UNIFORM CONVERGENCE OF THE FEM. APPLICATIONS TO STATE CONSTRAINED CONTROL PROBLEMS

EDUARDO CASAS AND MARIANO MATEOS

Abstract. In this paper we focus the numerical discretization of a state constrained control problem governed by a semilinear elliptic equation. Distributed and boundary controls are considered. We study the convergence of the discrete optimal controls to the continuous optimal controls in the weak and strong topologies. Previous to this analysis we obtain some results of convergence in the L^∞ norm of the approximations of the state equation by finite elements, which is essential to deal with the pointwise state constraints.

Comput. Appl. Math. 21 (2002), no. 1, 67 – 100

1. Introduction

In this paper, our goal is to prove some convergence theorems of the discretizations of state constrained optimal control problems governed by semilinear elliptic equations. The analysis is carried out for pointwise state constraints. Because of these constraints, to achieve our goal we need the uniform convergence of the numerical discretization of the state equation. Therefore the first part of the paper is devoted to prove some convergence results and L^∞ error estimates of the approximations of the state equation by finite elements. We consider the Dirichlet and Neumann cases, which correspond to distributed and boundary controls.

The presence of the state constraints introduces some difficulties in the analysis. Maybe the most important difficulty appears when we try to approximate an admissible control by a sequence of discrete controls admissible for the corresponding discrete problems. In particular the possibility of obtaining this approximation for the optimal control is essential to prove the convergence of the discretizations. This cannot be done for every control problem, some stability of the optimal cost functional with respect to small perturbations of the set of admissible states is necessary to have an approximating sequence of the optimal control formed by discrete admissible controls.

An advance of these ideas was presented by E. Casas in [2] for the case of a Dirichlet boundary value problem and distributed controls under the assumption of $H^2(\Omega)$-regularity of the states. In this paper we consider the case where only $W^{1,p}(\Omega)$-regularity of the states can be assured, which allows us to extend the results to the case of a Neumann boundary value problem and boundary controls. We prove some new error estimates and convergence results for the approximation of the state equations by finite elements. Then we use these results to establish the

---

1991 *Mathematics Subject Classification.* 49N99, 35J25, 65N30.

Key words and phrases. Elliptic partial differential equations, Finite element Method, Uniform error estimates, Optimal control problems, State constraints.

Research partially supported by Dirección General de Enseñanza Superior e Investigación Científica (Spain).

convergence of the so called stable control problems. We will also see that almost all problems are stable.

Error estimates for the control is not considered in this paper. It is an open problem how to obtain error estimates for the discretizations of the controls under the presence of pointwise state constraints. Second order optimality conditions have to be used in order to derive these estimates, but these conditions are not well known for the case of pointwise state constraints. In the absence of state constraints, some papers dealing with error estimates can be found in the literature; see N. Arada, E. Casas and F. Tröltzsch [1] and the references there cited.

The plan of the paper is as follows. In Section 2, the error estimates for the discretization of a semilinear elliptic equation is studied. The section is divided into two parts, the first one devoted to the Dirichlet case, and the second one dealing with the Neumann case. Section 3 is also divide into two parts, in the first part the numerical approximation of a distributed control problem is studied. In the second part, the case of a boundary control is considered.

2. Error estimates in the discretization of semilinear equations

There are some papers in the literature devoted to the study of uniform estimates for the error in the finite element method for linear and quasilinear elliptic equations; see R . Scott [14] or J. Frehse and R. Rannacher [9]. In these papers, the regularity required for the solution of the equation is stronger than that one we can assume for the optimal states of the pointwise state constrained optimal control problems. In the linear case, P.G. Ciarlet and P.A. Raviart [6] relaxed the regularity of the state by using a discrete maximum principle valid for some grids of finite elements, the so called triangulations of *nonnegative type*. They prove uniform convergence of $\|y - y_h\|_\infty \to 0$ for this type of triangulations under the assumption that $y \in W^{1,p}(\Omega)$. Also they derived some error estimates for states $y \in W^{2,p}(\Omega)$. Finally, by assuming $H^2(\Omega)$ regularity for y, there are some well known uniform error estimates for linear equations; see for instance P.G. Ciarlet [4, pags. 143–144].

In this section we will prove uniform convergence and error estimates for states in $H^2(\Omega)$ and $W^{1,p}(\Omega)$ and semilinear elliptic equations. As far as we know, these results are new. We start with the Dirichlet problem and we will finish with the Neumann problem. First we introduce some notation and hypotheses assumed in the whole paper.

Let Ω be a convex subset of R^N, $N = 2$ or $N = 3$, Γ its boundary and A an operator of the form

$$Ay = -\sum_{i,j=1}^{N} \partial_{x_j} \left[a_{ij}\partial_{x_i} y\right],$$

where $a_{i,j} \in C^{0,1}(\bar{\Omega})$ and such that there exist $m, M > 0$ satisfying

$$m\|\xi\|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq M\|\xi\|^2 \quad \forall \xi \in \mathbb{R}^N \text{ and } \forall x \in \Omega.$$

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, monotone decreasing in the second variable, with $f(\cdot, 0) \in L^{p/2}(\Omega)$, for some $p > N$, and satisfying the following local Lipschitz condition: for all $M > 0$ there exists $\phi_M \in L^2(\Omega)$ such that

$$(1) \qquad |f(x, y_1) - f(x, y_2)| \leq |\phi_M(x)||y_1 - y_2| \text{ for a.e. } x \in \Omega \text{ and } |y_1|, |y_2| \leq M.$$

In $\bar{\Omega}$ we consider a family of triangulations on $\{\mathcal{T}_h\}_{h>0}$. To each element $T \in \mathcal{T}_h$ let us associate two parameters: $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the greatest ball included in T. We will set $h = \max_{T\in\mathcal{T}_h} \rho(T)$. We will make the following assumptions on the triangulation:

- Regularity assumption: there exists $\sigma > 0$ such that $\frac{\rho(T)}{\sigma(T)} \leq \sigma \quad \forall T \in \mathcal{T}_h$ and $h > 0$.
- Inverse assumption: there exists $\rho > 0$ such that $\frac{h}{\rho(T)} \leq \rho \quad \forall T \in \mathcal{T}_h$ and $h > 0$.
- Set $\bar{\Omega}_h = \cup_{T\in\mathcal{T}_h} T$, Ω_h its interior and Γ_h its boundary. Then we will suppose that the vertexes of $\mathcal{T}_h$ placed on the boundary of Γ_h are points of Γ.

Consider the spaces

$$V_h = \left\{ y_h \in C(\bar{\Omega}) : \ y_{h|_T} \in P_1(T) \quad \forall T \in \mathcal{T}_h \ y_h = 0 \text{ in } \Omega \setminus \Omega_h \right\}$$

and

$$W_h = \left\{ y_h \in C(\bar{\Omega}_h) : \ y_{h|_T} \in P_1(T) \quad \forall T \in \mathcal{T}_h \right\},$$

where $P_1(T)$ is the space of polynomials of degree 1 on T. V_h is a vector subspace of $W_0^{1,p}(\Omega)$ and W_h is a subspace of $W^{1,p}(\Omega)$.

We will use the Lagrange interpolation operator

$$\Pi_h : \quad C(\bar{\Omega}) \quad \longrightarrow \quad W_h$$

being $\Pi_h z$ the unique element in W_h such that $\Pi_h z(x_i) = z(x_i)$ for every node of the triangulation x_i. In the case of a function z vanishing on Γ, we will extend $\Pi_h z$ to $\bar{\Omega}$ by zero and we will denote this extension by $\Pi_h z$ too. In the last case we have that $\Pi_h z \in V_h$.

2.1. **Dirichlet case.** Here our goal is to study the uniform approximation by the finite element method of the solution of the equation

$$\left\{ \begin{array}{rcll} Ay & = & f(\cdot, y) + f_0 & \text{in } \Omega \\ y & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{2}$$

where $f_0 \in W^{-1,p}(\Omega)$ for some $p > N$.

By classical arguments we can deduce from the monotonicity of f and (1) the existence of a unique solution of (2) in $H_0^1(\Omega) \cap C(\bar{\Omega})$; see G. Stampacchia [15] for the boundedness of the solution. Now if $f_0 = 0$ and $f(\cdot, 0) \in L^2(\Omega)$, we deduce from the convexity of Ω and the Lipschitz continuity of the coefficients a_{ij} that the solution is in $H^2(\Omega)$; see P. Grisvard [10]. If $f_0 \neq 0$ or $f(\cdot, 0) \notin L^2(\Omega)$, then the solution belongs only to the Sobolev space $W^{1,p}(\Omega)$ for some $p > N$, close to N; see D. Jerison and C. Kenig [11] and M. Mateos [12].

The discrete version of (2) is defined as follows. For every h, let us define $y_h \in V_h$ as the unique element that satisfies

$$a(y_h, z_h) = \int_\Omega f(x, y_h(x)) z_h dx + \langle f_0, z_h \rangle_{W^{-1,p}(\Omega)\times W_0^{1,p}(\Omega)} \quad \forall z_h \in V_h, \tag{3}$$

where

$$a(y_h, z_h) = \sum_{i,j=1}^{N} \int_\Omega a_{i,j}(x) \partial_{x_i} y_h(x) \partial_{x_j} z_h(x) dx.$$

The proof of the existence of a solution of (3) is well known. It is enough to apply, in a convenient way, Browder's fixed point theorem along with the monotonicity of f. Our purpose is to prove that $y_h \to y$ in $C(\bar{\Omega})$. We will distinguish the most regular case corresponding to $f_0 = 0$ from the general one $f_0 \neq 0$. In the regular case, we will derive the same error estimates than for the linear case. Before stating the results let us formulate a lemma whose proof can be found in P.G. Ciarlet and P.A. Raviart [5]; see also P.G. Ciarlet [3].

Lemma 1. *Set $m \geq 0$, $k \geq 0$, and $p, q \in [1, \infty]$. If the embeddings*

$$W^{k+1,p}(T) \hookrightarrow C^0(T)$$
$$W^{k+1,p}(T) \hookrightarrow W^{m,q}(T)$$

hold, then there exists a constant $C > 0$ independent of h such that

$$\|y - \Pi_T y\|_{W^{m,q}(T)} \leq C h^{N\left(\frac{1}{q}-\frac{1}{p}\right)+k+1-m} \|y\|_{W^{k+1,p}(T)}, \tag{4}$$

where $\Pi_T y$ is the restriction of $\Pi_h y$ to the element T.

