Testing the degree of overlap for the expected value of random intervals

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Abstract

Some hypothesis tests for analyzing the degree of overlap between the expected value of random intervals are provided. For this purpose, a suitable measure to quantify the overlapping grade between intervals is considered on the basis of the Szymkiewicz-Simpson coefficient defined for general sets. It can be seen as a kind of likeness index to measure the mutual information between two intervals. On one hand, an estimator for the proposed degree of overlap between intervals is provided and its strong consistency is analyzed. On the other hand, two tests are also proposed in this framework: a one-sample test to examine the degree of overlap between the expected value of a random interval and a given interval, and a two-sample test to check the degree of overlap between the expected value of two random intervals. To solve such hypothesis tests, two statistics are suggested and their limit distributions are studied by considering both asymptotic and bootstrap techniques. Their power has been also explored by means of local alternatives. In addition, some simulation studies are carried out to investigate the behavior of the proposed approaches. Finally, the performance of the tests is also reported in a real-life application.

Keywords: Overlapping index, Interval-valued data, Random intervals, Hypothesis testing, Bootstrap approach.

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1. Introduction

Interval data appear when dealing with different experimental studies involving ranges, fluctuations, subjective perceptions, censored and grouped data, among others (see, for instance, [2, 8, 13, 14, 17, 18, 23, 24, 28]).

The variables modeling such kind of imprecise information are called random intervals (or RIs, for short). Two different situations can be tackled in this context: on one hand, interval-data may appear as disjunctive sets representing incomplete information about a real variable (epistemic view, as stated in [6]); on the other hand, the experimental data may appear as essentially interval-valued data describing precise information (ontic view, in accordance with [6]). The study in this work is included in the second scenario.

Random intervals have been shown to be suitable in different settings. Regarding classification and discriminant analysis for interval data different works has been developed in [8, 24, 28, 32], to mention only a few. In addition, the problem of interval data in regression analysis has been tackled, for instance, in [4, 5, 10, 9, 29].

Concerning hypothesis tests for the expected value or Aumman expectation of random intervals (which is also an interval), the one-sample test, the two-sample test and the ANOVA have been previously developed in the literature [12, 16]. Due to the imprecision inherent to the intervals setting, the equality between intervals becomes a strict assumption. In addition, the lack of universal order between intervals makes the development of one-sided tests not a simple task. Thus, in order to both maintain consistency with such an imprecision and to tackle a kind of one-sided hypothesis tests problem, some tests aiming at relaxing strict equalities can be proposed and they can be used as ANOVA a posteriori tests once the strict equality has been discarded. Some of them are described below.

One option to slacken the strict equality is to consider a measure of inclusion. In this context, one-sample inclusion tests for the Aumann expectation of RIs have been addressed in [22] leading to the proposal of one-sided tests as a particular case in spite of the lack of order between interval data. The inclusion measure considered in that work was introduced by Sánchez [26] and is defined as a ratio between the measure of the intersection of two intervals (introduced in [27]) and the measure of the reference interval. In other words, the inclusion index shows how much content an interval is in another one that serves as a reference, so it is a kind of unidirectional measure.

Apart from an inclusion measure, it is possible to define bidirectional measures to compare intervals assigning the same importance to the elements that are compared. This is the case, for instance, of the similarity index used for developing the one-sample and two-sample similarity tests for the expected value of random intervals in [23] and [25]. Here the measure considered is based on the classical Jaccard similarity coefficient for classical convex sets firstly introduced in [15] and it is defined as a ratio between the measure of the intersection of the sets and the measure of their union. From an interpretive point of view, this measure serves to indicate how similar two intervals are and it is sensitive to differences in length of the intervals.

The first aim of this contribution is to define a new measure to quantify the mutual information between two intervals. For this purpose, a particularization of the Szymkiewicz-Simpson coefficient for general sets [31] to the intervals framework is proposed by computing the ratio between the measure of the intersection of the two intervals and the smallest of their measures. This index measures the degree of overlap between intervals and differs from the similarity grade in that whilst the latter considers the lengths of both intervals, the former is only related to the length of the smallest one. Note that the similarity index is always lower that or equal to the overlapping index. In that sense, the overlapping index is useful to quantify the information shared by the Aumman expectation of two variables (or by the Aumman expectation of one variable and a fixed interval) regardless of their lengths whereas the similarity index is useful whenever it is necessary to consider the lengths of both expectations. Finally, once the overlapping index is defined, its corresponding estimator will be also proposed and some of its properties will be analyzed.

