\hat{n}\cdot h_{\rm t}d{\rm t}\hat{\rm t} = 0,$$

n is the coordinate normal to C_T and t is tangential to C_T, then

$$\left.\lim_{\Delta t\to 0}\phi_e(n,t+\Delta t)\right|_{C_T} = \phi_e(n,t)|_{C_T} = c_{e,2},$$

being $c_{e,2}$ a constant, and the same follows $\phi_h = c_{h,2}$ on C_T.

Then, the HPWs are obtained when solving:

$$\boxed{\begin{array}{c}\Delta\phi_e = 0 \text{ on } S_T,\\ \frac{\partial\phi_e}{\partial n} = c_{e,1} \text{ on } C_T,\\ \phi_e = c_{e,2} \text{ on } C_T,\end{array}} \quad (29)$$

and

[3] Without loss of generality, the normal coordinate is supposed to be metric so its scale factor $h_n = 1$.

$$\boxed{\begin{array}{c}\Delta\phi_h = 0 \text{ on } S_T,\\ \frac{\partial\phi_h}{\partial n} = c_{h,1} \text{ on } C_T,\\ \phi_h = c_{h,2} \text{ on } C_T,\end{array}} \quad (30)$$

The constants $c_{e,1}$ and $c_{e,2}$, and $c_{h,1}$ and $c_{h,1}$ can not be zero for the equations (3.27) and (3.28) not be trivially solvable.

These equivalent problems in eqs. (3.29) and (3.30) are of the form of Cauchy problems seen as boundary value problems (BVPs) with Cauchy conditions on C_T, so the verify uniqueness by the Cauchy–Kowalewski theorem, [Zwi97], supposing C_T is a smooth (piecewise) curve (if C_T accepts a regular parameterization –as it is supposed to– then C_T is smooth).
Because of the solution of the BVP is unique and normalized, the constants must verify:

$$\boxed{\begin{array}{c}c_{e,1} = c_{h,1} \equiv c_1,\\ c_{e,2} = c_{h,2} \equiv c_2.\end{array}} \quad \textit{Uniqueness of the normalized BVPs (3.29) and (3.30)}$$

The solutions of the BVPs above need to be orthogonal with respect to other modes.

Remark 9. *It is important to detach that not only the BCs but also the class of the functions belonging the space in which the BVP is solved is determining for solving it.*

In addition, it is important to obtain the physical meaning of the BCs as well as the interpretation of the class of funtions the modes belong.
For the purpose of physical interpretation, the condition the TEM solutions based on Harmonic Scalar Potentials (HSP) (that is HPWs) have to verify is that S_T is multiply connected so S_T have at least one "hole", just as it has been proved in Sect. 2.3 in Chpt. 2. Equivalenty, if S_T is multiply connected, C_T is not a Jordan curve, or every point in the curve is not connected to another one.

The TEM mode based on HSP (the HPWs) is the mode $\xi = 0$ in eqs. (3.27) and (3.28) that verifies:

$$V^{\pm}_{HPW} \equiv V^{\pm}(\xi = 0) = \frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}k^2 I^{\pm}(\xi = 0) \equiv \frac{1}{\mp\gamma}\frac{1}{j\omega\varepsilon_{eq}}k^2 I^{\pm}_{HPW}, \text{ and} \tag{3.31}$$

$$I^{\pm}_{HPW} \equiv I^{\pm}(\xi = 0) = \frac{1}{\mp\gamma}\frac{1}{j\omega\mu}k^2 V^{\pm}(\xi = 0) \equiv \frac{1}{\mp\gamma}\frac{1}{j\omega\mu}k^2 V^{\pm}_{HPW}. \tag{3.32}$$

Combining eqs. (3.31) and (3.31), it leads to:

$$\gamma_{HPW} = jk = j\omega\sqrt{\mu\varepsilon_{eq}}, \text{ and} \tag{3.33}$$

$$Z_{0,HPW} \equiv \frac{V^{\pm}_{TEM}}{I^{\pm}_{TEM}} = \sqrt{\frac{\mu}{\varepsilon_{eq}}}, \tag{3.34}$$

which are the propagation constant and the characteristic impedance of TEM modes based on HSP, respectively, so the propagation constant and the characteristic impedance of HPWs, just as they have been obtained in eqs. (2.51) and (2.68) in Chpt. 2.

As a result of this analysis, the potential functions $\phi^{\pm}_e \equiv V^{\pm}_{HPW} f(\mathrm{t}_1, \mathrm{t}_2)$ and $\phi^{\pm}_h \equiv I^{\pm}_{HPW} f(\mathrm{t}_1, \mathrm{t}_2)$, in which $f(\mathrm{t}_1, \mathrm{t}_2)$ is a generic function which verifies the zero Laplacian (harmonic) on S_T, parameterized by $Z_{0,HPW}$ and γ_{HPW} leads to define the line parameters which appear in the equivalent *telegrapher's equations* by means of inverse characterization. This is explained in the following section, particularized to HPWs.

3.3 Equivalent waves. Telegrapher's equations

In the previous section it has been seen that potential functions may act as equivalent waves in the supposed *telegrapher's equations*. These potentials, which define the fields by differentiaton, are related by means of the characteristic impedance, and they present propagative behavior γ, which has been used a priori to parameterize propagative waves, but then is characterized by both constitutive parameters and BCs.
In particular, it has been seen the case of obtaining HPWs, which are TEM modes based in HSPs. This case serves as example of use of the inverse characterization.
Nevertheless, the process of completing the analysis is the same for any mode which is solution of *Maxwell equations* when they are decoupled using the dot product previously introduced.

In this section the process of inversely characterizing the line parameters of the equivalent TL is generically and briefly presented.

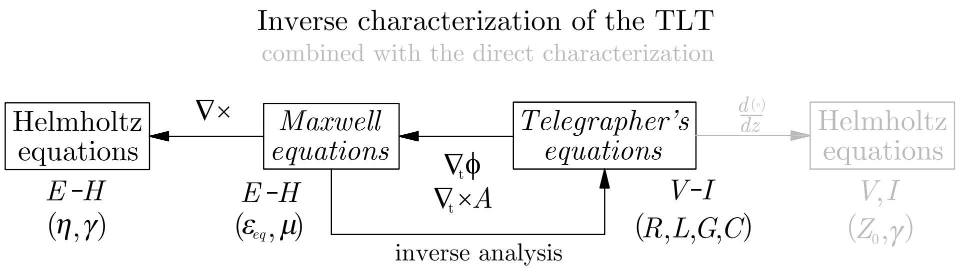


Fig. 3.4: Scheme that summarizes the transformations regarding the inverse characterization of the GTLT. The direct characterization from *telegrapher's equations* is added as the alternative way of solving the *Maxwell equations*, which results being a way much more simple and intuitive than decoupling these vector equations.

It starts assuming the potentials ϕ_e, ϕ_h, A_e, and A_h[4] are related by means of (generalized) *telegrapher's equations*:

$$\frac{\partial \upsilon}{\partial z} = -(\bar{R} + \jmath\omega\bar{L})\imath, \qquad (3.35)$$

$$\frac{\partial \imath}{\partial z} = -(\bar{G} + \jmath\omega\bar{C})\upsilon, \qquad (3.36)$$

in which

$$\upsilon \equiv \begin{cases} \phi_e \\ A_e \end{cases}, \quad \imath \equiv \begin{cases} \phi_h \\ A_h \end{cases},$$

so there are a total of four possible *telegrapher's equations* in which $\bar{R}$, $\bar{L}$, $\bar{G}$, and $\bar{C} \in \mathbb{C}$.

On the other hand, the potentials ϕ_e, ϕ_h, A_e, and A_h are related in certain way when solving *Maxwell equations*, for example by means of the amplitudes of ϕ_e and ϕ_h which characterize the HPWs in eqs. (3.31) and (3.32). This relation translates to impedances between the potentials, which depend on the constitutive parameters and those which parameterize the BCs, besides the "spectral parameters" ω and γ. In this sense, two equations lead to solve both the characteristic impedance between the modes and the propagation constant of the related modes, so the basic parameters.

From this point, the line parameters are obtained from basic parameters by means of inverse characterization. This could be done analytically or graphically by means of complex analysis in the

[4] A_e and A_h are the modulus of vectors $\boldsymbol{A_e} \equiv A_e\hat{z}$ and $\boldsymbol{A_h} \equiv A_h\hat{z}$, so they are scalar functions.

associated CTLT.
Then, the complete characterization of the TL under study and its physical interpretation in the associated TLT may be done, just as it has been carried out in 2.3.1 in Chpt. 2 for the lossy characterization of lossy TLs that support HPWs.

Remark 10. *By means of the inverse characterization the line parameters of the equivalent TL are obtained. Thereafter, the complete characterization of basic and wave parameters may be done. Since the line parameters are obtained from the parameters that characterize the potentials whose derivatives are solutions of Maxwell equations, this analysis is thoroughly supported in the sense that the equivalent telegrapher's equations comes from modes that represent real solutions in WGs.*

In next section, HPWs are inversely characterized from the parameters γ_{HPW} and $Z_{0,HPW}$. This characterization mainly parameterizes the constitutive parameters, besides the dependence on frequency, just as it is done in the previous chapter.

3.4 Guided waves particular cases

3.4.1 Inverse characterization of harmonic plane waves

In this section, the HPWs are classified among the possible solutions. Then line parameters of this type of waves are inversely characterized.
The following scheme allows for visualizing where the HPW solutions are within the "Universe of solutions" that this GTLT-v1 leads to:

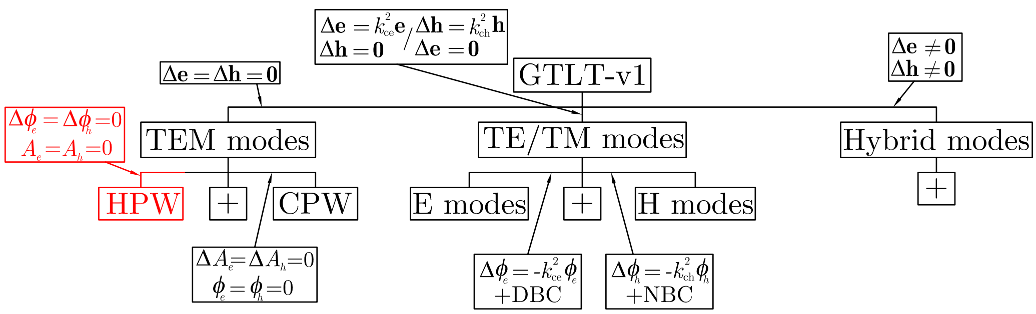


Fig. 3.5: Scheme of possible modes and particular cases that the GTLT-v1 lets to analyze, together with the conditions that lead to them. The HPWs are within the TEM modes (those modes that verify $\Delta \boldsymbol{e} = \Delta \boldsymbol{h} = 0$, together with the so called "curled" plane waves (CPW). Other well-known solutions are the E-modes and the H-modes, [Mar51], obtained when imposing the Dirichlet boundary conditions (DBC) and Neumann boundary conditions (NBC), respectively, over the scalar potentials which verify the Helmholtz equation.

Notice that the HPWs classified in Fig. 3.5 are only "a branch" among all the possible solutions. In fact, it is, together the "curled" plane waves (CPW) (that waves whose vector potential modulus are harmonic, so they represent fields whose field lines are closed in the cross section) the most basic solutions.

Now let's characterize the line parameters in based on the solutions obtained representing HPWs. For this purpose, the basic paremeters of the equivalent TL obtained from decoupling the (generalized) *telegrapher's equations* in eqs. (3.35) and (3.36):

$$\bar{\gamma} = \sqrt{\left(\bar{R} + j\omega\bar{L}\right)\left(\bar{G} + j\omega\bar{C}\right)}, \text{ and} \tag{3.37}$$

$$\bar{Z}_0 = \sqrt{\frac{\bar{R} + j\omega\bar{L}}{\bar{G} + j\omega\bar{C}}}; \tag{3.38}$$

are identifyed with the corresponding parameters obtanined for HPWs in eqs. (3.33) and (3.34), respectively:

$$\bar{\gamma} \Leftrightarrow \gamma_{HPW}, \tag{3.39}$$

$$\bar{Z}_0 \Leftrightarrow Z_{0,HPW}. \tag{3.40}$$

When operating for solving the line parametes in terms of the constitutive parameters, taking into account the form of the particular lossless cases in the process which is carried out in Appendix 3.C, the result is that

$$(r \equiv) \quad \frac{R}{\omega L} = \frac{\varepsilon''_{eq}}{\varepsilon'_{eq}}, \text{ and} \tag{3.41}$$

$$(g \equiv) \quad \frac{G}{\omega C} = \frac{\mu''}{\mu'}. \tag{3.42}$$

Notice that the line parameters in eqs. (3.41) and (3.42) are real values because the solutions of the indetification are found in these numbers.
The idetification means that: (i) $r \equiv \frac{R}{\omega L}$ and $g \equiv \frac{G}{\omega C}$ are the parameters which characterize the TL in terms of losses. This fact is used in the following chapter as part of the CTLT-v1; (ii) by means of the inverse characterization from basic parameters, the line parameters are cannot be solved completely but the lossy ratios r and g. However, this could be solved when going deeper into the *Maxwell equations* separating the source of losses, just as it is done at the end of Appendix 3.C, leading to:

$$\begin{cases} R = \omega\varepsilon''_{eq} \\ L = \varepsilon'_{eq} \end{cases} \text{, and } \begin{cases} G = \omega\mu'' \\ C = \mu' \end{cases} ; \tag{3.43}$$

and (iii) if the line parameters are chosen non frequency dependent, then from the identifications above $\varepsilon''_{eq} \cong \mu'' \cong c''/\omega$, and $\varepsilon'_{eq} \cong \mu' \cong c'$, in which c' and c'' are real constants.

Taking into account these facts, the parameterizations above are the same as those obtaned in Sect. 2.4.1 in Chpt. 2 when analyzing the lossy case within the LTLT. Thus, the subsequent analysis of the rest of TL prameters has been done there, and the inverse characterization is finally connected with this LTLT.

The same process which has been carried out for HPW may be done with the rest of solutions in a much more efficient way taking advantage of the GTLT-v1 presented in this chapter. An additional example of this inverse analysis characterizing E-mode solutions is presented among the **1** at the end of this thesis book.

3.5 Conclusions

In this chapter a generalized version of the TLT has been presented with the objective of gathering in the same framework different types of waves making them ready to be parameterized for their subsequent study in equivalent TLs.
For the purpose of generalizing the study of the voltage and current equivalent waves, the *Maxwell equations* are not solved by differetiation while imposing different conditions on them, just as it has been done in the previous chapter for obtaining HPWs. Instead of that, the solutions are possed in integral/summable form (in general, a **LC**), and the definition of a complete inner product lets to obtain each of them.
This generalized process, which is much more efficient than solving each of the possible case one by one is called the inverse characterization.
Each of those solutions in the **LC** is supposed to verify the *telegrapher's equations* which are posed generalized in terms of equivalent waves. The term "generalized" makes reference to consider the line parameters complex, and the equivalent waves of two possible types: scalar potentials and vector potentials. These assumptions extend the possible solutions which may be characterized under the same basic parameter form, achieving the resultant GTLT the goal of generalizing the analysis.

The inverse characterization in the GTLT assumes the waves are propagative, so γ acts as a parameter in a "propagative domain". This parameter, together with the characteristic impedance, needs to be characterized in terms of the constitutive parameters and the BCs imposed. This means that the BCs are put at the same level as the parameters that characterize the WG, which are both mapped in the line parameters of the equivalent TL.

The HPWs are only one of the possible solutions that this analysis lets to parameterize in equivalent TLs. In fact, this type of waves are the most basic solutions which come from the GTLT. The zero Laplacian of the vector eigenfunctions ($\mathbf{e}$ and $\mathbf{h}$) that characterize these waves makes the associated BCs have no direct influence in the line parameters. As a consequence, the line parameters of HPWs are real values.

This analysis is ready for parameterizing more solutions in WGs leading to the associated subsersions of this GTLT (just like the LTLT). An example of the inverse characterization of E-modes (see these solutions in the scheme in Fig. 3.5) is shown among the **Applications** of this analysis.

Chapter 3

Appendices

Appendix 3.A

In this section, the role of γ as a parameter explained as a function of k and the eigenvalues of the Laplacian in the cross section is shown.
For this purpose, the fields $\boldsymbol{E}(\mathbf{t}, z)$ and $\boldsymbol{H}(\mathbf{t}, z)$ are decoupled from *Maxwell equtions* leading to the vector Helmholtz equations:

$$\Delta \boldsymbol{E} + k^2 \boldsymbol{E} = 0, \text{ and} \tag{3.A.1}$$

$$\Delta \boldsymbol{H} + k^2 \boldsymbol{E} = 0. \tag{3.A.2}$$

The Laplacian in eqs. (3.A.1) and (3.A.2) is replaced using the operators particularized in the generaized coordinate system used for describing cylindrical WGs:

$$\Delta_{\mathrm{t}} \boldsymbol{E} + \frac{\partial^2 \boldsymbol{E}}{\partial z^2} + k^2 \boldsymbol{E} = \mathbf{0}, \text{ and} \tag{3.A.3}$$

$$\Delta_{\mathrm{t}} \boldsymbol{H} + \frac{\partial^2 \boldsymbol{H}}{\partial z^2} + k^2 \boldsymbol{H} = \mathbf{0}. \tag{3.A.4}$$

Now, if assuming the propagative behavior of fields using "propagative exponentials", eqs. (3.A.3) and (3.A.4) reduce to:

$$\Delta_{\mathrm{t}} \boldsymbol{E}_{\boldsymbol{a}}^{\pm} + (\gamma^2 + k^2) \boldsymbol{E}_{\boldsymbol{a}} = \mathbf{0}, \text{ and} \tag{3.A.5}$$

$$\Delta_{\mathrm{t}} \boldsymbol{H}_{\boldsymbol{a}}^{\pm} + (\gamma^2 + k^2) \boldsymbol{H}_{\boldsymbol{a}} = \mathbf{0}. \tag{3.A.6}$$

The amplitude fields $\mathbf{E}_a(\mathbf{t};\gamma)$ and $\mathbf{H}_a(\mathbf{t};\gamma)$ are first assumed to be expanded as a linear combinations, $\mathbf{LC}_{\boldsymbol{\gamma}_{ce}}$ and $\mathbf{LC}_{\boldsymbol{\gamma}_{ch}}$, of certain $\mathbf{e}(\mathbf{t};\boldsymbol{\gamma}_{ce})$ and $\mathbf{h}(\mathbf{t};\boldsymbol{\gamma}_{ch})$ orthogonal basis functions, respectively:

$$\mathbf{E}_a(\mathbf{t};\gamma) = \mathbf{LC}_{\boldsymbol{\gamma}_{ce}} \left[\epsilon(\boldsymbol{\gamma}_{ce};\gamma)\mathbf{e}(\mathbf{t};\boldsymbol{\gamma}_{ce})\right], \tag{3.A.7}$$

$$\mathbf{H}_a(\mathbf{t};\gamma) = \mathbf{LC}_{\boldsymbol{\gamma}_{ch}} \left[\eta(\boldsymbol{\gamma}_{ch};\gamma)\mathbf{h}(\mathbf{t};\boldsymbol{\gamma}_{ch})\right]. \tag{3.A.8}$$

Now, let's substitute the expansions (3.A.7) and (3.A.8) above into the eqs. (3.A.5) and (3.A.6):

$$\mathbf{LC}_{\gamma} \left\{\mathbf{LC}_{\boldsymbol{\gamma}_{ce}} \left[\epsilon(\boldsymbol{\gamma}_{ce};\gamma) \left(\Delta_t \mathbf{e}(\mathbf{t};\boldsymbol{\gamma}_{ce}) + \left(\gamma^2 + k^2\right) \mathbf{e}(\mathbf{t};\boldsymbol{\gamma}_{ce})\right)\right] e^{\mp\gamma z}\right\} = \mathbf{0}, \tag{3.A.9}$$

$$\mathbf{LC}_{\gamma} \left\{\mathbf{LC}_{\boldsymbol{\gamma}_{ch}} \left[\eta(\boldsymbol{\gamma}_{ch};\gamma) \left(\Delta_t \mathbf{h}(\mathbf{t};\boldsymbol{\gamma}_{ch}) + \left(\gamma^2 + k^2\right) \mathbf{h}(\mathbf{t};\boldsymbol{\gamma}_{ch})\right)\right] e^{\mp\gamma z}\right\} = \mathbf{0}. \tag{3.A.10}$$

For the moment, it is assumed that:

$$\Delta_t \mathbf{e}(\mathbf{t};\boldsymbol{\gamma}_{ce}) \equiv \mathbf{LC}_{\boldsymbol{\gamma}'_{ce}} \left[-\lambda^2(\boldsymbol{\gamma}'_{ce};\gamma_{ce})\mathbf{e}(\mathbf{t};\boldsymbol{\gamma}'_{ce})\right], \text{ and}$$

$$\Delta_t \mathbf{h}(\mathbf{t};\boldsymbol{\gamma}_{ch}) \equiv \mathbf{LC}_{\boldsymbol{\gamma}'_{ch}} \left[-\lambda^2(\boldsymbol{\gamma}'_{ch};\gamma_{ch})\mathbf{h}(\mathbf{t};\boldsymbol{\gamma}'_{ch})\right].$$

It is possible to select the set of basis functions in such a manner that $-\lambda^2(\boldsymbol{\gamma}'_{ce};\gamma_{ce})$ and $-\lambda^2(\boldsymbol{\gamma}'_{ch};\gamma_{ch})$ are the eigenvalues of the operators $\mathbf{LC}_{\gamma_{ce}}[\circ]$ and $\mathbf{LC}_{\gamma_{ch}}[\circ]$, that is diagonalizing them so:

$$\Delta_t \mathbf{e}(\mathbf{t};\boldsymbol{\gamma}_{ce}) \equiv -\lambda^2(\boldsymbol{\gamma}_{ce})\mathbf{e}(\mathbf{t};\boldsymbol{\gamma}_{ce}), \text{ and} \tag{3.A.11}$$

$$\Delta_t \mathbf{h}(\mathbf{t};\boldsymbol{\gamma}_{ch}) \equiv -\lambda^2(\boldsymbol{\gamma}_{ch})\mathbf{h}(\mathbf{t};\boldsymbol{\gamma}_{ch}), \tag{3.A.12}$$

and thus $\mathbf{e}(\mathbf{t};\gamma_{ce}) \in \text{Aut}\{\mathbf{E}_a(\mathbf{t};\gamma)\}$ and $\mathbf{h}(\mathbf{t};\gamma_{ch}) \in \text{Aut}\{\mathbf{H}_a(\mathbf{t};\gamma)\}$.
When replacing these eigenfunctions to eqs. (3.A.9) and (3.A.10) they result in:

$$\mathbf{LC}_{\boldsymbol{\gamma}_{ce}}\left\{\left(\mathbf{LC}_{\gamma}\left[\epsilon(\boldsymbol{\gamma}_{ce},\gamma)e^{\mp\gamma z}\right]\right)\left(-\lambda^2(\boldsymbol{\gamma}_{ce})+\gamma^2+k^2\right)\mathbf{e}(\mathbf{t};\boldsymbol{\gamma}_{ce})\right\}=\mathbf{0}, \tag{3.A.13}$$

$$\mathbf{LC}_{\boldsymbol{\gamma}_{ch}}\left\{\left(\mathbf{LC}_{\gamma}\left[\eta(\boldsymbol{\gamma}_{ch},\gamma)e^{\mp\gamma z}\right)\left(-\lambda^2(\boldsymbol{\gamma}_{ch})+\gamma^2+k^2\right)\right]\mathbf{h}(\mathbf{t};\boldsymbol{\gamma}_{ch})\right\}=\mathbf{0}, \tag{3.A.14}$$

(3.A.15)

in which also the $\mathbf{LC}_{\gamma}$ and $\mathbf{LC}_{\boldsymbol{\gamma}_{ce}}$, and the $\mathbf{LC}_{\gamma}$ and $\mathbf{LC}_{\boldsymbol{\gamma}_{ch}}$ order has been interchanged.
Scalar multiplying both sides of (3.A.13) and (3.A.14) by $\mathbf{e}(\mathbf{t};\boldsymbol{\gamma}_{ce0})$ and $\mathbf{h}(\mathbf{t};\boldsymbol{\gamma}_{ch0})$, respectively, leads to:

$$\boxed{\gamma^2=\lambda^2(\boldsymbol{\gamma}_{ce})-k^2,}$$

$$\boxed{\gamma^2=\lambda^2(\boldsymbol{\gamma}_{ch})-k^2,}$$

for every $\boldsymbol{\gamma}_{ce}$ and $\boldsymbol{\gamma}_{ch}$ because $\mathbf{LC}_{\gamma}\left[\epsilon(\boldsymbol{\gamma}_{ce},\gamma)e^{\mp\gamma z}\right]\neq 0$ and $\mathbf{LC}_{\gamma}\left[\eta(\boldsymbol{\gamma}_{ch},\gamma)e^{\mp\gamma z}\right]\neq 0$ unless the fields are trivially null.

As a consequence, the originally complex variable γ may be addressed as a complex parameter to the 2-dim space that are expanded as $\mathbf{E}_a(\mathbf{t})$ and $\mathbf{H}_a(\mathbf{t})$ in eqs. (3.A.7) and (3.A.8) from the orthogonal basis functions $\mathbf{e}(\mathbf{t},\boldsymbol{\gamma}_{ce})$ and $\mathbf{h}(\mathbf{t},\boldsymbol{\gamma}_{ch})$, respectively.
In addition, if such basis functions satisfy (3.A.11) and (3.A.12), then the result of applying the transversal Laplacian is in the same space as $\mathbf{e}(\mathbf{t},\boldsymbol{\gamma}_{ce})$ and $\mathbf{h}(\mathbf{t},\boldsymbol{\gamma}_{ch})$, and thus in the same 2-dim space as $\mathbf{E}_a(\mathbf{t})$ and $\mathbf{H}_a(\mathbf{t})$.

As a result, it might be said that γ plays the roll of a parameter which may be obtained from the eigenvalues $\lambda^2(\boldsymbol{\gamma}_{ce})$ and $\lambda^2(\boldsymbol{\gamma}_{ch})$ to the vector Laplacian operator applied to fields when they are expanded as a $\mathbf{LC}_{\gamma}$ of complex exponentials. This is equivalent to the time harmonic dependence, in which ω -being a factor in k- is seen as a parameter. Therefore, the complex analysis of γ as a parameter becomes crucial.

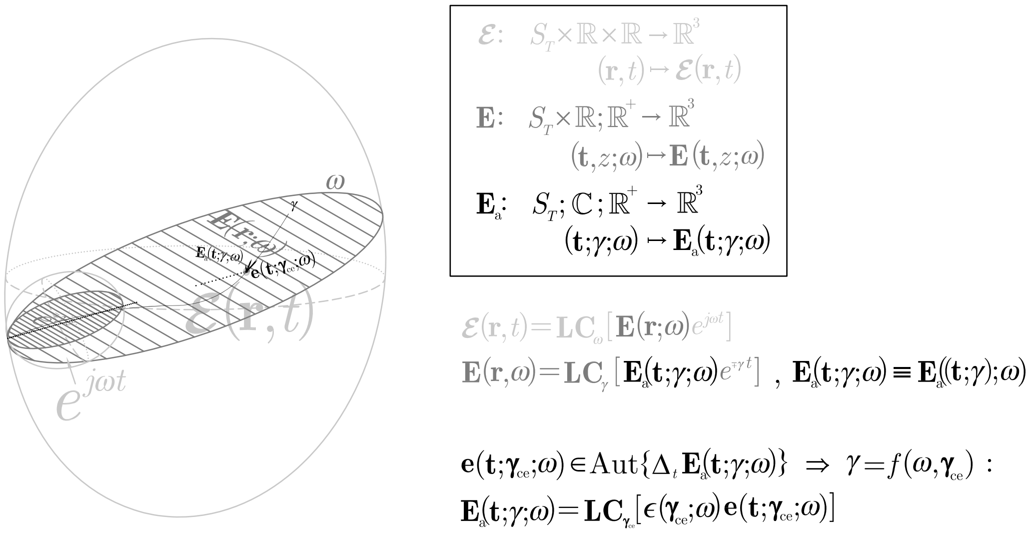


Fig. 3.6: General scheme of the E-field functions belonging different vector spaces: $\mathcal{E}(\mathbf{r}, t)$ belongs to the four-dimensional space $S_T \times \mathbb{R} \times \mathbb{R}$ represented as an ellipsoid, which may be obtained as a continuous superposition of different surfaces parameterized by ω; the coefficients that expand $\mathcal{E}(\mathbf{r}, t)$ by means of the base $e^{j\omega t}$, $\mathbf{E}(\mathbf{r}, \omega)$, are within the surfaces which represent the $S_T \times \mathbb{R}; \mathbb{R}^+$ space. These surfaces are, in turn, arrangements of lines parameterized by γ and expanded by $e^{\mp\gamma z}$; Their coefficients $\mathbf{E}_a(\mathbf{t}; \gamma)$ ($\equiv \mathbf{E}_a(\mathbf{t}; \gamma; \omega)$) are in the $S_T; \mathbb{C}; \mathbb{R}^+$ space; at the end, the line points may be pointed by the appropriate eigenfunction $\mathbf{e}(\mathbf{t}; \boldsymbol{\gamma}_{ce})$ which lives in the same space (surface) $\mathbf{E}(\mathbf{r}; \omega)$.

The H-field would follow the same scheme except for the line parameterized by $\mathbf{H}_a(\mathbf{t}; \gamma)$ which is different from $\mathbf{E}_a(\mathbf{t}; \gamma)$, although both they must be clearly related by means of Maxwell equations.

Appendix 3.B

The scalar product (dot product) of two vectorial functions depending on the transversal coordinates is defined here. The completeness of the induced metric space of vectorial functions is also proved so the resultant space is a Hilbert Space and every field in the space may be expressed in terms of a linear combination of a basis that, moreover, converges.

Recall the original space of vector functions is the Sobolev space, $W^{2,2}$, whose domain is S_T, that is the vector functions defined on S_T with finite energy components and belonging $\mathcal{C}^2$. Moreover, this functions are at least $\mathcal{C}^1$ on the boundary C_T. Using the generalized notation, [Her14], the dot product between $\mathbf{f}(t_1,t_2)$ and $\mathbf{g}(t_1,t_2)$

$$\langle \mathbf{f}(t_1,t_2), \mathbf{g}(t_1,t_2)\rangle = c \in \mathbb{C},$$

may be writen as

$$\langle \mathbf{f}, \mathbf{g}\rangle \equiv \iint_{S_T} \mathbf{f}\cdot\mathbf{g}^* dS, \tag{3.B.16}$$

$\mathbf{g}^*$ is the complex conjugate of $\mathbf{g}$, and the dependence of the functions on the transversal coordinates is omitted for the sake of simplicity.
Each vector function may be writen as the gradient of a scalar potential plus the curl of a vector potential[5], so

$$\begin{aligned}\mathbf{f} &= -\nabla_t\phi_f + \nabla_t\times A_f\hat{z}, \text{ and}\\ \mathbf{g} &= -\nabla_t\phi_g + \nabla_t\times A_g\hat{z},\end{aligned}$$

and the scalar product in eq. (3.B.16) equals

$$\begin{aligned}\iint_{S_T} \mathbf{f}\cdot\mathbf{g}^* dS &= \iint_{S_T} \left(-\nabla_t\phi_f + \nabla_t\times A_f\hat{z}\right)\cdot\left(-\nabla_t\phi_g^* + \nabla_t\times A_g^*\hat{z}\right) dS =\\ &= \underbrace{\iint_{S_T} \nabla_t\phi_f\cdot\nabla_t\phi_g^* dS}_{\text{(a)}} + \underbrace{\iint_{S_T} \nabla_t\phi_f\cdot\nabla_t\times A_g^*\hat{z} dS}_{\text{(b)}} +\\ &+ \underbrace{\iint_{S_T} \nabla_t\times A_f\hat{z}\cdot\nabla_t\phi_g^* dS}_{\text{(c)}} + \underbrace{\iint_{S_T} \nabla_t\times A_f\hat{z}\cdot\nabla_t\times A_g^*\hat{z} dS}_{\text{(d)}}.\end{aligned} \tag{3.B.17}$$

Using the Green's second identity in (a) and (d), as well as some vector identities[6] in (b)-(d), the sum in (3.B.17) is expanded as:

$$\begin{aligned}\iint_{S_T} \mathbf{f}\cdot\mathbf{g}^* dS =& \underbrace{-\iint_{S_T} \phi_f\Delta_t\phi_g^* dS + \oint_{C_T} \phi_f\nabla_t\phi_g^*\cdot\hat{n}dl}_{\text{(a)}} + \underbrace{\iint_{S_T} \left(\nabla_t\phi_f\times\nabla_t A_g^*\right) dS\hat{z}}_{\text{(b)}} -\\ &\underbrace{-\iint_{S_T} \left(\nabla_t A_f\times\nabla_t\phi_g^*\right) dS\hat{z}}_{\text{(c)}} \underbrace{-\iint_{S_T} A_f\Delta_t A_g^* dS + \oint_{C_T} A_f\nabla_t A_g^*\cdot\hat{n}dl}_{\text{(d)}},\end{aligned} \tag{3.B.18}$$

which is the definition of the scalar product using scalar functions and space vector operations. Notice that if $\phi_f \propto A_g^*$ and $A_f \propto \phi_g^*$, the terms (b) and (c) vanish since their gradient are parallel on S_T. Furthermore, A_f and ϕ_g, or ϕ_f and A_g, may be null so $\mathbf{f} = \nabla_t\phi_f$ and $\mathbf{g} = \nabla_t\times A_g$, or

[5] The vector potential has only $\hat{z}$ compontent to be its curl in S_T.
[6] $\nabla_t\times A\hat{z} = A\nabla_t\hat{z} + \nabla_t A\times\hat{z} = \nabla_t A\times\hat{z}$
. $\nabla_t\phi\cdot\nabla_t A\times\hat{z} = \hat{z}\cdot(\nabla_t\phi\times\nabla_t A)$ (scalar triple product)
. $(\nabla_t A_1\times\hat{z})\cdot(\nabla_t A_2\times\hat{z}) = (\nabla_t A_1\cdot\nabla_t A_2)(\hat{z}\cdot\hat{z}) - (\hat{z}\cdot\nabla_t A_2)(\nabla_t A_1\cdot\hat{z}) = (\nabla_t A_1\cdot\nabla_t A_2)$

$\mathbf{f} = \nabla_t \times A_f$ and $\mathbf{g} = \nabla_t \phi_g$, so that only the term (b) or (d) are different from zero. Moreover, it is possible that **f** and **g** are both the gradient of a scalar potential, or both the curl of a vector potential, that is $\mathbf{f} = \nabla_t \phi_f$ and $\mathbf{g} = \nabla_t \phi_g$, or $\mathbf{f} = \nabla_t \times A_f$ and $\mathbf{g} = \nabla_t \times A_g$, in a way that only the terms (a) or (d) survive. In any case, the terms that are not null fix the extra properties the orthogonal potentials have to fulfill in order to guarantee the orthogonality between vector functions.
It is possible to rewrite the terms (b) and (c) in eq. (3.B.18) using identities of vector product (cross product) and curl[7] together with the Stokes' theorem as:

$$\begin{aligned}\iint_{S_T} \mathbf{f}\cdot\mathbf{g}^* dS = &\underbrace{-\iint_{S_T} \phi_f \Delta_t \phi_g^* dS + \oint_{C_T} \phi_f \nabla_t \phi_g^* \cdot \hat{n} dl}_{\text{(a)}} + \underbrace{\oint_{C_T} \phi_f \nabla_t A_g^* \cdot (\hat{z}\times\hat{n}) dl}_{\text{(b)}} + \\ &+ \underbrace{\oint_{C_T} \phi_g^* \nabla_t A_f \cdot (\hat{z}\times\hat{n}) dl}_{\text{(c)}} \underbrace{- \iint_{S_T} A_f \Delta_t A_g^* dS + \oint_{C_T} A_f \nabla_t A_g^* \cdot \hat{n} dl}_{\text{(d)}}\,,\end{aligned} \tag{3.B.19}$$

with $\hat{z}\times\hat{n}$ the unit vector tangent to the curve C_T described clockwise. This is the way to define formally the scalar product between two vector functions.

The norm of a vector function **f**,

$$\|\mathbf{f}\| = n \in \mathbb{R},$$

is induced from the dot product

$$\|\mathbf{f}\| = \langle \mathbf{f}, \mathbf{f} \rangle^{\frac{1}{2}} = \left[\iint_{S_T} \mathbf{f}\cdot\mathbf{f}^* dS \right]^{\frac{1}{2}}. \tag{3.B.20}$$

By using the expansion in equation (3.B.18), the square of the norm is

$$\begin{aligned}\iint_{S_T} \mathbf{f}\cdot\mathbf{f}^* dS = &-\iint_{S_T} \phi_f \Delta_t \phi_f^* dS + \oint_{C_T} \phi_f \nabla_t \phi_f^* \cdot \hat{n} dl + \iint_{S_T} \left(\nabla_t \phi_f \times \nabla_t A_f^*\right) dS \hat{z} + \\ &+ \iint_{S_T} \left(\nabla_t \phi_f^* \times \nabla_t A_f\right) dS \hat{z} - \iint_{S_T} A_f \Delta_t A_f^* dS + \oint_{C_T} A_f \nabla_t A_f^* \cdot \hat{n} dl = \\ = &-\iint_{S_T} \phi_f \Delta_t \phi_f^* dS + \oint_{C_T} \phi_f \nabla_t \phi_f^* \cdot \hat{n} dl + \\ &+ 2\mathrm{Re}\left\{ \iint_{S_T} \left(\nabla_t \phi_f \times \nabla_t A_f^*\right) dS \hat{z} \right\} - \\ &- \iint_{S_T} A_f \Delta_t A_f^* dS + \oint_{C_T} A_f \nabla_t A_f^* \cdot \hat{n} dl = \\ = &-\iint_{S_T} \phi_f \Delta_t \phi_f^* dS + \oint_{C_T} \phi_f \nabla_t \phi_f^* \cdot \hat{n} dl + \\ &+ 2\mathrm{Re}\left\{ \oint_{C_T} \phi_f \nabla_t A_f^* \cdot (\hat{z}\times\hat{n}) dl \right\} - \\ &- \iint_{S_T} A_f \Delta_t A_f^* dS + \oint_{C_T} A_f \nabla_t A_f^* \cdot \hat{n} dl.\end{aligned} \tag{3.B.21}$$

And distance between two vector functions **f** and **g**

$$d\left(\mathbf{f}, \mathbf{f}\right) = d \in \mathbb{R},$$

[7] $\nabla_t \phi \times \nabla_t A = \nabla_t \times (\phi \nabla_t A)$

is induced from the norm induced from the scalar product

$$\begin{aligned} d(\mathbf{f}, \mathbf{g}) =& \|\mathbf{f} - \mathbf{g}\| = \langle \mathbf{f} - \mathbf{g}, \mathbf{f} - \mathbf{g} \rangle^{\frac{1}{2}} = \left[\iint_{S_T} (\mathbf{f} - \mathbf{g}) \cdot (\mathbf{f} - \mathbf{g})^* \, dS \right]^{\frac{1}{2}} = \\ &= \left[\iint_{S_T} \mathbf{f} \cdot \mathbf{f}^* dS - \iint_{S_T} \mathbf{f} \cdot \mathbf{g}^* dS - \left(\iint_{S_T} \mathbf{f} \cdot \mathbf{g}^* dS \right)^* + \iint_{S_T} \mathbf{f} \cdot \mathbf{f}^* dS \right]^{\frac{1}{2}} = \\ &= \left[\|\mathbf{f}\|^2 - 2\mathrm{Re}\left\{ \langle \mathbf{f}, \mathbf{g} \rangle \right\} + \|\mathbf{g}\|^2 \right]^{\frac{1}{2}} . \end{aligned} \tag{3.B.22}$$

If $\mathbf{f} \to \mathbf{g}$, for example defiing $\alpha\mathbf{g}$, $\alpha \in \mathbb{C}$, and making $\alpha \to 0$, then $d(\mathbf{f}, \mathbf{g}) \to 0$, so that the space equipped with this dot product is complete, so a Hilbert space.

Appendix 3.C

In this section, the algebra for obtaining the line parameters of HPWs in terms of the constitutive parameters is shown, serving also as example for parameterizing more modes within the GTLT-v1, which is useful studying the mode behavior in the subsequent version of the CTLT.

On one hand, the propagation constant and the characteristic impedance of HPWs are obtained in eqs. (3.33) and (3.34), respectively. On the other hand, the propagation constant and the characteristic impedance obtianed when decoupling the (generalized) *telegrapher's equations* are in eqs. (3.37) and (3.38), respectively. Identifying these parameters by pairs as:

$$\bar{\gamma} \Leftrightarrow \gamma_{HPW}, \text{ and} \tag{3.C.23}$$

$$\bar{Z}_0 \Leftrightarrow Z_{0,HPW}; \tag{3.C.24}$$

and written them in terms of line parameters and constitutive parameters, it make possible solving them.

Firstly, the line parameters defining the propagation constant are requied to be solved. That is because γ generically acts as a parameter for the characteristic impedance, although for the HPW charactarestic impedance does not.
The parameter γ_{HPW} is:

$$\begin{aligned}\gamma_{\mathrm{HPW}} &= j\omega\sqrt{\varepsilon_{eq}\mu} \equiv j\omega\sqrt{(\varepsilon'_{eq} - j\varepsilon''_{eq})(\mu' - j\mu'')} = \\ &= \sqrt{-\omega^2\mu'\varepsilon'_{eq} + \omega^2\mu''\varepsilon''_{eq} + j\omega^2(\mu'\varepsilon''_{eq} + \mu''\varepsilon'_{eq})},\end{aligned} \tag{3.C.25}$$

whereas γ is[8]:

$$\gamma = \sqrt{RG - \omega^2 LC + j\omega(RC + GL)}. \tag{3.C.26}$$

If γ is required to represent γ_{HPW} in the same way in terms of lossy parameterizations, then first of all,

$$\begin{cases} RG - \omega^2 LC = \omega^2(\mu'\varepsilon'_{eq} + \mu''\varepsilon''_{eq}) \\ RC + GL = \omega(\mu'\varepsilon''_{eq} + \mu''\varepsilon'_{eq}) \end{cases}, \tag{3.C.27}$$

and secondly, since the lossless case corresponds with $R = G = 0$ and $\varepsilon''_{eq} = \mu'' = 0$,

$$\mu'\varepsilon'_{eq} = LC, \tag{3.C.28}$$

then RG in the first equation in eq. (3.C.27) is

$$RG = \omega^2\varepsilon''_{eq}\mu''. \tag{3.C.29}$$

Solving μ' and μ'' from eqs. (3.C.28) and (3.C.28), and substituying them in the second equation in eq. (3.C.27), the following 2nd order equation in $r' = \omega\varepsilon''/\varepsilon'$ is obtained[9]:

$$r'2 - \left(\frac{R}{L} + \frac{G}{C}\right)r' + \frac{RG}{LC} = 0. \tag{3.C.30}$$

Solving this equation, it leads to:

$$r' = \frac{\omega\varepsilon''}{\varepsilon'} = \begin{cases} \frac{R}{L} \\ \frac{G}{C} \end{cases}.$$

[8]The notation with bars ($\bar{}$) for indicating complex parameters or parameters which result from complex parameters is simplified in this section writing the parameters normal. However, they are addressed in the same manner.

[9]The notation r' is used on purpose because it corresponds with the parameterization of conductor losses used in the variable frequecy domain in the CTLA. The same happens with the dielectric losses g'.

Now the process is repeated with the characteristic impedance taking into account that for the lossless case:

$$\frac{\mu'}{\varepsilon'} = \frac{L}{C}. \tag{3.C.31}$$

Among the two possible cases for r', only

$$r' \equiv \frac{\omega\varepsilon''}{\varepsilon'} = \frac{R}{L} \tag{3.C.32}$$

is possible, whereas

$$g' \equiv \frac{\omega\mu'}{\mu'} = \frac{G}{C}. \tag{3.C.33}$$

The way of solving each line parameter from the ratios r' and g' is analyzing the role of the constitutive parameters in *Maxwell equations* and the line parameters in the *telegrapher's equations*. Eqs. (3.31) and (3.32) are of the form of *telegrapher's equations* when fields are just integrated from *Maxwell equations*. If identifying the constitutive parameters in each equation with the line parameters of the generalized *telegrapher's equations*, it leads to the following additional equations

$$R + j\omega L = j\omega\varepsilon_{eq} = \omega\varepsilon''_{eq} + j\omega\varepsilon'_{eq}, \text{ and} \tag{3.C.34}$$

$$G + j\omega C = j\omega\mu = \omega\mu'' + j\omega\mu'. \tag{3.C.35}$$

Then,

$$\begin{cases} R = \omega\varepsilon''_{eq} \\ L = \varepsilon'_{eq} \end{cases}, \text{ and } \begin{cases} G = \omega\mu'' \\ C = \mu' \end{cases}. \tag{3.C.36}$$

In this way, the line parameters are written in terms of the constitutive parameters and frequency, ready for the analysis in the subsequent version of the CTLT.

Part II

Complex Analysis of the Transmission Line Theory

Chapter 4

The Complex Transmission Line Theory

4.1 Introduction

The TLT has thoroughly demonstrated its usefulness when parameterizing different structures in which EM fields propagate, [Mar51, Col90, Poz98, Col01]. These waveguide structures, which are invariant along the z-axis –just as they have been introduced in Chpt. 3 generalizing the mode propagation–, are fully equivalent to the parameterized TLs, once they are characterized in direct or inverse manner.
In this way, the equivalent TL is completely defined by the line parameters, which compose the basic parameters that characterize and relate the equivalent voltage and current single wave solutions in the TL. These single voltage and current waves are, in turn, equivalent to the electric and magnetic fields in the underlying TLT which they characterize, for example, the EM fields regarding harmonic plane waves in lossy media, studied under the LTLT. In any case, the basic parameters define both the kinematics (electrodynamics) of the equivalent waves that flow in each of the directions in the TL ($\pm\hat{z}$) by means of the propagation constant, and the relation between them in each direction by means of the characteristic impedance.
The total wave solution is the linear combination of the (partial) single wave solutions weighted by the coefficients with define the BCs along the z-axis. The wave parameters define the relations within and between the total voltage and current waves.
Thus, the analysis of the parameters defined in the equivalent TL: line parameters, basic parameters and wave parameters; determines the solutions in the parameterized waveguide.

For the generalized study of losses, it has been seen in Chpts. 2 and 3 that analyzing the solutions and the parameters in the *frequency domain* has some clear advantages: (i) losses are generalized in the sense that not only conductor losses but also those losses wich affect dispersive mediums can be considered; (ii) both *Maxwell equations* and "modified wave equations"[1] simplify to first order ODEs and Helmholtz-type equations, respectively; (iii) the parameters regarding the TL become complex. This fact, far from being a disadvantage, is a great advantage when analyzing the parameters of the TL. Notice that, for example, the attenuation constant, α, and the phase constant, β, are gathered together in the same complex expression: they are the real and imaginary parts of the (complex) propagation constant, γ, respectively; or the phase between the electric and magnetic field solutions is the phase of the (complex) characteristic impedance, Z_0; (iv) since the parameters are complex, they admit a graphical representations in their respective complex planes. The immediate consequence of using graphical analysis is geometrizing the underlying physical problem

[1] Remember: "modified wave equations" refers to the well-known wave equations to which a first order term –in this case parameterizing some source of losses– has been added.

regarding the wave propagation in TLs, and thus many physical interpretations are obtained by using geometrical concepts of classical Geometry and Differential Geometry, [MP77]; and (v) since these complex parameters are transformations between themselves, they may be seen as complex transformations of mappings, [BC90], between them. Moreover, this way of analyzing the transformations between the TL parameters has direct representation on complex planes, so not only the physical interpretations but also the direct and inverse characterizations introduced in previous chapters may be deduced from graphical analysis.

Remark 11. *Graphical analysis results natural when dealing with complex parameters which are characteristic in the frequency domain. This way to graphically see the parameters lets geometrizing the different characterizations regarding the TLT. As a result, the problem of characterizing the equivalent waves in the TL becomes geometrical. Moreover, physical interpretations regarding the propagative behavior of waves in TLs are directly mapped on the complex planes associated to each parameter, and those physical properties that could be hidden are found by applying concepts of Differential Geometry.*
In addition, since the way of characterizing the parameters is by means of transformations between them, the direct and inverse characterization of the equivalent TLs –and thus the way of solving the single waves– has direct representation in this type of analysis (based on transformations).

In addition, the graphical analysis has the advantage of "compressing" all the possible values for each parameter –and so the seeked solutions in the TL– in only one complex plane.
This idea of "universalizing" all the possible values of each TL parameter in one plane connects with the fact that either different particular cases come from a general expression, for example the lossless or non disperive cases come from the lossy case in the direct characterization introduced in Chpt. 2, or different line parameters come from the same propagative behavior and relation between electric and magnetic fields in the inverse characterization presented in Chpt. 3.
These graphical analysis in complex planes and the transformations between them are part of the methodology in the so called ***Complex Transmission Line Analysis*** (CTLA), [Gag01], in which the graphical analysis are "universally" parameterized depending on the characterization which is being carried out.

Remark 12. *In both the direct characterization and the inverse characterization of the TLT, the graphical analysis supposes a "universalization" of all the possible solutions in the planes of each parameter under study.*
The CTLA is based on parameterizing the transformations between complex planes, which is the way of studying them graphically, and so they are able to be analyzed geometrically.

Keeping in mind the interest of studying the TL parameters in the *frequency domain*, the parameterizations used in the direct characterization are the losses, whereas the different parameterizations of basic parameters are those which have special interest in the inverse characterization.
On the other hand, it is interesting to analyze how these characterizations vary in terms of frequency, and thus expanding/finding any solution in *time domain*. This analysis affects both the direct and inverse characterizations, but in different way: while the direct characterization describes how the basic parameters behave with frequency, the inverse characterization looks for the frequency which corresponds to the a priori defined basic parameters.
These analysis may be brought to/from (depending if the characterization is direct/inverse) the characterization of wave parameters. In this sense, it is interesting to analyze, for example, how the equivalent total voltage and current waves behave along the TL, which mainly reduces to the study of wave parameters along the TL (this is a direct characterization); or how the wave parameters vary with losses or frequency at any point of the TL (by definition, this is an inverse characterization).
All these (non trivial) analysis require studying the basic concepts (it refers to the basic transformations) regarding the direct inverse characterizations in the context of the CTLA. Having studied the basic transformations, these problems are finally been able to be solved as examples of use of the CTLA (the ones previosly cites are solves in Chpt. 5.

Remark 13. *The CTLA settles the basis of the methodology for analyzing future more complex problems.*
The physical meaning of the basic analysis in the CTLA is found in the type of parameterizations which are chosen to solve any particular problem.

For example, if the behavior of the TL is intended to be studied in terms of losses, the parameterizations should be the losses (this corresponds with a direct characterization). Or, if the losses are required to be studied in terms specifications on different TL parameters (for examples the basic parameters), the parameterizations should describe these parameters.
More interesting examples are the studies which parameterize the TL's length for the analysis along the TL, and also the parameterization of frequency for the variable frequency analysis.
Recall that the LTLT just as it has been presented in Chpt. 2 leaves a lot of open problems, some of them are very interesting to solve indeed since they can not be solved analitically, for example: (i) explain rigorously the low-losses approximation, that is, see its error in comparison with the lossy case when both frequency is fixed and variable; (ii) suggest a method for expanding the solutions from the *frequency domain* to *time domain* by analyzing how the TL parameters vary with frequency; (iii) answer which TLs produce, for example, the same phase difference between the electric and magnetic fields, that is the same phase on the characteristic impedance (this is basic analysis regarding the inverese characterization); (iv) analyze how the wave parameters vary with losses, frequency, or along the TL and, from this analysis, obtain any value which is of special interest. For example the real values of the wave impedance along the TL, if they exist; among others.

The ***Complex Transmission Line Theory*** (CTLT) arises out of the need of analyzing different generalizations of the TLT, and it is intended to characterize the equivalent TL under study both directly and inversely by means of CTLA.
The CTLT takes on: (i) selecting the appropriate normalizations depending on the analysis which is intended to be performed; (ii) normalizing the parameters under study in such a way that the parameterized curves "universalize" the analysis. In this sense the term "universalizing" refers to gather each parameterized analysis (losses, frequency, etc.) in only one complex plane; (iii) obtaining the seeked physical interpretations from graphical analysis or uncovering more by means of geometrical analysis; (iv) proposing practical uses of both the described analysis to solve more complex TL related problems, and the TLs under study by taking advantage of their phsyical properties, for example, losses combined with frequency could be suggested for matching purposes; and (v) applying the analysis to solve real electromagntic systems (see the **Future Lines** at the end of this thesis book).

Remark 14. *The CTLT is proposed as the theory which explains different versions of the TLT by using CTLA, that is, using resources of Complex Analysis to analyze TLs. In this way, the CTLT is intended to overcome the limitations of the mathematical analysis by transforming the original analytical problem giving graphical and so geometrical point of view, at the same time that it "universalizes" the analysis.*

The first generalization of the TLT is the LTLT presented in Chpt. 2. Recall that this generalization deals with HPWs in which losses are fully taken into account. Moreover, the same lossy HPWs are studied when particularizing the GTLT-v1 presented in Chpt. 3. Thus, the desired parameterizations in the associated first version of the CTLT (CTLT-v1) which give the main physical meaing are the losses regarding the equivalent TL, besides the frequency.
From this point, it is required to analyze a posteriori how the wave parameters vary wtih frequency and along the TL, serving as examples of use of the analysis presented in this chapter.
These analysis would complete the CTLT-v1, leading to the alternative of the (analytical) LTLT.

For the purposes regarding the CTLT enumerated above, the theoretical aspects are presented in this chapter by means of basic analysis which comprise the transformations between the parameterized curves in the normalized complex planes.
Next chapter, Chpt. 5, shows how to "merge" these basic analysis and the underlying analytical expressions to complete the CTLT.

In this chapter, the direct characterization of the lossy TL following the analysis regarding the LTLT introduced in Chpt. 2 is presented in Sect. 4.3. In this direct characterization, the basic parameters are analyzed as transformations of lossy parameterized curves both when frequency is fixed and variable, and the wave parameters are studied by means of parameterizations of basic parameters.
The inverse characterization of the lossy TL following the analysis ing the GTLT regarding HPWs introduced in Chpt. 3 is presented in Sect. 4.4. In the inverse characterization, the line parameters are analyzed by means of specifications in basic parameters which parameterize the resultant curves. On their behalf, these basic parameters are studied parameterized by wave parameters.
At the end, some conclusions regarding the theoretical aspects of the CTLT-v1 presented in this chapter are outlined in Sect. 4.5 while emphasizing their usefulness when studying TL related problems, which are presented as examples of use in Chpt. 5.

Nevertheless, a background to the analysis regarding the CTLA is presented in the following section prior to describe the associated CTLT-v1 rigorously.

4.2 Complex Transmission Line Analysis: previous works

The reader is referred to the text in [Gag01] to find some of the characterizations of the CTLA introduced for the first time. In this *CTLA handbook*, the presented analysis correspond to the (here named) direct characterization of the CTLT (which is only contextualized in analyis of TEM modes[2]):

- The basic parameters are studied by means of graphical analysis of the parameterized curves in the respective complex planes. This analysis lacks of several points:

  (i) the rigoruous mathematical description in terms of transformations between planes. In order to overcome this mathematical issue, it is required to define "the space of parameterizations" algebraically (see Appendix 4.A). From this "space", the "plane of parameterizations" is defined to mathematically characterize and graphically see the curves to be mapped into the basic parameter complex planes;

  (ii) the geometrical description of the resultant curves, which has special importance in parameterizing future more complex analysis;

  (iii) the physical interpretation of the depicted curves after their geometrical analysis; and

  (iv) the practical uses of these analysis in both solving more TL-related problems and being applied to solve other EM problems.

  The first point is special important when thinking about the inverse characterization: if a function which "maps" the lossy parameters (or other parameters, for example, frequency) into the basic parameters is not defined, the inverse characterization –given by the inverse function– could never have been thought within the context of the CTLT, limiting the description of more solutions in waveguides (as those ones presented in Chpt. 3) by means of studying their equivalent TLs. In fact, it is the inverse function (which is a complex variable function) the one which actually "sketches" the original mapping (seen as function) by means of different type of parameterizations, namely: real-imaginary parameterized parts and modulus-phase parameterizations.
  This way of inversely seeing the analysis is not only crucial to describe equivalent TLs but also to find the application in solving some problems, for example by using integral equations defined over this "space of parameterizations" (see the section **Applications** at the end of this thesis book).

- The wave parameters are studied both graphically and geometrically. However, the analysis lacks of:

  (i) the physical interpretation of the study detaching its importance in parameterizing lossy TLs and the limitations in the analysis.
  It is crucial to explain the role of angle of the characteristic impedance in the analysis φ_{Z_0}, also by its inverse characterization in terms of losses/frequency; and

  (ii) the practical uses, for example in analyzing the wave parameters along the TL, beyond its usefulness as a graphical tool, [GDG06].

  The limitations to this analyis when analyzing lossy TLs inspire the definition of the inverse characterization of basic parameters parameterized by the wave parameters, which is the dual analysis to the description of wave parameters parameterized by basic parameters introduced in the *CTLA handbook*.

Thus, regarding these previous works in the CTLA: (i) the analysis introduced in the *CTLA handbook* are conveniently contextualized within the direct characterization of the LTLT presented in Chpt. 4, as well as rigorously described in the basis of the CTLT; (ii) the inverse characterization is defined as the alternative/complementary analysis to the direct one; and (iii) both characterizations are proposed to complete the CTLT-v1 regarding lossy TLs, in which the mentioned open

[2] Transversal ElectroMagnetic modes.

problems: the analysis in terms of losses, frequency and along the TL; are solved as examples of use in next chapter by combining the characterizations.

4.3 Direct characterization of the Complex Transmission Line Theory (CTLT-v1.0a)

In this section, the direct characterization of the TL regarding the LTLT presented in Chpt. 2 is studied from the the CTLT point of view.
The analysis presented here exemplifies the methodology to be followed in the rest of characterizations regarding the different versions of the TLT, which emerge when (a) analyzing guided waves in different mediums, with (b) different boundary conditions, studied in (c) different domains (see the Table 2.1 introduced in Chpt. 2, which shows the classification of the TLT versions addressed in the Thesis)[3]. These analysis cover those issues detached in the introduction of this chapter regarding the subsequent achivement of the –in this case– CTLT-v1: (i) studying different types of parameterizations: lossy parameterization, parameterization of frequency, and length parameterization; (ii) proposing the normalizations for each parameterized analysis; (iii) doing the graphical analysis and characterizing the resultant curves geometrically; (iv) obtaining physical interpretations from the graphical and geometrical analysis; and (v) detaching the possible practical uses in both manufacturing circuits and analyzing real electromagnetic systems. An appropriate division in parts following these (i-v) points is presented for each parameter under study regarding the underlying – also in this case– LTLT.

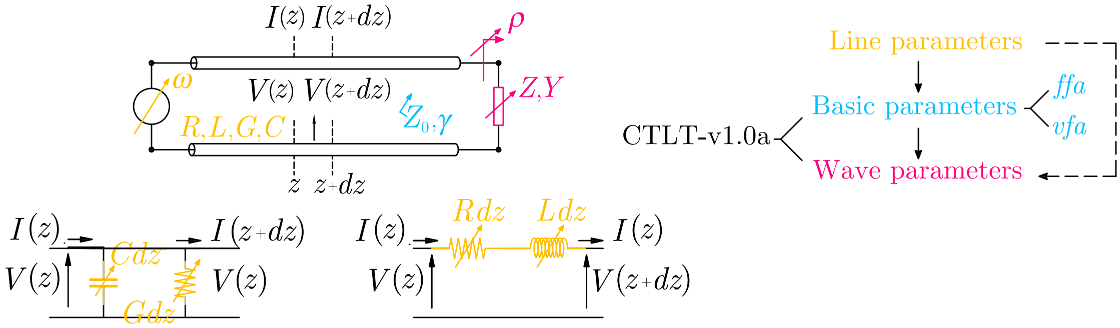


Fig. 4.1: Scheme of the lossy TL whose parameters: both the basic parameters in terms of losses and frequency (the line parameters), and the wave parameters in a fixed point on the TL; are analyzed in the CTLT-v1.0a.

This section is organized as follows: firstly, the basic parameters are characterized in terms of losses when frequency is fixed. Then, the basic parameters are characterized when frequency varies. Both the *fixed frequency analysis* (*ffa*) and *variable frequency analysis* (*vfa*) are basic[4] analysis which let to know how the individual voltage and current waves behave (see in Fig. 4.1 the line parameters which vary –marked in yellow– for characterizing the basic parameters –marked in blue–).
After this, the wave parameters are next characterized parameterizing the transformations between them, while inheriting the parameterizations of losses/frequency from the analysis of basic paremters (see in Fig. 4.1 the parameters which vary –the basic parameters marked in blue– for characterizing the wave parameters –marked in pink–). Thus, this latter analysis manages two types of parameterizations: the parameterization regarding the basic parameters and the parameterizations of wave parameters in themselves; but the true usefulness of this characterization is in fixing the parameterization regarding the basic parameters in order to analyze how the total waves vary when changing the parameterizations of wave parameters at any point of the TL, that is, when changing the BCs (in the TL's length) at any fixed point in the TL.
These basic transformations let to study a posteriori how the wave parameters vary in terms of line parameters, which is one of the objectives proposed in the CTLT analyzed in Chpt. 5.

[3]The items (a), (b), and (c) are different combined for posing different versions of the TLT.
[4]A particular analysis is called "basic" when it lies in one single complex transformation.

4.3.1 Direct characterization of basic parameters

Fixed frequency analysis

The *ffa* of basic parameters refers to the analysis in terms of losses, corresponding to the analysis in the *frequency domain* presented in Sect. 2.3 in Chpt. 2.

Remark 15. *When talking about the ffa, it is assumed to be operating in one specific subspace of coeffcients regarding the harmonic function –a basis representing the (inverse) Fourier Transform (Fourier expansion)– parameterized by ω. Thus, the ffa regarding the CTLT is the homologous to analyze the wave solutions (in this case HPWs) in the frequency domain regarding the underlying TLT (in this case the LTLT).*

In particular, the analysis of the lossy case in which the line parameters are not frequency dependent is presented in this first version of the CTLT regarding the direct characterization (CTLT-v1.0a), corresponding to the analysis introduced in Sect. 2.4.1 in Chpt. 2. Moreover, since this lossy case generalizes the lossless and the non dispersive cases, besides it is the origin of the low-losses approximation, the *ffa* lets to explain these cases at the same time it is described. The fact that all the particular cases can be analyzed by only one characterization reveals the importance of the CTLT as an efficient methodology, alternative to study the underlying analytical expressions.

***Parameterizations*:** The line parameters of the equivalent lossy TL represented by the schemes in Fig. 2.17 determine the basic parameters of the lossy TL.
These line parameters (R, L, G, C) should be parameterized taking into account that (i) in this study, frequency is a fixed parameter, and (ii) the objective here it to characterize how losses affect the TL. In this sense, notice that in eq. (2.89) the lossy constitutive parameters are mapped in R and G, and that is because the lossless case in eq. (2.91) correspond to $R = 0$ and $G = 0$. This suggests normalizing R and G with respect to ωL and ωC, respectively, which defines both the parameterizations of losses in the *ffa*:

$$\begin{cases} r = \frac{R}{\omega L} \in [0, \infty[\\ g = \frac{G}{\omega C} \in [0, \infty[\end{cases} ; \tag{4.1}$$

and the normalizations of basic parameters with respect to the respective losseless cases (they are next studied separately for the characteristic impedance and the propagation constant).
The parameters r and g in eq. (4.1) are called the conductor and dielectric losses, respectively, because they simulate the losses due to the conduction of the equivalent current $I(z)$ and the losses due to the presence of a dielectric between the contours "energized" with the potential difference $V(z)$[5].

Since the lossy parameterizations are not correlated (that is, they are independent because the real and imaginary parts of the constitutive parameters only appear mapped once in the (real) line parameters), it is possible to draw a (non Euclidean) plane whose axis refer to the lossy parameterizations r (acting as x-axis) and g (acting as y-axis). This plane is called the rg-plane, and it is especially useful when explaining the origin of the parameterizations in the direct characterization presented here, and to see graphically the analysis regarding the inverse characterization presented in Sect. 4.4 (see the origin and properties of the rg-plane in Appendix 4.A).

[5]This denomination is kept from previous studies in [Gag01], although it is somewhat confusing because the contours which support HPWs are PECs, so their conductivity $\sigma_{PEC} \to \infty$ and thus they do not present any resistane for the current to flow. However R in eq. (2.89) does not refer to the losses due to the material resistivity, but losses caused by magnetic dispersivity.

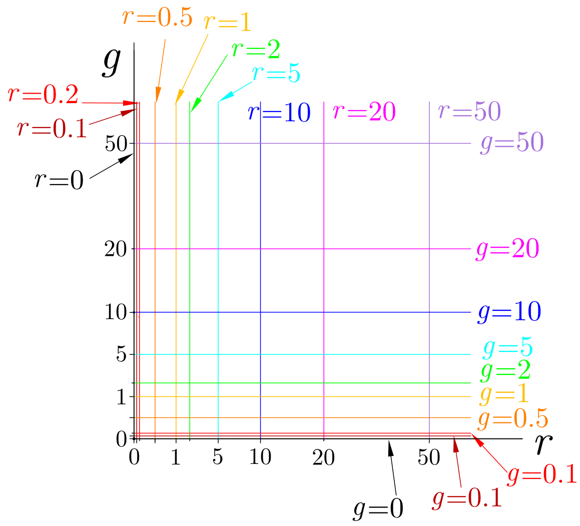


Fig. 4.2: Lossy parameterizations r and g in the rg-plane ($\equiv \mathbb{R}^2 \cup (0,0)$) for the *ffa* of basic parameters.

	Lossy parameterizations
Lossless case	$r = 0,\ g = 0$
Non dispersive case	$r = g$
Low-losses approximation	$r \to 0\ (r \ll),\ g \to 0\ (g \ll)$

Table 4.1: Values of the lossy parameterizations which define the particular cases: lossless and non dispersive cases; and the low-losses approximation from the parameterized lossy case.

For the *ffa*, the lossy parameterizations are the constant curves

$$\begin{cases} r = r_0 \in [0, \infty) \\ g = g_0 \in [0, \infty) \end{cases} \tag{4.2}$$

in the rg-plane, as depicted in Fig. 4.2. Notice that this plane is restricted to the first quadrant ($\mathbb{R}^2 \cup (0,0)$) for the lossy case studied in this version of the CTLT, besides both axis r and g are real, which allows the graphical representation of the rg-plane.
The r and g parameterizations completely define the basic parameters once they are accordingly normalized.
In addition, the particular cases and the approximation presented in Sect. 2.4 in Chpt. 2 can be mapped with these lossy parameterizations, just as it is presented in Table 4.1.

Characteristic impedance: The importance of the equivalent characteristic impedance regarding the direct characterization of the lossy TL when relating the individual wave solutions in both amplitude and phase difference is detached in the LTLT presented in Sect. 2.4.1 in Chpt. 2.

***Normalization*:** The expression of the characteristic impedance in terms of the line parameters for the lossy case in eq. (2.71) is normalized with repect to the lossless case in eq. (2.94), leading

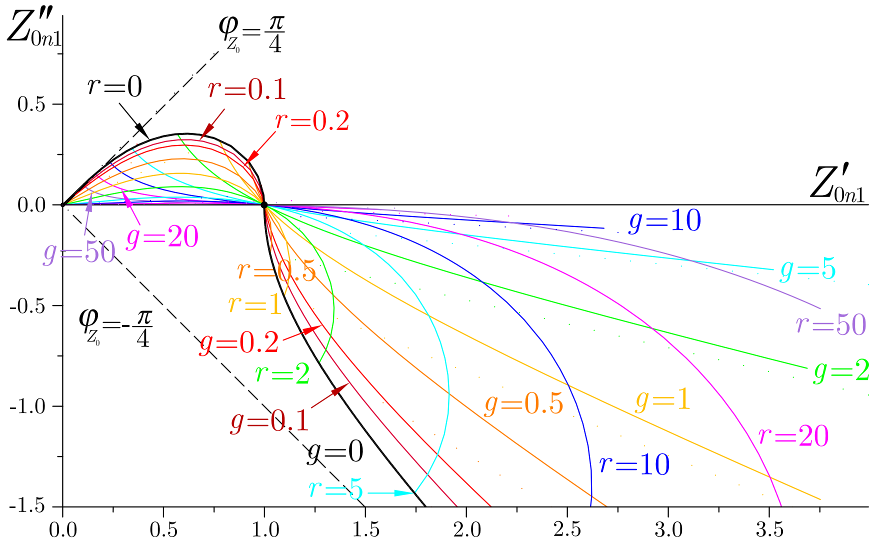


Fig. 4.3: Graphical analysis of the curves parameterized by losses in the Z_{0n1}-plane.

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to

$$Z_{0n1} = \frac{Z_0}{Z_{0,sp}} = \frac{\sqrt{\frac{R+j\omega L}{G+j\omega C}}}{\sqrt{\frac{L}{C}}} = \sqrt{\frac{1-j\frac{R}{\omega L}}{1-j\frac{G}{\omega C}}} = \sqrt{\frac{1-jr}{1-jg}} \in D_{Z_{0n1}} \subset \mathbb{C}, \tag{4.3}$$

having used the definition of the lossy parameterizations in eq. (4.1).
Notice that this normalization is done with respect to a real value so the angle of the normalized characteristic impedance is the same as in the original Z_0. This fact is of great importance when interpretating Z_{0n1} phisically, and also when taking the angle of the characteristic impedance as the parameter for future analysis.
The parameter Z_{0n1} is called the normalized characteristic impedance for the *ffa*, [Gag01]. It is parameterized by the conductor and dielectric losses so it could be characterized in terms of them. Eq. (4.3) may be rewritten in terms of the 2^{th} root of a quotient of two complex numbers:

$$Z_{0n1} = Z'_{0n1} + jZ''_{0n1} = |Z_{0n1}|e^{j\varphi_{Z_0}} \equiv \sqrt{\frac{n}{d}}, \text{ in which} \tag{4.4}$$

$$n = |n|e^{j\varphi_n} = 1 - jr,\ |n| \geq 1,\ \varphi_n \leq 0, \text{ and} \tag{4.5}$$

$$d = |d|e^{j\varphi_d} = 1 - jg,\ |d| \geq 1,\ \varphi_d \leq 0. \tag{4.6}$$

This expression is useful when separating the parameterization r (included in n) and g (included in d).
Moreover, only by using the particularizations in Table 4.1, it is possible to identify the particular cases and approximations regarding the characteristic impedance which are explained in Sect. 2.4 in Chpt. 2, which shows the capabilities of the r-g parameterizations for compressing the complete analysis.

***Graphical analysis*:** The most intuitive way to analyze Z_{0n1} in terms of lossy parameterizations is by means of graphical analysis in its associated complex plane. For the purpose of representing the curves parameterized by r, this parameterization is kept fixed while g varies in its whole range (described in eq. (4.2)), [Gag01]. Conversely, the g-parameterized curves are obtained when keeping fixed g and r varies.

Rigorously, this parameterized analysis may be interpreted as the complex mapping, [BC90], of the r and g constant curves from the rg-plane introduced in Fig. 4.2 to the Z_{0n1}-complex plane. In this sense, Z_{0n1} is seen as a complex function (in this case a real bi-variate complex function)[6]:

$$\begin{aligned} \boldsymbol{Z_{0n1}}: \quad & \mathbb{R}^{2+} \cup (0,0) \to D_{Z_{0n1}} \subset \mathbb{C} \\ & (r,g) \rightsquigarrow \boldsymbol{Z_{0n1}}(r,g) = \sqrt{\frac{1-jr}{1-jg}}. \end{aligned} \tag{4.7}$$

This is an analytic function in the rg-plane and so its Taylor series (power series) exist, [Bro96], which justify the existence of the low-losses approximation, but it is not conformal, [BC90] if seeing it as a mapping.
Nevertheless, using some properties of mappings: complex scalar multiplication and shifting, inversion, and 2^{th} root; [BC90], and concatenating them, it is possible to represent the parameterized curves in the Z_{0n1}-plane represented in Fig. 4.3.
In this way, all the possible values of the normalized characteristic impedance are represented in only one complex plane, which clearly shows the possibilities of graphical analysis in compressing the analysis of Z_{0n1} in terms of losses.

The curves in Fig. 4.3 are within the region (the range $D_{Z_{0n1}}$ of the function in eq. (4.7)) bounded by the curves parameterized by $r = 0$ and $g = 0$ ($r = 0 \cup g = 0$), and the real axis ($Z''_{0n1} = 0$). This region is, in turn, within the sector $\varphi_{Z_0} =]-\pi/4, \pi/4]$.
In particular, the g-curves present asymptotic behavior as $r \to \infty$, and they limit with the curve parameterized by $r = 0$ in the other end. On their behalf, the r-cuves are delimited by the origin of Z_{0n1}-plane ($Z_{0n1} = 0 + j0$) and the curve parameterized by $g = 0$.

***Geometrical analysis*:** The graphical analysis in terms of the r- and g-parameterized curves in the Z_{0n1}-complex plane in Fig. 4.3 immeditely suggests analyzing these resultant curves geometrically, in order to see both their geometrical properties and the alternative representation of the original equation of Z_{0n1} in eq. (4.3), and thus the original Z_0 in eq. (2.71).

The r-parameterized curves follow the general equation

$$\begin{aligned} |Z_{0n1}| &= \sqrt{|n| \cos (2\varphi_{Z_0} - \varphi_n)}, \\ \varphi_{Z_0} &\in \left[\frac{\varphi_n}{2}, \frac{\pi}{4} + \frac{\varphi_n}{2} \right[, \end{aligned} \tag{4.8}$$

which is written in polar form, having parameterized the (complex) variable n (which depends on r) defined in eq. (4.4) as a parameter.

The set of r-curves described by eq. (4.8) parameterized by n are of the form of quarter lemniscates of Bernouilli, [Law72].
Notice that each curve is a complex scalar transformation from another one. In particular, each curve is the scalar transformation of the curve parameterized by $r = 0$ (the upper limit in the Z_{0n1}-plane), obtained when multiplying it by the complex factor $\sqrt{|n| e^{j\frac{\varphi_n}{2}}}$. This fact is useful for characterizing the possible linear transformations of this set of curves only by means of transforming one of them, and also for obtaining some specific points on this curves. For example, the end

[6]In order to distinguish each parameter and the function which represent it, which defines a transformation, the functions are written in boldface. For example, while Z_{0n1} is addressed as the normalized characteristic impedance, $\boldsymbol{Z_{0n1}}$ refers to the function which transforms (r, g) points to normalized characteristic impedances, that is, Z_{0n1} points.

of the curve parameterized by $r = 0$ is on $Z_{0n1}|_{r=0,end} \equiv \boldsymbol{Z_{0n1}}(r = 0, g = 0) = 1 + j0$. Then, the end of the curve parameterized by $r = r_0$ characterized by the complex value $n = n_0$ is on $Z_{0n1}|_{r=r_0,end} \equiv \boldsymbol{Z_{0n1}}(r = r_0, g = 0) = \sqrt{|n_0|}e^{j\frac{\varphi_{n_0}}{2}}$.

The g-parameterized curves follow the general equation

$$|Z_{0n1}| = \frac{1}{\sqrt{|d| \cos\left(2\varphi_{Z_0} + \varphi_d\right)}}, \qquad \varphi_{Z_0} \in \left[-\frac{\varphi_d}{2} - \frac{\pi}{4}, -\frac{\varphi_d}{2}\right[, \tag{4.9}$$

which is written in polar form, having taken d in eq. (4.4) as a parameter.

The set of g-curves described by eq. (4.9) parameterized by d are of the form of quarter hyperbolas, [Law72].
Each hyperbola is a scalar complex transformation from another one. In particular, the factor $1/\left(\sqrt{|d|}e^{j\frac{\varphi_d}{2}}\right)$ multiplies the curve parameterized by $g = 0$ to obtain the set of curves in (4.9). In this way, any linear transformation of the g-curves can be done only by means of transfoming one of them.

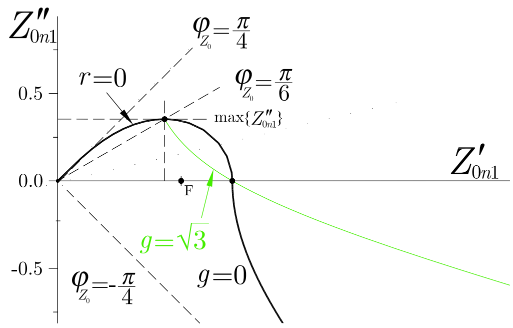

Fig. 4.4: Graphical representation of the maximum of Z''_{0n1} ($Z''_{0n1,max}$) in the Z_{0n1}-plane.

Since the general expression of the curves in the bi-parameterized Z_{0n1}-plane is known, some important geometrical properties of curves may be obtained by eliminating one parameter. For example, the maximum value of the imaginary part of Z_{0n1} ($\max\{Z''_{0n1}\} \equiv Z''_{0n1,max}$) represented in Fig. 4.4 is on the curve parameterized by $r = 0$, and it is obtained when $g = \sqrt{3}$, so $Z''_{0n1,max} = 1/\left(2\sqrt{2}\right)$ (see the calculus in Appendix 4.B).

***Physical interpretations*:** The Z_{0n1}-plane has straightforward physical interpretations: the modulus of Z_{0n1} fixes the relation between the amplitudes of the voltage and current waves (as long as the parameter which normalize Z_{0n1}, that is $Z_{0n1,sp}$, is known), whereas the phase is exactly the phase shifting between these waves (because the normalization Z_{0n1} does not affect angles). In this characterization, both the modulus and the phase included in Z_{0n1} are obtained in terms of their lossy dependence.

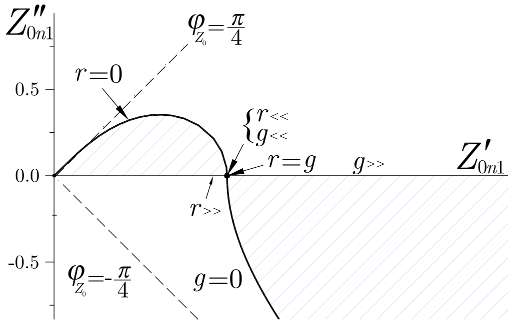

Fig. 4.5: The location of the lossless and non dispersive cases, and the low-losses and high-losses approximation in the Z_{0n1}-plane.

It is important to detach that the resultant curves in Fig. 4.3 are "universal", in the sense that each parameterized curve correspond with multiple TLs that have r or g as a parameter. Thus, each point in the Z_{0n1}-plane is not identified with an unique TL but a infinite set of TLs working at different (fixed) frequencies.
Among the points and the physical meaning of each of them, there are those which are of special interest because they correspond to the particular cases and the approximation introduced in Sect. 2.4 in Chpt. 2. These points are located in Fig. 4.5 labeled by the terminology introduced in Table 4.1. Notice how the points fit the particularized expressions of Z_0 for each case. In particular, the point $Z_{0n1} = 1 + j0$ corresponds to the non dispersive case –the losses case included– and the low-losses approximation. In this way, the function $\boldsymbol{Z_{0n1}}(r, g)$ in eq. (4.7) is clearly non injective in the bisector of the rg-plane.
Moreover, new particular cases or behaviors of lossy TLs may be deduced by examining the lossy characterization of Z_{0n1} in Fig. 4.3. For example, the high-losses approximations are obtained when making the values of r or g tend to infinity.

	Lossy parameterizations
high-losses approximation	$r \to \infty$ ($r \gg$) or $g \to \infty$ ($g \gg$)

Table 4.2: Values of the lossy parameterizations wich define the high-losses approximation from the parameterized lossy case.

These high-losses approximation produce real values of the characteristic impedance, as it can be seen depicted in Fig. 4.5.
Rememeber that low-losses approximation correspond to real TLs if and only if they are non dispersive, as it is reflected in Fig. 2.16 presented Sect. 2.4 in Chpt. 2. This also occurs with the high-losses approximation. Thus, the real axis in the Z_{0n1} is not contained in the allowed region except to the non dispersive point $Z_{0n1} = 1 + j0$.