The following inequality, whose proof can be found in P.G. Ciarlet [3, Theorem 17.2], gives us the equivalence constant between two Sobolev norms in a finite dimensional space:

$$\|y_h\|_{W^{m,q}(\Omega_h)} \leq C \frac{1}{h^{N \max\left\{0, \frac{1}{p}-\frac{1}{q}\right\}} h^{m-l}} \|y_h\|_{W^{l,p}(\Omega_h)}\ \forall v_h \in V_h,\ \text{if } l \leq m, \tag{5}$$

$C > 0$ being independent of h.

2.1.1. *Regular case.* Suppose now that $f_0 = 0$ and $f(\cdot, 0) \in L^2(\Omega)$. In a first step, we will also make a stronger assumption on f than that one fixed in (1): there exists a function $\phi \in L^2(\Omega)$ such that

$$|f(x, t_1) - f(x, t_2)| \leq |\phi(x)|\,|t_1 - t_2| \quad \forall t_1, t_2 \in \mathbb{R}, \quad \text{a.e. } x \in \Omega. \tag{6}$$

This restrictive condition of global type will be relaxed later and we will see that assumption (1) is enough. As mentioned before, equation (2) has a unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$ and it can be formulated variationally as follows

$$\left\{\ a(y, z) = (f(x, y), z)_{L^2(\Omega)}\ \forall z \in H_0^1(\Omega).\right. \tag{7}$$

The numerical approximation of y is now given by the variational equation

$$\left\{\begin{array}{l} \text{Find } y_h \in V_h \text{ such that} \\ a(y_h, z_h) = (f(x, y_h), z_h)_{L^2(\Omega)}\ \forall z_h \in V_h. \end{array}\right. \tag{8}$$

The following result is a generalization for semilinear equations of the well known Céa's Lemma

Lemma 2. *Let y and y_h be solutions of the variational problems* (7) *and* (8) *respectively. Then there exists a constant $C > 0$ independent of h such that*

$$\|y - y_h\|_{H^1(\Omega)} \leq C \|y - \Pi_h y\|_{H^1(\Omega)}.$$

Proof. The result is a consequence of the $H_0^1(\Omega)$-ellipticity of a, the monotonicity of f in the second variable, the Lipschitz condition imposed on f and the continuous embedding from $H^1(\Omega)$ in $L^4(\Omega)$:

$$\begin{aligned}
&\|y-y_h\|^2_{H^1(\Omega)} \leq Ca(y-y_h,y-y_h) \leq \\
&\leq C\left\{a(y-y_h,y-y_h)-(f(\cdot,y)-f(\cdot,y_h),y-y_h)\right\} = \\
&= C\left\{a(y-y_h,y-z_h)-(f(\cdot,y)-f(\cdot,y_h),y-z_h)\right\} \leq \\
&\leq C\left\{\|y-y_h\|_{H^1(\Omega)}\|y-z_h\|_{H^1(\Omega)}+\|\phi\|_{L^2(\Omega)}\|y-y_h\|_{L^4(\Omega)}\|y-z_h\|_{L^4(\Omega)}\right\} \leq \\
&\leq C\left\{\|y-y_h\|_{H^1(\Omega)}\|y-z_h\|_{H^1(\Omega)}+\|\phi\|_{L^2(\Omega)}\|y-y_h\|_{H^1(\Omega)}\|y-z_h\|_{H^1(\Omega)}\right\} \leq \\
&\leq C\|y-y_h\|_{H^1(\Omega)}\|y-z_h\|_{H^1(\Omega)} \text{ for all } z_h\in V_h.
\end{aligned}$$

Dividing by $\|y-y_h\|_{H^1(\Omega)}$ and taking $z_h=\Pi_h y$ we achieve to the desired result. □

Now we have the following lemma.

Lemma 3. *Let y and y_h be solutions of the variational problems* (7) *and* (8) *respectively. Then there exists a constant $C>0$ independent of h such that*

$$\|y-y_h\|_{H^1(\Omega)} \leq Ch\|y\|_{H^2(\Omega)}.$$

Proof. Using Lemma 2, the inequality

$$\|y\|_{H^1(\Omega\setminus\Omega_h)} \leq Ch\|y\|_{H^2(\Omega)}$$

(cf. P.A. Raviart and J.M. Thomas [13, Lemma 5.2-3]) and Lemma 1 with $m=1$, $q=2$, $k=1$ and $p=2$, we have that

$$\|y-y_h\|_{H^1(\Omega)} \leq C\|y-\Pi_h y\|_{H^1(\Omega)} \leq$$

$$C\left(\|y\|_{H^1(\Omega\setminus\Omega_h)}+\|y-\Pi_h y\|_{H^1(\Omega_h)}\right) \leq Ch\|y\|_{H^2(\Omega)},$$

and the proof is complete. □

To obtain the error estimate in $L^2(\Omega)$ let us introduce the function

$$\alpha(x)=\left\{\begin{array}{ll}\dfrac{f(x,y_h(x))-f(x,y(x))}{y(x)-y_h(x)} & \text{if } y(x)\neq y_h(x)\\ 0 & \text{in other case.}\end{array}\right. \tag{9}$$

Notice that $\alpha(x)\geq 0$.

We have that for all $\psi\in L^2(\Omega)$ there exists a unique $z_\psi\in H^2(\Omega)\cap H^1_0(\Omega)$ satisfying

$$\left\{\begin{array}{rcll} A^*z_\psi+\alpha(x)z_\psi &=& \psi & \text{in } \Omega\\ z_\psi &=& 0 & \text{on } \Gamma.\end{array}\right.$$

From (6) we deduce that $\|\alpha\|_{L^2(\Omega)}\leq\|\phi\|_{L^2(\Omega)}$, then there exists a constant $C>0$ independent of α such that $\|z_\psi\|_{H^2(\Omega)}\leq C\|\psi\|_{L^2(\Omega)}$. This Dirichlet problem can be formulated variationally as

$$a(z,z_\psi)+(\alpha z_\psi,z)=(\psi,z)\ \forall z\in H^1_0(\Omega), \tag{10}$$

and it can be approximated by

$$a(z_h,z_{\psi,h})+(\alpha z_{\psi,h},z_h)=(\psi,z_h)\ \forall z_h\in V_h. \tag{11}$$

We are going to apply a very similar technique to that of the linear case to find an error estimate for $y-y_h$ in $L^2(\Omega)$.

Lemma 4. *Let y and y_h be solutions of the variational problems* (7) *and* (8) *respectively. Then there exists a constant $C>0$ independent of h such that*

$$\|y-y_h\|_{L^2(\Omega)} \leq Ch^2\|y\|_{H^2(\Omega)}.$$

Proof. Take any $\psi \in L^2(\Omega)$. Using (10), the definition of $\alpha(x)$, (7) and (8), the continuity of a, Lipschitz's condition (6), and Sobolev's and Hölder's inequalities as in the previous proof, we have

$$
\begin{aligned}
&(\psi, y-y_h) = a(y-y_h, z_\psi) + (\alpha z_\psi, y-y_h) = \\
&= a(y-y_h, z_\psi - z_{\psi,h}) + a(y-y_h, z_{\psi_h}) + (\alpha z_\psi, y-y_h) = \\
&= a(y-y_h, z_\psi - z_{\psi,h}) + \int_\Omega (f(x,y)-f(x,y_h)) z_{\psi,h}\, dx + \\
&+ \int_\Omega \frac{f(x,y_h)-f(x,y)}{y-y_h} z_\psi (y-y_h)\, dx = \\
&= a(y-y_h, z_\psi - z_{\psi,h}) + \int_\Omega (f(x,y_h)-f(x,y))(z_\psi - z_{\psi,h})\, dx \le \\
&\le C\|y-y_h\|_{H^1(\Omega)}\|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} + \int_\Omega |\phi(x)|\,|y-y_h|\,|z_\psi - z_{\psi,h}|\, dx \le \\
&\le C\|y-y_h\|_{H^1(\Omega)}\|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} + \|\phi\|_{L^2(\Omega)}\|y-y_h\|_{L^4(\Omega)}\|z_\psi - z_{\psi,h}\|_{L^4(\Omega)} \le \\
&\le C\|y-y_h\|_{H^1(\Omega)}\|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} \le Ch\|y\|_{H^2(\Omega)} h\|z_\psi\|_{H^2(\Omega)} \le \\
&\le Ch^2\|y\|_{H^2(\Omega)}\|\psi\|_{L^2(\Omega)},
\end{aligned}
$$

where the last estimates follow from Lemma 3 and the usual estimates for finite elements. Thus

$$\|y-y_h\|_{L^2(\Omega)} = \sup_{\|\psi\|_{L^2(\Omega)}\le 1} (\psi, y-y_h) \le Ch^2\|y\|_{H^2(\Omega)},$$

and the proof is complete. □

Theorem 1. *Let y and y_h be solutions of the variational problems* (7) *and* (8) *respectively. Then there exists a constant $C > 0$, independent of h such that*

$$\|y-y_h\|_{L^\infty(\Omega_h)} \le Ch^{2-\frac{N}{2}}\|y\|_{H^2(\Omega)}.$$

Proof. We have that

$$\|y-y_h\|_{L^\infty(\Omega_h)} \le \|y-\Pi_h y\|_{L^\infty(\Omega_h)} + \|\Pi_h y - y_h\|_{L^\infty(\Omega_h)}. \tag{12}$$

Due to Lemma 1, taking $m = 0$, $q = \infty$, $k = 1$ and $p = 2$, we have that

$$\|y-\Pi_h y\|_{L^\infty(\Omega_h)} \le Ch^{2-\frac{N}{2}}\|y\|_{H^2(\Omega)}. \tag{13}$$

Applying (5) we have that

$$\|\Pi_h y - y_h\|_{L^\infty(\Omega_h)} \le Ch^{-\frac{N}{2}}\|\Pi_h y - y_h\|_{L^2(\Omega_h)}. \tag{14}$$

Again due to Lemma 1, taking $m = 0$, $q = 2$, $k = 1$ and $p = 2$, we get

$$\|\Pi_h y - y\|_{L^2(\Omega_h)} \le Ch^2\|y\|_{H^2(\Omega)}, \tag{15}$$

and due to Lemma 4

$$\|y-y_h\|_{L^2(\Omega_h)} \le Ch^2\|y\|_{H^2(\Omega)}. \tag{16}$$

From (15) and (16) it follows that

$$\|\Pi_h y - y_h\|_{L^2(\Omega_h)} \le \|\Pi_h y - y\|_{L^2(\Omega_h)} + \|y-y_h\|_{L^2(\Omega_h)}, \le Ch^2\|y\|_{H^2(\Omega)}.$$

This along with (14) implies that

$$\|\Pi_h y - y_h\|_{L^\infty(\Omega_h)} \le Ch^{2-\frac{N}{2}}\|y\|_{H^2(\Omega)},$$

which together with (12) and (13) complete the proof of the theorem. □

Let us see now how we can obtain the same results with less restrictive conditions on the growing of f in the second variable.