On the other hand, the second goal of this proposal is to develop a one-sample test for analyzing the degree of overlap between the Aumman expectation of a random interval and a fixed interval as well as a two-sample test for the degree of overlap between the expectations of two random intervals. The coefficient defined to quantify the degree of overlap between intervals varies between 0 and 1, where 0 means that the measure of the common points is 0 and 1 means that either one interval is completely included in the other one or both intervals are equal. Such cases 0 and 1 have been previously tackled in [22] so the interest will be centered in degrees included in the open interval $(0, 1)$.

To solve the proposed tests, two statistics are provided and their asymptotic and bootstrap limit distributions are theoretically analyzed. The proposed bootstrap approach grants the approximation to the sampling distribution of the statistic in practice, since the asymptotic one depends on unknown parameters. Additionally, some simulation studies are carried out to show the empirical behavior of the approaches and a real-life application is also gathered.

The manuscript is organized as follows: Section 2 includes some preliminaries about random intervals, the definition of the overlapping index and its corresponding estimators for the one and two sample cases, analyzing the strong consistency of those estimators. The description of the one-sample test for the degree of overlap between the Aumman expectation of a random interval and a fixed interval is presented in Section 3, whereas the corresponding two-sample test is addressed in Section 4. The power of the suggested tests is investigated in Section 5. Moreover, some simulations studies are included in Section 6 and the applicability of the approaches is illustrated in Section 7. To conclude, Section 8 comprises some remarks and open problems.

2. Preliminary concepts

Let $\mathcal{K}_c(\mathbb{R})$ be the family of non-empty closed and bounded intervals of $\mathbb{R}$. An interval $A \in \mathcal{K}_c(\mathbb{R})$ can be characterized by either its $(\inf, \sup)$-representation (i.e., $A = [\inf A, \sup A]$) or its $(\text{mid}\,, \text{spr}\,)$-representation, $A = [\text{mid}\,A \pm \text{spr}\,A]$, where $\text{mid}\,A \in \mathbb{R}$ is the mid-point or centre and $\text{spr}\,A \geq 0$ the spread or radius of A. The second characterization has been shown to be more operative than the first one and a valuable tool for different statistical purposes (see, for instance, [3, 7, 30]).

Given $A_1, A_2 \in \mathcal{K}_c(\mathbb{R})$ and $\lambda \in \mathbb{R}$, the usual arithmetic operations between intervals are based on the Minkowski's addition [20] and the product by a scalar, and they are expressed on terms of the $(\text{mid}\,, \text{spr}\,)$-representation as $A_1 + \lambda A_2 = [(\text{mid}\,A_1 + \lambda \text{mid}\,A_2) \pm (\text{spr}\,A_1 + |\lambda| \text{spr}\,A_2)]$.

The Lebesgue measure of an interval $A \in \mathcal{K}_c(\mathbb{R})$ is given by $\lambda(A) = 2\text{spr}\,A$. Besides, the Lebesgue measure of the intersection between A and B, for any $A, B \in \mathcal{K}_c(\mathbb{R})$, can be expressed as follows (see [27]):

$$\lambda(A \cap B) = \max\Big\{0, \min\Big\{ 2\text{spr}\,A, 2\text{spr}\,B, \text{spr}\,A + \text{spr}\,B - |\text{mid}\,A - \text{mid}\,B| \Big\}\Big\} \quad (1)$$

A measure of the degree of overlap between two intervals $A, B \in \mathcal{K}_c(\mathbb{R})$ can be defined according to the Szymkiewicz-Simpson coefficient [31], which is a similarity measure related to the Jaccard index introduced in [15]. It measures the degree of overlap between two sets that can be interpreted as the quantity of information that two sets have in common. It is computed by dividing the length of the intersection of the two sets by the smallest of their lengths. In the case of intervals we have that the degree of overlap can be defined as follows:

$$O(A,B) = \frac{\lambda(A \cap B)}{\min\{\lambda(A), \lambda(B)\}}, \tag{2}$$

where either A or B are assumed not to be reduced to a singleton. It is easy to see that $0 \leq O(A,B) \leq 1$, since $\lambda(A \cap B) \leq \min\{\lambda(A), \lambda(B)\}$. This overlap measure satisfies that $O(A,B) = 0$ iff $A \cap B = \emptyset$, $O(A,B) = 1$ if $A = B$, $A \subset B$ or $B \subset A$, and $O(A,B) \in (0,1)$ iff $A \cap B \neq \emptyset$ and $A \neq B$, $A \not\subset B$ and $B \not\subset A$. Figure 1 shows some examples of the degree of overlap of between two intervals.