***Practical uses*:** In this part, the most important practical uses obtained from the analysis of Z_{0n1} in terms of losses are outlined. These practical uses are focused on both the importance of this analysis for more complex lossy characterizations and the possible applications of losses in designing circuits.

The role of the phase of Z_{0n1} –and thus Z_0– when parameterizing the losses is here detached. This parameter "universalizes" the lossy parameterizations in a certain way, while it inherits the properties of the denormalized characteristic impedance at the same time. The importance of this parameter when characterizing the wave parameters based on changes between themselves (which is equivalent to change the BCs) is analyzed in Sect. 4.3.2 in this chapter.

For the analysis of wave parameters in terms of losses, this analysis results crucial since the wave parameters explicitly depend on the characteristic impedance. The analysis of wave parameters in terms of losses is developed in Chpt. 5, which is presented as an example of use of this analysis of Z_{0n1}, expressly.

In addition, the "universal" nature of the curves in the Z_{0n1}-plane makes the analysis in terms of losses a powerul method to expand the characteristic impedance of any case or approximation. For this purpose, Z_{0n1} is seen as a function of losses ($\boldsymbol{Z_{0n1}}(r,g)$ as defined in eq. 4.7, and sketched in the inverse analysis by using the inverse function $\boldsymbol{Z_{0n1}^{-1}}(r,g)$) may act as the kernel of an integral operator or integral equation (see this application regarding the deinition of the normalized characteristic impedance in **Applications**).

Also notice that losses may be used for matching purposes at any fixed point on the TL. In this way, the analysis of Z_{0n1} in terms of losses provides the limits for Z_{0n1} to do the matching.
This capability of lossy TLs has been used in [VG17-I], together with more properties of lossy TLs recalled in the following sections, in order to introduce a graphical chart for matching loads with losses in an easy way.

Propagation constant: The propagation constant determines the physical characteristics of propagation: attenuation and phase speed of the individual waves; just as it has been stated in the direct characterization regarding the LTLT presented in Sect. 2.4.1 in Chpt. 2.
The analysis of the propagation constant in terms of losses is developed in this section, dual to the analysis of the characteristic impedace presented before.

***Normalization*:** The natural normalization of the propagation constant regarding the lossy case in eq. (2.68) for its study in terms of losses is with respect to the lossless phase constant (β_{sp}), which define the lossless propagation constant in eq. (2.95):

$$\begin{aligned}\gamma_{n1} =& \frac{\gamma}{\beta_{sp}} = \frac{\sqrt{(R+j\omega L)\,(G+j\omega C)}}{\omega\sqrt{LC}} = j\sqrt{\left(1-j\frac{R}{\omega L}\right)\left(1-j\frac{G}{\omega C}\right)} = \\ =& j\sqrt{(1-jr)\,(1-jg)} \in D_{\gamma_{n1}} \subset \mathbb{C},\end{aligned} \tag{4.10}$$

having used the same definition of lossy parameterizations in eq. (4.1).
Notice that this normalization has been done with respect to a real value so the angle of the normalized propagation constant is kept the same as the original propagation constant γ. This fact is important when analyzing the curves in the γ_{n1}-complex plane and deducing physical interpretations of this angle, and also when using this angle to parameterize the losses in future analysis (see the example in which the TL is characterized both in terms of losses and along the TL presented in Chpt. 5).
The parameter γ_{n1} is called the normalized propagation constant for the *ffa*, [Gag01]. It is parameterized by the conductor and dielectric losses, which play equivalent role as factors in eq. (4.10). This fact means an important graphical reduction, which also concerns the physical interpretations of losses when parameterizing the propagation constant.
Notice that the normalization with respect to β_{sp} includes the frequency. Despite the frequency is a fixed parameter for this analysis, its inclusion in the normalization affects the interpretation in the meaning of the resultant "universal" curves.

Again it is also possible to rewrite eq. (4.10) in terms of the n and d complex factors as

$$\gamma_{n1} = \alpha_{n1} + j\beta_{n1} = |\gamma_{n1}|\, e^{j\varphi_\gamma} \equiv j\sqrt{n\cdot d}, \tag{4.11}$$

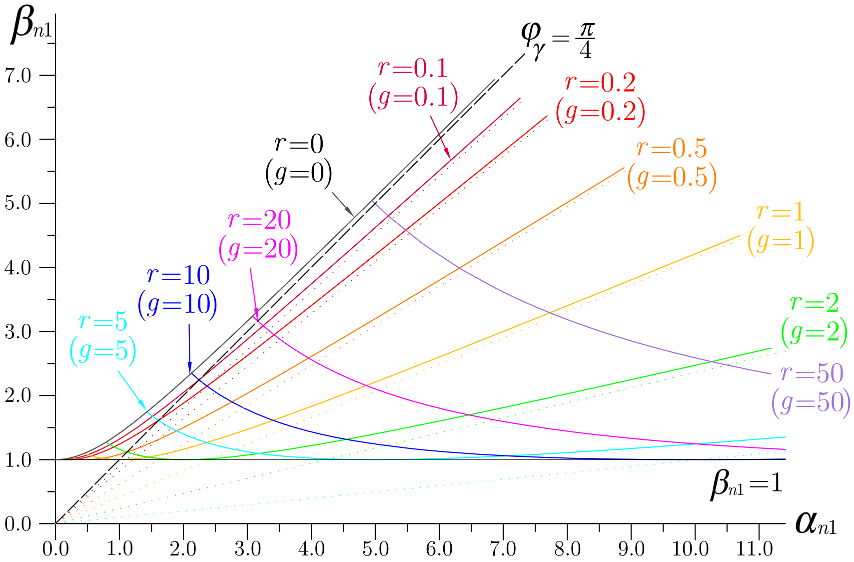

Fig. 4.6: Lossy parameterized curves in the γ_{n1}-plane. These curves are the result of transforming the constant curves which parameterize the losses in the rg-plane in Fig. 4.2.

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in which n and d are defined as in eqs. (4.5) and (4.6), repectively. Because of the equivalent role of r and g in eq. (4.10), n and d also play de same role as factors in eq. (4.11).
The normalized attenuation constant, α_{n1}, is

$$\alpha_{n1} = \sqrt{\frac{rg - 1 + \sqrt{(r^2+1)(g^2+1)}}{2}}, \tag{4.12}$$

whereas the normalized phase constant, β_{n1}, is

$$\beta_{n1} = \sqrt{\frac{1 - rg + \sqrt{(r^2+1)(g^2+1)}}{2}}. \tag{4.13}$$

By means of the particularizations of r and g in Table 4.1, it is possible to obtain the particularized expression of γ_{n1}, α_{n1}, and β_{n1} for each particular case or approximation.

***Graphical analysis*:** The normalized propagation constant is graphically studied by means of parameterized curves in its associated complex plane. For the purpose of interpreting the graphical analysis in an easy way, the parameter r or g is kept fixed while the other one: g or r, respectively; varies in its whole range, dedined in eq. (4.2). This leads to draw the r- and g-parameterized curves, also respectively. Since the parameterizations r and g in the expression of γ_{n1} in eq. (4.10) play the same role of factors in the square root, the resultant curves parameterized by r and g overlap.

For the rigorous interpretation of the graphical analysis, the normalized propagation constant is seen as a complex mapping which transforms the constant parametrizations in the rg-plane depicted in Fig. 4.2 to curves in the γ_{n1}-complex plane. In this sense, γ_{n1} has to be seen as a real bi-variate complex function:

$$\begin{aligned} \boldsymbol{\gamma_{n1}} : \quad & \mathbb{R}^{2+} \cup (0,0) \to D_{\gamma_{n1}} \subset \mathbb{C} \\ & (r,g) \rightsquigarrow \boldsymbol{\gamma_{n1}}(r,g) = j\sqrt{(1-jr)(1-jg)}. \end{aligned} \tag{4.14}$$

This function is analytic in the rg-plane so it can be locally expanded in Taylor series, whish justifies the existence of the low-losses approximation, but it is not conformal when seeing it as a mapping.

Using some properties of mappings: product with complex scalar, shifting, and 2^{th} root; and composing them, it is possible to draw the parameterized curves in the γ_{n1}-plane, which is represented in Fig. 4.6.
The γ_{n1}-complex plane gathers all the possible values of the normalized propagation constant in only one complex plane.

The curves parameterized by losses are in the region (denoted by $D_{\gamma_{n1}}$, which is the range of the function presented in eq. (5.62)) bounded by the lossless curves $r = 0$ or $g = 0$, and the constant line $\beta_{n1} = 1$. Recall that each curve is double parameterized because the equivalent role of r and g in the expression of γ_{n1}.
It may be seen that the curves adjoin the lossless curve on one side, while they present asymptotic behavior on the orther side. Moreover, every curve is tangent to the line β_{n1}=1 in one point. The intersection on the lossless curve, the angle of the asymptote, and the point in which each curve is tangent can be deduced by geometrical analysis.

***Geometrical analysis*:** The geometrical analysis of the curves depicted in Fig. 4.6 gives an alternative representation of the equation of γ_{n1} in eq. (4.10), and thus to the original definition of the propagation constant in eq. (2.68).

The r- and g- parameterized curves follow the same general equation[7]

$$|\gamma_{n1}| = \frac{\sqrt{|n|}}{\sqrt{-\cos\left(2\varphi_\gamma - \varphi_n\right)}} \quad \left(|\gamma_{n1}| = \frac{\sqrt{|d|}}{\sqrt{-\cos\left(2\varphi_\gamma - \varphi_d\right)}}\right), \qquad (4.15)$$
$$\varphi_\gamma \in \left]\frac{\pi}{4} + \frac{\varphi_n}{2}, \frac{\pi}{2} + \frac{\varphi_n}{2}\right] \quad \left(\varphi_\gamma \in \left]\frac{\pi}{4} + \frac{\varphi_d}{2}, \frac{\pi}{2} + \frac{\varphi_d}{2}\right]\right),$$

which are written in polar form having taken the complex quantity n (d) introduced in eq. 4.5 (eq. (4.6)) as a parameter.
These curves are of the form of hyperbolas, [Law72], in the γ_{n1}-plane.
Each hyperbola may be obtained by the complex scalar transormation from any another one. In particular, the factor $\sqrt{|n|}e^{j\frac{\varphi_n}{2}}$ ($\sqrt{|d|}e^{j\frac{\varphi_d}{2}}$) multiplies the curve parameterized by $r = 0$ ($g = 0$) resulting in the curve parameterized by n (d). In this way, any linear transformation of the set of curves parameterized by r (g) may be obtained by transforming only one on them, to then rotating the one which has been linearly transformed.

It is possible to use the scalar transformation property mentioned above to obtain: (i) the point in which each curve intersects the lossless limit, denoted by $\gamma_{n1,r-lim}$ ($\gamma_{n1,g-lim}$). Notice that the curve parameterized by $r = 0$ ($g = 0$) interects in $\gamma_{n1} = 0 + j1$ with the other lossless curve $g = 0$ ($r = 0$). This point is the same for each lossy curve after transforming it, so $\gamma_{n1,r-lim} = j\sqrt{n} \equiv \sqrt{|n|}e^{j\left(\frac{\pi}{2}+\frac{\varphi_n}{2}\right)}$ ($\gamma_{n1,g-lim} = j\sqrt{d} \equiv \sqrt{|d|}e^{j\left(\frac{\pi}{2}+\frac{\varphi_d}{2}\right)}$); and (ii) the angle of the radious which is asymptotic to each parameterized curve, denoted by $\varphi_{\gamma,asymp}$. Notice that the curve parameterized by $r = 0$ ($g = 0$) presents asymptotic behavior to the bisector of the γ_{n1}-plane. By using the complex transformation above, the angle of each asymptote is $\varphi_{\gamma,asymp} = \pi/4 + \frac{\varphi_n}{2}$ ($\varphi_{\gamma,asymp} = \pi/4 + \frac{\varphi_n}{2}$) for each r- (g-) parameterized curce.
In addition, since the general equation of each curves in the γ_{n1}-plane is known, it may be differentiated for obtaining the relative minimum in which it is tangent to the line $\beta_{n1} = 1$, denoted by

[7] An equation in brackets refers to an expression which has the same form as the equation which goes with, so their parameterizations are interchangeable. This notation is used throughout the Thesis book to address the equations in which the "orthogonal" parameterizations play the same role, for example r and g (and thus n and d) in the normalized propagation constant γ_{n1}.

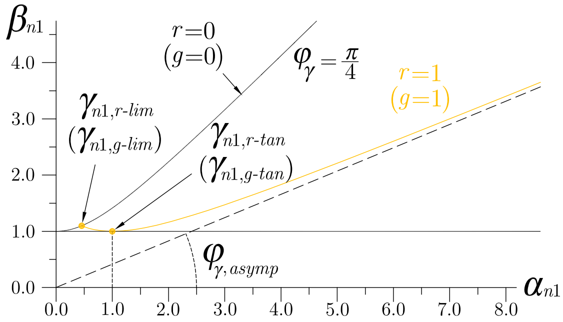

Fig. 4.7: Example of location of remarkable points of the curve parameterized by $r = 1$ $(g = 1)$.

$\gamma_{n1,r-tan}$ $(\gamma_{n1,g-tan})$, which is found at $\gamma_{n1,r-tan} = r + j$ $(\gamma_{n1,d-tan} = g + j)$ (see Appendix 4.C). A graphical example of the location of these points when $r = 1$ $(g = 1)$ is depicted in Fig. 4.7.

***Physical interpretations*:** The γ_{n1}-plane has clear physical interpretation when examining the real and imaginary parts of each point there.
The real part in the γ_{n1}-plane, α_{n1}, is proportional to the attenuation constant, so proportional to the way the voltage and current waves attenuate along the direction of propagation. Notice that since $\alpha_{n1} = \alpha/\beta_{sp} = \alpha\lambda_{sp}/(2\pi)$, the normalized attenuation constant gives an idea of how much the waves attenuate along in one (lossless) wavelength. The minimum attenuation is produced when the propagation constant is $\gamma_{n1,r-lim}$ $(\gamma_{n1,g-lim})$.
The imaginary part in the γ_{n1}-plane, β_{n1}, is proportional to the phase constant, so inversely proportional to the phase speed of the voltage and current waves along the direction of propagation. Notice that since $\beta_{n1} = \beta/\beta_{sp} = c_e\beta/\omega = c_e/v_p$, the normalized phase constant gives an idea of how is the phase velocity of the wave in comparison with the speed of light in the medium. The maximun speed is produced when the propagation constant is $\gamma_{n1,r-tan}$ $(\gamma_{n1,g-tan})$, which is the only case in which this velocity equals the speed of light.

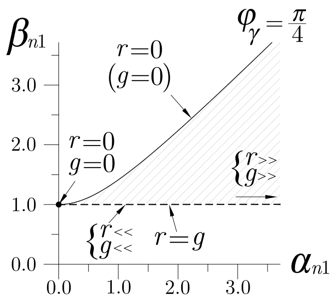

Fig. 4.8: The lossless and non dispersive cases, and the low-losses and high-losses approximations located in the γ_{n1}-plane.

The curves in the γ_{n1}-plane are also "universal" in the sense that each curve represent multiple TLs parameterized by r or g. As a consequence, each point in this plane represents those TLs with different line parameters working at different frequencies which keep r or g fixed.
Some of these points in the γ_{n1}-plane are of special interest because they correspond to those particular cases and the approximation explained in Sect. 2.4 in Chpt. 2: the lossles and non dispersive case, and the low-losses approximation presented in Table 4.1; and the high-losses approximation presented in Table 4.2. These cases are graphically located in the γ_{n1}-plane in Fig. 4.8.
Notice that the lower limit in the γ_{n1}-plane $(\beta_n 1 = 1)$ corresponds with the non dispersve case,

when $r = g$ and $\gamma_{n1,nd} = r + j1 \equiv \gamma_{n1,nd} = g + j1$, in which the waves present the highest phase velocity.
The lossless case is a particular case of the non dispersive case when $r = g = 0$, located at $\gamma_{n1} = 0 + j1 \equiv \gamma_{n1,sp}$.

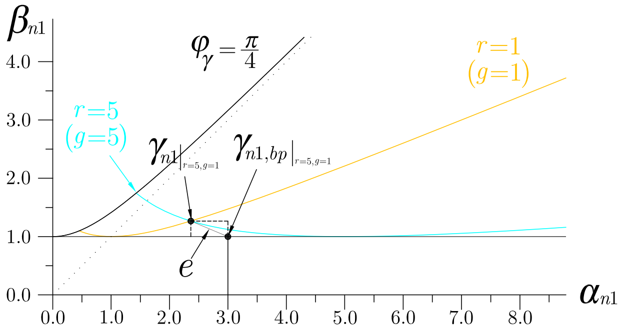

Fig. 4.9: Graphical example of the error analysis (e) between the low-losses approximation and the lossy case when $r = 5$ and $g = 1$ (which is the same as the case $r = 1$ and $g = 5$).

On its behalf, the low-losses approximation can be rigorously explained by means of the graphical analysis of the lossy case presented in Fig. 4.6. An example of the error analysis regarding the low-losses approximation when $r = 5$ and $g = 1$ (reciprocally the same when $r = 1$ and $g = 5$) is graphically analyzed in Fig. 4.9. Notice that, regarding the low-losses approximation,

$$\alpha_{n1,bp} = \frac{r + g}{2}, \tag{4.16}$$

so in the example in Fig. 4.9, $\alpha_{n1,bp}|_{r=5,g=1} = 3$, an thus $\gamma_{n1,bp}|_{r=5,g=1} = 3 + j1$, whereas $\gamma_{n1}|_{r=5,g=1} = j\sqrt{(1 - j5)\,(1 - j1)}$, which explains the error (depicted as e in Fig. 4.9.
From this example it may be generalized that: (i) the attenuation constant regarding the low-losses approximation is overestimated; and (ii) the phase constant is underestimated, so the phase velocity is overestimated; except in the non dispersive case, in which $\gamma_{n1,bp}|_{r=g} = \gamma_{n1}|_{r=g} \equiv \gamma_{n1,nd}$.

***Practical uses*:** The most important practical uses of the analysis of γ_{n1} in terms of losses are next detached focusing on both its importance in future analysis and the possible applications of this analysis.

The analysis of γ_{n1} in terms of losses results crucial when analyzing the variation of wave parameters along the TL in terms of losses, which is shown as example of use in Chpt. 5.
Moreover, this analysis supports the (more basic) analysis along the TL when the losses of the TL are fixed, also presented as example of use in Chpt. 5.
In both cases, the role the phase of γ_{n1} –and thus the phase of the denormalized propagation constant, γ– plays is crucial. This parameter determines every analysis along the TL once the appropriate normalization of the TL's length –the eletrical length– is chosen[8].

The "universal" nature of this analysis makes possible to expand any case or approximation. For this purpose, γ_{n1} is seen as a function of losses ($\boldsymbol{\gamma_{n1}}(r, g)$) which may act as the kernel of an integral operator or integral equation. The analysis of γ_{n1} in terms of losses gives the limits in the range of this function. This function is characterized and "sketched" by its inverse ($\boldsymbol{\gamma_{n1}^{-1}}(r, g)$) which is studied by means of the inverse characterization in Sect. 4.4.2.

[8]Depending the analysis that is being studied, the normalization of the TL's lenght will be accordingly chosen. For example, if the losses and the frequency on the TL are fixed the normalization of the TL's length is with respect to the lossy λ, so proportional to the lossy β.

The use of losses for load matching at any fixed point of the TL can be extended at any point of the TL taking into account the characterization of the propagation constant in terms of losses presented in this section.
The graphical chart presented in [VG17-I] is expressly introduced to facilitate the use of losses for the load matching along the TL.

Variable frequency analysis

The *vfa* of basic parameters refers to the analysis in terms of frequency for the TLs whose losses are fixed in certain way. In this sense, this analysis expands the solution from the harmonics in the *frequency domain* to the *time domain*, both presented Sect. 2.3 in Chpt. 2 regarding the LTLT.

Remark 16. *When talking about the vfa, it is clearly refering to the expansion of the ω-parameterized harmonics of any signal/solution in the TL. In particular, the characterization of the TL parameters when frequency is variable corresponds to either the spectral analysis of these parameters if seeing them as operators (some of them are operators, indeed, which relate the voltage and current waves) or the expansion of voltage or current functions in time domain.*
For example, for solving $v^+(z,t)$:

$$v^+(z,t) = \frac{1}{2} Re\left\{ \int_\omega V^+(\omega) e^{-\gamma(\omega)z} e^{j\omega t} d\omega \right\}, \tag{4.17}$$

for which, for example, the spectrum $V^+(\omega)$ is a Dirac delta ($\delta(\omega - \omega_0)$) and the integral has to be understood in distribution sense, if $v^+(z,t)$ represents a sinusoidal signal varying ω_0 times per second.

Thus, the analysis varying the frequency results indispensable for deparameterizing frequency and giving the solution in $[z,t]$-coordinates, not limiting the analysis to time harmonic regime.
As a result, this *vfa* may be seen as the complement of the *ffa*, in the sense that it can be used to answer how the basic parameters vary if isolating the variation with frequency, once the losses are fixed. Moreover, it supports the inverse analysis introduced in Sect. 4.4.2 regarding the basic parameters, in order to answer which changes on losses produces the same effect of changing the frequency.
As a consequence of the equivalence between changing losses and varying frequency, the analysis presented here can be used to explain the particular cases and approximations, but in terms of frequency. These explanations may be useful, for example, to exaplain asymptotic behaviors of waves with frequency, leading to introduce the asymptotic techniques, [Mil06].

***Parameterizations*:** In this case, the frequency, ω, determines the analysis of the basic parameters. Therefore, the frequency should be parameterized taking into account that (i) the resultant paremeterization has to be linear function of frequency, and (ii) it is seeking for a combination of line parameters (which are not frequency dependent) that allows this linear variation of frequency. If examining the rg-plane in Fig. 4.2, it is noticeable that the modulus in this plane ($|\tau| = \sqrt{r^2+g^2}$) is inverse linear function of frequency, since

$$|\tau| = \sqrt{r^2+g^2} = \sqrt{\left(\frac{r'}{\omega}\right)^2 + \left(\frac{g'}{\omega}\right)^2} = \frac{\sqrt{r'^2+g'^2}}{\omega}, \tag{4.18}$$

using the notation (τ) and geometrical interpretation of the rg-plane in the "space of parameterizations" presented in Appendix 4.A. This suggest $1/|\tau|$ be the parameterization of frequency which is being sought. As a consequence, the angle in this plane (θ_c as presented in Appendix 4.A, which is the angle which forms τ in eac ω-plane, so independent of frequency) is the parameter which determines the TLs which present the seeked linear variation of the parameterized frequency.

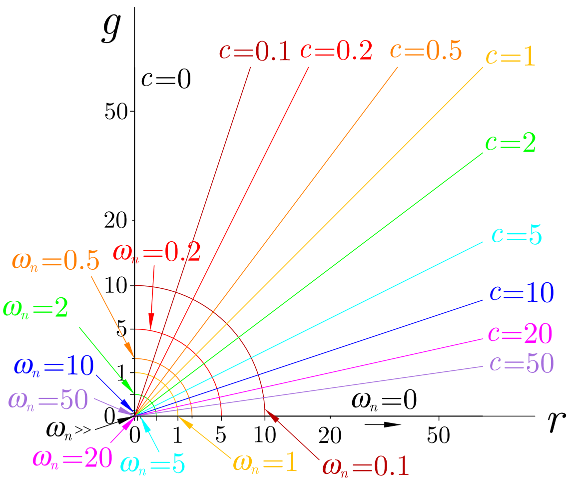

Fig. 4.10: Parameterizations ω_n and c in the rg-plane ($\equiv \mathbb{R}^2 \cup (0,0)$) for the *vfa* of basic parameters.

	Lossy parameterizations
Lossless case	$\omega_n \to \infty$ ($\omega_n \gg$)
Non dispersive case	$c = 1$
Low-losses approximation	$\omega_n \to \infty$ ($\omega_n \gg$)
High-losses approximation	$\omega_n \to 0$ ($\omega_n \ll$)

Table 4.3: Values of the parameterizations regarding the *vfa* which define the particular cases: lossless and non dispersive cases; and the approximations: low-losses and high-losses approximation.

Also notice that $1/|\tau| = \omega/\rho$ is represented by the angle $\varphi_\tau = \tan^{-1}(\omega/\rho)$ in the "space of parameterizations".
As a result, the parameterizations to be used for the *vfa* are:

$$\begin{cases} \omega_n \equiv \frac{1}{|\tau|} = \frac{\omega}{\sqrt{r'+g'}} = \frac{\omega}{\sqrt{\frac{R}{L}+\frac{G}{C}}} \in [0,\infty[\\ c \equiv \frac{1}{\tan(\theta_c)} = \frac{r'}{g'} = \frac{RC}{GL} \in [0,\infty[\end{cases} . \tag{4.19}$$

The parameter c allows the analysis with frequency, whereas ω_n fixes relative frequency scales over the frequency analysis. The parameters which identify the conductor and dielectric losses in the *vfa* are r' and g', wich are defined just as they have been used in eq. (4.19).
The parameters c and ω_n are called the "dispersivity constant" and the "normalized frequency", respectively, because their physical interpretation in the analysis. The first one, c, makes reference to how much dispersive the TL parameterized by c is, in the sense that it presents more or less variation in the phase constant in the face of changes in frequency. This will be noticeable in the size of the frequency scales. The second parameter, ω_n, is a normalization of frequency in itself. It is chosen with respect to the parameterizations of the normalized losses r' and g' used in eq. (4.19).

For the *vfa*, the parameterizations of frequency are the set of quarter of circumferences in the rg-plane, whereas the parameterization of the "dispersivity" are the radious in this plane.
The rg-plane is in the first quadrant just as for the *ffa*, but in the case of the *vfa* it is parameterized in modulus and phase (both real), as Fig.4.10 shows.
The parameters ω_n and c completely define the basic parameters once they are accordingly nor-

malized.
In addition, the particular cases and the approximations located in the *vfa* can be mapped using the dual particularizations of the *ffa* but expressed in terms of frequency parameterizations, as shown in Table 4.3.

Characteristic impedance: In this part, the meaning of the characteristic impedance for characterzing both the relation between the amplitudes and the phase difference between the electric and magnetic fields is analyzed when frequency varies.

***Normalization*:** The normalization of the characteristic impedance is done with respect to the non dispersive case, which does not depends explicitly on frequency so it is suitable for the purpose of normalizing:

$$Z_{0n2} = \frac{Z_0}{Z_{0,nd}} = \frac{\sqrt{\frac{R+j\omega L}{G+j\omega C}}}{\sqrt{\frac{L}{C}}} = \sqrt{\frac{\omega - j\frac{R}{L}}{\omega - j\frac{G}{C}}} = \sqrt{\frac{\omega - jr'}{\omega - jg'}} \in D_{Z_{0n2}} \subseteq \mathbb{C}, \tag{4.20}$$

having used the definition of the lossy parameterizations r' and g' which appear in eq. (4.19).
Since $Z_{0,sp} \equiv Z_{0,nd}$, the expresion of Z_{0n2} is equivalent as the one in eq. (4.3) used for the *ffa*, so

$$Z_{0n2} \equiv Z_{0n1}, \tag{4.21}$$

but, in the case of Z_{0n2}, separating the dependence on the frequency from the lossy parameterizations for its analysis. Thus, it is possible to use Z_{0n1} for both cases[9], as it is done in [Gag01]. The direct consequence of this analysis is that $D_{Z_{0n2}}$ in the *vfa* takes up the same in region in the Z_{0n2}-plane as Z_{0n1} in the *ffa*. Moreover, not only the angle φ_{Z_0} is the same as in the *ffa*, but also the modulus, which is important when transforming these curves in future analysis (see an example of these *ffa* and *vfa* transformations in Chpt. 5). As a result, those parameterizations (c, ω_n) corresponding with the pair (r, g) in the rg-plane lead to the same point in both Z_{0n1}-plane and the Z_{0n2}-plane.

Using the parameterizations for the *vfa* introduced in eq. (4.19), the normalized characteristic impedance may is rewritten as

$$Z_{0n2} = \sqrt{\frac{\omega_n - j\frac{r'}{\sqrt{r'^2+g'^2}}}{\omega_n - j\frac{g'}{\sqrt{r'^2+g'^2}}}} = \sqrt{\frac{\omega_n - j\cos(\theta_c)}{\omega_n - j\sin(\theta_c)}}, \text{ in which} \tag{4.22}$$
$$\omega_n \in [0, \infty[\,, \text{ and}$$
$$\theta_c \in \left[0, \frac{\pi}{2}\right] \;\; (c = 1/\tan(\theta_c) \in [0, \infty[).$$

This expression is useful for separating the parameterizations of the normalized frequency, ω_n and the dispersivity constant, c, paremeterized by the angle θ_c.

[9]The notation Z_{0n2} is expressly used to distinguish the normalized characteristic impedance in the *vfa* from the normalized characteristic impedance regarding the *ffa*, in contrast to use the same notation for both cases, just as it is done in [Gag01].

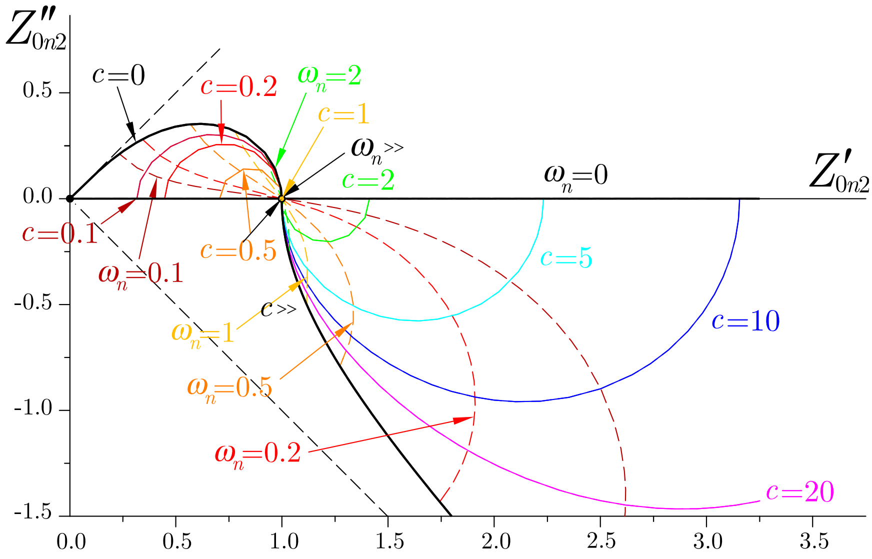

Fig. 4.11: Graphical analysis of the curves parameterized by frequency in the Z_{0n2}-plane.

***Graphical analysis*:** For the simple understanding of the graphical analysis, the representation is done by keeping fixed the parameter c when varying ω_n in its whole range, to draw the curves which represent Z_{0n2} in the whole frequency range. Reciprocally, the normalized frequency, ω_n, is kept fixed while varying c to see how any frequency which is of interest or bandwidth (representing a scale) change with the losses parameterized by c. These analysis leads to the c- and ω_n-parameterized curves, respectively.

This analysis is rigorously carried out by considering Z_{0n2} as complex function of two real variables ω_n and c,

$$\begin{aligned} \boldsymbol{Z_{0n2}} : \quad & \mathbb{R}^{2+} \cup (0,0) \to D_{Z_{0n2}} \subset \mathbb{C} \\ & (\omega_n, c) \rightsquigarrow \boldsymbol{Z_{0n2}}(\omega_n, c) = \sqrt{\frac{\omega_n - j\cos\left(\tan^{-1}(c)\right)}{\omega_n - j\sin\left(\tan^{-1}(c)\right)}}. \end{aligned} \tag{4.23}$$

The same properties of the function defined for the *ffa* are for this definition in the *vfa*, because it only changes the parametrizations but not the definition in itself.
In this case, the function is seen as a (non conformal) mapping of the curves parameterized by ω_n and c from the rg-plane to the Z_{0n2}-complex plane. Using some properties of mappings: product by complex scalar, shifting, inversion, and 2^{th} root; [BC90], and composing them, it is possible to draw the ω_n- and c-parameterized curves in the Z_{0n1}-plane represented in Fig. 4.11.

As it has been mentioned, the region in which Z_{0n1} varies is the same as in the *ffa*, but in this case the boundary curves are parameterized by $c = 0$ and $c \gg$, and the real axis of the Z_{0n2}-plane, which is parameterized by $\omega_n = 0$ except to the point $Z_{0n1} = 1 + j0$ which is parameterized by $\omega \ll$ (equivalently, $c = 1$).
The curves labeled by $c < 1$ have $Z''_{0n1} > 0$, whereas the curves with $c < 0$ have $Z''_{0n1} < 0$. These c-curves rise in $\omega_n = 0$ and death in $\omega_n \gg$.
The curves labeled by ω_n go from $c = 0$ to $c \gg$.

***Geometrical analysis*:** By geometrical analysis, it is possible to obtain the general expressions of the characteristic impedance for the *vfa*.

The curves parameterized by c follow the general equation:

$$\left(Z'^2_{0n2} + Z''^2_{0n2}\right)^2 = (c+1)\left(Z'^2_{0n2} - Z''^2_{0n2}\right) - c, \text{ with} \tag{4.24}$$

$$\begin{cases} Z'_{0n2} \geq 0,\ Z''_{0n2} \geq 0, \text{ if } c \leq 1 \\ Z'_{0n2} \geq 0,\ Z''_{0n2} \leq 0, \text{ if } c > 1 \end{cases}. \tag{4.25}$$

These curves are Cassini ovals, [Law72], parameterized by c. The general equation of them has been obtained in eq. (4.D.29) in Appendix 4.D, as well as its polar form

$$|Z_{0n1}| = \begin{cases} \sqrt{\left(\frac{c+1}{2}\right)\left[\cos\left(2\varphi_{Z_0}\right) + \sqrt{\left(\frac{c-1}{c+1}\right)^2 - \sin^2\left(2\varphi_{Z_0}\right)}\right]} \\ \text{if } \varphi_{Z0} > \frac{1}{2}\sin^{-1}\left(\frac{|c-1|}{c+1}\right) \\ \sqrt{\left(\frac{c+1}{2}\right)\left[\cos\left(2\varphi_{Z_0}\right) - \sqrt{\left(\frac{c-1}{c+1}\right)^2 - \sin^2\left(2\varphi_{Z_0}\right)}\right]} \\ \text{if } \varphi_{Z0} < \frac{1}{2}\sin^{-1}\left(\frac{|c-1|}{c+1}\right) \end{cases} \text{ for which} \tag{4.26}$$

$$\varphi_{Z_0} \in \left[0, \frac{\pi}{4}\right[$$

in eq. (4.D.30) in Appendix 4.D. This polar expression it is useful for parameterizing the modulus and phase of this equations and transforming them to other planes.
Notice that, if $c = 0$ the curve degenerates to that in eq. (4.8) in the *ffa* parameterized $r = 0$, that is the upper curve limit in the Z_{0n1}-plane (which coincides with the upper limit in the Z_{0n2}-plane). In this case, the ovals are not complex scalar transformation between themselves. Nevertheless, some interesting points may be obtained by using the general equation of ovals. As an example, the point in which the c-curves intersect the real axis ($Z''_{0n1} = 0$) is located at $Z'_{0n2} = \sqrt{c}$.

On its behalf, a closed general equation of the curves parameterized by ω_n can not be obtained.

***Physical interpretations*:** Recall the physical interpretation of the modulus and phase of Z_{0n2} in determining the relation between the amplitudes and the phase difference between the electric and magnetic fields, respectively. In this case, these properties are characterized in both the whole frequency band and at one particular (relative) frequency.

The c-parameterized curves describe how the characteristic impedance of a TL whose lossy parameters are related by c varies when changing the frequency. Moreover, the parameter c determines if the characteristic impedance is either inductive ($Z''_{0n2} > 0$) if $c < 1$, capacitive ($Z''_{0n2} > 0$) if $c > 1$, or pure resistive ($Z''_{0n2} = 0$) if $c = 1$.
The ω_n-parameterized curves fix relative frequency scales or bandwidths over the c-curves. This means that, once $Z_{0n2} \equiv Z_{0n1}$ is located in the plane by using the *ffa* (the frequency is given), the ω_n curves determine the multiples/submultiples of the original frequency. Alternatively, the ω_n parameterization gives the values of the characteristic impedance for a fixed frequency when losses vary but the same attenuation regarding the non dispersive case[10].
As a result, these parameterizations play a role in the *vfa* very similar to r- and g-curves in the

[10] Remember: $\alpha_{nd} \propto r' = g'$; so $\omega_{n,nd} \equiv \omega/r'$. If $\omega_{n,nd}$ is constant and also ω is, then α_{nd} is constant.

ffa, although their physical meaning is not as straighforward as the lossy parameterizations are. In this sense, c has to be interpreted as a "degree of dispersivity". This is deduced from how the length of the curve gaps, that are fixed by the interection between each c-curve and the ω_n curves, vary as long as c differs from unity. In the extreme case, $c = 1$, the characteristic impedance does not vary with frequency, so the corresponding TL is non disperive.

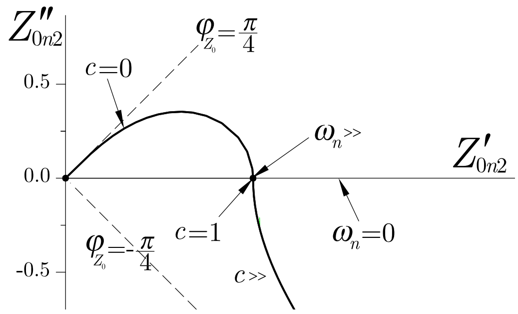

Fig. 4.12: Location of the lossless and non dispersive cases, and the low-losses and high-losses approximation in the Z_{0n2}-plane sung the parameterizations of the *vfa*.

Using the graphical analysis in Fig. 4.11 and the particular cases defined in Table 4.3 it is possible to locate those ones in the Z_{0n2}-plane. In Fig. 4.12, the Z_{0n2}-plane in which the particular cases and approximations are detached is depicted. Notice that the characteristic impedance in the non disperive case is asymptotically the same when frequency increases. In fact, this non dispersive characteristic, which is the same as the lossless one does not depend on frequency, contrary to the rest of cases.

***Practical uses*:** This analysis is equivalent to the lossy characterization of the characteristic impedance but in terms of frequency parameterizations. Thus, the same practical uses may be brought to this analysis, which adquires special importance when analyzing circuits in a freqency band.

The phase of the characteristic impedance also changes with frequency. Thus, the wave parameters at any point of the TL analyzed in Sect. 4.3.2 in terms of the this angle also change with frequency. In addition, the wave parameters at the load are analyzed in terms of frequency in Chpt. 5 as example of use of this analysis.

This "universal" analysis in terms of frequency results crucial for transforming the wave solutions from the *frequency domain* to the *time domain*. In this sense, the function $\boldsymbol{Z_{0n2}}(\omega_n, c)$ has been defined from the parameterizations in the rg-plane to the Z_{0n2}-plane, and it can be used as kernel of an integral operator or integral equation that expands Z_{0n2} in a bandwidth, that is, describing the TL by means of Z_{0n2} in *time domain*.

This analysis may be also useul for matching TLs with frequency, or for seeing how the matching at a point it the TL varies in some specific frequency bands.

Propagation constant: The analysis of the propagation constant in terms of frequency parameterizations is carried out in this part with the objective of characterizing how both the attenuation of individual waves and their phase speed behave when changing the frequency.

This frequency analysis also gives an idea of how the dispersivity affects waves in the bandwidth of operation.

***Normalization*:** When looking for a parameter that serves to normalize the propagation constant for the *vfa*, it is required for this parameter (i) to simplify the original expression of γ in eq. (2.68) to one which is able to be written in terms of the frequency parameterizations ω_n and c, besies that (ii) it does not depend on frequency. The only parameter which does not depend explicitly on frequency[11] and it is a particular existing solution (in the sense of real solution) of the equivalent Helmholtz equation is the attenuation constant in the non dispersive case, α_{nd} used in eq. (2.109)[12]. As a consequence, the normalized propagation constant is:

$$\begin{aligned}\gamma_{n2} = \frac{\gamma}{\alpha_{nd}} &= \frac{\sqrt{(R+j\omega L)(G+j\omega C)}}{R_{nd}\sqrt{\frac{C_{nd}}{L_{nd}}}} = \frac{\sqrt{(R+j\omega L)(G+j\omega C)}}{G_{nd}\sqrt{\frac{L_{nd}}{C_{nd}}}} = \\ &= j\frac{\sqrt{(\omega - jr')(\omega - jg')}}{r'_{nd}} = j\frac{\sqrt{(\omega - jr')(\omega - jg')}}{g'_{nd}} \in D_{\gamma_{n2}} \subset \mathbb{C},\end{aligned} \tag{4.27}$$

in which the subindex "$_{nd}$" in the line parameters and lossy paramterizations has been used to make explicit the difference between the parameters in the non dispersve case and those relative to the lossy case. In this case, the region of the compelx plane in which γ_{n2} expands, $D_{\gamma_{n2}}$, has to be determined by later graphical and geometrical analysis.

Using the frequency parameterizations introduced in eq. (4.19), and taking into account that the non dispersive case is with $\theta_c = \pi/4$ (c=1)[13], the expression in eq. (4.27) may be rewritten as

$$\begin{gathered}\gamma_{n2} = \alpha_{n2} + j\beta_{n2} = j\sqrt{2}\sqrt{(\omega_n - j\sin(\theta_c))(\omega_n - j\cos(\theta_c))}, \text{ in which} \\ \omega_n \in [0, \infty[\,, \text{ and } \theta_c \in \left[0, \tfrac{\pi}{2}\right].\end{gathered} \tag{4.28}$$

Just as for the characteristic impedance in this *vfa*, this latter expression is useful for separating the parameterizations ω_n and c (this latter coming from θ_c).

***Graphical analysis*:** Thanks to the expression of γ_{n2}, it is possible to separate the dependence of ω_n and c by parameterizing one of them while varying the other one. This form of proceeding lets drawing the ω_n- and c-parameterized curves in the γ_{n2}, as it is usual for graphical bi-variate analysis.

In order to be rigorous with the graphical analysis, the function $\boldsymbol{\gamma_{n2}}$ of real positive variables ω_n and c is defined as:

$$\begin{aligned}\boldsymbol{\gamma_{n2}}: \quad & \mathbb{R}^{2+} \cup (0,0) \to D_{Z_{\gamma_{n2}}} \subset \mathbb{C} \\ & (\omega_n, c) \rightsquigarrow \boldsymbol{\gamma_{n2}}(\omega_n, c) = j\sqrt{2}\sqrt{\left(\omega_n - j\cos\left(\tan^{-1}(c)\right)\right)\left(\omega_n - j\sin\left(\tan^{-1}(c)\right)\right)}.\end{aligned} \tag{4.29}$$

The function above may be seen as a mapping from the ω_n- and c-curves in the rg-plane and the γ_{n2}-complex plane. Since the curves in the rg-plane in Fig. 4.10 are orthogonal parameterizations,

[11] Since the line parameters neither depend on frequency in the case studied, referring the "explicit non dependence on frequency" is equivalently to talk about the non dependece on frequency at all.

[12] In [Gag01], the parameter which normalizes the analysis propagation contant for the *vfa* is α_{bp}. However, just as it has been defined this parameter in eq. (2.118), it parameterizes a non existing solution, just an approximation, unless this low-losses approximation is siuch that it coincides with the non-dispersive case. Only in this case, $\alpha_{bp} \equiv \alpha_{nd}$ the approximation is a real solution, and so it is valid for the normalization.

[13] This identity can be find thanks to the non dispersive case is a particular case (not a limit) in the parameterizaations posed in the rg-plane. Otherwise (if for example taking the low losses approximation), it can not be addressed rigorously from the definition of the parameterizations in the rg-plane.

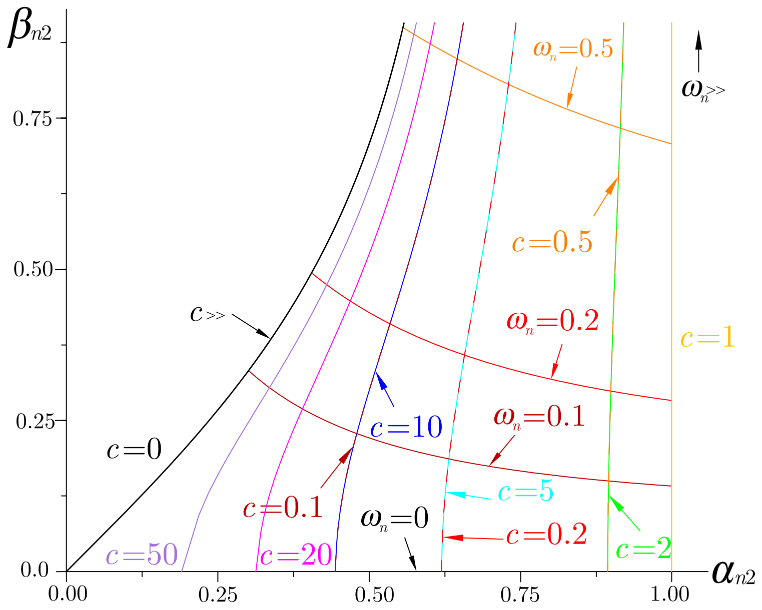

Fig. 4.13: Graphical analysis of the curves parameterizing the *vfa* in the γ_{n2}-plane.

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it is possible to transform them independently. For this purpose, some well-known properties of mappings: complex scalar transformations, shifting, product and square root; are used for obtaing the resultant curves in the γ_{n2}-plane.

In Fig. 4.13 the graphical analysis of γ_{n2} in its complex plane is represented parameterizing the curves for the *vfa*.
The curves are in the region delimited by the curves $c = 0$ $(c \gg)$ [14] on one hand, and $c = 1$ on the other hand.
Notice that the parameterizations c and $1/c$ lead to draw curves which overlap, due to the symmetric definition of the mapping in eq. (4.29).
In addition, the curves present asymptotic behavior to the line $\gamma_{n2} = 1$ as $\omega_n \gg$ (and this when $\omega \to \infty$). On the contrary, each c-curve starts in the line $\beta_{n2} = 0$ with different α_{n2} when $\omega_n = 0$ (and thus when $\omega = 0$), being α_{n2} always lower than the attenuation when $c = 1$.

Geometrical analysis: The general equations of the curves in the γ_{n2} for the *vfa* are here obtained.

The c-curves follow the general equation:

$$\begin{gathered}\left(\alpha_{n2}^2 + \beta_{n2}^2\right)^2 = \\ = 4\frac{\alpha_{n2}^2\beta_{n2}^2}{(\cos(\varphi_c)+\sin(\varphi_c))^2}\left[1 + \frac{\alpha_{n2}^2\beta_{n2}^2}{(\cos(\varphi_c)+\sin(\varphi_c))^2}\right] + 4\cos^2(\varphi_c)\sin^2(\varphi_c), \text{ in which} \\ \varphi_c = \tan^{-1}\left(\frac{1}{c}\right).\end{gathered} \tag{4.30}$$

This equation has been obtained in eq. (4.D.31) in Appendix 4.D. It describes a set of (unknown, in the sense that they are not classified) quartic curves in the γ_{n2}-plane, for which only specific

[14] The same notation with brackets is used here when the parameterizations regarding γ_{n2} are equivalent and so the curves overlap.

values of c correspond to classified curves. For example, when $c = 0$ ($c \gg$), the resultant curve limit is the so called Bullet nose, [Law72]; or when $c = 1$, the curve is the constant line parameterized by $\alpha_{n2} = 1$.
The equation before lets to solve the intersection of the c-curves with the real axis, that is the value of α_{n2} when $\omega = 0$, denoted by $\alpha_{n2,min} = \sqrt{2\cos(\varphi_c)\sin(\varphi_c)}$.

The ω_n curves follow the general equation:

$$\alpha_{n2}^2 - \beta_{n2}^2 = \frac{1}{\omega_n^2}\left(\alpha_{n2}^2\beta_{n2}^2\right) - 2\omega_n^2 - 1, \tag{4.31}$$

which has been obtained in eq. (4.D.32) in Appendix 4.D. This equation describes a set of quartic curves (also unkwnown) in the γ_{n2}-plane.
This lets to obtain obtain the intersections with any c-curve. For example, the curve parameterized by $c = 1$ ($\alpha_{n2} = 1$) is intersected in the minimum of β_{n2}, denoted by $\beta_{n2,min} = \sqrt{2}\omega_n$, so proportional to ω_n.

***Physical interpretations*:** Both the graphical analysis and the subsequent geometrical characterization concerning the study of the propagation constant in terms of frequency parameters leave some interesting and applicable physical interpretations.

The normalization γ_{n2} for the *ffa* has to be understood relative to the non dispersive attenuation constant:

On one hand, the normalized attenuation constant, α_{n2}, is the dimensionless quotient between the attenuation of lossy TLs and the associated non dispesive case (the cases that equalizes the lossy parameterizations). Also notice that the non dispersive case coincides with the low-losses approximation when this latter is an existing (real) solution in the equivalent TL. Thus, α_{n2} is a particular solution of the low-losses approximation ($\alpha_{n3} = \alpha/\alpha_{bp}$, used in [Gag01].
If looking at the graphical analysis in Fig. 4.13, it is noticeable that the attenuation constant for each curve parametrized by c is $\alpha_{n2} \geq \alpha_{n2,min}$ and $\alpha_{n2} < 1$, achieved when $\omega = 0$ and $\omega \gg$, respectively. This means that the attenuation constant is smaller than the one regarding the non dispersive case, and also the low-losses approximation, $\omega_n \gg$. In fact, this latter case corresponds to the asymptotic behavior of the propagation constant for each c-curve.
In addition, the non dispersive case corresponding to $c = 1$ is the only curve which presents regular ω_n-frequency scales, just as its denomination suggests.

On the other hand, the normalized propagation constant, β_{n2}, has to be understood as a relative ratio with respect to the ratio (angle) of the non dispersive case:

$$\beta_{n2} = \frac{\beta}{\alpha_{n2}} = \frac{\beta}{\alpha_2}\frac{\beta_{nd}}{\beta_{nd}} = \frac{\beta}{\beta_{nd}}\frac{\beta_{nd}}{\alpha_{nd}} = \frac{\beta}{\beta_{nd}}\tan\left(\varphi_{\gamma_{nd}}\right). \tag{4.32}$$

This parameter measures the difference of the phase constant between the non dispersive case and the lossy case.
Also notice that when $\omega_n \gg$, $\beta_{n2} \to \infty$, which suggests that the dipersivity increases wtith frequency.

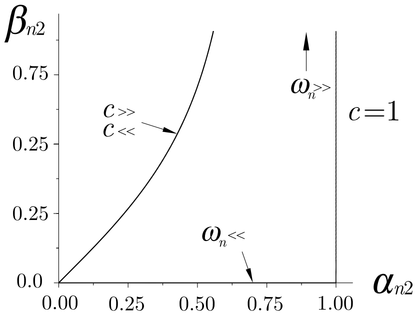

Fig. 4.14: Location of particular cases and approximation regarding the *vfa* of the propagation constant in its normalized complex plane.

The points corresponding to the particular cases and approximations are detached in Fig. 4.14. Notice that $\beta_{n2} \to 0$ when $\omega_n \to 0$ ($\omega \to 0$), which corresponds with the high-losses cases. This behavior does not mean that the waves are nor propagative, but the attenuation per wavelength is much more greater than the distance the wave moves. In fact, only in DC regime ($\omega_n \equiv \omega = 0$) the waves do not propagate, independently of losses are.
The low-losses approximation ($\omega_n \gg$) only coincides with the non dispersive case when $c = 1$ in the limit.

***Practical uses*:** The usefulness of this analysis is on describing how the physical properties of the individual waves vary with frequency.
Conversely, this analysis may be useful for determining the characteristics of the wave in a specific bandwidth.

In addition, this analysis is useful for analyzing the behavior of wave parameters along the TL in terms of frequency, which supposes, in turn, the complete characterization of lossy TLs for designing circuits using both frequency and length as variables.
For this purpose, notice that the angle of the propagation constant may be directly measured from the γ_{n2}-plane, which is the variable to be used to characterize the wave paramters along the TL, as it is used in one of the examples presented in Chpt. 5.

In addition, notice that, when expanding any voltage or current wave solution in frequency to obtain the wave in time domain, for example by means of eq. (4.17), the analysis of γ_{n2} in the *vfa* governs the integral equation. For example,

$$\begin{aligned} v^+(z,t) =& \frac{1}{2}\mathrm{Re}\left\{\int_\omega V^+(\omega)e^{-\gamma(\omega)z}e^{j\omega t}d\omega\right\} \equiv \\ \equiv& \frac{1}{2}\mathrm{Re}\left\{\int_{\omega_n=0}^{\infty} V^+(\omega_n)e^{-\gamma_{n2}(\omega_n;c)(\alpha_{nd}z)}e^{j\omega t}(d\omega_n/\left(\sqrt{r^2+g'^2}\right)\right\}, \end{aligned} \tag{4.33}$$

in which r' and g' are fixed, so also both α_{nd} and c are, and c parameterizes the propagation normalizes constant γ_{n2}.
This example shows the capabilities of the present analysis to analyze, for example, different circuits in a frequency band. This useulness is detached among the **Applications**.

4.3.2 Direct characterization of wave parameters

The direct characterization of wave parameters refers to the analysis in the *frequency domain* parameterizing them at any point of the TL.

The wave parameters have been introduced in Sect. 2.3 in Chpt. 2 to define the relations of the total voltage and current waves which propagate in the equivalent TL. In particular, the lossy case in which the line parameters do not depend on frequency is presented on this version of the CTLT (CTLT-v1.0), which corresponds to the particular analysis introduced in Sect. 2.4 in Chpt. 2.
In particular, the analysis to be presented here will be focused on describing the wave parameters when specific conditions on thenselves and the basic parameters are parameterized. Essentialy, the direct characterization regarding each wave parameter supposes analyzing it parameterizing another, besides a parameterization regarding the basic parameters.

Remark 17. *The direct characterization of wave parameters lies in double parameterized analysis: one parameterization is "exterior" and real, based on the basic parameters; and the other one is an "inner" parameterization and complex (which is, in practise, bi-real), based on the wave parameters.*

Physically, this parameterized analysis is equivalent to fix, on one hand, the BCs at any point of the and, on the other hand, the losses and frequency of the TL to characterize the wave parameters.

The direct characterization of wave parameters supposes a considerable increase on the complexity of the analysis. Firstly because the number of parameterizations increases: here not only the losses of the TL are parameterized by also the BCs are taken into account; but also because the number of the parameters to be studied is consequently greater.
This means that, if the analysis are required to be basic transformations[15], some degree os precision in the analyis is assumed to be missed, which supposes reducing the complexity but increasing the difficulty in the understanding the of both their analytical and physical interpretation.

Remark 18. *When increasing the number of parameters involved for a specific analysis, its direct interpretations are hidden. Namely, the physical interpretarions of the analysis in question are not as obvious as in the analysis that parameterizes the physical parameters directly.*
This is inevitable in the analysis of wave parameters: if both the parameterizations of losses and the BCs are addressed at the same time, then the parameterizations used in the analysis do not deal specifically with them. As a result, the analysis become a priori "more analytical than physical".

It is a priori remarkable the usefulness of the analysis to be presented in this section in both designing and understanding circuits based on TLs. The graphical analysis have led to useful graphical tools, as for example the *Generalized Smith Chart* (GSC) presented in [GDG06], among others explained in [Gag01].

This analysis of wave parameters is especially helpful when analyzing TLs in which the losses and frequency are fixed, for example if the wave parameters are required to be described along the TL. This analysis along the TL is presented as example of use of the direct characterization of wave parameters in Ex. 01 in Sect. 5.2 in Chpt. 5. Nevertheless, this latter analysis gives a partial view of the total wave solutions since both the losses and the frequency are fixed to the TL under study. Its generalization (and thus, because of the reason remarked above, the analysis would miss the physical interpretation of the study in terms of either losses/frequency or the TL's length) presented in Ex. 03 in Sect. 5.4 in Chpt. 5 completes this intended analysis.

Keeping the purposes of the direct charactrization of wave parameters in mind, the analysis is presented following the steps concerning the CTLT: the normalizations are obtained from the original expressions of the wave parameters when taking into account the fixed parameterizations regarding the basic and wave parameters to be used (the sheme in Fig. 4.1 lets to see which parameterzations concern this analysis); the parameterized graphical analysis of the involved normalized basic parameters. As mentioned above, the greater complexity of this analysis supposes increasing both the number of parameterizations and the parameters to take into account. Then the number of mappings between complex planes becomes greater; the geometrical analysis of the

[15]Remember that basic transformations are those involving two planes, in order to see them as mappings.

resultant curves; the physical interpretations of the analysis; and the possible practical uses of them.

Recall that the graphical analysis are thoroughly described in the *CTLA Handook*, [Gag01]. Thus, only an example of graphical analysis is here presented. Nevertheless, this example serves to reference the use of parameterizations and normalizations, as well as obtaining the possible physical interpretations regarding both the graphical and geometrical analysis. Moreover, the practical uses of this analysis are detached at the end of the section, especially its use for analyzing some of the examples presented in Chpt. 5.
In any case, a very similar analysis concerning the inverse characterization of basic parameters is presented in detail in Sect. 4.4.2, which serves to see how the graphical analysis in this section should be completed (just as it is done in [Gag01]).

***Parameterizations*:** Recalling the original expressions of the wave parameters regarding the lossy case presented in 2.4.1 in Chpt. 2: the reflection coefficient; the wave impedance, and the wave admittance, defined either from the generator in eqs. (2.80), (2.78), and (2.78), respectively, or from the load the load in eqs. (2.87), (2.85), and (2.85), also respectively (both definitions follow the same form if considering any point of the TL); which are expressed as transformations between themselves if the characteristic impedance, Z_0, is parameterized.
The following normalizations are defined:

$$Z_{0n} = \frac{Z_0}{|Z_0|} = e^{\varphi_{Z_0}} \equiv c_0 + js_0, \text{ in which} \tag{4.34}$$

$$c_0 = \cos(\varphi_{Z_0}), \text{ and } s_0 = \sin(\varphi_{Z_0}),$$

$$Z_{n0} = \frac{Z}{|Z_0|} = Z'_{n0} + jZ''_{n0} = |Z_{n0}|e^{j\varphi_Z}, \tag{4.35}$$

and Z generically describes the wave impedance at any point of the TL, that is either $Z(z)$ or $Z(l)$. Notice the notation difference in eqs. 4.34 and 4.35, which is explained in Appendix 4.E (also outlined at the beginning of the Thesis book).
Moreover, to be coherent with the definitions of the rest of wave parameters: Y and ρ; their normalizations are:

$$Y_{n0} = Y|Z_0| = Y'_{n0} + jY''_{n0} = |Y_{n0}|e^{j\varphi_Y}, \tag{4.36}$$

$$\rho_{n0} \equiv \rho, \tag{4.37}$$

respectively.

Remark 19. *Since in the analysis of the wave parameters the parameterized BCs are parameterizations of the wave parameters in themselves, each normalization to be described is exactly the same as the parameterization used for describing the rest of wave parameters. As a consequence, once the normalization is chosen, it forzes the normalization for the rest of wave parameters.*

Some properties concerning these normalizations have to be detached: (i) the normalized reflection coefficient follows the same expression as the original one, ρ. This fact reveals the "universal" nature of the reflection coefficient, which goes beyond the normalization. This, in turn, supposes that ρ is the parameter which describes future analysis which combine this direct characterization with the inverse characterization introduced in Sect. 4.4.2; (ii) since the modulus of the characteristic impedance, $|Z_0|$, depends on losses and frequency (see the analysis in the section before), their respective explicit parameterizations are missed in this analsysis. As a result, only the angle of the characteristic impedance, which explicitly appears in eq. (4.34), inherits the dependence on losses; and (iii) since the normalizations are done with respect to $|Z_0|$, the angles of wave parameters are the same as their denormalized versions. This fact is also useful when combining characterizations, and thus for using these characterizations with practical-design purposes.

Notice that the normalized wave parameters are complex. If any of these normalizations have to act as parameterizations, either both the real and imaginary parts, or both the modulus and phase are required to be parameterized for the complete parameterization of the analysis. Thus, it is about parameterizing the a priori complex parameterizations by splitting the real and imginary parts, or the modulus and phase.
Nevertheless, there is a way to relate the bi-real parameterizations in only one complex parameterization: an specific (physical) analysis. For example, the analyis along the TL, which is developed in Ex. 03 in Sect. 5.4 in Chpt. 5, relates the modulus and phase of the complex parameterizations of the reflection coefficient, leading to one (complex) parameterization.

Remark 20. *The nature of the parameterizations of wave parameters is complex. In order to address them, they are separated by its real and imaginary parts, or modulus and phase. Nevertheless, each specific analysis in the TL, for example the analysis of wave parameters along the TL, leads to assemble them in only one (complex) parameterization. Thus, it is the physical interpretation of the problem under study which provides sense to the complex parameterizations at the same time that it reduces the number of parameterizations.*

***Normalizations*:** The normalizations regarding the wave parameters are directly defined by taking into account the definition of the parameterizations, because they may be seen as transformations between themselves, apart from the parameterization of losses included in the angle of the characteristic impedance, φ_{Z_0}. As a result, the analysis are addressed from the parameterizations in the complex plane of each normallized wave parameter, [Gag01].Thus:

(i) From the parameterizations of the normalized wave impedance, Z_{n0} in eq. (4.35), the normalized wave parameters to be analyzed are the normalized wave admittance,

$$Y_{n0} = \frac{1}{Z_{n0}}, \tag{4.38}$$

and the (normalized) reflection coefficient,

$$\rho = \frac{Z_{n0} - Z_{0n}}{Z_{n0} + Z_{0n}} = \frac{Z_{n0} - e^{j\varphi_{Z_0}}}{Z_{n0} + e^{j\varphi_{Z_0}}}, \tag{4.39}$$

as defined in (4.36) and eq. (4.37), respectively.

(ii) From the parameterizations of the normalized wave admittance, Y_{n0} in eq. (4.36), the normalized wave parameters to be analyzed are the normalized wave impedance,

$$Z_{n0} = \frac{1}{Y_{n0}}, \tag{4.40}$$

and the (normalized) reflection coefficient,

$$\rho = \frac{1 - Y_{n0}Z_{0n}}{1 + Y_{n0}Z_{0n}} = \frac{1 - Y_{n0}e^{j\varphi_{Z_0}}}{1 + Y_{n0}e^{j\varphi_{Z_0}}}, \tag{4.41}$$

as defined in (4.35) and eq. (4.37), respectively.

(iii) From the parameterizations of the (normalized) reflection coefficient, ρ in eq. (4.37), the normalized wave parameters to be analyzed are the normalized wave impedance,

$$Z_{n0} = Z_{0n}\frac{1+\rho}{1-\rho} = e^{j\varphi_{Z_0}}\frac{1+\rho}{1-\rho}, \tag{4.42}$$

and the normalized wave admittance,

$$Y_{n0} = \frac{1}{Z_{0n}}\frac{1-\rho}{1+\rho} = e^{-j\varphi_{Z_0}}\frac{1-\rho}{1+\rho}, \tag{4.43}$$

as defined in (4.35) and eq. (4.36), respectively.

These definitions directly define complex mappings, which may be graphically and geometrically analyzed.

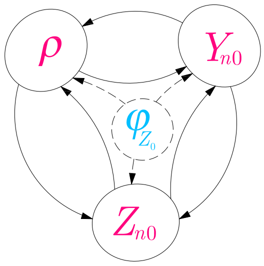

Fig. 4.15: Scheme of transformations between the normalized wave parameters regarding the direct characterization of the CTLT. The continuous arrows indicates the transformations between the normalized wave parameters (complex parameterized), whereas the dashed arrows indicate the extra "partial" (in the sense that it has no direct physical meaning) parameterization φ_{Z_0}, inherited from the basic parameter analysis.

***Graphical analysis*:** Eqs. (4.40)-(4.43) define the complex transformations between the normalized wave parameters including the extra parameterization of the phase of the characteristic impedance, φ_{Z_0} (see the scheme in Fig. 4.15). These transformations may be seen as complex mappings or complex variable functions fitting the generic form

$$T: \quad \begin{array}{c} D_{w_p} \subset \mathbb{C} \rightarrow D_{W_p} \subset \mathbb{C} \\ w_p \rightsquigarrow T(w_p) = W_p \end{array}. \tag{4.44}$$

The function $T_{\varphi_{Z_0}}(\circ)$ designates any of the transformations in eqs. (4.40)-(4.43), while w_p designates the normalized wave parameter which is parameterized, and W_p is the parameter to be characterized by the transformation $T_{\varphi_{Z_0}}(\circ)$. The transformation $T_{\varphi_{Z_0}}$ are denoted as the characterized parameter.

Example 4.3.1. *The transformation from the Z_{n0}-plane to the ρ-plane is given by the expression in eq.* (4.42)*. In this case, the generic notation in eq.* (4.45) *reduces to:*

$$\rho_{\varphi_{Z_0}}: \quad \begin{array}{c} D_{Z_{n0}} \subset \mathbb{C} \rightarrow D_{\rho} \subset \mathbb{C} \\ Z_{n0} \rightsquigarrow \rho\,(Z_{n0}]) = \rho \end{array}. \tag{4.45}$$

Notice that every mapping defined as in the example before is (extra) parameterized by the phase of the characteristic impedance, so $T_{\varphi_{Z_0}}(\circ)$ describes a set of complex functions[16].

For depicting the transformations defined in eq. (4.45), two different useful types of parameterizations in the domain of the transformation are defined: the paramaterized real-imaginary parts (a-b), and the modulus-phase parameterizations (m-p)[17]; are employed to define the complex parameterization. These parameterizations really helps drawing the curves in the complex plane of the parameter under study, especially the modulus-phase parameterizations, because the importance of angles in complex transformations, which go beyond the normalizations considered for the analysis.

[16] Here $T_{\varphi_{Z_0}}(\circ)$ is considered a set of complex functions of complex variable –parameterized by φ_{Z_0}– because these functions operates with complex values at any fixed point of the TL. However, if the wave parameters are written as functions of a physical parameter, for example functions of the coordinate describing the TLs length, the set $T_{\varphi_{Z_0}}[\circ]$ referes to complex operators –parameterized by φ_{Z_0}. Notice the slight difference in the notation in this latter case using square brackets.

[17] The hypenation ("-") is again used in the notation here to indicate the bi-real parameterizations which go together to define a complex parameterization.

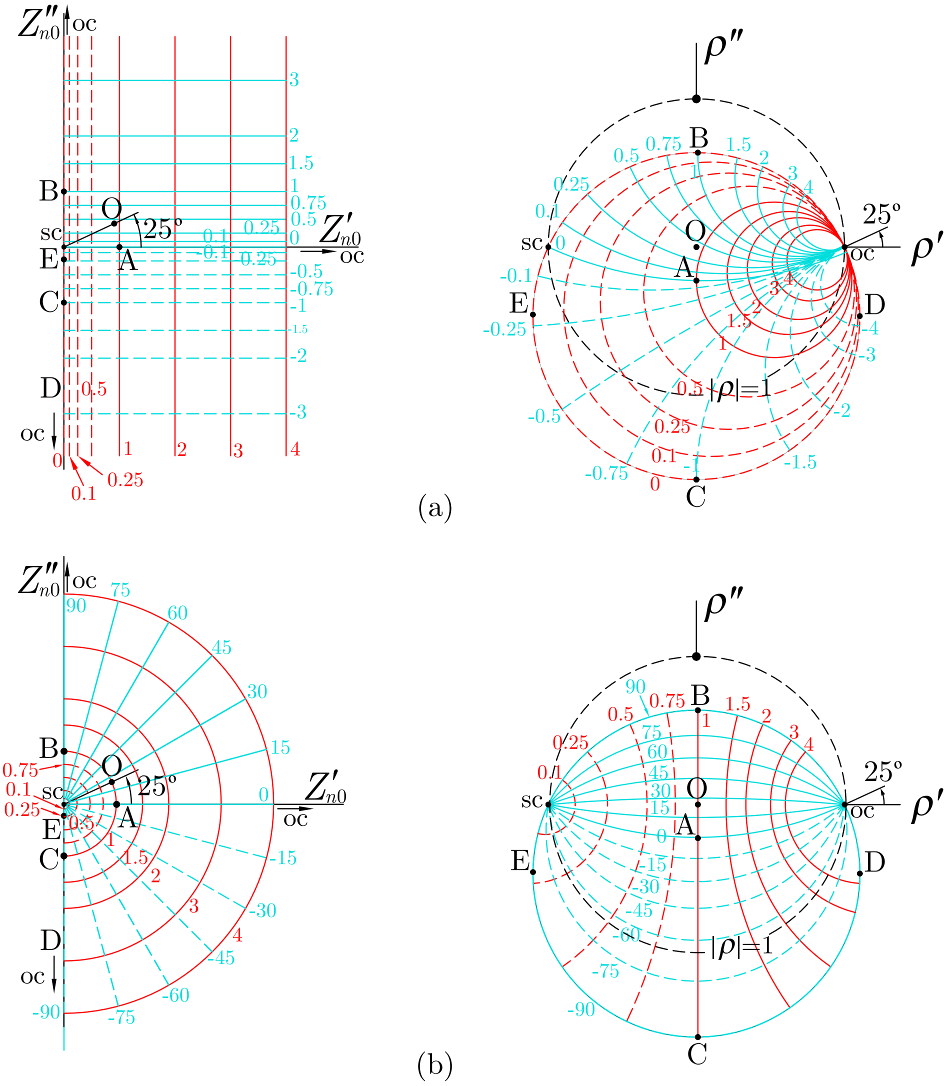

Fig. 4.16: Complex transformations from the Z_{n0}-plane parameterizing (a) its real-imaginary parts, and (b) its modulus-phase, to the ρ-plane, when $\varphi_{Z_0} = 25°$.

Example 4.3.1 (cont.). *The real-imaginay parameterized parts in the Z_{n0}-plane are denoted by*

$$\begin{cases} Z'_{n0} = a \\ Z''_{n0} = b \end{cases} ,$$

whereas the modulus-phase parameterized parts in the Z_{n0}-plane are denoted by

$$\begin{cases} |Z_{n0}| = m \\ \varphi_{Z_{n0}} \equiv \varphi_Z = p \end{cases} ,$$

to be transformed to the ρ-plane.

Notice that each part of the parameterizations may be studied independently to be geometrically characterized as a curve in the range of the parameter under study.
Moreover, for being the transformations completely characterized, it is required to specify the

domains of each parameter. Since the parameters are mutually connected (see Fig. 4.15) the definition of the domain of one of them directly defines the range of the transformation, which is, in turn, the domain of the inverse transformation. Taking this fact into account, the first definition is in the domain of Z_{n0}, whose real part is $Z'_{n0} \geq 0$ (as it is depicted in Fig. 4.16). Thus, the domain in the Y_{n0}- and ρ-planes is bounded by the transformation of the curve $Z'_{n0} = a = 0$ into these planes.

By using properties of complex transformations: complex scalar products, shiftings, and quotients; point by point, it is possible to draw the resultant curves.
In Fig. 4.16, an example of graphical analysis of the transformation introduced in Example 4.3.1 when $\varphi_{Z_0} = 25°$ is represented (in which some remarkable points and particular cases are located to be then analyzed). Specifically, the ρ-plane represented in Fig. 4.16 is the GSC, [GDG06], particularized to $\varphi_{Z_0} = 25°$. This example of graphical analysis may be found (completed) in [VG16-I], and a complete graphical analysis of all the possible transformations in eqs. (4.40)-(4.43) is documented in [Gag01]. These graphical analysis are similar to those relative to the inverse characterization of basic parameters, presented in Sect. 4.4.2 in detail.

The graphical analysis helps the geometrical characterization of the curves of the transformation in the task of solving some specific values of the parameter under study and, conversely, which parameterizations lead to those interesting values.

***Geometrical analysis*:** The complex transformations in eqs. (4.40)-(4.43) are geometrically characterized in order to: (i) facilitate their representation; (ii) find alternative expressions to the complex functions, which lead to locate some remarkable points or interesting particular cases in the involved complex planes; and (iii) obtain physical interpretations and practical uses of this analysis, some of them "hidden" in the underlying original equations or complex transformations.

Let's analyze the "conformability" (that is the properties of a map to be conformal, [BC90]) of the transformations from each parameterized plane, separately. For this purpose, it is firstly required to analyze the regions in which each plane is defined:
Because of its physical meaning, the Z_{n0}-plane includes those $Z_{n0} \in \bar{\mathbb{C}} \backslash Z'_{n0} \geq 0$[18]. From this definition and taking into account the transformation of the parameterizations from the Z_{n0}-plane to the Y_{n0}-plane in eq. (4.40), the Y_{n0}-plane takes up the same complex region. On its behalf, the ρ-plane is bounded by the transformation of the limits in the Z_{n0}-plane by using the expression in eq. (4.42). As it may be seen in the Fig. 4.16, this region is within the circumference which passes through $\rho = 1 + j0$ and $\rho = -1 + j0$.
Now, it may be said that:

(i) From the parameterizations in the Z_{n0}-plane, the transformation to the Y_{n0}-plane in eq. (4.40) is conformal in the Z_{n0}-plane except for $Z_{n0} = \infty$ (because $1/0$ exists in the Y_{n0}-plane but the derivarive of the transformation is null in $Z_{n0} = \infty$, which is a critical point); and the transformation to the ρ-plane in eq. (4.42) is also conformal in the Z_{n0}-plane except for $Z_{n0} = \infty$ (because the denominator of the transformation can not be null in the domain of Z_{n0}, but the derivarive is null in $Z_{n0} = \infty$).

(ii) Similarly, from the parameterizations in the Y_{n0}-plane, the transformations in eqs. (4.38) and (4.43) to the Z_{n0}- and the ρ-planes , respectively, are also conformal except for $Y_{n0} = \infty$.

(iii) From the parameterizations in the ρ-plane, the transformations in eqs. (4.39) and (4.3.2) are conformal in this plane.

[18] $\bar{\mathbb{C}}$ denotes the extended complex plane: $\bar{\mathbb{C}} \equiv \mathbb{C} \cup \infty$, [BC90]

From this analysis, it may be concluded that the transformations in eqs. (4.40)-(4.43) are, in practice, conformal, because the measure of the angles can not be done at infinity.
In fact, these transformations are of the type of Möbius transformations, [Apo90], which map lines and circumferences into circumferences. These circumferences are completely dermined by geometrically locating their center and radious.

Example 4.3.1 (cont.). *The transformations from the real (a)-imaginary(b) parameterized parts in the Z_{n0}-plane to the ρ-plane are described by the expressions, [Gag01]:*

$$\begin{cases} \left(\rho' - \frac{a}{a+c_0}\right)^2 + \left(\rho'' + \frac{s_0}{a+c_0}\right)^2 = \left(\frac{1}{a+c_0}\right)^2 \\ \left(\rho' - \frac{b}{b+s_0}\right)^2 + \left(\rho'' - \frac{c_0}{b+s_0}\right)^2 = \left(\frac{1}{b+s_0}\right)^2 \end{cases}.$$

These are the equations of circumferences with centers in $\rho_a = c'_a + jc''_a = a/(a+c_0) - js_0/(a+c0)$ and $\rho_b = c'_b + jc''_b = b/(b+s_0) + jc_0/(b+s_0)$, respectively, and radious $r_a = 1/(a+c_0)$ and $r_b = 1/|b+s_0|$, also respectively.
The expression of these curves can be condensed using the following notation

$$(c', c'') : r,$$

so the circunferneces in the ρ-plane obtained from the transformations of the real-imaginary parts in the Z_{n0}-plane are condensely denoted as:

$$\begin{cases} \left(\frac{a}{a+c_0}, -\frac{s_0}{a+c_0}\right) : \frac{1}{a+c_0} \\ \left(\frac{b}{b+s_0}, \frac{c_0}{b+s_0}\right) : \frac{1}{|b+s_0|} \end{cases}.$$

The transformations from the modulus (m)-phase(p) parameterizations in the Z_{n0}-plane to the ρ-plane are condensely described by, [Gag01]:

$$\begin{cases} \left(\frac{m^2+1}{m^2-1}, 0\right) : \frac{2m}{|m^2-1|} \\ \left(0, -\frac{1}{\tan\left(p-\varphi_{Z_0}\right)}\right) : \frac{1}{\left|\sin\left(p-\varphi_{Z_0}\right)\right|} \end{cases}.$$

The continuation of Example 4.3.1 above shows the simplicity of the resultant curves. The mathematical analyis id shown in [Gag01], together with the geometrical characterization of the rest of possible transformations. A similar analysis to that one followed in [Gag01] is developed in the inverse characterization of basic parameters, presented in Sect. 4.4.2.

Both the graphical and geometrical analysis let to: (i) find some remarkable points as the ones detached in Fig. 4.16 labeled with **A**-**D**, and **O**. The location on these points only depends on the parameterization of the angle φ_{Z_0}. For example, the point **A** is in the intersection between the curves parameterized by $Z'_{n0} = a = 1$ and $Z''_{n0} = b = 0$, or equivalentely, $|Z_{n0}| = m = 1$ and $\varphi_Z = p = 0$. Taking for example the parameterizations *m*-*p*, the curve parameterized by $m = 1$ is the imaginary axis in the ρ-plane, while the *p*-curve is there represented by the circumference parameterized by $(0, c_0/s_0) : 1/|s_0|$, which is centered in the same axis. Thus, **A** is $1/|s_0|$ separated from the center of this circunference, on the imaginary axis, so $\rho_{\mathbf{A}} = 0 + j(c_0 - 1)/s_0$; and (ii) characterize some regions or domains regarding the involved complex plane. For example, as said before, the domain of ρ is bounded by the transformation of the imaginary axis from the Z_{n0}-plane (or the Y_{n0}-plane, because there the imaginary axis is mapped in the same region –although it is reversed), so the curve characterized by $\varphi_Z = 90°$. As a result, the boundary curve which limits the domain of the ρ-plane follow the form $(0, -s_0/c_0) : 1/c_0 \equiv (0, -\tan(\varphi_{Z_0})) : 1/\cos(\varphi_{Z_0})$.
The complete analysis of the location of the remarkable points for each transformation and complex plane is detailed in [Gag01].
In particular, notice that the ρ-plane supposes a contraction with respect to the Z_{n0}- or Y_{n0}-planes,

and for this reason a useful graphical tool, or GSC. Notice that this φ_{Z_0}-parameterized chart is, in turn, topologically the same as the original *Smith Chart* (SC), which has first introduced in [Smi39, Smi44].

***Physical interpretations*:** The direct characterization of wave parameters should be interpreted as the way for determining one specific wave parameter at a specific point in the TL when both the characteristic impedance is fixed and any other wave parameter, which sets the BCs at this point, is known. In these sense, this characterization uses both conditions –the characteristic impedance and another wave parameter– to "universalize" the values of the wave parameter to be characterized. For example, as it has been seen by means of the graphical analysis in Fig. 4.16b, the reflection coefficient is solved in a specific point of the TL if the phase of the characteristic impedance, the phase of the impedance at this point and the ratio between the modulus of the wave impedance and the modulusof the characteristic impedance, $|Z_{n0}|$, are known[19].

Keeping these ideas in mind, the real value of the graphical analysis presented above is in representing all the possible values of the normalized wave parameters in only one complex plane. For this purpose, the influence of losses/frequency is reduced to the parameterization of φ_{Z_0}. In this way, if the losses/frequency parameterizations are known, the basic parameters are also known by the analysis in Sect. 4.3.1, and thus φ_{Z_0} is known, which completely determines the transformations between the wave parameters. This is also possible because the process of normalizing the wave parameter complex planes leaves the planes "isomorphic" (the normalized wave impedance and wave admittance planes takes up the same region as those regarding the original parameters, so the curves preserve the form; and ρ is the same as in its normalized version).