Theorem 2. *Suppose that* (1) *holds and that* $f(x,0) \in L^2(\Omega)$. *Then the conclusions of Lemmas 3 and 4 and of Theorem 1 remain valid.*

Proof. Set $M = \|y\|_{L^\infty(\Omega)} + 1$ and

$$f_M(x,t) = \begin{cases} f(x,-M) & \text{if } t < -M \\ f(x,t) & \text{if } |t| \le M \\ f(x,M) & \text{if } t > M. \end{cases}$$

We have that for all $x \in \Omega$, $f_M(x,y(x)) \equiv f(x,y(x))$. And therefore we have that

$$\begin{cases} Ay &= f_M(x,y) \text{ in } \Omega \\ y &= 0 \text{ on } \Gamma. \end{cases}$$

Take y_h^M the solution of the discrete variational problem

$$\begin{cases} \text{Find } y_h^M \in V_h \text{ such that} \\ a(y_h^M, z_h) = (f_M(x,y_h^M), z_h) \ \forall z_h \in V_h. \end{cases}$$

From Theorem 1 we have that

$$\|y - y_h^M\|_{L^\infty(\Omega_h)} \le Ch^{2-\frac{N}{2}}\|y\|_{H^2(\Omega)},$$

therefore for all h less than a certain h_0 we have that $\|y - y_h^M\|_{L^\infty(\Omega_h)} \le 1$, and then $\|y_h^M\|_{L^\infty(\Omega_h)} \le \|y\|_{L^\infty(\Omega)} + 1 = M$, which implies that $f_M(x,y_h^M) = f(x,y_h^M)$ and consequently y_h^M is the solution of the problem (8) and the desired estimates hold. □

2.1.2. *Non regular case.* Now we suppose that $f_0 \ne 0$, $f_0 \in W^{-1,p}(\Omega)$. Under this situation $y \in W_0^{1,p}(\Omega)$ for some $p > N$. As before, we will start assuming that the global condition (6) holds. First we prove convergence in $H^1(\Omega)$.

Lemma 5. *Let y and y_h be the solutions of equations* (2) *and* (3) *respectively. Then*

$$\lim_{h\to 0} \|y - y_h\|_{H^1(\Omega)} = 0.$$

Proof. Obviously Cea's Lemma 2 remains valid, then

$$\lim_{h\to 0} \|y - y_h\|_{H^1(\Omega)} \le \lim_{h\to 0} C\|y - \Pi_h y\|_{H^1(\Omega)} \le \lim_{h\to 0} C\|y - \Pi_h y\|_{W^{1,p}(\Omega)} = 0,$$

the last equality being a well known result of the interpolation theory in Sobolev spaces; see P.G. Ciarlet[3]. □

A convergence result in $L^2(\Omega)$ can also be proved.

Lemma 6. *Let y and y_h be the solutions of equations* (2) *and* (3) *respectively. Then*

$$\lim_{h\to 0} \frac{\|y - y_h\|_{L^2(\Omega)}}{h} = 0.$$

Proof. Take $\psi \in L^2(\Omega)$. Following exactly the proof of Lemma 4 we obtain

$$(\psi, y - y_h) \le C\|y - y_h\|_{H^1(\Omega)}\|z_\psi - z_{\psi,h}\|_{H^1(\Omega)} \le Ch\|y - y_h\|_{H^1(\Omega)}\|\psi\|_{L^2(\Omega)}.$$

So

$$\frac{1}{h}\|y - y_h\|_{L^2(\Omega)} \le C\|y - y_h\|_{H^1(\Omega)}$$

and applying Lemma 5 we obtain the desired limit. □

Lemma 7. *Let $y \in W^{1,p}(\Omega)$ with $p > N$. Then*

$$\lim_{h\to 0} \frac{\|y - \Pi_h y\|_{L^p(\Omega_h)}}{h} = 0.$$

Proof. We use Lemma 1 and the fact that $\Pi_h y \in W^{1,p}(\Omega_h)$ to deduce that

$$\|y - \Pi_h y\|_{L^p(\Omega_h)} = \|y - \Pi_h y - \Pi_h(y - \Pi_h y)\|_{L^p(\Omega_h)} \leq Ch\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)}$$

and the result follows dividing by h and using again the convergence of the interpolation in $W^{1,p}(\Omega)$. $\square$

Now we can prove the uniform convergence in dimension 2.

Theorem 3. *Suppose $N = 2$. Let y and y_h be the solutions of equations* (2) *and* (3) *respectively. Then*

$$\lim_{h\to 0} \|y - y_h\|_{L^\infty(\Omega)} = 0.$$

Proof. If we apply the triangular inequality, Lemma 1, the inequality (5) and that $N = 2$ we obtain

$$\begin{aligned}
\|y - y_h\|_{L^\infty(\Omega_h)} &\leq \|y - \Pi_h y\|_{L^\infty(\Omega_h)} + \|\Pi_h y - y_h\|_{L^\infty(\Omega_h)} \\
&\leq C\left[h^{1-\frac{2}{p}}\|y\|_{W^{1,p}(\Omega)} + h^{-1}\|\Pi_h y - y_h\|_{L^2(\Omega_h)}\right] \\
&\leq C\left[h^{\frac{p-2}{p}}\|y\|_{W^{1,p}(\Omega)} + \frac{\|\Pi_h y - y\|_{L^2(\Omega_h)}}{h} + \frac{\|y - y_h\|_{L^2(\Omega_h)}}{h}\right]
\end{aligned}$$

Taking into account $p > N = 2$, Lemmas 6 and 7 and the continuous embedding $L^p(\Omega) \subset L^2(\Omega)$, we deduce that the last three terms converge to zero.

Finally, notice that since $y \in C(\bar{\Omega})$ and $y(x) = 0$ on Γ, then $\|y\|_{L^\infty(\Omega\setminus\Omega_h)}$ tends to zero when $h \to 0$, so the proof is complete. $\square$

To give a result in dimension 3 we must make two extra assumptions:

(H1) Function ϕ given in (6) belongs to $L^r(\Omega)$ for some $r > 2$

(H2) The triangulation is of non negative type; see P.G. Ciarlet and P.A. Raviart [6] or P.G. Ciarlet [3] for this definition.

In the case of a non negative type triangulation $\mathcal{T}_h$, if $p > N$, $a_{i,j} \in L^\infty(\Omega)$ and y_h is the solution of the discrete problem

$$a(y_h, z_h) = \langle g, z_h\rangle \quad \text{for all } z_h \in V_h,$$

with $g \in W^{-1,p}(\Omega_h)$, then the discrete maximum principle holds [6]:

$$\|y_h\|_{L^\infty(\Omega_h)} \leq C\|g\|_{W^{-1,p}(\Omega_h)}. \tag{17}$$

Using this principle we prove the following theorem.

Theorem 4. *Suppose that the coefficients $a_{i,j} \in L^\infty(\Omega)$, and let y and y_h be the solutions of the equations* (2) *and* (3) *respectively. Then, if the triangulation is of non negative type*

$$\|y - y_h\|_{L^\infty(\Omega_h)} \leq Ch\|y\|_{W^{2,p}(\Omega)} \quad \text{if } y \in W^{2,p}(\Omega),\ p > 2N \tag{18}$$

and

$$\lim_{h\to 0} \|y - y_h\|_{L^\infty(\Omega)} = 0 \quad \text{if } y \in W^{1,p}(\Omega),\ p > N. \tag{19}$$

Proof. Notice first that in order to have the solution in $W^{1,p}(\Omega)$ it is sufficient for the coefficients $a_{i,j}$ to belong to $C(\bar{\Omega})$. On the other hand, the $W^{2,p}(\Omega)$ regularity can be assured if the coefficients are in $C^{0,1}(\bar{\Omega})$, Γ is of class $C^{1,1}$ and $f(\cdot, y)$ and f_0 are in $L^p(\Omega)$.