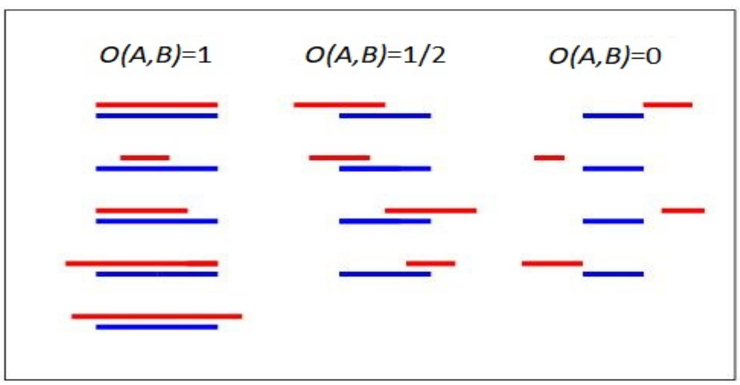

Figure 1: Different representations for the degree of overlap between A (in red) and B (in blue)

The random variables that model those situations in which the corresponding outcomes are intervals on $\mathcal{K}_c(\mathbb{R})$ are called *random intervals* (RIs for short). Given a probability space $(\Omega, \mathcal{A}, P)$, an RI is a Borel measurable mapping $X : \Omega \to \mathcal{K}_c(\mathbb{R})$ w.r.t. the Hausdorff metric on $\mathcal{K}_c(\mathbb{R})$ (see [19]). Equivalently, X is an RI if both $\operatorname{mid} X, \operatorname{spr} X : \Omega \to \mathbb{R}$ are real-valued random variables and $\operatorname{spr} X \geq 0$ a.s.-$[P]$.

The expected value of an RI X in the Aumann's sense (also called Aumann mean, see [1]) can be defined in terms of classical expectations as $E([\operatorname{mid} X \pm \operatorname{spr} X]) = [E(\operatorname{mid} X) \pm E(\operatorname{spr} X)]$, whenever $\operatorname{mid} X, \operatorname{spr} X \in L^1(\Omega, \mathcal{A}, P)$.

Let $\{X_i\}_{i=1}^n$ be a simple random sample drawn from X. The sample expectation (or sample mean) of X is defined coherently in terms of the interval arithmetic (based on the above mentioned Minkowski's addition) as $\overline{X_n} = (1/n) \sum_{i=1}^n X_i$, and it satisfies that $\overline{X_n} = [\text{mid}\, \overline{X_n} \pm \text{spr}\, \overline{X_n}]$.

2.1. Estimators for the degree of overlap

Given an RI X and a simple random sample drawn from X, say $\{X_i\}_{i=1}^n$, the corresponding estimator for the degree of overlap between $E(X)$ and a given interval A has the following expression:

$$\widehat{O}(\overline{X_n}, A) = \frac{\lambda(\overline{X_n} \cap A)}{\min\{\lambda(\overline{X_n}), \lambda(A)\}}. \tag{3}$$

Analogously, given two RIs X and Y, and two random samples $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ drawn from them, the estimator for the degree of overlap between $E(X)$ and $E(Y)$ is

$$\widehat{O}(\overline{X_n}, \overline{Y_n}) = \frac{\lambda(\overline{X_n} \cap \overline{Y_n})}{\min\{\lambda(\overline{X_n}), \lambda(\overline{Y_n})\}}. \tag{4}$$

The main properties associated with the statistical reliability of the estimators defined above are the unbiasedness and their strong consistency with respect to the corresponding population measures. Since $\overline{X_n}$ and $\overline{Y_n}$ are unbiased estimators for the parameters $E(X)$ and $E(Y)$, and min, max and $|\cdot|$ are continuous functions, it is straightforward to derive that $E(\widehat{O}(\overline{X_n}, A)) = O(E(X), A)$ and $E(\widehat{O}(\overline{X_n}, \overline{Y_n}) = O(E(X), E(Y))$.

On the other hand, the strong consistency of the estimators defined above is provided in Theorem 1. It can be assured directly from the classical strong consistency of $\text{mid}\, \overline{X_n}$, $\text{mid}\, \overline{Y_n}$, $\text{spr}\, \overline{X_n}$ and $\text{spr}\, \overline{Y_n}$ w.r.t. $\text{mid}\, E(X)$, $\text{mid}\, E(Y)$, $\text{spr}\, E(X)$ and $\text{spr}\, E(Y)$, by taking into account that the functions min, max and $|\cdot|$ are continuous.