Remark 21. *The angle of the characteristic impedance, φ_{Z_0}, which parameterizes both the graphical (including the set of functions which define the transformatins) and the geometrical analysis, determines the influence of losses in the (direct) characterization of wave parameters. In this way, this parameter becomes the really useful parameterization of losses for the analysis of wave parameters, and thus the total waves in the equivalent TL. For this reason, its inverse analysis (presented as the inverse characterization of line parameters): which losses correspond to φ_{Z_0}; is of special interest when looking for the physical meaning of the analysis.*

Some particular cases especially important in the analysis of wave parameters are the short circuit, denoted as **sc**, the open circuit, denoted as **oc**, and the matching load, denoted as **O**, because they are located at fixed points in some of the wave parameter complex planes.
In Fig. 4.16, these points are detached. In general: the **sc** is fixed at $Z_{n0,\mathbf{sc}} = 0 + j0$ in the Z_{n0}-plane, and at $\rho_{\mathbf{sc}} = -1 + j0$ in the ρ-plane (at infinity in the Y_{n0}-plane); the **oc** is fixed at $Y_{n0,\mathbf{oc}} = 0+j0$ in the Y_{n0}-plane, and at $\rho_{\mathbf{oc}} = 1+j0$ in the ρ-plane (at infinity in the Z_{n0}-plane); the **O** point is at $\rho_{\mathbf{O}} = 0+j0$ in the ρ-plane, at $Z_{n0,\mathbf{O}} = e^{j\varphi_{Z_0}}$ in the Z_{n0}-plane, and at $Y_{n0,\mathbf{O}} = e^{-j\varphi_{Z_0}}$ in the Y_{n0}-plane.
Moreover, the point labeled with **A** represents the same load as the lossless and non dispersive case, and also the low-losses approximation. This point is fixed at $Z_{n0,\mathbf{A}} = 1+j0$ in the Z_{n0}-plane, and at $Y_{n0,\mathbf{A}} = 1 + j0$ in the Y_{n0}-plane, whereas it is at $\rho_{\mathbf{A}} = 0 + j(c_0 - 1)/s_0$ in the ρ-plane (as mentioned before among the geometrical analysis).

These facts reveal different meanings/usefulnesses of the direct characterization regarding the wave parameter complex planes in locating points: the true physical interpretation of wave parameters is in both the Z_{n0}- and Y_{n0}-planes, because the points with true physical meaning, for example **sc** or **oc**, and the points which represent particular cases of the TL, for example **A**, are fixed. On the

[19] This example clarifies the idea of "universalizing the analysis". The immediate consequence of "universalizing the analysis" is in reducing the number of parameters that should be known (in the example described above the ratio between the modulus –but not both the modulus of the wave impedance and the characteristic impedance– should be known). This means that each point in the resultant curves regarding the graphical analysis does not represent one specific scenaio but infinite problems with different parameters.

other hand, the ρ-plane is useful for practical-design purposes, because points with practical uses, for example the matching load, **O**, are fixed.

Remark 22. *The true physical meaning of wave parameters is in the wave impedance/ admittance, whereas the relection coefficient is a "mathematical tool" which connects the basic parameters and wave parameters. This statement is based on (i) the "universal" definition of the particular cases regarding TLs as fixed points in the wave impedance/admittance comple planes, in contrast to the their variable location in the ρ-plane; and (ii) the dimensional nature of the wave impedance/admittance, in contrast to ρ, which is dimensionless.*

***Practical uses*:** The direct characterization of wave parameters presents many interesting practical uses when decribing these parameters when losses/frequency are fixed by means of the phase of the characteristic impedance, φ_{Z_0}.
When φ_{Z_0} is fixed, the transformations between the wave parameters complex planes are determined by the graphical analysis, for example the graphical analysis in Fig. 4.16, for which $\varphi_{Z_0} = 25°$. In this context, the changes on the load affect the reflected wave, and so the matching measured by the reflection coefficent, which can be analyzed by using the GSC, for example the one represented in Fig. 4.16a.
On the other hand, the changes on the reflection coefficient produce variation on the wave impedance/admittance, which can be analyzed by using the inverse transformations from the ρ-plane to the Z_{n0}-plane, [Gag01]. A clear and very important example of this latter analysis happens when analyzing the wave parameters along the TL. For example, using the expression of ρ parameterized by the length from the load introduced in eq. (2.83) in Chpt. 2, its complex parameterizations from the ρ-plane leads to graphically analyze the wave impedance/admittance in their respective complex planes. This analysis, which is presented as example of use in Ex. 01 in Sect. 5.2 in Chpt. 5, supposes extending the analysis of the wave parameters along the whole TL, and thus characterize graphically the total waves as functions (signals) of the length, having an alternative to the analysis of the original expressions.

However, this characterization is not efficient for analyzing changes on the characteristic impedance, because it supposes denormalizing the wave parameters for each change. Nevertheless, it may be useul for verifying how change some particular cases when varying the basic parameters, for example the matching point, **O**.

4.4 Inverse Characterization of the Complex Transmission Line Theory (CTLT-v1.0b)

In this section the inverse characterization of the TL related to the LTLT presented in Chpt. 3 is studied as the alternative to the direct characterization of the CTLT while detaching its advantages, in particular those concerning its physical interpreation and also, more specifically, its practical uses.
This analysis is easily applicable for characterizing several solutions in waveguides –not only HPWs– when taking different particularizations of the GTLT introduced in Chpt. 3. Nevertheless, the analysis presented in this section refers to the characterization of HPWs, in order to be compared with the direct characterization of HPW presented before, also acting together to solve some TL-related problems presented as examples of use in Chpt. 5.

The inverse characterization of the CTLT presents opposite parameterizations and analysis when comparing it with the direct characterization[20].
On one hand, instead of describing the basic parameters in terms of different type of parameterizations, here the line parameters corresponding to these parameterizations are studied in terms of basic parameters, leading to important physical interpretations and practical uses of this type of characterization.
On the other hand, instead of transforming the wave parameters between themselves (parameterized by the angle of the characteristic impedance), the transformations between the characteristic impedance and the reflection coeffient (this latter parameter should be considered as basic parameter from the inverse characterization point of view) are studied (parameterized by the angle of the wave impedance).
This fact, far from being a disadvantage, makes this analysis to be "the perfect complement" of the direct characterization, which can be demostrated by analyzing some examples using both characterizations together. Some of these examples, which are especially important to see the advantages of the CTLT in comparison with the TLT, are presented in Chpt. 5.
Furthermore, this characterization is specially useful for the application of the CTLT for the analysis on any EM problem whose solution can be achieved basing on TLs.

For the inverse characterization, the definition of the so called isocomplex numbers in Appendix 4.A results crucial for both the "geometrical" interpretation of the parameterizations concerning this analysis and also for the algebraic rigor is required for the analysis in the CTLT.

Remark 23. *Isocomplex numbers arise from the need of having a "space of parameters" which explains the characterizations in the CTLT. Concretely, this "space of parameters" is really useful for adding the physical interpretation to the CTLA as it has been presented: involving parameterized transformations between the planes associated to those TL parameters under study. Thus, the "space of parameters" is conciously defined for achieving the issue of parameterizing several physical parameters such as different type of losses and frequency (these parameters have immediate representation in the "space") and also some more that are hidden in different parameterizations, e.g. the length of the TL represented by curves which keep the angle of the propagation constant the same.*
Furthermore, this "space" lets to parameterize different mode solutions by inverse analysis, thanks to its definition in the most generalized way.

The rg-plane is the region of the "space of parameters" in which the curves parameterizing the basic parameters are plotted, leading to the graphical analysis.
Apart from the graphical representations, it is explained among the practical uses of the rg-plane in Appendix 4.A that the inverse characterization of basic parameters leads to obtain the inverse

[20]This is the main reason of the "inverse" denomination, although some more reasons of this name can be found, for example in the origin of the equivalent voltage and current waves, which differentially define the EM waves (regarding this inverse characterization), in contrast to definition of these equivalent waves by integrating the EM fields (concerning the direct characterization).

function which completely defines these parameters,which is even more interesting than simply seeing how the basic parameters vary in terms of different types of parameterizations. As a consequence of this inverse characterization in the "space of parameters", any TL is parameterized in an unique way.
This approach is much more interesting when trying to parameterize more solutions in different TLs, although in this version of the CTLT (CTLT-v1.0b) it comes to: (i) validate the LTLT and the direct characterization as it has been contextualized in Chpt. 2 and developed in previous Sect. 4.3 in this chapter; and, which is maybe more important, (ii) complete the characterization of the CTLT, which is shown in Chpt. 5 by means of examples of use for the application of both characterizations together.

The methodology of study is presented in the same way as the direct characterization of HPWs regarding the CTLT (CTLT-v1): (i) the main parameterizations of this study are described. This includes studying both the parameterization of basic parameters for the inverse characterization of line parameters and those concerning the wave parameters for the inverse characterization of basic parameters. The parameterizations regarding the basic parameters are "universalized", which means that any normalization can be used interchangeably by simply rescaling the parameterizations of those resultant curves that parameterize the real or imagiary parts or the modulus of these parameters in the rg-plane, while the parameterization of the angles remind the same. This is due to normalizations in the direct characterization are always done with respect to real values that "zoom" the complex planes associated to each parameter, but they conserve the magnitude (and sense) of the angles (the normaliations and the transformations between them are conformal). In order to exemplify this important property of the inverse characterization, take for example the propagation constant: the one normalized for the *ffa*, γ_{n1}, and that one normalized for the *vfa*, γ_{n2}. The parameters which normalize each case are different, but in both cases, they are real numbers. As a consequene, the same point in the rg-plane parameterized by (r, g) or (ω_n, c), describes different modulus in the γ_{n1}- and γ_{n2}-planes, respectively, but the same angle, so the points transform rescaled in each cited plane. Conversely, the same parameterization of the modulus in the γ_{n1}- and the γ_{n2}-planes leads to rescaled parameterizations in the rg-plane, provided that the phase is the same in both cases. This is proved more rigorously in Appendix 4.F. This fact is essential to reduce the inverse analysis of basic parameters to only one plane. Nevertheless, and this should be understood, each rg-plane has to be referenced to any parameterization for being the values of the parameterizations contextualized, and appropriately rescaled if needed; (ii) the normalizations of line parameters are, consequently, those coming from the definition in rg-planes, whereas the normalizations of basic parameters are studied referred to the wave parameter for the inverse characterization; (iii) the corresponding graphical analysis are done by previuosly solving any of the forms of the inverse function regarding each parameter under study when it is feasible, and analyzing the resultant curves geometrically. Some of them are specially useful to be analyzed, for example those that parameterize angles, for the subsequent application to other analysis; (iv) the physical interpretation of the inverse characterizations is direct because, it answers which parameterizations of the TL lead to specific behaviors of basic parameters. On the other hand, the inverse characterizations answers how the basic parameters vary when they are parameterized by wave parameters, which may be also provided with physical meaning when for example fixing the load and varying the basic parameters accordingly a specific physical behavior, for example, varying the characteritic impedance with frequency; and (v) the mention of the practical uses both when characterizing the parameters of the lossy TL and when solving related physical problems.
These points (i-v) are addresed for each parameter under study, dual to the manner the direct characterization of the lossy TL has been presented in Sects. 4.3.1 and 4.3.2.

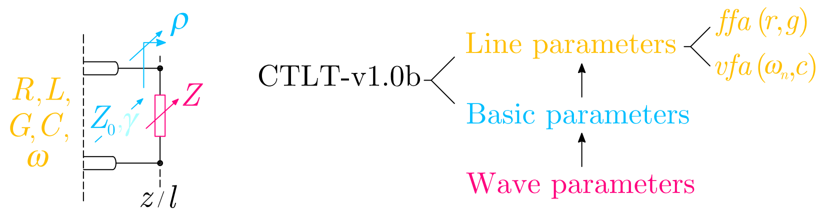

Fig. 4.17: Scheme of parameterizations regarding the basic parameters –marked in blue– to determine the line parameters described for both the *ffa* and the *vfa* –marked in yellow), and the parameterizations of wave parameters –marked in pink– to determine the basic parameters; both corresponding to the CTLT-v1.0b.

This section is organized as follows: firstly, the parameterizations of specific behavios of basic parameters are depicted. This means characterizing the line parameters inversely. The behaviors studied parameterize these analysis in rg-planes, and they consist in describing the parameterizations of the real and imaginary parts, and the modulus and phase of both basic parameters: the characteristic impedance and the propagation constant (in Fig. 4.17, see these parameters –colored in blue– which vary to characterize the line parameters –colored in yellow).
Notice that the parameterizations of the real-imaginary parts of basic parameters represented in the axis of the assoaciated complex planes are the reciprocal of r and g for the *vfa* in the direct characterization, because they are represented in the axis of the rg-plane, whereas the modulus-phase parameterizations of basic parameters are the reciprocal of the frequency parameterizations in the *vfa* regarding the direct characterizations, becuase they represent the modulus and phase in the rg-plane. Thus, the inverse characterization of basic parameters splits into real-imaginary parametrizations and modulus-phase parameterizations, reciprocally to the division for the *ffa* and the *vfa* concerning the direct characterization.
Secondly, the basic parameters: in this case, the characteristic impedance and the reflection coefficient (the reflection coefficient is here addressed as basic parameter, just as it has been presented in Chpt. 3); are analyzed as transformations between themselves extra parameterized by the wave parameters, namely by the wave impedance (in Fig. 4.17, see these parameters – colored in pink– which keeps constant to characterize the basic parameter transformations).

4.4.1 Inverse characterization of line parameters

The inverse characterization of line parameters is intended to define the functions which relate the line parameters in such a way that different parameterizations of basic parameters: real-imaginary parameterized parts and modulus-phase parameterizations; are described by these functions. This is equivalent to "sketch" the parameterizations of basic parameters over planes which represent the line parameters.
The underlying analysis of this characterization is the one introduced in Chpt. 3, in which the definition of harmonic potential functions as equivalent waves in the TL requires to solve the parameters of the assumed equivalent TL in which they propagate.

For the purpose of solving the line parameters, it is neccesary to take into account that, as it has been detached in the direct characterization, the appropriate normalizations of basic parameters lead to define the subsequent parameterizations of line parameters, which are arranged by pairs, for example r and g in the *ffa* regarding the direct characterization. This fact together with the property of the parameterizations to be rescaled accordingly to the normalization which has been chosen (in order to do not affect the graphical analysis: see Appendix 4.F), makes the analysis "universal" when it is doing over rg-planes.
As a consequence, there are two ways to proceed: (i) if the direct characterization concerning a particular analysis has been done (just as it happens in this case of analyzing harmonic plane waves

	ic-CTLT: method #1		ic-CTLT: method #2	
Parameters	*Given*: Z_0	Z_0^{-1}	*Given*: $Z_{mode} \equiv \eta$	Z_0^{-1}
	↓	↑	↓	↑
Functions	$\boldsymbol{Z_{0n1}}(r,g)$ ⟶	$\boldsymbol{Z_{0n1}^{-1}}(r,g)$	$\boldsymbol{Z_{mode}}(\mu,\varepsilon,\text{etc.})$ ⟶	$\boldsymbol{Z_{0n1}^{-1}}(r,g)$

Fig. 4.18: Example of the possible methods for inversely characterizing the characteristic impedance of the TL: the method #1 is based on solving, for example, the inverse function $\boldsymbol{Z_{0n1}^{-1}}$ regarding the one defined for the direct characterization, $\boldsymbol{Z_{0n1}}$; whereas the method #2 is based on solving this inverse function from the impedance of the mode parameterizing the constitutive parameters with line parameters.

described in terms of losses: CTLT-v1.0a in Sect. 4.3, concerning the LTLT), the inverse function of those defined for the graphical analysis are solved in any form, if possible (see Fig. 4.18, in which the method #1 is the one described in this point); otherwise (ii) either the analytical expression of the impedance, Z_{mode}, and the propagation constant, γ_{mode}, regarding a wave solution, or their graphical analyis in complex planes, just as it is done in [Rie98], are used for parameterizing the basic parameters supposing a complex function which characterize them (see Fig. 4.18, in which the method #2 is the one described in this point).
In this section, finding the inverse functions of those ones defined in the direct characterization (that is, the method (i) (#1) as it has just been described above and shematized in Fig. 4.18) is the method to be followed, whereas an example of obtaining the line parameterizations regarding superior modes (see Chpt. 3 for their introduction) is presented within the **Applications** and **Future Lines**, showing the great usefulness of the inverse characterization in parameterizing more solutions, not only HPWs.

Real-Imaginary parameterized parts

In this section, the real-imaginary parts of basic parameters: the characteristc impedance and the propagation constant; are parameterized in any of their normalized complex planes to be then transformed to the rg-plane.
For each parameter under study: (i) the parameterizations are mathematically described. In relation to this step, it is important to specify in which normalization of basic parameters the inverse function is going to be obtained for labeling the resultant curves accordingly; (ii) the curves are graphically depicted and geometrically analyzed; (iii) some physical interpretations for each analysis are obtained; and (iv) the practical uses of these inverse characterizations in both the solution of new problems and the possible applications in modelling and designing circuits are detached.

The analysis presented here parameterizes the real and imaginary parts concerning the normalizations of basic parameters with respect to the lossless case, that is, those normalizations used in the *ffa* regarding the direct characterization, presented in Sect. 4.3.1. Thus, the labels in the rg-plane wil be referred to the parameterizations of basic parameters normalized with respect to the lossless case.
If for example the parameterizations are required to be referred to the non dispersive case used for the *vfa*, the values of the parameterizations in the rg-plane have to be rescaled $\cdot 1$ (not rescaled) for characterizing the real-imaginary parts of the characteristic impedance, and $\cdot \beta_{sp}\alpha_{nd}$ in case of characterizing the real-imaginary parts of the propagation constant.

Characteristic impedance: Characterizing the real-imaginary parameterized parts of the characteristic impedance has mainly analytical and geometrical interest. In this sense, these parameterizations can be used for determining directly those parameterizations of losses/frequency which lead to specific values of real or imaginary parts of the characteritic impedance.

***Normalizations and parameterizations*:** As it has been mentioned, the normalization of the characteristic impedance, whose inverse function is intended to be analyzed, is that one used for the *ffa*, which is normalized with respect to the lossless case.
Thus, it is about obtaining the inverse function of $\boldsymbol{Z_{0n1}}$ defined in eq. (4.7), that is solving $\boldsymbol{Z_{0n1}^{-1}}$, in order to transform the real-imaginary parameterized parts for the subsequent graphical analysis.

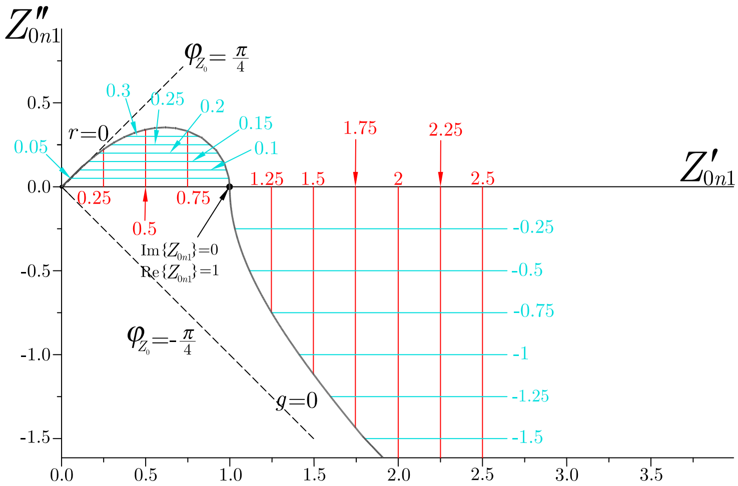

Fig. 4.19: Parameterizations of the real-imagninary parts in the Z_{0n1}-plane used for the inverse characterization in the rg-plane.

The parameterizations of the real and imaginary parts of Z_{0n1} are denoted by

$$\begin{cases} Z'_{0n1} = a \\ Z''_{0n1} = b \end{cases} , \tag{4.46}$$

which are depicted in Fig. 4.19, it its respective (bounded) complex plane.

The easiest way to define the inverse function for each parametrization is solving r as a function of g (see the mathematical notes of the development in Appendix 4.G), leading to the expressions

$$\begin{cases} r = g - 2a\left(1+g^2\right)\cdot\left(ga - \sqrt{a^2\left(1+g^2\right)-1}\right) \\ r = g + 2b\left(1+g^2\right)\cdot\left(gb - \sqrt{b^2\left(1+g^2\right)+1}\right) \end{cases} , \tag{4.47}$$

for each parameterized part in eq. (4.46).

These r vs. g expressions are able to be graphically represented and geometrically analyzed in the rg-plane.

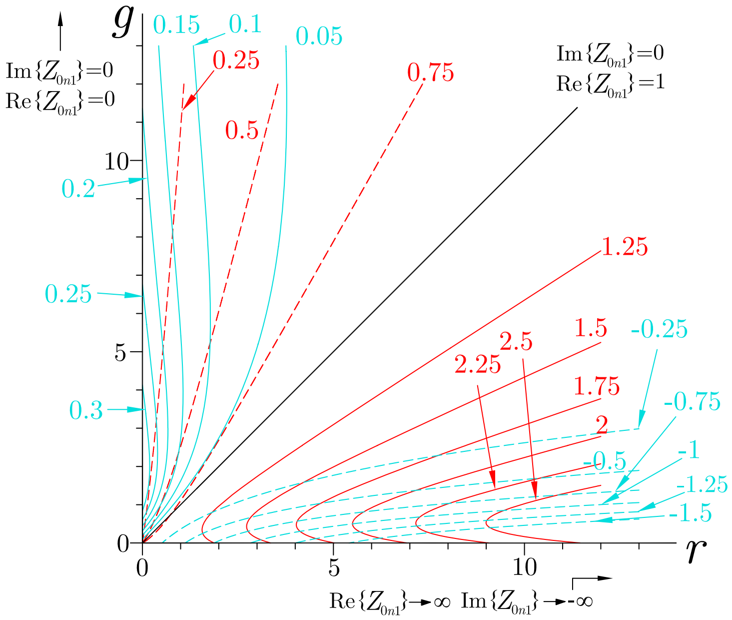

Fig. 4.20: Transformations of the real-imaginary parameterized parts of Z_{0n1} in the rg-plane.

***Graphical and geometrical analysis*:** Since r is expressed as function of g parameterizing the real-imaginary parts of the charateristic impedance, the graphical analysis in rg-planes is straighforward.

The resultant curves may be seen as the complex transformation from the Z_{0n1}-plane to the rg-plane given by the seeked inverse function $\boldsymbol{Z_{0n1}^{-1}}$, which is rigorously defined as:

$$\begin{aligned} \boldsymbol{Z_{0n1}^{-1}}: \quad & D_{Z_{0n1}} \subset \mathbb{C} \to \mathbb{R}^{2+} \cup (0,0) \\ & Z_{0n1} \rightsquigarrow \boldsymbol{Z_{0n1}^{-1}}(Z_{0n1}) = (r,g) \ (\equiv \tau_{Z_{0n1}} \in \mathbb{H}). \end{aligned} \tag{4.48}$$

Notice that this function is defined in complex variable Z_{0n1}.

Remark 24. *The inverse characterization establishes an approach regarding complex variable function is here in the inverse characterization of the expressions which are naturally complex due to their definition in the frequency domain. This fact, far from being an anecdote in the analysis, will bring important conclusions concerning future analysis in complex variable, in the sense that each problem defined in a complex domain is the inverse of the original problem defined in real variable when it is analyzed by means of complex functions parameterized by complex parameters.*

In Fig. 4.20 the graphical analysis of the characteristic impedance parameterized by its real-imaginary parts are depicted in the rg-plane. This representation "sketches" the inverse complex variable funtion $\boldsymbol{Z_{0n1}^{-1}}$ in the rg-plane. Thus, the resultant curves in the rg-plane may be seen as (complex) "curve levels" of the function $\boldsymbol{Z_{0n1}}$.

If analyzing the curves in the rg-plane geometrically, they can be classified as sets of hyperbolas, [Law72], parameterized by the real-imaginary parts of the characteristic impedance.
Some interesting properties that these curves show in the rg-plane may be detached: (i) the curves parameterizing the imaginary part of the charactristic impedance are hyperbolas with different orientation depending on the sign of the parameterization: when the parameterization is positive, the curve goes closing to $g = \sqrt{3}$, which produces the maximum value of the imaginary part of the

characteristic impedance, whereas when the parameterizations of the imaginary part tend to 0, the curves open to the bisector in the rg-plane. The curvature of these curves makes that there are two values of g which produces the same imaginary part of the characteristic impedance. On the other hand, when the parameterization is negative, the curves tend to be closer to the r-axis, as much as the parameterization is more negative. Moreover, for each especific negative parameterization of the imaginary part, the minimum r is given by $r_{min}|_{Z''_{0n1}=b<0} = -2b\sqrt{b^2+1}$, achieved when substituying $g = 0$ in the second equation in (4.47); (ii) the curves parameterizing the real part of the characteristic impedance present similar geometrical behavior. In this case, the curves whose parameterization is greater than the unity lead to two different values of g that produce the same real part; and (iii) the mentioned "double parameterization" (the same parameterizations for two different values in the rg-plane) also happens if analyzing the curves varying the frequency.

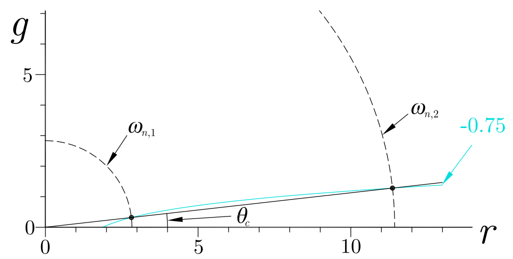

Fig. 4.21: Example of frequency analysis in the rg-plane which shows the "double parameterization" for an unique parameterization of the imaginary part of the characteristic impedance.

The graphical example above serves to show this phenomenon when varying the frequency; among the infinity of geometrical properties that could be extracted by combining the graphical analysis and the analytical expressions in eq. (4.47).

In any case, those most interesting geometrical properties are always related to the physical properties to be analyzed, or the practical uses which are intended to be obtained.

***Physical interpretation*:** The inverse characterization of the basic parameters serves to answer –and this happens in general in the inverse characterization– which parameterizations of losses or frequency lead to specific parametrizations of these parameters.

In this particular case of splitting the real and imagiary parts of the characteristic impedance, the only physical interest is when interpreting the terms of the mean transmited power separately (see the expression of the transmited power in lossy TLs in Sect. 2.4.1 in Chpt. 2). Otherwise, the interpretation of this inverse characterizations lacks of physical meaning, despite its mathematical interest.

***Practical uses*:** The main practical uses of this characterization are in taking advantage of losses for matching purposes. In this sense, if the load at any point is given, the TL that matches the load at this point is that one which has the same real and imaginary parts, making ρ in eq. (2.84) zero. Thus, once the real and imaginary parts are known, it is possible to solve –by means of this inverse characterization– which lossy parameterizations at a fixed frequency do the matching, or conversely at which frequency the matching produces when losses are fixed.

Propagation constant: Studying the real-imaginary parameterizations of the propagation constant to obtain the parameterizations of losses/frequency which lead to them has special interest because of the physical interpretation these parameterized parts have in the wave propagation: the real part –the attenuation constant– fixes how the waves decay along the TL, whereas the imaginary part –the phase constant– establishes the speed of the propagative waves. In this sense, specific values of both the attenuation constant and the phase constant may be analyzed by means of the analysis presented here.

***Normalizations and parameterizations*:** In this case, the analysis of the parameterizations of the propagation constant is carried out by taking again the normalizations with respect to the lossless case, γ_{n1}, which has been used for the *ffa* in the direct characterization of the propagation constant. Thus, it is about obtaining the inverse function of $\boldsymbol{\gamma_{n1}}$ defined in eq. (4.10), that is solving $\boldsymbol{\gamma_{n1}^{-1}}$ so that the real-imaginary parts of the propagation constant characterize the parameterizations of losses/frequency in the subsequent mathematical and graphical analysis.

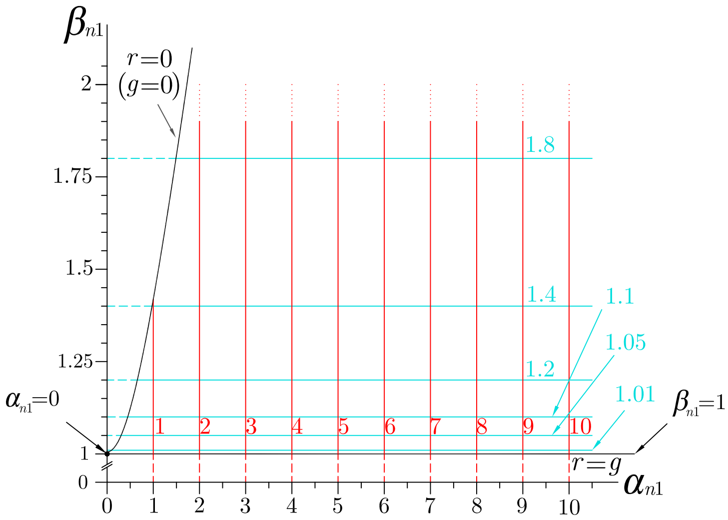

Fig. 4.22: Parameterizations of the real-imagninary parts in the γ_{n1}-plane used for the inverse characterization in the rg-plane.

The real-imaginary parts are parameterized as follows:

$$\begin{cases} \gamma'_{n1} = \alpha_{n1} = a \\ \gamma''_{n1} = \beta_{n1} = b \end{cases} . \tag{4.49}$$

These parameterizations are graphically depicted in Fig. 4.22 in the γ_{n1}-plane. Notice that the depicted γ_{n1}-planes does not have the same scale for both axis. The fact of not having a regular grid in the γ_{n1}-plane meets the purpose of having equispaced curves in the subsequent graphical analysis in the rg-plane.

The way to define the inverse function for each parameterization is solving r as a function of g (or g as a function of r because the equivalent role the r and g parameterizations play in γ_{n1}), in a similar way of how the curves parameterizing the real-imaginary parts of the characteristic impedance have been obtained in Appendix 4.G.

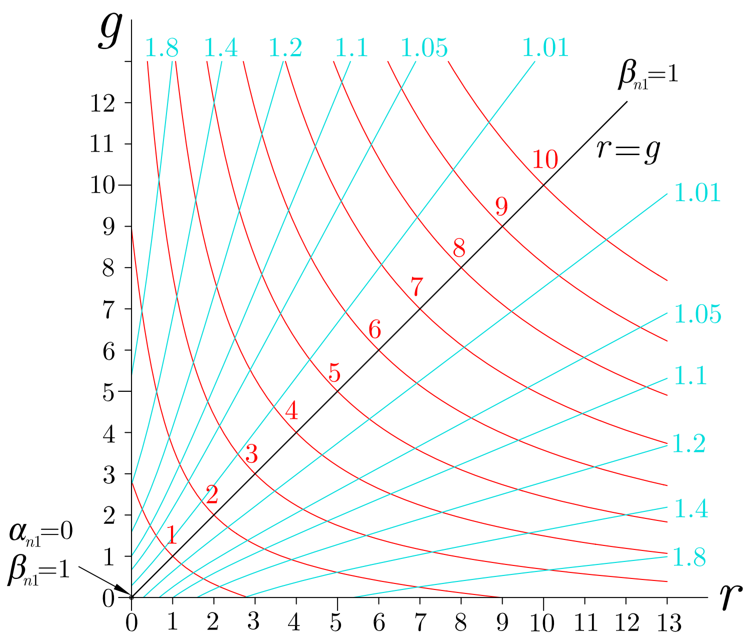

Fig. 4.23: Transformations of the real-imaginary parameterized parts of γ_{n1} in the rg-plane.

The following curves parameterized by the real-imaginray part of the propagation constant have been obtained:

$$\begin{cases} r = -g\left(2a^2+1\right) + 2a\sqrt{\left(1+g^2\right)\left(a^2+1\right)} \\ r = g\left(2b^2-1\right) \pm 2b\sqrt{\left(1+g^2\right)\left(b^2-1\right)} \end{cases} , \tag{4.50}$$

in which $g \in [0, \infty[$ for both equations,

or equivalently,

$$\begin{cases} g = -r\left(2a^2+1\right) + 2a\sqrt{\left(1+r^2\right)\left(a^2+1\right)} \\ g = r\left(2b^2-1\right) \pm 2b\sqrt{\left(1+r^2\right)\left(b^2-1\right)} \end{cases} , \tag{4.51}$$

in which $r \in [0, \infty[$ in both equations.

These curves are going to be graphically represented and geometrically analyzed in the rg-plane, while emphasizing their most important poperties.

***Graphical and geometrical analysis*:** The way of obtainig the parameterized curves in eq. (4.50) (or eq. (4.51)) allows for the direct graphical representation in rg-planes.

The resultant curves "sketch" the function $\boldsymbol{\gamma_{n1}}$ from the inverse function $\boldsymbol{\gamma_{n1}^{-1}}$, which is rigorously defined as:

$$\begin{aligned} \boldsymbol{\gamma_{n1}^{-1}} : \quad & D_{\gamma_{n1}} \subset \mathbb{C} \to \mathbb{R}^{2+} \cup (0,0) \\ & \gamma_{n1} \rightsquigarrow \boldsymbol{\gamma_{n1}^{-1}(\gamma_{n1})} = (r,g) \; (\equiv \tau_{\gamma_{n1}} \in \mathbb{H}). \end{aligned} \tag{4.52}$$

Notice that this inverse function is defined in complex variable γ_{n1}.

In Fig. 4.23, the graphical analysis of the propagation constant parameterized by the real-imaginary parts is depicted in the rg-plane. This representation in the rg-plane is one of the ways of obtaining the seeked inverse function $\boldsymbol{\gamma_{n1}^{-1}}$. The resultant curves may be seen as (complex) "curve levels" of the function $\boldsymbol{\gamma_{n1}}$.

The curves in the rg-plane are hyperbolas, [Law72], which are parameterized by the real-imaginary parts of the propagation constant.
If analyzing the curves geometrically, some interesting properties can be detached: (i) the curves are symmetric with respect to the bisector in the rg-plane. This is due to the parameterizations r and g are interchangeable because the equivalent role of them when defining γ_{n1}; (ii) the curves parameterizing the imaginary part of the propagation constant are unfolded. This means that for each r (or g) there are two values of g (or r) that lead to the same imaginary part. Moreover, the minimun r (or the minimun g) for each curve parameterized by the impaginary part is $r_{min}|_{\gamma''_{n1}=b} = 2b\sqrt{b^2-1}$ (or $g_{min}|_{\gamma''_{n1}=b} = 2b\sqrt{b^2-1}$). These curves present asymptotic behavior when $r \gg$ (or $g \gg$), for which the asymptote is the ray whose slope is $m_{asymp} = 2b^2-1+2b\sqrt{b^2-1}$ (or $m_{asymp} = 2b^2-1-2b\sqrt{b^2-1}$); and (iii) the hyperbolas parameterizing the real part of the propagation contant have their vertex in the bisector of the rg-plane. Moreover they cut the r-axis (or the g-axis) in the maximun r (or the maximun g), $r_{max}|_{\gamma'_{n1}=a} = 2a\sqrt{a^2+1}$ (or $g_{max}|_{\gamma'_{n1}=a} = 2a\sqrt{a^2+1}$).
Some geometrical properties are connected with the physical interpretations of the curves in the rg-plane.

Physical interpretations: As mentioned before, the inverse characterization of the propagation constant parameterizing the real and imaginary parts has special interest from the physical point of view because the direct relation between the parameterizations and the characteristics of the wave which propagates.
In any case, this analysis should be interpreted as any in the rg-plane. Particularly, this analysis answers which TLs lead to specific attenuation or wave speed of the waves which propagate along.

Regarding the attenuation constant, it may be said that: (i) for a fixed frequency (that is a circunference in the rg-plane) the minimum attenuation is produced when the TL is non dispersive; (ii) for a TL characterized by c (a ray inclined $\theta_c = \tan^{-1}(1/c)$) the attenuation decreases when ω increases; and (iii) when r (or g) is fixed, the minimum attenuation is achieved when r (or g) is the $r_{max}|_{\gamma'_{n1}=a}$ (or $g_{max}|_{\gamma'_{n1}=a}$), so that $\alpha_{n1,min}|_{r=r_0} = \sqrt{(-1+\sqrt{1+r_0^2})/2}$ (or $\alpha_{n1,min}|_{g=g_0} = \sqrt{(-1+\sqrt{1+g_0^2})/2}$).

Regarding the propagation constant, it is neccesary to detach that: (i) when frequency increases (the circunferences in the rg-plane are smaller), the phase constant tends to be the one characteristic of the non dispersive case; but (ii) when frequency decreases (the circunerences are bigger), the phase constant tends to be linear with frequency, because the asymptotic behavior of the phase constant curves; and (iii) when r (or g) is fixed, the maximum value of the propagations constant (minimum speed) is achieved when that r (or g) is the $r_{\min}|_{\gamma''_{n1}=b}$ (or $g_{\min}|_{\gamma''_{n1}=b}$), so that $\beta_{n1,max}|_{r=r_0} = \sqrt{(1+\sqrt{1+r_0^2})/2}$ (or $\beta_{n1,max}|_{g=g_0} = \sqrt{(1+\sqrt{1+g_0^2})/2}$).

In conclussion, seeing the graphical analysis in Fig. 4.23 supported by the geometrical analysis, it may be said that: when losses increase or frequency decreases the TL tends the non dispersive case. The speed of the wave is maximum in the non dispersive case, at the same time that the attenuation is minimum in this case when the lossy parameterizations ($r = g$) are fixed.

***Practical uses*:** The practical uses of this characterization mainly come when describing the physical behaviors of the equivalent waves that it parameterizes: attenuation and speed; varying the parameters of the TL, for example analyzing the TL when changing the frequency and seeing how the physics of the wave varies.

Modulus-phase parameterizations

In this section, the modulus-phase parameterizations of basic parameters: the charateristic impedance and the propagation constant; are studied in order to be characterized in rg-planes. In this way, an interesting representation for analyzing TL-related problems (alternative to the study of the real-imaginary parameterized parts) is obtained by means of this analysis, which shows special usefulness in characterizing the wave parameters, both along the TL and in terms of losses/frequency as it is seen by means of different examples presented in Chpt. 5.
In this sense, the inverse characterization of the angles of both the characteristic impedance and the propagation contant becomes especially important. In addition, recall that these angular parameterizations are "universal" because they are not affected by the normalizations of basic parameters. Thus, as it is shown in Chpt. 5, the angles of basic parameters completely characterize the behavior of the TL, so studying which losses/frequency leads to these angles is a crucial issue to be solved by means of this inverse characterization.

In the same way as the real-imaginary parameterized parts, the inverse characterization of basic parameters is studied in terms of: (i) the parameterizations regarding the normalization of basic parameters. As mentioned above, the angles are not scaled in any case, so the parameterizations of the angles are the same for each normalization of basic parameters or the denormalized versions; (ii) the graphical representation of the parameterized curves as well as their geometrical analysis; (iii) the physical interpretations of the characterization described inversely; and (iv) the practical uses of this characterization, detaching those that determine the analysis of wave parameters.

The analysis presented here is performed with respect to the lossless case, useful in the *ffa* of basic parameters. This only affects the parameterizations regarding the modulus, because the conservation of the angles after each normalization, as explained before. Thus, using different normalizations of basic parameters only affects in changing the label of the curves parameterizing the modulus, just as it is explained in Appendix 4.F, so that the subsequent definion of functions $\boldsymbol{Z_{0n1}}$ and $\boldsymbol{\gamma_{n1}}$ over the rg-planes is valid for every normalization by simply rescaling the modulus parameterizations. In this sense, the rg-plane in which the "curve levels" are plotted will be always referred to the normalization which describes them (in this case, the lossless case).
In any case, the inverse characterization of functions Z_{0n1} and γ_{n1} in rg-plane is helpful for any type of analysis: losses, frequency, etc., by means of the same graph, which is one of the most important usefulnes of this analysis.

Characteristic impedance: The inverse characterization of the characteristic impedance parameterizing its modulus-phase is essential for analyzing lossy TLs.
On one hand, this characterization has special physical meaning because the underlying phisycal interpretation regarding the modulus and and phase of the characteristic impedance: the modulus represents the relation in amplitude of the equivalent voltage and current waves, whereas the phase represents the phase shifting between these waves. Moreover, this latter parameter has direct relation with losses. Thus, by means of this inverse characterization, it is directly answered which TLs –which losses/frequency– present specific behaviors regarding these physical properties.
On the other hand, the inverse characterization of the characteristic impedance provides a representation of the function $\boldsymbol{Z_{0n1}}$ which is especially useful for analyzing TLs. In this sense, recall,

for example, that the GSC is parameterized by the angle of the characteristic impedance. Solving which TLs are able to be studied by the same parameterized GSC is of special interest in the analysis of wave parametrs or, conversely, explaining which angle parameterization should be used in the GSC in terms of the losses/frequency of the TL.
The main analytical result of this analysis is on defining and "sketching" the complex function $\boldsymbol{Z_{0n1}}$ as defined in eq. (4.7) (and Z_0, in general) when spliting its modulus and phase.

***Normalizations and parameterizations*:** The Z_{0n1}-complex plane regarding the *ffa* is the one selected for inversely characterized Z_{0n1} in terms of line parameters in the rg-plane. Thus, it is about obtaining the inverse function $\boldsymbol{Z_{0n1}^{-1}}$ which maps the modulus-phase parameterizations from the Z_{0n1}-plane to the rg-plane.

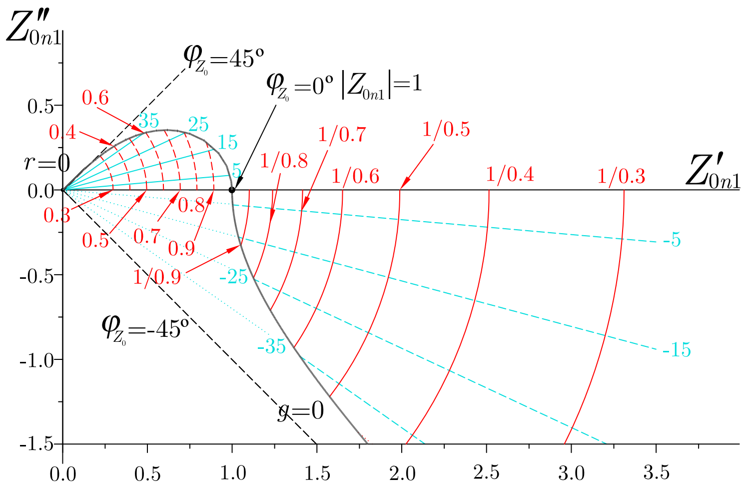

Fig. 4.24: Parameterizations of modulus-phase in the Z_{0n1}-plane used for the inverse characterization in the rg-plane.

The modulus-phase parameterizations of the characteristic impedance are denoted as:

$$\begin{cases} |Z_{0n1}| = m \in \begin{cases} \left]0, \sqrt{\cos 2\varphi_{Z_0}}\right] & \text{if } \varphi_{Z_0} > 0 \\ \left[\frac{1}{\sqrt{\cos 2\varphi_{Z_0}}}, \infty\right[& \text{if } \varphi_{Z_0} < 0 \end{cases} \\ \varphi_{Z_{0n1}} \equiv \varphi_{Z_0} = p \in \begin{cases} \left[0, \cos^{-1}\left(\frac{1}{2}|Z_{0n1}|^2\right)\right] & \text{if } |Z_{0n1}| < 1 \\ \left[\cos^{-1}\left(\frac{1}{2|Z_{0n1}|^2}\right), 0\right] & \text{if } |Z_{0n1}| > 1 \end{cases} \end{cases} . \tag{4.53}$$

The simplest way to obtain the modulus-phase parameterized curves in the rg-plane is by writing r as a function of g. In this sense, the resultant curves are (the mathematical procedure for obtaining these curves is developed in Appendix 4.H):

$$r = \sqrt{m^4(1+g^2)-1}, \text{ for which } g \in \left[\sqrt{\frac{1}{m^4}-1}, \infty\right[, \tag{4.54}$$

$$r = \frac{g - \tan(2p)}{1 + g\tan(2p)}, \text{ for which } \begin{cases} g \in [\tan(2p), \infty[& \text{if } g \geq 0 \\ g \in [0, |\tan(2p)|] & \text{if } g < 0 \end{cases} . \tag{4.55}$$

These curves have direct graphical representation in the rg-plane, where they can be geometrically analyzed.

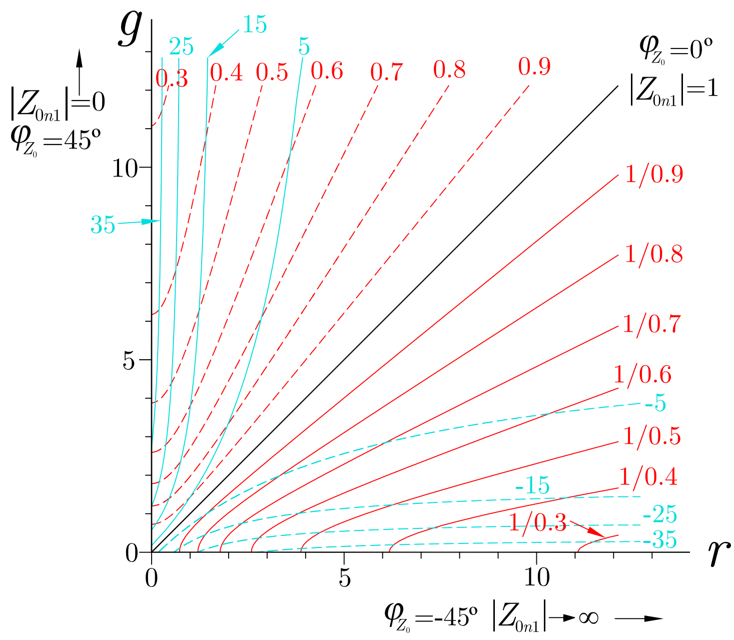

Fig. 4.25: Transformations of the modulus-phase parameterizations of Z_{0n1} in the rg-plane.

***Graphical and geometrical analysis*:** The curves in eqs. (4.54) and (4.55) parameterizing the modulus and phase of the characteristic impedance are represented in the rg-plane in Fig. 4.25. The resultant curves may be seen as the complex transformations of the from the Z_{0n1} complex plane to the rg-plane given by the inverse function $\boldsymbol{Z_{0n1}^{-1}}$ defined in eq. (4.48) (in complex variable, Z_{0n1}) when the modulus and phase are parameterized.

Some interesting geometrical properties of the curves depicted in the rg-plane concerning the modulus-phase parameterizations of the characteristic impedance may be detached: (i) both the modulus and phase parameterized curves are symmetric with respect to the bisector in the rg-plane. Moreover, each symmetric curve is parameterized by the inverse value in case of modulus parameterizations, or by the inverse sign in case of the phase parameterizations; (ii) the minimum value of r (or g) for each parameterization of the modulus is $r_{min}|_{|Z_{0n1}|=m} = \sqrt{m^4-1}$, for which $m > 1$ (or $g_{min}|_{|Z_{0n1}|=m} = \sqrt{m^4-1}$, for which $m < 1$). Moreover, each parameterization of the modulus is asymptotic to the ray whose slope is $m_{asymp} = 1/m^2$[21]; and (iii) the mimum value of r (or g) for each parameterization of the phase is $r_{min}|_{|\varphi_{Z_0}=p|} = -\tan(2p)$ (or $g_{min}|_{|\varphi_{Z_0}=p|} = \tan(2p)$. Each p-curve is asymtotic to $r_{asymp} = 1/\tan(2p)$ in case of $p > 0$, and $g_{asymp} = -1/\tan(2p)$ in case of $p < 0$.

The most important physical interperetations of this analysis are related to the geometrical properties of the curves in the rg-plane which parameterize the characteristic impedance in modulus-phase.

***Physical interpretations*:** In this case, the parameterized curves plotted in the rg-plane have to be interpreted in terms of which TLs lead to the specific values of the modulus or the phase of the characteristic impedance.

[21] The term m_{asymp} refers to the slope of the ray, not the modulus parametrization in itself.

If interpreting the graphical analysis in terms of losses, it may be said that the symmetries of the curves are produced when changing the values of losses, leading to the inverse modulus and the opposite phase, and so the inverse relation between the amplitudes and the contrary phase shifting regarding the voltage and currents waves.
Moreover, given a value of the conductor losses $r = r_0$, the maximum value of the modulus is $|Z_{0n1}|_{max}|_{r=r_0} = \sqrt[4]{r_0^2+1}$. Given a value dielectric losses $g = g_0$, the minimum value of the modulus is $|Z_{0n1}|_{min}|_{r=g_0} = \sqrt[4]{g_0^2+1}$. These values let to solve the maximum and minimum relations between the electric and magnetic fields.
On the other hand, the maximum phase difference for each parameterization of the conductor losses $r = r_0$ is $\varphi_{Z_0,min}|_{r=r_0} = \tan^{-1}(r_0)/2$ (for the dielectric losses g_0 is $\varphi_{Z_0,min}|_{r=g_0} = \tan^{-1}(g_0)/2$).

If interpreting the graphical analysis in terms of frequency, it may be said that, for each pair of losses whose ratio is c, the modulus of the characteristic impedance when the normalized frequency $\omega_n \ll$ is $|Z_{0n1}||_{\omega_n \ll} = \sqrt{c}$, whereas when $\omega_n \gg$, the modulus is $|Z_{0n1}||_{\omega_n \gg} = 1$.

***Practical uses*:** The most important application of the inverse characterization of the characteristic impedance is on describing the wave parameters both in terms of losses/frequency and along the TL.
Remember that the angle of the characteristic impedance fixes the GSC (see Fig. 4.16) which represents the transformations from the normalized wave impedance complex plane to the reflection coefficient complex plane.
If the losses of the TL are known, the inverse characterization in the rg-plane directly offers the phase of the charactersitic impedance.
Otherwise, if for example the frequency changes, the variation in the rg-plane turns into a proportional change on the modulus (ω_n) in the rg-plane, and so the change on the modulus and the phase (which is especially important for the parameterization of wave parameters) of the characteristic impedance.

Propagation constant: The inverse characterization of the propagation constant parameterizing its modulus-phase reults crucial for analyzing lossy TL, especially the wave parameters along the TL.
Contrary to the parameterizations of the real-imaginary parts, the modulus-phase parameterizations of the propagation constant do not have clear physical meaming. However, these parameterizations have special interest when describing the wave parameters both in terms of losses, and along the TL. Here it is foreseen that the phase of the propagation constant identifies a TL characterized along its extension (see Ex. 01 in Sect. 5.2 in Chpt. 5). Thus, the modulus of the propagation constant acts as a the parameter which describes the TL's length when the appropriate normalization is chosen (see Ex. 03 in Sect. 5.4 in Chpt. 5).
As a result of this characterization, a representation of the complex function $\boldsymbol{\gamma_{n1}}$ introduced by means of eq. (5.62) when splitting the modulus and phase is here obtained, allowing for "sketching" the function, in an alternative way of the analysis parameterizing the real-imaginary parts.

***Normalizations and parameterizations*:** The γ_{n1}-complex plane regarding the *ffa* serves to inversely characterize γ_{n1} in terms of line parameters in the rg-plane. From this point, it is about solving the inverse function $\boldsymbol{Z_{0n1}^{-1}}$ which maps the modulus-phase parametrizations from the γ_{n1}-plane to the rg-plane:

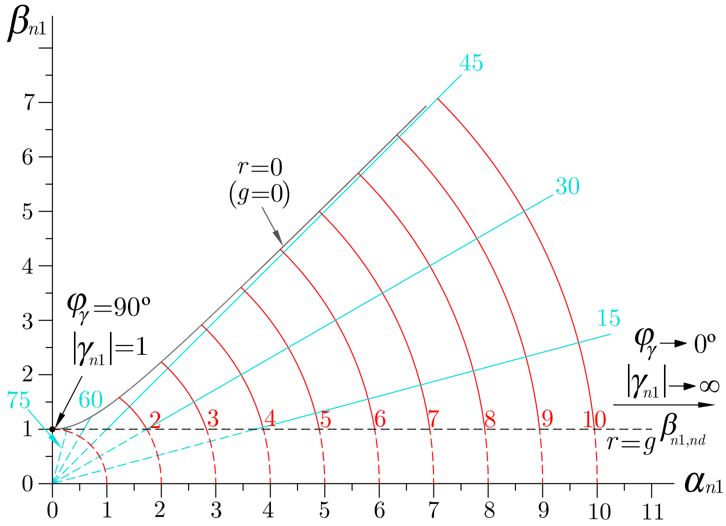

Fig. 4.26: Parameterizations of modulus-phase in the γ_{n1}-plane used for the inverse characterization in the rg-plane.

The modulus-phase parameterizations of the propagation constant are denoted as:

$$\begin{cases} |\gamma_{n1}| = m \in \begin{cases} \left[\frac{1}{\sin(\varphi_\gamma)}, \infty\right[\text{ if } \varphi_\gamma \in \left]0, \frac{\pi}{4}\right] \\ \left[\frac{1}{\sin(\varphi_\gamma)}, \frac{1}{\sqrt{-\cos(2\varphi_\gamma)}}\right] \text{ if } \varphi_\gamma = \left]\frac{\pi}{4}, \frac{\pi}{2}\right] \end{cases} \\ \varphi_{\gamma_{n1}} \equiv \varphi_\gamma = p \in \left[\sin^{-1}\left(\frac{1}{|\gamma_{n1}|}\right), \frac{1}{2}\cos^{-1}\left(\frac{-1}{|\gamma_{n1}|^2}\right)\right] \end{cases} . \tag{4.56}$$

Obtaining the modulus-phase parameterized curves in the rg-plane requires solving r (or g, because of the equivalent role of this parameter in the expression of γ_{n1} introduced in eq. (4.10) written as a function of g (or r). The mathematical procedure for solving in one or the other way is similar to the one presented in Appendix 4.H for the characteristic impedance. The resultant curves are:

$$r = \sqrt{\frac{m^4}{1+g^2} - 1}, \text{ for which } g \in \left[0, \sqrt{m^4 - 1}\right], \tag{4.57}$$

$$r = \frac{g + \tan(2p)}{g\tan(2p) - 1}, \text{ for which } g \in \left]\frac{1}{\tan(2p)}, \infty\right], \tag{4.58}$$

or, equivalentely:

$$g = \sqrt{\frac{m^4}{1+r^2} - 1}, \text{ for which } r \in \left[0, \sqrt{m^4 - 1}\right], \tag{4.59}$$

$$g = \frac{r + \tan(2p)}{r\tan(2p) - 1}, \text{ for which } r \in \left]\frac{1}{\tan(2p)}, \infty\right]. \tag{4.60}$$

These curves can be directly represented in the rg-plane leading to graphical representation of the modulus-phase parameterizations of the propagation constant.

***Graphical and geometrical analysis*:** The curves in eqs. (4.57) and (4.58) parameterizing the modulus and phase of the propagation constant are represented in Fig. 4.27.
The resultant curves may be seen as the complex transformations of the modulus-phase parameterizations from the γ_{n1}-plane to the rg-plane following the inverse function $\boldsymbol{\gamma_{n1}^{-1}}$ defined in eq. (4.52).

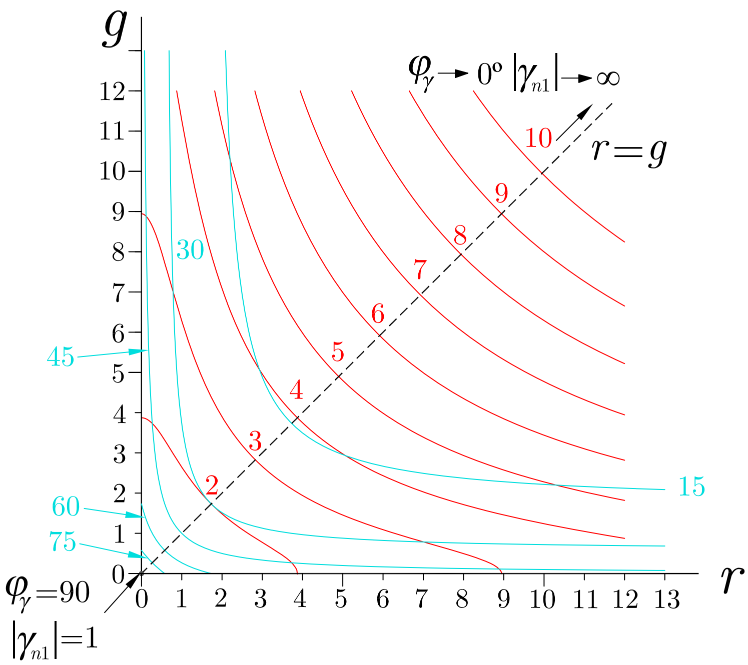

Fig. 4.27: Transformations of the modulus-phase parameterizations of γ_{n1} in the rg-plane.

Some geometrical properties of the curves which parameterized the modulus and phase of the propagation constant in the rg-plane may be detached: (i) the curves are completely symmetric with respect to the bisector in the rg-plane, and they all are hyperbolas, [Law72]; (ii) for the curves parameterizing the modulus, the maximum value of r (or g) for each parameterization is $r_{max}|_{|\gamma_{n1}=m|} = \sqrt[4]{m^4-1}$ ($g_{max}|_{|\gamma_{n1}=m|} = \sqrt[4]{m^4-1}$); and (iii) for the curves parameterizing the phase, the maximum value of r (or g) for each parameterization is $r_{max}|_{\varphi_\gamma=p} = \tan(2p)$ if $p > \pi/4$ (otherwise the maximum is at $+\infty$ and the curves are asymptotic to the line $g = 1/\tan(2p)$) ($g_{max}|_{\varphi_\gamma=p} = \tan(2p)$ if $p > \pi/4$, and $+\infty$ if $p \leq \pi/4$, and the curves are asymptotic to the line $r = 1/\tan(2p)$).

These geometrical properties are useful for detaching some physical interpretations regarding the modulus-phase parameterizations of the propagation constant.

Physical interpretations: Regarding this case, the parameterized curves in the rg-plane sould be understood in terms of which TLs lead to specific values of the modulus or the phase of the propagation constant.

When interpreting the graphical analysis, it may be said that the curve symmetries are because of the equivalent role of lossy parameterizations in the expression of the propagation constant.
Moreover, given a value of the conductor losses $r = r_0$ (or the dielectric losses $g = g_0$), the minimum value of the modulus of the propagation constant is $|\gamma_{n1}|_{min}|_{r=r_0} = \sqrt[4]{r_0^4+1}$ ($|\gamma_{n1}|_{min}|_{g=g_0} = \sqrt[4]{g_0^4+1}$).
On the other hand, given a value of the conductor losses $r = r_0$ (or the dielectric losses $g = g_0$), the maximum value of the phase of the propagation constant is $\varphi_{\gamma,max}|_{r=r_0} = \tan^{-1}(r_0)/2$ as ($\varphi_{\gamma,max}|_{r=r_0} = \tan^{-1}(r_0)/2$), and the mimum is $\varphi_{\gamma,min}|_{r=r_0} = \tan^{-1}(1/r_0)/2$ ($\varphi_{\gamma,min}|_{g=g_0} = \tan^{-1}(1/g_0)/2$).

It important to detach that the labels regarding the modulus parameterized curves change accordingly in case of analyzing the propagation constant in terms of frequency in the rg-plane. Nevertheless, the phase keeps the same for both the *ffa* and the *vfa* and, concerning this latter analysis, it may be said that the phase passes from $\varphi_\gamma = 0$ to $\varphi_\gamma = \pi/2$, when going from $\omega_n \ll$ to $\omega_n \gg$.

***Practical uses*:** The inverse characterization of the modulus and phase parameterized curves regarding the propagation constant shows great usefulness when analyzing the wave parameters along the TL in terms of losses. This analysis is presented as an example of use of this characterization in Ex. 03 in Sect. 5.4 in Chpt. 5.

It may be foreseen that the parameter which decribes the analyis of wave parameters along the TL is the angle of the propagation constant, once the appropriate normalization of the TL's length is chosen. For this reason, the analysis of those parameterizations of losses/frequency that lead to the same φ_γ becomes so important. Furthermore, this means obtaining those parameterizations which, keeping fixed φ_γ, parameterize the length of the TL (those complex pairs which vary the modulus), as presented among the **Applications**.

4.4.2 Inverse characterization of basic parameters

The most common analysis regarding the wave parameters deals with the transformations between themselves, so that they result completely charaterized by the corresponding parameterized analysis (explained in Sect. 4.3.2). Regarding this analysis, the graphical representations have shown true usefulness on describing –in only one (complex) graph– all of these parameterized transformations, while also giving geometrical interpretation to the analysis which is addressed.
The most common example of this kind of parameterized graphical analysis is the usual SC, [Smi39, Smi44], which is no more than the graph of the tranformations of the real-imaginary parameterized parts of the wave impedance (normalized with respect to the lossless characteristic impedance) to the reflection coefficient complex plane.
Different versions regarding the SC has arisen in the literature. For example, the GSC, [GDG06], (see Fig. 4.16a) is extra-parameterized by the phase of the characteristic impedance, φ_{Z_0}, which represents, in turn, different combinations of losses/frequency (those ones described by the curves regarding the inverse characterization when parameterizing φ_{Z_0}). Or, for example, in [WLH09], a generalized Smith Chart based on defining the transformation from the wave impedance plane to the reflection coefficient plane in the most generalized form (as parameterized Möbius transformation, [Apo90]) is presented in order to, precisely, generalize different versions of the SC. Nevertheless, the particular understanding of the GSC (and all the SCs) seen as "only" one of all the possible complex transformation between the wave parameters is detached in [Gag01]. This viewpoint opens the analysis to the study of all the possible complex planes interconnected, especially for the losses/frequency characterization, as outlined in the direct characterization of wave parameters presented in Sect. 4.3.2.

However, using the GSC – or any transformation involving the wave parameter complex planes at the same level– for characterizing the wave parameters in terms of losses/frequency –which is the goal of the CTLT– is quite a lot inefficient because any change on the parameterizations also changes the graph in both the normalizations and the extra-parameterization of angle φ_{Z_0}.
On the other hand, the BCs on the TL are often supposed to be fixed (normally at the end of the TL). These BCs could be given by the wave impedance at the load, Z_L, its inverse: the wave admittance at the load, Y_L; or the the reflection coefficient at the load, ρ_L. Among of them, it is usual to consider the impedance at the load fixed, because of its physical meaning implies its

realizability in circuits.
Keeping these ideas in mind, the subsequent natural question is: *How the reflection coefficient varies when changing the characteristic impedance if the impedance at the load is given?*
In order to answer this new question (up to know the characteristic impedance has been considered fixed for each analysis), the idea of transforming the characteristic impedances from the associated complex plane to the reflection coefficient complec plane when the wave impedance (normally at the load) is fixed naturally arises in the context of the CTLT. In particular, the transformation between the real-imaginary parameterized parts of the characteristic impedance to the reflection coefficient complex plane would play the role of the "*inverse Generalized Smith Chart* (iGSC)". Nevertheless, as in [Gag01], it should be taken into account all the possible parameterizations transforming these parameters in both directions (from the characteristic impedance to the reflection coefficient, and viceversa).
All of these transformations leads to the inverse characterization of basic parameters presented in this section.

There are some important points to take into accont for this analysis a priori: (i) firstly notice that, as shown in Fig. 4.17, the reflection coefficient is addressed at the same level as the charactersitic impedance. Thus, in this case, the reflection coefficient plays the role of "basic parameter"; (ii) since the transformations are at a fixed point of the TL, the transformations are only between the characteristic impedance and the reflection coefficient. On its behalf, the analysis of the propagation constant in terms of the characteristic impedance or the reflection coefficient could be obtained by means of lossy/frequency parameterizations, being an example of use of both the direct and inverse chracterizations a posteriori; (iii) both the limits in the Z_0-plane (obtained when denormalizing, for example, the Z_{0n1}-plane) and the Z-plane (obtained when dernormalizing the Z_{n0}-plane) concerning their respective direct characterizations should be taking into account, providing the inverse characterization of line parameters with the required physical sense and also making this inverse characterization of basic parameters the complement to the direct characterization of wave parameters. In this sense, it may be foreseen that ρ will play the role of link between both characterizations, being a wave parameter or a "basic parameter" depending on the type of analysis: direct or inverse; respectively; and (iv) this analysis is extends the inverse characterization of line parameters presented in Sect. 4.3, in the sense that the parameterizations of the characteristic impedance to be transformed to the reflection coefficient complex plane may be seen as the lossy/frequency parameterizations given by this latter inverse characterization.

As a result of the inverse characterization introduced by means of this section, the problem of describing the wave parameters in terms of frequency/losses will be able to be solved in an easy way. The related solution is presented in Ex. 02 in Sect. 5.3 in Chpt. 5 as example of use of this inverse characterization, acting together with the direct characterization of Z_{0n1}.

For the understanding of the analyis, the inverse characterization is splitted into two different analysis depending on the origin of the parameterizations: parameterizations of the normalized characteristic impedance and parameterizations of the reflection coefficient; in a similar to how it is done in [Gag01]. For this purpose, the normalizations and the subsequent parameterizations are first motivated to be then defined. Then the graphical and the geometrical analysis are described in detail. At the end, the practical uses for each parameterization are detached.

***Normalizations and parameterizations*:** As it has been introduced in this section, the normalizations for this inverse characterization are those which allows for analyzing the characteristic impedance and the reflection coefficient parameterized by the wave impedance, which is fixed for the analysis at any point of the TL.
Despite this analysis could be referred to any point of the TL, the impedance which parameterizes the analysis will be the one at the load, Z_L. This does not neccesary means fixing the impedance at the end of the TL, but any fixed point where the load is known. Moreover, this nomeclature is

helpful for the notation definitions[22] (see the notation expressly used in Appendix 4.E).

Recalling the expressions of the characteristic impedance, Z_0, in eq. (2.51), and the wave parameters: the reflection coefficient at the load, ρ_L, in eq. (2.84), and the supposed wave impedance, Z_L; defined in the lossy case in Sect. 2.4.1 in Chpt. 2, the following normalizations are defined:

$$Z_{Ln} = \frac{Z_L}{|Z_L|} = e^{\varphi_{Z_L}} = c_L + js_L, \text{ in which} \tag{4.61}$$
$$c_L = \cos(\varphi_{Z_L}), \text{ and } s_L = \sin(\varphi_{Z_L}), \text{ and}$$
$$Z_{0nL} = \frac{Z_0}{|Z_L|} = Z'_{0nL} + Z''_{0nL} = |Z_{0nL}|e^{j\varphi_{Z_0}}. \tag{4.62}$$

Regarding these definitions it is important to detach the differences in the notation with respect to the normalizations concerning the direct characterization of wave parameters, both explained in Appendix 4.E.

In accordance with the definition of ρ_L as function of both the characteristic impedance and the wave impedance, its normalized expression results the same as the denormalized one:

$$\rho_{Ln} = \frac{Z_{Ln} - Z_{0nL}}{Z_{Ln} + Z_{0nL}} = \frac{e^{j\varphi_{Z_L}} - Z_{0nL}}{e^{j\varphi_{Z_L}} + Z_{0nL}} \equiv \frac{Z_L - Z_0}{Z_L + Z_0} = \rho_L. \tag{4.63}$$

Some important facts concerning the definition of these normalizations should be detached: (i) as it has been seen, the reflection coefficient is the same after the normalization. This, besides revealing the "universal" nature of this parameter in the sense that it represents multiple TLs different loaded, is an important property for combining this inverse characterization with the direct characterization by using the same parameter and so the same associated complex ("only" the physical interpretation varies between the direct and inverse characterizations); and (ii) the angle of the impedance at the load, φ_{Z_L}, determines the dependence on the load in this characterization. As a result, it can be said that there exists a clear parallelism between the direct characterization and the inverse characterization: the reflection coefficient is the link between them, and each characterization is extra-parametrized by angles. This connection leads to solve the most important analysis to completely characterize the CTLT, and many other TL-related problems.

From these normalizations and the definition of ρ_L in eq. (4.63), the inverse transformation from Z_{0nL} to the ρ_L is solved:

$$Z_{0nL} = Z_{Ln}\frac{1-\rho_L}{1+\rho_L} = e^{j\varphi_{Z_L}}\frac{1-\rho_L}{1+\rho_L}, \tag{4.64}$$
$$\text{in which } \varphi_{Z_L} \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

Eqs. 4.63 and 4.64 define the transformations between the ρ-plane (ρ_L-plane) and the Z_{0nL}-plane in both senses, when parameterizing the normalized parameters ρ_L and Z_{0nL} in both its real-imaginary parts and modulus-phase.
The graphical and geometrical analysis are presented together for each type of parameterization (real-imaginary or modulus-phase) regarding each parameter (Z_{0nL} or ρ_L).

***Graphical and geometrical analysis*:** If analyzing the expressions which define the transformations in eqs. (4.63) and 4.64, it is possible to realize that they are conformal mappings, [BC90], and, in particular, Möbius-type, [Apo90]. Thus, analogously to the analysis of wave parameters

[22]The subscript "$_L$" refers to the point in which the load is known.

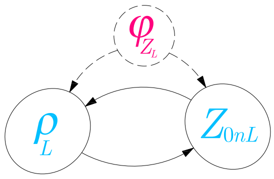

Fig. 4.28: Scheme of transformations between the normalized basic parameters regarding the inverse characterization of the CTLT referred at the load. The continuous arrows indicates the transformations between the normalized wave parameters (complex parameterized), whereas the dashed arrows indicate the extra parameterization of the impedance at the load, φ_{Z_L}.

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regarding the direct characterization, the lines and circunferences in the plane which acts as origin of parameterizations transform into circumferences in the other one, which suppose a strong reduction in the complexty of the related graphical and geometrical analysis.
Due to the simplicity of the a priori resultant curves, it is better to study the transformations geometrically before analyzing them graphically.
As said before, the transformations are studied from each complex plane: first the normalized characteristic impedance, and then the reflection coefficient; parameterizing on one hand the real-imaginary parts, and on the other hand the modulus-phase.

Tranformations from the Z_{0nL}-complex plane

The transformations from the Z_{0nL}-complex plane to the ρ-plane follow eq. (4.63).
In order to be rigorous when dealing with the transformations, the following complex function is defined:

$$\begin{aligned} \boldsymbol{\rho_{\varphi_{Z_L}}} : \quad & D_{Z_{0nL}} \subset \mathbb{C} \to D_{\rho_L} \subset \mathbb{C} \\ & Z_{0nL} \rightsquigarrow \boldsymbol{\rho_{\varphi_{Z_L}}}(Z_{0nL}) = \frac{e^{j\varphi_{Z_L}} - Z_{0nL}}{e^{j\varphi_{Z_L}} + Z_{0nL}}. \end{aligned} \tag{4.65}$$

This function $\boldsymbol{\rho_{\varphi_{Z_L}}}$ is actually a set of functions parameterized by φ_{Z_L}, and so the transformations have to be addressed in the same way.
The function is defined in the domain $D_{Z_{0nL}}$. The limits of this domain are obtained by either the direct characterization of the characteristic impedance (renormalized) or the inverse characterization (which does not have to be renormalized because the "universal" nature of angles). If analyzing, for example, the phase parameterizations of the characteristic impedance in the rg-plane in eq. 4.55 (depicted in Fig. 4.25), it may be seen that the limits of the parameterized curves are in $\varphi_{Z_0} = \pm\pi/4$, achieved at zero or infiniy, respectively (for the plus or minus), regarding any parameterization of the modulus. Thus, those Z_{0nL} values contained in the region bounded by the rays $\varphi_{Z_{0nL}} \equiv \varphi_{Z_0} = \pm\pi/4$, define the domain of the functions $\boldsymbol{\rho_{\varphi_{Z_L}}}$.

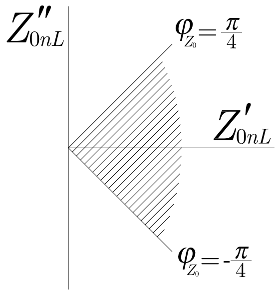


Fig. 4.29: Domain for the definition of parameterizations to be transformed from the Z_{0nL}-plane to the ρ-plane.

In the domain depicted in Fig. 4.29 the real-imaginay parameterized parts of Z_{0nL}, or the modulus and phase parameterizations are taken to be tranformed to the ρ-plane.
Once the transformations are defined (parameterized by the angle φ_{Z_L}), the range of ρ_L in $\mathbb{C}$ will be solved suppoted by the corresponding graphical analysis.

Real-imaginary parameterized parts: The Z_{0nL}-plane is parameterized in the real-imaginary parts as follows:

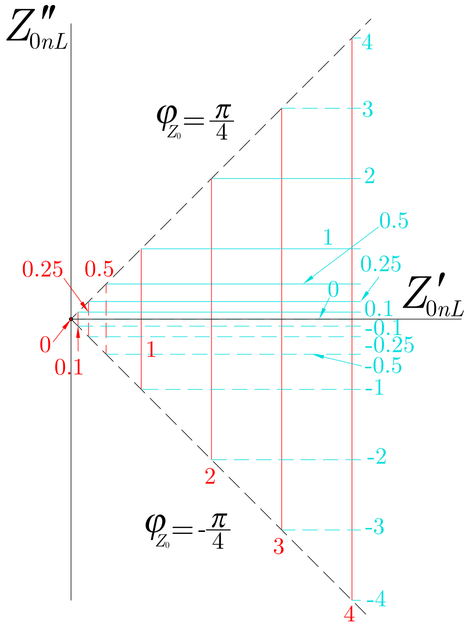


Fig. 4.30: Real-imaginary parameterized curves in the Z_{0nL}-plane.

$$\begin{cases} Z'_{0nL} = a, \text{ for which } Z''_{0nL} \in [-a, a]\text{ , and} \\ Z''_{0nL} = b, \text{ for which } Z'_{0nL} \in [b, \infty[\end{cases} . \tag{4.66}$$

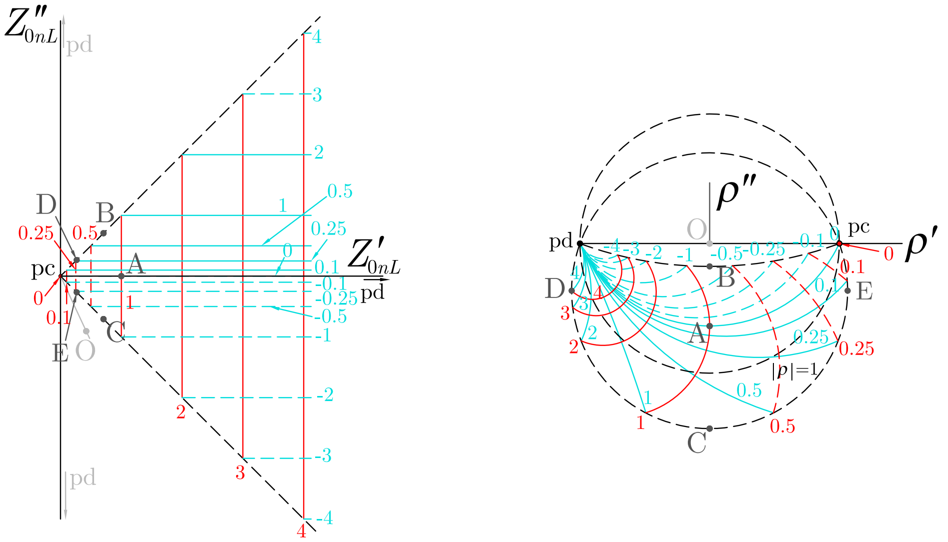


Fig. 4.31: Tranformations of the real-imaginary parameterized parts from the Z_{0nL}-plane to the ρ-plane when $\varphi_{Z_L} = -65°$.

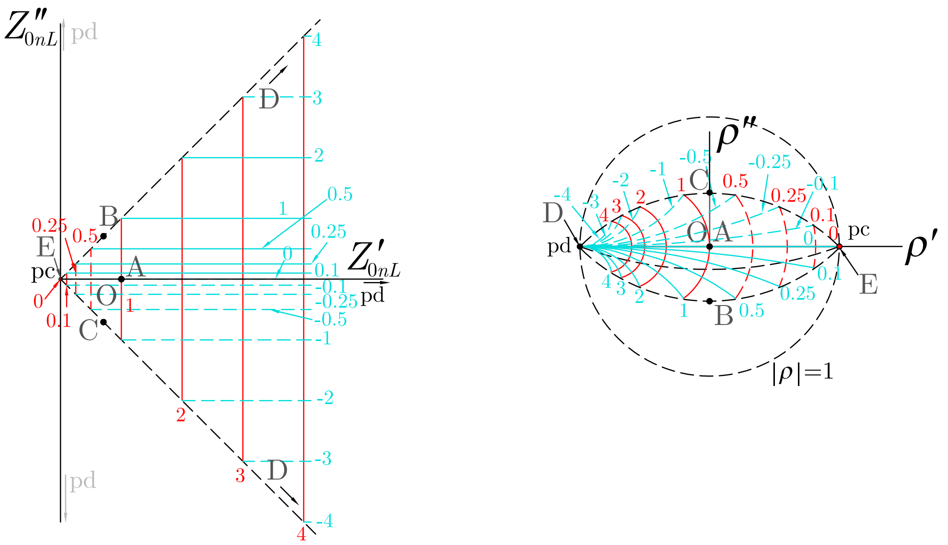


Fig. 4.32: Tranformations of the real-imaginary parameterized parts from the Z_{0nL}-plane to the ρ-plane when $\varphi_{Z_L} = 0°$.

The tranformations are analytically studied in Appendix 4.I by taking into account the geometrical properties of Möbius transformations: the resultant curves are circumferences (in fact arcs of circumferences because of the limits of the curves in the Z_{0nL}-plane). For this purpose, it is needed to find the inverse function of $\boldsymbol{\rho_L}$ in eq. (5.43), which is equivalent to solve Z_{0nL} from eq. (4.63) in terms of ρ, so exactly the same to obtain eq. (4.64).

Remark 25. *Those functions whose expression is defined as a (parameterized) linear fractional transformation have inverse, which is direct to obtain. In fact, the inverse function, which is required to solve the expression of the parameterized curves ("curve levels") in the plane of the transformation, is that one defined for the transformation in the opposite direction.*

The resultant circunferences parameterizing the real-imaginry parts are (see Appendix 4.I):

$$\begin{cases} \left(-\frac{a}{a+c_L}, \frac{s_L}{a+c_L}\right) : \frac{1}{a+c_L} \\ \left(-\frac{b}{b+s_L}, -\frac{c_L}{b+s_L}\right) : \frac{1}{|b+s_L|} \end{cases}, \tag{4.67}$$

respectively.

In Figs. 4.31 and 4.32 the graphical analysis of the transformations of the curves parameterizing the real-imaginary parts when $\varphi_{Z_L} = -65°$ and $\varphi_{Z_L} = 0°$, respectively, are depicted. In addition, some remarkable points are detached in both planes, which are explained in the subsequent geometrical analysis.

Notice that the tansformations described in this analysis play the inverse role of the transformation in the GSC. Thus, this particular transformation (from the real-imaginary parameterized parts in the Z_{0nL}-plane to the ρ_L-plane) definesthe so called "*inverse Generalized Smith Chart*" (iGSC).

Before analyzing the graphical results geometrically, the same transformation parameterized by modulus-phase is next described.

Modulus-phase parameterizations: When the Z_{0nL}-plane is parameterized in its modulus and phase:

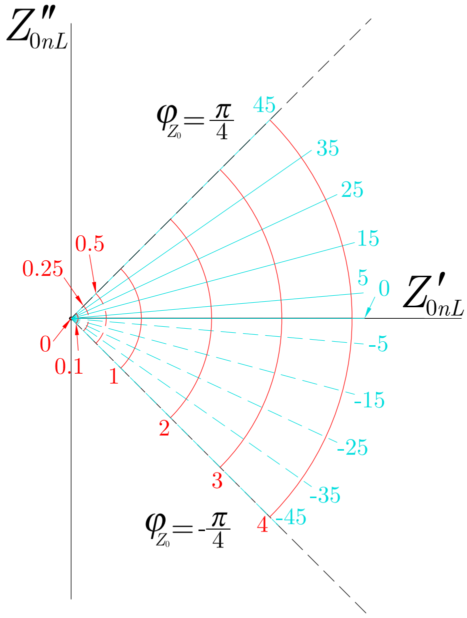


Fig. 4.33: Modulus-phase parameterized curves in the Z_{0nL}-plane.

$$\begin{cases} |Z_{0nL}| = m, \text{ for which } \varphi_{Z_0} \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right], \text{ and} \\ \varphi_{Z_0} = p, \text{ for which } |Z_{0nL}| \in \left]0, \infty\right[\end{cases} . \tag{4.68}$$

The transformations are analytically studied in Appendix 4.H, equivalently to the real-imaginary parameterizations.
The resultant circumferences parameterizing the modulus-phase are (see them in Appendix 4.I):

$$\begin{cases} \left(-\frac{m^2+1}{m^2-1}, 0\right) : \frac{2m}{m^2-1} \\ \left(0, \frac{1}{\tan(p-\varphi_{Z_L})}\right) : \frac{1}{|\sin(p-\varphi_{Z_L})|} \end{cases} , \tag{4.69}$$

respectively.