Let $y \in W_0^{1,p}(\Omega)$ and $y_h \in V_h$ be solutions of the problems (2) and (3) respectively. We have that $y_h - \Pi_h y$ is the unique element of V_h that satisfies

$$a(y_h - \Pi_h y, z_h) = a(y - \Pi_h y, z_h) + (f(x, y_h) - f(x, y), z_h)\ \forall z_h \in V_h. \tag{20}$$

Let us study the norm of the operator

$$T : W_0^{1,p'}(\Omega_h) \longrightarrow \mathbb{R}$$
$$Tz = a(y - \Pi_h y, z) + (f(x, y_h) - f(x, y), z),$$

where p' is the conjugate exponent of p. Due to Hölder's inequality, we know that

$$a(y - \Pi_h y, z) \leq C\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)}\|z\|_{W_0^{1,p'}(\Omega_h)} \quad \forall z \in W_0^{1,p'}(\Omega_h).$$

We have that $W^{1,p}(\Omega_h) \hookrightarrow H^1(\Omega_h) \hookrightarrow L^6(\Omega_h)$. If we suppose that $p \leq 3 + \varepsilon$, with ε small enough, then $W^{1,p'}(\Omega_h) \hookrightarrow L^s(\Omega_h)$, with $s < 3$, as close to 3 as we precise. So s can be chosen in such a way that

$$\frac{1}{r} + \frac{1}{6} + \frac{1}{s} = 1.$$

Thus, using Hölder's inequality and Cèa's generalized lemma (Lemma 2)

$$\begin{aligned}
\left|\int_{\Omega_h} (f(x, y_h) - f(x, y))z\,dx\right| &\leq \int_{\Omega_h} |\phi(x)|\,|y - y_h|\,|z|\,dx \\
&\leq \|\phi\|_{L^r(\Omega)}\|y - y_h\|_{L^6(\Omega_h)}\|z\|_{L^s(\Omega_h)} \\
&\leq C\|y - y_h\|_{H^1(\Omega_h)}\|z\|_{W^{1,p'}(\Omega_h)} \\
&\leq C\|y - \Pi_h y\|_{H^1(\Omega_h)}\|z\|_{W^{1,p'}(\Omega_h)} \\
&\leq C\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)}\|z\|_{W^{1,p'}(\Omega_h)}.
\end{aligned}$$

Therefore

$$\|T\|_{W^{-1,p}(\Omega_h)} \leq C\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)}$$

But, applying maximum principle (17) if $3 < p \leq 3 + \varepsilon$ to equation (20) we have that there exists a constant $C > 0$ independent of h such that

$$\|y_h - \Pi_h y\|_{L^\infty(\Omega_h)} \leq C\|T\|_{W^{-1,p}(\Omega_h)} \leq C\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)},$$

and using that $W^{1,p}(\Omega_h) \hookrightarrow L^\infty(\Omega_h)$, we get to:

$$\begin{aligned}
\|y - y_h\|_{L^\infty(\Omega_h)} &\leq \|y - \Pi_h y\|_{L^\infty(\Omega_h)} + \|y_h - \Pi_h y\|_{L^\infty(\Omega_h)} \quad \leq \\
&\leq C\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)} \to 0 \text{ when } h \to 0,
\end{aligned} \tag{21}$$

which proves (19).

If $y \in W^{2,p}(\Omega)$, applying Lemma 1 we have that

$$\|y - \Pi_h y\|_{W^{1,p}(\Omega_h)} \leq Ch\|y\|_{W^{2,p}(\Omega)}, \tag{22}$$

which along with (21) implies (18). $\square$

Theorem 5. *Under the assumptions of Lemmas 5 and 6 and Theorems 3 and 4 and replacing the hypothesis* (6) *by* (1)*, then the conclusions of these Lemmas and Theorems remain valid.*

The proof of this theorem is analogous to that of Theorem 3.

2.2. **Neumann case.** We will suppose for Neumann's problem that Γ is polygonal or polyhedrical. In this case $\Omega_h = \Omega$. Let us consider now $a_0 \in L^{\frac{Np}{N+p}}(\Omega)$, $a_0 \geq 0$, $a_0 \not\equiv 0$ in Ω, $f_0 \in (W^{1,p'}(\Omega))'$ and $v \in L^\infty(\Gamma)$. We want to study the uniform approximation by the finite element method of the solution of the equation

$$\left\{\begin{array}{rcll} Ay + a_0 y & = & f(\cdot, y) + f_0 & \text{in } \Omega \\ \partial_{n_A} y & = & v & \text{on } \Gamma. \end{array}\right. \tag{23}$$

For each h, let us define $y_h \in W_h$ as the unique element that satisfies

(24)

$$\sum_{i,j=1}^{N} \int_\Omega a_{i,j}(x) \partial_{x_i} y_h(x) \partial_{x_j} z_h(x) dx + \int_\Omega a_0(x) y_h(x) z_h(x) dx =$$
$$\int_\Omega f(x, y_h(x)) z_h dx + \langle f_0, z_h \rangle_{W^{1,p'}(\Omega))' \times W^{1,p'}(\Omega)} + \int_\Gamma v(s) z_h(s) ds \quad \forall z_h \in W_h.$$

As for the Dirichlet case, this variational equation has a unique solution y_h. Our objective is to show that $y_h \to y$ in $L^\infty(\Omega)$. We will get advantage of these results in next section to study a control problem, where v will stand for the control. Generally $v \notin H^{\frac{1}{2}}(\Gamma)$ and therefore we cannot obtain $H^2(\Omega)$ regularity for the state y. The maximum regularity we can deduce is $y \in W^{1,p}(\Omega)$ for some $p > N$; see M. Dauge [7]. First we state the following lemma.

Lemma 8. *Let y and y_h be respectively the solutions of the equations* (23) *and* (24)*. Then*

$$\lim_{h \to 0} \|y - y_h\|_{H^1(\Omega)} = 0$$

and

$$\lim_{h \to 0} \frac{\|y - y_h\|_{L^2(\Omega)}}{h} = 0.$$

Proof. The first equality is deduced in the same way than in the proof of Lemma 5. For the second identity we can follow the same procedure as in the proof of Lemma 6 by taking for every $\psi \in L^2(\Omega)$, $z_\psi \in H^2(\Omega)$ as the unique solution of

$$\left\{\begin{array}{rcll} A^* z_\psi + a_0 z_\psi + \alpha(x) z_\psi & = & \psi & \text{in } \Omega \\ \partial_{n_{A^*}} z_\psi & = & 0 & \text{on } \Gamma, \end{array}\right. \tag{25}$$

with $\alpha(x)$ defined as in (9). $\square$

Finally, adapting, in the obvious way, the proofs of Theorems 3, 4 and 5 and using Lemma 8 we obtain the next two theorems.

Theorem 6. *Suppose that $N = 2$. Let y and y_h be the solutions of the equations* (23) *and* (24) *respectively. Then*

$$\lim_{h \to 0} \|y - y_h\|_{L^\infty(\Omega)} = 0.$$

Theorem 7. *Suppose that the coefficients $a_{i,j} \in L^\infty(\Omega)$, and let y and y_h be the solutions of the equations* (23) *and* (24) *respectively. Then, if the triangulation is of non negative type,*

$$\|y - y_h\|_{L^\infty(\Omega)} \leq Ch \|y\|_{W^{2,p}(\Omega)} \quad \text{if } y \in W^{2,p}(\Omega),\ p > 2N \tag{26}$$

and

$$\lim_{h \to 0} \|y - y_h\|_{L^\infty(\Omega)} = 0 \quad \text{if } y \in W^{1,p}(\Omega),\ p > N. \tag{27}$$

3. Numerical approximation of optimal control problems

This section is devoted to the study of the discretization of a control problem. In the first part we study a distributed control problem governed by a semilinear equation with Dirichlet boundary conditions and in the second part a boundary control problem governed by an equation with Neumann boundary conditions will be considered.

3.1. **Distributed control.** We will follow the notation introduced in Section 2. The control problem is defined as follows.

$$(\mathrm{P}_\delta)\left\{\begin{array}{l}\min J(u)=\int_\Omega L\left(x, y_u(x), u(x)\right) dx \\ u \in K, \quad g\left(x, y_u(x)\right) \leq \delta \quad \forall x \in \bar{\Omega},\end{array}\right.$$

where

(1) K is a convex, weakly* closed, bounded and non empty subset of $L^\infty(\Omega)$.
(2) $L : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a Carathéodory function, convex in the third variable and satisfying

$$\forall M > 0\ \exists \psi_M \in L^1(\Omega) \text{ such that } |L(x, y, u)| \leq \psi_M(x) \tag{28}$$

for a.e. $x \in \Omega$ and $|y|$, $|u| \leq M$.
(3) $g : \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.
(4) (y, u) satisfies the state equation

$$\left\{\begin{array}{rcll} Ay_u & = & f(x, y_u) + u & \text{in } \Omega \\ y_u & = & 0 & \text{on } \Gamma. \end{array}\right. \tag{29}$$

Since $f(\cdot, 0) + u \in L^{p/2}(\Omega)$ for some $p > N$, then (29) has a unique solution $y_u \in W_0^{1,p}(\Omega)$ for some $p > N$ small enough; see M. Mateos [12]. Moreover, the boundedness and the estimates for y_u provided in [12] imply that there exists $C_K > 0$, independent of u, such that

$$\|y_u\|_{W^{1,p}(\Omega)} \leq C_K \quad \forall u \in K. \tag{30}$$

Moreover if $\{u_j\}_{j=1}^\infty \subset L^\infty(\Omega)$ and $u_j \to u$ weakly* in $L^\infty(\Omega)$, then $y_{u_j} \to y_u$ strongly in $W^{1,p}(\Omega)$. Finally, if $f(\cdot, 0) \in L^2(\Omega)$, then $y_u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Concerning the existence of a solution for problem (P_δ), we have the following result.

Theorem 8. *There exists a number $\delta_0 \in \mathbb{R}$ such that problem (P_δ) has at least one solution for every $\delta \geq \delta_0$, and (P_δ) has no admissible controls for $\delta < \delta_0$.*

Proof. From (30) we deduce that there exists a constant C such that $\|y_u\|_{L^\infty(\Omega)} \leq C$ for every $u \in K$. Let M and m be the respectively the supremum and the infimum of g in $\bar{\Omega} \times [-C, C]$. Then it is obvious that (P_δ) does not have admissible controls for $\delta < m$ and all the elements of K are admissible controls for $\delta \geq M$. Let δ_0 be the infimum of the values δ for which (P_δ) has admissible controls. Then $m \leq \delta_0 \leq M$ and (P_δ) has not admissible controls for $\delta < \delta_0$. Let us prove that there exists at least an admissible control for (P_{δ_0}). Let $\{\delta_j\}$ be a decreasing sequence converging to δ_0 and $\{u_j\} \subset K$ a sequence of controls such that every $\{u_j\}$ is admissible for (P_{δ_j}). Since K is bounded, we can take a subsequence, denoted in the same way and weakly* convergent in $L^\infty(\Omega)$ to an element $u_0 \in K$. Then $y_{u_j} \to y_{u_0}$ in $W_0^{1,p}(\Omega)$, therefore $\{y_{u_j}\}$ converges uniformly to y_{u_0} and hence

$$g(x, y_{u_0}(x)) = \lim_{j\to\infty} g(x, y_{u_j}(x)) \leq \lim_{j\to\infty} \delta_j = \delta_0 \text{ for all } x \in \bar{\Omega}.$$

Therefore u_0 is an admissible control for (P_{δ_0}).