Theorem 1. *Let X and Y be two RIs. The estimators $\widehat{O}(\overline{X_n}, A)$ and $\widehat{O}(\overline{X_n}, \overline{Y_n})$ defined in (3) and (4), respectively, are strongly consistent w.r.t. the measures $O(E(X), A)$ and $O(E(X), E(Y))$, i.e.,*

a) $\widehat{O}(\overline{X_n}, A) \stackrel{n\to\infty}{\longrightarrow} O(E(X), A)$ *a.s.-[P].*

b) $\widehat{O}(\overline{X_n}, \overline{Y_n}) \stackrel{n\to\infty}{\longrightarrow} O(E(X), E(Y))$ *a.s.-[P].*

3. One-sample test for the degree of overlap between the expected value of an RI and a fixed interval

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $X : \Omega \longrightarrow \mathcal{K}_c(\mathbb{R})$ be an RI such that $\text{spr}\, E(X) > 0$, and $A \in \mathcal{K}_c(\mathbb{R})$ such that $\text{spr}\, A > 0$. In order to avoid some trivial cases and to assure the existence of the moments involved in the theoretical developments, consider the RI X belonging to the following class of random intervals:

$$\begin{aligned}\mathcal{P} = \{X : \Omega \to \mathcal{K}_c(\mathbb{R}) \,|\, &\sigma^2_{\mathrm{mid}\,X} < \infty \,, 0 < \sigma^2_{\mathrm{spr}\,X} < \infty \text{ and} \\ &\sigma^2_{\mathrm{mid}\,X,\,\mathrm{spr}\,X} \neq \sigma^2_{\mathrm{mid}\,X}\sigma^2_{\mathrm{spr}\,X}\},\end{aligned} \tag{5}$$

where $\sigma^2_{\mathrm{mid}\,X}$, $\sigma^2_{\mathrm{spr}\,X}$ and $\sigma_{\mathrm{mid}\,X,\,\mathrm{spr}\,X}$ are the usual real variances and covariance for the real variables $\mathrm{mid}\,X$ and $\mathrm{spr}\,X$, respectively.

Given a degree $d \in (0, 1]$, the aim is to test

$$H_0 : O(E(X), A) \geq d \quad \text{vs.} \quad H_1 : O(E(X), A) < d. \tag{6}$$

We have chosen the study of the one-sided test in (6) since it seems to be the more appealing in practical applications. Nevertheless, the corresponding two-sided test and the other one-sided test can be analogously studied.

It should be noticed that the corresponding one-sided test $\leq$ for case $d = 1$ makes no sense since the degree of overlap between two values is always lower than or equal to 1. On the other hand, the two-sided test for the case $d = 0$ is the same that the one-sided test $\leq$ for the same case, and it is equivalent to test the empty intersection between $E(X)$ and A, which has been previously studied in [22]. Again, the corresponding one-sided test $\geq$ in case $d = 0$ has no sense since the degree of similarity between two values is always greater than or equal to 0. Therefore, we will focus the attention in the case $d \in (0, 1]$.

From (1) and (2) it is easy to show that the hypotheses of the test (6) can be equivalently expressed as

$$\begin{aligned} H_0 &: \max\Big\{ \min\big\{O_1(d), O_2(d)\big\}, \min\big\{O_3(d), O_4(d)\big\}\Big\} \leq 0; \\ H_1 &: \max\Big\{ \min\big\{O_1(d), O_2(d)\big\}, \min\big\{O_3(d), O_4(d)\big\}\Big\} > 0, \end{aligned} \tag{7}$$

where

$$\begin{aligned} O_1(d) &= \mathrm{mid}\,E(X) - \mathrm{mid}\,A + (2d-1)\,\mathrm{spr}\,E(X) - \mathrm{spr}\,A, \\ O_2(d) &= \mathrm{mid}\,E(X) - \mathrm{mid}\,A - \mathrm{spr}\,E(X) + (2d-1)\,\mathrm{spr}\,A, \\ O_3(d) &= -\mathrm{mid}\,E(X) + \mathrm{mid}\,A + (2d-1)\,\mathrm{spr}\,E(X) - \mathrm{spr}\,A, \\ O_4(d) &= -\mathrm{mid}\,E(X) + \mathrm{mid}\,A - \mathrm{spr}\,E(X) + (2d-1)\,\mathrm{spr}\,A. \end{aligned} \tag{8}$$