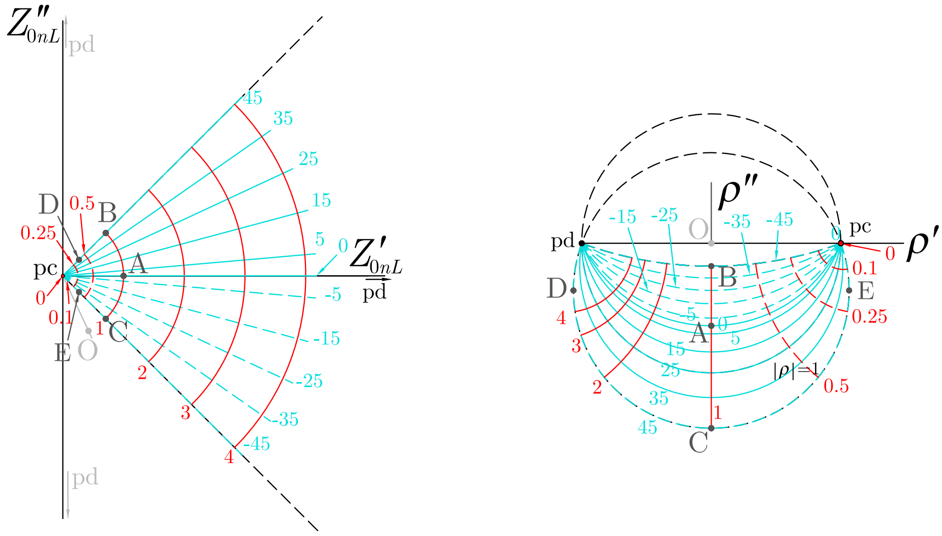


Fig. 4.34: Tranformations of the modulus-phase parameterizations from the Z_{0nL}-plane to the ρ-plane when $\varphi_{Z_L} = -65°$.

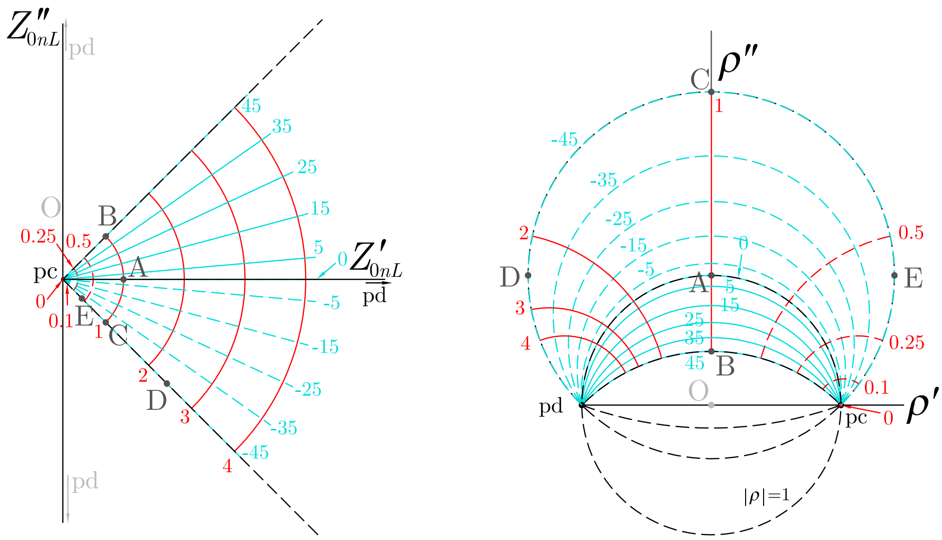


Fig. 4.35: Tranformations of the modulus-phase parameterizations from the Z_{0nL}-plane to the ρ-plane when $\varphi_{Z_L} = 90°$.

In Fig. 4.34 and 4.35, the graphical analysis of the transformations of the curves parameterizing the modulus-phase when $\varphi_{Z_L} = -65°$ and $\varphi_{Z_L} = 90°$, respectively, are depicted.

With respect to the geometrical analysis, overlapping the curves in the graphical analysis are the same remarkable points as in the real-imaginary parameterizations: the point marked as (**A**) characterizes $Z_{0nL} = 1 + j0 \equiv 1e^{j0}$, which is in the imaginary axis in the ρ-plane:

$$\rho''_{(\mathbf{A})} = \frac{1 - \cos(\varphi_{Z_L})}{\sin(\varphi_{Z_L})}; \tag{4.70}$$

the points (**B**) and (**C**) are those points characterizing $Z_{0nL} = 1/\sqrt{2} \pm j1/\sqrt{2} \equiv 1e^{\pm j\pi/4}$. These points transform to the maximum and minimum values on the imaginary axis in the ρ-plane:

$$\begin{cases} \rho_{(\mathbf{B})} \equiv \rho''_{max} = \frac{1-\cos\left(-\frac{\pi}{4}-\varphi_{Z_L}\right)}{\sin\left(-\frac{\pi}{4}-\varphi_{Z_L}\right)} \\ \rho_{(\mathbf{C})} \equiv \rho''_{min} = \frac{1-\cos\left(\frac{\pi}{4}-\varphi_{Z_L}\right)}{\sin\left(\frac{\pi}{4}-\varphi_{Z_L}\right)} \end{cases}; \tag{4.71}$$

the points (**D**) and (**E**) are respectively the minimum and maximum values of the real part of ρ. Despite obtaining these points in the Z_{0nL}-plane is object of the opposite tranformation (from the ρ-plane to the Z_{0nL}-plane), solving them is an interesting example of how the graphical analysis helps the geometrical operations: As it might be seen by means of Figs. 4.31-4.35, the points (**D**) and (**E**) are over the curve parameterized by $p = -45°$ if $\varphi_{Z_L} > 0$ or $p = 45°$.

It is important to detach that the transformations from the Z_{0nL}-plane to the ρ-plane (parameterized by $\varphi_{Z_{Ln}}$) regarding this inverse characterizations follow the same form as those from the Y_{n0}-plane to the ρ-plane (parameterized by φ_{Z_0}) regarding the direct characterization if applying the following changes:

$$\begin{cases} \varphi_{Z_L} \to -\varphi_{Z_0} \\ Z_{0nL} \to Y_{n0} \end{cases}, \tag{4.72}$$

and viceverse when changing from the direct characterization to the inverse characterization.
This fact helps the use of both transformations -direct and inverse- together, for example changing between the GSC and the iGSC.

Tranformations from the ρ-complex plane

The transformations from the ρ-plane to the Z_{0nL}-plane follow eq. (4.64).
In order to be rigorous when dealing with the transformations, the following complex function (in complex variable) is defined:

$$\begin{aligned} \boldsymbol{Z_{0nL,\varphi_{Z_L}}}: \quad & D_{\rho_L,\varphi_{Z_L}} \subset \mathbb{C} \to D_{Z_{0nL}} \subset \mathbb{C} \\ & \rho_L \rightsquigarrow \boldsymbol{Z_{0nL,\varphi_{Z_L}}}(\rho_L) = e^{j\varphi_{Z_L}} \frac{1-\rho}{1+\rho}. \end{aligned} \tag{4.73}$$

This function is actually a set of functions parameterized by φ_{Z_L}, which supposes a final rotation in the curves prior to be transformed to the Z_{0nL}-plane. Thus, the transformations following Z_{0nL} have to be addressed as a set parameterized by φ_{Z_L}.
The function Z_{0nL} is defined in the domain $D_{\rho,\varphi_{Z_L}}$. This domain depends on φ_{Z_L}, and that because the concordance to be imposed with the function $\rho_{\varphi_{Z_L}}$ defined in eq. (5.43), whose range has to be the domain of Z_{0nL} in orther to be them inverse functions. As it may be seen by the graphical analysis of the transformation $\rho_{\varphi_{Z_L}}$, its range in the ρ-plane is delimited by the transformations of the curves parameterized by $\varphi_{Z_0} = \pm\pi/4$, includying the point (**A**):

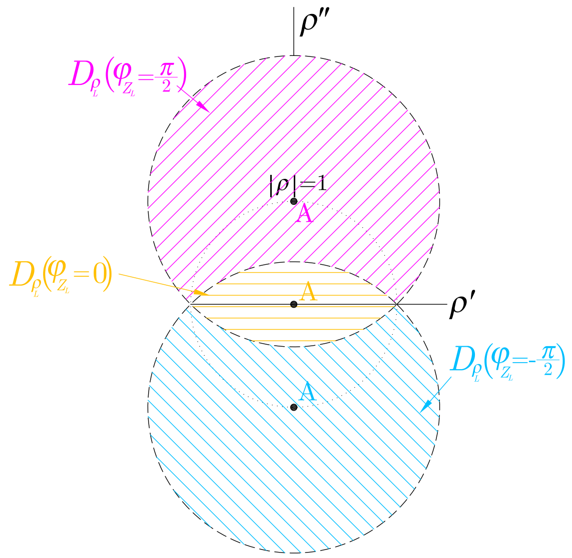


Fig. 4.36: Different domains of the $\rho_{L,\varphi_{Z_L}}$ function: $D_{\rho_L,\varphi_{Z_L}=\pm\pi/2}$ and $D_{\rho_L,\varphi_{Z_L}=0}$; in which different parameterized curves are defined to be transformed to the Z_{0nL}-plane.

For example, in case $\varphi_{Z_L} = \pi/2$, the upper limit the circumfernce $(0,1) : \sqrt{2}$, which intersects with the lower limit in $\rho = -1 + j0$ and $\rho = 1 + j0$; the cirfumference $(0,-1) : \sqrt{2}$.

For analyzing the function Z_{n0L} defined in eq. (5.39) graphically, each region $D_{\rho_L,\varphi_{Z_L}}$ is parameterized in its real-imaginary parts or its modulus-phase.

Real-imaginary parameterized parts: The ρ-plane is firstly parameterized in its real-imaginary parts as follows:

$$\begin{cases} \rho' = a \in \left[-\frac{1}{|\sin\left(\frac{\pi}{4}-\varphi_{Z_L}\right)|}, \frac{1}{|\sin\left(\frac{\pi}{4}-\varphi_{Z_L}\right)|}\right] \\ \text{, for which } \rho'' = \pm\frac{\sqrt{1-a^2\sin^2\left(\frac{\pi}{4}-\varphi_{Z_L}\right)}-\cos\left(\frac{\pi}{4}-\varphi_{Z_L}\right)}{\sin\left(\frac{\pi}{4}-\varphi_{Z_L}\right)} \\ \rho'' = b \in \left[-\frac{1}{|\sin\left(\frac{\pi}{4}-\varphi_{Z_L}\right)|}, \frac{1}{|\sin\left(\frac{\pi}{4}-\varphi_{Z_L}\right)|}\right] \\ \text{, for which } \rho' = \pm\sqrt{1-b^2-\frac{2b}{\tan\left(\frac{\pi}{4}-\varphi_{Z_L}\right)}} \end{cases}. \tag{4.74}$$

The tranformations to the Z_{0nL}-plane are studied using the inverse mapping in eq. (4.63), parameterizing the real imaginary parts, similarly to the procedure which have been followed in Appendix 4.I for the contrary transformation.
Since the transformations are also Möbius-type, they result in circunferences in the Z_{0nL}-plane parameterized as:

$$\begin{cases} \left(-c_L\frac{a}{a+1}, -s_L\frac{a}{a+1}\right) : \frac{1}{|a+1|} \\ \left(\frac{-s_L+bc_L}{b}, \frac{-s_L-bc_L}{b}\right) : \frac{1}{|b|} \end{cases}. \tag{4.75}$$

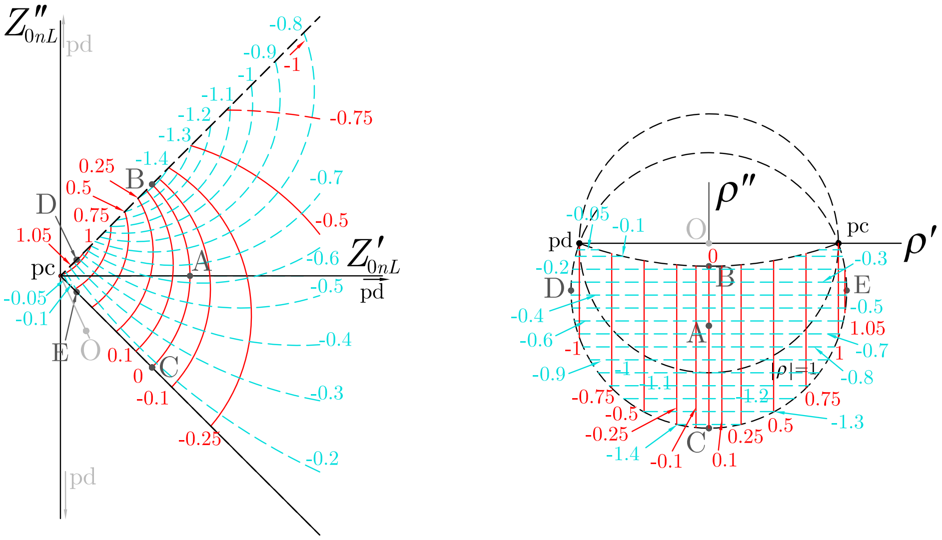


Fig. 4.37: Tranformations of the real-imaginary parameterized parts from the ρ-plane to the Z_{0nL}-plane when $\varphi_{Z_L} = -65°$.

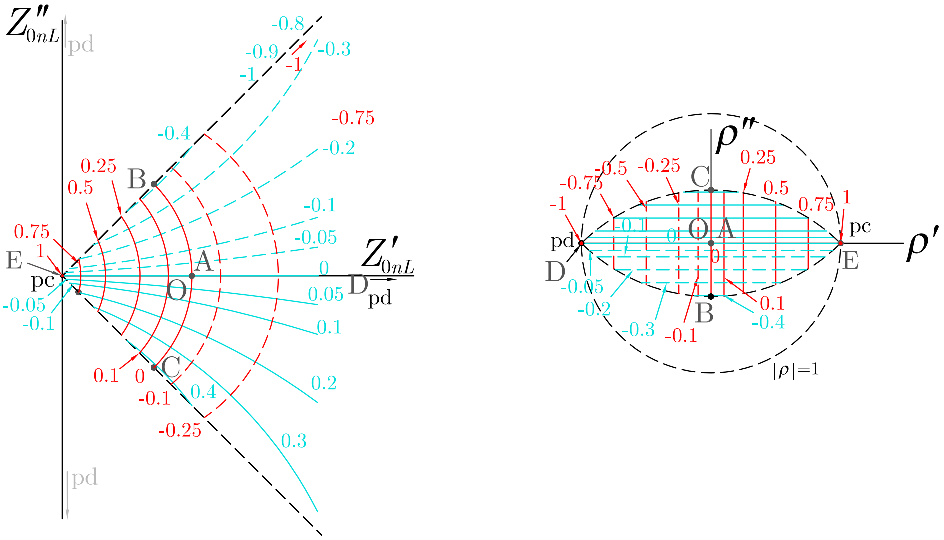


Fig. 4.38: Tranformations of the real-imaginary parameterized parts from the ρ-plane to the Z_{0nL}-plane when $\varphi_{Z_L} = 0°$.

In Figs. 4.37 and 4.38 the transformations of the curves parameterized by the real-imaginary parts from the ρ-plane to the Z_{0nL}-plane when $\varphi_{Z_L} = -65°$ and $\varphi_{Z_L} = 0°$, respectively, are graphically represented.

Modulus-phase parameterizations: Now the ρ-plane is parameterized in its modulus-phase:

$$\begin{cases} |\rho| = m \\ \varphi_\rho = p \end{cases} . \tag{4.76}$$

The tranformations to the Z_{0nL}-plane are also studied using the inverse mapping in eq. (4.63), but parameterizing the modulus and phase of ρ, following a similar mathematica procedure to the one developed in Appendix 4.I for the opposite transformation.
Since the transformations are also Möbius-type, they result in circunferences in the Z_{0nL}-plane parameterized as:

$$\begin{cases} \left(-c_L \frac{m^2+1}{m^2-1}, -s_L \frac{m^2+1}{m^2-1}\right) : \frac{2m}{|m^2-1|} \\ \left(-\frac{s_L}{\tan(p)}, \frac{c_L}{\tan(p)}\right) : \frac{1}{|\sin(p)|} \end{cases} . \tag{4.77}$$

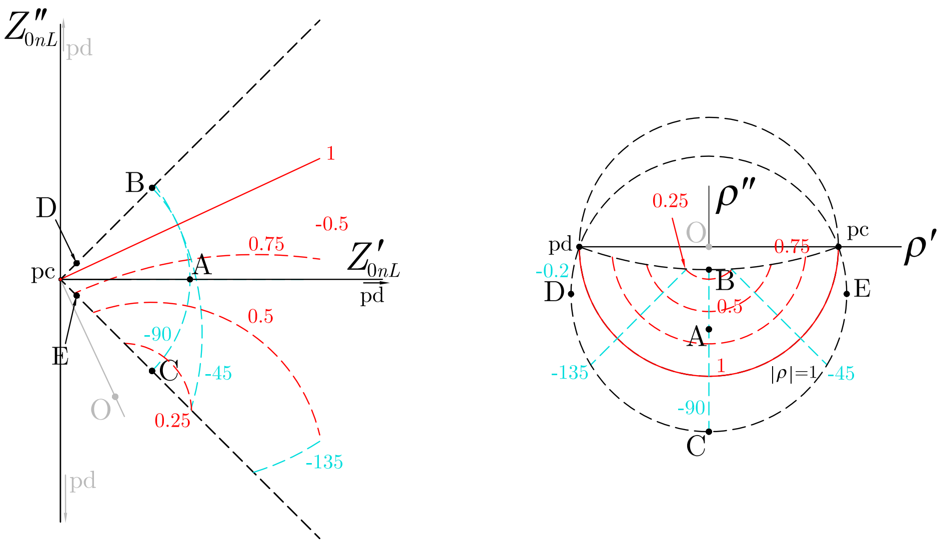


Fig. 4.39: Tranformations of the modulus-phase parameterizations from the ρ-plane to the Z_{0nL}-plane when $\varphi_{Z_L} = -65°$.

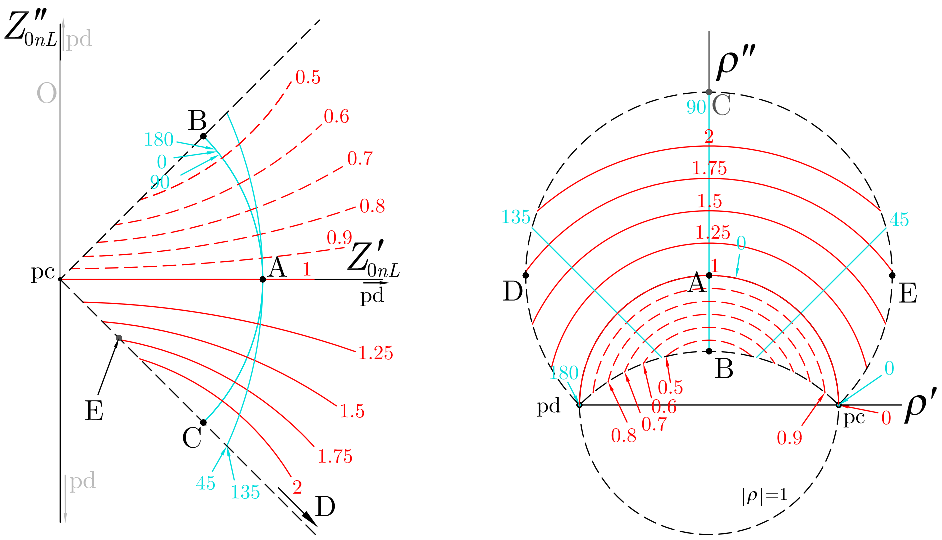


Fig. 4.40: Tranformations of the modulus-phase parameterizations from the ρ-plane to the Z_{0nL}-plane when $\varphi_{Z_L} = 90°$.

In Figs. 4.39 and 4.40 the transformations of the curves parameterized by the modulus-phase from the ρ-plane to the Z_{0nL}-plane are represented when $\varphi_{Z_L} = -65°$ and $\varphi_{Z_L} = 90°$, respectively.

***Physical interpretations*:** The inverse characterization of basic parameters should be interpreted as the analysis of the changes affecting the line parameters when the wave impedance at any point of the TL is known. Since the wave impedance parameterizes the analysis, all the parameters under this inverse study are referred to the load, although this does not mean studying the parameters at the end of the TL.

Typically, the wave parameters are analyzed as a function of basic parameters along the TL, which is the objective of the direct characterization. These parameters detrmine the relation of the total equivalent waves, an so the physical meaning. For this purpose, ρ is consiered part of wave parameters, because it composes the total voltage and current waves separately. In this case, the characteristic impedance (besides the propagation constant) is considered contant, something which agrees the definitions regarding the original equations.
But, in the case of the inverse characterization, the analysis deals with study of the variations on the basic parameters when they change. In this way, it is about knowing how the individual wave solutions change, and so the phyical meaning is somewhat missed in this sense. Nevertheless, this analysis gains in the physical interpetation of how the variations of wave parameters affect the total wave when analyzing ρ, for example when frequency varies.
Thus, the role of ρ should be in reinterpreted in certain way. Here it does not act as a paramter to compose the wave impedance as final objective, but the parameter which determines the matching when the characteristic impedance varies. As a consequence, this analysis leaves room to the possibility of varying the characteristic impedance with frequency for example, something that is inherent to lossy TLs.

The mentioned reinterpretation for the role of ρ makes that some particular cases and remarkable points in its associated complex plane also do. The points $\rho = 1 + j0$ and $\rho = -1 + j0$ (which correspond to the open circuit (**oc**) and the short circuit (**sc**), respctively, regarding the direct characterization) correpond to the perfect conductor (**pc**) and the perfect dielectric (**pd**), that is $Z_0 = 0$ and $Z_0 = \infty$, respectively, in this inverse characterization. Equivalently, if using the inverse characterization of line parameters, it means that the point (**pc**) in the ρ-plane is associated to ($r = 0$,$g \to \infty$) in the rg-plane, whereas the point (**pd**) is associated to ($r \to \infty$,$g = 0$) in the rg-plane.

On one hand, the transformation from the Z_{0nL}-plane to the ρ-plane has to be understood in terms of which reflection coefficients produce different parameterizations of the characteristic impedance. On the other hand, the transformation from the ρ-plane to the Z_{0nL}-plane should be interpreted in terms of which characteristic impedance produce different parameterizations of the reflection coefficient.
Both types of (complex) transformations (the transformations in both senses) do not have explicit physical meaning until the curves of any of the domains are connected, leading to the effective practical uses of this analysis, for example, when connecting the curves in the Z_{0nL} describing how the characteristic impedance (in comparison with the impedance at the load, which fixes the normalization) varies in the whole frequency band.

In any case, it is convinient to detach the role of φ_{Z_L} concerning the inverse characterization. Just like φ_{Z_0} in the direct characterizations, the parameter φ_{Z_L} determines the characterization of the TL at the load. Thus, this is the parameter which characterizes the load. For example, if the load is purely capacitive $\varphi_{Z_L} = -90°$; if the load is resistive $\varphi_{Z_L} = 0°$, and the iGSC to be used in this case is the one depicted in Fig. 4.32; if the load is purely inductive $\varphi_{Z_L} = 90°$, for which the plane to be used is the one respresented in Fig. 4.35 (in this case, this is not a iGSC becuase it parameterizes the modulus-phase of Z_{0nL}), and if the load is mixed (resistive and capacitive/iductive) the iGSC to be used is, for example, the one represented in Fig. 4.31.

***Practial uses*:** As it has been foreseen, the inverse characterization of basic parameters is especially useful when analyzing TLs at the load.
For example, the complex transformations from the Z_{0nL}-plane to the ρ-plane serve to analyze how the reflection coefficient behaves when the characteristic impedance changes with losses or frequency. These significant analysis are analyzed in Ex. 02 in Sect. 5.3 in Chpt. 5 as example of use. These example reveals the usefulness of combining the direct characterization and the inverse characterization, both referred to the basic parameters.

Regarding the transformation from the ρ-plane to the Z_{0nL}, it may be used to characterize which characteristic impedances lead to specific values of the reflection coefficient.
An interesting application of this analysis comes when analyzing the characteristic impedances which lead to decribe ρ along the TL. Normally the wave impedance is described along the TL through ρ, when it is defined along the TL (this analysis is developed in Ex. 01 in Sect. 5.2 in Chpt. 5). Nevetheless, it is possible to think inversely solving "Z_{0nL} along the TL" from ρ described along the TL, and fixing the wave impedance at the load. Then, by using the inverse characterization of line parameters it is possible to see which losses/frequency parameterizations respresent the length of the "inverse" TL.

Remark 26. *The inverse analysis not only solves directly which TLs (which losses/frequency) are associated with different specifications on basic parameters. It is also the analysis which parameterizes different behaviors of basic and wave parameters with the lossy/frequency parameterizations, which is the true advantage of the inverse characterization when parameterizing different physical properties, for example the TL's length; or more solutions, for example, superior modes.*
For the purpose of parameterized different TL-related problems, the a priori real parameterizations turn to be complex (see the ***Applications*** *for having a reference of how complex parameterizations explain, for example, the behavior of different mode solutions).*

4.5 Conclusions

In this chapter, the CTLT regarding HPWs which propagate in lossy media has been presented exemplifying the methodology of analysis based on Complex Analysis, which is an interesting alternative to describe in detail the underlying equations regarding the LTLT.

Although the presented analysis of HPW in the *frequency domain* parameterized by losses is only an example of how to proceed for analyzing more mode solutions (related to different BCs), domains, and parameterizations, the methodology of analysis has been described in detail, which leaves many interesting conclusions to detach, serving as guide for future analysis, and also contributing with interesting analytical resources.
Before explaining the results regarding the behavior of HPWs in lossy TLs, it is crucial to outline the main contributions and interpretations obtained regarding generalized analysis of TLs.

The analysis of TLs is focused on the parameters which describe the propagative solutions. Thus, characterizing these parameters in the context of the analysis which is intended to be described (for example, in terms of losses) is the way to obtain the interpretation of the solutions. However, these (physical) interpretations are preceded by the definition of the (complex) transformations between the parameters involved. Although these transfomations –which have special intuitive representation in (complex) planes– lack physical meaning, they suppose the basis for analyzing any TL-related problem, and thus they have been called "basic transformations".
In this context, the inverse characterization, which is presented here to support the GTLT (in particular the GTLT-v1 introduced in Chpt. 3) for analyzing the influence of losses in TLs, is the correct way to proceed for any analysis before providing the physical interpretations.

Remark 27. *Regarding the LTLT, the inverse characterization arises from answering: "Which line parameters parameterize the TL which simulates the propagation of HPWs in lossy media?"; but the same question could be posed for the interest of parameterizing the TL of different modes, in different domains, and under different parameterizations.*

This inverse characterization (which often will not be achieved easily) requires a solid algebraic definition of the "space of parameters" (or "space of parameterizations"), which is met in the definition of isocomplex numbers (see this definition in Appendix 4.A).
Based on this definition, the analysis of the TL parameters as complex transformations is rigorously posed: the transformations are defined as complex functions which related complex parameters, so they are complex variable functions. Thus, these functions are addressed as mappings relate to Complex Analysis, as well as their properties (for example, conformability, which is crucial for characterizing the transformations geometrically). The mappings are generically defined as transformations of the real-imaginary parameterized parts or the modulus-phase parameterizations, in abscense of providing the physical interpretations to the analysus yet.
Moreover, going back in the inverse characterization lets to answer "which parameterizations (of basic parameters) come from the total waves (parameterized by the wave parameters)". In this latter process, the reflection coefficient loses its original meaning as wave parameter to be considered as basic parameter for the inverse characterization.

Remark 28. *The reflection coefficient becomes the parameter which connect the analysis of basic parameters and wave parameters for any characterization thanks to the duality here suggested playing the role of wave or basic parameter depending on the analysis that is being carried out.*

With the definition and pursposes of the inverse characterization in mind, the direct characterization provides the physical sense to the parameterized analysis. Moreover, the transformations involved in the direct characterization are able be studied within the context of mappings.
The direct characterization refers to all the possible analysis that connect the complex parameterizations of the inverse characterization, giving it physical sense, for example, the analyis of basic parameters in terms of losses.

Direct characterization	Inverse characterzation
• Physical parameterizations (real): losses, frequency, length, etc.	• Complex parameterizatons: real-imaginary/modulus-phase.
• Complex parameters described by real parameterizations.	• Complex parameters (sometimes particularized to real parameters) described by complex parameterizations.
• It describes one physical (real) behavior	• It parameterizes the specifications (complex) parameters
• The phase φ_{Z_0} is the parameter which characterizes losses/frequency	The phase φ_{Z_L} is the parameter which characterizes the total waves

Table 4.4: Comparison between the properties of the direct and inverse characterizations.

Furthermore, both the direct and inverse characterizations are based on parameterizations which are described by angles in any plane or space. This is direct consequence of normalizing the parameters involved in the transformations, making the analysis "universal", which means that any point in the normalized complex planes or in the space of parameters represents multiple TLs which keep the value of this parameter constant.
In this sense, the rg-plane defined from the "space of parameters" gathers all the definition of the TL parameters based on angles in only one graph: losses, frequency, the phases of basic parameters, etc.; that is, all the parameterizations which "universalize" the analysis.

When looking at the results of the analysis in terms of losses and frequency regarding the direct characterization, it may be concluded that: (i) the graphical analysis of the resultant parameterized curves leads to analyze the TL parameters geometrically, and thus having an alternative representation of them as parameterized curves, besides the underlying complex expressions; (ii) the geometrical analysis lets to characterize more complex problems in the same way (geometrically), avoiding the limitations of studying the complex expressions analytically. Related to the geometrical analysis and the idea of "universalizing" the characterizations by means of angles, the phases φ_{Z_0} and φ_{Z_L} have special importance to the parameterizations of losses/frequency for the direct characterization, and the parameterization of wave parameters (the BCs in any z along the TL) for the inverse characterization. These parameters determine the (extra) parameterizations providing the physical meaning or solvability in each case (see Table 4.4).
The analysis in terms of losses and frequency (the direct characterization) explains the particular cases and approximations, and it is especially important for showing the limits in which the parameters vary while providing physical meaning to the solutions.

For finishing the characterization of HPWs within the LTLT, the wave parameters representing the total waves have to be analyzed in the variables they depend on: the length of the TL and frequency (direct characterization). For this purpose, the analysis presented in this chapter are applied to solve these TL-related problems within the context of the CTLT, serving as examples of use, which are presented in next chapter.

Chapter 4

Appendices

Appendix 4.A

The complex "space" of parameters for the TL characterization is here presented detaching: (i) its analytical properties based on its mathematical definition; (ii) its graphical and geometrical representation, including some interesting projections useful for the TL analysis, for example the rg-plane; (iii) its physical interpretation in terms of the parameters of the TL; and (iv) its practical uses in characterizing any TL.
Although most of these properties are presented particularized to the analysis of harmonic plane waves in lossy media in the *frequency domain*, this "space" is useful for studying any type of parameterizations concerning guided waves.
The definition of the "space" of parameters is crucial for understanding and defining the complex transformations as direct or inverse functions.

***Definition*:** The orgin of this "space" of parameters is direct consequence of the possibility of "universalizing" the curves which represent the TL parameters (in particular the basic parameters) in their respective normalized complex planes, without affecting the physical meaning of the parameters involved. In this sense, it has been seen that the parameterizations reduce the complexity of analyis taking advantage of its physical meaning. For example, the characteristic impedance regarding the lossy case, Z_0, introduced in eq. (2.51), depends on four parameters: R, L, G, and C; besides the frequency, ω. If the intended analysis consists in characterizing Z_0 in terms of lossy parameters, an appropriate normalization with respect to the losseless characteristic impedance, $Z_{0,sp}$, introduced in eq. (2.94), leads to reduce the parameterizations to two (the frequensy is included within these couple of parameterizations). So, in this case, the analysis interpreted in terms of losses is appropriately reduced, as it has been developed in Sect. 4.3.1 for the *ffa*.
These facts –and the experience of studying multiple parameterized analysis which have been developed in the same way– lead to think that any of the proposed analysis could be done by means of the appropriate parameterizations of TL parameters, which motivates the definition and use of the "space" of parameters introduced here.

For the deinition of the "space" of parameters, firstly start considering tuples, τ, of three elements:

$$\begin{aligned} \tau \equiv (r', g', j\omega)\,, \\ r', g', j\omega \in \mathbb{C}, \end{aligned} \tag{4.A.1}$$

in which $r' = R/L$, $g' = G/C$, and $j\omega = j\omega = j\,(\omega' + j\omega'') = j\omega' - \omega''$. These definitions are generic for multiple purposes: the a priori real parameters R, L, G, and C, and also ω are complexified, and they particularize accordingly the study that is being carried out.
In the *frequency domain* $\omega \equiv \omega' \in \mathbb{R}^+ \cup \{0\}$, so the third element of the tuple in eq. (4.A.1) is $j\omega \equiv j\omega' \in \mathbb{I}$[23].

For the graphical understanding of the the operations which are going to be defined involving τ-tuples, the components r' and g' are considered real positive or zero. This case corresponds with

[23]The imaginary part of ω, ω'', would be the alternative form of considering losses if the solutions of wave equations were seeking in the "*phase constant domain*".

parameterizations of HPWs:

$$\tau_{HPW} \equiv \tau \equiv (r', g', j\omega)\,, \qquad (4.A.2)$$
$$r', g' \in \mathbb{R}^+ \cup \{0\}, \text{ and } j\omega \in \mathbb{I}^+ \cup \{0\}.$$

Nevertheless, this particularization could be generalized in case r' and g' were complex by using hipercomplex numbers accordingly (see for example the documents in [Ola00] for having a full reference of the definitions and operations regarding hipercomplex numbers).

Some operations involving τ-tuples are defined:
The sum of two τ-tuples, τ_1 and τ_2 is defined as

$$\tau_1 + \tau_2 = (r'_1, g'_1, j\omega_1) + (r'_2, g'_2, j\omega_2) = (r'_1 + r'_2, g'_1 + g'_2, j\omega)\,, \qquad (4.A.3)$$
$$\text{for which is mandarory that } \omega_1 = \omega_2 = \omega,$$

otherwise the sum (+) is not defined.
Because two tuples must have the same third component, which is a imaginary number ($\in \mathbb{I}$) in the *frequency domain*, these numbers are called *isocomplex numbers*.
Under this premise, the τ-tuples ($\in \mathbb{H} \equiv \mathbb{C}^2 \times \mathbb{I}^+ \subset \mathbb{C}^3$) form an Abelian group $((\mathbb{H}, +))$[24].
For the practical uses of these isocomplex numbers, it is neccesary to emphasize the particular sums:

$$\tau_r = (r', 0, j\omega) + (0, 0, j\omega)\,, \text{ and} \qquad (4.A.4)$$
$$\tau_g = (0, g', j\omega) + (0, 0, j\omega)\,, \qquad (4.A.5)$$

so $\tau_{i,+} = (0, 0, j\omega)$ acts as the sum identity for isocomplex numbers whose third component is $j\omega$. The $\tau_{i,+}$-tuple is written simplified as $j\omega$.
Notice that the numbers τ_r and τ_g represent, separately, complex numbers, but their sum can not be interpreted in terms of the addition of complex numbers.
Basically, the sum operation defined in eq. (4.A.3) serves to define isocomlex numbers that merge r' and g' components, so any isocomplex number τ may be separated into its τ_r and τ_g parts, so that $\tau{=}\tau_r{+}\tau_g$.

An alternative representation for isocomplex numbers is the polar form. The polar form of isocomplex numbers follows the notation

$$\tau \equiv \rho_\tau \angle \theta_\tau + j\omega_\tau, \qquad (4.A.6)$$
$$\text{in which } \rho_\tau = \sqrt{r'^2 + g'^2} \in \mathbb{C}\text{ , and } \theta_\tau = \tan^{-1}(g'/r') = \tan^{-1}(1/c) \in \mathbb{C}.$$

The sum operation with the sum identity ($j\omega_\tau$) is used in this notation to emphasize the dependence on ω of the addend $\rho_\tau \angle \theta_\tau + j\omega_\tau \equiv (r', g', j\omega)$.
The modulus ρ_τ and the polar angle θ_τ are generally defined in $\mathbb{C}$. Nevertheless, for HPW they are (positive or zero) real numbers, $\rho_\tau \in [0, \infty[$ and $\theta_\tau \in [0, \pi/2]$.
The tuples are obtained by transforming the polar form with the following changes:

$$\begin{cases} r' = \rho_\tau \cos(\theta_\tau) \\ g' = \rho_\tau \sin(\theta_\tau) \\ \omega \equiv \omega_\tau \end{cases}, \qquad (4.A.7)$$

in which the functions cos(∘) and sin(∘) have in general arguments in $\mathbb{C}$ (but in $\mathbb{R}^+ \cup 0$ in case of parameterizing HPWs).
The polar form is especially useful for defining the product between isocomplex numbers below.

[24]Notice that for the τ-tuples representing harmonic plane waves are within an Abelian group (with inverse of the sum), it is neccesary that r', $g' \in \mathbb{R}$, not in $\mathbb{R}^+$, as it has been particularized in this appendix. Nevertheless, the inverse element of the sum is not used in practice.

The product of two τ-tuples is defined in porlar form as

$$\begin{aligned}\tau_1 \cdot \tau_2 &= (\rho_{\tau_1}\angle\theta_{\tau_1} + j\omega_1)\cdot(\rho_{\tau_2}\angle\theta_{\tau_2} + j\omega_2) = \\ &= (\rho_{\tau_1}\rho_{\tau_2} - \omega_1\omega_2)\angle(\theta_{\tau_1} + \theta_{\tau_2}) + j(\omega_1\rho_2 + \omega_2\rho_1),\end{aligned} \tag{4.A.8}$$

so the modulus of the product in its polar form is $\rho_{\tau_1\cdot\tau_2} = \rho_{\tau_1}\rho_{\tau_2} - \omega_1\omega_2$, the angle $\theta_{\tau_1\cdot\tau_2} = \theta_{\tau_1} + \theta_{\tau_2}$, and the third component is the "beat" of the frequencies of the isocomplex factors weighted by the modulus, $\omega_{\tau_1\cdot\tau_2} = \omega_{\tau_1}\rho_{\tau_2} + \omega_{\tau_2}\rho_{\tau_1}$.
Notice that, in this case, the third components of the factors do not have to be the same (as for the sum), although they will be when using isocomplex numbers for practical purposes.
In this case the product indentity is

$$\tau_{i,\cdot} = 1\angle 0 + j0 \; (\equiv (1 + j0)\angle(0 + j0) + j0), \tag{4.A.9}$$

and, by definition, $\tau = \tau \cdot \tau_{i,\cdot}$.
From the definition of $\tau_{i,\cdot}$, the inverse of an isocomplex number is defined as

$$\tau^{-1} \equiv \frac{1}{\tau} = \left[\left(\frac{\rho_\tau}{\omega_\tau^2 + \rho_\tau^2}\right)\angle(-\theta_\tau)\right] - j\left(\frac{\omega_\tau}{\omega_\tau^2 + \rho_\tau^2}\right). \tag{4.A.10}$$

The isocomplex numbers form a Abelian group with the product $(\cdot)$ defined above, $((\mathbb{H}, \cdot))$.

Proof. It has been seen that: (i) the product of two isocomplex numbers is, by definition, another isocomplex number (closure); (ii) there is an identity element, $\tau_{i,\cdot}$, defined in eq. (4.A.9); and (iii) the inverse for each isocomplex number (except for $\tau_0 = 0\angle\theta_{\tau_0} + j0$) is denoted as τ^{-1}, as introduced in eq. (4.A.10).
The proof of the commutativity is direct by using the conmmutativity between complex numbers.
The associativity is proved as follows: for all τ_1, τ_2, and τ_3 in $\mathbb{H}$, the associativity would mean:

$$\tau_1 \cdot (\tau_2 \cdot \tau_3) = (\tau_1 \cdot \tau_2)\,\tau_3 \equiv \tau_3 \cdot (\tau_1 \cdot \tau_2). \tag{4.A.11}$$

The l.h.s. in equation before is developed as

$$\begin{aligned}\tau_1 \cdot (\tau_2 \cdot \tau_3) &= (\rho_{\tau_1}\angle\theta_{\tau_1} + j\omega_{\tau_1})\left[(\rho_{\tau_2}\rho_{\tau_3} - \omega_{\tau_2}\omega_{\tau_3})\angle(\theta_{\tau_2} + \theta_{\tau_3}) + j(\omega_{\tau_1}\rho_{\tau_2} + \omega_{\tau_2}\rho_{\tau_1})\right] = \\ &= (\rho_{\tau_1}\rho_{\tau_2}\rho_{\tau_3} - \rho_{\tau_1}\omega_{\tau_2}\omega_{\tau_3} - \omega_{\tau_1}\rho_{\tau_2}\omega_{\tau_3} - \omega_{\tau_1}\rho_{\tau_3}\omega_{\tau_2})\angle(\theta_{\tau_1} + \theta_{\tau_2} + \theta_{\tau_3}) + \\ &+ j(\omega_{\tau_1}\rho_{\tau_2}\rho_{\tau_3} - \omega_{\tau_1}\omega_{\tau_2}\omega_{\tau_3} + \rho_{\tau_1}\rho_{\tau_2}\omega_{\tau_3} + \rho_{\tau_1}\rho_{\tau_3}\omega_{\tau_2}).\end{aligned}$$

The r.h.s is

$$\begin{aligned}\tau_3 \cdot (\tau_1 \cdot \tau_2) &= (\rho_{\tau_3}\angle\theta_{\tau_3} + j\omega_{\tau_3})\left[(\rho_{\tau_1}\rho_{\tau_2} - \omega_{\tau_1}\omega_{\tau_2})\angle(\theta_{\tau_1} + \theta_{\tau_2}) + j(\omega_{\tau_1}\rho_{\tau_2} + \omega_{\tau_2}\rho_{\tau_1})\right] = \\ &= (\rho_{\tau_3}\rho_{\tau_1}\rho_{\tau_1} - \rho_{\tau_3}\omega_{\tau_1}\omega_{\tau_2} - \omega_{\tau_3}\rho_{\tau_1}\omega_{\tau_2} - \omega_{\tau_3}\rho_{\tau_2}\omega_{\tau_1})\angle(\theta_{\tau_1} + \theta_{\tau_2} + \theta_{\tau_3}) + \\ &+ j(\omega_{\tau_3}\rho_{\tau_1}\rho_{\tau_2} - \omega_{\tau_1}\omega_{\tau_2}\omega_{\tau_3} + \rho_{\tau_3}\rho_{\tau_1}\omega_{\tau_2} + \rho_{\tau_3}\rho_{\tau_2}\omega_{\tau_1}).\end{aligned}$$

If comparing the developed l.h.s. and r.h.s. term by term, it can be seen they are the same, so the associativity is proved. □

However, the isocomplex numbers forming a group with the sum and product as defined above $((\mathbb{H}, +, \cdot))$ is not possible because the heterogeneity of these operations causes that the distributive property can not be verified. Nevertheless, this property is not crucial for the practical purposes of isocomplex numbers.

The main usefulness of isocomplex numbers is when using them in the product. In this sense, it is useful to known the closed expression of the square root of an isocomplex number, $\sqrt{\tau_1} = \tau_2$.
Using the 2^{nd} power:

$$\tau_1 = (\pm\tau_2)^2,$$

indentifying the modulus, phase and the term in j, and solving those concerning τ_2, it leads to

$$\begin{cases} \rho_{\tau_2} = \pm\sqrt{\frac{\rho_{\tau_1}+\sqrt{\rho_{\tau_1}^2+\omega_{\tau_1}^2}}{2}} \\ \theta_{\tau_2} = \frac{\theta_{\tau_1}}{2} \\ \omega_{\tau_2} = \pm\frac{\omega_{\tau_1}}{\sqrt{2}\sqrt{\rho_{\tau_1}+\sqrt{\rho_{\tau_1}^2+\omega_{\tau_1}^2}}} \end{cases}.$$

Thus, the square root of $\tau = \rho\angle\theta j\omega$ is:

$$\sqrt{\tau} = \pm\sqrt{\frac{\rho + \sqrt{\rho^2+\omega^2}}{2}}\angle\theta/2 \pm j\frac{\omega}{\sqrt{2}\sqrt{\rho+\sqrt{\rho^2+\omega^2}}}, \tag{4.A.12}$$

for which the sign is accordingly chosen in terms of the domain of interest, for example only the positive solution is valid for describing parameters of HPWs in the *frequency domain.*

***Graphical representation and geometrical interpretation*:** The so called isocomplex numbers can be represented in a complex "space". Strictly speaking, this "space" of numbers is not an Euclidean space, althoufh the graphical representation may help the understading of the definition of these numbers, and also some of their properties based on geometrical analysis. Moreover, these geometrical properties have clearly correspondence with the study of parameterizations in complex functions representing the TL parameters, and thus the representation shows a lot of usefulnesses in the TLT.

The representation of isocomplex numbers is, on one hand, clearly inspired by the representation of complex numbers in a complex plane but, in the other hand, influenced for the physical meaning of the parameterizations in use in the CTLT, and how they parameterize (by pairs, besides the frequency) the TL parameters.
In this section, the graphical representation of isocomplex numbers in the "space" $\mathbb{H}$ is introduced together with the graphical/geometrical interpetation of the operations between themselves.
Since $\mathbb{H}$ is in $\mathbb{C}^3$, the most general graphical representation of isocomplex numbers (that would be in a space of 3×2 coordinates) is not possible. However, the particularization of isocomplex numbers for representing the real lossy parameterizations and frequency regarding HPWs can be afforded in a space affine to $\mathbb{R}^{3+}$, which is able to be depicted. The generalization would be done by considering each axis as complex.

The tuple representing τ may be plot in a 3-axis space as:

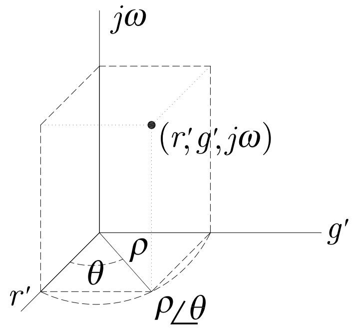


Fig. 4.41: Gaphical representation of the tuple $(r', g', j\omega)$ which represents an isocomplex number in its "space".

Notice that the components are represented ortogonally, making those independent. Moreover, the modulus of the polar form corresponds with the modulus, ρ, in the $[r', g']$-plane, whereas the

phase, θ, is the angle between these componentes, measured from the r'-axis.
This represenation allows for defining the isocomplex numbers in an intuitive way. In addition, the graphical representation lets to define the modulus of an isocomplex number[25], $|\tau|$, as the "length" of the ray which goes between the origin and the point locating τ in the "space", whereas the angle of the isocomplex number[26], φ_τ, is the angle which forms this ray with the $[r', g']$-plane:

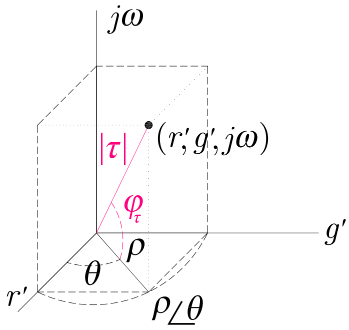


Fig. 4.42: Graphical definition of the modulus and phase of an isocomplex number represented within its "space".

As said before, the isocomplex numbers τ_r and τ_g defined in eqs. (4.A.4) and (4.A.5), respectively, are of special interest. They are represented as:

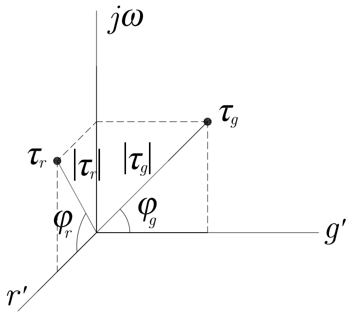


Fig. 4.43: Representation of the isocomplex numbers τ_r and τ_g in the "space" of parameters. The angles of these tuples are $\varphi_r = \tan^{-1}(1/r)$ and $\varphi_g = \tan^{-1}(1/g)$, in which r and g are the parameterizations of losses for the *ffa*.

If these tuples τ_r and τ_g are added with the rules of the sum ($\omega_{\tau_r} = \omega_{\tau_g}$), the result is represented as follows:

The resultant tuple is the τ_c-tuple, which is also of great usefulness in the analysis of TL parameters when frequency is variable.
This is also an example of how to add isocomplex numbers graphically.

The isomplex numbers τ_r and τ_g can form a "plane": the rg-plane. This "plane" arises when adding the τ_r and τ_g convenientely weighted by coefficients α_r and α_g, for which "coneveniently" means that $\alpha_r\tau_r$ and $\alpha_g\tau_g$ must have the same ω-component in order to be able to be added according to the definition of the sum in eq. (4.A.3).

A generic coefficient α is defined in the form of iscomplex numbers as:

$$\alpha = \alpha\angle 0 + j0,\ \alpha \in \mathbb{C}.$$

[25]Notice the difference between the modulus of an isocomplex number, $|\tau|$, and the modulus of the polar form, $\rho \equiv \rho_\tau$.

[26]Notice that the angle of the iscomplex number, φ_τ, is completely different to the angle of the polar form, θ_τ.

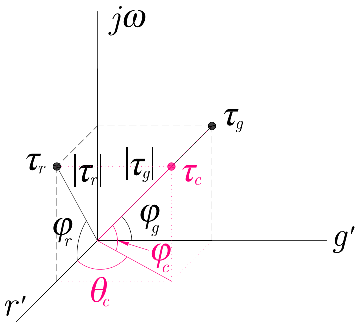


Fig. 4.44: Representation of the sum of the isocomplex numbers τ_r and τ_g in the "space" of parameters. The angle of the resultan tuple is $\varphi_c = \tan^{-1}((r'+g')/\omega) \equiv \tan^{-1}(1/\omega_n)$, whereas the angle of the polar form is $\theta_c = \tan^{-1}(g'/r') = \tan^{-1}(1/c)$. These angles are the frequency parameterizations used in the *vfa*.

Notice that this definition makes that any ray in the "space", for example that one which represents τ_c in Fig. 4.44, can be represented as

$$c\text{-ray:} \quad \{\tau \in \mathbb{H} \backslash \tau = \alpha\,(\tau_r + \tau_g)\}\,,. \tag{4.A.13}$$

The "unitary" isocomplex numbers, $\hat{\tau_r}$ and $\hat{\tau_g}$ which represent τ_r and τ_g, respectively, with coefficients $\alpha_r = r'\angle 0 + j0$ and $\alpha_g = g'\angle 0 + j0$, also respectively, are:

$$\begin{cases} \hat{\tau_r} = 1\angle 0 + j\frac{\omega}{r'} \equiv 1\angle 0 + j/r \\ \hat{\tau_r} = 1\angle \frac{\pi}{2} + j\frac{\omega}{g'} \equiv 1\angle \frac{\pi}{2} + j/g \end{cases} . \tag{4.A.14}$$

Thus, the rg-plane is defined as:

$$rg\text{-plane:} \quad \{\tau \in \mathbb{H} \backslash \tau = \alpha_r \hat{\tau_r} + \alpha_g \hat{\tau_g}\}\,, \tag{4.A.15}$$

, in which $\alpha_r = r'\angle 0 + j0$, and $\alpha_g = g'\angle 0 + j0$.

Notice that, graphically in the "space", the rg-plane is the horizontal plane elevated ω:

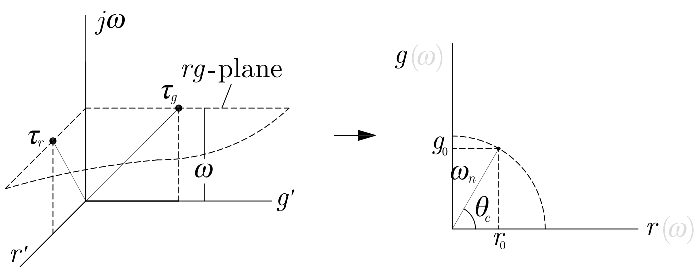


Fig. 4.45: The rg-plane defined from the sum of the isocomplex numbers $\tau_r = \alpha_r\hat{\tau_r}$ and $\tau_g \alpha_g \hat{\tau_g}$, elevated ω.

The rg-plane is referred to ω and the modulus of the components α_r and α_g should be referred to ω leading to the coordinates (r, g) on this plane (or (ω_n, θ_c) if writing them in polar form) which implicitly depend on ω (except to θ_c).
Thus, rg-plane is not and Euclidean plane but a plane in which different angles are "measured": the parameters r, g, and ω_n represent angles in the "space", whereas c represents the same angle in each horizontal plane parameterized by ω.
Another way to understand the coordinates in the rg-plane is seeing them as the inverse of the third component of the unitary tuples $\hat{\tau_r}$ and $\hat{\tau_g}$ defined in eq. (4.A.14).

Concerning the graphical representation of the product between isocomplex numbers, it is important to detach the properties when interpreting this operation in terms modulus ($|\tau|$) and phase (φ_τ).
The modulus of the product of two isocomplex numbers is the product of their modulus, whereas the phase of the product is the sum of the phases of the isocomplex numbers involved.
Analogously, this can be extended to obtain the modulus-phase relations of product with the inverse, the power, and the roots.

Proof. The identities with modulus and phase regarding the product of isocomplex numbers are proved. The properties regarding the rest of operations based on the product could be proved following a similar procedure.

The product of two isocomplex numbers τ_1 and τ_2 has been defined in eq. (4.A.8). The modulus of this product is:

$$|\tau_1 \cdot \tau_2| = \sqrt{\rho_{\tau_1\cdot\tau_2}^2 + \omega_{\tau_1\cdot\tau_2}^2} = \sqrt{\left(\rho_{\tau_1}\rho_{\tau_2} - \omega_{\tau_1}\omega_{\tau_2}\right)^2 + \left(\rho_{\tau_1}\omega_{\tau_2} + \rho_{\tau_2}\omega_{\tau_1}\right)^2} =$$
$$= \sqrt{\rho_{\tau_1}^2\rho_{\tau_2}^2 + \omega_{\tau_1}^2\omega_{\tau_2}^2 + \rho_{\tau_1}^2\omega_{\tau_2}^2 + \rho_{\tau_2}^2\omega_{\tau_1}^2} = \sqrt{\left(\rho_{\tau_1}^2 + \omega_{\tau_1}^2\right)}\sqrt{\left(\rho_{\tau_2}^2 + \omega_{\tau_2}^2\right)} = |\tau_1| \cdot |\tau_2|,$$

which is equal to the product of the modulus, as it has been noted.

On the other hand, the phase of the product of τ_1 and τ_2 is:

$$\varphi_{\tau_1\cdot\tau_2} = \tan^{-1}\left(\frac{\rho_{\tau_1}\omega_{\tau_2} + \rho_{\tau_2}\omega_{\tau_1}}{\rho_{\tau_1}\rho_{\tau_2} - \omega_{\tau_1}\omega_{\tau_2}}\right) = \tan^{-1}\left(\frac{\frac{\omega_{\tau_1}}{\rho_{\tau_1}} + \frac{\omega_{\tau_2}}{\rho_{\tau_2}}}{1 - \frac{\omega_{\tau_1}}{\rho_{\tau_1}}\frac{\omega_{\tau_2}}{\rho_{\tau_2}}}\right) =$$
$$= \tan^{-1}\left(\frac{\tan\left(\varphi_{\tau_1}\right) + \tan\left(\varphi_{\tau_2}\right)}{1 - \tan\left(\varphi_{\tau_1}\right)\tan\left(\varphi_{\tau_2}\right)}\right) = \tan^{-1}\left(\tan\left(\varphi_{\tau_1} + \varphi_{\tau_2}\right)\right) = \varphi_{\tau_1} + \varphi_{\tau_2},$$

having applied the definition of the tangent of the angle sum, to conclude that the phase of the product is the sum of the phases of the factors in it. □

These properties regarding the product and the similarities with the product of complex (or bicomplex) numbers, let to define the process of "complexifying" isocomplex numbers, ζ, as

$$\begin{aligned} \zeta:\ & \mathbb{H} \to \mathbb{C}\ (\mathbb{C}^2) \\ & \tau \rightsquigarrow z = \zeta(\tau) = |\tau| + j\varphi_\tau, \end{aligned} \tag{4.A.16}$$

and the process of "isocomplexifying" complex numbers (that is, generating isocomplex numbers), δ_θ,

$$\begin{aligned} \delta:\ & \mathbb{C}\ (\mathbb{C}^2) \to \mathbb{H} \\ & z \rightsquigarrow \tau = \delta_\theta(z) = |z|\angle\theta + j\varphi_z. \end{aligned} \tag{4.A.17}$$

Graphically:

Notice that the process of "isocomplexifying" complex numbers is associated to the "complexification" as its inverse, so it inherits θ (δ_θ), or explicitly defines θ by convention:

Complex number	Associated θ [rad] (in the polar form)
$r' + j\omega$	0
$g' + j\omega$	$\pi/2$
$j1\ (\equiv j)$	0

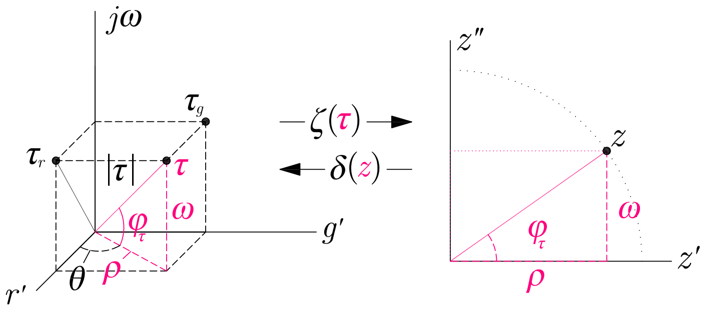


Fig. 4.46: Graphical representation of the "complexification" of isocomplex numbers ($z = \zeta(\tau)$), and "isocomplixification" of complex numbers ($\tau = \delta_\theta(z)$).

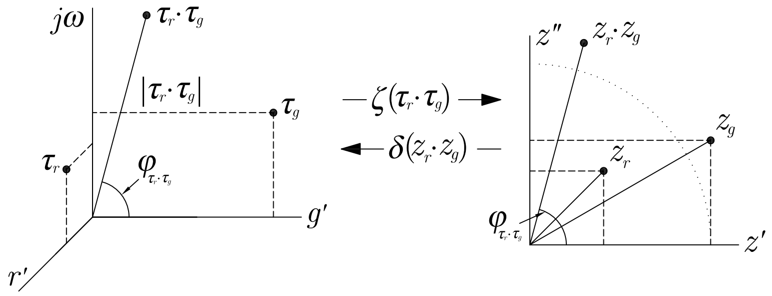


Fig. 4.47: Graphical representation of the product of two isocomplex numbers τ_r and τ_g ($\tau_r \cdot \tau_g$) and the corresponding operation ($z_r \cdot z_g$) in the complex plane.

Keeping these properties and rules in mind, the multuplication of two isocomplex numbers results much more intuitive.
The following example serves to illustrate the product of two isocomplex numbers and the correspondence in the complex plane.

The isocomplex numbers form a group with the product with some interesting properties to split the dependence on the parameterizations, and also for exaplaining graphically their influence for composing any expression in terms of their product.
Moreover, the direct relation with complex numbers allows for defining an alternative way to represent complex values, very useful for the graphical analysis.

***Physical interpretations*:** The definition of isocomplex numbers is clearly related to the parameterizations of the TL. In fact, these numbers split up for parameterizing the TL, making use of the geometrical interpretation underlying their definition an the intuitive graphical representation. However, the use of isocomplex numbers should be explained in the contexto of both the direct and inverse characterization of the TLT.
This section is intended to explain the relationship between the definition of isocomplex numbers together with the geometry in the "space" of isocomplex numbers and the physical interpretation based on the parameters of the TL. Making this connection explicit is crucial for using the isocomplex numbers with practial purposes in the analysis of TLs. As a result of this relation, the space of isocomplex numbers is addressed as the space of parameters.

As it has been presented in the CTLT, the "universal" analysis of basic parameters in terms of parameterizations of losses or frequency parameterizations for the *ffa* and the *vfa*, respectively, can be afforded by two parameterizations for each case: r and g, and c and ω_n.
As it has been seen in the definition and geometrical analysis of isocomplex numbers, these pa-

rameterizations has direct representation as angles in the "space" of isocomplex numbers (see Figs. 4.43 and 4.44):

$$r \leftrightarrow \varphi_r,$$
$$g \leftrightarrow \varphi_g,$$
$$c \leftrightarrow \theta_c,$$
$$\omega_n \leftrightarrow \varphi_c.$$

Thus, it can be said that angles in the space of isocomplex numbers parameterize different analysis of the TL, and so this "space" is addressed as the space of parameterizations.
The reason of these angle parameterizations is found in the parameterization of the analysis in the *frequecy domain*. Since the problem is parameterized by ω, any parameter so it is, with the exception of that parameter which parameterizes the TL in the whole frequency band (c). This is the reason for stablishing $j\omega$ as orthogonal axis in the "space", in order to have the possibility of being each angle parameterization shared with frequency.
In this sense, notice that angles in general universalize the study in themselves because they fix the relation between two parameterizations, for example conductor losses and frequency in r (φ_r).

The "space" of isocomplex numbers shows all the possible combinations of line parameters which lead to characteize the TL uniquely. Each one of the two parameterizations regarding any TL analysis is represented by a surface in the space of parameters. For example, the parameterization of the conductor losses is represented as the "plane" inclined φ_r:

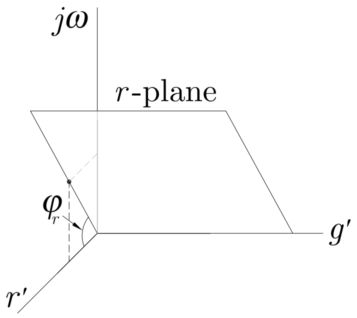


Fig. 4.48: Example of graphical representation of the r-plane, that is the points in the space of parameters numbers parameterized by r.

Since the parameterizations concerning any TL analysis are defined by pairs representing surfaces, their intersection is a curve which represents all the TLs parameterized by both parameterizations. For example, a straight line arising from the origin of the space of parameterizations represents all the TLs parameterized by r and g (the intersection between the r- and g-planes) for the *ffa*:

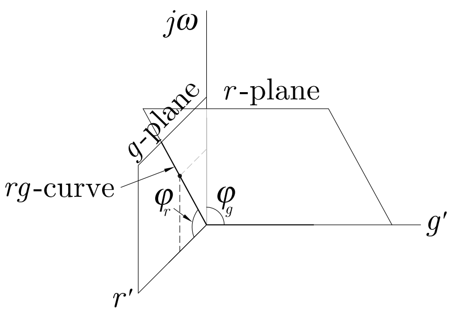


Fig. 4.49: Example of graphical representation of the r- and g-planes, and the intersection denoted as the rg-curve which represents all the possible TLs parameterized by r and g in the *ffa*.

In this example, the curve which results by the intersection of the r- and g-planes is denoted as the rg-curve.
This geometrical interpreation of the parameterizations in the space may be extended for every pair of parameterizations. In particular, the c-planes are those planes which are folded θ_c from the $g = 0$-plane. Since the $j\omega$-axis is completely contained within these c-planes, c is the parameterization used for describing all the TLs in the whole frequency band. In this sense, the sum of the tuples parameterizing r and g weighted by the same coefficients leads to the curves (the c-rays in eq. (4.A.13)) which define the *vfa*.

The arrangement of all the curves intersectioning the surfaces can be plotted in a plane whose axis are the parameterizations (then angles) under study. Following the example above, the rg-plane is the "plane" whose axis are the parameterizations of losses for the *ffa*. Rememeber that this plane (see it in Fig. 4.A.15) is algebraically defined by the linear combination of the unitary tuples following eq. (4.A.14) –in this example when parameterizing the TL with r and g–, whose third components are the inverse of the parameterizations of losses in the *ffa*.

Furthermore, the definition of the basic parameter of the TL in terms of different types of parameterizations for each intended analysis using complex quantities, together with the "isocomplexification" of these complex parameters, lets to define the parameters of the TL in terms of isocomplex numbers in the same space of parameters. The process of "complexifying" these isocomplex numbers is, esentially, the way to see the parameters of the TL as transformations.
The main advantage of this physical interpretation is on seeing some physical properties mapped on the parameters graphically, for example the parameters (including frequency) which leads to the same phase of the characteristic impedance:

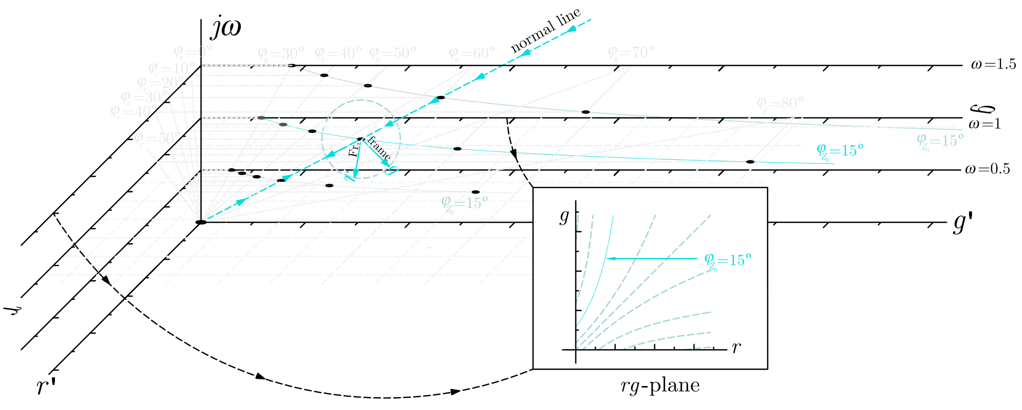


Fig. 4.50: Example of graphical analysis of the curves in the space of parameterizations which keep the angle of the characteristic impedance constant and its compact representation in the rg-plane.

In the graphical example above, the points that conserve the difference between the φ_r and φ_g constant represent the TLs with the same φ_{Z_0}.
These interpretations reveals some interesting practical uses of isocomplex numbers when characterizing TLs.

Practical uses The isocomplex numbers, as they have been defined and interpreted playing the role of parameterizations of any TL, show great advantages when parameterizing it.
In this section the practical uses of these numbers in relation to the characterization of TLs are detached.

The definition of isocomplex numbers results crucial when seeing the basic parameters as complex transformations from different parameterizations in the "space" of these numbers.
Moreover, these numbers are important for extending the analysis to the characterization of wave parameters, as it is exemplified in Chpt. 5, by means of examples of use of them in the inverse characterization, at the same time the analysis concerning the direct characterization is used.
In this sense, any curve in the space of parameters, or equivalently in the rg-plane, can be transformed by means of the definition of TL parameters as functions whose domain is these regions.
In particular, the rg-plane as defined in eq. (4.A.15) and represented in Fig. 4.45, just as it has been interpreted in the previous section in terms of the parameterizations for characterizng the TL, is especially useful for depicting in a simple plane the curves which are the origin of the transformations in the direct characterization. Thus, this representation offers a "compact tool" for characterizing TL.

Because of the physical interpretation the rg-plane has in terms of parameterizations, it explains geometrically the origin different kinds of them. Thus, this plane (in fact, it is an hyperplane if spliting the axis in its real and imaginary parts) is useful for determining which parameterizations lead to specific behaviors of TL parameters, for example, which parameterizations make constant the phase of the characteristic impedance, as it is graphically presented in Fig. 4.50. In this sense, the greatest usefulness of these numbers is in characterizing the TL inversely.

Related to the inverse characterization, the rg-plane is the region in which not only the parameterizations appear explained, but also from which the TL parameters, especially the basic parameters, are characterized as functions.
For the purpose of characterizing the basic parameters of the TL in the rg-plane, the curves representing orthogonal parameterizations of the basic parameters, e.g. real and imaginary parts of basic parameters, are depicted on this plane playing the role of curve levels of the functions which represent them.
This inverse characterization can be done by solving the inverse function related to the one originaly defined in the direct characterization for each parameter under study. This anlysis is done in the inverse characterization of the CTLT (CTLT-v1.0b) in Sect. 4.4.2 presented in this chapter.

Since the rg-plane may act as the domain of the complex functions parameterized in different ways, for example parameterizing the losses in the *ffa*, or the frequency in the *vfa*, it gives raise to define functions which, acting together with those ones related to the inverse analysis of the parameters of the TL, solve integrally a particular problem.
This can be seen clearly when presenting practical uses of the inverse characterization, but here the role of the rg-plane is intended to be detached as the domain where doing these calculus: not only integral, but also differential, geometrical, etc.

Moreover, both the graphical and geometrical analysis help solving the inverse function, while seeing graphically the properties of the TLs at the same time.
Many times finding the solution related to a problem is not feasible by analytical procedures but graphical or geometrical. For example, the definition of the rg-plane in the analysis of HPWs lets to solve the inverse function of basic parameters in an easy way when representing g vs. r in the rg-plane, which is based on solving the related functions in the *ffa*, to be then extended to any type of parameterizations becuase their geometrical reprsentation in rg-planes is well known as a consequence of the physical interpretation.

In addition, multiple physical interpretations of TLs which are a priori disconnected with the rg-plane could be mapped on it. For example, the TL's length is able to be "simulated" by means of the imaginary part of the axis in the rg-plane when complexifying them. This is a clear example of the usefulness and application (see it in **Applications**) of the "space of parameterizations" (and, in particular, the rg-plane) when parameterizing the TL completely.

Remark 29. *The rg-plane is the region defined from the "space" of isocomplex numbers useful for the analysis of TLs in different ways: (i) it explains the parameterizations of losses or frequency, and many others like anglular parameterizations, or even the length of the TL; (ii) it is the domain for defining the (direct) functions representing different TL parameters regarding the direct characterization; and (iii) it is the region of the range of the defined inverse functions for the inverse characterization. Moreover, it presents some graphical and geometrical properties regarding the definitions of TL parameters both directly and inversely, which makes it a very interesing "analytical tool" for supporting the CTLT.*

Appendix 4.B

The maximum value of Z_{0n1} is on the curve parameterized by $r = 0$ so $n = 1$. Particularizing the general equation in eq. (4.8):

$$|Z_{0n1}|\,|_{r=0} = \sqrt{\cos\left(2\varphi_{Z_0}\right)}, \tag{4.B.18}$$

so

$$Z''_{0n1} = \sqrt{\cos\left(2\varphi_{Z_0}\right)}\sin\left(\varphi_{Z_0}\right). \tag{4.B.19}$$

If differentiating eq. (4.B.19), the maximum is obtained when it verifies

$$\cos\left(2\varphi_{Z_0}\right)\cos\left(\varphi_{Z_0}\right) = \sin\left(2\varphi_{Z_0}\right)\sin\left(\varphi_{Z_0}\right). \tag{4.B.20}$$

Now applying trigonometric identities, it leads to the equivalent condition for obtaining the maxium of Z''_{0n1} when

$$\tan\left(\varphi_{Z_0}\right) = \frac{1}{3}, \tag{4.B.21}$$

so $\varphi_{Z_0} = \pi/6$ for Z''_{0n1} to be the maximum. Substituying this value of φ_{Z_0} in eq. (4.B.19), the maximum of Z_{0n1}, $Z_{0n1,max} = 1/\left(2\sqrt{2}\right)$.
On the curve parameterized by $r = 0$ it means that $g = \sqrt{3}$ so in eq. (4.3), this makes $Z_{0n1} = 1/\sqrt{1-j3}$.

Appendix 4.C

The minimum of $\beta_{n1} = 1$. It is interesting to known at which angle φ_γ this minimum happens. The expression of β_{n1} in terms of φ_γ and one of the parameterizations of losses (remember that this expression has the same form for the r- and g-parameterized curves) is

$$\beta_{n1} = \frac{\sqrt{|n|}}{\sqrt{-\cos(2\varphi_\gamma - \varphi_n)}} \sin(\varphi_\gamma), \tag{4.C.22}$$

having choose the parameterization for r, for example.

If differentiating 4.C.22, the minium is obtained when

$$\sin(2\varphi_\gamma - \varphi_n)\sin(\varphi_\gamma) = -\cos(2\varphi_\gamma - \varphi_n)\cos(\varphi_\gamma) \tag{4.C.23}$$

verifies.
Now if applying some trigonometric identities, the equation before reduces to

$$-\cos(\varphi_\gamma - \varphi_n) = \cos(\varphi_\gamma - \varphi_n), \tag{4.C.24}$$

which is fulfilled if and only if

$$\varphi_\gamma = \frac{\pi}{2} + \varphi_n. \tag{4.C.25}$$

As a consequence, the value of γ_{n1} for the curve parameterized by n (d) in which β_{n1} is minimum is $\gamma_{n1,r-tan} = \tan(|\varphi_n|) + j \equiv r + j$ ($\gamma_{n1,g-tan} = \tan(|\varphi_d|) + j \equiv g + j$).

Appendix 4.D

The general equation of the curves which result from the geometrical analysis of basic parameters regarding the *vfa* is here obtained.

Starting with the expression of the characteristic impeance in eq. (4.20), the square of Z_{0n1} leads to identify the real and imaginary parts,

$$\begin{cases} Z'^2_{0n1} - Z''^2_{0n1} = \frac{\omega_n^2+\sin(\theta_c)\cos(\theta_c)}{\omega_n^2+\sin^2(\theta_c)} \\ 2Z'_{0n1}Z''_{0n1} = \frac{\omega_n(\sin(\theta_c)-\cos(\theta_c))}{\omega_n^2+\sin^2(\theta_c)} \end{cases}. \tag{4.D.26}$$

From the first case, ω_n^2 is easily solved as

$$\omega_n^2 = \frac{\sin(\theta_c)\cos(\theta_c) - \sin^2(\theta_c)\left(Z'^2_{0n1} - Z''^2_{0n1}\right)}{Z'^2_{0n1} - Z''^2_{0n1} - 1}, \tag{4.D.27}$$

to be replaced in the expression of the 4^{th} power of the modulus of Z_{0n1}:

$$\left(Z'^2_{0n1} + Z''^2_{0n1}\right)^2 = \frac{\omega_n^2 + \cos^2(\theta_c)}{\omega_n^2 + \sin^2(\theta_c)}, \tag{4.D.28}$$

which, after operating and substituying $\tan(\varphi_c) = 1/c$ leads to obtain the general expression of the characterisic impedance parameterized by c

$$\left(Z'^2_{0n1} + Z''^2_{0n1}\right)^2 = (c+1)\left(Z'^2_{0n1} - Z''^2_{0n1}\right) - c. \tag{4.D.29}$$

Eq. (4.26) is the general equation of the Cassini ovals [Law72]. It can be written in polar form as

$$|Z_{0n1}| = \begin{cases} \sqrt{\left(\frac{c+1}{2}\right)\left[\cos(2\varphi_{Z_0}) + \sqrt{\left(\frac{c-1}{c+1}\right)^2 - \sin^2(2\varphi_{Z_0})}\right]} \\ \text{if } \varphi_{Z0} > \frac{1}{2}\sin^{-1}\left(\frac{|c-1|}{c+1}\right) \\ \sqrt{\left(\frac{c+1}{2}\right)\left[\cos(2\varphi_{Z_0}) - \sqrt{\left(\frac{c-1}{c+1}\right)^2 - \sin^2(2\varphi_{Z_0})}\right]} \\ \text{if } \varphi_{Z0} < \frac{1}{2}\sin^{-1}\left(\frac{|c-1|}{c+1}\right) \end{cases} \text{for which} \quad \varphi_{Z_0} \in \left[0, \frac{\pi}{4}\right[. \tag{4.D.30}$$

On the other hand, the expression of the general equation of the propagation constant parameterized by c from eq. (4.28) may obtained by similar procedure: identifying the real and imaginary parts of the square of γ_{n2} for solving ω_n to be substituyed into the 4^{th} power of its modulus; leading to

$$\left(\alpha_{n2}^2 + \beta_{n2}^2\right)^2 = 4\frac{\alpha_{n2}^2\beta_{n2}^2}{(\cos(\theta_c)+\sin(\theta_c))^2}\left[1 + \frac{\alpha_{n2}^2\beta_{n2}^2}{(\cos(\theta_c)+\sin(\theta_c))^2}\right] + 4\cos^2(\theta_c)\sin^2(\theta_c), \text{ in which} \quad \varphi_c = \tan^{-1}\left(\frac{1}{c}\right). \tag{4.D.31}$$

The general equation of γ_{n2} parameterized by c does not fit any set of (quartic) curves, but specific curves for particular cases of c. For example, the curve parametrized by $c = 0$ ($c \gg$), is the so called Bullet nose, [Law72].

The expression of the general equation of the propagation constant parameterized by ω_n is obtained by solving c from the system of the separated real and imaginary parts of $gamma_{n2}$ square, leading to

$$\alpha_{n2}^2 - \beta_{n2}^2 = \frac{1}{\omega_n^2}\left(\alpha_{n2}^2 \beta_{n2}^2\right) - 2\omega_n^2 - 1. \tag{4.D.32}$$

This equation describes a set of quartic curves parameterized by ω_n which are unknown in the literature regarding planar curves.

Appendix 4.E

An exaplanation of the notation used for the normalizations using in this chapter (and througout the Thesis book) is given. The purpose of explaining the symbols in the notation will serve to the better understanding the text.

generic normalization

$$X_{(\mathrm{x})nY}$$

Fig. 4.51: Generic notation for a normalized parameter $X_{(\mathrm{x})}$ whose normalization is done with respect to Y.

In Fig. 4.51, a generic typography of the notation of a normalized parameter is presented to explain each of the particular notations used in this book:

(i) The symbol $X_{(\mathrm{x})}$ refers to the parameter to be normalized (the "normalized"). The subscript is sometimes ommited (e.g. the generic wave impedance, Z), but normally it refers to one specific parameter (e.g. the characteristic impedance, Z_0).

(ii) The subscript n ("$_n$") has to be read as "normalized with respect to" (or "normalized over") to be the notation well interpreted.

(iii) The subscript Y ("$_Y$") refers to the "normalizer" parameter (the "normalizer", which is always real in the analysis presented in this book for the angle conservation of the "normalized"). It may refer to: (i) a particular case of the "normalized" (in this case the "normalizer" is represented by a number different from "0", e.g. "1" refers to the lossless case); (ii) another parameter (in this case the "normalizer" is a letter or "0", e.g., "L" refers to the normalization with respect to the modulus of the impedance at the load); or (iii) the modulus of the "normalized" (in this case the "normalizer" is omitted).

The followng chart lists the possible and useful values of the "normalized" and the "normalizer", and their meanings:

Normalized ($X_{(\mathrm{x})}$)	Normalizer (Y)
Z_0 (characteristic impedance); γ (propagation constant); α (attenuation constant); β (phase constant); Z (wave impedance); Y (wave admittance); Z_L (wave impedance at the load)	1: lossless case; 2: (real part of) non dispersive case; 3: (real part of) low losses approx.; 0: modulus of the characteristic impedance; L: modulus of the impedance at the load; (omitted): modulus of the "normalized";

Table 4.5: Summary of the "normalized" and "normalizer" parameters used in the CTLT-v1.

Some examples of use of this notation:
Z_{0n1}: The characteristic impedance (Z_0) normalized with respect to (n) the lossless case (1);
γ_{n2}: The propagation constant (γ) normalized with respect to (n) the non dispersive case (2);
Z_{0n}: The characteristic impedance (Z_0) normalized with respect to (n) its modulus (omitted);
Z_{Ln0}: The wave impedance at the load (Z_L) normalized with respect to (n) the modulus of the characteristic impedance (0);
among others used throughout the analysis.

Appendix 4.F

It is required to prove the following statement: "Given the modulus parameterization regarding a parameter (lets name it as χ) normalized with respect to a (real) parameter (lets call it $A \in \mathbb{R}$, so χ normalized with respect to a is χ_{nA}) which is inversely characterized in the rg-plane, the inverse analysis of another normalization (with respect to $B \in \mathbb{R}$, so χ normalized with respect to B is χ_{nB}), follows the same form but accordingly rescaled".