To conclude the proof, we must establish the existence of an optimal control for every $\delta \geq \delta_0$. Let $\{u_k\} \subset K$ be a minimizing sequence for (P_δ), this is $J(u_k) \to \inf(\mathrm{P}_\delta)$. We can take a subsequence, denoted again in the same way, which converges weakly* in $L^\infty(\Omega)$ to an element $\bar u \in K$. Arguing in a similar way to the previous paragraph, we can check that $g(x, y_{\bar u}(x)) \leq \delta$ for every $x \in \bar\Omega$. So $\bar u$ is an admissible control for problem (P_δ). Let us check that $J(\bar u) = \inf(\mathrm{P}_\delta)$. To do that we use Mazur's Theorem (see, for instance, Ekeland and Temam [8]): there exists a sequence of convex combinations $\{v_k\}_{k\in\mathbb{N}}$,

$$v_k = \sum_{j=k}^{n(k)} \lambda_{k,j} u_j, \quad \text{with} \quad \sum_{j=k}^{n(k)} \lambda_{k,j} = 1 \quad \text{and} \quad \lambda_{k,j} \geq 0,$$

such that $v_k \to \bar u$ strongly in $L^p(\Omega)$. Then, using the convexity of L with respect to the third variable, the dominated convergence theorem and (28), we get

$$J(\bar u) = \lim_{k\to\infty} \int_\Omega L(x, y_{\bar u}(x), v_k(x))dx \leq$$

$$\limsup_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} \int_\Omega L(x, y_{\bar u}(x), u_j(x))dx \leq \limsup_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} J(u_j) +$$

$$\limsup_{k\to\infty} \int_\Omega \sum_{j=k}^{n(k)} \lambda_{k,j} |L(x, y_{u_j}(x), u_j(x)) - L(x, y_{\bar u}(x), u_j(x))|dx =$$

$$\inf(\mathrm{P}_\delta) + \limsup_{k\to\infty} \int_\Omega \sum_{j=k}^{n(k)} \lambda_{k,j} |L(x, y_{u_j}(x), u_j(x)) - L(x, y_{\bar u}(x), u_j(x))|dx,$$

where we have used the convergence $J(u_k) \to \inf(\mathrm{P}_\delta)$. To check that the second summand of the previous expression tends to zero, we just have to notice that for every fixed x, the function $L(x, \cdot, \cdot)$ is uniformly continuous on bounded sets of $\mathbb{R}^2$, that the sequences $\{y_{u_j}(x)\}$ and $\{u_j(x)\}$ are uniformly bounded an that $y_{u_j}(x) \to y_{\bar u}(x)$ when $j \to \infty$. Therefore

$$\lim_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} |L(x, y_{u_j}(x), u_j(x)) - L(x, y_{\bar u}(x), u_j(x))| = 0 \quad \text{for a.e. } x \in \Omega.$$

Using again the dominated convergence theorem we deduce that

$$\limsup_{k\to\infty} \int_\Omega \sum_{j=k}^{n(k)} \lambda_{k,j} |L(x, y_{u_j}(x), u_j(x)) - L(x, y_{\bar u}(x), u_j(x))|dx = 0,$$

and the proof is complete. □

Now let us proceed to discretize the control problem. Consider the space

$$U_h = \left\{ u_h \in L^\infty(\Omega) : \; u_{h|T} \in P_0(T) \quad \forall T \in \mathcal{T}_h \right\}.$$

For all $u_h \in U_h$ we will denote by $y_h(u_h)$ the unique element in V_h that satisfies

$$\sum_{i,j=1}^{N} \int_\Omega a_{i,j} \partial_{x_i} y_h(u_h) \partial_{x_j} z_h dx = \int_\Omega (f(x, y_h(u_h)) + u_h) z_h dx \quad \forall z_h \in V_h. \tag{31}$$

For every $h > 0$ we take a convex, closed, bounded and non empty subset K_h of U_h in such a way that $\{K_h\}$ constitutes an internal approximation of K in the following sense

(1) For all $u \in K$ there exists $u_h \in K_h$ with $u_h \to u$ in $L^1(\Omega)$.
(2) If $u_h \in K_h$ and $u_h \to u$ weakly* in $L^\infty(\Omega)$, then $u \in K$.
(3) The sets $\{K_h\}_{h>0}$ are uniformly bounded in $L^\infty(\Omega)$.

Let us formulate the following finite dimensional problem.

$$(\mathrm{P}_{\delta h}) \left\{ \begin{array}{l} \min J_h(u_h) = \int_{\Omega_h} L\left(x, y_h(u_h)(x), u_h(x)\right) dx \\ u_h \in K_h, \quad g\left(x_j, y_h(u_h)(x_j)\right) \leq \delta \quad \forall j \in I_h, \end{array} \right.$$

where $\{x_j\}_{j=1}^{n(h)}$ is the set of vertexes of $\mathcal{T}_h$ and I_h is the set of indexes corresponding to the interior vertexes: $x_j \in \Omega$ if $j \in I_h$ and $x_j \in \Gamma$ if $j \notin I_h$.

It is the purpose of this section to show that the solutions of the discrete problems converge to the solution of the continuous problem. To do that, it is necessary to prove the fact that if $u_h \to u$ weakly* in $L^\infty(\Omega)$, then $y_h(u_h) \to y_u$ uniformly in Ω. Observe that we are not exactly in the case of the previous section, because there we proved that $y_h = y_h(u)$ converges to y_u uniformly.

Theorem 9. *For every $h > 0$ let us take $u_h \in U_h$ so that $u_h \to u$ weakly* in $L^\infty(\Omega)$. Then*

$$\lim_{h \to 0} \|y_h(u_h) - y_u\|_{L^\infty(\Omega)} = 0$$

if one of the following three assumptions is satisfied:

A1): $f(\cdot, 0) \in L^2(\Omega)$.
A2): $N = 2$.
A3): *Function ϕ_M introduced in* (1) *belongs to $L^r(\Omega)$, $r > 2$ and the triangulation is of non negative type.*

Proof. First let us assume that **A1)** holds. The weak* convergence of $\{u_h\}_{h>0}$ implies the boundedness of the family in $L^\infty(\Omega)$, then there exists a constant C such that

$$\|y_{u_h}\|_{H^2(\Omega)} \leq C \quad \forall h > 0. \tag{32}$$

Now using Theorem 1

$$\|y_h(u_h) - y_u\|_{L^\infty(\Omega)} \leq \|y_h(u_h) - y_{u_h}\|_{L^\infty(\Omega)} + \|y_{u_h} - y_u\|_{L^\infty(\Omega)} \leq$$
$$Ch^{2-\frac{N}{2}} + \|y_{u_h} - y_u\|_{L^\infty(\Omega)} \to 0.$$

To prove the same result under assumption **A2)** or **A3)** we will use the following two lemmas.

Lemma 9. *For all $h > 0$, all $u \in L^\infty(\Omega)$ and all $u_h \in U_h$ there exists $C > 0$ independent of h such that*

$$\|y_h(u_h) - y_h(u)\|_{H_0^1(\Omega)} \leq C\|u - u_h\|_{H^{-1}(\Omega)}.$$

Proof. From the monotonicity of f and the $H_0^1(\Omega)$ ellipticity of $a(\cdot, \cdot)$, we have that

$$m\|y_h(u_h) - y_h(u)\|^2_{H_0^1(\Omega)} \leq a(y_h(u_h) - y_h(u), y_h(u_h) - y_h(u)) =$$
$$(f(x, y_h(u_h)) - f(x, y_h(u)), y_h(u_h) - y_h(u)) + (u_h - u, y_h(u_h) - y_h(u)) \leq$$
$$\leq (u_h - u, y_h(u_h) - y_h(u)) \leq \|u - u_h\|_{H^{-1}(\Omega)} \|y_h(u_h) - y_h(u)\|_{H_0^1(\Omega)}.$$

Therefore

$$\|y_h(u_h) - y_h(u)\|_{H_0^1(\Omega)} \le \frac{1}{m}\|u - u_h\|_{H^{-1}(\Omega)}.$$

□

Lemma 10. *Let us assume that $u_h \to u$ in $L^\infty(\Omega)$ weakly*. Then the following equalities hold*

(1) $\lim_{h\to 0} \|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega_h)} = 0$.
(2) $\lim_{h\to 0} \|y_h(u_h) - y_{u_h}\|_{H^1(\Omega)} = 0$.
(3) $\lim_{h\to 0} \frac{1}{h}\|y_h(u_h) - y_{u_h}\|_{L^2(\Omega)} = 0$.
(4) $\lim_{h\to 0} \frac{1}{h}\|y_{u_h} - \Pi_h y_{u_h}\|_{L^p(\Omega_h)} = 0$.

Proof. 1.- First we write

$$\|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega_h)} \le \|y_{u_h} - y_u\|_{W^{1,p}(\Omega_h)} +$$
$$\|y_u - \Pi_h y_u\|_{W^{1,p}(\Omega_h)} + \|\Pi_h y_u - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega_h)}.$$

Now the first summand converges to zero because of the continuity of the solution of the state equation with respect to the control and the second one due to the convergence of the interpolation in the Sobolev space $W^{1,p}(\Omega)$; see P.G. Ciarlet [3]. The third one also goes to zero because of the uniform boundedness of $\|\Pi_h\|$ in $W^{1,p}(\Omega)$.

2.- Let us write

$$\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)} \le \|y_{u_h} - y_u\|_{H^1(\Omega)} +$$
$$\|y_u - y_h(u)\|_{H^1(\Omega)} + \|y_h(u) - y_h(u_h)\|_{H^1(\Omega)}.$$

Once again the first summand converges to zero due to the continuity of the state with respect to the control and the second one due to Lemma 5. For the third summand we use Lema 9 and the fact that the weak* convergence of u_h in $L^\infty(\Omega)$ implies the strong convergence in $H^{-1}(\Omega)$.

3.- Since $y_h(u_h)$ and y_{u_h} are the discrete and continuous states associated to the same control, we can use the inequality obtained in the proof of Lemma 6

$$\frac{1}{h}\|y_{u_h} - y_h(u_h)\|_{L^2(\Omega)} \le C\|y_{u_h} - y_h(u_h)\|_{H^1(\Omega)}.$$

Now it is enough to use the already proved statement (2) of this lemma.