Let $\{X_i\}_{i=1}^n$ be a sample of random intervals independent and identically distributed as X, and let $\overline{X_n}$ be the associated sample mean. The corresponding test statistic is defined as follows:

$$T_X(d) = \sqrt{n}\max\Big\{ \min\big\{\widehat{O}_1(d), \widehat{O}_2(d)\big\}, \min\big\{\widehat{O}_3(d), \widehat{O}_4(d)\big\}\Big\}, \tag{9}$$

where

$$\begin{aligned}
\widehat{O}_1(d) &= \operatorname{mid}\overline{X_n} - \operatorname{mid} A + (2d-1)\operatorname{spr}\overline{X_n} - \operatorname{spr} A,\\
\widehat{O}_2(d) &= \operatorname{mid}\overline{X_n} - \operatorname{mid} A - \operatorname{spr}\overline{X_n} + (2d-1)\operatorname{spr} A,\\
\widehat{O}_3(d) &= -\operatorname{mid}\overline{X_n} + \operatorname{mid} A + (2d-1)\operatorname{spr}\overline{X_n} - \operatorname{spr} A,\\
\widehat{O}_4(d) &= -\operatorname{mid}\overline{X_n} + \operatorname{mid} A - \operatorname{spr}\overline{X_n} + (2d-1)\operatorname{spr} A.
\end{aligned} \tag{10}$$

Let us consider the bivariate normal distribution $Z = (z_1, z_2)^T \equiv \mathcal{N}_2\left(\vec{0}, \Sigma\right)$, where Σ is the covariance matrix for the random vector $(\operatorname{mid} X, \operatorname{spr} X)$. As X belongs to the class $\mathcal{P}$ defined in (5), the no singularity of the covariance matrix Σ is assured.

On the other hand, since the limit distribution of the statistic $T_X(d)$ for cases $d = 1$ and $d \in (0,1)$ in all the different situations under H_0 is analyzed in the following subsections by means of aysmptotic and bootstrap approaches.

3.1. Case d=1

Firstly, the limit distribution of the statistic $T_X(1)$ is provided in Lemma 1 and the proof is included in the Appendix.

Lemma 1. *For $n \in \mathbb{N}$, let $X_1, \ldots, X_n$ be n RIs independent and equally distributed from X, and defined on the probability space $(\Omega, \mathcal{A}, P)$. Let $T_X(d)$ be defined as in (9) and $X \in \mathcal{P}$. Whenever $O_1(1) = O_2(1) = O_3(1) = O_4(1) = 0$ (which means that $|\operatorname{mid} E(X) - \operatorname{mid} A| = |\operatorname{spr} E(X) - \operatorname{spr} A|$), it is fulfilled that*

$$T_X(1) \xrightarrow{\mathcal{L}} \max\left\{\min\{z_1 + z_2, z_1 - z_2\}, \min\{z_2 - z_1, -z_1 - z_2\}\right\} \tag{11}$$

as $n \to \infty$.

The consistency and the correctness of the test is stated as follows. If $d = 1$, $\alpha \in [0,1]$ and $k_{1-\alpha}$ is the $(1-\alpha)$-quantile of the asymptotic distribution of $T_X(1)$, and if H_0 in (29) is true, then it is easy to prove that

$$\limsup_{n\to\infty} P\left(T_X(1) > k_{1-\alpha}\right) \leq \alpha \ ,$$

taking into account Lemma 1. In addition, the equality is achieved whenever $O_1(1) = O_2(1) = O_3(1) = O_4(1) = 0$. On the other hand, if H_0 is not true, then it is also easy to verify that

$$\lim_{n\to\infty} P\left(T_X(1) > k_{1-\alpha}\right) = 1.$$

Then the test rejecting H_0 in (29) at the significance level α whenever $T_X(1) > k_{1-\alpha}$ is asymptotically correct and consistent.

The asymptotic distribution of $T_X(1)$ is not easy to handle in practice since it depends on unknown parameters. For this reason, a residual bootstrap approach is proposed.