Lets state this sentence analytically: Both the direct and inverse functions are supposed to be known: $\chi = f(r,g) \in \mathbb{C}$ and $(r,g) = f^{-1}(\chi) \in \mathbb{H}$ (being $\mathbb{H}$ the space of isocomplex numbers), respectively.
Moreover, the definitions of χ_{nA} and χ_{nB} are writte as:

$$\chi_{nA} = \frac{\chi}{A} \text{ and} \tag{4.F.33}$$

$$\chi_{nB} = \frac{\chi}{B}, \tag{4.F.34}$$

respectively; so, their modulus are:

$$|\chi_{nA}| = \frac{|\chi|}{A} \text{ and} \tag{4.F.35}$$

$$|\chi_{nB}| = \frac{|\chi|}{B}, \tag{4.F.36}$$

also respectively.
As the statement above formulates the function

$$f^{-1}(\chi_{nA}) \equiv f^{-1}(|\chi_{nA}|e^{j\varphi_\chi}) \quad \forall |\chi_{nA}| \tag{4.F.37}$$

is known, so there exists a function which relates r and g in such a way that they lead to $|\chi_{nA}|$:

$$f_{|\chi_{nA}|}(r,g) = |\chi_{nA}| = \tfrac{B}{A}|\chi_{nB}|, \tag{4.F.38}$$

in which the latter equality has been obtained by relating eqs. 4.F.35 and 4.F.36.

The questions is: is $f^{-1}(\chi_{nB})$ a function such that $f_{|\chi_{nB}|}(r,g)$ is a scaling of $f_{|\chi_{nA}|}(r,g)$? In order to prove this, it is assumed by defintion that:

$$f^{-1}(f_{|\chi_{nA}|}(r,g)e^{j\varphi_\chi}) = (r,g)|_{|\chi_{nA}|}, \tag{4.F.39}$$

in which $(r,g)|_{|\chi_{nA}|}$ are those (r,g) in the rg-plane which lead to $|\chi_{nA}|$ constant.
Then,

$$f^{-1}(\chi_{nB}) \equiv f^{-1}(|\chi_{nB}|e^{j\varphi_\chi}) = f^{-1}(\tfrac{A}{B}|\chi_{nA}|e^{j\varphi_\chi}) = f^{-1}(\tfrac{A}{B}f_{|\chi_{nA}|}(r,g)e^{j\varphi_\chi}), \tag{4.F.40}$$

and, since f^{-1} is defined in eq. (4.F.37) for all $|\chi_{nA}|$, then there exists

$$f_{\frac{A}{B}|\chi_{nA}|}(r,g) \tag{4.F.41}$$

such that

$$f^{-1}(f_{\frac{A}{B}|\chi_{nA}|}(r,g)e^{j\varphi_\chi}) = (r,g)|_{\frac{A}{B}|\chi_{nA}|} \equiv (r,g)|_{|\chi_{nB}|}. \tag{4.F.42}$$

Since is required that $(r,g)|_{|\chi_{nA}|}$ are the same (r,g) tuples that $(r,g)|_{|\chi_{nB}|}$, then

$$f_{\frac{A}{B}|\chi_{nA}|}(r,g) \equiv f_{|\chi_{nB}|}(r,g) = \tfrac{A}{B}f_{|\chi_{nA}|}(r,g), \tag{4.F.43}$$

which is a scaling of the original curve $f_{|\chi_{nA}|}(r,g)$.
Notice that $\varphi_{\chi_{nA}} = \varphi_{\chi_{nB}} \equiv \varphi_\chi$ is used throughout the proof.

Let's see an example of this scaling to change the parameterization of the curves from the normalized γ for the *ffa*, γ_{n1} as defined in eq. (4.10) to the normalized γ for the *vfa*, γ_{n2} as defined in eq. (4.27). The generic parameters and functions are:

$$\begin{cases} A \equiv \beta_{sp} = \omega\sqrt{LC} \\ B \equiv \alpha_{nd} = \frac{R\sqrt{C}}{\sqrt{L}} \\ \chi_{nA} \equiv \gamma_{n2} = \frac{\gamma}{\alpha_{nd}} \\ f_{|\chi_{nA}|}(r,g) \equiv f_{|\gamma_{n1}|}(r,g) = |\gamma_{n1}| = \sqrt[4]{(1+r^2)(1+g^2)} \end{cases} . \tag{4.F.44}$$

The labels of the normalized modulus $|\gamma_{n2}|$ are obtained from the labels of $|\gamma_{n1}|$ as:

$$|\gamma_{n2}| = |\gamma_{n1}|\frac{\omega L}{R}. \tag{4.F.45}$$

Appendix 4.G

The mathematical procedure for solving the inverse function of real and imaginary parts lies in separating these parts from the original parameterized (direct) funtions.
Here the expression function which relates r with g parameterized by the real part of the normalized characteristic impedance, Z_{0n1}, is obtained, but the similar procedure can be followed to obtain this relation for the imaginary parameterization, and also the ones for characterizing the real-imaginary parameterizations regarding the propagation constant (which are even more interesting because of its physical interpretation).

The square of Z_{0n1} leads to

$$Z_{0n1}^2 = Z_{0n1}'^2 - Z_{0n1}'^2 + j2Z_{0n1}' Z_{0n1}'^2 = \frac{1 - jr}{1 - jg} = \frac{1 + rg + j(g - r)}{1 + g^2}. \quad (4.G.46)$$

Identifying the real-imaginary parts, it leads to:

$$\begin{cases} Z_{0n1}'^2 - Z_{0n1}'^2 = \frac{1+rg}{1+g^2} \\ 2Z_{0n1}' Z_{0n1}'^2 = \frac{g-r}{1+g^2} \end{cases}. \quad (4.G.47)$$

Solving r as function of Z_{0n1}', $Z_{0n1}'^2$ and g from the second equation in the system of functions above, the following equation is obtained

$$r = g - 2(1 + g^2) Z_{0n1}' Z_{0n1}'. \quad (4.G.48)$$

In order to solve Z_{0n1}'', r in eq. (4.G.48) is substituyed into the second equation of system in (XIII), leading to the 2^{nd} order equation which relates Z_{0n1}', Z_{0n1}'' and g. Solving Z_{0n1}'' from this equation,

$$Z_{0n1}'' = gZ_{0n1}' + \sqrt{Z_{0n1}'^2 (1 + g^2) - 1}, \quad (4.G.49)$$

and substituying it into eq. (4.G.48), it leads to

$$r = g - 2Z_{0n1}' \left(1 + g^2\right) \left(gZ_{0n1}' + \sqrt{Z_{0n1}'^2 (1 + g^2) - 1}\right), \quad (4.G.50)$$

which is ready to be parameterized making $Z_{0n1}' = a$.

Appendix 4.H

The mathematical procedure for solving the inverse function of modulus and phase of basic parameters lies in separating them from the original definition of them as functions.
Here the expressions which relates r with g parameterized by, on one hand the modulus and, on the other hand the phase of the normalized characteristic impedance, Z_{0n1}, are obtained.
A similar procedure can be followed to obtain these expressions regarding the propagation constant, parameterized in modulus and phase, which are as interesting as the ones decribed here for the study and parameterization of lossy TL, especially the wave parameters.

On one hand, the 4^{th} power equals

$$|Z_{0n1}|^4 = \frac{1+r^2}{1+g^2}. \tag{4.H.51}$$

Solving r from the expression above, it leads to

$$r = \sqrt{|Z_{0n1}|^4 \left(1+g^2\right) - 1} \geq 0, \tag{4.H.52}$$

which is the expression r written as a function of g parameterized by the modulus of the characteristic impedance.
For being $r \geq 0$, $g \geq \sqrt{(1/m^4) - 1}$.

On the other hand, the tangent of the double-angle of the characteristic impedance is:

$$\tan(2\varphi_{Z_0}) = \frac{g-r}{1+gr}. \tag{4.H.53}$$

By solving r from this equation, the following equation which realtes r to g parameterized by the angle of the characterstic impedance is obtained:

$$r = \frac{g - \tan(2\varphi_{Z_0})}{1 + g\tan(2\varphi_{Z_0})} \geq= 0. \tag{4.H.54}$$

For being $r \geq 0$: $g \geq \tan(2\varphi_{Z_0})$ if $\varphi_{Z_0} > 0$; and $g < 1/\tan(2\varphi_{Z_0})$ if $\varphi_{Z_0} < 0$.

Appendix 4.I

Here mathematical developments for obtaining the parameterized curves in the ρ-plane from: (i) the real-imaginary parameterized parts; and (ii) the modulus-phase parameterizations; in the Z_{0nL}-plane are explained.

Regarding the real-imaginary parameterized parts, the inverse function (from ρ to Z_{0nL}), $Z_{0nL}(\rho)$, is splitting into its real and imaginary parts, which are parameterized as in eq. (4.66):

$$\begin{cases} a = \frac{c_L\left(1-\rho'^2-\rho''^2\right)+2\rho'' s_L}{(1+\rho')^2+\rho''^2} \\ b = \frac{s_L\left(1-\rho'^2-\rho''^2\right)-2\rho'' c_L}{(1+\rho')^2+\rho''^2} \end{cases}, \tag{4.I.55}$$

in which c_L and s_L are defined as in eq. (4.61).

Operating the first equation in eq. (4.I.55) algebraically, and grouping the resultant terms, it leads to:

$$\left(\rho' + \frac{a}{a+c_L}\right)^2 + \left(\rho'' - \frac{s_L}{a+c_L}\right)^2 = \left(\frac{1}{a+c_L}\right)^2, \tag{4.I.56}$$

which is the equation of a circumference parameterized by a, c_L, and s_L:

$$\left(-\frac{a}{a+c_L}, \frac{s_L}{a+c_L}\right) : \frac{1}{a+c_L}, \tag{4.I.57}$$

written by using the habitual reduced notation "(center):radious".

Operating the second equation in eq. (4.I.55) in the same way as in eq. (4.I.58), the resultant equation is:

$$\left(\rho' + \frac{b}{b+s_L}\right)^2 + \left(\rho'' + \frac{c_L}{b+s_L}\right)^2 = \left(\frac{1}{b+s_L}\right)^2, \tag{4.I.58}$$

which is the equation of a circumference parameterized by b, c_L, and s_L:

$$\left(-\frac{b}{b+s_L}, -\frac{c_L}{b+s_L}\right) : \frac{1}{|b+s_L|}. \tag{4.I.59}$$

On the other hand, regarding the modulus-phase parameterizations, the inverse function (from ρ to Z_{0nL}), $Z_{0nL}(\rho)$ is written spliting its modulus and phase, which are parameterized as in eq. (4.68):

$$\begin{cases} m^2 = \frac{\left(1-\rho'\right)^2+\rho''^2}{(1+\rho')^2+\rho''^2} \\ p = \varphi_{Z_L} - \tan^{-1}\left(\frac{\rho''}{1-\rho'}\right) - \tan^{-1}\left(\frac{\rho''}{1+\rho'}\right) \end{cases}. \tag{4.I.60}$$

Operating the first equation in eq. (4.I.60) algebraically, and grouping the terms accompanying ρ' and ρ'', it leads to:

$$\left(\rho' + \frac{m^2+1}{m^2-1}\right)^2 + \rho''^2 = \left(\frac{2m}{m^2-1}\right)^2, \tag{4.I.61}$$

which is the equation of a cicumference centered in the real axis, which may be written compactly as:

$$\left(-\frac{m^2-1}{m^2+1}, 0\right) : \frac{2m}{m^2-1}. \tag{4.I.62}$$

Now operating the second equation in eq. (4.I.60) algebraically: first solving $p - \varphi_{Z_L}$ to then calculate the tangent on both sides applying the angle sum identity; it leads to:

$$\rho'^2 + \left(\rho''^2 - \frac{1}{\tan\left(p - \varphi_{Z_L}\right)}\right)^2 = \left(\frac{1}{|\sin\left(p - \varphi_{Z_L}\right)|}\right)^2, \tag{4.I.63}$$

which is the equation of a cicumference centered in the imaginary axis, which may be compactly written parameterized by p and φ_{Z_L} as:

$$\left(0, \frac{1}{\tan\left(p - \varphi_{Z_L}\right)}\right) : \frac{1}{|\sin\left(p - \varphi_{Z_L}\right)|}. \tag{4.I.64}$$

Chapter 5

Examples of use of the Complex Transmission Line Theory

5.1 Introduction

The CTLT presented in Chpt. 4 to rigorously analyze the effects of losses on TLs which support the propagation of HPWs (CTLT-v1.0) has set up the basis of the methodology based on complex transformations between the parameters characterizing these TLs at different levels: the line parameters, basic parameters, and wave parameters.
The intuitive idea underlying the CTLT is connecting these groups of parameters in such a way that they allow for characterizing the propagative solutions in the TLs under study.

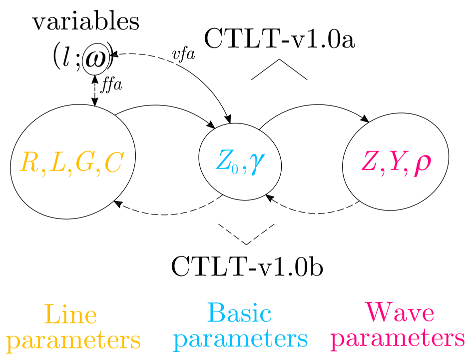


Fig. 5.1: Scheme of characterizations described within the context of the complex analysis of the LTLT regarding HPWs (CTLT-v1.0): the direct characterization (CTLT-v1.0a, in continuous lines) and the inverse characterization (CTLT-v1.0b, in dashed lines).

Two ways of "guiding" the analysis of the cited transformations have been presented concurrently: the direct and the inverse characterizations of the CTLT (CTLT-v1.0a and CTLT-v1.0b, respectively, within the CTLT-v1.0).
The direct characterization refers to the analysis describing the TL parameters in terms of losses. Its physical meaning is direct (this gives its name): it is the direct characterization of the TL parameterizing the losses; and, because this physical interpretation, it is clearly connected with the type of waves (HPWs) which are studied.
On its behalf, the inverse characterization refers to the analysis "guided" for the characterization of losses in terms of specifications on the TL parameters. Although this definition is clearly based on the inverse interpretation of the direct characterization (and this is enough for the comprehen-

sion of its role within the CTLT-v1.0), the inverse characterization is the basis and origin for the analysis of different mode solutions in equivalent TLs, which lets to define the GTLT-v1 in Chpt. 3. For this extended purpose, its "analytical" inverse connotation (more than the physical) is clear when trying to answer "which values parameterize a specific wave solution". This characterization is "more universal" than the inverse characterization in the sense that it gathers different mode solutions within the same analysis. However, each analysis has its own particularizations, as for example the inverse characterization of line parameters regarding HPWs in the CTLT-v1.0b, which takes into account that the parameterizations should be positive real or zero.

In Fig. 5.1, a scheme representing the transformations between the groups of TL parameters is shown. These transformations are represented by arrows starting at the group of parameters to be transformed and finishing at the transformed group.
Both the direct and the inverse characterization are represented in the same scheme by arrows in opposite direction.
For the study of the parameter transformations it is required for the analysis to be bi-parameterized in such a way that each of those transformations can be described both analytically and graphically by keeping fixed one parameter while varying the other one, and viceversa. This fact allows for analyzing the transformations in planes, an thus geometrically characterizing the resultant plane curves.
For the purpose of bi-parameterizing the transformations, the appropriate normalizations have to be chosen depending on the transformations to be analyzed and the physical interpretations the analysis is expected to describe.

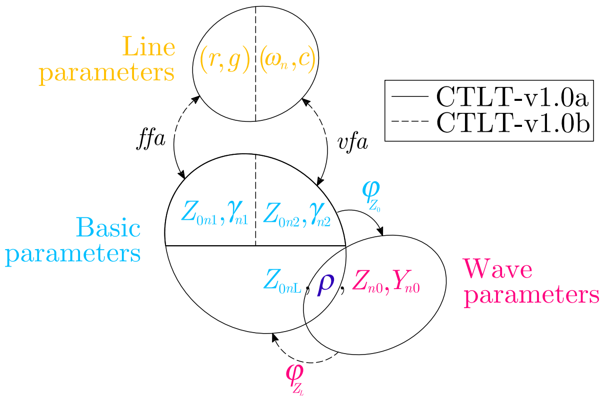


Fig. 5.2: Scheme of groups of normalized parameters and the transformations between them, included in the direct normalization (continuous arrows) and in the inverse normalization (dahed arrows).

In Fig. 5.2, a scheme representing the normalizations for describing the groups of parameters and the transformations between them is shown. It should be noted that: (i) the line parameters are differently parameterized depending on the *ffa*/*vfa*. However, these parameterizations are directly related in the plane which describes the line parameters: the rg-plane, where they are directly represented by the cartesian and the polar coordintates, repectively; (ii) on one hand, the basic parameters parameterize the wave parameters by means of the phase of the characteristic impedance, φ_{Z_0}, that is, the direct characterization of wave parameters. The remaining parameterization is inner: it is the complex parameterization (real-imaginary or modulus-phase) of any of the wave parameters. As a result, the direct characterization of wave parameters involves half transformations in the sense that the analysis is based on keeping fixed the parameterization regarding the basic parameters while varying the complex parameterizations of wave parameters. On the other hand, the wave parameters statically (this means at a fixed point in the TL, so the kinematics –studied by means of the propagation constant– are not characterized) parameterize the basic parameters by means of the phase of the wave impedance at the load, φ_{Z_L}, that is, the inverse characterization

of the basic parameters. The remaining parameterization is inner: it is the complex parameterization (real-imaginary or modulus-phase) of any of the (by definition, static) basic parameters; and (iii) the duality in the role of the reflection coefficient, ρ, which may be addressed in the group of wave paramters or as a basic parameter depending on the characterization is direct or inverse, respetively.
These facts (i-iii) should be taken into account for the subsequent extensions of the analysis, when solving TL-related problems, and also when dealing with the applications in the analysis or circuit design based on the results of the CTLT.

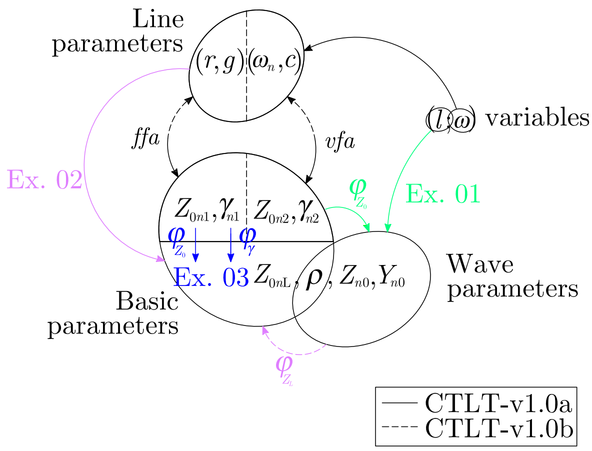


Fig. 5.3: Scheme of groups of normalized parameters and the transformations between them includying those that represent the examples of use analyed in this chapter.

The basic transformations presented in Fig. 5.2 within the context of the CTLT-v1 lets to analyze all the possible problems which are interesting to be solved as examples of use of the CTLA.
Among of them, there are some of special interest because the analytical expressions of the underlying LTLT is not able to answer them directly (without numerical computation). In this sense, the CTLT would be able to overcome this limitation and, which is even more important, provide the appropriate physical meaning to the analysis.
The following questions leave some interesting open problems which are required to be solved in the context of the CTLT:

(i) *How the TL parameters behave along the TL?*
The TL's length, l, is the main variable of the problem (the unique in the *ffa*). Thus, it is important to characterize all the parameters of the TL when l varies.
By definition (regarding the achieved solution of the problem), the only parameters which depend on the TL's length are the wave parameters (see the equations of $\rho(l)$, $Z(l)$, and $Y(l)$ in eqs. (2.83), (2.85), and (2.86), respectively). Notice that, among of them, the expressions of $Z(l)$ and $Y(l)$ present great complexity and, for example, they do not have immediate inverse function to solve the l at which a given Z or Y produces, if any (nevertheless, in [VG17-I], a trigonometric complex function whose argument is a logarithmic reparameterization of $\rho(l)$ ($\rho_{\log}(l)$) has been solved for describing Z and Y, so it may be used inversely to solve the $\rho_{\log}(l)$). Thus, it is required to characterize this variation of wave parameters along the TL in the context of the CTLT in order to solve the problem analytically.
Notice that the problem of characterizing the wave parameters is strictly a direct characterization. That is because it is intended to describe the parameters along the TL's lenght, so its nature is clearly physical, and thus typical of direct characterizatons.
In Fig. 5.3 this characterization is framed among the transformations involved in the CTLT (marked as Ex. 01).

(ii) *How the wave parameters behave in terms of losses and frequency?*
The immediate (in the sense of direct) analysis in terms of losses/frequency affects the ba-

sic parameters. But now the question is how the wave parameters are affected by these lossy/frequency parameterizatons.
In order to make this dependence explicit, the analysis should be referred at any fixed point along the TL. In this way, the parameterization of the wave impedance is fixed and, on the other hand, the initially complex parameterization of basic parameters is physically realizable when describing it in terms of any type of losses or frequency.
Moreover, just as this problem has been posed here, it combines both the direct characterization of the complex parameterizations regarding the basic parameters, and the inverse characterization of these basic parameters in terms of the (fixed) wave impedance.
In Fig. 5.3 this analysis is framed among the transformations defined in the CTLT (marked as Ex. 02).

(iii) *Which TLs keep the phase of the characteristic impedance fixed?*
It has been seen in Chpt. 4 that the angle φ_{Z_0} is crucial for the direct characterizaton of wave parameters when losses are fixed. Moreover, it has been seen that the inverse characterization of this angle lead to the losses/frequencies that parameterize it. But, how the rest of parameters: the propagation constant and the wave parameters; behave in terms of these changes?
And reciprocally: *Which TLs keep the phase of the propagation constant fixed?*
Consequently: *How can these phases combined for characterizing the TL?*
It might be advanced that these angles will serve to parameterize how the losses affect the wave parameters along the TL once the appropriate parameterizations of them are chosen. Of course, the scenario in this problem is much less specific than the two Ex. 01 and Ex. 02 before, which means a loss of specificity on the analysis in favor of the "universalization" of the analysis (two analysis are carried out at the same time).
This analysis combines the inverse characterization used when analyzing ρ (playing the role of basic parameter) in terms of angles, and the direct characterization when obtaining the rest of wave parameters from ρ (changing its role to wave parameter, previously).
In Fig. 5.3 this analysis is contextulized within the transformations of in the CTLT (marked as Ex. 03).

In this chapter the examples explained above: Ex. 01, Ex. 02, and Ex. 03; are presented.
Notice that these examples are the combination of basic transformations analyzed in the context of the CTLT which together describe a physical problem.
Each example is presented emphasizing: (i) the definitions and parameters (normalizations included) that lead to the intended analysis; (ii) the mathematical analysis of the problem, guided to the subsequent graphical analysis; (iii) the graphical analysis in itself. This analysis is presented gathering all the involved planes in order to be then compared. Sometimes the analysis in one plane is represented by a point, but in many cases the analysis is a complex set of curves, depending on the type of problem to be described. The planes which are more significative to the study are properly highlighted in size; (iv) the geometrical analysis of the curves based on both the mathematical notes and the graphical representations; (v) the physical interpretations regarding the analysis, while emphasizing the conclusions which are found in the physical behavior of TLs analyzed as in the example in question; and (vi) the possible practical uses for the analysis of TL-related problems and the applicability with practical-design purposes.

5.2 Analysis along the TL when losses and frequency are fixed

5.2.1 Definitions and parameters

The most immediate issue the CTLT has to deal with is the description of the total voltage and current waves along the TL, that is, the study of the wave equation solutions in terms of the TL's length, l.
The expressions of these total voltage and current waves as functions of z have been obtained in eqs. 2.72 and 2.73, respectively. For obtaining the expressions in terms of l, the following change is proposed:

$$z = D - l, \tag{5.1}$$

in which D (expressed in [m]) is the total length of the TL measured from the generator ($z = 0$[m], so $l = D$[m]) to the load ($z = D$[m], so $l = 0$[m]), so the total voltage and current waves result:

$$v(l) = V_L^+ e^{\gamma l} + V_L^- e^{-\gamma l}, \text{ and} \tag{5.2}$$
$$i(l) = I_L^+ e^{\gamma l} - I_L^- e^{-\gamma l}, \tag{5.3}$$

being $V_L^{\pm}$ and $I_L^{\pm}$ the phasors of the incident ($^+$) and reflected ($^-$) voltage and current waves, respectively.

Remember that the reflection coefficient defined as a function of l in eq. (2.83), $\rho(l)$, fixes the relation between the phasors of the incident and reflected waves.
Moreover, the wave impedance defined as a function of l in eq. (2.85), $Z(l)$, fixes the relation between the total voltage wave in eq. (5.2) and the total current in eq. (5.3); whereas the wave admittance, $Y(l)$, fixes the inverse relation.
The expressions of these wave parameters are here recalled for their subsequent analysis:

refl. coeff.	$\rho(l) = \frac{v^-(l)}{v^+(l)} = \rho_L e^{-2\gamma l} \left(\equiv \frac{i^-(l)}{i^+(l)}\right)$, $\rho_L = \frac{V_L^-}{V_L^+} = \frac{I_L^-}{I_L^+}$,	(4)
wave imp.	$Z(l) = \frac{v(l)}{i(l)} = \frac{V_L^+ e^{\gamma l} + V_L^- e^{-\gamma l}}{I_L^+ e^{\gamma l} - I_L^- e^{-\gamma l}} = Z_0 \frac{1+\rho(l)}{1-\rho(l)}$, $Z_0 = \frac{V_L^+}{I_L^+} = \frac{V_L^-}{I_L^-}$,	(5)
wave adm.	$Y(l) = \frac{1}{Z(l)} = Y_0 \frac{1-\rho(l)}{1+\rho(l)}$, $Y_0 = \frac{1}{Z_0}$.	(6)

The wave parameters described along the TL: $\rho(l)$, $Z(l)$, and $Y(l)$; are used for determining the voltage and current waves in the TL.

Remark 30. *The wave parameters described along the TL completely determine the incident and reflected voltage and current waves if the basic parameters are known.*
In fact, the characteristic impedance is not necessary to be known since the inverse characterization of basic parameters leads to solve it if ρ *and* Z *at the load* ($\rho_L \equiv \rho(l = 0)$ *and* $Z_L \equiv Z(l = 0)$) *are know.*

Thus, the issue of characterizing the voltage and current waves focuses on studying the wave parameters in eqs. (5.4)-(5.6).
The starting point is the data that characterizes the TL: (i) the line parameters when the frequency is fixed, which, in turn, determine the basic parameters by means of the direct characterization; as well as (ii) the impedance at the load, which sets the BCs at $l = 0$.

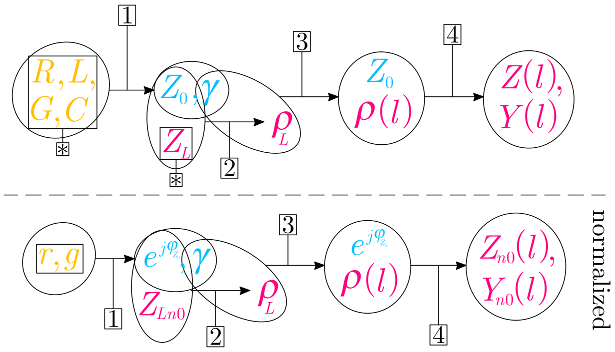


Fig. 5.4: Scheme of parameters and transformations (1-4) to be followed for finally achieving the characterization of the wave parameters along the TL. The parameters which are the data for this example of analysis are marked with the * symbol.

Once the basic parameters are known and fixed (for any l) the wave parameters are accordingly normalized with respect to them. Thus, the modulus of the characteristic impedance (which is fixed in this analysis) normalizes the wave impedance and wave admittance as in the direct characterization of wave parameters presented in Sect. 4.3.2 in Chpt. 4, leading to:

$$Z_{n0}(l) = \frac{Z(l)}{|Z_0|} = e^{j\varphi_{Z_0}} \frac{1+\rho(l)}{1-\rho(l)}, \text{ and} \tag{5.7}$$

$$Y_{n0}(l) = Z_0 \cdot Y(l) = e^{-j\varphi_{Z_0}} \frac{1-\rho(l)}{1+\rho(l)}, \tag{5.8}$$

while $\rho(l)$ remains invariant, as in eq. (5.4).
On the other hand, $Z_L \equiv Z(l=0)$ is also normalized when particularizing eq. (5.7) in such a way that:

$$Z_{Ln0} = \frac{Z_L}{|Z_0|} \equiv Z_{n0}(l=0), \tag{5.9}$$

and consequently ρ_L in eq. (5.4) is defined as:

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0} \equiv \frac{Z_{Ln0} - e^{j\varphi_{Z_0}}}{Z_{Ln0} + e^{j\varphi_{Z_0}}}. \tag{5.10}$$

The characterization of ρ_L lets studying the expression of $\rho(l)$ in eq. (5.4), which is also included in the expression of $Z_{n0}(l)$ and $Y_{n0}(l)$ in eqs. (5.7) and (5.8), respectively, leading to their study and so to fully characterize the wave parameters along the TL.

In Fig. 5.4, a scheme gathering all the parameters and the steps which "guide" the analysis along the TL is shown.

5.2.2 Mathematical analysis

The mathematical expressions of the total voltage and current waves in eqs. 5.2 and (5.3) are functions of l, rigorously defined as:

$$\begin{aligned} v,\, i: \quad [0, D] \subset \mathbb{R}^+ \cup \{0\} &\to \mathbb{C} \\ l &\mapsto v(l),\, i(l). \end{aligned}$$

These functions $v(l)$ and $i(l)$ are –at least– in $\mathcal{C}^2$, because they are a linear combination of solutions to the wave equation.

Likewise, the functions $\rho(l)$, $Z(l)$, and $Y(l)$, which come from the quotient of continuous functions, are also functions in $\mathcal{C}^2$, except in those points that make zero the denominator. The problem of solving the points l at which the wave parameter complex functions are not continuous is an interesting exercise of application of some Complex Analysis theoremas (in particular Cauchy-Riemann equations and), [BC90], which has been addressed in Appendix 5.A. The result of this analysis is such that the continuity of the wave parameter complex functions is verified if $D < \infty$ and (being $l > 0$), which is an obvious condition for the physical interpretation of the TL.
Moreover, it is possible to pose the problem of finding the discontinuities of a function that gives the l's at which some of the wave parameter complex functions equals a specific value.

Example 5.2.1. *A trivial example of using the Cauchy-Riemann equations for determining the l's at which a lossless TL behaves as an open circuit is shown. For this purpose, the $\rho(l)$ function is used, which is set as*

$$\rho(l) = e^{-2\gamma l}, \tag{5.11}$$

for the sake of simplicity and solvavility of the problem reviewed here as example.
The values at which the TL determined by $\rho(l)$ particularized to the lossless case is equivalent to an open circuit ($\rho(l) = 1$) are $l = k\pi/\beta \equiv k\lambda/2$.
However, in this example, an alternative method based on solving the discontinuities regarding a defined "characteristic function" $f(\alpha, \beta; l)$ is shown.
The characteristic function $f(\alpha, \beta; l)$ is, in this case, defined as

$$\begin{aligned} f(\alpha, \beta; l) =& \frac{1}{\rho(l) - 1} = \frac{1}{e^{-2\gamma l} - 1} = \\ =& \frac{e^{-2\alpha l}\cos(2\beta l) - 1}{e^{-4\alpha l} + 1 - 2e^{-2\alpha l}\cos(2\beta l)} + j\frac{e^{-2\alpha}\sin(2\beta l)}{e^{-4\alpha l} + 1 - 2e^{-2\alpha l}\cos(2\beta l)} =. \\ =& u(\alpha, \beta; l) + jv(\alpha, \beta; l) \end{aligned} \tag{5.12}$$

The Cauchy-Riemman equations particularized to this lossless case ($\alpha = 0$) are:

$$\begin{cases} u_\alpha|_{\alpha=0} = 0 \doteq v_\beta|_{\alpha=0} = 4\beta\left[-1 + \cos(2\beta l)\right] \\ u_\beta|_{\alpha=0} = 4\beta\left[-1 + \cos(2\beta l)\right] \doteq -\left. v_\alpha\right|_{\alpha=0} = 0 \end{cases}. \tag{5.13}$$

The points in which the singularties are able to occur are those in which the characteristic function verify the Cauchy-Riemann equations. Those are the predicted $l = k\pi/\beta \equiv k\lambda/2$.

As it may be foreseen, the complexity of applying this analytical method increases with lossy TLs and the examination of other wave parameter functions.

The alternative analysis lies in addressing the wave parameter complex functions as curves, which is possible thanks to the isomorphism between the complex numbers ($\mathbb{C}$) and the real numbers in the plane ($\mathbb{R}^2$).
In this context, the wave parameters should be seen as curves in the plane parameterized by l:

$$\begin{aligned} \rho, Z, Y\colon\quad (0, D) \subset \mathbb{R}^+ \cup \{0\} &\rightarrow \mathbb{R}^2\ (\mathbb{C}) \\ l &\mapsto (\rho', \rho''), (Z', Z''), (Y', Y''). \end{aligned}$$

Notice that the curves are defined over a open interval $l \in (0, L)$, for the rigorous definition of them as a regular parameterization of curve, [MP77].

On the other hand, the expressions of the wave impedance and wave admittance in eqs. (5.5) and (5.6), and also its normalized versions in eqs. 5.7 and 5.8, respectively, may be seen as the transformations of the curve $\rho(l)$.
These transformations are rigorously defined as:

$$\begin{aligned} T_Z\colon\quad \mathbb{R}^2\ (\mathbb{C}) &\rightarrow \mathbb{R}^2\ (\mathbb{C}) \\ \rho(l) &\mapsto T_Z\left[\rho(l)\right] = Z_0\frac{1 + \rho(l)}{1 - \rho(l)} = Z(l), \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} T_Y\colon\ \ \mathbb{R}^2\ (\mathbb{C}) \to& \mathbb{R}^2\ (\mathbb{C}) \\ \rho(l) \mapsto& T_Y\left[\rho(l)\right] = \frac{1}{Z_0}\frac{1-\rho(l)}{1+\rho(l)} = Y(l), \end{aligned} \tag{5.15}$$

for representing the wave impedance and wave admittance, respectively.
These transformations are conformal mappings, [BC90], studied in the context of complex transformations introduced in Sect. 4.3.2 in Chpt. 4.

Remark 31. *The study of the wave parameters along the TL is addressed in the context of the curve transformations taking advantage of the complex transformations defined for the direct characterization of wave parameters.*

Keeping this idea in mind, it is about characterizing the reflection coefficient along the TL in order to be then transformed by means of the transformations T_Z and T_Y defined in eqs. (5.14) and (5.15) to obtain the wave impedance and wave admittance, respectively, also described along the TL.

Remark 32. *The reflection coefficient described along the TL corresponds with its direct characterization in terms of the TL's length, which is equivalent to characterize the basic parameters in terms of losses, for example, and thus the same analysis based on: (i) obtaining the normalizations and parameterizations; (ii) the graphical and geometrical analysis; and (iii) the subsequent physical interpretations; may be done.*
As said before, (iv) the practical uses mainly come when describing the rest of wave parameters by means of complex transformations from the characterization of ρ.

Direct characterization of the reflection coefficient along the TL

In this section, the normalizations and parameterizations for the characterization of ρ along the TL, followed by the graphical and geometrical analysis of the curves representing ρ, to finally obtain the most important physical interpretations regarding this characterizations are outlined.
The purpose is "preparing" the characterization of the wave impedance and wave admittance along the TL, when see these latter parameters as complex transformations from ρ.

***Normalizations and parameterizations*:** As it has been seen several times, the reflection coefficient does not hold any normalization for the analysis of wave parameters in neither the direct characterization nor the invere characterization. Thus, the expression which describes ρ in terms of the TL's length is the one in eq. (5.4).

Nevertheless, the parameter which parameterizes ρ along the TL (l) may be normalized in different ways leading to different parameterizations of the TL's length.
Among the possible parameterizations of the length, it is common to take the electrical length og the TL (denoted as l_e), which is no more than the physical length referred to the wavelength which characterizes the propagative waves (denoted as λ), so

$$l_e = \frac{l}{\lambda} = \frac{\beta l}{2\pi}. \tag{5.16}$$

As a result, the expression of the reflection coefficient parameterized by the electrical length, l_e, is

$$\begin{aligned} \rho(l_e) = \rho_L e^{-\frac{4\pi\gamma l_e}{\beta}} = \rho_L e^{-4\pi\frac{\alpha}{\beta}l_e} e^{-j4\pi l_e} \in \mathbb{C}, \\ \text{in which } \rho_L = |\rho_L| e^{j\varphi_{\rho_L}} \in D_\rho \subset \mathbb{C}. \end{aligned} \tag{5.17}$$

Notice that l_e, just as it has been defined in eq. (5.16), is dimensionless.
Moreover, ρ_L is given by the previous analysis of Z_L in the ρ-plane (the transformation of Z_L to the ρ-plane by usng the GSC).
For the (complex) parameterization of ρ along the TL, it is interesting to split it in its modulus and phase:

$$\begin{cases} (|\rho| \equiv)\ |\rho|(l_e) = |\rho_L| e^{-4\pi \frac{\alpha}{\beta} l_e} \\ (\varphi_\rho \equiv)\ \varphi_\rho(l_e) = -4\pi l_e + \varphi_{\rho_L} \end{cases} . \tag{5.18}$$

Notice the complex parametric expression of $\rho(l_e)$ in eq. (5.17), or the one written in polar form in eq. (5.18), are parameterized by the electrical length l_e and ρ_L, besides the ratio α/β, which plays the role of the extra parameterization. In this sense, recall that for the complete characterization of ρ, a pair of parameterizations is required. For the analysis along the TL, one of them must be a parameterization of the TL's length, for instance, the electrical length. The other one is fixed by the specification of the line parameters in the TL and the impedance at the load, which are reflected in the definition of ρ_L and the ratio α/β.
The issue of finding a couple of parameterizations that parameterize the analysis of wave parameters in terms of losses and along the TL is specifically addressed in Ex. 03.

For the next basic graphical and geometrical analysis of the curves regarding ρ along the TL, ρ_L is supposed to be known.

***Graphical and geometrical analysis*:** A graphical example of the curve representing $\rho(l)$ (or $\rho(l_e)$, since this expression is a reparameterization of the original equation that conserves its shape) is shown below:

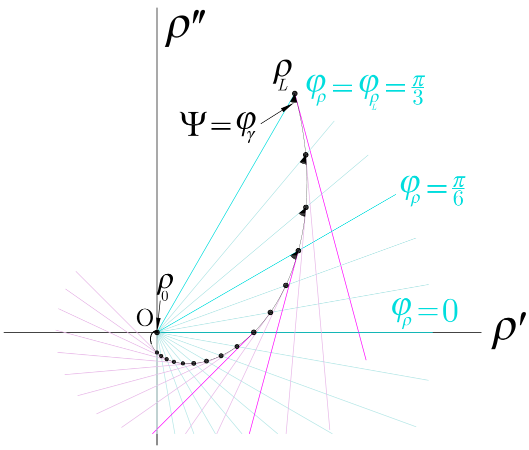


Fig. 5.5: Graphical example of the curve which represents ρ along the TL when $\rho_L = 1e^{j\frac{\pi}{3}}$ and $\alpha/\beta = 1/\tan(\varphi_\gamma) = 1/\tan(\Psi) = 1$.

This curve is a logarithmic spiral, [Law72], represented in the ρ-plane. Since the coefficient ρ_L is arbitary chosen in the example above, the spiral may have one end (parameterized by $l = l_e = 0$) in any point within the domain D_ρ. The other end tends to $\rho = 0$ as long as $l = l_e \to \infty$.

The general equation of this logarithmic spiral is obtained when eliminating the parameterization, for example, l_e in eq. (5.18) by solving l_e from the phase φ_ρ and substituying it into the modulus $|\rho|$:

$$|\rho| = |\rho_L| e^{-\frac{\alpha}{\beta}(\varphi_{\rho_L} - \varphi_\rho)}, \tag{5.19}$$
in which $\varphi_\rho \leq \varphi_{\rho_L}$.

This expression will be useful for parameterizing the reflection coefficient in modulus-phase to be then transformed to the wave impedance and admittance.

Notice that the general equation of the logarithmic spiral is determined by the ratio α/β once the scaling provided by ρ_L is known.
This ratio equals the cotangent of the phase of the propagation constant, so

$$\frac{\alpha}{\beta} = \frac{1}{\tan(\varphi_\gamma)}. \tag{5.20}$$

When analyzing the logarithmic spiral which represents ρ along the TL geometrically (see Appendix 5.B), it can be concluded that the angle between the radious and the tangent of the logaritmic spiral (denoted as Ψ) is exactly the phase of the propagation constant:

$$\Psi \equiv \varphi_\gamma. \tag{5.21}$$

In Fig. 5.5 this significant angular indentity is detached and used for representing the spiral graphically.
As a reault, it may be stated that the angle of the propagation constant determines the variation of ρ along the TL.

Remark 33. *Just as φ_{Z_0} is the parameter which determine the influence of losses at any fixed point in the TL, the parameter φ_γ determines the movement of wave parameters along the TL. As a consequence, both angles φ_{Z_0} and φ_γ seem to determine the dependence of the TL on both losses and the TL's length for the complete characterization of the wave parameters.*

***Physical interpretations*:** In this section, the main physical interpretations regarding the spiral that represents the reflection coefficient along the TL are detached, giving special emphasis to the geometrical properties and its usefulness for characterizing the rest of wave parameters.

At the load, the reflection coefficient is completely determined by the impedance and the characteristic impedance or, equivalently, the normalized impedance (at the load) and the phase of the characteristic impedance (see the equivalence in eq. (5.10)).
In addition, once the reflection coefficient at the load is known, its variation along the TL is determined by the phase of the propagation constant, which puts into relation the attenuation constant and the phase constant as in eq. (5.20).
As a consequence, there are multiple TLs (parameterized by different line parameters and frequency) with the same variation of ρ along the TL (those given by the inverse characterization of line parameters parameterized φ_γ), but different starting points and shiftings (given by ρ_L).

Moreover, the reflection coefficient tends to be zero when $l = l_e \to \infty$, provided that $\alpha \neq 0$ (and so $\varphi_\gamma \neq \pi/2$). This means that the (theoretically) infinite lossy TL matches the characteristic impedance at the beginning.
Thus, reciprocally, the TL only mathes the characteristic impedance at the load if the load is just the characteristic impedance.

As shown in Appendix 5.B, the arc length of the lossy TL ($\alpha \neq 0$) is not proportional to neither the physical length nor the electrical length. However, the phase of the reflection coefficient measured from the starting point of the curve is proportional to the electrical length (see the phase φ_ρ in eq. 5.18) and so to the physical length (because their linear relation defined in eq. (5.16)).

***Practical uses*:** From the general equation of the logarithmic spiral that represents ρ along the TL, the modulus and phase parameterizations of ρ are physically related, as well as the transformations to the wave impedance and wave admittance complex planes by means of an intuitive graphical analysis.
Then, the resultant transformations are geometrically analyzed, also based on this direct characterization of the reflection coefficient. For example, the length scales over the resultant curve transfomations are given by means of the transformation of the phase constant curves in the ρ-plane.

5.2.3 Graphical and geometrical analysis

For the graphical analysis of wave parameters, the transformations from the modulus-phase parameterizations of the curve that represents ρ along the TL in the ρ-plane concerning the direct characterization of wave parameters presented in Sect. 4.3.2 in Chpt. 4 are used in last resort (see this step marked as $\boxed{4}$ in the scheme that summarizes the analysis of this example in Fig. 5.4).
Prior to the parameterized transformations from the ρ-plane, the reflection coefficient at the load is numerically calculated or graphically located in the ρ-plane using the transformation from the Z_{n0}-plane, that is, the GSC, [GDG06], once the basic parameters are calculated by means of their direct characterization presented in Sect. 4.3.1 in Chpt. 4.
These analysis (summarized in the scheme in Fig. 5.4) are presented simultaneously in their respective complex planes for their proper comparison and the appropriate understanding of all of them together.

The datum for this example (marked with the symbol $\boxed{99}$ in Fig. 5.4) are the line parameters and frequency, and the wave impedance at the load. In order to see the differences in the graphical analyis when taking different combinations of these parameters, several examples are presented.

On the other hand, the modulus-phase parameterizations directly parameterize the general equation written in polar form of ρ along the TL (the modulus of ρ as a function of the angle of ρ: $|\rho|\,(\varphi_\rho)$) presented in eq. (5.19) in the following form:

$$\begin{cases} \varphi_\rho = p \\ |\rho| = e^{-\frac{\alpha}{\beta}(p_L - p)} = m \end{cases}, \tag{5.22}$$

$$\text{in which } m_L = |\rho_L|,\ p_L = \varphi_{\rho_L},$$
$$\text{and } p \in \left[p_L - \frac{4\pi L}{\lambda}, p_L\right], \text{ so } p \leq p_L.$$

The transformations of the m-p parameterized curves from the ρ-plane are the circunferences[1]

$$\begin{cases} \left(c_0 \frac{1+m^2}{1-m^2}, s_0 \frac{1+m^2}{1-m^2}\right) : \frac{2m}{|1-m^2|} \\ \left(\frac{-s_0}{\tan(p)}, \frac{c_0}{\tan(p)}\right) : \frac{1}{|\sin(p)|} \end{cases} \tag{5.23}$$

in the Z_{n0}-plane, and

$$\begin{cases} \left(c_0 \frac{1+m^2}{1-m^2}, -s_0 \frac{1+m^2}{1-m^2}\right) : \frac{2m}{|1-m^2|} \\ \left(\frac{s_0}{\tan(p)}, \frac{c_0}{\tan(p)}\right) : \frac{1}{|\sin(p)|} \end{cases} \tag{5.24}$$

in the Y_{n0}-plane, [Gag01], whose intersection following the relation given by eq. (5.22) lead to the curves representing Z_{n0} and Y_{n0} along the TL, respectively.
The parametric expression of the intersection of these curves results hard-to-analyze. Nevertheless,

[1] The well-known notation for refering the center:radious of circumferences, which has been employed throughout this Thesis book, is again used here.

in [VG17-I], a compact expression of these curves is given by using complex functions when the length of the TL is "complexified" (see this explanation in the **1**).

Some graphical examples are studied throughout the following pages. Then the appropriate conclusions based on geometrical analysis are pointed out.

5.2.4 Some examples of graphical analysis of wave parameters along the lossy TL

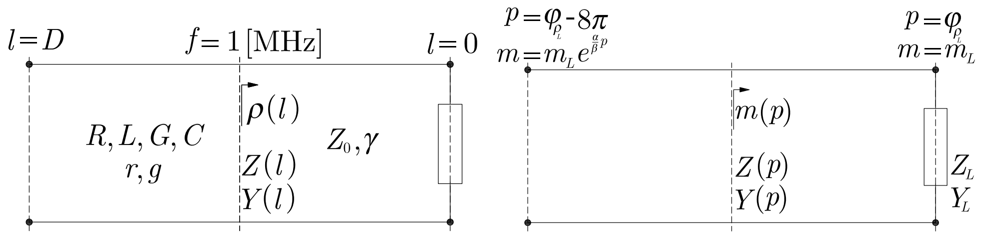
Example I

Numerical values of the transmission line parameters

Physical parameters	*Line parameters*	
	Param.	Normalizations
	$R = 75[\Omega]$ $L = 0.1[\text{mH}]$ $G = 10[\Omega^{-1}]$ $C = 1[\text{uF}]$	$r = 0.1194$ $g = 1.5915$
	Basic Parameters	
	Param.	Normalizations
$f = 1[\text{MHz}]$ $\lambda = 8.6018[\text{cm}]$ $c_e = 8.6018 \cdot 10^4[\text{m·s}^{-1}]$ $D/\lambda = 2$	$Z_0 = 6.6053 + j3.1542 =$ $= 7.3198\angle 25.5256°$ $\gamma = 46.2349 + j73.0446 =$ $= 86.4476\angle 57.6675°$	$Z_{0n1} = 0.6605 + j0.3154 =$ $= 0.7320\angle 25.5256°$ $\gamma_{n1} = 0.7359 + j1.1625 =$ $= 1.3759\angle 57.6675°$
	Wave parameters at the load	
	Param.	Normalizations
	$\rho_L = 0.7540 - j0.0543 =$ $= 0.7560\angle - 4.1175°$	-
	$Z_L = 50 + j10 =$ $= 50.9902\angle 11.3099°$	$Z_{Ln0} = 6.8308 + j1.3662 =$ $= 6.9661\angle 11.3099°$
	$Y_L = 0.0192 - j0.0038 =$ $= 0.0196\angle - 11.3099°$	$Y_{Ln0} = 0.1408 - j0.0282 =$ $= 0.1436\angle - 11.3099°$

Table 5.1: Numerical values of line parameteres, basic parameters, and wave parameters at the load. All of them parameterize the analysis of wave parameters along the transmission line analyzed by means of this first example I.

Transmission line scheme and parameterizations



(a) (b)

Fig. 5.6: Transmission line scheme with (a) the line parameters and (b) the parameterizations.

Direct characterization of basic parameters

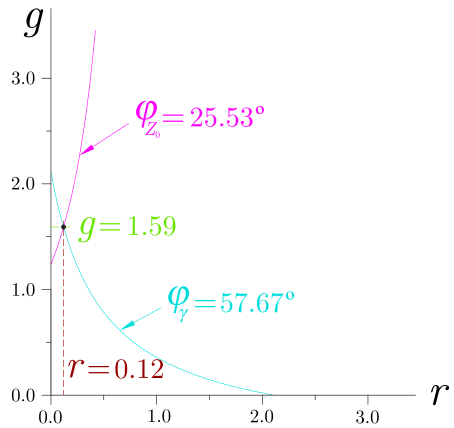
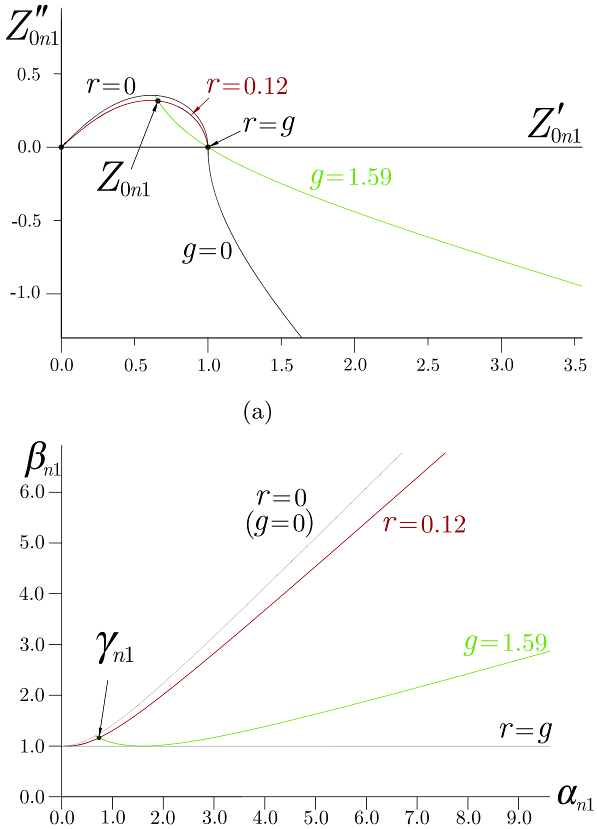


Fig. 5.7: Lossy parameterizations $r = 0.12$ and $g = 1.59$ in the rg-plane.



(a)

(b)

Fig. 5.8: The (a) Z_{0n1}- and (b) γ_n1-planes parameterized by $r = 0.12$ and $g = 1.59$.

Direct characterization of wave parameters along the TL

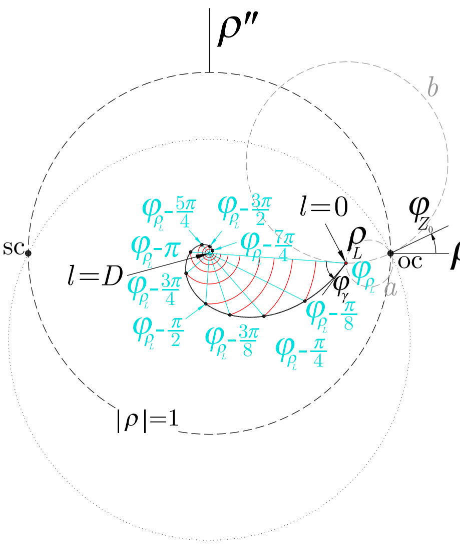
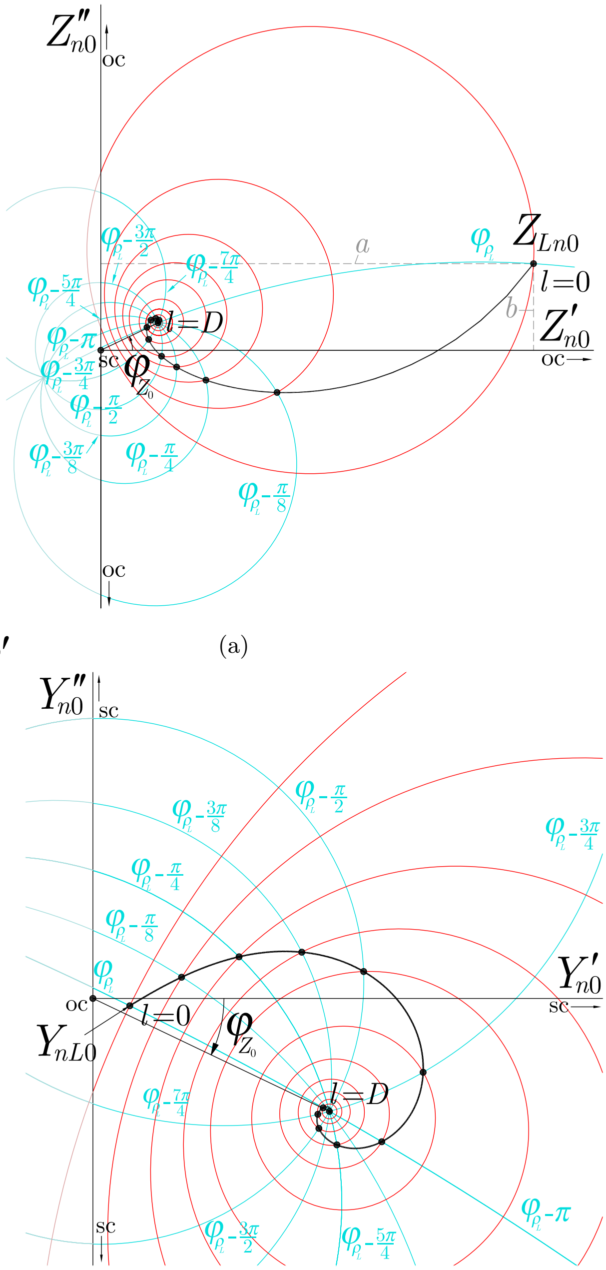


Fig. 5.9: Analysis of ρ along the TL.



(a)

(b)

Fig. 5.10: Graphical analysis of (a) Z_{n0} and (b) Y_{n0}, both along the TL.

Example II

Numerical values of the transmission line parameters

Physical parameters	*Line parameters*	
	Param.	Normalizations
	$R = 75[\Omega]$ $L = 0.1[\text{mH}]$ $G = 10[\Omega^{-1}]$ $C = 1[\text{uF}]$	$r = 0.0597$ $g = 0.7957$
	Basic Parameters	
	Param.	Normalizations
$f = 2[\text{MHz}]$ $\lambda = 4.7322[\text{cm}]$ $c_e = 9.4644 \cdot 10^4[\text{m}\cdot\text{s}^{-1}]$ $D/\lambda = 3.6347$	$Z_0 = 8.4416 + j2.6694 =$ $= 8.8537\angle 17.5482°$ $\gamma = 50.8711 + j132.7752 =$ $= 142.1869\angle 69.0363°$	$Z_{0n1} = 0.8442 + j0.2669 =$ $= 0.8854\angle 17.5482°$ $\gamma_{n1} = 0.48048 + j1.0566 =$ $= 1.1315\angle 69.0363°$
	Wave parameters at the load	
	Param.	Normalizations
	$\rho_L = 0.7052 - j0.0274 =$ $= 0.7057\angle - 2.2282°$ $Z_L = 50 + j10 =$ $= 50.9902\angle 11.3099°$ $Y_L = 0.0192 - j0.0038 =$ $= 0.0196\angle - 11.3099°$	- $Z_{Ln0} = 5.6474 + j1.1295 =$ $= 5.7592\angle 11.3099°$ $Y_{Ln0} = 0.1703 - j0.0341 =$ $= 0.1736\angle - 11.3099°$

Table 5.2: Numerical values of line parameteres, basic parameters, and wave parameters at the load. All of them parameterize the analysis of wave parameters along the transmission line analyzed by means of this second example II.

Transmission line scheme and parameterizations

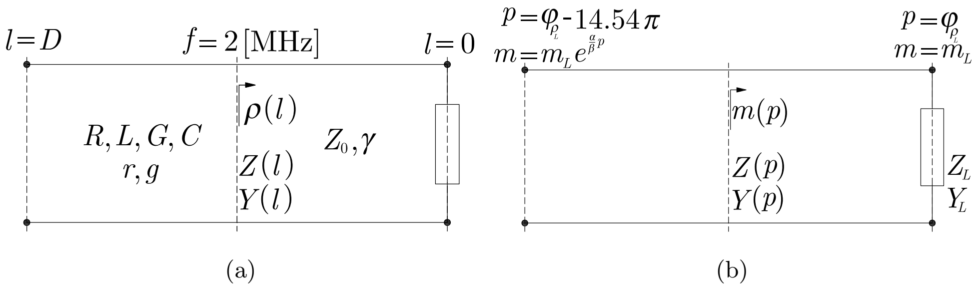


Fig. 5.11: Transmission line scheme with (a) the line parameters and (b) the parameterizations.

Direct characterization of basic parameters

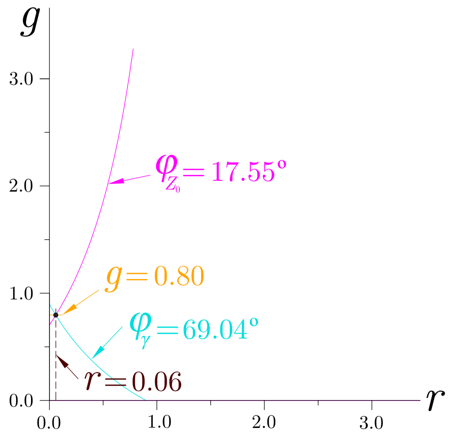


Fig. 5.12: Lossy parameterizations $r = 0.06$ and $g = 0.80$ in the rg-plane.

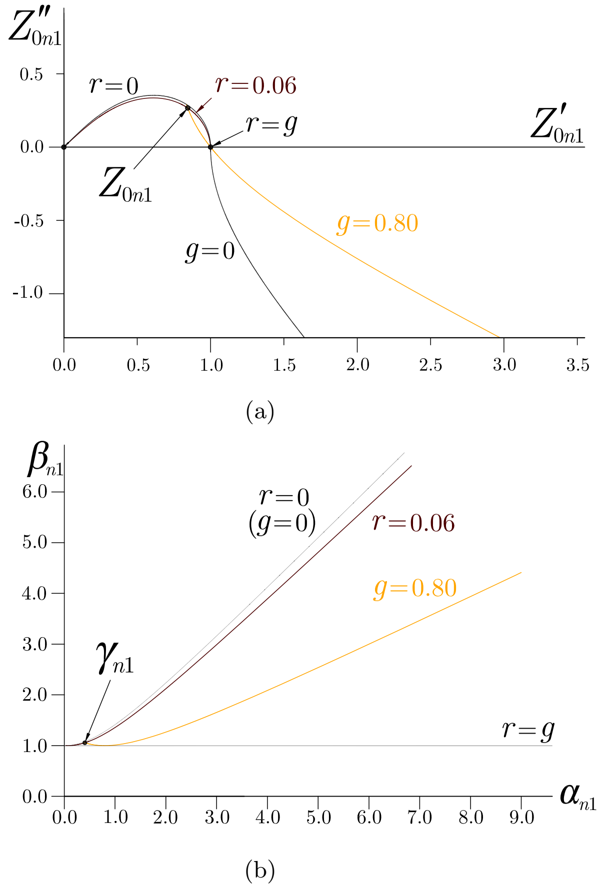


Fig. 5.13: The (a) Z_{0n1}- and (b) $\gamma_n 1$-planes parameterized by $r = 0.06$ and $g = 0.80$.

Direct characterization of wave parameters along the TL

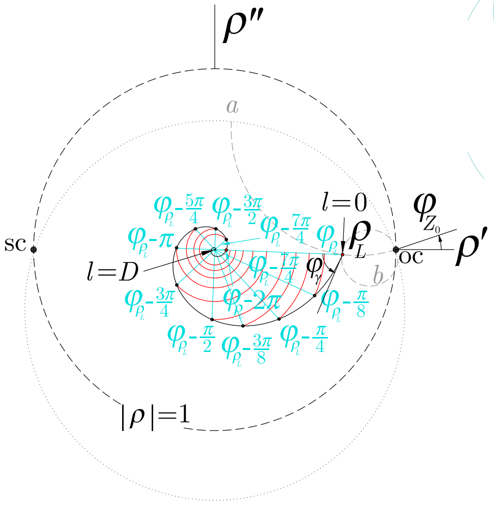


Fig. 5.14: Analysis of ρ along the TL.

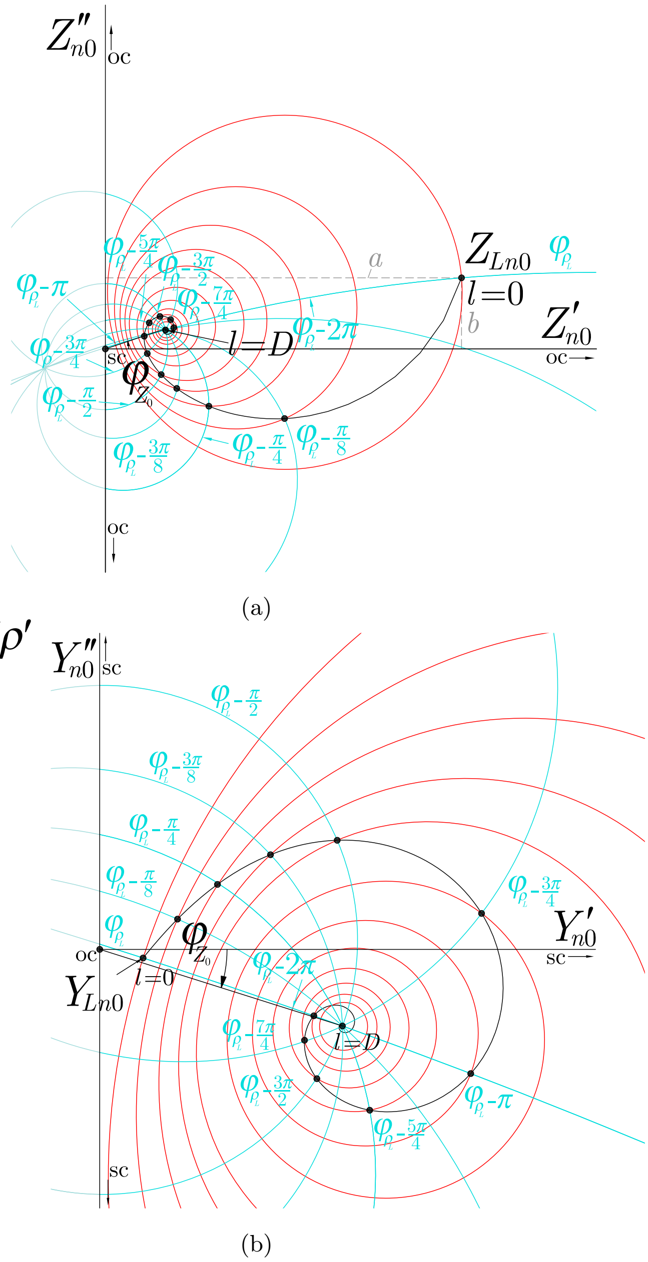


Fig. 5.15: Graphical analysis of (a) Z_{n0} and (b) Y_{n0}, both along the TL.

Example III

Numerical values of the transmission line parameters

Physical parameters	*Line parameters*	
$f = 2$[MHz] $\lambda = 4.7322$[cm] $c_e = 9.4644 \cdot 10^4$[m·s^{-1}] $D/\lambda = 3.6347$	Param.	Normalizations
	$R = 75[\Omega]$ $L = 0.1$[mH] $G = 10[\Omega^{-1}]$ $C = 1$[uF]	$r = 0.0597$ $g = 0.7957$
	Basic Parameters	
	Param.	Normalizations
	$Z_0 = 8.4416 + j2.6694 =$ $= 8.8537\angle 17.5482°$ $\gamma = 50.8711 + j132.7752 =$ $= 142.1869\angle 69.0363°$	$Z_{0n1} = 0.8442 + j0.2669 =$ $= 0.8854\angle 17.5482°$ $\gamma_{n1} = 0.48048 + j1.0566 =$ $= 1.1315\angle 69.0363°$
	Wave parameters at the load	
	Param.	Normalizations
	$\rho_L = 0.5011 - j0.6231 =$ $= 0.7996\angle -51.1921°$	-
	$Z_L = 10 - j15 =$ $= 18.0278\angle -56.3099°$	$Z_{Ln0} = 1.1295 - j1.6942 =$ $= 2.0362\angle -56.3099°$
	$Y_L = 0.0308 + j0.0462 =$ $= 0.0555\angle 56.3099°$	$Y_{Ln0} = 0.2724 - j0.4086 =$ $= 0.4911\angle 56.3099°$

Table 5.3: Numerical values of line parameteres, basic parameters, and wave parameters at the load. All of them parameterize the analysis of wave parameters along the transmission line analyzed by means of this third example III.

Transmission line scheme and parameterizations

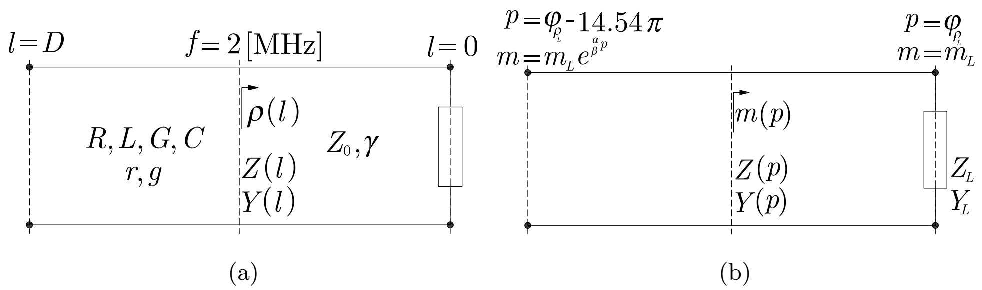


Fig. 5.16: Transmission line scheme with (a) the line parameters and (b) the parameterizations.

Direct characterization of basic parameters

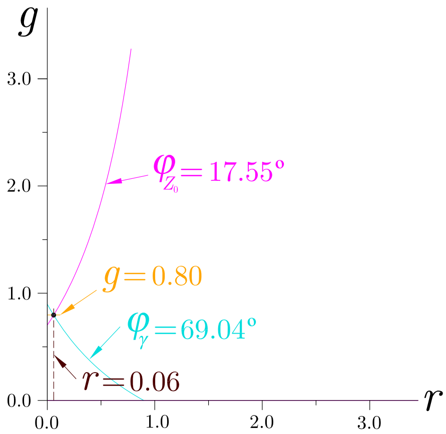


Fig. 5.17: Lossy parameterizations $r = 0.06$ and $g = 0.80$ in the rg-plane.

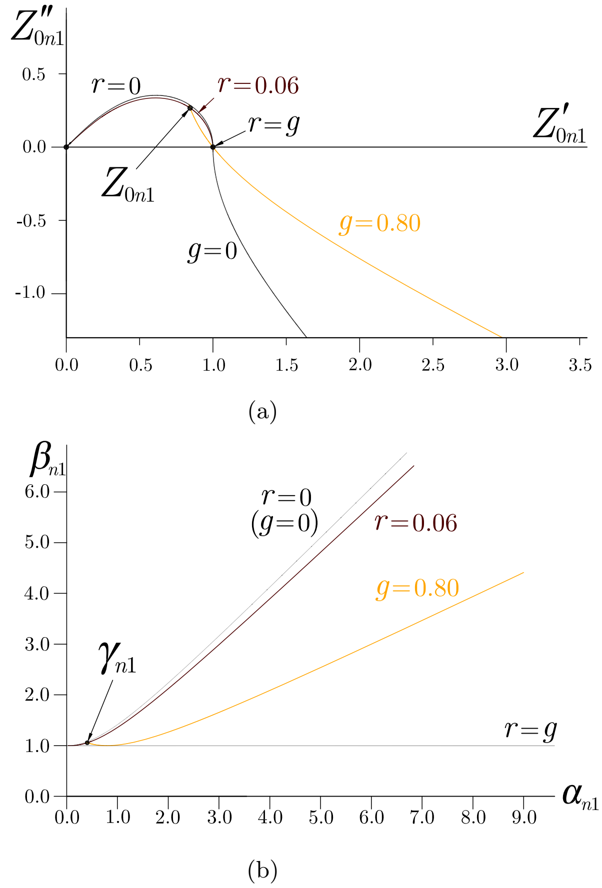


Fig. 5.18: The (a) Z_{0n1}- and (b) γ_n1-planes parameterized by $r = 0.06$ and $g = 0.80$.

Direct characterization of wave parameters along the TL

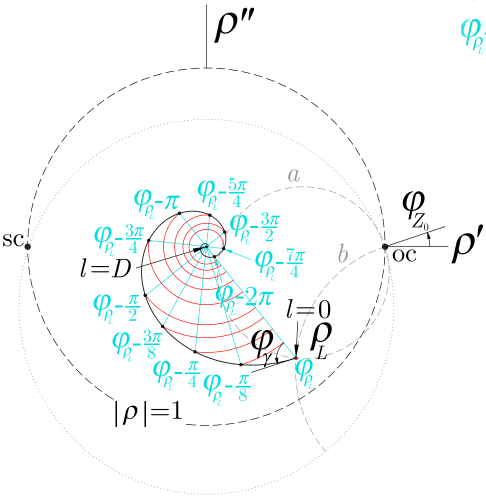
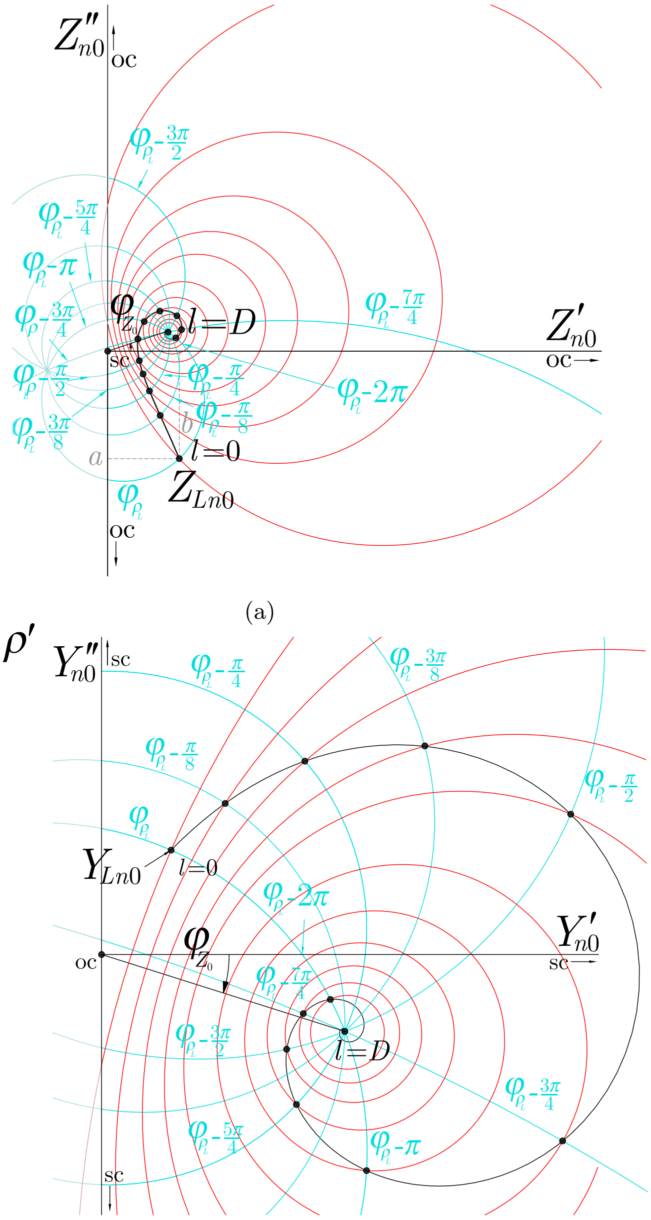


Fig. 5.19: Analysis of ρ along the TL.



(a)

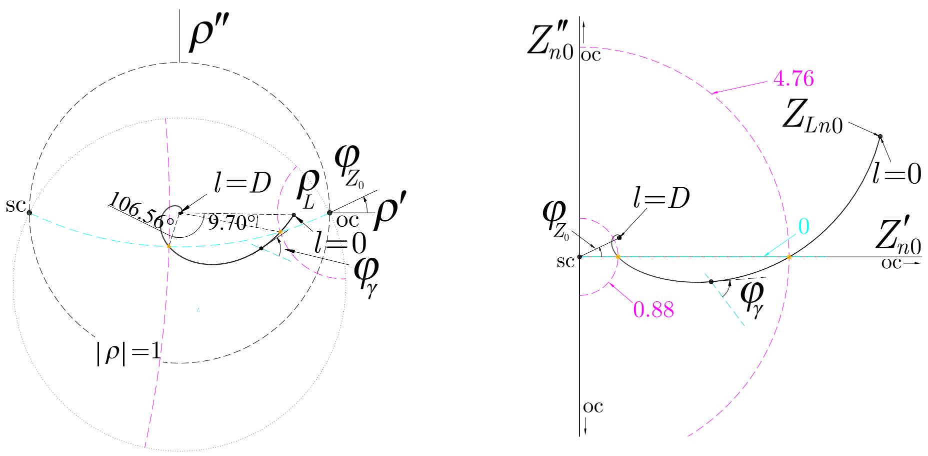
(b)

Fig. 5.20: Graphical analysis of (a) Z_{n0} and (b) Y_{n0}, both along the TL.

Three different examples (examples I-III in Sect. 5.2.4-5.2.4, respectively) has been graphically presented varying some specific parameters summarized in the following table:

	Example I	Example II	Example III
Frequency, f	1[MHz]	2[MHz]	2[MHz]
Load, Z_L	50+j10[Ω]	50+j10[Ω]	10-j15[Ω]

These a priori minor changes on frequency (from example I to example II) and the impedance at the load (from example II to example III) produce large variations on the basic and wave parameters.
For example, varying the frequency keeping the rest of parameters fixed (also the impedance at the load) makes the basic parameters vary, affecting both the propagative behavior of the equivalent waves and the relation between them. In this sense, notice how the TL becomes electrically larger when doubling the frequency, for instance. Moreover, the basic parameters afect the analysis of wave parameters in both the matching at the load and the behavior along the TL (although comparing the examples I and II, this change is hardly appreciable).
It is more significative the change when the impedance ath the load varies. In this case, even the sign of curvature of the curve in the Z_{n0}- or Y_{n0}-plane may change depending on the load (in order to be the resultant curve a representation of a real case: $Z'_{n0} \geq 0$ and $Y'_{n0} \geq 0$).



(a) (b)

Fig. 5.21: Example of locating the real values of the wave impedance in the Z_{n0}-plane and the transformations to the ρ-plane, at $l_1 = 2.9793$[mm] ($\equiv |\varphi_{\rho_L} - \varphi_\rho| = 9.70°$) and $l_2 = 32.7293$[mm] ($\equiv |\varphi_{\rho_L} - \varphi_\rho| = 106.76°$) on the curve that represents ρ along the TL.

Notice that the transformed curves in the Z_{n0}- and Y_{n0}-planes are also spiral-type, which are asymptotic to Z_{0n} and Y_{0n}, respectively. This is just as expected because the physical interpretations regarding the behavior of the reflection coefficient discussed before.

The graphical analysis lets to see at first sight some interesting points regarding the direct characterization of wave parameters.
For example, in examples I and II two real values of the wave impedance and wave admittance are expected to be achieved along the TL, whereas in example III only one real value (pure resistive) of the wave impedance or wave admittance can be found.
Furthermore, the distances where these real values can be solved, for instance, by using the transformation from the wave impedance complex to the ρ-plane, in which the angle measured from the load is proportional to the seeked electrical length corresponding to these points. In Fig. 5.21, the location of the points where the wave impedance is resistive is shown in the ρ-plane regarding the example I, which lets to measure the lengths at which these values are met.

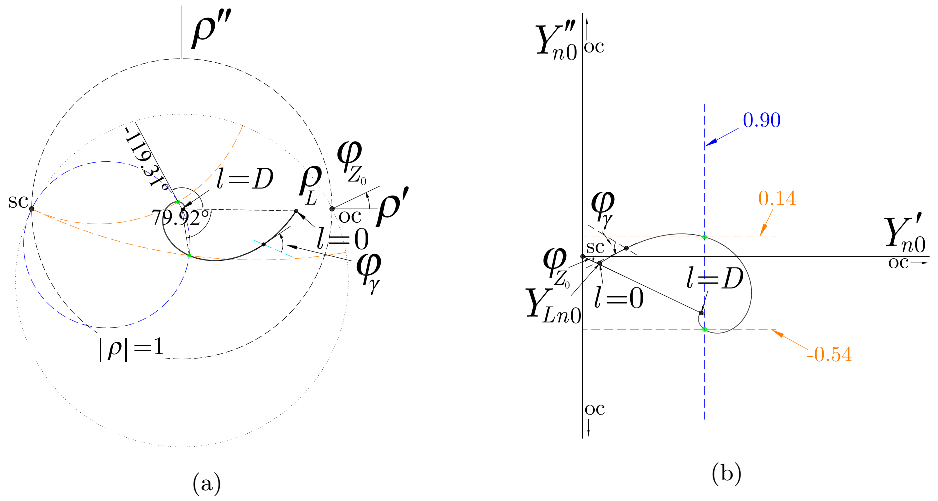


Fig. 5.22: Example of locating the admittances at which the imaginary part is the same as in the characteristic admittance: $Y_{n0,1} = 0.9024 + j0.1442$ and $Y_{n0,2} = 0.9024 - j0.5386$; whose corresponding lengths are measured in the ρ-plane: $l_1 = 24.5470$[mm] ($\equiv |\varphi_{\rho_L} - \varphi_\rho| = 79.92°$) and $l_2 = 73.9266$[mm] ($\equiv |\varphi_{\rho_L} - \varphi_\rho| = 240.69°$), respectively.

In the same sense, another interesting example of use (for practical purposes) of this graphical analysis could be locating those points at which the real part of the wave admittance is the same as the real part of the characteristic admittance. In Fig. 5.22, this analysis concerning the data of the example I has been carried out.

Remark 34. *The graphical analysis is really useful to analyze the wave parameters along the TL. On one hand, it lets to characterize the wave impedance and wave admittance in an easy way by using transformations from the reflection coefficient expressed along the TL, which is described by a well-known logarithmic spiral.*
On the other hand, it serves to solve the length where some interesting points related to the wave impedance and wave admittance behaviors. For this purpose, the angular measurements in the reflection coefficient complex plane directly gives the seeked lengths.

Based on the properties of the transformations from the ρ-plane, some important geometrical characteristics of the curves in the Z_{n0}- and Y_{n0}-planes are detached.
Since the transformations in eqs. (5.14) and (5.15) are conformal mappings, the angles are preserved after their application. On the other hand, the angle Ψ is preserved along the TL in the ρ-plane. Thus, this angle between radius and the tangent in each point of the logarithmic spiral is preserved (in magnitude and orientation) between the transformation of the radious and the transformation of the tangent. Moreover, both transformations are circumferences passing through $Z_{n0} = e^{j\varphi_{Z_0}}$ and (the non existing) $Z_{n0} = -e^{j\varphi_{Z_0}}$ in the Z_{n0}-plane, and $Y_{n0} = e^{-j\varphi_{Z_0}}$ and (the non existing) $Y_{n0} = -e^{-j\varphi_{Z_0}}$ in the Y_{n0}-plane, which has been easily proved in Appendix 5.C, so the center of them are in the radious whose angle is $\varphi_{Z_0} + \pi/2$ in the Z_{n0}-plane, and $-\varphi_{Z_0} + \pi/2$ in the Y_{n0}-plane. The tangents of these circumferences also preserve the angle Ψ between them, and the tangent of the circumference which represents the tangent in the ρ-plane is the tangent of the transformed spiral.
In Fig. 5.23 a graphical example of the curve in the Z_{n0}-plane representing the wave impedance along the TL is shown while detaching the angle between the tangent of the curve and the tangent of the transformations of the radious in the ρ-plane.