4.- By using Lemma 1 we get

$$\|y_{u_h} - \Pi_h y_{u_h}\|_{L^p(\Omega_h)} = \|y_{u_h} - \Pi_h y_{u_h} - \Pi_h(y_{u_h} - \Pi_h y_{u_h})\|_{L^p(\Omega_h)} \le$$
$$Ch\|y_{u_h} - \Pi_h y_{u_h}\|_{W^{1,p}(\Omega_h)}.$$

Now the result follows from statement (1). □

Now we conclude the proof of Theorem 9. Let us assume that $N = 2$.

$$\|y_h(u_h) - y_u\|_{L^\infty(\Omega_h)} \le \|y_h(u_h) - \Pi_h y_{u_h}\|_{L^\infty(\Omega_h)}$$
$$+\|\Pi_h y_{u_h} - y_{u_h}\|_{L^\infty(\Omega_h)} + \|y_{u_h} - y_u\|_{L^\infty(\Omega_h)}.$$

The third summand converges to zero due to the the continuity of the state with respect to the control. The convergence to zero of the second summand is a consequence of Lemma 10-(1). To estimate the first summand, let us take into account (5) and the fact that $N = 2$

$$\|y_h(u_h) - \Pi_h y_{u_h}\|_{L^\infty(\Omega_h)} \le \frac{C}{h}\|y_h(u_h) - \Pi_h y_{u_h}\|_{L^2(\Omega_h)} \le$$

$$C\left(\frac{\|y_h(u_h)-y_{u_h}\|_{L^2(\Omega)}}{h}+\frac{\|y_{u_h}-\Pi_h y_{u_h}\|_{L^2(\Omega_h)}}{h}\right).$$

Now we can apply Lemma 10 to deduce that this quantity converges to zero. So we have proved that

$$\lim_{h\to 0}\|y_h(u)-y_u\|_{L^\infty(\Omega_h)}=0.$$

Notice that $y_u \in C(\bar\Omega)\cap H_0^1(\Omega)$, then $\|y_u\|_{L^\infty(\Omega\setminus\Omega_h)}$ tends to zero when h decreases. The proof is complete under assumption **A2)**.

Let us conclude the proof of theorem by considering the last case: assumption **A3)**. We know that y_u and y_{u_h} are elements of $W^{1,p}(\Omega)$ for every $u\in K$ and $u_h\in K_h$. Let us write

$$\|y_h(u_h)-y_u\|_{L^\infty(\Omega)}\le \|y_h(u_h)-y_h(u)\|_{L^\infty(\Omega)}+\|y_h(u)-y_u\|_{L^\infty(\Omega)}.$$

The second summand converges to zero as a consequence of Theorem 4. Let us study the first summand. We know that $y_h(u_h)-y_h(u)$ solves the discrete problem

$$a(y_h(u_h)-y_h(u),z_h)=(f(x,y_h(u_h))+u_h-f(x,y_h(u))-u,z_h)\quad \forall z_h\in V_h.$$

In this case we can apply the discrete maximum principle (17), and we get

$$\|y_h(u_h)-y_h(u)\|_{L^\infty(\Omega)}\le C\|f(x,y_h(u))+u-f(x,y_h(u_h))-u_h\|_{W^{-1,p}(\Omega)}\le$$

$$\le C\left(\|f(x,y_h(u))-f(x,y_h(u_h))\|_{W^{-1,p}(\Omega)}+\|u-u_h\|_{W^{-1,p}(\Omega)}\right).$$

The weak* convergence in $L^\infty(\Omega)$ of u_h implies the strong convergence in $W^{-1,p}(\Omega)$, so the second summand converges to zero. On the other side, arguing as in the proof of Theorem 4 we get

$$\|f(x,y_h(u))-f(x,y_h(u_h))\|_{W^{-1,p}(\Omega)}\le\|\phi\|_{L^r(\Omega)}\|y_h(u)-y_h(u_h)\|_{H_0^1(\Omega)}.$$

Due to Lemma 9

$$\|y_h(u_h)-y_h(u)\|_{H_0^1(\Omega)}\le C\|u-u_h\|_{H^{-1}(\Omega)}.$$

Once again the weak* convergence of the u_h implies the strong convergence in $H^{-1}(\Omega)$. Therefore the states converge uniformly. □

By using Theorem 9, we can prove the following lemma which is essential in the proof of the converge of the discretization of (P_δ).

Lemma 11. *Let us suppose that one of the assumptions* **A1)**, **A2)** *or* **A3)** *of Theorem 9 holds and let* $\{u_{h_k}\}_{k=1}^\infty$ *be a sequence, with* $h_k\to 0$, *such that* $u_{h_k}\to u$ *weakly* in* $L^\infty(\Omega)$. *Then*

$$J(u)\le\liminf_{k\to\infty}J_{h_k}(u_{h_k}).$$

Proof. We know that there exists a sequence v_{h_k} of finite convex combinations of u_{h_k} converging strongly to u in $L^p(\Omega)$ for some $p\in(1,\infty)$:

$$v_{h_k}=\sum_{j=k}^{n(k)}\lambda_{k,j}u_{h_j},\quad\text{with }\lambda_{k,j}\ge 0\ \text{ and }\ \sum_{j=k}^{n(k)}\lambda_{k,j}=1,\ \lim_{k\to\infty}v_{h_k}=u\text{ in }L^p(\Omega).$$

So we can write

$$J(u)=\int_\Omega L(x,y_u,u)dx=\lim_{k\to\infty}\int_{\Omega_{h_k}}L(x,y_u,v_{h_k})dx\le$$

$$\leq \liminf_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} \int_{\Omega_{h_k}} L(x, y_u, u_{h_j})dx \leq$$

$$\limsup_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} \int_{\Omega_{h_k}} (L(x, y_u, u_{h_j}) - L(x, y_{h_j}(u_{h_j}), u_{h_j}))dx +$$

$$\liminf_{k\to\infty} \sum_{j=k}^{n(k)} \lambda_{k,j} \int_{\Omega_{h_j}} L(x, y_{h_j}(u_{h_j}), u_{h_j})dx.$$

The second summand coincides with $\liminf_{k\to\infty} J_{h_k}(u_{h_k})$. Thus the lemma will be proved if we establish that the first summand is zero. But this is obtained by using the uniform convergence $y_{h_k}(u_{h_k}) \to y_u$ (see Theorem 9) and arguing like at the end of the proof of Theorem 8. □

In order to analyze the convergence of the discretizations of the control problem, we have to introduce some stability concept to deal with the state constraints.

Definition 1. We will say that control problem (P_δ) is weakly stable on the left if

$$\lim_{\delta' \nearrow \delta} \inf (\mathrm{P}_{\delta'}) = \inf (\mathrm{P}_\delta).$$

Notice that weak stability on the right

$$\lim_{\delta' \searrow \delta} \inf (\mathrm{P}_{\delta'}) = \inf (\mathrm{P}_\delta) \tag{33}$$

is always true. Indeed take u_δ a solution of (P_δ). Since K is bounded, we can deduce the existence of a sequence $\{\delta_j\}$ such that $\delta_j \searrow \delta$ when $j \to \infty$ and $\lim_{j\to\infty} u_{\delta_j} = \bar{u}$ weakly* in $L^\infty(\Omega)$ for some $\bar{u} \in K$, being u_{δ_j} a solution of (P_{δ_j}). If y_j and $\bar{y}$ are the associated states to u_{δ_j} and $\bar{u}$ respectively, we have that $y_j \to \bar{y}$ uniformly in $\bar{\Omega}$. Therefore $\bar{u}$ is an admissible control for (P_δ). Now, using the admissibility of u_δ for each ($\mathrm{P}_{\delta'}$), with $\delta' > \delta$ and arguing as in the proof of Theorem 8, we obtain

$$\inf (\mathrm{P}_\delta) \leq J(\bar{u}) \leq \liminf_{j\to\infty} J(u_{\delta_j}) = \lim_{\delta' \searrow \delta} \inf (\mathrm{P}_{\delta'}) \leq J(u_\delta) = \inf (\mathrm{P}_\delta),$$

which proves (33).

From (33) it follows that the weak stability of ($\mathrm{P}_{\bar{\delta}}$) is equivalent to the continuity of mapping $\delta \to \inf (\mathrm{P}_\delta)$ at the point $\bar{\delta}$.

There are some problems not weakly stable on the left. Let us see two examples. The first one is a finite dimensional problem which will help us to illustrate geometrically that the lack of weak stability on the left implies that the problem is numerically ill posed.

Example 1. Consider the problem

$$(Q_\delta) \begin{cases} \text{Minimize } x^2 + (y-1)^2 \\ -5 \leq x \leq 5 \\ 0 \leq y \leq 1 \\ \frac{1}{5}x^3 + \frac{3}{5}x^2 - y + 2 \leq \delta. \end{cases}$$

Problem (Q_δ) is not weakly stable on the left for $\delta = 1$. In fact, $\inf(Q_1) = 0$, reaching the solution at the point $(0, 1)$. If we take $\delta' < 1$, then $1 \geq y > (1/5)x^3 +$

$(3/5)x^2+1 = x^2(x+3)/5+1$, and therefore we have that $x+3<0$, or what is the same $x<-3$. From here we deduce that

$$\lim_{\delta'\nearrow 1} \inf(Q_{\delta'}) \geq 9 > \inf(Q_1).$$

Observe that the admissible region of problem (Q_1) has an isolated point, and it is the point where the minimum is attained.

Next we introduce a control problem not weakly stable on the left.

Example 2. Take $\Omega = B(0,1)$ in $\mathbb{R}^N$, $N=2$ or $N=3$, and Γ its boundary. the state equation is given by

$$\begin{cases} -\Delta y_u &= & u & \text{in } \Omega \\ y_u &= & 0 & \text{on } \Gamma. \end{cases}$$

Let us take $z(x) = 2(1-\|x\|^2)$. It is easy to check that z satisfies the partial differential equation

$$\begin{cases} -\Delta z &= & 4N & \text{in } \Omega \\ z &= & 0 & \text{on } \Gamma, \end{cases}$$

and $z(0)=2$. Set

$$g(t) = \begin{cases} t & \text{if } t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}$$

We define the control problem

$$(P_\delta) \begin{cases} \min J(u) = \displaystyle\int_\Omega (u-4N)^2\, dx \\ |u(x)| \leq 5N \text{ a.e. } x \in \Omega, \quad g(y_u(x)) \leq \delta \text{ in } \bar{\Omega}. \end{cases}$$

Let us see that our example is not weakly stable on the left for $\delta = 1$.

The solution to (P_1) is attained by taking $u_1 = 4N$. Indeed $y_{u_1} = z$, and we have that $g(y_{u_1}(x)) = g(z(x)) \leq 1$ and $J(u) = 0$.

Take $\delta' < 1$. Let $u_{\delta'}$ be a solution of $(P_{\delta'})$ and $y_{\delta'} = y_{u_{\delta'}}$ the associated state. Necessarily $y_{\delta'}(0) < 1$ and therefore $1 < \|y_{\delta'} - z\|_{L^\infty(\Omega)}$ since both $y_{\delta'}(x)$ and $z(x)$ are continuous functions. Moreover $y_{\delta'} - z$ solves the problem

$$\begin{cases} -\Delta\left(y_{\delta'} - z\right) &= & u_{\delta'} - 4N & \text{in } \Omega \\ y_{\delta'} - z &= & 0 & \text{on } \Gamma, \end{cases}$$

and we obtain the inequality

$$1 < \|y_{\delta'} - z\|_{L^\infty(\Omega)} \leq C\|y_{\delta'} - z\|_{H^2(\Omega)} \leq C\|u_{\delta'} - 4N\|_{L^2(\Omega)} = C\sqrt{J(u_{\delta'})}.$$

where C is a constant that does not depend on δ'. Therefore, for all $\delta' < 1$

$$\inf(P_{\delta'}) > \frac{1}{C^2} > 0$$

and the weak stability on the left is not satisfied.

Nevertheless, almost all problems are weakly stable on the left.

Theorem 10. *Take δ_0 as in Theorem 8. Then, for all $\delta > \delta_0$ but at most a numerable set, problem* (P_δ) *is weakly stable on the left.*

Proof. Let δ_0 be the number obtained in Theorem 8. If we define $\varphi : [\delta_0, +\infty) \to \mathbb{R}$ with $\varphi(\delta) = \inf(\mathrm{P}_\delta)$, then φ is a monotone decreasing function, and therefore it is continuous at every point of $[\delta_0, +\infty)$ but at most a countable number of them. But, as we have already seen, weak stability on the left is equivalent to the continuity of φ in δ, and so the theorem is proved. $\square$

Now we can prove the convergence theorem.

Definition 2. Given a family of elements $\{u_h\}_{h>0}$, with $u_h \in K_h$ for every $h > 0$, we will say that u is an accumulation point of $\{u_h\}_{h>0}$ if there exists a subsequence $\{u_{h_k}\}_{k=1}^\infty$, with $h_k \to 0$, such that $u_{h_k} \to u$ weakly* in $L^\infty(\Omega)$.

Theorem 11. *Let δ_0 be as in Theorem 8 and $\delta > \delta_0$. If (P_δ) is weakly stable on the left and one of the assumptions* **A1)**, **A2)** *or* **A3)** *of Theorem 9 holds, then there exists $h_0 > 0$ such that $(\mathrm{P}_{\delta h})$ has at least a solution u_h for $h \le h_0$. Moreover, each accumulation point u of $\{u_h\}_{h\le h_0}$ is a solution of (P_δ). Finally*

$$\lim_{h\to 0} J_h(u_h) = \inf(\mathrm{P}_\delta). \tag{34}$$

Proof. Since every K_h is compact and J_h is continuous, the existence of a solution of $(\mathrm{P}_{\delta h})$ will be established if we prove that the set of admissible controls for $(\mathrm{P}_{\delta h})$ is not empty. To do that let us take $u_0 \in K$ an admissible control for problem (P_{δ_0}) and take $u_{0h} \in K_h$ in such a way that $u_{0h}(x) \to u_0(x)$ a.e. $x \in \Omega$. Since $u_{0h} \to u$ in every $L^p(\Omega)$, $1 \le p < \infty$, then $y_h(u_{0h}) \to y_{u_0}$ uniformly in $\bar\Omega$; see Theorem 9. Since $g(x, y_{u_0}(x)) \le \delta_0$ for every $x \in \bar\Omega$, we can deduce from the uniform convergence and the relation $\delta > \delta_0$ the existence of $h_0 > 0$ such that $g(x, y_h(u_{0h})) \le \delta$ for all $x \in \bar\Omega$ and each $h \le h_0$. So we conclude that $(\mathrm{P}_{\delta h})$ has a solution for every $h \le h_0$.

Now let $u_{\delta h}$ be a solution of $(\mathrm{P}_{\delta h})$, $h \le h_0$, and denote by $y_{\delta h}$ the associated state. Since $\{K_h\}_{h\le h_0}$ is uniformly bounded in $L^\infty(\Omega)$, we can extract a subsequence $\{u_{\delta h_k}\}$ of $\{u_{\delta h}\}_{h\le h_0}$ such that $h_k \to 0$ and $u_{\delta h_k} \to \bar u$ weakly* in $L^\infty(\Omega)$ for some $\bar u \in K$. Let us prove that $\bar u$ is a solution of (P_δ). Let $\bar y$ be the associate state to $\bar u$. From Theorem 9 we obtain that $y_{\delta h_k} \to \bar y$ uniformly in $\bar\Omega$. Using this convergence along with the uniform continuity of function $x \to g(x, \bar y(x))$ in $\bar\Omega$, the density of the vertices of the triangulations in $\bar\Omega$ and that $g(x_j, y_{\delta h_k}(x_j)) \le \delta$ for each vertex of the triangulation $\mathcal{T}_{h_k}$, it is easy to conclude that $g(x, \bar y(x)) \le \delta$ for every $x \in \bar\Omega$, and therefore $\bar u$ is an admissible control for (P_δ).

Let us take $\delta' \in (\delta_0, \delta)$ and let $u_{\delta'}$ be a solution of $(\mathrm{P}_{\delta'})$. For every $h \le h_0$ let us take $u_{\delta' h} \in K_h$ such that $u_{\delta' h}(x) \to u_{\delta'}(x)$ a.e. in $x \in \Omega$. From the uniform convergence $y_h(u_{\delta' h}) \to y_{u_{\delta'}}$ and the relation $g(x, y_{\delta'}(x)) \le \delta' < \delta$ for every $x \in \bar\Omega$, we deduce the existence of $h_{\delta'} > 0$ such that $g(x, y_h(u_{\delta' h})(x)) \le \delta$ for all $x \in \bar\Omega$ and all $h \le h_{\delta'}$, therefore $u_{\delta' h}$ is an admissible control for $(\mathrm{P}_{\delta h})$ always that $h \le h_{\delta'}$. This along with the fact that $u_{\delta h_k}$ is a solution of $(\mathrm{P}_{\delta h_k})$ implies that $J_{h_k}(u_{\delta h_k}) \le J_{h_k}(u_{\delta' h_k})$ for each k big enough. Using now Lemma 11 it follows that

$$J(\bar u) \le \liminf_{k\to\infty} J_{h_k}(u_{\delta h_k}) \le \liminf_{k\to\infty} J_{h_k}(u_{\delta' h_k}) = J(u_{\delta'}) = \inf(\mathrm{P}_{\delta'}).$$

Finally the stability condition on the left allows us to conclude

$$\inf(\mathrm{P}_\delta) \le J(\bar u) \le \lim_{\delta' \nearrow \delta} \inf(\mathrm{P}_{\delta'}) = \inf(\mathrm{P}_\delta),$$

which, together with the admissibility of $\bar u$ for (P_δ) proves that $\bar u$ is a solution of (P_δ). Identity (34) is immediate. $\square$

Remark 1. If the solution of the problem is unique, we have that the whole sequence $\{u_h\}_{h\le h_0}$ converges weakly* to the solution of the problem.

Theorem 12. *Let us suppose that the assumptions of the previous theorem hold and that L is of class C^2 in the third variable and*

$$\exists\alpha>0 \text{ such that } \frac{\partial^2 L}{\partial u^2}(x,y,u)\ge\alpha>0 \text{ for a.e. } x\in\Omega \text{ and all } y,u\in\mathbb{R},$$

$$\forall M>0\ \exists\varphi_M\in L^1(\Omega) \text{ such that } \left|\frac{\partial L}{\partial u}(x,y,u)\right|+\left|\frac{\partial^2 L}{\partial u^2}(x,y,u)\right|\le\varphi_M(x)$$

a.e. $x\in\Omega$ and $|u|,|y|\le M$. For every $h\le h_0$ let u_h be a solution of $(P_{\delta h})$ and let $\bar u$ be an accumulation point of $\{u_h\}$ with $u_{h_k}\to\bar u$ weakly in $L^\infty(\Omega)$. Then*

$$\lim_{k\to\infty}\|\bar u-u_{h_k}\|_{L^2(\Omega)}=0.$$

Proof. On one hand

$$\int_\Omega (L(x,y_{h_k}(u_{h_k}),u_{h_k})-L(x,\bar y,\bar u))dx=$$
$$(J_{h_k}(u_{h_k})-J(\bar u))+\int_{\Omega\setminus\Omega_{h_k}} L(x,y_{h_k}(u_{h_k}),u_{h_k})dx.$$

The first summand converges to zero due to the previous theorem and the second one because $\{L(x,y_{h_k},u_{h_k})\}$ is dominated by a function $\psi_M\in L^1(\Omega)$ and the measure of $\Omega\setminus\Omega_{h_k}$ goes to zero. So

$$\lim_{k\to\infty}\int_\Omega (L(x,y_{h_k}(u_{h_k}),u_{h_k})-L(x,\bar y,\bar u))dx=0. \tag{35}$$

On the other hand

$$\begin{aligned}&\int_\Omega (L(x,y_{h_k}(u_{h_k}),u_{h_k})-L(x,\bar y,\bar u))dx=\\ &\int_\Omega (L(x,y_{h_k}(u_{h_k}),u_{h_k})-L(x,\bar y,u_{h_k}))dx+\int_\Omega (L(x,\bar y,u_{h_k})-L(x,\bar y,\bar u))dx.\end{aligned} \tag{36}$$

As in the proof of Theorem 8

$$\lim_{k\to\infty}\int_\Omega (L(x,y_{h_k}(u_{h_k}),u_{h_k})-L(x,\bar y,u_{h_k}))dx=0. \tag{37}$$