The property of preserving the angles in conformal mappings lets to draw the curves in the wave parameter complex planes in an easy way. For this purpose, both φ_γ and φ_{Z_0} should be used, once

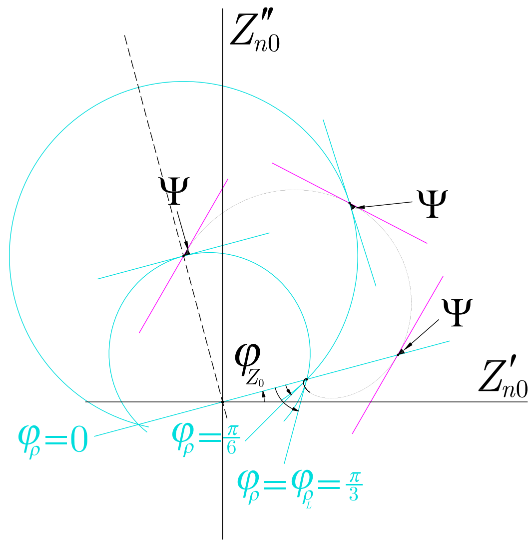


Fig. 5.23: Graphical example of the curve which represents Z_{n0} along the TL constructed by the preservation of $\Psi = \pi/4$ between the transformation of the radious and the transformation of the tangent from the ρ-plane.

the impedance or the admittance at the load are located in the Z_{n0}-plane or Y_{n0}-plane.

Remark 35. *The graphical analysis and the geometrical properties of this graphical analysis let to see that the angles φ_{Z_0} and φ_γ suffice to determine the wave parameters along the TL once the impedance at the load is known.*

The fact that φ_{Z_0} and φ_γ completely characterize the wave parameters along the TL when the load is fixed is crucial for posing the analysis of the wave parameters along the TL in terms of losses in Ex. 03 in Sect. 5.4.
Moreover, this analysis is supported by the inverse characterization of line parameters, which allows for knowing which losses are with the angles of the basic parameters which appear in the study reviewed here.

5.2.5 Physical interpretations

The analysis of wave parameters along the TL should be understood in the context of describing the equivalent voltage and current waves along the TL. For this purpose, the basic parameters have to be known, just as it has been considered in this example.
The reflection coefficient along the TL serves to describe how the total voltage or current waves are expanded from the individual waves. On their behalf, the wave impedance and wave admittance let to obtain the total current and total voltage waves from the total voltage and total current, respectively. In this sense, since the wave impedance and admittance relate the total waves, only these wave parameters have true physical meaning in the description of waves along the TL, whereas the reflection coefficient is a "mathematical tool" for obtaining each of these total waves.
In this sense, recall an important result for the physical interpretation of lossy TLs, which is possible thanks to the combination of graphical analysis and Complex Analysis theoremas: in Appendix 5.A it is concluded that the short circuit or open circuit are only possible at the load if the TL is loaded like that. Otherwise, these particulr behaviors can not be found at any point along the TL. That is because these points belong to the limit in the ρ-plane (for example, the GSC). As a consequence, these particular behaviors are not possible along the lossy TL.

Remark 36. *The short circuit and open circuit can not be found at any point of the lossy TL, unless they are loaded. In this latter case, these particular behaviors of lossy TLs can only be found at the load.*
This result disagrees with the case of the lossless TL, in which any reactive load can be transformed to a short circuit or an open circuit in less than $\lambda_{sp}/2$*, and periodically it repeats every* $\lambda_{sp}/2$*.*

Nevertheless, the reflection coefficient may be used for analyzing the true physical behaviors of total voltage and current waves –and their relation by means of the impedance/admittance– if this parameter is interpreted as the transformation from the rest of wave parameter complex planes. For example, in Fig. 5.21 it has been shown how the points in which the wave impedance is real (and thus, the phase between the total voltage and current waves is null) transform into some specific points in the curve which represents ρ along the TL, and with that these points adquire physical meaning.

In addition, it should be detached the capability of ρ for the length measurements.
Recall that the angle measured from the reflection coefficient at the load, $\varphi_m \equiv \varphi_{\rho_L} - \varphi_\rho$, is proportional to both the electrical length and physical length in the following form:

$$\varphi_m = 4\pi l_e = 4\pi l/\lambda \equiv 2\beta l. \tag{5.25}$$

Thus, this angle φ_m may be directly used for measuring lengths, something which is not possible in the wave impedacnce or wave admittance complex planes.

On the other hand, the fact that the description of the reflection coefficient along the TL directly depends on the phase of the propagation constant, and so the ratio between the attenuation constant and the phase constant (instead of the absolute values of them), suggests that measuring the attenuaton per wavelength is more effective that do it in metres, for the appropriate physical interpretation of this parameter.

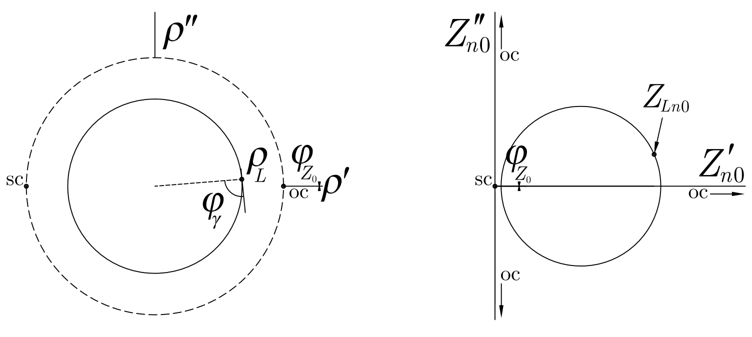


Fig. 5.24: Example of wave parameter graphical analysis along the TL when the TL is lossless ($r = 0$, $g = 0 \to \varphi_{Z_0} = 0$, $\varphi_\gamma = \pi/2$). The curves that represent the wave parameters are circumferences in each wave parameter complex plane (also in the Y_{n0}-plane).

Moreover, notice that the analysis presented before concerning lossy TLs differs significantly from the one particularized for the lossless case represented in Fig. 5.24 above. This particularization results much more easy to study because the resultant curves are circumferences.
Nevertheless, the geometrical analysis presented before allows for addressing the curves in the same graphical way, but generalizing the presence of losses (something which is required in the CTLT analysis) for the rigorous application.

5.2.6 Practical uses

The direct characterization of wave parameters along the TL has been presented emphasizing the graphical results and the underlying geomtry based on the properties of conformal transformations. It has been seen that the complex transformations between the wave parameter complex planes serve to describe these parameters along the TL in a different way, which suppose an efficient alternative to the analysis of the underlying mathematical expressions.
In this sense, the curves are geometrically characterized in such a way that it could be represented for any analysis along the TL. For example, it is common to use the analysis along the TL for load matching, using parallel stubs and/or adjusting the length of the TL. For these pruposes, the analysis presented in this CTLA example set the basis for the requiried characterization along the TL.
In addition, this example adds a new variable in the analysis which can be also used in the design of TLs and stub tuning: the losses. These losses ("controlled" by the angle of the basic parameters) have demonstrated being useful for matching loads, [VG17-I], [VG17-II], [VG17-III].

Furthermore, it has been seen the influence of the the phase of basic parameters in the graphical and geomtrical analysis of the wave parameters along the TL. The fact that angles directly appear in the graphical analysis can be used to obtain the line parameters of the equivalent TL by using the inverse characterization (in order to see an example of this inverse characterization refer to the rg-planes in Figs. 5.7, 5.12, and 5.17). Moreover, since the TL is completely characterized by the phases of the basic parameters, besides the load, by only knowing two values of, for example, the reflection coefficient at different positions along the TL (for example, at the load ($l = 0$) and at the generator ($l = D$)) suffices for characterizing the equivalent lossy TL.

But probably the most interesting result regarding this example is that the phase of both basic parameters determines the behavior of the wave parameters along the TL once the losses and the impedance at the load are fixed.
On one hand, notice that fixing the losses means fixing the normalization of wave parameters and the GSC by means of φ_{Z_0}, which are useful for locating the reflection coefficient at the load. Then, on the other hand, fixing that losses supposes fixing the propagation constant, and so the variation of the reflection coefficient along the TL and the rest of wave parameters, characterized by φ_γ.
This result suggests parameterizing the TL by means of these phases, which leads to establish a new useful parameterization based on angles analyzed in Ex. 03 in Sect 5.4.

5.3 Analysis at the load in terms of losses and frequency

5.3.1 Definitions and parameters

In this example, the influence of the variation of losses and frequency over the parameters that characterize the lossy TL is explicitly studied.
This analysis in terms of losses and frequency is intended to complete the direct characterization of basic parameters for the *ffa* and the *vfa*, both presented in Sect. 4.3.1 in Chpt. 4. Thus, the parameters that make explicitly the lossy and frequency analysis are those used for the *ffa*: r and g, defined in eq. (4.1); and those used in the *vfa*: ω_n and c defined in eq. (4.19); so the subsequent characterizations in this example are splitted into the same way.
In addition, the interpretations regarding the analysis in this example should be understood as for the direct characterization of basic parameters: (i) the parameterizations of losses or frequency are defined to "universalize" the *ffa* or the *vfa*, respectively, so that multiple TLs are represented by the same parameterization which describes any specific behavior of the parameters under study (not only the basic parameters); (ii) the graphical analysis should be interpreted in terms of the transformations from the plane of parameterizations: the rg-plane depicted in Figs. 4.2 and 4.10 for the *ffa* and the *vfa*, respectively; to the planes that represent the normalized version of those parameters under study, for example the Z_{0n1}-plane. The resultant analysis in this example are not "basic" transformations (in the sense of there is not a single function which directly maps the lossy/frequency parameterizations to the complex plane under study) but composed transformations (multiple transformations are used). As a consequence, the geometrical characterizations that supports the graphical analysis should be studied taking into account the transformation composition; and, in the same way, the (iii) physical interpretations and possible practical uses have to be understood in terms of the lossy/frequency parameterizations used: the *ffa* is the analysis in the *frequency domain* in which the harmonic characterized by ω normalizes the parameterizations, whereas the *vfa* makes reference to the spectral analysis, [Her14], for the subsequent expansion of the harmonics in the *time domain*.

Remark 37. *Despite the analysis in terms of losses and frequency are addressed as transformations from the same space of parameters to the complex planes associated to the parameters which are of interest, each analysis should be interpreted in a different way: the analysis in terms of losses refers to the influence of losses in the description of the TL parameters, whereas the analysis in terms of frequency has to be understood as the spectral analysis.*
Each analysis also has its own uses. For example, the analysis in terms of losses is useful for studying some particular cases or approximations regarding how they affect the TL parameters, whereas the analysis varying the frequency is useful for describing how the voltage and current waves behave in the frequency domain.

Both the anlysis in terms of losses and frequency must be referred at the load (or any fixed point along the TL), whose impedance is supposed known for the analysis carried out by this example. This premise lets to eliminate the dependence of voltage and current waves on the length of the TL, l, something that is study in detail in Ex. 01 in Sect. 5.2.
In this way, after setting $l = 0$, the total voltage and current waves to be studied are:

$$V = V_L^+ + V_L^- = V_L^+ \left(1 + \rho_L\right), \text{ and} \tag{5.26}$$
$$I = I_L^+ + I_L^- = I_L^+ \left(1 - \rho_L\right), \tag{5.27}$$

for the *ffa*, and

$$V(\omega) = V_L^+(\omega) + V_L^-(\omega) = V_L^+(\omega) \left(1 + \rho_L(\omega)\right), \text{ and} \tag{5.28}$$
$$I(\omega) = I_L^+(\omega) + I_L^-(\omega) = I_L^+(\omega) \left(1 - \rho_L(\omega)\right), \tag{5.29}$$

for the *vfa*, in the dependence on ω is explicitly pointed out.

	ffa	*vfa*
charact. imped.	$Z_0 = \frac{V_L^+}{I_L^+} = \frac{V_L^-}{I_L^-}$ (30)	$Z_0(\omega) = \frac{V_L^+(\omega)}{I_L^+(\omega)} = \frac{V_L^-(\omega)}{I_L^-(\omega)}$ (32)
reflect. coeff.	$\rho_L = \frac{V_L^-}{V_L^+} = \frac{I_L^-}{I_L^+}$ (31)	$\rho_L(\omega) = \frac{V_L^-(\omega)}{V_L^+(\omega)} = \frac{I_L^-(\omega)}{I_L^+(\omega)}$ (33)

The TL parameters which let to describe the voltage and current waves for both the *ffa* and the *vfa* are:

The impedance at the load and the admittance at the load belong to the datum for this study, which are not affected by neither the parameterization of losses nor the frequency parameterization in the *ffa* or the *vfa*, respectively.

Remark 38. *The fact that the wave impedance at the load is not affected by neither the variation of losses for the frequency says a lot about this parameter regarding its physical interpretation and also as BC in any specific point along the TL: it determines the relation between the total voltage and current waves and so the characteristic impedance and the reflection coefficient.*

As a consequence, the wave impedance at the load, which does not depend on neither the losses nor the frequency, (extra) parameterizes the *ffa* or the *vfa*.
As said before, the rest of the parameterizations used depend on the type of analysis.

The modulus of the wave impedance normalizes the parameters under study (those in eqs. (5.30)-(5.32)), just as it is done for the inverse characterization of the basic parameters presented in Sect. 4.4.2 in Chpt. 4, leading to:

$$Z_{0nL} = \frac{Z_0}{|Z_L|} = e^{j\varphi_{Z_0}} \frac{1-\rho}{1+\rho} \text{ , for the } \textit{ffa}\text{, or} \tag{5.34}$$

$$Z_{0nL}(\omega) = \frac{Z_0(\omega)}{|Z_L|} = e^{j\varphi_{Z_0}(\omega)} \frac{1-\rho(\omega)}{1+\rho(\omega)} \text{ , for the } \textit{vfa}, \tag{5.35}$$

whereas ρ and $\rho(\omega)$ remain invariant, as it is usual because of their role relating the characteristic impedance and wave impedance in a linear fractional transformation, which makes the common normalizations result simplified.

In this case, ρ and $\rho(\omega)$ are addressed as basic parameters, just as it has been introduced for the inverse characterization.

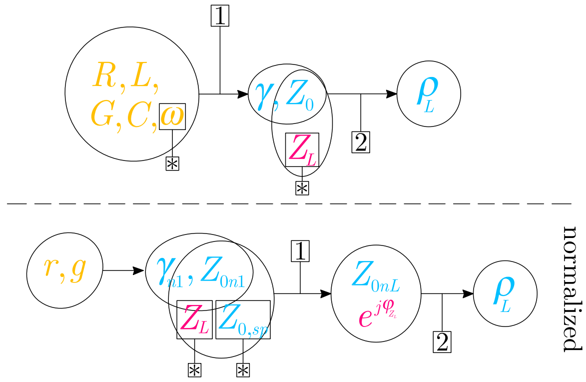
In addition, Z_0 in eq. (5.34) characterized in terms of losses may be obtained when denormalizing the direct characterization of Z_{0n1} in eq. (4.3) studied for the *ffa*. Thus Z_{0nL} for the *ffa* may be rewritten in terms of Z_{0n1} as:

$$Z_{0nL} = \frac{Z_{0,sp}}{|Z_L|} Z_{0n1}, \tag{5.36}$$

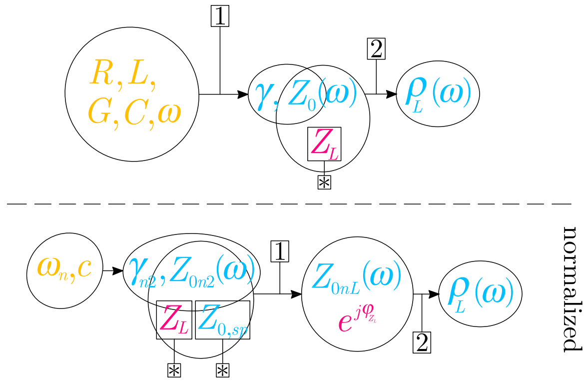
which only supposes a real scaling of Z_{0n1}.
Reciprocally, $Z_0(\omega)$ in eq. (5.35) described in terms of frequency may be obtained from the denormalization of $Z_{0n2}(\omega_n) \equiv Z_{0n2}(\omega)$ defined in eq. (4.20) for the *vfa*. In this case, $Z_0(\omega)$ may be rewritten as in terms of $Z_{0n2}(\omega)$ as:

$$Z_{0nL}(\omega) = \frac{Z_{0,nd}}{|Z_L|} Z_{0n2}(\omega) \equiv \frac{Z_{0,sp}}{|Z_L|} Z_{0n2}(\omega), \tag{5.37}$$

which is a real scaling of $Z_{0n2}(\omega)$.



(a)



(b)

Fig. 5.25: Schemes of parameters and transformations (1-2) to be followed for finally achieving the characterization of the basic parameters in terms of (a) losses (for the *ffa*) and (b) frequency (for the *vfa*). The parameters which are the data in each analysis are marked with the 99 symbol.

On the other hand, although the propagation constant may be studied for both the *ffa* and the *vfa*, its characterization is not significant in this case in which the analysis are carried out in a fixed point of the TL, making the analysis in this example "static" (notice that γ is not present in eqs. (5.26)-(5.29)).

In Fig. 5.25 two schemes gathering all the parameters which are involved in the analysis in terms of both losses (Fig. 5.25a) and frequency (Fig. 5.25b) are shown together with the normalizations to be used in each analysis.
Notice the step in which (and how) the normalized characteristic impedance appears for each case. The direct characterization of the characteristic impedance presented in Sect. 4.3.1 in Chpt. 4 lets to characterize the mathematical analysis of the curves in the respective Z_{0n1}- and Z_{0n2}-complex planes. From this analysis, the main new characterization in terms of losses or frequency affects the reflection coefficient, whereas the rest of wave parameters are invariant and so single points in their respective complex planes.

5.3.2 Mathematical analysis

When analyzing the problem of characterizing the TL parameters in terms of losses or frequency for the *ffa* or the *vfa*, which relate the equivalent voltage and current waves, the parametric analysis of the expressions involved in the underlying LTLT are mainly addresed. In this sense, notice that

the expressions V and I in eqs. (5.26) and (5.27) used for the *ffa* and those $V(\omega)$ and $I(\omega)$ in eqs. (5.28) and (5.28) used for the *vfa* are not explicit functions of one or more variables[2] but expressions parameterized by losses/frequency.
Nevertheless, in the direct characterization of the basic parameters introduced in Sect. 4.3 in Chpt. 4, in particular for the interpretation of the graphical analysis as complex transformations, the appropriate functions representing the basic parameters whose domain is in the space of parameterizations, in particular in the rg-plane, have been rigorously defined.
This is the way to mathematically follow the analysis presented in this example.
Instead of operating with the voltage and current functions defined over the rg-plane, it is better to deal with the parameters of the TL, seen as functions, in the same way as in the direct characterization of basic parameters.

Firstly notice that, by definition, the wave imepdance and wave admittance in this problem (at the load) do not depend on none of the variables in the rg-plane, and so their respective normalized functions are constant in this domain[3]:

$$\begin{aligned}
\boldsymbol{Z_{Ln}}, \boldsymbol{Y_{Ln}}\colon\ & \mathbb{R}^+ \cup \{0\} \subset \mathbb{H} \to \mathbb{C} \\
& (r,g) \mapsto \begin{cases} \boldsymbol{Z_{Ln}}(r,g) = \frac{Z_L}{|Z_L|} = e^{j\varphi_{Z_L}} = c_L + js_L \\ \boldsymbol{Y_{Ln}}(r,g) = Y_L|Z_L| = e^{-j\varphi_{Z_L}} = c_L - js_L \end{cases}, \\
& (\omega_n,c) \mapsto \begin{cases} \boldsymbol{Z_{Ln}}(\omega_n,c) = \frac{Z_L}{|Z_L|} = e^{j\varphi_{Z_L}} = c_L + js_L \\ \boldsymbol{Y_{Ln}}(\omega_n,c) = Y_L|Z_L| = e^{-j\varphi_{Z_L}} = c_L - js_L \end{cases}, \\
& \text{in which } c_L = \cos(\varphi_{Z_L}), \text{ and } s_L = \sin(\varphi_{Z_L}).
\end{aligned} \tag{5.38}$$

On the other hand, the function $\boldsymbol{Z_{0nL}}$ is defined from $\boldsymbol{Z_{0n1}}(r,g)$ in eq. (4.7) and $\boldsymbol{Z_{0n2}}(\omega_n,c)$ in eq. (4.23), used in the direct characterization of basic parameters for the *ffa* and the *vfa*, respectively:

$$\begin{aligned}
\boldsymbol{Z_{0nL}}\colon\ & \mathbb{R}^+ \cup \{0\} \subset \mathbb{H} \to \mathbb{C} \\
& (r,g) \mapsto \boldsymbol{Z_{0nL}}(r,g) = \frac{Z_{0,sp}}{|Z_L|}\boldsymbol{Z_{0n1}}(r,g), \\
& (\omega_n,c) \mapsto \boldsymbol{Z_{0nL}}(\omega_n,c) = \frac{Z_{0,nd}}{|Z_L|}\boldsymbol{Z_{0n2}}(\omega_n,c) \equiv \frac{Z_{0,sp}}{|Z_L|}\boldsymbol{Z_{0n2}}(\omega_n,c).
\end{aligned} \tag{5.39}$$

Notice that, for both the *ffa* and the *vfa*, each function is point by point (in the rg-plane) the same. This means that $\boldsymbol{Z_L}$ (also $\boldsymbol{Y_L}$) and $\boldsymbol{Z_{0nL}}$ behave in the same way in the (r,g) or the (ω_n,c) coordinates provided that they represent the same point in the rg-plane.
The main difference between these analysis comes from the parameterization of the curves in the rg-plane.

The usual way to see these functions in their respective complex planes is as curves, represented when keeping fixed one parameter and varying the other one, leading to the curve parameterized by the parameter which is kept fixed. For example, the curve which represents Z_{0nL} in the whole frequency band results from keeping fixed c while varying ω_n (equivalently,varying ω). In this example, the resultant curve is rigorously defined as:

$$\begin{aligned}
Z_{0nL}\colon\ & (0,\infty) \equiv \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^2\ (\mathbb{C}) \\
& \omega_n \mapsto (Z'_{0nL}(\omega_n), Z''_{0nL}(\omega_n)),
\end{aligned}$$

[2]Even in the expressions $V(\omega)$ and $I(\omega)$ used for the *vfa* the dependence on ω is not clear. For example, despite ω appears in $\gamma \equiv \gamma(\omega)$, this parameter does not have any influence in the volatage and current waves analyzed at the load, and thus analyzing what happens with the total waves in terms of frequency where the load is fixed supposes an interesting problem here addressed.

[3]As usual throughout this Thesis book, the functions defined over the isocomplex numbers are written boldfaced.

in which ω_n plays the role of the parameter for the regular parameterization of the planar curve which characterizes Z_{0nL} in the whole frequency band.

The following transformation, T_ρ, transforms the curves from the Z_{0nL}-plane (parameterized by either losses or frequency) to the ρ-plane:

$$\begin{aligned} T_\rho\colon\quad \mathbb{R}^2\ (\mathbb{C}) &\rightarrow \mathbb{R}^2\ (\mathbb{C}) \\ Z_{0nL}(\text{par.}) &\mapsto T_\rho\left[Z_{0nL}(\text{par.})\right] = \frac{Z_{Ln} - Z_{0nL}(\text{par.})}{Z_{Ln} + Z_{0nL}(\text{par.})}, \end{aligned} \tag{5.40}$$

in which "par." indicates any of the possible parameterizations r, g, ω_n, or c, all of them defined in $\mathbb{R}^+$ (analogously to how "l" parameterizes the length of the TL in Ex. 01).

Since the curves in the Z_{0nL}-plane are real scalings of the curves in the Z_{0n1}-plane and the Z_{0n2}-plane, their graphical and geometrical analysis, as well as their physical interpretations have already been studied in detail in the direct charactrization presented in Sect. 4.3.1 in Chpt. 4.
The resultant general equations of Z_{0nL} written in polar form used for the *ffa* analysis are:

$$\begin{aligned} |Z_{0nL}| &= \frac{Z_{0,sp}}{|Z_L|}\sqrt{|n|\cos\left(2\varphi_{Z_0} - \varphi_n\right)}, \\ \varphi_{Z_0} &\in \left[\frac{\varphi_n}{2}, \frac{\pi}{4} + \frac{\varphi_n}{2}\right[, \\ \text{in which } |n| &= \sqrt{1+r^2}, \text{ and } \varphi_n = -\tan^{-1}(r), \end{aligned} \tag{5.41}$$

for the curves parameterized by r, and

$$\begin{aligned} |Z_{0nL}| &= \frac{Z_{0,sp}}{|Z_L|}\frac{1}{\sqrt{|d|\cos\left(2\varphi_{Z_0} + \varphi_d\right)}}, \\ \varphi_{Z_0} &\in \left[-\frac{\varphi_d}{2} - \frac{\pi}{4}, -\frac{\varphi_d}{2}\right[, \\ \text{in which } |d| &= \sqrt{1+g^2}, \text{ and } \varphi_d = -\tan^{-1}(g), \end{aligned} \tag{5.42}$$

for the curves parameterized by g.
In both cases, the phase of the characteristic impedance, φ_{Z_0}, may act act as the parameter which describes the curves.
For the *vfa*, only the curves parameterized by c are described by a general equation in polar form. Nevertheless, these c-curves, which represent the TL in the whole frequency band, may be obtained from the intersection of those r- and g-curves that keep the ratio $c = r/g$ constant, making unncesary reparameterizing the analysis.

From this point, the complete graphical analysis can be done by using the complex parameterizations presented in the CTLT-v1. In particular, the inverse analysis of basic parameters lets to describe the curves parameterized by r and g, or ω_n and c, in the ρ-plane. As a result, the function:

$$\begin{aligned} \boldsymbol{\rho_L}\colon\quad \mathbb{R}^+ \cup \{0\} \subset \mathbb{H} &\rightarrow \mathbb{C} \\ (r,g) &\mapsto \boldsymbol{\rho_L}(r,g) = \boldsymbol{T}_\rho[\boldsymbol{Z_{0nL}}(r,g)], \\ (\omega_n, c) &\mapsto \boldsymbol{\rho_L}(\omega_n, c) = \boldsymbol{T}_\rho[\boldsymbol{Z_{0nL}}(\omega_n, c)], \end{aligned} \tag{5.43}$$

in which $\boldsymbol{T}_\rho$ would be strictly an operator defined over the function $\boldsymbol{Z_{0nL}}$ to get $\boldsymbol{\rho}$ in eq. (5.43), will be analyed by means of the subsequent graphical examples.

5.3.3 Graphical and geometrical analysis

In this section, a complete graphical analysis of losses and frequency for the *ffa* and the *vfa*, respectively, is carried out, to be then characterized from the geometrical point of view.

On one hand, recall the basic parameters which have been described in terms of losses and frequency in Sect. 4.4.2 in Chpt. 4. From these analysis, the appropriate normalizations concerning the characteristic impedance are proposed in eqs. (5.36) and (5.37), in order to obtain the curves in the Z_{0nL}-plane. For this purpose, both $Z_{0,sp}$ and Z_L are supposed be part of the datum (marked with the symbol □ in Fig. 5.25) for each of the examples which are graphically solved.
On the other hand, once Z_{0nL} is characterized for both the *ffa* and the *vfa*, it is transformed by using the modulus-phase parameterizations of the inverse characterization of basic parameters, in particular the transformation from the Z_{0nL}-plane to the ρ-plane. In this way, Z_{0nL} is parameterized as:

$$\begin{cases} \varphi_{Z_0} = p \\ |Z_{0nL}| = Z_{0nL,sp}\sqrt{|n|\cos\left(2p - \varphi_n\right)} \end{cases} , \tag{5.44}$$

$$\text{in which } Z_{0nL,sp} = \frac{Z_{0,sp}}{|Z_L|},\ |n| = \sqrt{1+r^2}, \text{ and } \varphi_n = -\tan^{-1}(r),$$

for the curves parameterizing r, and

$$\begin{cases} \varphi_{Z_0} = p \\ |Z_{0nL}| = \frac{Z_{0nL,sp}}{\sqrt{|d|\cos\left(2\varphi_{Z_0}+\varphi_d\right)}} \end{cases} , \tag{5.45}$$

$$\text{in which } |d| = \sqrt{1+g^2}, \text{ and } \varphi_d = -\tan^{-1}(g),$$

for the g-parameterized curves.
The c-parameterized curves are given by the intersection of the r- and g-cuves which verify $r/g = c$. In addition, the wave parameters Z_n and Y_n, which are part of the datum in this example, are represented in their respective complex planes as ("fixed") single points.

The following examples, parameterized by different loads and lossless characteristic impedances, serve to see the process of the curve construction based on the transformations from the Z_{0nL}-plane to the ρ-plane for the *ffa*, constituting an example of use of the inverse characterization.
From this lossy characterization, the intersections between those curves that keep the ratio c constant lead to the graphical analysis for the *vfa*.
Varying the parameterizations of losses r and g and gathering the intersections in c-curves, a complete analysis of the TL parameters in terms of losses and frequency is graphically obtained.

5.3.4 Some examples of graphical analysis of basic parameters in terms of losses

Example I

Numerical values of the transmission line parameters

Physical parameters	*Line parameters*	
	Param.	Normalizations
	$R = 0,\ L,$ $G \neq 0,\ C$	$r = 0,\ g = 1,$ $c = 0.5,\ c = 2$
	Basic Parameters	
	Param.	Normalizations
	$Z_{0,sp} = 50[\Omega]$	-
$f,\ \lambda,\ c_e,\ D/\lambda$	$Z_0 = 38.8443 + j16.0899 =$ $= 42.0448\angle 22.5000°$	$Z_{0nL} = 0.7769 + j0.3218 =$ $= 0.8409\angle 22.5000°[\Omega]$
	γ	$\gamma_{n1} = 0.4551 + j1.0987 =$ $= 1.1892\angle 67.5000°[\mathrm{m}^{-1}]$
	Wave parameters at the load	
	Param.	Normalizations
	ρ_L	-
	$Z_L = 50 + j0 = 50\angle 0°$	$Z_{Ln} = 1 + j0 = 1\angle 0°$
	$Y_L = 0.0200 + j0.0000 =$ $= 0.0200\angle 0°$	$Y_{Ln} = 1 + j0 = 1\angle 0°$

Table 5.4: Numerical values of the TL parameters: $Z_{0,sp}$ and Z_L; for the analysis in this first example.

The values of the normalized basic parameters, Z_{0nL} and γ_{n1}, result from the intersection of the curves $r = 0$ and $g = 1$, which are studied as examples of curve transformation from the Z_{0nL}-plane to the ρ-plane in the *ffa*.

Fixed frequency analysis

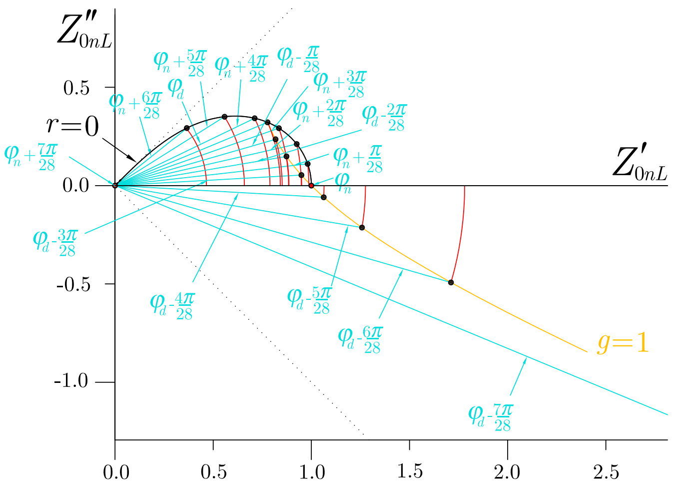


Fig. 5.26: Modulus and phase parameterizations of the $r = 0$ and $g = 1$ curves in the Z_{0nL}-plane.

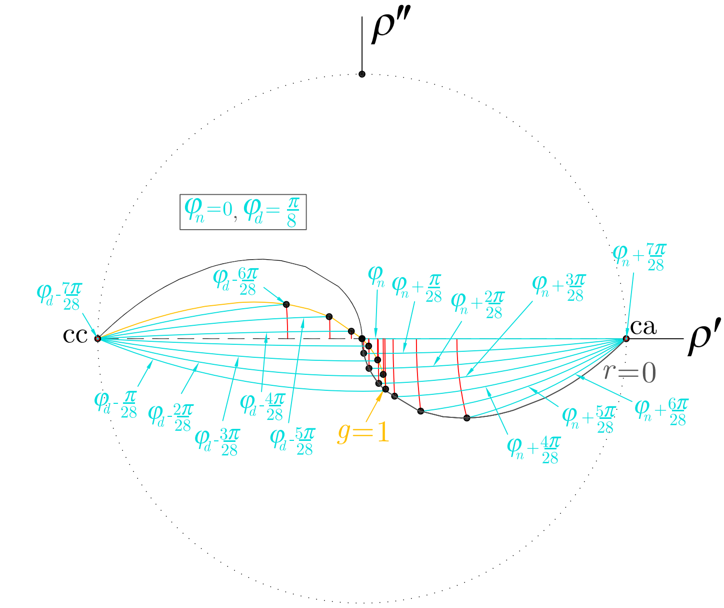


Fig. 5.27: Transformations of the $r = 0$ and $g = 1$ curves to the ρ-plane.

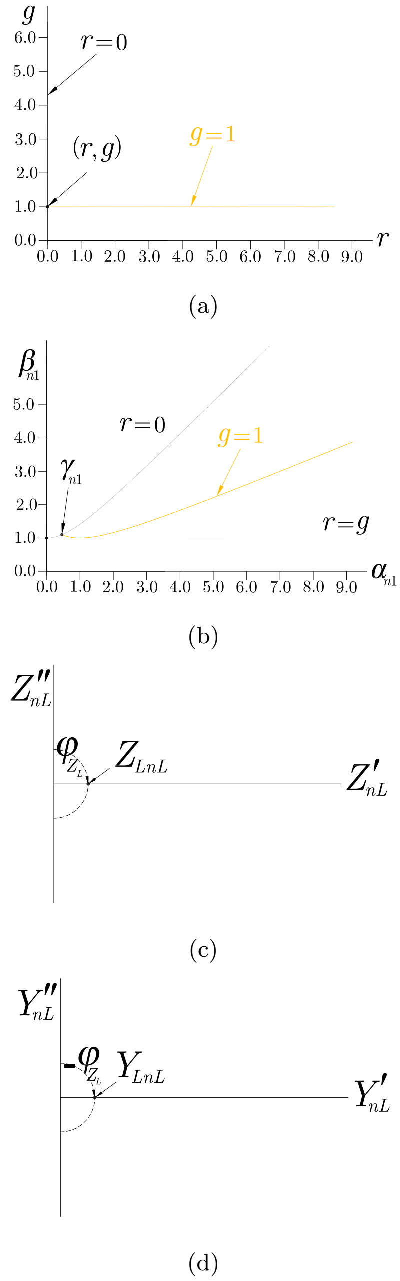


Fig. 5.28: The supportive (a) rg-, (b) γ_{n1}-, (c) Z_{Ln}-, and Y_{Ln}-planes.

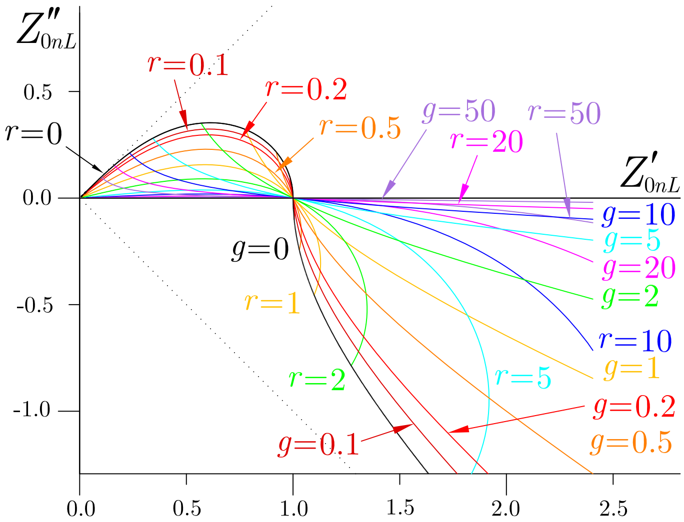


Fig. 5.29: Curves paramterizing the confuctor and dielectric losses in the Z_{0nL}-plane

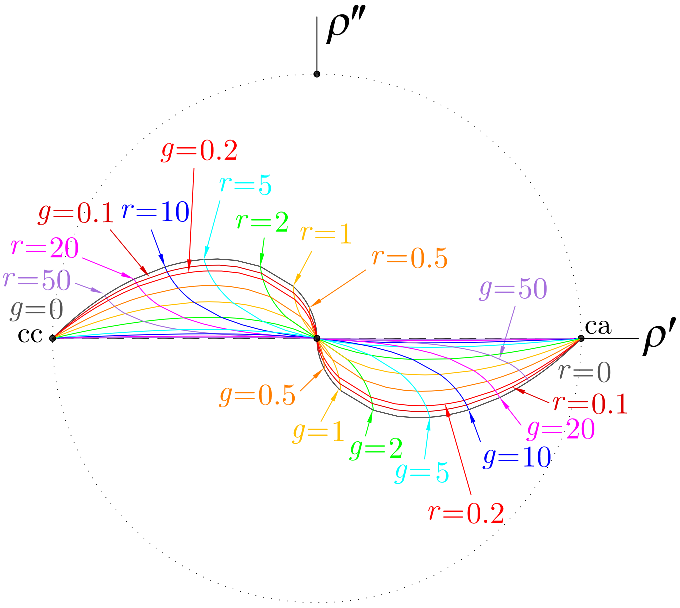
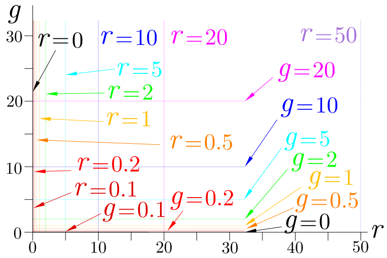
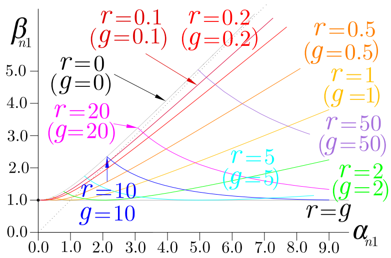


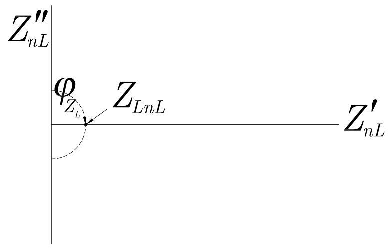
Fig. 5.30: Transformation of the curves parametrizing losses to the ρ-plane.



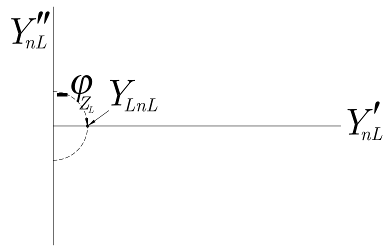
(a)



(b)



(c)



(d)

Fig. 5.31: The supportive (a) rg-, (b) γ_{n1}-, (c) Z_{Ln}-, and Y_{Ln}-planes.

Variable frequency analysis

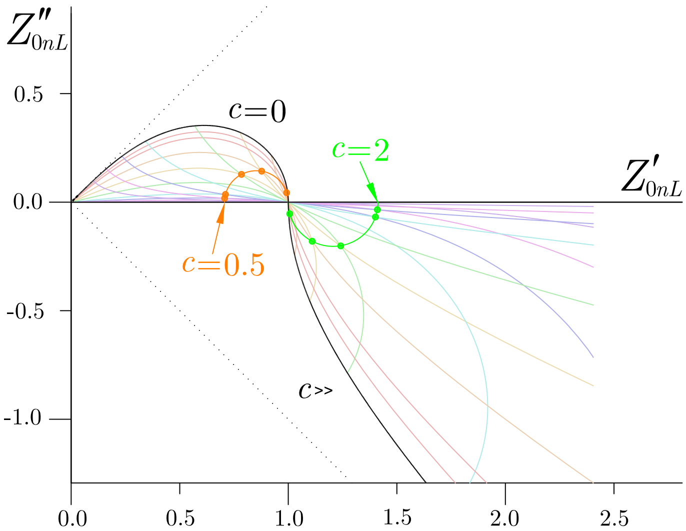


Fig. 5.32: Curves with constant $c = 0.5$ and $c = 2$ ratio from the intersections of the curves with parameterized by conductor and dielectric curves in the Z_{0nL}-plane.

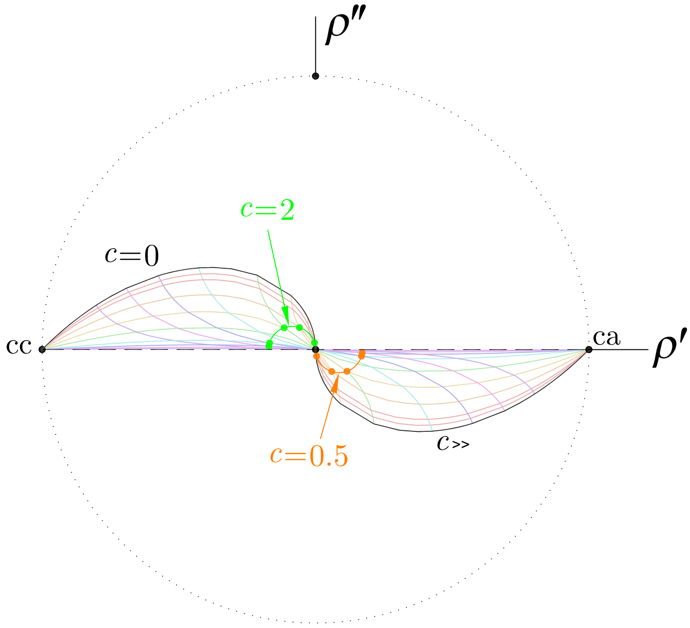


Fig. 5.33: Transformation of the interections of lossy curves with constant $c = 0.5$ and $c = 2$ lossy ratio to the ρ-plane.

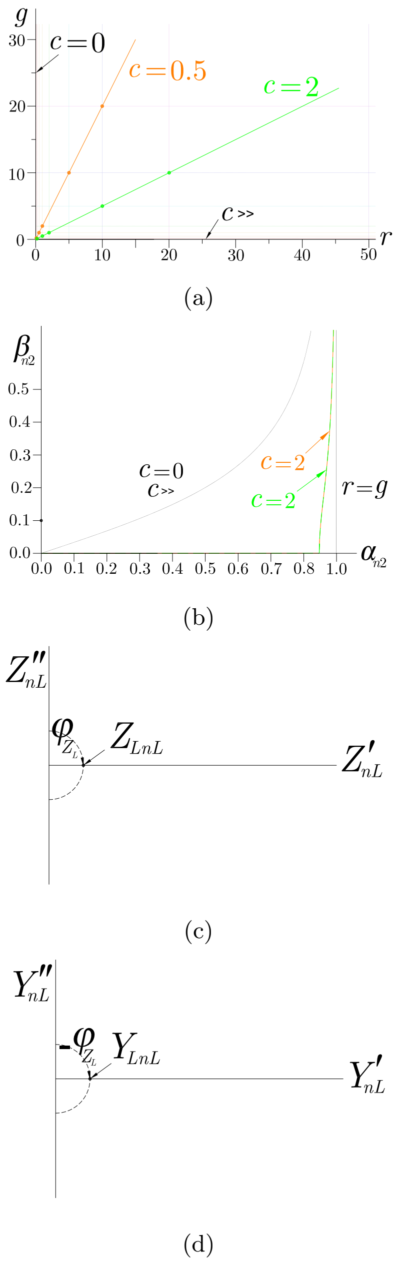


Fig. 5.34: The supportive (a) rg-, (b) γ_{n2}-, (c) Z_{nL}-, and Y_{nL}-planes.

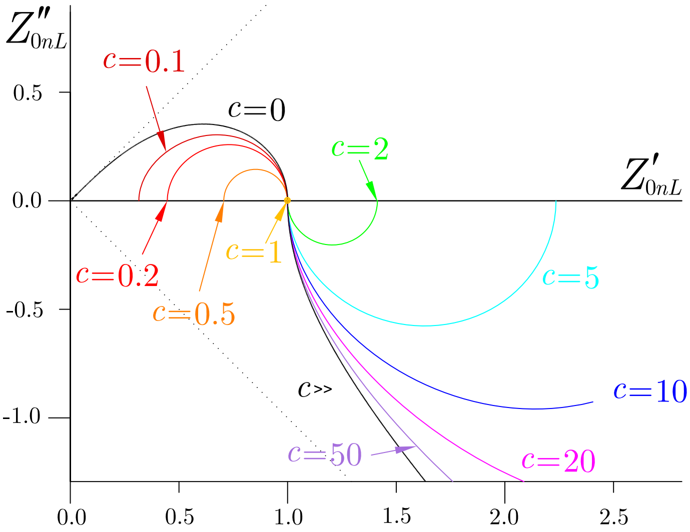


Fig. 5.35: Curves parameterized by c in the Z_{0nL}-plane.

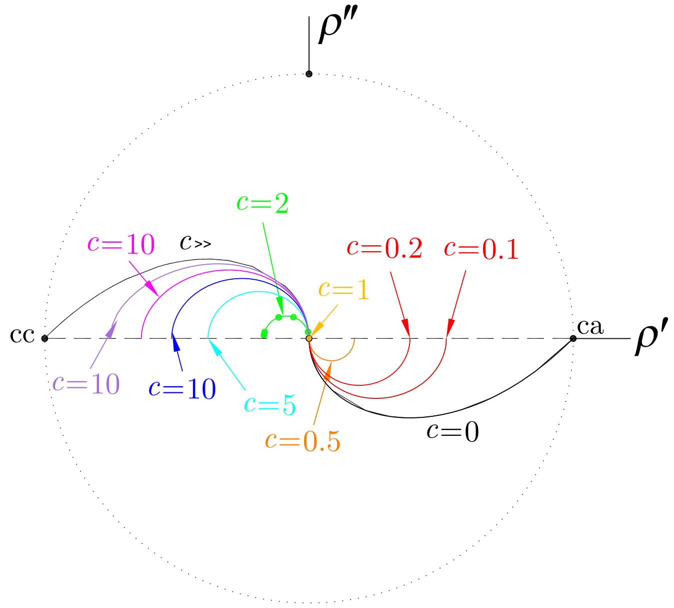


Fig. 5.36: Transformation of the curves with constant c ratio in the ρ-plane.

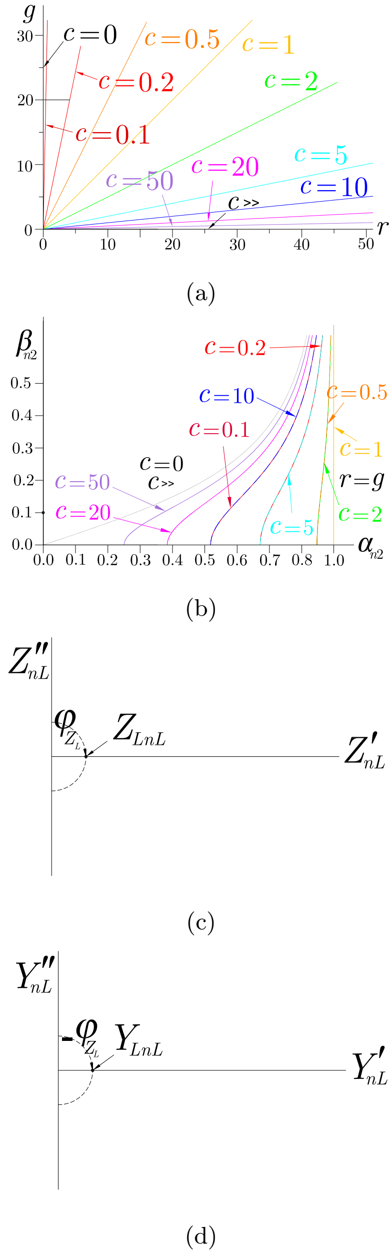


Fig. 5.37: The supportive (a) rg-, (b) γ_{n2}-, (c) Z_{nL}-, and Y_{nL}-planes.

Example II

Numerical values of the transmission line parameters

Physical parameters	*Line parameters*	
	Param.	Normalizations
	$R \neq 0$, L, $G = 0$, C	$r = 5$, $g = 0$, $c = 0.2$, $c = 5$
	Basic Parameters	
	Param.	Normalizations
f, λ, c_e, D/λ	$Z_{0,sp} = 50[\Omega]$ $Z_0 = 87.3142 - j71.5805 =$ $= 112.9050\angle - 39.3450°$ γ	- $Z_{0nL} = 0.8731 - j0.7158 =$ $= 1.1291\angle 112.9050°[\Omega]$ $\gamma_{n1} = 1.4316 + j1.7463 =$ $= 2.2581\angle 50.6550°[\text{m}^{-1}]$
	Wave parameters at the load	
	Param.	Normalizations
	ρ_L $Z_L = 0 + j100 = 100\angle 90°$ $Y_L = 0.0000 - j0.0100 =$ $= 0.0100\angle - 90°$	- $Z_{Ln} = 0 + j1 = 1\angle 90°$ $Y_{Ln} = 0 - j1 = 1\angle - 90°$

Table 5.5: Numerical values of the TL parameters for the analysis in this second example: $Z_{0,sp}$ and Z_L.
The values of the normalized basic parameters, Z_{0nL} and γ_{n1}, result from the intersection of the curves $r = 5$ and $g = 0$, which are studied as examples of curve transformation from the Z_{0nL}-plane to the ρ-plane in the *ffa*.

Fixed frequency analysis

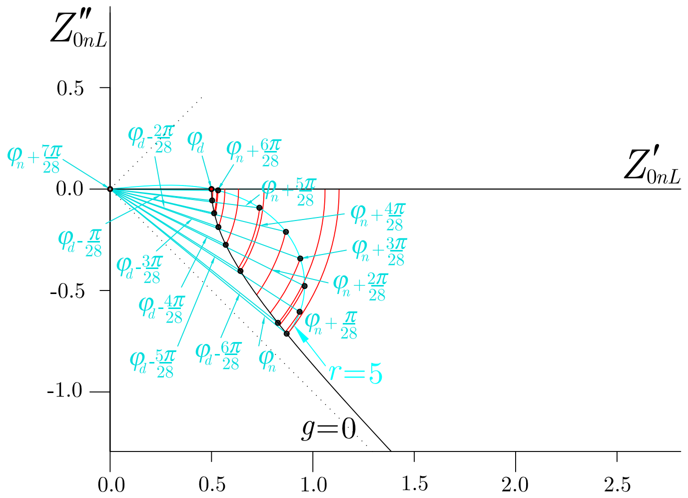


Fig. 5.38: Modulus and phase parameterizations of the $r = 5$ and $g = 0$ curves in the Z_{0nL}-plane.

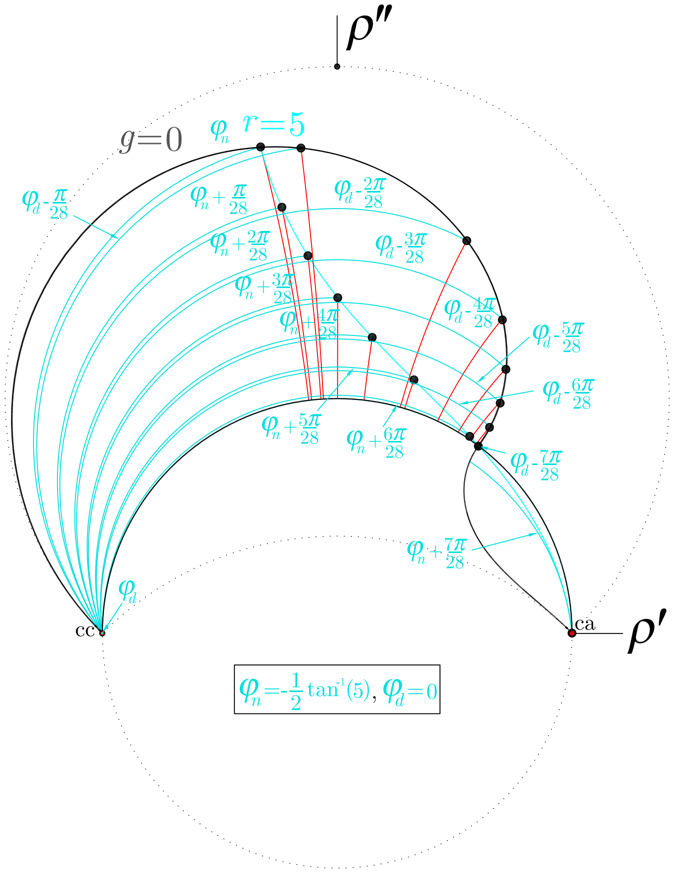


Fig. 5.39: Transformations of the $r = 5$ and $g = 0$ curves to the ρ-plane.

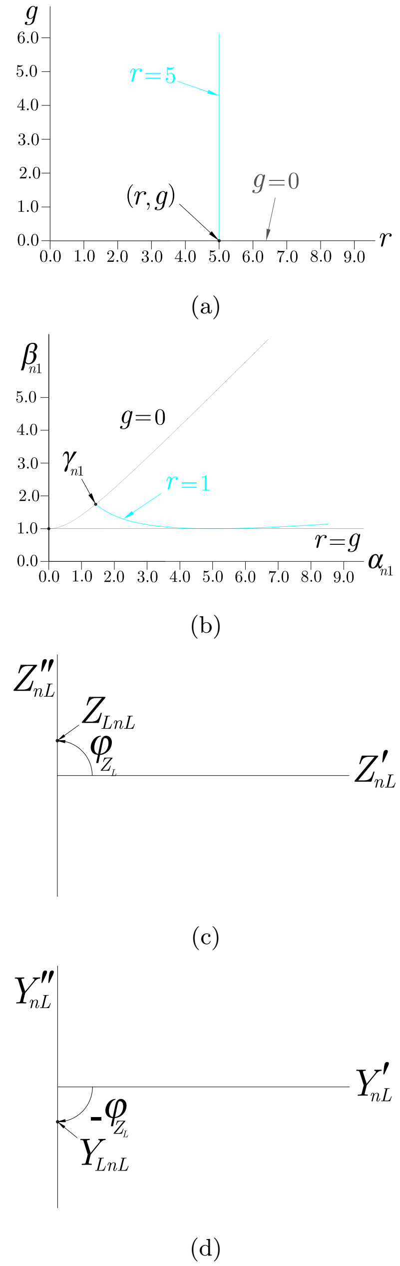


Fig. 5.40: The supportive (a) rg-, (b) γ_{n1}-, (c) Z_{nL}-, and Y_{nL}-planes.

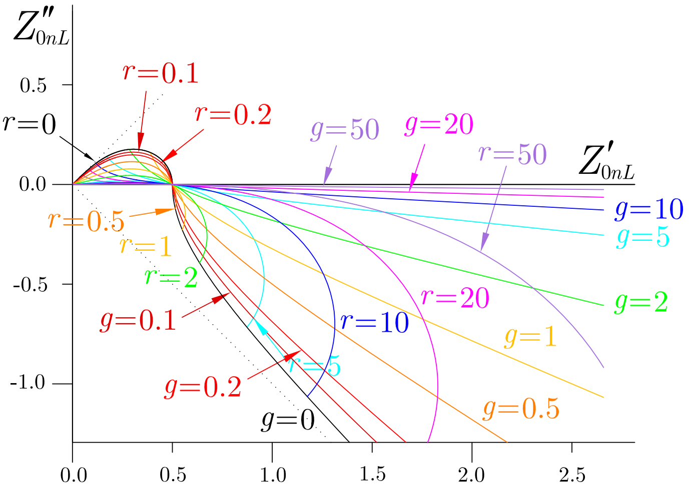


Fig. 5.41: Curves paramterizing the confuctor and dielectric losses in the Z_{0nL}-plane

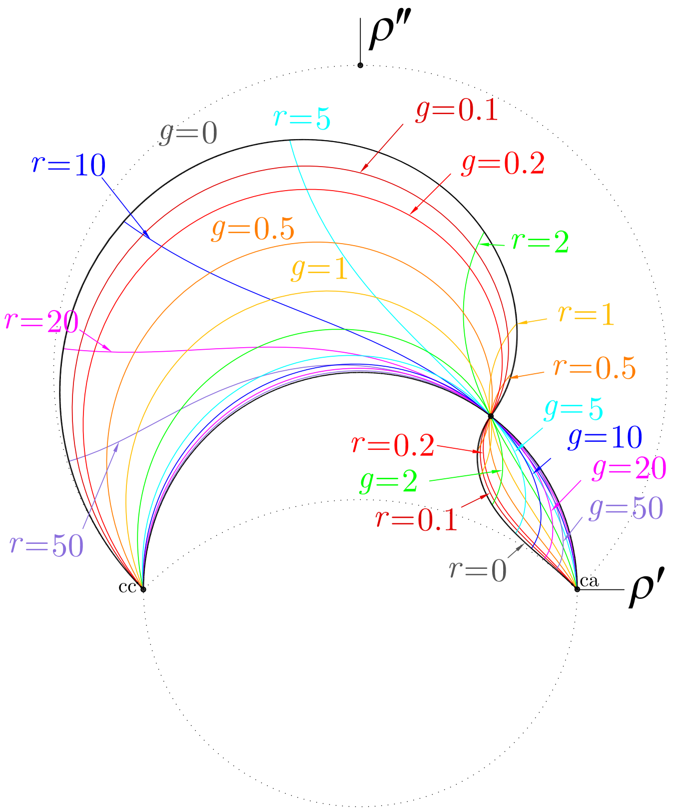


Fig. 5.42: Transformation of the curves parametrizing losses to the ρ-plane.

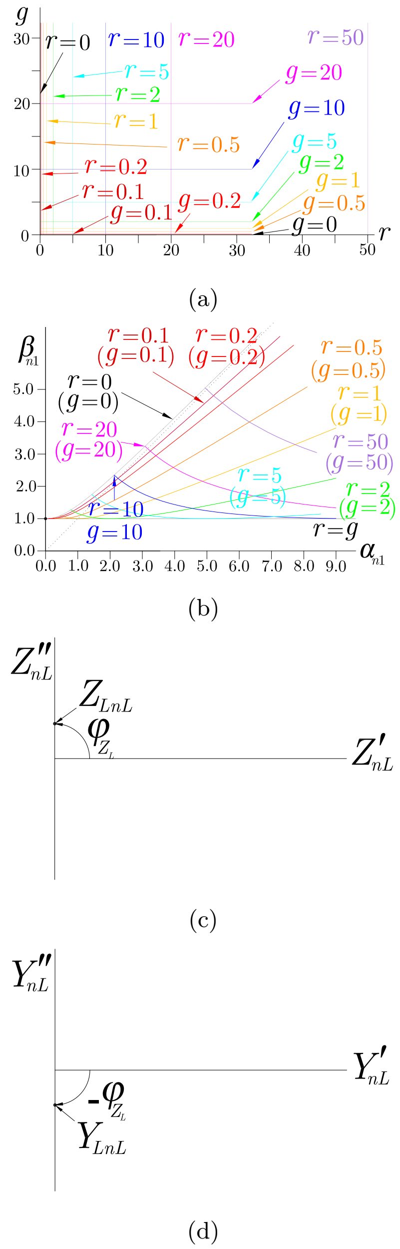


Fig. 5.43: The supportive (a) rg-, (b) γ_{n1}-, (c) Z_{nL}-, and Y_{nL}-planes.

Variable frequency analysis

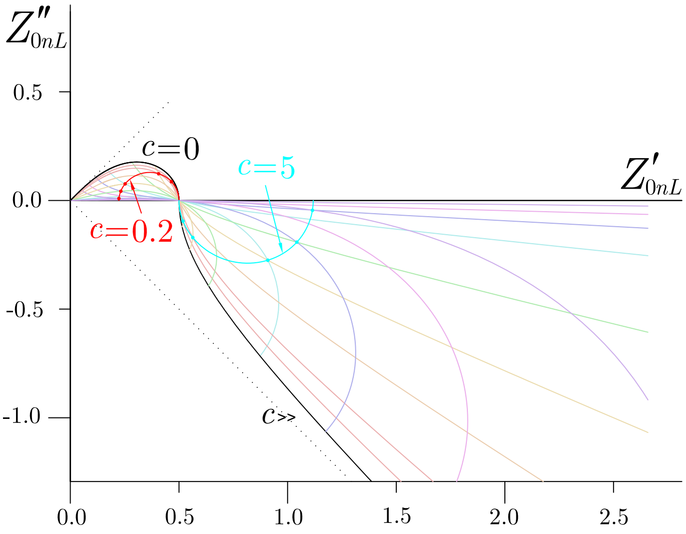


Fig. 5.44: Curves with constant $c = 5$ and $c = 0.2$ ratios.

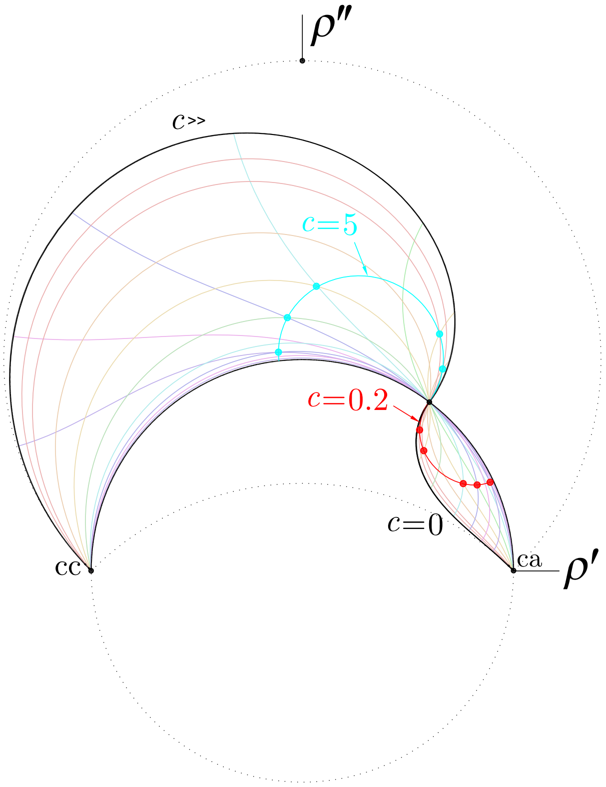


Fig. 5.45: Transformation of the interections of lossy curves with constant $c = 5$ and $c = 0.2$ lossy ratio to the ρ-plane.

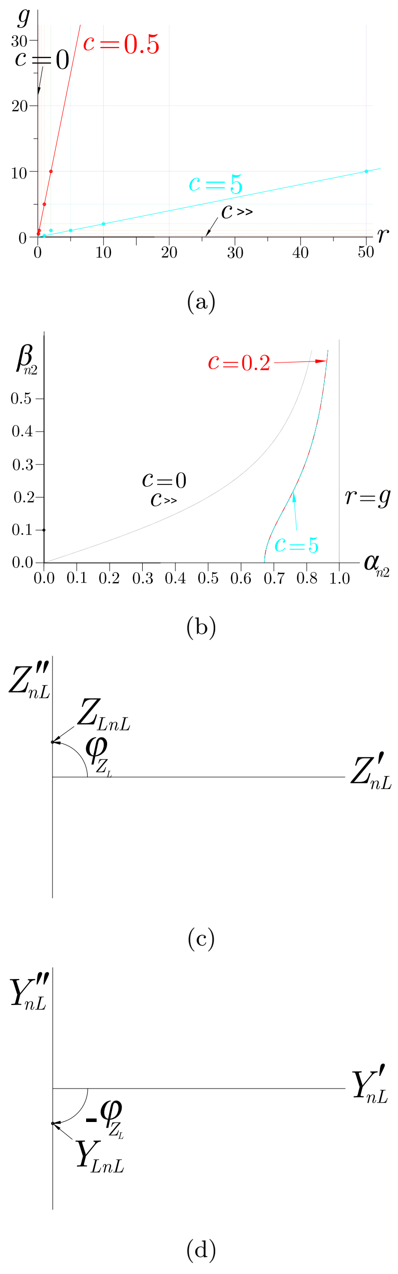


Fig. 5.46: The supportive (a) rg-, (b) γ_{n2}-, (c) Z_{nL}-, and Y_{nL}-planes.

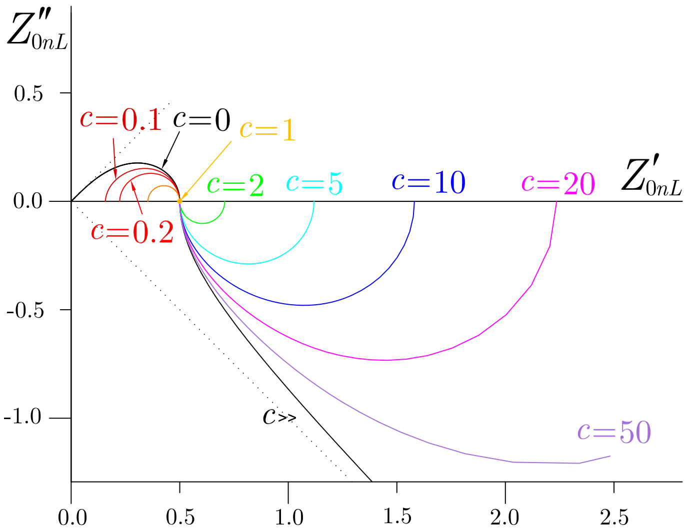


Fig. 5.47: Curves parameterized by c in the Z_{0nL}-plane.

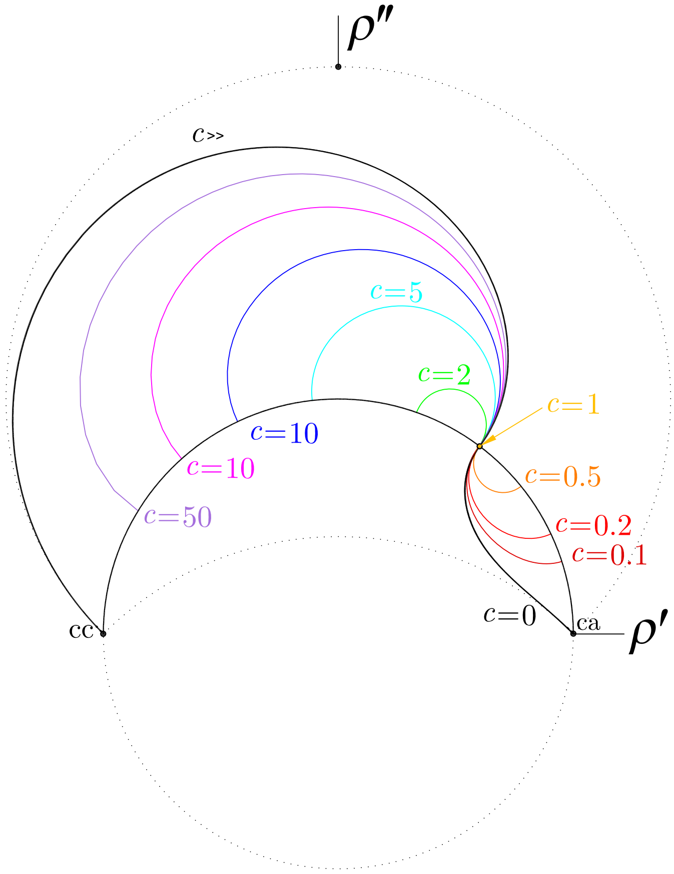


Fig. 5.48: Transformation of the curves with constant c ratio in the ρ-plane.

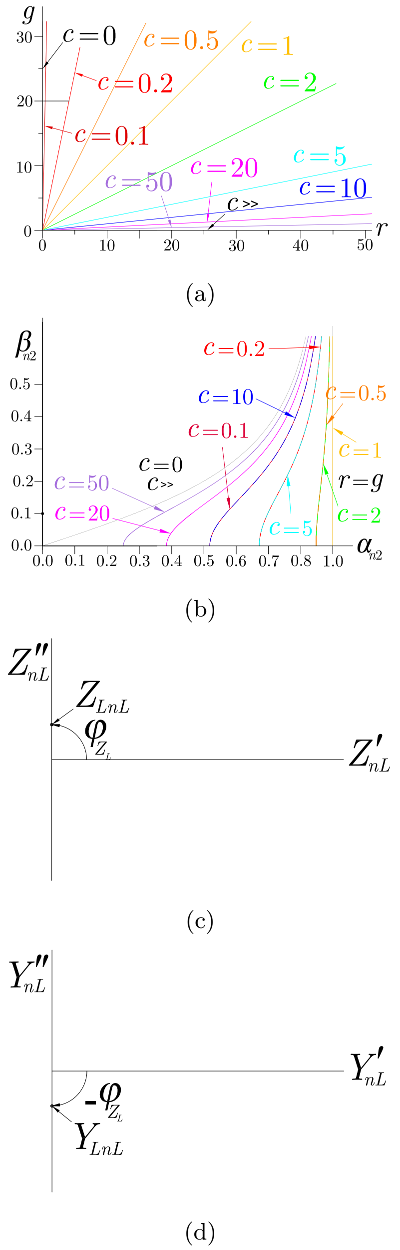


Fig. 5.49: The supportive (a) rg-, (b) γ_{n2}-, (c) Z_{nL}-, and Y_{nL}-planes.

Example III

Numerical values of the transmission line parameters

Physical parameters	*Line parameters*	
	Param.	Normalizations
	$R = 0$, L, $G = 0$, C	$r = 0$, $g = 0$, $c>>$, c=0
	Basic Parameters	
	Param.	Normalizations
f, λ, c_e, D/λ	$Z_{0sp} = 75[\Omega]$ $Z_0 = 75 + j0 =$ $= 75\angle 0°$ γ	- $Z_{0nL} = 3 + j0 =$ $= 3\angle 0°[\Omega]$ $\gamma_{n1} = 0 + j1 =$ $= 1\angle 90°[\mathrm{m}^{-1}]$
	Wave parameters at the load	
	Param.	Normalizations
	ρ_L $Z_L = 17.6777 - j17.6777 =$ $= 25\angle - 45°$ $Y_L = 0.0283 + j0.0283 =$ $= 0.0400\angle 45°$	- $Z_{Ln} = 0.7071 - j0.7071 =$ $= 1\angle - 45°$ $Y_{Ln} = 0.7071 + j0.7071 =$ $= 1\angle 45°$

Table 5.6: Numerical values of the TL parameters for the analysis in this third example: $Z_{0,sp}$ and Z_L.

The values of the normalized basic parameters, Z_{0nL} and γ_{n1}, result from the intersection of the curves $r = 0$ and $g = 0$, which are studied as examples of curve transformation from the Z_{0nL}-plane to the ρ-plane in the *ffa*.

Fixed frequency analysis

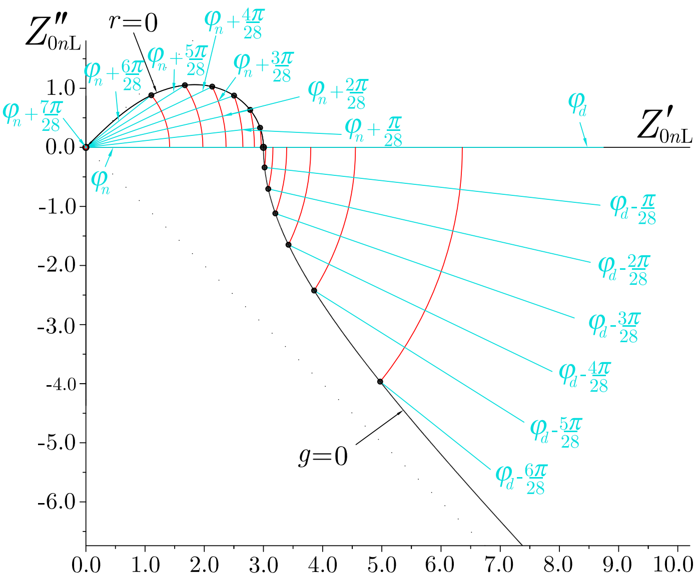


Fig. 5.50: Modulus and phase parameterizations of the $r = 0$ and $g = 0$ curves in the Z_{0nL}-plane.

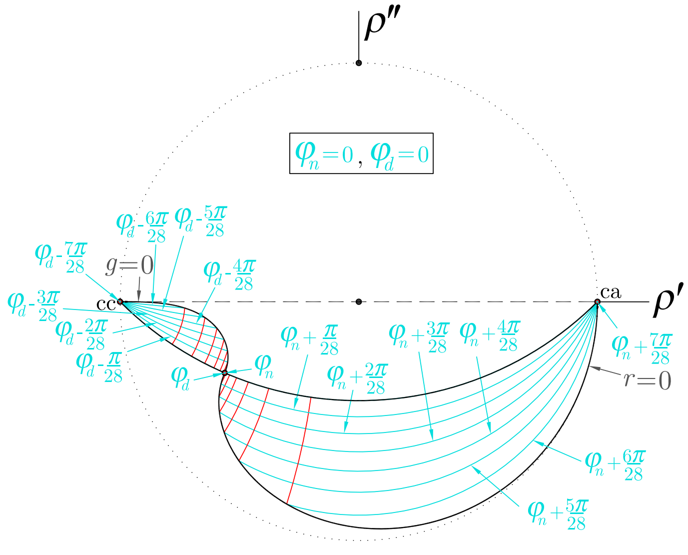


Fig. 5.51: Transformations of the $r = 0$ and $g = 0$ curves to the ρ-plane.

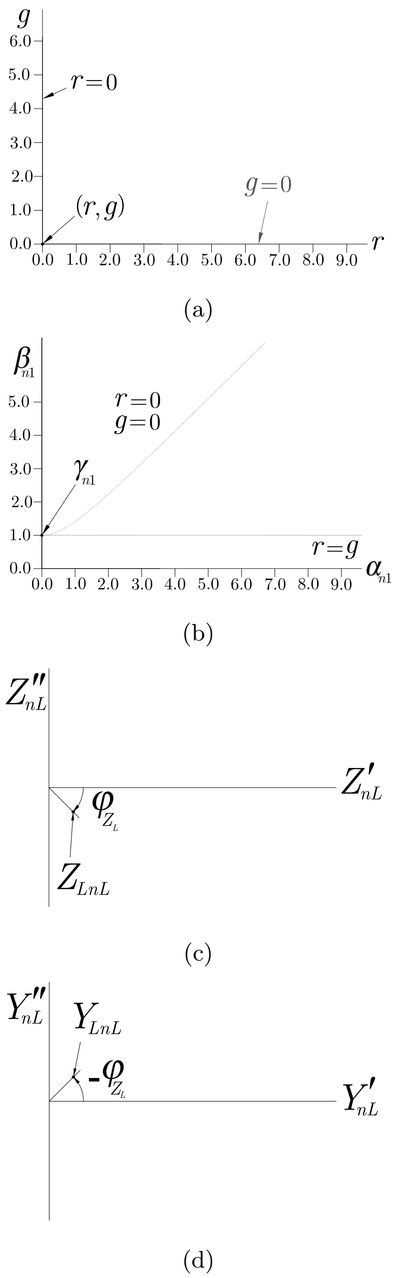


Fig. 5.52: The supportive (a) rg-, (b) γ_{n1}-, (c) Z_{nL}-, and Y_{nL}-planes.

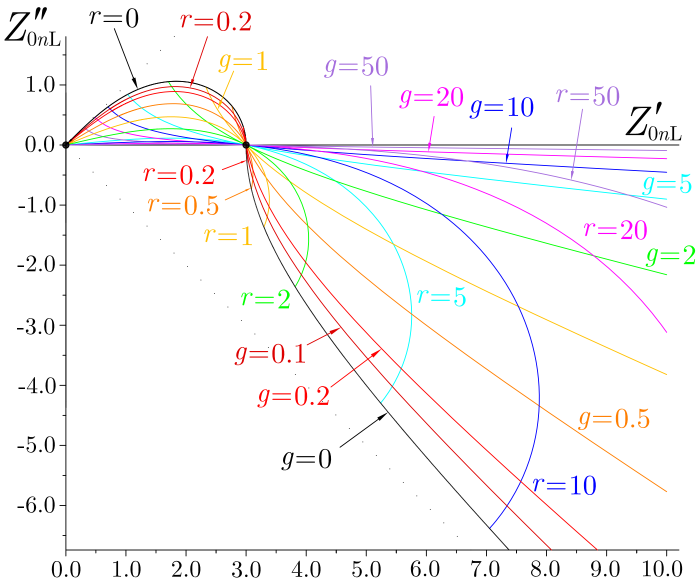


Fig. 5.53: Curves paramterizing the confuctor and dielectric losses in the Z_{0nL}-plane

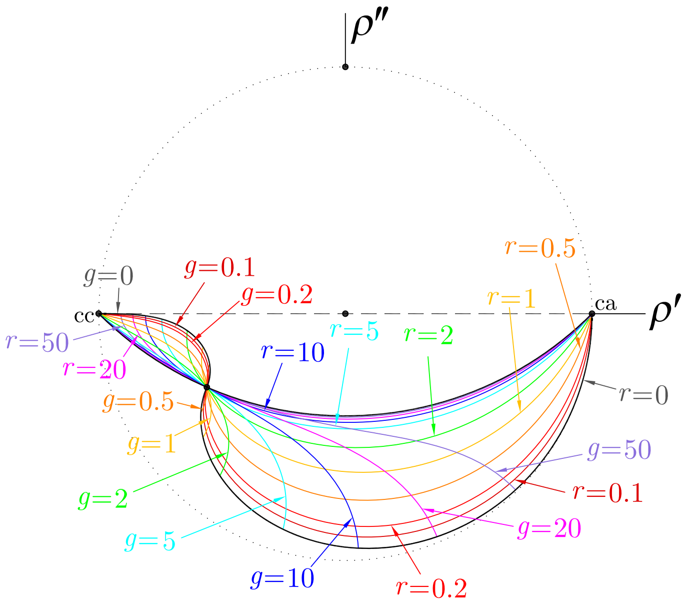


Fig. 5.54: Transformation of the curves parametrizing losses to the ρ-plane.

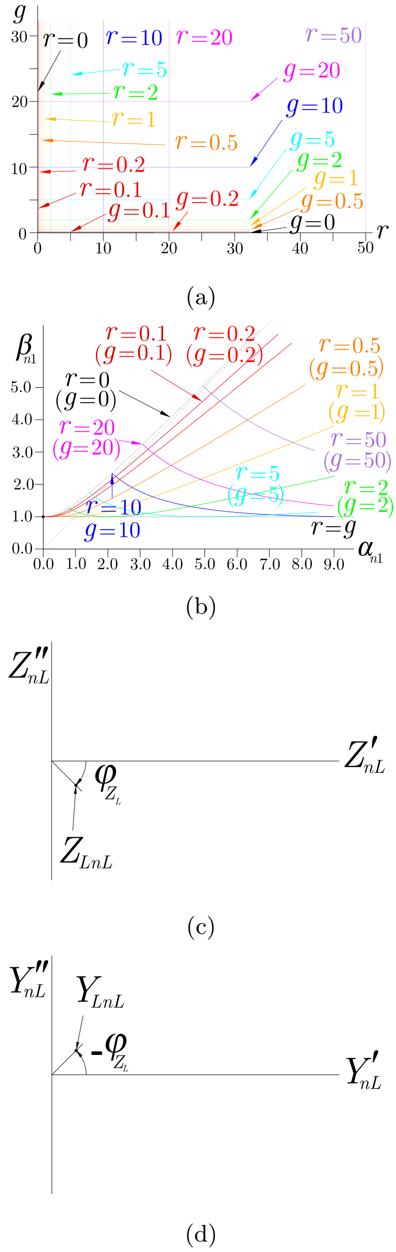


Fig. 5.55: The supportive (a) rg-, (b) γ_{n1}-, (c) Z_{nL}-, and Y_{nL}-planes.