As a consequence of (35)–(37) we have that

$$\lim_{k\to\infty}\int_\Omega (L(x,\bar y,u_{h_k})-L(x,\bar y,\bar u))dx=0. \tag{38}$$

Making now a Taylor expansion of order two we obtain that

$$\int_\Omega (L(x,\bar y,u_{h_k})-L(x,\bar y,\bar u))dx=$$
$$\int_\Omega \frac{\partial L}{\partial u}(x,\bar y,\bar u)(u_{h_k}-\bar u)dx+\frac{1}{2}\int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar y,v_k)(u_{h_k}-\bar u)^2dx,$$

where $v_k(x)=\bar u(x)+\theta_k(x)(u_{h_k}(x)-\bar u(x))$ for some measurable function $0\le\theta_k\le 1$. Since u_{h_k} converges weakly* to $\bar u$, the first summand converges to zero:

$$\lim_{k\to\infty}\int_\Omega \frac{\partial L}{\partial u}(x,\bar y,\bar u)(u_{h_k}-\bar u)dx=0. \tag{39}$$

Finally we have that

$$\frac{1}{2}\int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar y, v_k)(u_{h_k}-\bar u)^2 dx \geq \frac{\alpha}{2}\|\bar u - u_{h_k}\|^2_{L^2(\Omega)}.$$

Therefore we can write

$$\frac{\alpha}{2}\|\bar u - u_{h_k}\|^2_{L^2(\Omega)} \leq \int_\Omega (L(x,\bar y, u_{h_k}) - L(x,\bar y,\bar u))dx - \int_\Omega \frac{\partial L}{\partial u}(x,\bar y,\bar u)(u_{h_k}-\bar u)dx$$

which converges to zero due to (38) and (39). So $\|\bar u - u_{h_k}\|_{L^2(\Omega)}$ converges to zero and the proof is complete. □

3.2. **Neumann case.** In this section we consider the following boundary control problem

$$(\mathrm{P}_\delta)\begin{cases} \min J(u) = \int_\Gamma \ell(s, y_u(s), u(s))ds \\ u\in K, \quad g(x, y_u(x)) \leq \delta \quad \forall x\in\bar\Omega, \end{cases}$$

where

(1) K is a convex, weakly* closed, bounded and non empty subset of $L^\infty(\Gamma)$.
(2) $\ell : \Gamma\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is a Carathéodory function, convex in the third variable and satisfying

$$\forall M>0\ \exists \psi_M \in L^1(\Gamma) \ \text{ such that } |\ell(x,y,u)| \leq \psi_M(x) \tag{40}$$

for a.e. $x\in\Gamma$ and $|y|$, $|u|\leq M$.
(3) $g : \bar\Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.
(4) (y,u) satisfies the state equation

$$\begin{cases} Ay + a_0 y &= f(\cdot, y) \quad \text{in } \Omega \\ \partial_{n_A} y &= u \quad\quad\ \ \text{on } \Gamma, \end{cases} \tag{41}$$

with $a_0 \in L^{\frac{Np}{N+p}}(\Omega)$, $a_0\geq 0$, $a_0 \not\equiv 0$ in Ω, $p>N$.
(5) Ω is a polygonal or polyhedral convex domain.

By using the regularity results by M. Dauge [7] we can obtain the same result as for the Dirichlet case: there exists $p>N$ small enough such that (41) has a unique solution $y_u \in W^{1,p}(\Omega)$ assuming that $f(\cdot,0)\in L^{p/2}(\Omega)$. Moreover the weak convergence of the controls $u_j \to u$ in $L^\infty(\Gamma)$ implies the strong convergence $y_{u_j}\to y_u$ in $W^{1,p}(\Omega)$. By using this fact, the proof of Theorem 8 can be repeated for the boundary control problem (P_δ), with the obvious modifications.

Now the discrete version of (P_δ) is defined as follows. As in Section 2.2, for every $h>0$ we consider a regular triangulation of Ω. Because of the assumption on Ω we have that $\Omega_h = \Omega$. For this triangulation we define U_h as the set of elements $u_h \in L^\infty(\Gamma)$ such that u_h is constant on every boundary side $T\cap\Gamma$ (face if $N=3$) for any $T\in\mathcal{T}_h$. For each $u_h\in U_h$, let us define $y_h(u_h)\in W_h$ as the unique element that satisfies

$$\begin{aligned}\sum_{i,j=1}^N \int_\Omega a_{i,j}(x)\partial_{x_i} y_h(u_h)(x)\partial_{x_j} z_h(x)dx + \int_\Omega a_0(x) y_h(u_h)(x) z_h(x) dx = \\ \int_\Omega f(x, y_h(u_h)(x)) z_h(x) dx + \int_\Gamma u_h(s) z_h(s) ds \quad \forall z_h\in W_h.\end{aligned} \tag{42}$$

Now the discrete control problem is formulated in the following way

$$(\mathrm{P}_{\delta h})\begin{cases} \min J_h(u_h) = \int_\Gamma \ell(s, y_h(u_h)(s), u_h(s))\, ds \\ u_h\in K_h, \quad g(x_j, y_h(u_h)(x_j)) \leq \delta \quad 1\leq j\leq n(h), \end{cases} \tag{43}$$

where $\{K_h\}_{h>0}$ constitutes an internal approximation of K, as in the distributed control case, and $\{x_j\}_{j=1}^{n(h)}$ is the set of vertexes of the triangulation $\mathcal{T}_h$.

If we change the weak* convergence of $\{u_h\}$ to u in $L^\infty(\Omega)$ by the weak* convergence in $L^\infty(\Gamma)$, then Theorem 9 is valid under one of the assumptions **A2)** and **A3)**. The proof is almost the same, taking into account that Lemma 10 is still true and the inequality of Lemma 9 must be replaced by

$$\|y_h(u_h) - y_h(u)\|_{H^1(\Omega)} \leq C\|u - u_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Assumption **A1)** is not applicable to the boundary control case because the states y_u are not elements of $H^2(\Omega)$ due to the fact that $u \notin H^{1/2}(\Gamma)$.

Analogously to Lemma 11, it can be proved in the same way that

$$u_{h_k} \to u \text{ weak* in } L^\infty(\Gamma) \Rightarrow J(u) \leq \liminf_{k\to\infty} J_{h_k}(u_{h_k}).$$

Finally, using again the concept of weak stability on the left introduced in Definition 1, we can prove that the solutions of the discrete problems converge to the solutions of the continuous problem in the same way as in Theorems 11 and 12, what is summarize in the following result.

Theorem 13. *Let δ_0 be as in Theorem 8 and $\delta > \delta_0$. If (P_δ) is weakly stable on the left, then there exists $h_0 > 0$ such that (P_δ) has at least a solution u_h for every $h \leq h_0$. Moreover, each accumulation point u of $\{u_h\}_{h\leq h_0}$ is a solution of (P_δ) and*

$$\lim_{h\to 0} J_h(u_h) = \inf\,(\mathrm{P}_\delta).$$

Finally, if ℓ is of class C^2 in the third variable and

$$\exists \alpha > 0 \text{ such that } \frac{\partial^2 \ell}{\partial u^2}(x, y, u) \geq \alpha > 0 \text{ for a.e. } x \in \Gamma \text{ and all } y, u \in \mathbb{R},$$

$$\forall M > 0\ \exists \varphi_M \in L^1(\Gamma) \text{ such that } \left|\frac{\partial \ell}{\partial u}(x, y, u)\right| + \left|\frac{\partial^2 \ell}{\partial u^2}(x, y, u)\right| \leq \varphi_M(x)$$

a.e. $x \in \Gamma$ and $|u|, |y| \leq M$, and $\bar{u}$ is an accumulation point of $\{u_h\}$ with $u_{h_k} \to \bar{u}$ weakly in $L^\infty(\Gamma)$, then*

$$\lim_{k\to\infty} \|\bar{u} - u_{h_k}\|_{L^2(\Gamma)} = 0.$$

References

[1] N. Arada, E. Casas, and F. Tröltzsch, *Error estimates for the numerical approximation of a semilinear elliptic control problem*, (To appear).

[2] E. Casas, *Finite element approximations for some state-constrained optimal control problems*, Mathematics of the Analysis and Design in Process Control (Amsterdam) (P. Borne, S.G. Tzafestas, and N.E Radhy, eds.), North Holland, 1992, pp. 293–301.

[3] P.G. Ciarlet, *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978.

[4] P.G. Ciarlet and J.L. Lions, *Handbook of numerical analysis*, North-Holland, Amsterdam, 1991.

[5] P.G. Ciarlet and P.A. Raviart, *General Lagrange and Hermite interpolation in R^n with applications to finite element methods*, Arch. Rational Mech. Anal. **46** (1972), 177–199.

[6] ———, *Maximum principle and uniform convergence for the finite element method*, Comput. Methods Appl. Mech. Engrg. **2** (1973), 17–31.

[7] M. Dauge, *Neumann and mixed problems on curvilinear polyhedra*, Integral Equations Operator Theory (1992), 227–261.

[8] I. Ekeland and R. Temam, *Analyse convexe et problèmes variationnels*, Dunod-Gauthier Villars, Paris, 1974.

[9] J. Freshe and R. Rannacher, *Asymptotic L^∞-error estimates for linear finite element approximation of quasilinear boundary value problems*, SIAM J. Numer. Anal. **5** (1978), no. 2, 418–431.
[10] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston-London-Melbourne, 1985.
[11] D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130** (1995), 161–219.
[12] M. Mateos, *Optimal control problems governed by semilinear elliptic equations with integral constraints on the gradient of the sate*, Ph.D. thesis, University of Cantabria, 2000.
[13] P.A. Raviart and J.M. Thomas, *Introduction à l'analyse numérique des equations aux dérivées partielles*, Masson, Paris, 1983.
[14] R. Scott, *Optimal L^∞ estimates for the finite element method on irregular meshes*, Math. Comp. **30** (1965), 681–697.
[15] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), 189–258.

Departamento de Matemática Aplicada y Ciencias de la Computación, E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria, 39005 Santander, Spain
E-mail address: `eduardo.casas@unican.es`

Departamento de Matemáticas, EUIT Industriales, Universidad de Oviedo, C/Manuel Llaneza, 33204 Gijón, Spain
E-mail address: `mmateos@orion.ciencias.uniovi.es`