Variable frequency analysis

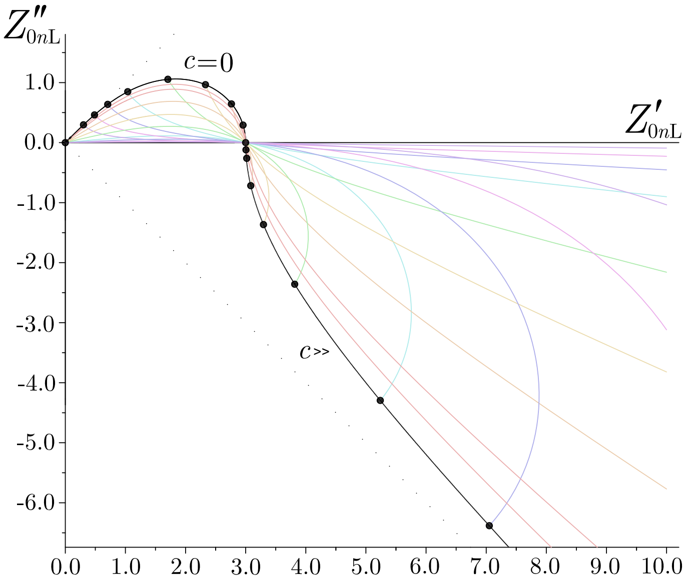


Fig. 5.56: Curves with constant $c = 0$ and $c_{>>}$ratios.

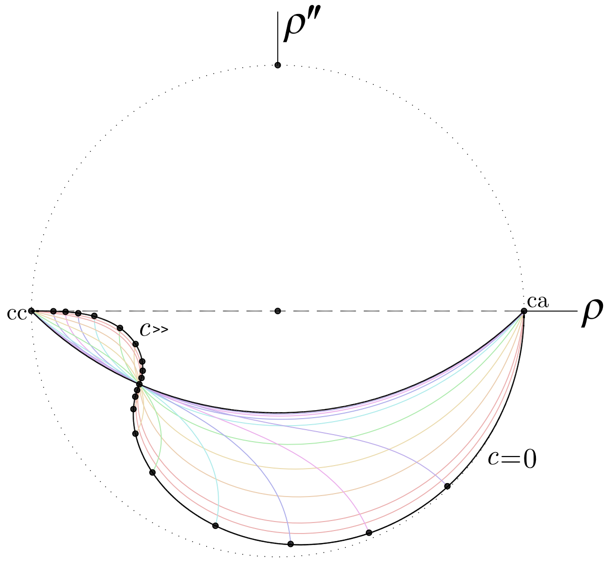


Fig. 5.57: Transformation of the interections of lossy curves with constant $c = 0$ and $c_{>>}$lossy ratio to the ρ-plane.

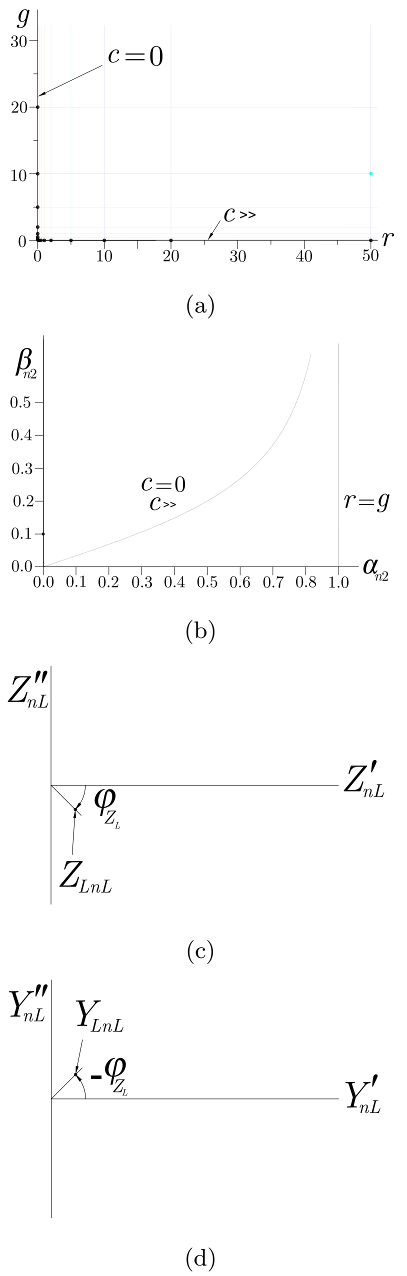


Fig. 5.58: The supportive (a) rg-, (b) γ_{n2}-, (c) Z_{nL}-, and Y_{nL}-planes.

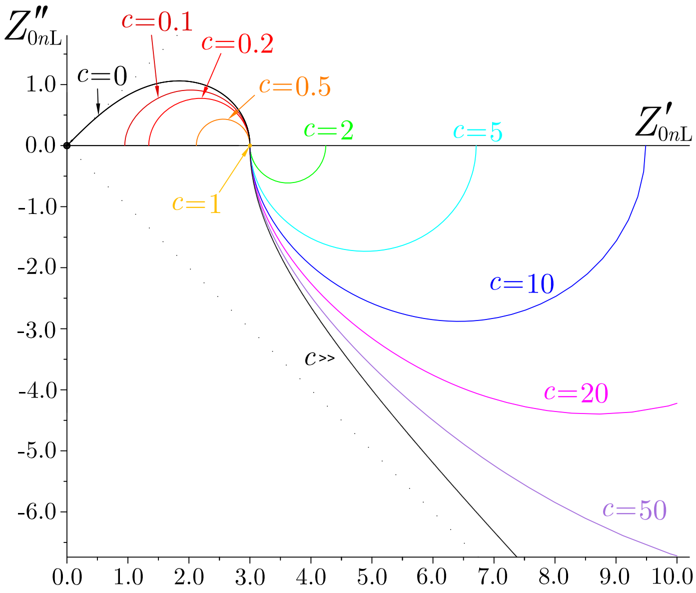


Fig. 5.59: Curves parameterized by c in the Z_{0nL}-plane.

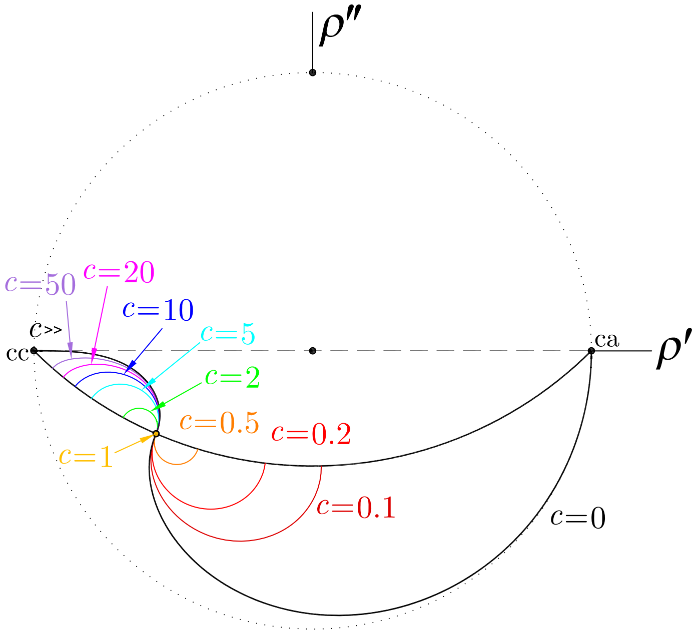


Fig. 5.60: Transformation of the curves with constant c ratio in the ρ-plane.

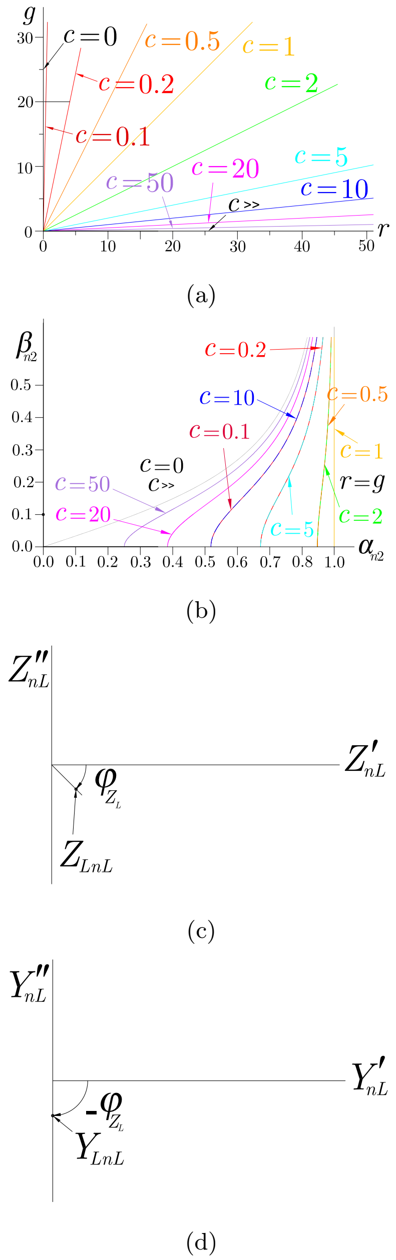


Fig. 5.61: The supportive (a) rg-, (b) γ_{n2}-, (c) Z_{nL}-, and Y_{nL}-planes.

Three different examples (examples I-III in Sects. 5.3.4-5.3.4, respectively) has been presented, varying both the lossless characteristic impedance and the impedance at the load in the following manner:

	Example I	Example II	Example III
$Z_{0,sp}$	$50[\Omega]$	$50[\Omega]$	$75[\Omega]$
Z_L	$50 + j0[\Omega]$	$0 + j100[\Omega]$	$25e^{-j\frac{\pi}{4}}[\Omega]$

Notice how "slight" changes on the impedance at the load makes that characteristic impedance and the reflection coefficient vary a lot.
In particular, if the ratio $Z_{0nL,sp} = Z_{0,sp}/|Z_L|$ is kept constant, Z_{0nL} would be the same, but not ρ_L, which depends individually on $Z_{0,sp}$ and Z_L (with the phase φ_{Z_L} included).

If analyzing the properties of the graphical analysis geometrically, it is possible to characterize the resultant curves and parameters when varying the losses and frequency.

On one hand, the curves in the Z_{0nL}-plane parameterized by losses are lemniscates of Bernouilli, [Law72], just as they have been classified and studied in the direct characterization of Z_{0n1}, but in this case scaled by the factor $Z_{0nL,sp}$. The curves parameterized by c used in the *vfa* are Cassini ovals, [Law72], just as they are presented in the direct characterization of Z_{0n2}, but multiplyed by the same factor Z_{0nL}. The general equations of these curves are well-known, [Law72].
Moreover, the curves in the γ_{n1}-plane and γ_{n2}-plane for the *ffa* and the *vfa* remain the same, because they are not affected by the "static" normalization with respect to the load.

On the other hand, the most interesting geometrical results to be described in the analysis in terms of losses and frequency concern the reflection coefficient at the load.
By means of the graphical examples presented in this section (Examples I-III), it can be seen at first sight that the curves in the ρ-plane are topologically equal to those in the Z_{0nL}-plane. In fact, the transformation between these planes supposes a strong contraction with respect to the original analysis of the characteristic impedance in terms of losses or frequency. In fact, this transformation given by T_ρ in eq. (5.40) is conformal, so the angles between the curves are preserved in magnitude and orientation.
As a result of this contraction, the curves in the ρ-plane are delimited by, on one hand, the same curves parameterized by $g = 0$ and $r = 0$ in the *ffa* or $c = 0$ and $c \gg$ in the *vfa* and, on the other hand, by the curves parameterized by $r \gg$ and $g \gg$ or $\omega_n = 0$ and $\omega_n \gg$ for the *ffa* or the *vfa*, respectively. These latter limits coincide with the curve parameterized by $\varphi_{Z_0} = 0$ ("universal", because angles in general do not depend on the normalization), which transforms to the arc of the circumference given by

$$\left(0, \frac{1}{\tan(\varphi_{Z_L})}\right) : \frac{1}{|\sin(\varphi_{Z_L})|}, \tag{5.46}$$

which only depends on the phase of the load that parameterizes this analysis, φ_{Z_L}.
Notice that this analysis in terms of losses/frequency reduces the limits that the GSC originally predicts, just as it can be seen in Figs. 5.30, 5.42 and 5.54 for the *ffa*, or equivalently in Figs. 5.36, 5.48 and 5.60 for the *vfa*.

Furthermore, some remarkable points, which meet some specific physical behaviors, are located on the ρ-plane, being these useful for further geometrical analysis in the ρ-plane.
These points are the perfect conductor, (**pc**), the perfect dielectric, (**pd**), and the non dispersive case, (**nd**)[4]. Notice that this latter non dispersive case generalizes the lossless case.
The (**pc**) is located at $\rho = \rho_{\rm pc} = 1 + j0$, whereas the (**pd**) is at $\rho = \rho_{pd} = -1 + j0$, whatever the

[4] These names are explained by the physical interpretation of the reflection coefficient in terms of some specific behaviors of the characteristic impedance, just as it has been explained in the inverse characterization of the basic parameters presented in Sect. 4.4.2 in Chpt. 4.

impedance at the load would be. These are the only points which reamin fixed in the ρ-plane when varying the impedance at the load.
As it has been said, the analysis of ρ_L in terms of losses and frequency (generally) depends on both the load and the lossles characteristis impedance, and so the (**nd**) does.
In Appendix 5.D, the location of (**nd**) in the ρ-plane is solved using geometrical analysis. As a result of this analysis, it is known that this point is located at

$$\rho = \rho_{\text{nd}} = |\rho_{\text{nd}}| e^{j\varphi_{\text{nd}}}, \tag{5.47}$$

$$\text{in which} \begin{cases} \varphi_{\rho_{\text{nd}}} = \tan^{-1}\left(\frac{2Z_L''}{|Z_L|^2 - Z_{0,sp}^2}\right) \\ |\rho_{\text{nd}}| = \frac{1}{\tan(\varphi_{Z_L})}\left(\sin(\varphi_{\rho_{\text{nd}}}) + \sqrt{\tan^2(\varphi_{Z_L}) + \sin^2(\varphi_{\rho_{\text{nd}}})}\right) \end{cases}.$$

The (**nd**) point, which is common to every curve in the *ffa* and the *vfa*, may be used for locating any point in the Z_{0nL}-plane to be transformed to the ρ-plane.
For this purpose, it is about "triangulating" the point which is interesting the be transformed using the angle φ_{Z_0} and the angle which forms the ray which has its origin in the (**nd**) point, $\varphi_{Z_{0,\text{nd}}}$, in the Z_{0nL}-plane.

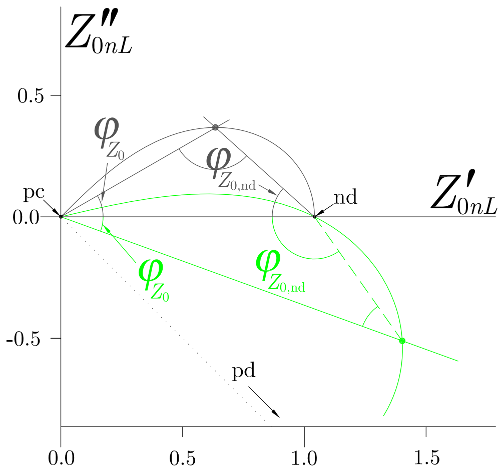


Fig. 5.62: Example of "angular trinagulation" using the angles φ_{Z_0} and $\varphi_{Z_{0,\text{nd}}}$ in the Z_{0nL}-plane for locating some random points on it.

In Fig. 5.62, an example of use of the angles φ_{Z_0} and $\varphi_{Z_{0,\text{nd}}}$ is depicted.
Using the "angular triangulation" (named phase-phase parameterization) instead of the modulus-phase parameterizations has two main advantages: (i) the scaling produced by $Z_{0nL,sp}$ does not affect the angles at all, so this phase-phase parameterization is "universal" in the Z_{0nL}-plane; and (ii) the phase-phase parameterization takes full advantage of the conformability of the transformation from the Z_{0nL}-plane to the ρ-plane. In this way, the points in the ρ-plane belonging the curves depicted for the analysis in terms of losses and frequency can be explained.

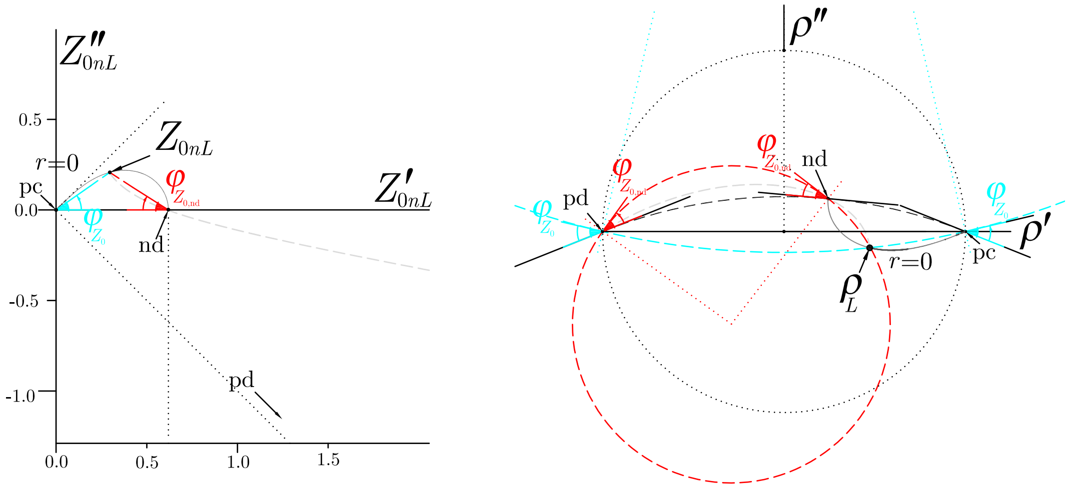


Fig. 5.63: Example of transformation of a point belonging the curve parametrized by $r = 0$ by using the phase-phase parameterization from the Z_{0nL}-plane to the ρ-plane.

In Fig. 5.63 above, an example of use of the phase-phase paramterization for locating a point in the ρ-plane is represented. Regarding this transformation, some interesting geometrical facts should be detached: (i) since the tansformation from the Z_{0nL}-plane to the ρ-plane is Möbius-type, the transformations of the rays characterized by φ_{Z_0} and $\varphi_{Z_{0,\mathrm{nd}}}$ are circumferences in the ρ-plane; moreover, (ii) since $Z_{0nL} = \infty$ is located at $\rho_{\mathrm{pd}} = -1 + j0$ in the ρ-plane, the angles φ_{Z_0} and $\varphi_{Z_{0,\mathrm{nd}}}$ "measured" at infinity can be geometrically represented in the ρ-plane; (iii) both angles φ_{Z_0} and $\varphi_{Z_{0,\mathrm{nd}}}$ are measured from the curve limit in eq. (5.46), and the orientation is preseved from the references in the Z_{0nL}-plane; and (iv) since φ_{Z_0} is measured from both the (**pd**) and (**pc**) points, the center of the circumference which represents the ray characterized by φ_{Z_0} is in the imaginary axis, which confirms the geometrical representation based on circunferences parameterized by the phase in the inverse characterization of basic parameters.

By repeating the process of parameterizing each of the points belongin the curve which is intended to be transformed, the resultant curve in the ρ-plane is depicted and geometrically characterized based on angular transformations.

Remark 39. *The main advantages of using the phase-phase parameterization instead of the modulus-phase parameterization are in: (i) reducing the parameterizations regarding load to the process of locating the non dispersive point (**nd**) in the ρ-plane; and (iii) keeping the analysis purely geometrical based on angular translations.*
In this way, the resultant curves in the ρ-plane are geometrically characterized.

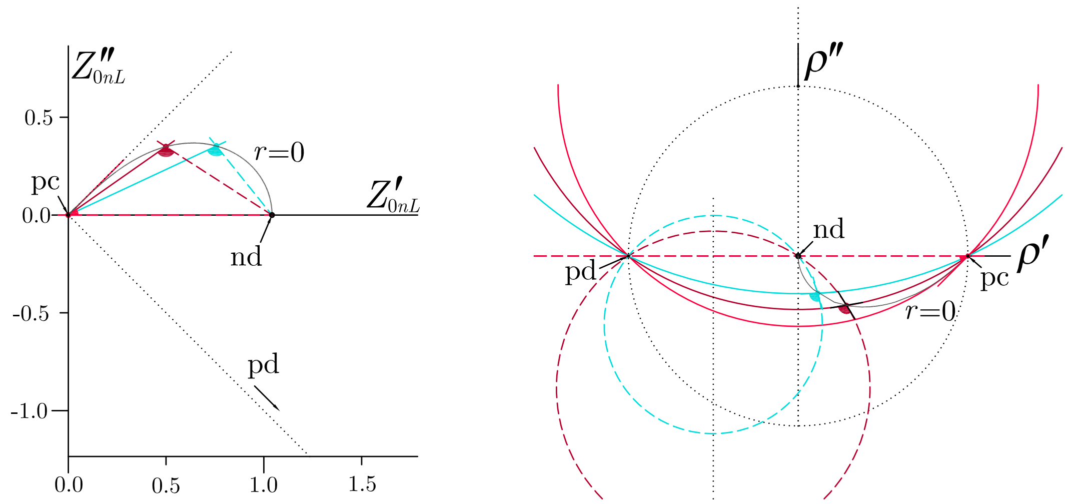


Fig. 5.64: Example of construction of the curve parameterized by $r = 0$ when $Z_{0,sp} = Z_L \in \mathbb{R}^+$, so $Z_{0nL,sp} = 1$ and $\varphi_{Z_L} = 0$.

In Fig. 5.64, an example of use of the phase-phase parameterization for representing the curve parameterized by $r = 0$ with the data in Example I (Sect. 5.3.4) is shown.

On their behalf, the points in the Z_{Ln}- and Y_{Ln}-planes are geometrically characterized by the phases φ_{Z_L} and $-\varphi_{Z_L}$, respectively.

5.3.5 Physical interpretations

The analysis presented in this example should be interpreted in terms of varying the losses or the frequency at any point of the TL, loaded by the wave impedance Z_L, to see how the TL parameters consequently change.

This analysis is equivalent to the one presented in Example I but, instead of varying the length of the TL, here the losses and frequency vary for the *ffa* and the *vfa*.
Thus, the true physical interpretation comes with the parameterizations used to describe the analysis in terms of losses/frequency which, in turn, clearly delimit the original analysis: the characteristic impedance and the reflection coefficient at any fixed point along the TL is given by this analysis, which has physical meaning by itself, and it is within the a priori possible values of them. For example, the original limit is parameterizd by the angle φ_{Z_0}. However, not all the possible values of Z_0 are physically realizable, and the limits of Z_0 are given by means of this characterization in terms of the physical parameters –losses or frequency– under study.

This analysis reveals the parameters which have true physical meaning. The parameters which change with losses/frequency at any fixed point along the TL are not true physical parameters. In fact, those paramaters which change with losses/frequency may be addressed as "analytical parameters" which vary according to the parameters which are fixed in this analysis, because they have true physical meaning. Moreover, the parameters which varies with losses/frequecy describe only certain parts of the total waves, and so they are potentially dimensionless[5].

On the other hand, this analysis serves to explain the particular cases or approximations, just as it has been done for the characteristic impedance and the propagation constant in the direct

[5]Even the normalized characteristic impedance is dimensionless.

characterization of basic parameters.
The lossless and the non dispersive cases, as well as the low losses approximation, are located in the (**nd**) point in the ρ-plane, which depends on the load, just as it has been explained among the geometrical analysis.
Moreover, two additional particular cases arise from this analysis:

	Lossy parameterizations
Perfect conductor (**pc**)	$r = 0$ and $g \to \infty$ ($g \gg$)
Perfect dielectric (**pd**)	$g = 0$ and $r \to \infty$ ($r \gg$)

Table 5.7: Values of the parameterizations regarding the *ffa* which define the particular cases: perfect conductor and perfect dielectric.

The perfect conductor (**pc**) and the perfect dielectric (**pd**) cases, which mix the lossless and high lossy cases of each parameterization of losses, are fixed points in the ρ-plane. Notice that the (**pd**) point corresponds to $Z_{0nL} = \infty$ in the extended Z_{0nL}-complex plane, so the ρ-plane is the way for making this point collapse to the fixed point (**pd**).

5.3.6 Practical uses

The analysis introdced by means of this example presents many potential practical uses in the analysis of TLs: some of them are directly related to the application of the achieved analytical results, also together with the analysis along the TL presented in Example I, and other ones would be related to describe the behavior of real electromagnetic systems, basing the analysis on characterizing their equivalent TL.

Firstly notice the capacity of ρ described at any fixed point along the TL in terms of losses in "varying" the reflection coefficient, which revals the possibility of using losses for matching load purposes.
However, only if the (**nd**) point (regarding this analysis at the load) is located at the origin in the ρ-plane the TL would match the load. That is because the signaled point (**nd**) represents the point in which $|Z_L| = |Z_0|$ and, if this point is at the origin in the ρ-plane, it means that $\varphi_{Z_L} = \varphi_{Z_0}$, leading to the desired matching.
As a consequence, the analysis of ρ in terms of losses/frequency is not useful for matching loads by itself, but it hints the capacity of using losses/frequency combined with the analysis along the TL for this purpose. In effect, this is shown in [VG17-I], and explained as application in **1**.

According to this latter interpretation of losses, frequency may be specifically used to achieve the matching by using the GSC in combination with the iGSC, because it is a controllable parameter in practice when the TL works in time harmonic regime, just as it is done in RF and microwave Engineering.
In any case, the analysis "along the frequency" for a fixed ratio c involving the line parameters is given by the analysis presented in this section, becoming especially important those geometrical analysis presented before in this example.

Finally, it should be detached the possibilities of this analysis when dealing with the TL parameters seen them as functions (in this case functions of losses/frequency).
For example, let's look at the possibilities of the defined functions $\boldsymbol{Z_{0nL}}(r, g)$ and $\boldsymbol{\rho_L}(r, g)$, or $\boldsymbol{Z_{0nL}}(\omega_n, c)$ and $\boldsymbol{\rho_L}(\omega_n, c)$ for characterizing any electromagnetic system (circuit) in which HPWs propagate.

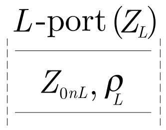


Fig. 5.65: Scheme of the port L (L-port) loaded by Z_L representing any circuit in which the parameters Z_{0nL} and ρ_L are to be characterized.

The circuit in Fig. 5.65 is characterized at a port L (L-port) loaded with Z_L (physically measureable), as schematized in the figure above.
Then, at the L-port, the circuit presents a characteristic impedance and a reflection coefficient measured with respect to the characterstic load Z_L, so it has its own Z_{0nL} and ρ_L.
These parameters are given in integral form as:

$$Z_{0nL} = \int_{rg\text{-plane}} \boldsymbol{f}(r,g)\boldsymbol{Z_{0nL}}(r,g)d\boldsymbol{\mu}(r,g) \text{ , and} \tag{5.48}$$

$$\rho_L = \int_{rg\text{-plane}} \boldsymbol{f}(r,g)\boldsymbol{\rho}(r,g)d\boldsymbol{\mu}(r,g), \tag{5.49}$$

in which the functions $\boldsymbol{Z_{0nL}}(r,g)$ and $\boldsymbol{\rho}(r,g)$ are the ones studied for this examples in eqs. (5.39) and (5.43), and they play the role of the kernel of the integral operator acting in the rg-plane with the measure $\boldsymbol{\mu}(r,g)$ (to be defined) based on the definition of $\boldsymbol{f}(r,g)$, which is the characteristic function of the circuit to analyze.
A trivial example of this circuit would be the any lossy TL:

Example 5.3.1. *A lossy TL parameterized by $r = r_0$ and $g = g_0$ is generically defined by:*

$$\begin{cases} \boldsymbol{f}(r,g) = \delta(r - r0, g - g0) \\ d\mu(r,g) = drdg \end{cases} ,$$

in which $\delta(r,g)$ would be the bivariate Dirac's delta, so the integrals in eqs. (5.48) *and* (5.49) *above have to be understood within the frame of the Theory of Distributions, [Sch51], or the Theory of Generalized Functions, [GS64].*
As a result, Z_{0nL} and ρ_L are defined as $\boldsymbol{Z_{0nL}}(r_0,g_0)$ and $\boldsymbol{\rho_L}(r_0,g_0)$, respectively, or equivalently by the intersection of the curves parameterized by $r = r_0$ and $g = g_0$ in the graphical analysis described in this example.

Reciprocally, the analysis could be done in terms of frequency, becoming much more intuitive for practical purposes.
In this sense, the equivalent Z_{0nL} and ρ_L at the L-port could be measured in a specific bandwidth, for instance, by keeping fixed c while varying ω_n accordingly.

5.4 Analysis along the TL in terms of losses and frequency

Before dealing with the analysis in this example, it is highly convenient to indroduce it taking into account the results achieved by means of the two previuous examples, which are part of the conclusions of the analysis of lossy TLs studied under the context of the CTLT.
At the same time, the objectives related to this example are intended to be clearly posed.

On one hand, by means of Ex. 01 presented in Sect. 5.2, it has been seen that the phase of the propagation constant, φ_γ, determines the analysis along the TL once the reflection coefficient at the load is known.
It means that all the possible combinations of losses that lead to the same φ_γ, present the same variation of the reflection coefficient along the TL.
In this way, the phase of the propagation constant adquires great relevance in those analysis which are posed along the TL.

On the other hand, it has been seen by means of Ex. 02 presented in Sect. 5.3 that the phase of the characteristic impedance, φ_{Z_0}, has special influence on describing the curves parameterized by losses or frequency at any fixed point along the TL where the impedance is known.
In particular, it has been seen that φ_{Z_0} may be used as the parameterization for the curves parameterized by losses/frequency (see, for example, the curves parameterized by the conductor losses in eq. (5.41)). Furthermore, by means of the direct characterization of wave parameters, for example in the GSC, it has been seen that φ_{Z_0} determines the influence of losses on the wave parameter transformations.
In this way, the phase of the characteristic impedance results crucial for parameterizing losses/frequency at any fixed point along the TL.

Keeping in mind the behaviors of the TL regarding the phases of the basic parameters, the issue of studying how the parameters of the TL behave when φ_γ or φ_{Z_0} are constant and what type of problems these analysis would solve result of special interest. In this way, the questions posed at the beginning of this chapter are totally motivated by the previous examples of use of the CTLT in regards to the analysis along the TL and in terms of losses.

Remark 40. *The analysis along the TL and in terms of frequency/losses reveal that they are determined by the phases of basic parameters. Conversely, if studying the influence of these angles in different scenarios, both the analysis along the TL and in terms of frequency/losses may be completely characterized.*

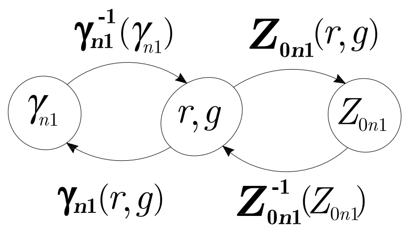


Fig. 5.66: Sheme of possible transformations between the normalized basic parameters based on the definition of the direct and inverse functions from the domain of parameterizations (r, g).

From this point forward, the study of the TLs parameterized by the phase of both the characteristic impedance and the propagation constant will determine the analysis along the TL in terms of losses/frequency.
Notice that this study is feasible thanks to the definition of basic parameters in terms of line parameters, which leads to the subsequent definition of their respective functions in terms of line parameters and thus the inverse functions for the inverse characterization of line parameters.

In Fig. 5.66 the transformations between the basic parameters having the domain of parameterizations (r,g) as the link of them are shown. This connection between basic parameters may be explicitly accomplished by parameterizing their phase, for example obtaining those pairs (r, g) that lead to the same φ_{Z_0} (denoted as $(r, g)_{\varphi_{Z_0}} \equiv \boldsymbol{Z^{-1}_{0n1,\varphi_{Z_0}}}(Z_{0n1})$), so that the TL which keep this angle constant is characterized by the propagation constant $\gamma = \boldsymbol{\gamma_{n1}}[(r, g)_{\varphi_{Z_0}}] \equiv \gamma[\boldsymbol{Z^{-1}_{0n1,\varphi_{Z_0}}}(Z_{0n1})]$.

Remark 41. *The fact that the basic parameters may be seen as functions of the line parameters (functions defined on the rg-plane) makes that all the TL parameters are mutually connected. This allows, in turn, their analysis in terms of the phase of each basic parameter, that is, the analysis using "cross angular parameterizations".*

For the issue of parameterizing the analysis in terms of angles, notice the great advantage the angular parameterizations have when normalizing the parameters under study: they are not affected by any of the proposed normalizations.

Remark 42. *The fact that angles are able to parameterize the analysis both in terms of losses/frequency and along the TL makes the analysis "universal" in itself, because the angles trascend the (real) normalizations proposed throughout the CTLT.*

The seeked parameterizations are explained for different purposes regading the analysis in terms of the phase of basic parameters.
Then the subsequent mathematical analysis are addressed to obtain the expressions of the TL parameters written in terms of the angular parameterizations.
The graphical and geometrical analysis are presented in the usual way: comparing the planes associated to each parameter under study, and indicating the fixed parameters in each case.
In this case, the physical interpretations underlying the analysis have even further importance because (i) the parameterizations of angles have non direct physical meaning in the description of losses/frequency and the TL's length, and (ii) the results are often non trivial to be understood, at least at first sight from graphical analysis, which is also a consequence of the use of angles as parameterizations.
At the end, the practical uses of this example are outlined, emphasizing the conclusions and the manner the angles can generalize the TL analysis. Moreover, the main limitations to this analysis are pointed out, suggesting at the same time the appropriate alternatives to overcome them, if the same basis of this study are intended to be recycled for future analysis or applications.

5.4.1 Definitions and parameters

As it has been mentioned in the introduction of this example, the angles of basic parameters, φ_{Z_0} and φ_γ, may act as the parameterizations for the analysis combining the influence of losses/frequency and the studies along the TL.

Moreover, an important fact to take into account for the subsequent mathematical analysis is that any of the normalizations regarding the basic parameters used in the CTLT is valid for the analysis "crossing" the parameterizations of their angles. That is because the angle parameterizations do not depend on the normalization chosen, as it has been explained before.
Nevertheless, studying the influence of angles from the normalizations regarding the lossless case (used for the *ffa* in terms of losses) has clear advantages with respect to rest of normalizations: (i) the basic parameters presents compact expressions in terms of r and g, and they are given by the product or quotient of the complex values n and d, as they have been originally defined in eqs. (4.5) and (4.6), respectively; and (ii) the inverse characterization of line parameters in terms of the phases of the line parameters has been done using the (r, g) coordinates.
In any case, this "lossless normalization" may be easily denormalized a posteriori, to be then renormalized as the subsequent analysis requiere.

The first question to be addressed in order solve the problem of parameterizing the losses/frequency and the TL's length by means of the phases φ_{Z_0} and φ_γ is related to answer which losses/frequency and lengths corresponds to these angles.

Remark 43. *The original question of which TL's leads to the same phase of each of the basic parameters is posed inversely to see which losses correspond to these angles. The reason for posing the inverse problem is founded in the previous analysis either along the TL or in terms of losses/frequency, which suggest that the phases of basic parameters serve to describe these analysis provided that the appropriate BCs are setted on.*
As a consequence, the inverse characterization appears as the required method for solving these issues on the parameterization.

As suggested, the inverse characterization of line parameters in terms of the phases of the basic parameters presented in Sect. 4.4.1 in Chpt. 4 directly answers which parameterizations of losses and frequency correspond to these angles.
On the other hand, it about solving which lengths along the TL (measured from the load) correspond to the angles of basic parameters.
Answering this question directly depends on (i) the normalization of the length which is chosen and (ii) how it is embedded in the parameters which describe the TL along its length.
As it has been studied in Ex. 01 presented in Sect. 5.2, the parameter from which the analysis along the TL are addressed is the reflection coefficient, expressed as a function of the TL's length in eq. (5.4) or in case of normalizing the TL's legth in eq. (5.17).
Moreover, one of the most important conclusions regarding the analysis along the TL is that the angle φ_γ determines the analysis of ρ along the TL when the reflection coefficient at the load is known. Also notice that the (positive real) normalization of the length is a factor accompanying the propagation constant. These facts suggest that there exists a normalization of the TL's length for which the expression of the reflection coefficient depends explicitly on the angle of the propagation constant.
The normalization in question is[6]:

$$l_n = \frac{|\gamma| l}{2\pi}. \tag{5.50}$$

Notice that this normalization is the same as the electrical length defined in eq. (5.16) only in case the TL is lossless.
The resultant reflection coefficient along the TL is reparameterized as:

$$\rho(l_n) = \rho_L e^{-2\pi e^{j\varphi_\gamma}|\gamma| l} = \rho_L e^{-4\pi e^{j\varphi_\gamma} l_n} \equiv \rho_L e^{-4\pi \cos(\varphi_\gamma) l_n} e^{-j4\pi \sin(\varphi_\gamma) l_n}, \tag{5.51}$$

in which $\rho_L \in$ is supposed to be known.
Notice that this expression represents the same spiral as in eq. (5.19) in the ρ-plane, so only the dependence on φ_γ has been made explicit by means of eq. (5.51).
By means of the definition of l_n in eq. (5.50), it is clear that varying the modulus of the propagation constant (or any normalization related to this) is equivalent to change l, and so "moving" the reflection coefficient along the TL, provided that ρ_L is fixed. This means that the curves parameterized by φ_γ, which are represented by keeping fixed φ_γ while $|\gamma|$ (or any normalization of this) varies, are able to represent "shiftings" along the TL, providing that (i) the appropriate BCs are satisfied (for example, in the expression of $\rho(l_n)$, ρ_L must be fixed) and (ii) the analysis is properly interpreted, which has implicit knowing the limitations[7].
In any case, using losses for (partially) representing the TL's length has implicit "universalizing" the analysis of the TL parameters along the TL in terms of losses, which is the purpose of the example presented here.

[6]The usual notation employed in this Thesis book is extended for this case. Then "l_n" is "the length normalized with respect to its modulus".

[7]These limitations are studied when physically interpreting the analysis, but anyone can already realize the modulus of γ can not be from 0 to ∞ if it is intepreted in terms of (real) losses. This suggests, in turn, introduceing the complex losses for having a complete parameterization of length in the (complex) rg-plane.

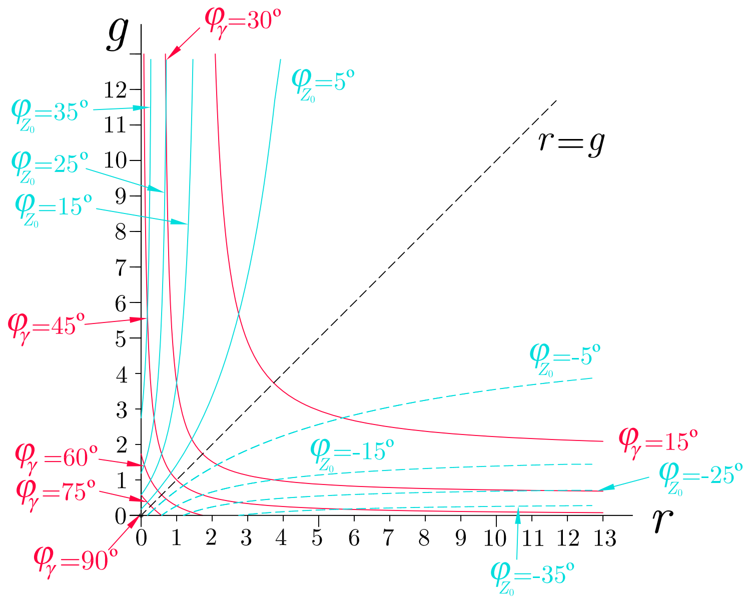


Fig. 5.67: The rg-plane parameterized by the phases of the basic parameters.

In Fig. 5.67 the hyperbolas parameterized by the phases of the basic parameters are shown (remember this study has been done in Sect. 4.4.1 in Chpt. 4). These curves form an non orthogonal coordinate system which is useful for parameterizing the basic and wave parameters along the TL in terms of losses/frequency.
Reciprocally, this analysis gives the physical interpretation of the analysis parameterized by the angles of the basic parameters based on line parameters.
Also notice that the analysis in the rg-plane gives the limits of the phase of one of the basic parameters when the other angle is fixed, and thus the restrictions in the analysis along the TL in terms of losses. In particular, each curve parameterized by φ_γ represents the variation along the TL which, in turn, are described in terms of losses. In this sense, φ_{Z_0} may be used for representing the TL's length on the coordinate system in the rg-plane in Fig. 5.67. However, neither all the values of losses are represented by the curve parameterized by φ_γ in the rg-plane, nor all the possible phases φ_{Z_0} decribe this curve. This means that the analysis in terms of angular parameterizations supposes a partial analysis of both losses and lengths, at least when using real parameterizations in the rg-plane.

Remark 44. *The analysis in terms of angular parameterizations has clear limitations in the analysis along the TL in terms of losses/frequency, which mainly come with the initial assumption that lossy parameterizations are real, which is, on the other hand, natural because the physical interpretation of losses.*
Nevertheless, the analysis in terms of the angles of the basic parameters turns out to be the appropriate way to address the analysis of all the possible parameterizations: losses/frequency and length; at the same time, and the rg-plane gridded by the curves representing the phases the origin of the parameterizations in which the limits of this analysis are found.

In any case, this analysis is an adequate starting point for combining the analysis along the TL and the analysis in terms of losses/frequency.
The subsequent mathematical analysis are aimed to describe the rest of the TL parameters in terms of angular parameterizations.

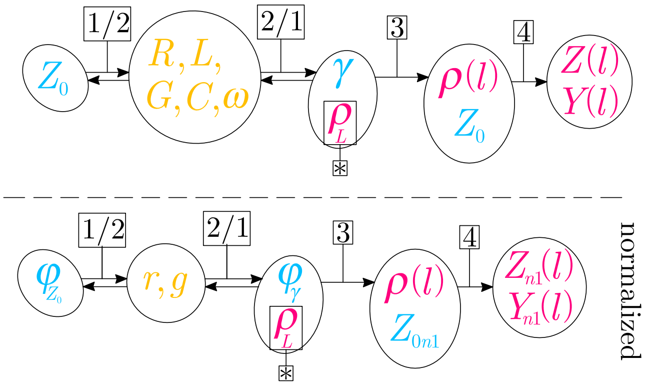


Fig. 5.68: Scheme of parameters and transformations (1-4) to be followed for finally achieving the characterization of the TL parameters both along TL and in terms of losses/frequency. The parameters which are the data for this example of analysis are marked with the 99 symbol.

Specifically, for the direct graphical representation of the reflection coefficient along the TL, it is neccesary to know its value at the load, ρ_L, which is the data in this example (see the scheme that summarizes the trasformations used in this example in Fig. 5.68).
Moreover, the use of the phase of the characteristic impedance for describing the reflection coefficient along the TL (Z_0 is not constant), besides the fact of forcing ρ_L to be fixed, makes the wave impedance and wave admittance vary along the TL in a different way of how they are described along the TL when Z_0 is fixed in Ex. 01. Notice that for this final purpose of obtaning the wave parameters from the reflection coefficient (step 4 in Fig. 5.68), their normalization is chosen with respect to the lossless case:

$$Z_{n1} = \frac{Z}{Z_{0,sp}}, \text{ and} \tag{5.52}$$

$$Y_{n1} = Z_{0,sp}Y; \tag{5.53}$$

because (i) this normalization does not depend on the angle φ_{Z_0}, which describes the analysis, and (ii) the characteristic impedance normalized with respect to the lossless case is the parameter chosen for solving the (lossy) parameterizations which characterize the angle φ_{Z_0}.

5.4.2 Mathematical analysis

This section is intended to obtain and define the functions and transformations to characterize every TL parameter in terms of the phases of the basic parameters.
In many occasions, the inverse characterization of line parameters will help this analyis. In particular, this occurs in step 1/2 in the scheme Fig. 5.68 (which is equivalent to find the functions schematized in Fig. 5.66), in which it is required solving the line parameters which lead to fixed angles φ_{Z_0} and φ_γ in order to replace them in γ_{n1} and Z_{0n1}, respectively, for obtaining the basic parameters "cross parameterized" by angles.

Due to both the normalized characteristic impedance and the normalized propagation constant used for the *ffa*, Z_{0n1} and γ_{n1}, are described by the complex quantities n and d defined in eqs. (4.5) and (4.6), respectively, their phases are the quotient and the product of the phases of these complex numbers, also respectively.
In Appendix 5.E the general polar equations of Z_{0n1} and γ_{n1} describes in terms of angles are

obtained, leading to:

$$|Z_{0n1}| = \sqrt{\frac{\sin(\varphi_\gamma - \varphi_{Z_0})}{\sin(\varphi_\gamma + \varphi_{Z_0})}}, \tag{5.54}$$

$$\text{in which } \begin{cases} \varphi_\gamma - \frac{\pi}{2} \leq \varphi_{Z_0} \leq \frac{\pi}{2} - \varphi_\gamma & \text{if } \varphi_\gamma \geq \frac{\pi}{4} \\ -\varphi_\gamma < \varphi_{Z_0} < \varphi_\gamma & \text{if } \varphi_\gamma < \frac{\pi}{4} \end{cases},$$

and

$$|\gamma_{n1}| = \sqrt{\frac{1}{\sin(\varphi_\gamma - \varphi_{Z_0})\sin(\varphi_\gamma + \varphi_{Z_0})}}, \tag{5.55}$$

$$\text{in which } \begin{cases} -\varphi_{Z_0} < \varphi_\gamma \leq \varphi_{Z_0} + \frac{\pi}{2} & \text{if } \varphi_{Z_0} < 0 \\ \varphi_{Z_0} < \varphi_\gamma \leq \frac{\pi}{2} - \varphi_{Z_0} & \text{if } \varphi_{Z_0} \geq 0 \end{cases},$$

rescpetively.
In the mathematical developments, notice how the inverse characterization is crucial for obtaining the limits of the phase of each parameter in terms of the cross angle parameterization.
Also from the mathematical notes it is important to detach the two conditions than the phases of the basic parameters simultaneosly verify:

$$0 < \varphi_{Z_0} + \varphi_\gamma \leq \frac{\pi}{2}, \text{ and} \tag{5.56}$$

$$-\frac{\pi}{2} \leq \varphi_{Z_0} - \varphi_\gamma < 0, \tag{5.57}$$

which are called the "realizability conditions", [VG17-I], of the TL in which HPWs propagate.

Notice that, by means of eqs. (5.E.21) and (5.E.27) the basic parameters may be completely characterized crossing the angular parameterizations, just as the steps $\boxed{1/2}$ in the analysis scheme presented in Fig. 5.68 suggest.
From this point, the angular parameterizations are used for complete the analysis of wave parameters.

On one hand, for the analysis of the reflection coefficient along the TL, $\rho(l_n)$, in eq. (2.83), φ_γ is fixed, which gives the limits of φ_{Z_0} by means of eq. (5.E.21). Consequenlty, these limits affect the variation of γ_{n1} in eq. (5.E.27), which determines the denormalized propagation constant γ and thus the normalized length l_n in eq. (5.50).
Notice that this normalized length is universal in the sense that it gathers different γ's and l's (physical lengths) under the same value, which makes possible the analysis along the TL and in terms of losses/frequency at the same time. Thus, the analysis of ρ both along the TL and in terms of losses/frequency is limited by means of the angular characterization of φ_{Z_0}.
In this sense, the critical value is the lossless TL in which $\varphi_\gamma = \pi/2$, so the phase of the characteristic impedance is always $\varphi_{Z_0} = 0$. Consequently, γ_{n1} is fixed so the analysis along the TL in terms of losses (only the lossy case is possible in this case) results in a single point in the ρ-plane.
The process of obtaining the reflection coefficient along the TL in terms of losses corresponds with the step $\boxed{3}$ in Fig. 5.68.

On the other hand, once ρ is characterized, the transformations to Z_{n1} and $Y_n 1$ defined in eqs. (5.52) and (5.53), respectively, are given by the linear fractional transformations:

$$Z_{n1} = Z_{0n1}\frac{1+\rho}{1-\rho}, \text{ and} \tag{5.58}$$

$$Y_{n1} = \frac{1}{Z_{n1}} = \frac{1}{Z_{0n1}}\frac{1-\rho}{1+\rho}, \tag{5.59}$$

also respectively.
These transformations are conformal so the graphical analysis facilitates their understading when $\rho(l)$ is parameterized in modulus-phase, for instance, to be transformed to the Z_{n1}- and Y_{n1}-complex planes.
The process of obtaining the wave impedance and wace admittance which leads to the analysis along the TL in terms of losses corresponds with the step $\boxed{4}$ in Fig. 5.68.

5.4.3 Graphical and geometrical analysis

As is it usual in the CTLA, the description of the TL parameters is graphicall addressed transforming the parameterized curves between the associated complex planes.

In this case, the curves in the Z_{0n1}- and γ_{n1}-planes parameterized by angles follow eqs. (5.E.21) and (5.E.27), respectively.
Rigorously, the curves are generaically defined as:

$$\begin{aligned} Z_{0n1}\colon\quad & D_{\varphi_{Z_0}(\varphi_\gamma)} \in \mathbb{R} \to \mathbb{C} \\ & \varphi_{Z_0} \mapsto Z_{0n1,\varphi_\gamma}(\varphi_{Z_0}) = \sqrt{\frac{\sin(\varphi_\gamma - \varphi_{Z_0})}{\sin(\varphi_\gamma + \varphi_{Z_0})}} e^{j\varphi_{Z_0}}, \end{aligned} \tag{5.60}$$

which is the curve of Z_{0n1} characterized by φ_γ whose descriptive parameter is φ_{Z_0}; and

$$\begin{aligned} \gamma_{n1}\colon\quad & D_{\varphi_\gamma(\varphi_{Z_0})} \in \mathbb{R} \to \mathbb{C} \\ & \varphi_\gamma \mapsto \gamma_{n1,\varphi_{Z_0}}(\varphi_\gamma) = \sqrt{\frac{1}{\sin(\varphi_\gamma - \varphi_{Z_0})\sin(\varphi_\gamma + \varphi_{Z_0})}} e^{j\varphi_\gamma}, \end{aligned} \tag{5.61}$$

which is the curve of γ_{0n1} characterized by φ_{Z_0} whose descriptive parameter is φ_γ; which may be directly depicted in the Z_{0n1}- and γ_{n1}-planes, respectively.

The curves in the ρ-plane come from the expression in eq. (5.51). These curves are parameterized by φ_γ, so φ_{Z_0} is the parameter which describes the losses and the TL's length.
For this purpose, the parameter φ_{Z_0} in the curve γ_{n1} in eq. (5.61) changes its role to be the descriptive parameter, whereas φ_γ characterizes the analysis.
This is equivalent to define the bivariate function:

$$\begin{aligned} \boldsymbol{\gamma_{n1}}\colon\quad & D_{(\varphi_{Z_0},\varphi_\gamma)} \in \mathbb{R}^{2+} \subset \mathbb{H} \to \mathbb{C} \\ & (\varphi_{Z_0},\varphi_\gamma) \mapsto \boldsymbol{\gamma_{n1}}(\varphi_{Z_0},\varphi_\gamma) = \sqrt{\frac{1}{\sin(\varphi_\gamma - \varphi_{Z_0})\sin(\varphi_\gamma + \varphi_{Z_0})}} e^{j\varphi_\gamma}, \end{aligned} \tag{5.62}$$

in which is $D_{(\varphi_{Z_0},\varphi_\gamma)}$ coincides with the rg-plane depicted in Fig. 5.67, every points in this plane verifies the "realizability conditions"; and parameterize φ_γ to define the curve:

$$\begin{aligned} \gamma_{n1}\colon\quad & D_{\varphi_{Z_0}} \in \mathbb{R} \to \mathbb{C} \\ & \varphi_{Z_0} \mapsto \gamma_{n1,\varphi_\gamma}(\varphi_{Z_0}) \equiv \boldsymbol{\gamma_{n1}}(\varphi_{Z_0};\varphi_\gamma) \end{aligned}. \tag{5.63}$$

From eq. (5.63), the curve which represents the reflection coefficient is defined as:

$$\begin{aligned} \rho\colon\quad & D_{\varphi_{Z_0}} \in \mathbb{R} \to \mathbb{C} \\ & \varphi_{Z_0} \mapsto \rho = \rho_L e^{-2\gamma_{n1,\varphi_\gamma}(\varphi_{Z_0})\beta_{sp} l} \equiv \rho_L e^{-2\left(e^{j\varphi_\gamma}\right) l_n(\varphi_{Z_0})} \end{aligned}. \tag{5.64}$$

In eq. 5.64 ρ is described along the TL varying l_n, which depends on the angle φ_{Z_0}, while φ_γ is kept fixed. This is equivalent to vary losses in the whole rg-plane. As a result, ρ is being characterized both along the TL and in terms of losses.

In order to obtain the rest of wave parameters, notice that the transformations in eqs. (5.58) and (5.59) are of the same type of those for the direct characterization of wave parameters, [Gag01], described from the ρ-plane to the Z_{n0}-plane and the Y_{n0}-plane.
Taking into account that:

$$Z_{n0} = \frac{Z_{n1}}{|Z_{0n1}|}, \text{ and} Y_{n0} = |Z_{0n1}| Y_{n1}; \tag{5.65}$$

the modulus-phase parameterizations of the curve representing ρ:

$$\begin{cases} m = |\rho| = |\rho_L| e^{-2\cos(\varphi_\gamma) l_n} \\ p = \varphi_\rho = -2\sin(\varphi_\gamma) l_n + \varphi_{\rho_L} \end{cases}; \tag{5.66}$$

transform to the circumferences:

$$\begin{cases} \left(Z'_{n1} - c_0 |Z_{0n1}| \frac{m^2-1}{m^2+1}\right)^2 + \left(Z''_{n1} - s_0 |Z_{0n1}| \frac{m^2-1}{m^2+1}\right)^2 = \left(\frac{|Z_{0n1}| 2m}{m^2-1}\right)^2 \\ \left(Z'_{n1} + \frac{s_0 |Z_{0n1}|}{\tan(p)}\right)^2 + \left(Z''_{n1} - \frac{c_0 |Z_{0n1}|}{\tan(p)}\right)^2 = \left(\frac{|Z_{0n1}|}{\sin(p)}\right)^2 \end{cases}, \tag{5.67}$$

and

$$\begin{cases} \left(Y'_{n1} - \frac{c_0}{|Z_{0n1}|} \frac{m^2-1}{m^2+1}\right)^2 + \left(Y''_{n1} + \frac{s_0}{|Z_{0n1}|} \frac{m^2-1}{m^2+1}\right)^2 = \left(\frac{2m}{|Z_{0n1}|(m^2-1)}\right)^2 \\ \left(Y'_{n1} - \frac{s_0}{|Z_{0n1}|(\tan(p))}\right)^2 + \left(Y''_{n1} - \frac{c_0}{|Z_{0n1}|(\tan(p))}\right)^2 = \left(\frac{1}{|Z_{0n1}|(\sin(p))}\right)^2 \end{cases}, \tag{5.68}$$

in the Z_{n1}- and the Y_{n1}-planes, respectively.
These are the circumferences:

$$\begin{cases} \left(c_0 |Z_{0n1}| \frac{m^2-1}{m^2+1}, s_0 |Z_{0n1}| \frac{m^2-1}{m^2+1}\right) : \frac{|Z_{0n1}| 2m}{|m^2-1|} \\ \left(-\frac{s_0 |Z_{0n1}|}{\tan(p)}, \frac{c_0 |Z_{0n1}|}{\tan(p)}\right) : \frac{|Z_{0n1}|}{|\sin(p)|} \end{cases}, \tag{5.69}$$

and

$$\begin{cases} \left(\frac{c_0}{|Z_{0n1}|} \frac{m^2-1}{m^2+1}, -\frac{s_0}{|Z_{0n1}|} \frac{m^2-1}{m^2+1}\right) : \frac{2m}{|Z_{0n1}||m^2-1|} \\ \left(\frac{s_0}{|Z_{0n1}|(\tan(p))}, \frac{c_0}{|Z_{0n1}|(\tan(p))}\right) : \frac{1}{|Z_{0n1}||\sin(p)|} \end{cases}, \tag{5.70}$$

when they are written using the compact notation of circumferences.
Making use of these transformations from the ρ-plane, the curves in the Z_{n1}-plane and the Y_{n1}-plane can be represented.

An example of the analysis of the TL parameters characterized in terms of angles for the simultaneous charactrization of the TL along its length and interms of losses is next presented.
Accompaniying the example is some numerical data, which serves to see how different lossy and length parameterizations given by this analysis lead to the same TL parameter, which demonstrates its duality for the TL characterization.

5.4.4 Example of the graphical analysis along the TL in terms of losses

Example I

Numerical values of the transmission line parameters

Complete analysis	*Particular case analysis*				
	$\varphi_\gamma = \frac{\pi}{4} - \frac{\pi}{16} = \frac{3\pi}{16}$				
	Particular case numerical values				
	φ_{Z_0}[rad]	(r, g)	$l[\mu\text{m}]$	$\rho(l)$	$Z_{n1}(l)[\Omega]$
$\rho_L = 0.8 + j0.2$	0.000	$(1.497, 1.497)$	3.398	$0.779 + j0.177$	$4.511 + j4.412$
	0.044	$(1.217, 1.836)$	3.344		$3.490 + j4.164$
	0.087	$(1.014, 2.366)$	3.197		$2.678 + j3.857$
	0.131	$(0.821, 3.294)$	2.898		$1.861 + j3.390$
	0.175	$(0.668, 5.031)$	2.463		$1.200 + j2.809$
	0.219	$(0.539, 9.803)$	1.828		$0.651 + j2.041$

Table 5.8: Parameter ρ_L which characterizes the analysis along the TL in terms of losses in this example and some numerical values which characterize the analysis when $\varphi_\gamma = 3\pi/16$.

Complete analysis along the TL in terms of losses

Transmission line scheme and parameterizations

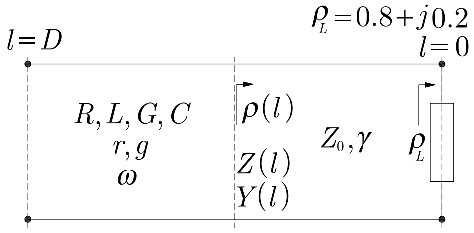


Fig. 5.69: Scheme of the TL for the parameter analysis along the length and in terms of losses.

Inverse characterization in terms of the phase of the propagation constant

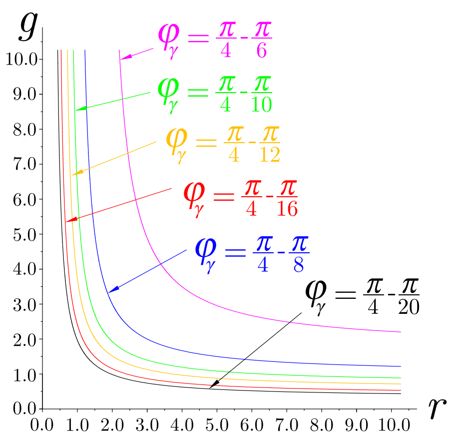
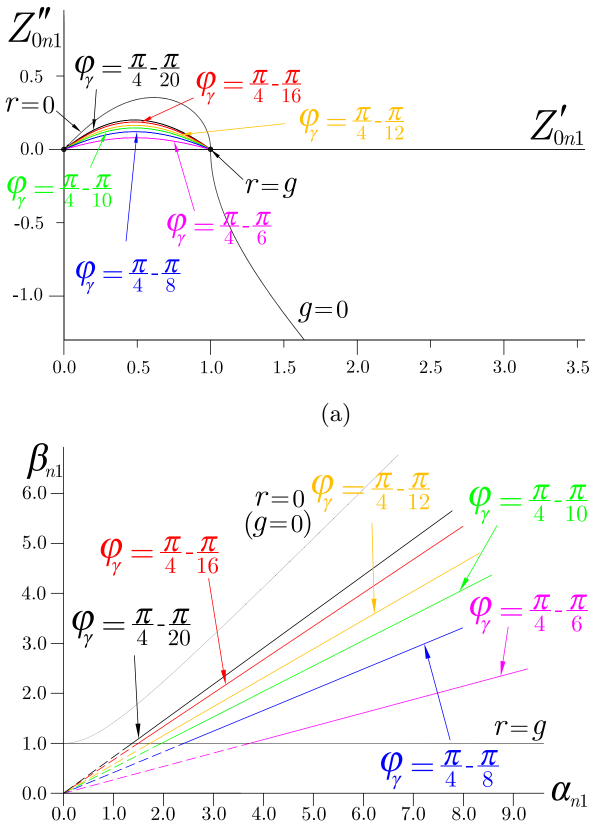


Fig. 5.70: The rg-plane parameterized by φ_γ.



(b)

Fig. 5.71: The (a) Z_{0n1}- and (b) γ_{n1}-planes parameterized by φ_γ.

Direct characterization of wave parameters along the TL in terms of losses

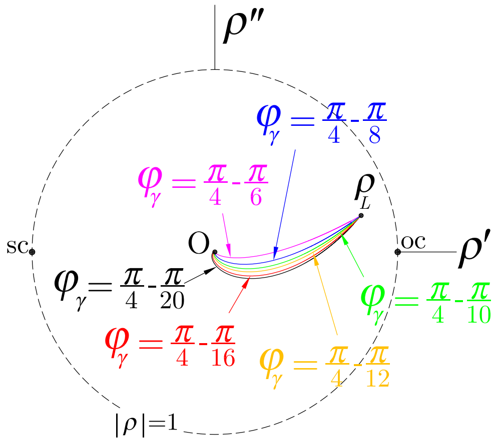
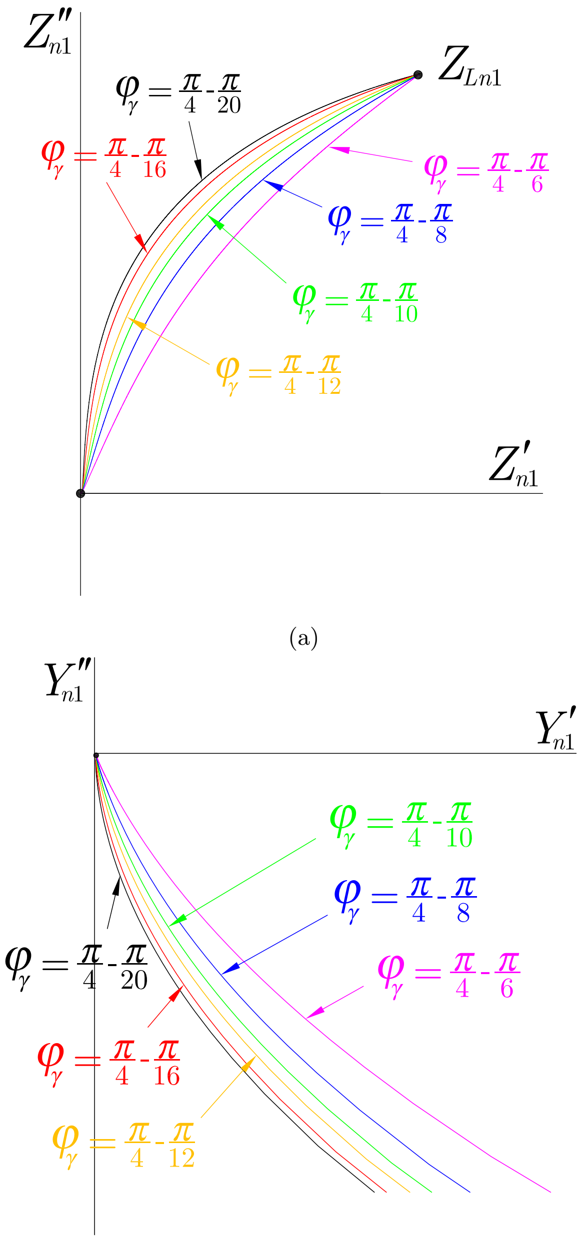


Fig. 5.72: Analysis of ρ along the TL in terms of losses (parameterized in φ_γ).



(b)

Fig. 5.73: Graphical analysis of (a) Z_{n1} and (b) Y_{n1} along the TL and in terms of losses (parameterized in φ_γ.

In Example I, a graphical analysis when $\rho_L = 0.8 + j0.2$ is shown in the context of the CTLA. Moreover, some numerical values which characterize the curve $\varphi_\gamma = 3\pi/16$ have been outlined in Table 5.8.

The angle φ_{Z_0}, which serves to describe losses and the TL's length at the same time, varies along the φ_γ-curves.
Some of those values of (r, g) and l which lead to the same reflection coefficient along the TL are gathered in Table 5.8 together with the wave impedance.
In Fig. 5.71a the Z_{0n1}-plane is parameterized by φ_γ. This analysis is the graphical representation of the curves obtained by means of eq. (5.E.21). This Z_{0n1}-plane together with the rg-plane serve to characterize the TL graphically, under the context of the inverse characterization.
Then, from this inverse charactrization, ρ is a direct transformation of the curves (lines) in the γ_{n1}-plane, given by the exponential in eq. (5.51).
At the end, the conformal transformations from the ρ-plane to the Z_{n1}- and Y_{n1}-planes are carried out by means of the intersection of those circumferences which result from parameterizing the spirals in the ρ-plane.

Notice that the graphical example is presented for those values of $\varphi_\gamma < \pi/4$. That is because the lines in the γ_{n1}-plane parameterized by these φ_γ's tend to inifity, which makes possible represent $\rho \to 0 + j0$ for any lossy TL.
Moreover, since the φ_γ-parameterized curves in the Z_{0n1}-plane pass through $Z_{0n1} = 0 + j0$, then the these curves also passes through $Z_{n1} = 0 + j0$, something which brings important physical consequences.
In addition, the non dispersive point in the Z_{n1}-plane is also common for every φ_γ-parameterized curve, which means that there is a load common for these curves, Z_{Ln1}, which is also obtained when $l = 0$ (and thus the notation using the subscript $_L$, read as "at the load").

Undoubtely, the possibilty of mixing losses and the analysis along the TL in the same analysis makes the analysis more difficult to understand from the physical point of view. Thus, the following interpretations result helpful to this issue.
But, on the other hand, this "shared analysis" suggest new practical uses and applications in which this study may help achieving ans/or interpreting other results.

5.4.5 Physical interpretations

The analysis presented by means of this example answers which TL's –characterized by specific line parameters (losses and frequency included)– lead to the same variation along the length which describes them. This is equivalent to group all the possible lossy TLs in classes, in such a way that each of the values of the parameters that characterize the TL describe multiple TLs at differents lengths.

Taking into account the reduction in the analysis, which is "multiparameterized" in origin, it is obvious that the physical interpretations are not as direct as in previous examples, but also in favour of the compaction in the graphical analysis.
In this sense, the analysis presented by means of the rg-plane results crucial when supporting graphically the analysis in the rest of the planes for its physical interpretation. The analysis of the curves parameterized by φ_γ meets in the graphical analysis in the rg-plane the possible combinations of lossy/frequency parameterizations which present the same variation of the rest of parameters along the TL.

This analysis is clearly limit by the physical interpretation of the parameters involved in the analysis. For example, while the basic parameters are the same for each point along the TL, the reflection coefficient is not. Conversely, this means that each point in the basic parameter complex planes admits any parameterization of the length. However, each point in the wave parameter complex planes does not admit any parameterization of length. Just as the possible values of the lossy parameterizations are given by the inverse characterization in the rg-plane, those possible values of the TL's length are given by the analysis in any plane which represents γ (in the example presented in this section the γ_{n1}-plane) and in particular the parameterizations with φ_γ constant. One immediate example which serves to prove that not every point in the ρ-plane parameterizes any length is the reflection coefficient at the load. At the load, there is no possible combination of losses that makes the modulus of the propagation constant zero, so none of them makes the normalized length, l_n, zero, because $|\gamma_{n1}| \geq 1$. As a result, the value of the (fixed) reflection coefficient at the load only is achieved at the load (which is trivial indeed, but it proves the statment regarding the limits of this analaysis).

There is an additional physical limitation to this analysis, which is about the "realizability" of fixing a reflection coefficient (which has no physical meaning in itself) at the load, and keep it non dependent on the characteristic impedance, for instance.
The "trick" of this analysis is precisely on varying accordingly the wave impedance at any point along the TL for obtaining the ρ at this point together with the characteristic impedance parameterized by certain losses. In this way, also the characteristic impedance would be referred at certain point (besides the losses) when looking for a specific reflection coefficient when the impedance is fixed at this point. This is because the effects of varying the TL's length are translated to variations on losses, and so changes on the characteristic impedance which would be able to change along the TL in certain way under this interpretation.

As it may be seen, this analysis may have multiple interpretations depending on both the parameter which is analyzed and those which parameterize it. Depending on the practical uses for which this analysis is intended to, one viewpoint or another will be chosen.

5.4.6 Practical uses

The analysis presented by means of this example gathers both the analysis in terms of losses/frequency and the analysis along the TL. In this way, one analysis or the other can be analyzed under the same context. For example, if the losses of the TL are fixed, the γ (and thus the φ_γ) is fixed, and the analysis along the TL is found in the ρ-plane, over the curve characterized by φ_γ. Otherwise, if losses are variable, the analysis changes between the parameterized curves. Thus, there is a duality in the analysis when seeing on one hand, the analysis in terms of losses and, on the other hand, the analysis along the TL.

This analysis reveals an interesting property of lossy TLs: each point in the ρ-plane is not uniquely parameterized by one pair of losses-length, but infinite. This means that it is possible to think about a TL which keeps the reflection coefficient along the TL when both losses and the wave impedance vary in certain way (see the numerical example in Table 5.8, for instance).
Conversely, imagine a TL in which the pair (r, g) varies along the TL in certain way (for example imagine that the dielectric varies along the TL's length). Then, there is a ρ constant along TL and a specific function $Z_{n1}(l)$ which leads to that profile of losses along the TL. As a result, this analysis can be used for characterizing non homogeneous lossy TLs.

Finally, the limitations to this analysis along the TL in terms of losses suggest looking for new parameterizations of losses that avoid them. Since φ_γ is the parameter which characterizes the losses along the TL, these new parameterizations of losses must keep this angle constant.
These useful parameterizations are found in complex numbers.

5.5 Conclusions

In this chapter the analysis which the CTLA of HPWs had been left in previous chapter for finally establishing the CTLT-v1 has been carried out as examples of use of the analysis presented in the previous chapter. In this way, the CTLT-v1 is complete. Here a summary of what the CTLT-v1 means is given outlining the differences with the classic analytical procedure:
Recall that this version of the CTLT (CTLT-v1) refers to the study of HPWs (those are a type of TEM modes) in lossy media in the *frequency domain* by means of equivalent TL in which the influence of arbitary losses is fully considered. The underlying Theory which comes to be reinterpreted is the LTLT (presented in Chpt. 2 and obtained particularized from the GTLT-v1 presented in Chpt. 3).
With this objective of offering an alternative viewpoint of the LTLT, the analysis and the subsequent physical interpretations of them have been carried out in this chapter together with the "basic" characterizatios in Chpt. 4.
Despite the analytical expressions of the paramters that characterize the lossy TL have been obtained in the LTLT, a thorough analysis has been possible thanks to the CTLT-v1 culminated in this chapter.
In this sense, it is important to change the conception of the analysis. For this purpose, some equivalences between the classical analysis and the complex analysis of TLs are given as examples for the right understanding of this Theory: (i) when giving the expressions of the paremeters along a specific TL (those $\rho(l)$, $Z(l)$, and $Y(l)$ when Z_0 and γ are fixed), they have to be understood as in Ex. 01; (ii) when giving the expressions of the TL parameters at the load (those ρ_L, Z_0, and γ, when Z_L and Y_L are fixed), they have to be understood as in Ex. 02; and (iii) when the expressions are set in general form (those Z_0 and γ as well as $\rho(l)$, $Z(l)$, and $Y(l)$) the corresponding general analysis is in Ex. 03.
In any case, it is important to detach that all the planes regarding the parameters involved in the study of the TLs have to be understood together as a part of the methodology of the CTLT based on CTLA. Moreover, that planes in which the associated parameters is a point fix the type of analysis that is being done, for example in each of the examples in Ex. 01 the basic parameters are represented by a single point because the TL is known, whereas in Ex. 02 the impedance and admittance are the planes in which the represented points parameterize the analysis.

Remark 45. *When giving the solutions of a TL-related problem (in this case, the HPWs) in the context of the CTLT (in this case, the CTLT-v1), those do not have to be understood as an isolated value or function but a points (if the analysis is parameterized) or, in general, curves in the planes associated to the TL parameters, all of them visualized together.*

This viewpoint lets to see all the planes interconnected. In fact, they are by the transformation between them studied in the CTLT-v1.

Remark 46. *Is not possible to see planes associated to each parameter in the CTLT disconnected. Each point in the allowed regions of any plane has an image in another plane. This justifies the transformations between the TL parameters.*

This graphical and geometrical conception of the interconnected planes lets to see the influence of the parameters in the analysis, something that is absolutely non trivial when analyzing the mathematical expressions individually.
In this sense, some interesting conclusions may be obtained from the CTLT-v1 viewpoint: the angle of the propagation constant, φ_γ determines the analysis along the TL, so the influence of the TL's length, whereas the angle of the characteristic impedance, φ_{Z_0}, parameterizes the losses. Thus, the use of the angles of basic parameters serve to characterize the TL along its length and in terms of its losses and frequency.

These angles have inverse representation in terms of lossy parameterizations, which reveals the rg-plane (and in general the "space of parameterizations") as the domain from which all the possible parameterizations (the TL's length included) come from.
In this way, the subsequent versions of the CTLT will use the same "domain" for parameterizing the solutions and analyzing them by means of the CTLA following the same steps of the examples presented in this chapter.

Chapter 5

Appendices

Appendix 5.A

The Cauchy-Riemann equations,[BC90], are neccesary conditions for any complex variable function to be analytic. Since these equations are necessary (but not sufficient) to fulfilled for a function to be continuous, conversely they can be used for determining the values in which any function is not differentiable, and so exploring the function singularities.
Remember that the singularties of $Z(l)$ and $Y(l)$ come from the denominator so this may be is isolated in a fuction $D(l)$:

$$D^{\mp}(l) = \frac{1}{1 \mp \rho(l)} = \frac{1}{1 \mp \rho_L e^{-2\gamma l}}$$

for the $Z(l)$ and $Y(l)$ cases, respectively. Thus, the singularities of $D^{\mp}(l)$ are the singularities of $Z(l)$ and $Y(l)$.

The function $D(l)$ is then complexified as:

$$\begin{aligned} D(l) \to D : \mathbb{C} &\to \mathbb{C} \\ \gamma &\to D(\gamma; l). \end{aligned}$$

In abuse of notation, the function denoted as $D^{\mp}$ is also in this complexified version, but now it differs on the complex variable. The original length variable l is used as a parameter in the complexified version.
Separating the real and imaginary parts denoted as $u(\gamma) \equiv u(\alpha, \beta)$ and $v(\gamma) \equiv v(\alpha, \beta)$ (both parameterized by the real parameter l), respectively:

$$\begin{aligned} u(\alpha, \beta) &= \frac{1 \mp |\rho_L| e^{-2\alpha l} \cos(2\beta l - \varphi_{\rho_L})}{1 \mp 2|\rho_L| e^{-2\alpha l} \cos(2\beta l - \varphi_{\rho_L}) + |\rho_L|^2 e^{-4\alpha l}}, \\ v(\alpha, \beta) &= \mp \frac{|\rho_L| e^{-2\alpha l} \sin(2\beta l - \varphi_{\rho_L})}{1 \mp 2|\rho_L| e^{-2\alpha l} \cos(2\beta l - \varphi_{\rho_L}) + |\rho_L|^2 e^{-4\alpha l}}, \end{aligned}$$

the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial u(\alpha, \beta)}{\partial \alpha} &= \frac{\partial v(\alpha, \beta)}{\partial \beta} \\ \frac{\partial u(\alpha, \beta)}{\partial \beta} &= -\frac{\partial v(\alpha, \beta)}{\partial \alpha}, \end{aligned}$$

have to be fulfilled at those points at which $D(\gamma; l)$ is differentiable and therefore continuous. Thus, the points that not verify the Cauchy-Riemann conditions are singularities.

After operating, every (α, β) verifies the Cauchy-Riemann equations for every l except to those points in which $|D(\gamma; l)|$ is not defined, which happens when $\rho(l) = \pm 1$.
As a consequence, only the lossy TLs that passes throught the open circuit (**oc**) and the short circuit (**sc**) presents singularties in $Z(l)$ and $Y(l)$, respectively.
Although this seems to be a trivial result it will bring important conclusions: only the lossy TLs which are loaded by a (**oc**) or a (**sc**) present that (**oc**) or (**sc**) at the load. Otherwise, the (**oc**) or (**sc**) can not be achieved at any point along the TL, something that is physically explainable.

Appendix 5.B

Here the geometrical properties of the (logarithmic-type) spiral in eq. (5.19) which represents the reflection coefficient along the TL are analyzed.

The expression in eq. (5.19) may be rewritten as:

$$|\rho| = |\rho_L| e^{-\frac{\alpha}{\beta}\varphi}, \quad \text{in which } \varphi = \varphi\rho_L - \varphi_\rho \in [0, 2\beta D], \tag{5.B.1}$$

being D the physical length of the TL.

The tangential angle (denoted by $\phi(\varphi)$) of the logarithmic spiral is differentially defined as $\phi(\varphi) = \varphi$, [Law72].

The length of the spiral (denoted as $s(\varphi)$, measured from the load: $\varphi = 0$) is:

$$s(\varphi) = |\rho_L| \left(\sqrt{\left(\frac{\beta}{\alpha}\right)^2 + 1} \right) \left(1 - e^{-\frac{\alpha}{\beta}\varphi}\right), \tag{5.B.2}$$

so the total length of the curve which represents ρ along the TL is $L = |\rho_L| \left(\sqrt{\left(\frac{\beta}{\alpha}\right)^2 + 1} \right) \left(1 - e^{-2\alpha D}\right)$.
Notice that if $\alpha \to \infty$ the spiral degenerates to the radious which goes from the origin to $|\rho| = |\rho_L|$, so the length of the spiral is, accordingly to definition of the arc length, $s(\varphi) = |\rho_L|$.

The curvature (denoted as $\kappa(\varphi)$) is defined as:

$$\kappa(\varphi) = \frac{d\phi(\varphi)}{ds(\varphi)} \equiv \frac{\frac{d\phi(\varphi)}{d\varphi}}{\frac{ds(\varphi)}{d\varphi}} = \frac{e^{\frac{\alpha}{\beta}\varphi}}{|\rho_L| \left(\sqrt{\left(1 + \frac{\alpha}{\beta}\right)^2} \right)}. \tag{5.B.3}$$

The most important property of the spiral in eq. 5.B.1 is that th angle between the radious and the tangent is constant:

$$\Psi = \tan^{-1}\left(\frac{\rho}{-\frac{d|\rho|(\varphi)}{d\varphi}} \right) = \frac{1}{\alpha/\beta} = \frac{\beta}{\alpha}, \tag{5.B.4}$$

which means that this angle is the phase of the propagation constant:

$$\Psi \equiv \varphi_\gamma. \tag{5.B.5}$$

As it may be foreseen, this latter geometrial result will bring many interpretations to the graphical and physical characterization of the wave parameters along the TL, as well as the possibility of using the angle of the propagation constant for parameterizing the TL in future analysis.

Appendix 5.C

In this section, a couple of points for locating the circumferences in the Z_{n0}- and Y_{n0}-planes, which are the transformations from the radious in the ρ-plane, are found.

On one hand, the transformation from the Z_{n0}-plane to the ρ-plane is given by the following expression:

$$\rho = \frac{Z_{0n} - e^{\varphi z_0}}{Z_{0n} + e^{\varphi z_0}}. \tag{5.C.6}$$

Two points for any of the radious in the (extended) ρ-plane[8] are $\rho = 0$ and $\rho = \infty$. These points correspond to the points $Z_{n0} = e^{\varphi z_0}$ and $Z_{n0} = -e^{\varphi z_0}$, which are the zero and the pole of the expression in eq. (5.C.6).
Thus, these points belong to the circunference which is the transformation of any of the radious in the ρ-plane to the Z_{n0}-plane.

On the other hand, the transformation from the Y_{n0}-plane to the ρ-plane is given by the following expression:

$$\rho = \frac{1 - Y_{n0} e^{\varphi z_0}}{1 + Y_{0n} e^{\varphi z_0}}. \tag{5.C.7}$$

Two points for any of the radious in the (extended) ρ-plane are $\rho = 0$ and $\rho = \infty$. The points $\rho = 0$ and $\rho = \infty$ correspond to the points $Y_{n0} = e^{-\varphi z_0}$ and $Y_{n0} = -e^{-\varphi z_0}$ in the Y_{n0}-plane, which are the zero and the pole of the expression in eq. (5.C.7).
Thus, these points belong to the circunference which is the transformation of any of the radious in the ρ-plane to the Y_{n0}-plane.

As a consequence, the center of the circumference which represents the transformation of the radious is on the bisector of the segment which goes from the solved points in the Z_{n0}- and the Y_{n0}-planes.

[8]The extended ρ-plane refers to the extended complex plane $\bar{\mathbb{C}} \equiv \mathbb{C} \cup \infty$.

Appendix 5.D

In this section, the location of the reflection coefficient which corresponds to the non dispersive case, $\rho_{\rm nd}$ (labeled as (**nd**) in the Z_{0nL}-plane and the ρ-plane), is solved by combining geometrical characterizations and some basic calculus.
It shoud be that this non dispersive case includes the lossless case, and also the low losses apprxi-mation. Moreover, $\rho_{\rm nd}$ is the asymptotic value of ρ when $\omega \to \infty$.

The point (**nd**) in the ρ-plane is given by its phase, $\varphi_{\rho_{\rm nd}}$, and the modulus, $|\rho_{\rm nd}|$.
Lets start with solving the phase: $\rho_{\rm nd}$ is the result of:

$$\rho_{\rm nd} = \frac{Z_L - Z_{0,nd}}{Z_L + Z_{0,nd}} = \frac{\frac{Z_L}{|Z_0|} - 1}{\frac{Z_L}{|Z_0|} + 1} \equiv \frac{Z_{Ln0} - 1}{Z_{Ln0} + 1}, \tag{5.D.8}$$

when using the normalization of the impedance at the load, Z_{Ln0}, of the direct characterization of wave parameters.
Graphically, the operation in eq. (5.D.8) in the Z_{Ln0}-plane:

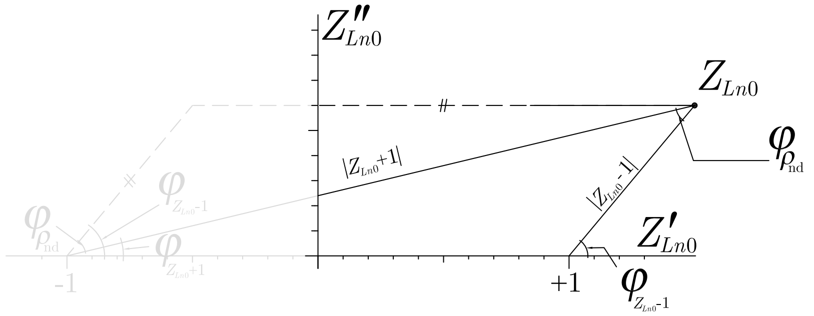


Fig. 5.74: Graphical representation of the geometrical analysis concerning angles and modulus involved in the obtaining of $\rho_{\rm nd}$ in the Z_{0nL}-plane.

Geometrically analyzing the figure above, it leads to the seeked angle :

$$\varphi_{\rho_{\rm nd}} = \varphi_{Z_{Ln0}-1} - \varphi_{Z_{Ln0}+1} = \tan^{-1}\left(\frac{2Z''_{Ln0}}{|Z_{Ln0}|^2 - 1}\right) \equiv \tan^{-1}\left(\frac{2Z''_L}{|Z_L|^2 - Z^2_{0,sp}}\right). \tag{5.D.9}$$

The modulus of $\rho_{\rm nd}$ is obtained by the intersection of the line whose slope is $\varphi_{\rho_{\rm nd}}$ in eq. (5.D.9) and the arc of the circunference in eq. (5.46) which is the limit in the ρ-plane and contains the (**nd**):

$$\begin{cases} \rho'' = \tan(\varphi_{\rho_{\rm nd}})\rho' \\ \rho'^2 + \left(\rho'' - \frac{1}{\tan(\varphi_{Z_L})}\right)^2 = \frac{1}{\sin(\varphi_{Z_L})^2} \end{cases}. \tag{5.D.10}$$

Substituying ρ'' of the line into the eqution of the circunference and operating algebraically, it leads to solve $\rho'_{\rm nd}$:

$$\rho'_{\rm nd} = \frac{\cos(\varphi_{\rho_{\rm nd}})}{\tan(\varphi_{Z_L})}\left(\sin(\varphi_{\rho_{\rm nd}}) + \sqrt{\tan^2(\varphi_{Z_L}) + \sin^2(\varphi_{\rho_{\rm nd}})}\right). \tag{5.D.11}$$

Moreover, since:

$$\rho'_{\rm nd} = |\rho_{\rm nd}|\cos(\varphi_{\rho_{\rm nd}}), \tag{5.D.12}$$

the modulus of the reflection coefficient for the non dispersive case results:

$$|\rho_{\mathrm{nd}}| = \frac{1}{\tan(\varphi_{Z_L})}\left(\sin(\varphi_{\rho_{\mathrm{nd}}}) + \sqrt{\tan^2(\varphi_{Z_L}) + \sin^2(\varphi_{\rho_{\mathrm{nd}}})}\right), \tag{5.D.13}$$

written as a function of the solved $\varphi_{\rho_{\mathrm{nd}}}$.

Appendix 5.E

The phase of the characteristic impedance and the phase of the propagation constant can be written in terms of the phases of the complex n and d defined in eqs. (4.5) and (4.6) as:

$$\varphi_{Z_0} = \frac{1}{2}\varphi_n - \frac{1}{2}\varphi_d, \text{ and} \tag{5.E.14}$$

$$\varphi_\gamma = \frac{\pi}{2} + \frac{1}{2}\varphi_n + \frac{1}{2}\varphi_d, \tag{5.E.15}$$

respectively.

Adding φ_{Z_0} and φ_γ in eqs. (5.E.14) and (5.E.15), it is possible to solve φ_n as:

$$\varphi_n = \varphi_{Z_0} + \varphi_\gamma - \frac{\pi}{2}. \tag{5.E.16}$$

Reciprocally, if substracting eqs. (5.E.14) and (5.E.15), it leads to solve φ_d as:

$$\varphi_d = \varphi_\gamma - \varphi_{Z_0} - \frac{\pi}{2}. \tag{5.E.17}$$

Looking at the expressions in eqs. (5.E.16) and (5.E.17), two important inequalities useful for the TL analysis may be deduced taking into account that $\pi/2 < \varphi_n\ (\varphi_d) \leq 0$:

$$0 < \varphi_{Z_0} + \varphi_\gamma \leq \frac{\pi}{2}, \tag{5.E.18}$$

$$-\frac{\pi}{2} \leq \varphi_{Z_0} - \varphi_\gamma < 0, \tag{5.E.19}$$

which are "realizability conditions" for TLs in which HPWs propagate.

On one hand, the modulus of the normalized characteristic impedance in eq. (4.3) can be written in terms of both the phase of the complex numbers n and d as:

$$|Z_{0n1}| \equiv \sqrt{\frac{\cos(\varphi_d)}{\cos(\varphi_n)}}. \tag{5.E.20}$$

Now substituying φ_n and φ_d in eq. (5.E.20) before, it leads to expression of the modulus of Z_{0n1}:

$$|Z_{0n1}| \equiv \sqrt{\frac{\cos(\varphi_\gamma - \varphi_{Z_0} - \frac{\pi}{2})}{\cos(\varphi_\gamma + \varphi_{Z_0} - \frac{\pi}{2})}} = \sqrt{\frac{\sin(\varphi_\gamma - \varphi_{Z_0})}{\sin(\varphi_\gamma + \varphi_{Z_0})}}. \tag{5.E.21}$$

Notice that the "realizability conditions" guarantee the arguments of the cosines in eq. (5.E.21) both are in $]-\pi/2, 0]$, which means that the equivalent sines are both negative and so their signs cancel.

Furthermore, the closed analytical expression of modulus $|Z_{0n1}|$ in terms of the angles φ_{Z_0} and φ_γ, (5.E.21) is intended to be used for analyzing the characteristic impedance when φ_γ is seen as a parameter. Thus, it is required knowning in which range φ_{Z_0} varies when parameterizing φ_γ. For this purpose, φ_{Z_0} is solved from the parametric eq. (5.E.16) leading

$$\varphi_{Z_0} = \frac{\pi}{2} + \varphi_n - \varphi_\gamma = \frac{\pi}{2} - \tan^{-1}(r) - \varphi_\gamma. \tag{5.E.22}$$

The next step is to relate the range in which the parameter r varies to φ_γ when seeing this a parameter.

The inverse characterization of line parameters parameterized by φ_γ facilitates this purpose. Recalling the parametric equation of r obtained in eq. (4.58):

$$r = \frac{g - \tan(\pi - 2\varphi_\gamma)}{g\tan(\pi - 2\varphi_\gamma) - 1} = \frac{g + \tan(2\varphi_\gamma)}{g\tan(2\varphi_\gamma) - 1}; \tag{5.E.23}$$

two different cases when analyzing the limit of r may be distinguished:

$$\begin{cases} r \in [0, \tan(\pi - 2\varphi_\gamma)] & \text{if } \varphi_\gamma \geq \frac{\pi}{4} \\ r \in \left(-\frac{1}{\tan(\pi - 2\varphi_\gamma)}, \infty\right) \equiv \left(-\tan\left(2\varphi_\gamma - \frac{\pi}{2}\right), \infty\right) & \text{if } \varphi_\gamma < \frac{\pi}{4} \end{cases}. \tag{5.E.24}$$

Substituying the r limits for each case of eq. (5.E.24) into eq. (5.E.22), it leads to the limits of φ_{Z_0}:

$$\begin{cases} \varphi_\gamma - \frac{\pi}{2} \leq \varphi_{Z_0} \leq \frac{\pi}{2} - \varphi_\gamma & \text{if } \varphi_\gamma \geq \frac{\pi}{4} \\ -\varphi_\gamma < \varphi_{Z_0} < \varphi_\gamma & \text{if } \varphi_\gamma < \frac{\pi}{4} \end{cases}. \tag{5.E.25}$$

The modulus of Z_{0n1} seen as function of the angle φ_{Z_0} and parameterized by the angle φ_γ in eq. (5.E.21), together with the limits for φ_{Z_0} also parameterized by φ_γ, leads to complete the mathematical characterization of the characteristic impedance when the phase of the propagation constant is fixed.
Since the normalization of the characteristic impedance is the same for both the *ffa* and the *vfa*, the curves on the Z_{0n1}-plane parameterized by φ_γ would be exactly the same as in the Z_{0n2}-plane.

On the other hand, the modulus of the normalized propagation constant in eq. (4.10) can be written as:

$$|\gamma_{n1}| \equiv \sqrt{\frac{1}{\cos(\varphi_d)\cos(\varphi_n)}}. \tag{5.E.26}$$

Substituying φ_n and φ_d in eq. (5.E.26), it leads to:

$$|\gamma_{n1}| \equiv \sqrt{\frac{1}{\cos(\varphi_\gamma - \varphi_{Z_0} - \frac{\pi}{2})\cos(\varphi_\gamma + \varphi_{Z_0} - \frac{\pi}{2})}} = \sqrt{\frac{1}{\sin(\varphi_\gamma - \varphi_{Z_0})\sin(\varphi_\gamma + \varphi_{Z_0})}}. \tag{5.E.27}$$

The expression of modulus $|\gamma_{n1}|$ in terms of the angles φ_γ and φ_{Z_0}, (5.E.27) is intended to be used for analyzing the propagation constant when φ_{Z_0} is seen as a parameter. Thus, it requires knowing in which range φ_γ varies when φ_{Z_0} is parameterized. For this purpose, φ_γ is solved from the parametric eq. (5.E.16), for instace, leading to:

$$\varphi_\gamma = \frac{\pi}{2} + \varphi_n - \varphi_{Z_0} = \frac{\pi}{2} - \tan^{-1}(r) - \varphi_{Z_0}. \tag{5.E.28}$$

The inverse characterization of line parameters in terms of the phase φ_{Z_0} gives the range in which r in eq. (5.E.22) may vary. Recalling the expression of r parametrized by φ_{Z_0} introduced in eq. (4.55):

$$r = \frac{g - \tan 2\varphi_{Z_0}}{1 + g\tan 2\varphi_{Z_0}}. \tag{5.E.29}$$

Two different cases may be distinguished:

$$\begin{cases} r \in \left[0, \frac{1}{\tan(2\varphi_{Z_0})}\right) & \text{if } \varphi_{Z_0} < 0 \\ r \in [-\tan(2\varphi_{Z_0}), \infty) \equiv \left(-\tan\left(2\varphi_\gamma - \frac{\pi}{2}\right), \infty\right) & \text{if } \varphi_{Z_0} \geq 0 \end{cases}. \tag{5.E.30}$$

Substituying the r limits for each case of eq. (5.E.30) into eq. (5.E.28), it leads to the limits of φ_γ:

$$\begin{cases} \varphi_{Z_0} < \varphi_\gamma \leq \varphi_{Z_0} + \frac{\pi}{2} & \text{if } \varphi_{Z_0} < 0 \\ \varphi_{Z_0} < \varphi_{Z_0} \leq \frac{\pi}{2} - \varphi_{Z_0} & \text{if } \varphi_{Z_0} \geq 0 \end{cases}. \tag{5.E.31}$$

The limits of φ_γ together with the expression of the modulus in eq. (5.E.27), both paramterized by the angle of the characteristic impedance, lead to the analysis of the propagation constant when the phase of φ_{Z_0} is fixed.
This analysis has been carried out when the normalization regarding the propagation constant is considering the frequency fixed. From this analysis, it is relatively easy to generalize to the variable frequency case. Notice that while the angle of every normalizations of γ is the same, the modulus is a transformation depending on this angle and the parameterized φ_{Z_0}. In this way, the modulus of the propagation constant when frequency is variable:

$$\begin{aligned} |\gamma_{n2}| =& \frac{|\gamma_{n1}|}{r} = \tan(\varphi_\gamma + \varphi_{Z_0}) \sqrt{\frac{1}{\sin(\varphi_\gamma - \varphi_{Z_0}) \sin(\varphi_\gamma + \varphi_{Z_0})}} = \\ =& \frac{1}{\cos(\varphi_\gamma + \varphi_{Z_0})} \sqrt{\frac{\sin(\varphi_\gamma + \varphi_{Z_0})}{\sin(\varphi_\gamma - \varphi_{Z_0})}} \equiv \frac{1}{\cos(\varphi_\gamma + \varphi_{Z_0})} \frac{1}{|Z_{0n1}|}, \end{aligned} \tag{5.E.32}$$

for which the limits in (5.E.31) have been taken into account, and r has been written in terms of the angles using eq. (5.E.16).

The graphical analysis of these mathematical results lets to explain them in an intuitive way as well as showing the usefulness of the inverse analysis of line parameters.

Part III

Applications

Applications

The following sections refer to the possible applications of the CTLA to solve TL-related problems at the same time they complete the analysis based on complex parameterized transformations. The applications are splitted into four fields of use:

(i) the graphical tools which arise from the TL comlex analysis, just as they have been presented within the context of the CTLT but detaching the practical uses and how to work with them;

(ii) the extension of the parameterizations to complex values for different purposes: "simulating" the TL's length, parameterize higher modes, etc.;

(iii) the analysis of non trivial and non pure analytical EM problems in waveguides, combining numerical analysis; and

(iv) the exemplification the CTLT supposes to the GSST, and the possible contributions to future versions, mainly focused on finally achieving Complex GSST.

These application may be object of study in future research, as well as they have been/may be scientific contributions.

5.6 Graphical tools based on the CTLA

In this section, the use of the most representative graphical analysis in the CTLA are outlined while emphasizing their practical purposes and the advantage they have in comparison with analytical and nummerical techniques.

5.6.1 The logarithmic reparameterization of the GSC: the log-GSC

Recalling the analysis of the TL parameters along the TL's length, the logarithmic spirals are the curves that represent the reflection coefficient in this variable. Thus, a logarithmic reparameterization of the reflection coefficient along the TL presented in [VG17-I][9] serves to transform these spirals into lines making the analysis much more affordable in graphs.

Following the steps in the analysis, the graphical representation of the logarithmic of ρ, denoted as $\rho_{\log}$, in one of the branch of the complex logarithmic function, leads to the $\rho_{\log}$-plane. In this plane, the representation of the transformation of the real-imaginary parts from the normalized impedance leads to the curves that sketches the log-GSC.

In the $\rho_{\log}$-plane, the reflection coefficient along the TL is represented by means of lines whose slope is directly related to the angle of the propagation constant. As a consequence, this plane is really useful to connect points along the TL in an easy way, and so matching loads using lossy TLs, [VG17-I, VG17-III].
Moreover, the representation of the impedance along the TL is a different spiral analytically represented by the complex cotangent function taking its argument on the $\rho_{\log}$-plane, and so a complex transfroamtions from this plane given by this well-known complex function.

[9]This paper is available online in http://www.jpier.org/PIERL/pierl68/08.17022009.pdf.

5.6.2 Using the direct and inverse characterizations in the ρ-plane combined for analyzing the lossy TL

In this section, the fundamentals of combining the direct and inverse characterizations in the ρ-plane for analyzing lossy TL's in terms of both the TL's length and frequency are outlined. The analysis presented here reduces to graphical operations in the ρ-plane, so it is about of extending the use of the GSC using the direct and inverse characterizationn together. This possibility is due to the reflection coefficient is invariant when representing different normalizations of wave and basic parameters, and thus the asociated plane adquires the "universal" nature that makes it the most appropriate when combining different analysis.
A very simple but representative example of this analysis will be analyzed graphically.

Mathematical analysis

Recall the normalizations chosen in the direct and inverse normalizations of wave parameters and basic parameters, respectively.
On one hand, the direct characterization of the wave impedance and admittance along the lossy TL supposes Z_0 is fixed an so the normaliations are $Z_{n0} = Z/|Z_0| \equiv Z(l)/|Z_0|$ and $Y_{n0} = Y|Z_0| \equiv Y(l)|Z_0|$, respectively. In this case, the angle of the characteristic impedance, φ_{Z_0}, determines the analysis along the TL.
On the other hand, the inverse characterization of the characteristic impedance in terms of frequency suppoes the load Z_L fixed and so the normalization used is $Z_{0nL} = Z_0/|Z_L| \equiv Z_0(\omega)/|Z_L|$. In this case, the angle of the impedance at the load, φ_{Z_L}, determine the analysis in terms of frequncy when the losses of the TL are known. Also notice that the notation Z_L (and φ_{Z_L} is only a name to refer the impedance at fixed point along the TL, ans so it may be extended to any Z (and φ_Z).

In the ρ-plane, both the real-imaginary parameterized parts and modulus-phase paramterizations of the wave parameters in the direct characterization and the characteristic impedance in the inverse characterization, are circumferences that have been characterized in their respective analysis.
When comparing the normalized parameters it may be seen that:

$$|Z_{0nL}| = |Y_{Ln0}|, \tag{I}$$

and so the modulus parameterizations of the characteristic impedance for the inverse analysis are mapped onto the same curves as the modulus parameterizations of the wave admittance for the direct characterization.
Furthermore, the curves which parameterize the phase of the wave impedance in the direct characterization (determined by φ_{Z_0}) are the same as those parameterized by the phase of the characteristic impedance in the inverse characterization (determined by φ_{Z_L}). They follow the circumferences:

$$\left(0, \frac{1}{\tan\left(\varphi_{Z_0} - \varphi_{Z_L}\right)}\right) : \frac{1}{|\sin\left(\varphi_{Z_0} - \varphi_{Z_L}\right)|}, \tag{II}$$

for which φ_{Z_0} and φ_{Z_L} are parameterizations which depend on either frequency or the TL's length, respectively.

As it has been outlined, the ρ-plane lets stu study both analysis at the same time without rescaling the reflection coefficient. In addition, because of the identity between the modulus and phase parameterizations of the normalized characteristic impedance and the normalized admittance, these parameterizations are "universal" and very useful for the analysis in terms of frequency and along the TL carried out at the same time.
When it is required to sum series impedances or parallel admittances, the modulus-phase paramterizations should swith to the real-imaginay parameterized parts in the ρ-plane, that is, the GSC.

Example of graphical analysis using direct and inverse characterizations

The following TL scheme represents the example to be solved by using the direct and inverse characterizations in the ρ-plane:

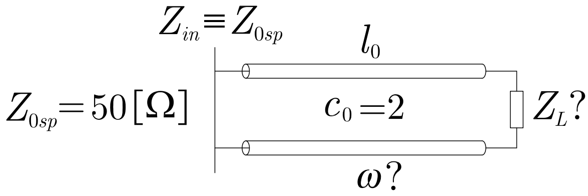


Fig. I: Scheme of the TL-related example solved using graphical analysis.

It is about finding the load Z_L separated $l_0 = 0.1[\mu\text{m}]$ from a lossless medium characterized by $Z_{0,sp}$ which provides matching in this lossless medium. In this case, the frequency is not specified, but the lossy ratio of the lossy TL which separates the lossless medium and the load is $c = 2$. The phase velocity in the lossless medium is $v_p = 10^7[\text{m}\cdot\text{s}^{-1}]$.

The reflection coefficient at the boundary between the lossless medium and the lossy TL, ρ_{in}, analyzed in terms of losses when the impedance at this input, Z_{in}, is the desired $Z_{0,sp} = 50[\Omega]$, so $Z_{in} \equiv Z_{0,sp}$, is analyzed as in Ex. 02 presented in Chpt. 5:

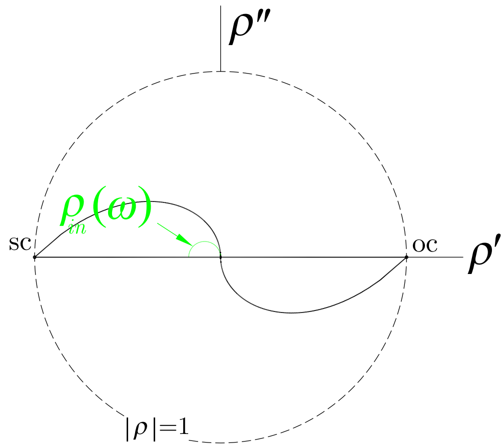


Fig. II: Variable frequency analysis of ρ_{in} when $Z_{in} = Z_{0,sp} = 50[\Omega]$.

In Fig. II, the analysis of the reflection coefficient at the boundary in terms of frequency, $\rho_{in}(\omega)$ is represented. Since there are no restrictions in the frequency of operation, $\omega_0 = 1[\text{MHz}]$ is selected, for instance, and so a single $\rho_{in} \equiv \rho_{in}(\omega_0)$.

Once the frequency is chosen and ρ_{in} is known, φ_{Z_0} is completely determined. The curve which passes through the selected ρ_{in} determines this angle.

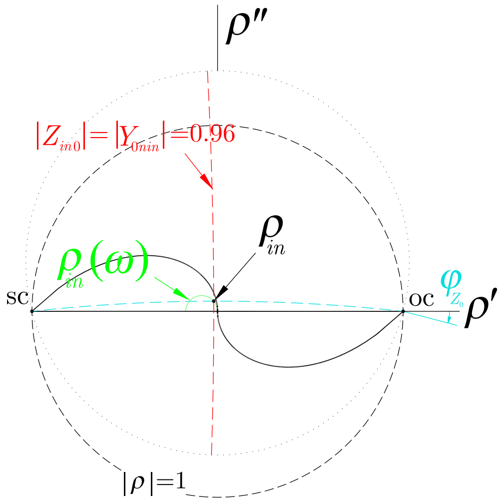


Fig. III: Analysis of ρ_{in} and the equivalent Z_0 when $\omega = \omega_0 = 1$[MHz].

Moreover, φ_γ is also determined by the lossy ratio c of the lossy TL and the normalized frequency ω_n. Thus, the tangential angle of the spiral and so the analysis along the TL:

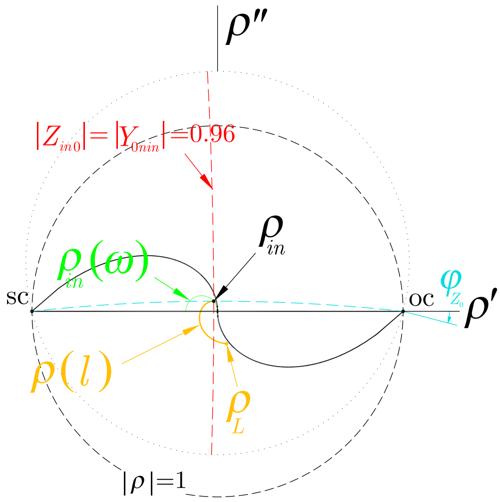


Fig. IV: Analysis of ρ along the TL from the input to the load.

Notice that the analysis is reverse with respect to the one carried out in Ex. 03 presented in Chpt. 5, in the sense is that this analysis goes from the input to the load, and so the spiral rotates counterclockwise.

By means of simple graphical and geometrical analysis it is possible to obtain the circunferenes which represent the real and imaginary parts of of Z_L in the associated GSC:

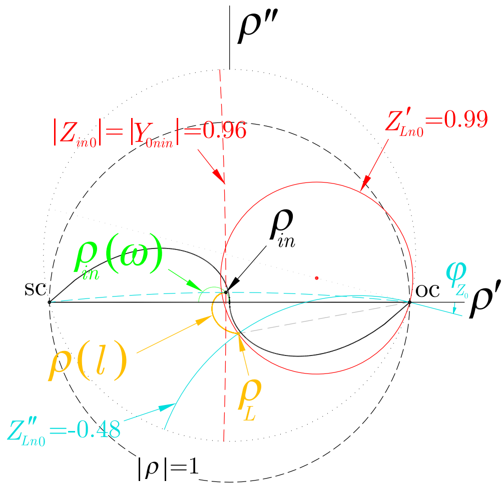


Fig. V: Analysis of Z_L in the GSC.

Although this example is quite simple, it clearly reveals that lossy TLs are useful for matching loads, and specially in practice using the frequency at the same time as the length of the TLs when losses are fixed.

5.7 Complex parameterizations in the CTLA

In this section, the use of complex parameterizations is presented with two main purposes: (ii.a) parameterizing the length of the TL into the "space of parameterizations"; and (ii.b) parameterizing mode solutions different from HPWs into this same "space".
The parameterization of these TL behaviors is expected to ease the analysis at the same time it explains and exemplifies the use of the "space of parameterizations", already used in Chpt. 4 (the rg-plane) and formally defined in Appendix 4.A in that chapter. As a result, it may be continued stating that this "space" is the origin of every parameterization useful for the TL analysis, so not only losses are parameterized there[10].

Notice that the use of complex parameterizations makes the real rg-plane insufficient for the graphical representations of new complex curves. Nevertheless, whenever the graphical analysis are possible they will be used for the sake of clariying the complex and possible geometrical analysis.

5.7.1 Using complex parameterizations of losses for parameterizing the TL's length

It has been seen that the phase of the propagation constant, φ_γ, determines the analysis along the TL. As a consequence, all the parameterizations of losses that keep this angle constant lead to the same variation of wave parameters along the TL. The term "variation" makes reference to a relative reference, which supposes knowing the differential, or geometrically the tangent of the curve which represents the analysis of any parameter along the TL. However, only when imposing the BCs along the TL, the analysis is complete.
This may be clearly seen in the analysis of the reflection coefficient along the TL in terms of losses, analyzed in Ex. 03 presented in Sect. 5.4 in Chpt. 5, in which the reflection coefficient at the load

[10]This has been already proved when parameterizing, for example, the relative frequency scales or the phase of basic parameters in the rg-plane.

is supposed to be (a priori) known (playing the role of BC along the TL). In that example of use of the CTLA for characterizing the TL in terms of losses and along its extension at the same time, inverse analysis of φ_γ in terms of losses lets to solve the losses that, keeping this phase fixed, vary the modulus of γ. As a consequence, the normalized length, l_n, defined in eq. (5.50) varies so that the analysis along the TL in terms of losses was able to be done.

This incipient example, although it was revealing for the possibility of combning the analysis in terms of losses and those carried out along the TL, and in particular the use of lossy parameterizations for simulating the TL's length, it shows a lot of important limitations: (i) the grade of abstraction is high because the resultant curves in the ρ-plane parameterized by φ_γ "universalize" within the same curve losses and the TL's length, in the sense that each point belonging to the φ_γ-curve represents multiple TLs parameterized by different losses at multiples lenghts measured from the load. This also happens with the curves which represent the basic parameters in the *vfa* in which losses and frequency are combined within the same analysis; (ii) supposing the reflection coefficient at the load as BC is not physically realistic but mathematically efficient for this analysis. Instead of that, the impedance at the load is more feasible as (physical) BC, just as it is considered in Ex. 01 presented in Sect. 5.2 in Chpt. 5, but the analysis along the can not be "unversalized" in terms of losses; and (iii) not all the possible values of l_n are possible since it depends on the modulus of the propagation constant. This fact clearly limits the analysis along the complete TL.

With the aim of: (i) overcoming these limitations; and (ii) showing that the "space of parameterizations" also holds the length parameterization in certain way; the following complex parameterization is proposed for studying the inflcuence of losses over the TL's length.

Mathematical analysis

Recall the expression of the normalized propagation constant with respect to the lossless case, γ_{n1}, in eq. (4.10) used for the *ffa*. From this expression, the complexified propagation constant is:

$$\bar{\gamma_{n1}} = j\sqrt{(1-j\bar{r})(1-j\bar{g})}, \text{ in which} \qquad \text{(III)}$$
$$\bar{r},\, \bar{g} \in \mathbb{C}.$$

Because the equivalent role of complex conductor and dielectric losses, $\bar{r}$ and $\bar{g}$, respectively, in eq. (III), it is assumed $\bar{g} = g_0 \in \mathbb{R}^+ \cup \{0\}$, for instance, the parameter which determines the source of losses without lost of generality. This would help the general analysis and, especially, the representation.
Thus, the conductor losses have been complexified in the following form:

$$r \in \mathbb{R}^+ \cup \{0\} \rightarrow \bar{r} = r_R + jr_I \in \mathbb{C}\backslash r_R \equiv r \geq 0, \qquad \text{(IV)}$$

with the purpose[11] of describing the complex parameterizations that lead to the same φ_γ so that they represent different modulus of γ and thus different normalized lengths l_n.
In these terms, the phase of the propagation constant is written as:

$$\varphi_\gamma = \frac{\pi}{2} - \tan\left(\frac{r_R}{r_I+1}\right) - \tan\left(g_0\right). \qquad \text{(V)}$$

Solving r_I in terms of r_R from this equation, it leads to the following expression:

$$r_I = \left(\frac{1}{\tan\left(\pi - 2\varphi_\gamma - \tan^{-1}(g_0)\right)}\right) r_R - 1, \text{ in which} \qquad \text{(VI)}$$
$$r_R \in [0,\infty);$$

[11]When a complexification of any a priori real parameter (because of its physical interpretation) is performed, the underlying purpose (a priori analytical) is required to be outlined.

which is a expression of a line in the $\bar{r}$-complex plane[12].
Notice that the slope of the line in eq. (VI) depends on φ_γ and $g = g_0$. All the combinations of these parameters that keep the slope invariant, describe the same curve along the TL in terms of losses. In fact, if this parameterization is seen de-complexified, the slope

$$\frac{1}{\tan\left(\pi - 2\varphi_\gamma - \tan^{-1}(g_0)\right)} = r \equiv r_R, \qquad \text{(VII)}$$

leding to $r_I = 0$, as it is originally supposed and physically makes sense.
Thus, this complexification has to be seen as an extension of the real case: a deformation of the original parameters line parameter for parameterizing the length into them.

Graphical analysis and related physical interpretations

A graphical analysis imposing the BC on the reflection is shown. Then this is physically interpreted briefly for the appropriate understanding of the graphs and what they represent within the context of the CTLT.
When imposing the BCs on the relfection coefficient at any point of the TL (referred as "at the load"), the expression of the reflection coefficient is the same as in eq. (5.51), in which ρ_L is known. In this case, the graphical transformation is from the "complex lines" in the rg-space following eq. (VI) to the ρ-plane:

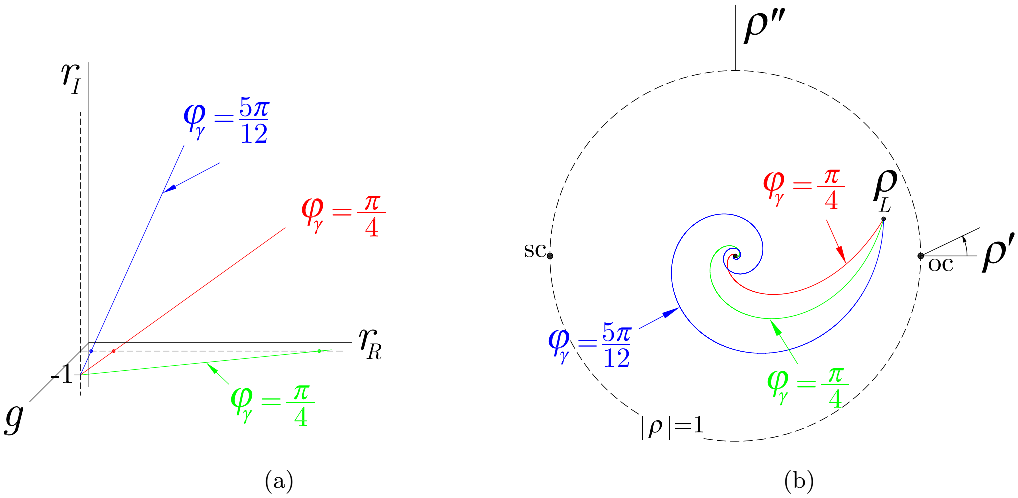


Fig. VI: Complex parameterizations of losses in the (a) rg-space ($g = 0.1$) and the transformations to the ρ-complex plane for the study of the reflection coefficien along the TL.

In Fig. VI, different φ_γ-parameterized curves, which represent both losses and the TL's length by means of the complexification of r when $g = 0.1$ is kept fixed in the rg-complex space, are transformed to the ρ-plane "simulating" the variation of the reflection coefficient along the TL.
Although the curves are in the real constant g-plane in the rg-space, the same behavior of ρ may be obtained in the real contant r-plane, just because the equivalent role of $\bar{r}$ and $\bar{g}$ in the expression of $\bar{\gamma}$.
Notice that the curves starts in $\bar{r} = 0 - j$ (equivalently, $\bar{g} = 0 - j$), and thus:

$$\bar{r} \in \mathbb{C} \backslash r_R \geq 0,\ r_I \geq -1 \qquad (\text{symmetrically, } \bar{g} \in \mathbb{C} \backslash r_R \geq 0,\ r_I \geq -1) \qquad \text{(VIII)}$$

This point represents the point $l_n = \equiv l = 0$, that is the position of the load.

With this analysis: (i) it is possible to see directly which losses are parameterized in the ρ-plane, only by examining the rg-space of parameterizations, instead of doing the inverse analysis of γ

[12] This plane is the complex extension of the r-axis in the rg-plane.

in the rg-plane as in Ex. 03 presented in Sect. 5.4 in Chpt. 5; (ii) having parameterized all the possible values of l_n and thus l from the load to infinity, leading to no restrictions in the analysis as a consequence of real lossy parameterizations, just as they are found in Ex. 03; and, additionally, (iii) proving that in the "space of parameterizations" the TL's length can be parameterized using equivalent losses.

This analysis could be extended to the analysis of the wave impedance when imposing BCs on the load. For this purpose, the losses on the chracteristic impedance would "simulate" the TL's length. However, this analysis is not as direct as the one with BC imposed on the reflection coefficient.

5.7.2 Using complex parameterizations of losses for characterizing high order mode solutions

In Chpt. 5, it has been seen how generalizing the equivalent *telegrapher's equations* by means of the complexification of the a priori real line parameters, the rest of mode solutions (beyond HPWs belonging to TEM modes) could be parameterized using the same form of the expressions of the characteristic impedance and the propagation constant. In other words, it has been seen that it is not necessary to solve the individual propagative waves for obtaining the mode impedance and the propagation constant. Furthermore, it was assumed working in the so called "propagative domain", which is no more than parameterizing the propagation constant γ from the (Laplace) complex exponentials that form a basis of propagative waves in the waveguide (in the same way as using the frequency ω for parameterizng time exponentials). Nevertheless, since the nature of γ seen as a parameter is complex, this needs to be characterized in its respective complex plane in terms of losses, BCs, and frequency. In this sense, recall the equivalent characterization of γ in the *ffa* done in the CTLT-v1 presented in Sect. 4.3.1 in Chpt. 4, in which the analysis of HPWs in terms of losses and frequency is carried out. In this case, the BCs does not appear explicitly. This does not mean that they are not present but they are normalized as a part of the line parameters. This fact is direct consequence of imposing zero Laplacian, which is equivalent to consider a null eigenvalue for the Laplacian that acts over the potential functions in eqs. (3.25) and (3.26).
Keeping these particular results in mind, it may be said that the eigenvalues of the Laplacian are addressed like the losses gathered in k. Thus, the Laplacian eigenvalues (denoted by the square of k_c, as usually cited in books for the particular analysis of E- and H-modes, [Mar51], and in general TM and TE modes, respectively) are considered as additional sources of losses.

Remark 47. *Generalizing the influence of losses allows for interpreting the eigenvalues of the Laplacian affecting the pontentials employed in the generalized* **LC** *of modes used when integrating Maxwell equations. In this sense, studying one isolated mode supposes analyzing one particular case, in the same way as for example studying the lossless case as a particular case within the general lossy case.*
As a result, dealing wtith BCs and the associated modes follows the same scheme as the analysis of the influence of losses in TL's, so the CTLT may be used to describe this mode analiysis.

In order to exemplify the analysis of E- and H-modes, the inverse analysis of the propagation constant in terms of losses and BCs is next presented. Recall that γ acts as a parameter in the "propagative domain", in which it should be characterized in terms of generalized losses[13]. The following analysis is carried out when only conductive losses represented by finite non zero conductivity σ affects the complex constitutive parameters. This assumption is taken for the sake of simplicity.

[13]The term ·"generalized losses" refer to those losses produced by constitutive parameters and losses due to BCs and the eigenvalues they generate.

Mathematical analysis

The general expression of γ for E- and H-modes in a conductive medium, [Mar51], characterized by the eigenvalues k_c is

$$\gamma = \sqrt{k_c^2 - k^2}, \tag{IX}$$

in which $k_c \in \mathbb{N} \cup \{0\}$, $k = \omega\sqrt{\mu\varepsilon} \in \mathbb{C}$, and in this case $\varepsilon_{eq} \equiv \varepsilon' - j\frac{\sigma}{\omega} \in \mathbb{C}$, and $\mu \equiv \mu' \in \mathbb{R}$.
As it has been mentioned, the eigenvalue $k_c = 0$ corresponds to the study of HPWs.

The expression of γ in eq. (IX) expands as:

$$\gamma = \sqrt{k_c^2 - \omega^2(\mu'\varepsilon') + j\omega\sigma\mu'}. \tag{X}$$

On the other hand, the generalized expression of γ obtained from the generalized *telegrapher's equations* in eq. (3.37) may be written as:

$$\gamma = \sqrt{\bar{R}\bar{G} - \omega^2\bar{L}\bar{C} + j\omega(\bar{L}\bar{G} + \bar{R}\bar{C})}. \tag{XI}$$

The coefficients of the polynomials within the square root in eqs. (IX) and (XI) are identified as follows:

$$\begin{cases} \bar{R}\bar{G} = k_c^2 \\ \bar{L}\bar{C} = \mu'\varepsilon' \\ \bar{L}\bar{G} + \bar{R}\bar{C} = \mu'\sigma \end{cases} . \tag{XII}$$

Operating with them, the following system is obtained:

$$\begin{cases} \frac{\bar{R}}{\bar{L}} + \frac{\bar{G}}{\bar{C}} \equiv \bar{g}' + \bar{r}' = \frac{\sigma}{\varepsilon'} \\ \frac{\bar{R}}{\bar{L}}\frac{\bar{G}}{\bar{C}} \equiv \bar{g}'\bar{r}' = \frac{k_c^2}{\mu'\varepsilon'} \end{cases} . \tag{XIII}$$

Notice that r' and g' are the conductor and dielectric losses used in the *vfa*, so scaling them with respect to ω leads to the r and g used in the "space of parameters".
Solving $\bar{r}'$ and $\bar{g}'$ from the system in eq. (XIII), it leads to:

$$\bar{r}' = \frac{\frac{\sigma}{\varepsilon'} + \sqrt{\left(\frac{\sigma}{\varepsilon'}\right)^2 - 4\frac{k_c^2}{\varepsilon'\mu'}}}{2}, \text{ and} \tag{XIV}$$

$$\bar{g}' = \frac{\frac{\sigma}{\varepsilon'} - \sqrt{\left(\frac{\sigma}{\varepsilon'}\right)^2 - 4\frac{k_c^2}{\varepsilon'\mu'}}}{2}. \tag{XV}$$

Some interesting results and physical behaviors may be deduced from analyzing the expressions of $\bar{r}$ and $\bar{g}$: (i) When the conductivity is zero, that is when the medium is a perfect dielectric, the higher order modes are represented by pure imaginary parameterizations of losses; (ii) when the expression is particularized for $k_c = 0$, the parameterizations reduce to real values, as it is expected in case of dealing with HPWs. In this case, $\bar{r}' \equiv r' = 0$, because of the approrpaite selection of the sign in the expressions above; and (iii) in general $\bar{r}'$ and $\bar{g}'$ are complex conjugate.

As a result, it may be foreseen that the H- and E- modes use complex conjugate parameterizations of losses to be described using the CTLA. The corresponding analysis and physical interpretations of this (complex conjugate) parameterized analysis would lead to the associated CTLT of these modes.

5.8 The CTLA combined with numerical analysis

The definition of the space of parameterizations, and in particular the rg-plane, and the functions which transform the parameterizations in these domains to the complex planes associated to each parameter under study, allows for discretizing them and integrating the seeed solution.

In this way, notice that the integral solution to a problem may be posed in a Green's function inspired form, [Sta79]. Nevertheless, the concept changes significantly:

(i) Any problem which is solved in an integral way using the associated Green's function, needs to solve previously the so called Green's problem, which lies in finding the corresponding Green's function. This problem is associated to find the impulse response of an inverse operator by using the delta of Dirac, from the SST perspective.
From the CTLT point of view, each analyzed TL parameter in terms of the parameterizations in the rg-plane would play the role of a Green's function. Equivalentely, a physical problem would be represented by integrating the behavior of multiple lossy TL's.

(ii) In this case, the operator is not clearly represented by a mathematical expression but for a physical problem/behavior which is described by the parameter involved.

(ii) The integrals are not defined a priori in the rg-plane but the inverse characterization serves to define the paths of integrations depending on the problem to be solved.

Let's see a simple but representative example of this integral description in the CTLT.
The physical problem under study is about solving the reflective wave in a fixed point ($l = 0$) in which an incident wave comes to a impedance which is different to the characteristic impedance of the lossy medium where the incident wave initially propagates (characterized by r' and g', and thus a $c = r'/g'$).
The scheme of the problem is:

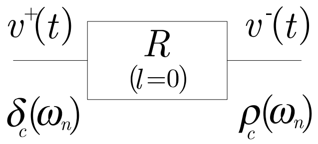


Fig. VII: Scheme of the problem which is about solving the reflective wave, $v^-(t)$ from the incident wave v^+(t).

In this case, the reflection coefficient characterized by c and analyzed when the frequency varies (so the normalized frequency ω_n also does) is useful to solve the problem of finding the reflective wave $v^{-1}(t)$ from an incident wave $v^+(t)$. Notice that the vfa of ρ is the impulse responsonse of this operator. However, this parameter is adressed in a different domain (the rg-plane) so it comes from an inverse analysis. Thus, the solution uses $\rho_c(\omega_n)e^{j\omega_n t}$ as the Green's function of the problem, which lets to find the reflective wave:

$$v^-(t) = \mathrm{Re}\left[\int_{\omega_n} V^+(\omega_n)e^{j\omega_n t}\rho_c(\omega_n)\sqrt{r'^2 + g'^2}d\omega_n\right], \tag{XVI}$$

in which $V^+(\omega_n)$ is the spectrum of the incident wave.

This method would help the solution of more TL-related problems when paraticularizing different behaviors. For example, one intersting problem to solve is soling the wave impedance when integrating different modes. Since it has been proved that modes may be addressed as complex losses, the integral would be in curves in complex domains.

5.9 The CTLT as example of use of the GTLT

It has been seen that the CTLT may be explained in different aspects from the GTLT.
Furthermore, the GTLT uses some solid definitions generalizations, e.g. the scalar product defined from potentials, that may extend the current version of the GSST.
Thus, it is expected that the analysis in both the GTLT and the CTLT serve as example of applications of future versions of the GSST.

General Conclusions

In this Thesis, the ***Complex Transmission Line Theory*** (CTLT) has been presented as a solid methodology of analysis alternative to the classic ***Transmission Line Theory*** (TLT), at the same time that it has been provided with the general mathematical definitions which serve to complete any of the possible versions of the CTLT under the same theoretical framework.
In particular, the first version of the CTLT (denoted as CTLT-v1), which has been originally posed to reinterpret the analysis of the ***Lossy Transmission Line Theory*** (LTLT) (i.e. the rigorous characterization of wave solutions in lossy Transmission Lines (TLs)) has served to exemplify the use of the ***Complex Transmission Line Analysis*** (CTLA) while obtaining the appropriate physical interpretations from the complex analysis to validate and remplace the underlying –in this case– LTLT.
However, the CTLT should not be exclusively set aside for studying a specific type of solutions under certain conditions (e.g. the LTLT) but for analyzing any type of EM system, either theoretical in origin or real systems modeled theoretically. This is an ambicious but actually achievable goal by using of the CTLT (subject to the increasing of the complexity in the analytical developments and physical interpretations).

An emerging theory requires us to redefine our mindset...

The interpretation of the CTLT requires a change of mindset, something which has been thoroughly exaplined in this thesis book through different definitions, graphs, schemes, remarks, and examples of use. Among of them, the most important and general interpreatations are next outlined and summarized as conclusions in order to base this new perspective of analysis:

(i) Since the solutions in the equivalent TLs are completely define by means of their parameters, studying them is as acceptable as giving the expressions of voltage and current waves.
In addition, despite these latter mathematical expressions of waves are a relatively compact manner of giving the solutions in the TL, they lack of the analysis of the parameters involved on them and thus the related physical interpretations, so analyzing the TL parameters is definitely more "revealing" than simply giving a closed mathematical expression of waves. Thus, the CTLT bases its analysis on the characterization of the TL parameters.

(ii) Keeping this idea of focusing the analyis on the TL parameters in mind, it is neccesary to see all the parameters gathered together in order to give an idea of the behavior of the solutions, prior to be physically interpeted. In this sense, each value of any TL parameter is also mapped on the rest, so they all are connected. This fact inherently introduces the idea of transformation between the TL parameters involved in the analysis. Thus, the CTLT is focused on the study of those transformations between the TL parameters.
Each version of the CTLT is considered analytically complete when all of those possible transformations of the underlying TLT are characterized.
These transformations may be seen:

(ii.1) analytically as complex transformations;

(ii.2) graphically involving the planes associated to each parameter; and

(ii.3) geometrically as plane curves equally parameterized (e.g. the circumferences parameterized by the real and imaginary parts of the wave impedance describing the reflection coefficient: the GSC, [GDG06]).

For the purpose of defining the transformations from these viewpoints, it has been needed to define the domains of the parameters over which the tranformations operates from/to. There are two ways to deal with this issue:

a. using the complex planes which naturally appear when operating with complex parameters (e.g. the plane associated with the characteristic impedance), or

b. conveniently defining new domains from the (ii.1) analytical (algebraic), (ii.2) graphical, and (ii.3) geometrical perspectives (e.g the rg-plane in which the line parameters are mapped).

(iii) While defining the transformations between the TL parameters lets to study all of them together leading to their complete analytical characterization, the general study still lacks of the physical meaning. Thus, the CTLT is finally complete when selecting the curves which meet specific physical criteria in the plane of parameterizations (the rg-plane) and transforming them to the rest of complex planes associated to the TL parameters. The selection of these curves manily meets two purposes:

(iii.1) describing the parameters physically, i.e. studying their physical behavior in terms of the variation of the physics of the EM system parameterized into the equivalent TL (e.g. the waveguide losses, sizes, etc.), and

(iii.2) describing the solutions in terms of the original variables: time[14] and length; in certain way by means of the parameters in use.

A lot of times, these purposes of the CTLT are not tackled independently (e.g. the analysis along the TL's length and losses may be coped together by using the parameterizations of angles of basic parameters).

(iv) In order to keep the analysis using complex parameterizations, in complex planes, and regarding plane curves, the adequate normalizations of the TL parameters are chosen, also depending on the goal of the analysis (e.g. for the analysis of the TL parameters in terms of losses, i.e. a physical analysis of the parameters, the noramalizations are chosen with respect to the lossless case). In this way, the normalizations lead to "universalize" the behavior of the paramteters, which means grouping the parameters in terms of the parameterizations used (analytically, it means forming equivalence classes), representing all the possible values of the normalized parameter onto the same parameterized point, and using the parameterizations for representing "universal" plane curves.

Under these considerations the CTLT-v1 has been satisfactorily posed. The following general conclusions regarding the resultant analysis may be outlined:

(i) The underlying LTLT in which the CTLT-v1 settles its basis leads to known the parameters under study and how they are directly connected. Nevertheless, it has been seen that the most efficient manner to obtain these relations for the CTLA is neither following the usual order: (from) line parameters → basic parameters → wave parameters (i.e. the so called direct characterization); even for the simplest mode solutions (HPWs within the TEM modes); nor characterizing the parameters inversely: (until) line parameters ← basic parameters ← wave parameters (i.e. the so named inverse characterization). This latter characterization, although it works fine for charactrizing a single mode (e.g. HPWs), is not efficient for obtaining the parameters of a set of solutions.
The most efficient way to obtain the parameters of the TL for the CTLA lies in combining both the direct and inverse characterizations: (generalized) line parameters ← basic parameters → wave parameters[15]. In this way, each of the mode solutions is inversely mapped onto the (generalized) line parameters, from which the direct characterization lets to obtain the wave parameters. As a result, not only the constitutive parameters but also the BCs are tackled as part of (generalized) line parameters in certain way, and so the analysis in terms of losses exemplified in this book adquires greater relevance.

[14]The way of parameterizing time is, indirectly, by means of the frequency, analyzed in the *variable frequency analysis*.
[15]The complete procedure would be:

(generalized) line parameters ← basic parameters (from potentials)
↓
line parameters → basic parametrs → wave parameters.

And that's not all. The parameterizations for the CTLT become such relevant that the original variables are addressed as part of these parameterizations in two different ways:

(i.1) Limiting the analysis to specific domains, which means operating in the domain of the coefficients that expands the solutions in certain basis set (e.g. the expressions in the *frequency domain* belong to the set of coefficients when time exponentials $e^{j\omega t}$ are chosen as basis). In this sense, the spectral variable transforms to a parameter together with its corresponding analysis (in the *frequency domain*, there is a corresponding *variable frequency analysis*); and/or

(i.2) Complexificating and/or geometrizing the problem (e.g. the angle of the propagation constant, φ_γ, determines the variation along the TL so the modulus may be used to describe the TL's length under their relative normalization).

Inspired in this efficient description of the parameters by combining direct and inverse analysis, the first version of the ***Generalized Transmission Line Theory*** (GTLT-v1) arises up as the Theory which better supports the CTLT for the parameter characterization. This GTLT does not deal with solving the waves but with characterizing the basic parameters of the solutions in terms of losses, BCs, etc. In particular, HPWs have been object of the study in the GTLT-v1 showing that the inverse characterization completely remplaces the direct characterization originally used in the LTLT.
In addition, the inverse characterization has demonstrated being very useful for finding alternative parameterizations of losses and the TL's length based on the angles of basic parameters.

(ii) The curves in the (not Euclidean) rg-plane and in general in the "space of parameterizations", both defined and explained algebraically, graphically and geometrically, completely define the transformations which are of interest in the CTLT:

(ii.1) the curves with contsant r and g in the rg-plane[16] define the lossy parameterizations for the *fixed frequency analysis*;

(ii.2) the curves with modulus constant (ω_n) in the rg-plane and the curves with constant angle (θ_c), define the parameterizations for the *variable frequency analysis*; and

(ii.3) different sets of hyperbolas in the rg-plane define the complex parameterizations of basic parameters.

As a consequence, the rg-plane gathers all the parameterizations which are of interest in the CTLT for the full desciption of the TL parameters.
Moreover, the complex parameterizations are tackled as only one, instead of splitting them into its real and imaginary parts or modulus and phase. That is because there is always a physical parameter in the analysis regarding the CTLT which relates the a priori separated parameterizations (e.g. the parameterizations in the ρ-plane are related by means of the TL's length for the analysis along the TL).

One important final result regarding the analysis of the transformation involved is that the angles of basic parameters, φ_γ and φ_{Z_0}, completely define the behavior of the TL parameters in terms of losses/frequency and along the TL, provided that the appropriate normalizations are chosen for each parameter under study depending on the type of study –in terms of the losses/frequency/TL's length– it is being analyzed. That is mainly because of the angles go beyond the normalizations (they keep the same after the normalizations) for each of those analysis in the TL.

In the same sense, the reflection coefficient, ρ, is the only parameter which keeps the same after the normalizations for each study. Thus, ρ is proved to be the parameter which describes the analysis in terms of losses, frequency and along the TL at the same time, and thus in the same complex plane. This makes the GSC (the ρ-plane complex parameterized

[16]Since the rg-plane is not Euclidean they are not parallel curves in the "space of parameterizations"

by the wave impedance/admittance) and the iGSC (the ρ-plane parameteized by the characteristic impedance) the most useful graphical tools for dealing with that duality the analysis along the TL and in terms of losses/frequency suppose in the CTLT.

Remark. *The planes which "universally" gather all the parameterizations which are useful for describing the CTLT are, on one hand the rg-plane and, on the other hand, the ρ-plane.*

In conclussion, the CTLT just as it has been presented and exemplifyed in the present Thesis supposes a novel way to see the usual TLT and also generalized versions of the TLT. The methodology of analysis based on CTLA analysis the CTLT uses analytic methods which result greatly intuitive once the mindset has been changed for understanding the parameters as transformations between planes. This persepective leads to geometrize the problem. Thus, the graphical representations adquire important relevance for both explaining the analysis and solving TL related problems.

... but the change may become really worth when dealing with rigorous analysis.

Future Lines

The most important research that it has not tackled in the Thesis is left as possible future lines to continue the analysis.
On one hand, some of these possible future analysis are related to the **Applications** of the CTLA which have been previously presented. In this sense, the analysis presented in the Thesis will serve as example of use of the methodology based on complex transformations for the full description of the TL parameters. For the purpose of extending the CTLA to different mode solutions, the GTLT-v1 presented in Chpt. 3 has been posed. Thus, it is about particularizing this GTLT to different modes in order to do the corresponding CTLA obtaining the subsequent physical interpretations.
On the other hand, future versions of the GTLT could be posed leading to different perspectives of the guided wave analysis.
Thus, the process of bulding up future versions for the TL analysis seems to be clearly guided. Generaly speaking, it would follow the scheme:

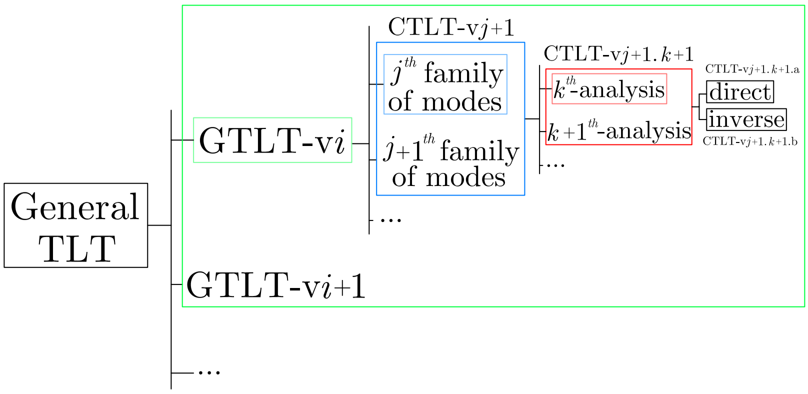


Fig. VIII: Scheme of succesive versions to be follow for the achievement of the General TLT analysis. Any TLT should be study in generalized versions of the TLT (GTLT). Notice that the GTLT-vi is within the GTLT-v$i+1$. The last GTLT would represent the General TLT. Moreover, different families of modes are studied as particular cases of each GTLT. In turn, different analysis within a family of modes may be carried out. These analysis are characterized directly and inversely. The analysis presented in the Thesis correspond to the CTLT-v1.0-a and CTLT-v1.0b applied to the characterization of HPWs obtained from particularizng the GTLT-v1.

Keeping in mind this scheme, the following analysis are proposed as future research lines:

(1) The most immediate analysis would be part of the characterizations of HPWs within the context of the GTLT-v1, and so witihin the CTLT-v1. This analysis is suggested to deal with frequency dependent line parameters. This mainly affects the vfa.
Notice that the lossy case particularized in Chpt. 2 takes ε'' and μ'' as inverse functions of frequency, whereas σ is linear function of frequency, both in order to make the line parameters non frequency dependent. Thus, the line parameters would have to be generalized to functions of frequency, mainly polynomials on frequency, to deal with all the possible cases of frequency dependent constitutive parameters. In this way, the analysis would become more realistic and so capable to be applied to real EM systems.
On the other hand, and related to the frequency dependence, the analysis in terms of frequency at the load, would have to take into account the frequency dependence of impedance/admittance at the load. This would also make the analysis more realistic. For example, a capacitor which varies its admittance linearly with frequency would be able to be analyzed if the frequency dependence of the load is rigorously analyzed.
The associated CTLA considering the frequency dependence of line parameters would lead to the CTLT-v1.1.

(2) Different mode solutions using the GTLT-v1 will be studied using CTLA. Among of them, it is interesting to particularize the complex analysis to: (i) those waves that come from the curl of vector potentials instead of the gradient of scalar potentials, and analyze the differences with repect to this latter case. This case could be added to the existing CTLT-v1.0, which generally would refer TEM modes; (ii) those waves which come from potentials that form a basis of eigenfunctions of the Laplacian. These are the E- and H-modes whose first approach has been introduced among the **Applications**. Depending on the space of numbers the eigenvalues belong to, the corresponding subversions of the CTLT-v2 will be addressed, for example, as CTLT-v2.1 for integers, CTLT-v2.2 for real eigenvaleus, etc. Notice that each of these versions would involve the previous ones, as well as the CTLT-v1. As a result the CTLT-v2 would refer TE and TM modes; and (iii) those modes that combine different assumtions posed over the Laplacian, as for example zero Laplacian for electric scalar potetials ($\Delta\phi_e = 0$) at the same time that the magnetic scalar potentials are eigefunctions of the Laplacian ($\Delta\phi_h = -k_{c,h}^2\phi_h$). These "hybrid modes" (HM) could be gather in the CTLT-v3.

(3) Finally, posing future versions of the GTLT in order to make the general analysis of the TLT more affordable would be the method to extend the Theory. To achieve this goal, different domains (not only the *frequency domain*) could be used to expand the previous solutions while bringing the analysis of more. This step supposes a reinterpretation of both the way to deal with the TLT from *Maxwell equations* and the use of the CTLA which, in any case, appears bound to the general analysis either in the use of complex functions/parameters or the complex variable.
From this point, it is suggested working in the *velocity domain*; in fact, the *phase velocity domain*. In this domain the waves would be grouped by the phase velocity, so this is the way of relating first time derivatives and the derivatives in z (the direction of propagation). That is the key of this analysis, so it is not about finding the eigefunctions of time derivatives (complex exponentials parameterized by frequency) but a constant relations between time and z- derivatives. Thus, the concept of analysis changes completely.
One additional possibility is working in the *propagative domain* from the beginning. Recall that the *propagative domain* is referred in Chpt. 3 when dealing with those solutions whose propagative term $e^{\mp\gamma z}$ is assumed as basis. In this domain, which is tackled within the frequency domain, γ is taken as a parameter and, as such, it has been characterized in terms of frequency. Then, conversely, the idea of working in the *propagative domain* from the beginning (out of the *frequency domain*) lies in using $\mp\beta \in \mathbb{R}$ as the parameter in z-exponentials $e^{\mp\beta z}$ which would describe the analysis a priori. As a consequence, $\omega \in \mathbb{C}$ is addressed as complex frequency, which would have to be characterized a posteriori in terms of constitutive parameters and $\mp\beta$. A brief example of use of this generalization based on operating in the *propagative domain* is next presented to clarify the underlying idea.

Example. Let's transform the *Faraday's law* to the *propagative domain*.
For this purpose, on the l.h.s. the curl of the electric field in general cylindrical (one unitary vector is $\hat{z}$) coordintes may be written as:

$$\begin{aligned} \nabla\times\mathcal{E} = \frac{1}{h_1h_2}\begin{vmatrix} h_1\hat{\mathrm{t}_1} & h_2\hat{\mathrm{t}_2} & \hat{z} \\ \frac{\partial}{\partial \mathrm{t}_1} & \frac{\partial}{\partial \mathrm{t}_2} & \frac{\partial}{\partial z} \\ h_1\mathcal{E}_{\mathrm{t}_1} & h_2\mathcal{E}_{\mathrm{t}_2} & \mathcal{E}_z \end{vmatrix} = \hat{\mathrm{t}_1}\left[\frac{1}{h_2}\frac{\partial\mathcal{E}_z}{\partial \mathrm{t}_2} - \frac{\partial\mathcal{E}_{\mathrm{t}_2}}{\partial z}\right] + \\ +\hat{\mathrm{t}_2}\left[-\frac{1}{h_1}\frac{\partial\mathcal{E}_z}{\partial \mathrm{t}_1} + \frac{\partial\mathcal{E}_{\mathrm{t}_1}}{\partial z}\right] + \frac{1}{h_1h_2}\hat{z}\left[\frac{\partial h_2\mathcal{E}_{\mathrm{t}_2}}{\partial \mathrm{t}_1} - \frac{\partial h_1\mathcal{E}_{\mathrm{t}_1}}{\partial \mathrm{t}_2}\right] \end{aligned}; \qquad \text{(XVII)}$$

in which t_1 and t_2 are the coordinates on the transverse of the structure that is z-invariant (the waveguide), and h_1 and h_2 are their respective scale factors.

In the *propagative domain* the z-derivatives transform as:

$$\frac{\partial(\circ)}{\partial z} \rightarrow \mp j\beta,$$

so the curl of $\boldsymbol{E}$ (the electric field expressed in the *propagative domain*) may be written as:

$$\nabla \times \boldsymbol{E} = \hat{\mathrm{t}_1}\left[\frac{1}{h_2}\frac{\partial E_z}{\partial \mathrm{t}_2} \pm j\beta E_{\mathrm{t}_2}\right] + \hat{\mathrm{t}_2}\left[-\frac{1}{h_1}\frac{\partial E_z}{\partial \mathrm{t}_1} \mp j\beta E_{\mathrm{t}_1}\right] + \frac{1}{h_1 h_2}\hat{z}\left[\frac{\partial h_2 E_{\mathrm{t}_2}}{\partial \mathrm{t}_1} - \frac{\partial h_1 E_{\mathrm{t}_1}}{\partial \mathrm{t}_2}\right]. \quad \text{(XVIII)}$$

On the other hand, the r.h.s. of the *Faraday's law* may be written in the *propagative domain* as:

$$-\frac{\partial \boldsymbol{B}}{\partial t} \equiv -\frac{\partial \mu(\beta)\boldsymbol{H}}{\partial t} \equiv -\mu(\beta)\frac{\partial H_{\mathrm{t}_1}}{\partial t}\hat{\mathrm{t}_1} - \mu(\beta)\frac{\partial H_{\mathrm{t}_2}}{\partial t}\hat{\mathrm{t}_2} - \mu(\beta)\frac{\partial H_z}{\partial t}\hat{z}; \quad \text{(XIX)}$$

in which $\mu(\beta) \equiv \mu \in \mathbb{C}$ is the magnetic permeablity of the medium, which depends on z in the *time domain* and so in β in the *propagative domain.*

The l.h.s. and r.h.s. are matched component by component, so that:

$$\begin{cases} \frac{1}{h_2}\frac{\partial E_z}{\partial \mathrm{t}_2} \pm j\beta E_{\mathrm{t}_2} = -\mu\frac{\partial H_{\mathrm{t}_1}}{\partial t} \\ \frac{1}{h_1}\frac{\partial E_z}{\partial \mathrm{t}_1} \pm j\beta E_{\mathrm{t}_1} = \mu\frac{\partial H_{\mathrm{t}_2}}{\partial t} \\ \frac{1}{h_1 h_2}\left[\frac{\partial h_2 E_{\mathrm{t}_2}}{\partial \mathrm{t}_1} - \frac{\partial h_1 E_{\mathrm{t}_1}}{\partial \mathrm{t}_2}\right] = -\frac{\partial H_z}{\partial t} \end{cases}. \quad \text{(XX)}$$

Notice that the z-components of the system above may be obtained from the transversal components.
For the sake of simplicity in this example, let's consider the seeked wave solution is plane, so $E_z = H_z = 0$. Imposing this condition, and adding the first and second equations of the system in eq. (XX) leads to:

$$\begin{cases} \pm j\beta\left(-E_{\mathrm{t}_2} + E_{\mathrm{t}_1}\right) = \mu\frac{\partial\left(H_{\mathrm{t}_1} + H_{\mathrm{t}_2}\right)}{\partial t} \\ \frac{1}{h_1 h_2}\left[\frac{\partial h_2 E_{\mathrm{t}_2}}{\partial \mathrm{t}_1} - \frac{\partial h_1 E_{\mathrm{t}_1}}{\partial \mathrm{t}_2}\right] = 0 \end{cases}. \quad \text{(XXI)}$$

Using vector identities, the first equation in the system in eq. (XXI) may be rewritten as:

$$\pm j\beta\left(\hat{z} \times \boldsymbol{E}_{\mathrm{t}}\right) = \mu\frac{\partial \boldsymbol{H}_{\mathrm{t}}}{\partial t} \quad \text{(XXII)}$$

On its behalf, the second equation in eq. (XXI) indicates that $\nabla_t \times \boldsymbol{E}_{\mathrm{t}} = \mathbf{0}$, so that $\boldsymbol{E}_{\mathrm{t}}$ comes from a scalar potential, and so $\boldsymbol{H}_{\mathrm{t}}$ does. Thus, eq. (XXII) is of the form of *telegrapher's equations* but in the *propagative domain* and thus the derivative is with respect to the time. It may be foreseen that the pair of *telegrapher's equations* would be:

$$\varepsilon\frac{\partial v(t)}{\partial t} + \sigma v(t) = \pm j\beta i(t), \quad \text{(XXIII)}$$

$$\frac{\partial i(t)}{\partial t} = \pm j\frac{\beta}{\mu}v(t). \quad \text{(XXIV)}$$

When decoupling these equations by differentiating the first one and substituying the second equation on it, it leads to a 2nd order ODE for $v(t)$. Solving this ODE, the volatage wave results in:

$$v(t) = e^{j\omega_1 t} + e^{j\omega_2 t}, \text{ in which} \quad \text{(XXV)}$$

$$\omega_1 = \frac{-\frac{\sigma}{\varepsilon} + \sqrt{\left(\frac{\sigma}{\varepsilon}\right)^2 - 4\frac{\beta^2}{\mu\varepsilon}}}{2}, \text{ and } \omega_2 = \frac{-\frac{\sigma}{\varepsilon} - \sqrt{\left(\frac{\sigma}{\varepsilon}\right)^2 - 4\frac{\beta^2}{\mu\varepsilon}}}{2},$$

so the pair of frequencies ω_1 and ω_2 are complex conjugate in general.
These solutions should be analyzed by CTLA and interpreted in the corresponding version of the CTLT.

Studying the CTLT from both different particularizations of the GTLT-v1 and also new versions of the GTLT will lead to new research on the topic presented in the Thesis.

Bibliography

EM Theory of Guided Waves and Transmission Line Theory

[Mar51] N. Marcuvitz, *Waveguide Handbook*, Vol. 10, McGraw-Hill, 1951.

[Col90] R.E. Collin, *Field Theory of Guided Waves*, 2nd Ed., IEEE Press, 1990.

[Cle96] P.C. Clemmow, *The Plane Wave Spectrum Representation of Electromagnetic Fields*, IEEE Press, 1996.

[Poz98] D.M. Pozar, *Microwave Engineering*, John Wiley & Sons. 1998.

[Col01] R.E. Collin, *Foundations for Microwave Engineering*, 2nd Ed., IEEE Press, 2001.

[HY02] G.W. Hanson, A.B. Yakovlev, *Operator Theory for Electromagnetics*, 1st Ed., Springer-Verlag, 2002.

Complex Transmission Line Analysis and Complex Transmission Line Theory

[Rie98] J. Riego Martínez, *Analysis and Parameterization of Guided Systems with Arbitrary Losses*, Final Degree Project at University of Valladolid (available at *Miguel Delibes* Library), 1998.

[Gag01] E. Gago-Ribas, *Complex Transmission Line Analysis Handbook*, Vol. GW-I, "Electromagnetics & Signal Theory Notebooks" series, GREditores, 2001.

[GVH15] E. Gago-Ribas, P. Vidal-García, J. Heredia Juesas, "Complex Analysis and Parameterization of the Lossy Transmission Line Theory and its Application to Solve Related Physical Problems", Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA'15), available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/7297091`, pp. 141-144, Oct. 2015.

[VG16-I] P. Vidal-García, E. Gago-Ribas, "Complex Analysis of the Transmission Line Theory: Anslytical Characterization and Examples of Use", Proceedings of Progress in Electromagnetic Rerearch Symposium 2016, available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/7735278`, pp. 3262-3266, Oct. 2016.

[VG16-II] P. Vidal-García, E. Gago-Ribas, "A Generalized Complex Transmission Line Theory: Characterization and some Examples", XXXI National Symposium of the International Union of Radio Science (URSI 2016), Madrid (Spain), Sept. 2016.

Graphical Tools and Charts of Complex Transmission Line Analysis

[Smi39] P.H. Smith, "Transmission Line Calculator", Electronics, Vol. 12, No. 1, pp. 29-31, 1939.

[Smi44] P.H. Smith, "An Improved Transmission Line Calculator", Electronics, Vol. 17, No. 1, p. 131, 1944.

[GDG06] E. Gago-Ribas, C. Dehesa-Martínez, M.J. González-Morales, "Complex Analysis of the Lossy-Trasmission Line Theory: A Generalized Smith Chart", Turkish Journal of Electrical Enginerring & Computer Sciences (Elektrik), Vol. 14, No. 1, pp. 173-194, 2006.

[WLH09] Y. Wu, Y. Liu, H. Huang, "Extremely Generalized Planar Smith Chart based on Möbius Transformations", Microwave Optical Letters, Vol. 51, pp. 1164-1167, 2009.

[VG17-I] P. Vidal-García, E. Gago-Ribas, "A Logarithmic Version of the Complex Generalized Smith Chart", Progress in Electromagnetic Research Letters, Vol. 68, pp. 53-58, 2017.

[VG17-II] P. Vidal-García, E. Gago-Ribas, "The Logarithmic Generalized Smith Chart: Theoretical Analysis", Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA'17), available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/8065638`, pp. 1766-1769, Oct. 2017.

[VG17-III] P. Vidal-García, E. Gago-Ribas, "The Logarithmic Generalized Smith Chart: Examples of Use", Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA'17), available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/8065636`, pp. 1758-1761, Oct. 2017.

Complex Analysis

[BC90] J.W. Brown, *Complex Variables and Applications*, McGraw-Hill, 1990.

[Ola00] S. Olariu, "Complex Numbers in n Dimensions", arXiv online database, available in `https://arxiv.org/abs/math/0011044`, 2000.

Differential Geometry

[Law72] J.D. Lawrence, *A Catalog of Special Plane Curves*, Dover Publications, 1972.

[MP77] R.S. Millman, G.D. Parker, *Elements of Differential Geometry*, Prentice-Hall, 1977.

Differential and Integral Equations

[Sta79] I. Stakgold, *Green's Functions and Boundary Value Problems*, Wiley & Sons, 1979.

[Eva97] L.C. Evans, *Partial Differential Equations*, 1st Ed., Vol. 19, American Mathematical Society, 1997.

[Zwi97] D. Zwillinger, *Handbook of Differential Equations*, 3rd Ed., Academic Press, 1997.

General Mathematical Analysis and Calculus

[Sch51] L. Schwartz, *Théorie des distributions*, Hermann, 1951.

[GS64] I.M. Gelfand, G.E. Shilov, *Generalized Functions*, 2nd Ed., Vols.I, II, III, Academic Press, 1964.

[HP80] V. Hutson, J.S. Pym, *Applications of Functional Analysis and Operator Theory*, Academic Press, 1980.

[Apo90] T.M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, 2nd ed., Springer-Verlag, 1990.

[Bro96] A. Browder, *Mathematical Analysis* Springer, 1996.

[Mil06] P.D. Miller, *Applied Asymptotic Analysis*, Vol. 75, American Mathematical Society, 2006.

Signals and Systems Theory and Generalizations

[OWY97] A.V. Oppenheim, A.S. Willsky, I.T. Young, *Signals and Systems*, 2nd Ed., Prentice-Hall, 1997.

[Lin04] D.K. Linder, *Introduction to Signals and Systems*, McGraw-Hill, 1999.

[Gag09] E. Gago-Ribas, "A Scheme to Generalize Signal Theory and Its Application in Electromagnetics", Proceedings on IEEE International Symposion on Antennas & Propagation, and USNC/URSI National Radio Science Meeting (IEEE TAP/URSI 2009), available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/5171929`, pp. 2009.

[Her14] J. Heredia Juesas, *Generalized Signals and Systems Theory for its Application to the Description, Resolution and Parameterization of Electromagnetic Problems*, PhD. Thesis available at University of Oviedo Repository (`http://hdl.handle.net/10651/25579`), 2014.

[HGV15] J. Heredia Juesas, E. Gago-Ribas, P. Vidal-García, "Application of the Rigged Hilbert Spaces into the Generalized Signals & Systems Theory", Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA'15), available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/7297341`, pp. 1365-1368, Oct. 2015.

[HGV16] J. Heredia Juesas, E. Gago-Ribas, P. Vidal-García,, "Application of the Rigged Hilbert Spaces into the Generalized Signals and Systems Theory: Practical example", Proceedings of Progress in Electromagnetic Rerearch Symposium 2016, available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/7735737`, pp. 4728-4733, Oct. 2016.

Complex Spaces in EM

[GGD97] E. Gago-Ribas, M.J. González Morales, C. Dehesa Martínez, "Analytical parameterization of a 2D real propagation space in terms of complex electromagnetic beams", Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA'97), Transactions on Electronics, Special issue on Electromagnetic Theory - Scattering and Diffraction (IEICE) Vol. E80C, No. 11, pp. 1434-1439, 1997.

[GG99] E. Gago-Ribas, M.J. González Morales, "Complex distances and complex angles in beam propagation and scattering", Proceedings on 4th Conference on Electromagnetics and Light Scattering by Non-spherical Particles: Theory and Applications, pp. 103-110, Vigo (Spain), 1999.

[GG00] M.J. González Morales, E. Gago-Ribas, "On the Application of the analytical continuation from real to complex spaces to some EM field solutions", Proceedings on 5th International Symposium on Antennas, Propagation and EM Theory (ISAPE2000), pp. 61-64, Beijing (China), 2000.

[HF01] E. Heyman, L.B. Felsen, "Gaussian beam and pulsed-beam dynamics: complex-source and complex-spectrum formulations within and beyond paraxial asymptotics", Journal of the Optical Society of America A, Vol. 68, No. 7, pp. 1588-1611, 2001.

[GG97] E. Gago-Ribas, M.J. González Morales, "2D Complex Spaces in Electromagnetism", Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA'97), pp. 191-194, Torino (Italy), 2002.

[MGD07] R. Mahíllo-Isla, M.J. González Morales, C. Dehesa-Martínez, "2D complex beam representation in terms of cylindrical harmonics", Progress in Electromagnetic Research Symposium (PIERS 2007), Prague (Czech Republic), 2007.

[MGD09] R. Mahíllo Isla, M.J. González Morales, C. Dehesa Martínez, "Plane wave spectrum of 2D complex beams", Journal of Electrimagnetic Waves and Applications (JEMWA), Vol. 23, pp. 1123-1131, 2009.

Summary of scientific contributions and conference papers

Jounal Scientific Contributions:

i. Published:

[1] P. Vidal-García, E. Gago-Ribas, ”A Logarithmic Version of the Complex Generalized Smith Chart”, Progress in Electromagnetic Research Letters, Vol. 68, pp. 53-58, 2017. (Available online in: `http://www.jpier.org/PIERL/pier.php?paper=17022009`).

ii. Unpublished (available on request):

[1] P.Vidal-García, E. Gago-Ribas, ”Complex Analysis of the Lossy Transmission Line Theory: Generalized Mathematical Analysis and Parameterizations”.

[2] P.Vidal-García, E. Gago-Ribas, ”Complex Analysis of the Lossy Transmission Line Theory: Examples of Use”.

Conference papers:

i. International:

[1] E. Gago-Ribas, P. Vidal-García, J. Heredia Juesas, ”Complex Analysis and Parameterization of the Lossy Transmission Line Theory and its Application to Solve Related Physical Problems”, Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA’15), available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/7297091`, pp. 141-144, Oct. 2015.

[2] P. Vidal-García, E. Gago-Ribas, ”Complex Analysis of the Transmission Line Theory: Anslytical Characterization and Examples of Use”, Proceedings of Progress in Electromagnetic Rerearch Symposium 2016, available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/7735278`, pp. 3262-3266, Oct. 2016.

[3] P. Vidal-García, E. Gago-Ribas, ”The Logarithmic Generalized Smith Chart: Theoretical Analysis”, Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA’17), available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/8065638`, pp. 1766-1769, Oct. 2017.

[4] P. Vidal-García, E. Gago-Ribas, ”The Logarithmic Generalized Smith Chart: Examples of Use”, Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA’17), available at IEEE Xplore Online in `https://ieeexplore.ieee.org/document/8065636`, pp. 1758-1761, Oct. 2017.

ii. National:

[1] P. Vidal-García, E. Gago-Ribas, "A Generalized Complex Transmission Line Theory: Characterization and some Examples", XXXI National Symposium of the International Union of Radio Science (URSI 2016), Madrid (Spain), Sept. 2016. (Available on request).

Activities within the thesis framework

University internships:

i. National:

a) Polytechnic University of Valencia

1. (03/26/2018 - 05/07/2018); Introduction to numerical analysis in waveguides; at Instituto de Telecomunicaciones y Aplicaciones Multimedia (iTeAM).
2. (05/25/2018 - 07/04/2018); Numerical algorithms for the resolution of scattering problems in waveguides; at Instituto de Telecomunicaciones y Aplicaciones Multimedia (iTeAM).

ii. International:

a) Northeastern University (Boston, U.S.)

1. (08/02/2018 - 09/07/2018); Design and measurements in waveguides; at Sica-LAB; (under grant "Movilidad para la Formación de PDI de la Universidad de Oviedo").

Teaching experience:

i. Classes:

2016-2017:
1. Practical classes of *Signal proccesing* (24 hours), Polytechnic School of Engineering, University of Oviedo, Gijón (Spain).
2. Practical classes of *Propagation and Radiation* (36 hours), Polytechnic School of Engineering, University of Oviedo, Gijón (Spain).

2017-2018:
1. Practical classes of *Signals and Systems* (36 hours), Polytechnic School of Engineering, University of Oviedo, Gijón (Spain).
2. Practical classes of *Communication Theory* (24 hours), Polytechnic School of Engineering, University of Oviedo, Gijón (Spain).

ii. Final Degree Projects tuitions:

2017: "Analysis, design, and development of web applications provided with scientific contribution management systems", Industrial and Informatics School of Engineering, University of León, León (Spain).

2018: "Complex analysis of propagative TEM modes on Transmission Lines with frequency dependent parameters", School of Sciences, University of Oviedo, Oviedo (Spain).