Let X be an RI such that $\operatorname{spr} E(X) > 0$, and let $\{X_i\}_{i=1}^n$ be a simple random sample drawn from X. Consider a bootstrap sample $\{X_i^*\}_{i=1}^n$ being chosen randomly and with replacement from $\{X_i\}_{i=1}^n$. The bootstrap statistic is defined as follows:

$$\begin{aligned} T_X^*(1) = \max\Big\{ & \min\Big\{\sqrt{n}\left(\operatorname{mid}\overline{X_n^*} - \operatorname{mid}\overline{X_n} + \operatorname{spr}\overline{X_n^*} - \operatorname{spr}\overline{X_n}\right) \\ & \qquad \sqrt{n}\left(\operatorname{mid}\overline{X_n^*} - \operatorname{mid}\overline{X_n} - \operatorname{spr}\overline{X_n^*} + \operatorname{spr}\overline{X_n}\right)\Big\}, \\ & \min\Big\{\sqrt{n}\left(-\operatorname{mid}\overline{X_n^*} + \operatorname{mid}\overline{X_n} + \operatorname{spr}\overline{X_n^*} - \operatorname{spr}\overline{X_n}\right) \\ & \qquad \sqrt{n}\left(-\operatorname{mid}\overline{X_n^*} + \operatorname{mid}\overline{X_n} - \operatorname{spr}\overline{X_n^*} + \operatorname{spr}\overline{X_n}\right)\Big\}. \end{aligned} \tag{12}$$

The asymptotic distribution of $T_X^*(1)$ is given in Lemma 4. The result can be proved analogously to Lemma 1 by applying in this case the Bootstrapped CLT (see [11]).

Lemma 2. *Let X in $\mathcal{P}$. Whenever $O_1(1) = O_2(1) = O_3(1) = O_4(1) = 0$, it is fulfilled that*

$$T_X^*(1) \xrightarrow{\mathcal{L}} \max\{\min\{z_1 + z_2, z_1 - z_2\}, \min\{z_2 - z_1, -z_1 - z_2\}\} \quad as - [P]. \tag{13}$$

The consistency of the bootstrap procedure is straightforwardly derived. The distribution of $T_X^*(1)$ is approximated in practice by means of the Monte Carlo method as follows:

Algorithm for the bootstrap approach:

Step 1. For the simple random sample $\{X_i\}_{i=1}^n$, compute the value of the statistic $T_X(1)$ in (9).

Step 2. By resampling from $\{X_i\}_{i=1}^n$, get a bootstrap sample $\{X_i^*\}_{i=1}^n$ and compute the value of the bootstrap statistic $T_X^*(1)$ defined in (12).

Step 3. Repeat Step 2 a large number B of times to get a set of B values of the bootstrap statistic denoted by *Boot*.

Step 4. Compute the bootstrap p-value for the Test (29) as the proportion of values in *Boot* which are greater than or equal to $T_X(1)$.

3.2. Case $d \in (0,1)$

Lemma 3 provides the limit distribution of $T_X(d)$ in different situations under H_0 in (29) when $d \in (0,1)$. The proof is contained in the Appendix.

Lemma 3. *For $n \in \mathbb{N}$, let $X_1, \ldots, X_n$ be n RIs independent and equally distributed from X, and defined on the probability space $(\Omega, \mathcal{A}, P)$. Let $T_X(d)$ be defined as in (9). If $X \in \mathcal{P}$, then:*

a) Whenever $O_1(d) = O_2(d) = 0$ (which means that $\operatorname{mid} E(X) = \operatorname{mid} A + 2(1-d)\operatorname{spr} A$ and $\operatorname{spr} E(X) = \operatorname{spr} A$), it is fulfilled that

$$T_X(d) \xrightarrow{\mathcal{L}} \min\{z_1 + (2d-1)z_2, z_1 - z_2\} \quad \text{as } n \to \infty. \tag{14}$$

b) Whenever $O_3(d) = O_4(d) = 0$ (which means that $\operatorname{mid} E(X) = \operatorname{mid} A + 2(d-1)\operatorname{spr} A$ and $\operatorname{spr} E(X) = \operatorname{spr} A$), it is fulfilled that

$$T_X(d) \xrightarrow{\mathcal{L}} \min\{-z_1 + (2d-1)z_2, -z_1 - z_2\} \quad \text{as } n \to \infty. \tag{15}$$

c) Whenever $O_1(d) = 0$ and $O_2(d) > 0$ (which means that $O_3(d) < 0$, $O_4(d) < 0$), it is fulfilled that

$$T_X(d) \xrightarrow{\mathcal{L}} z_1 + (2d-1)z_2 \quad \text{as } n \to \infty. \tag{16}$$

d) Whenever $O_1(d) > 0$ and $O_2(d) = 0$ (which means that $O_3(d) < 0$, $O_4(d) < 0$), it is fulfilled that

$$T_X(d) \xrightarrow{\mathcal{L}} z_1 - z_2 \quad \text{as } n \to \infty. \tag{17}$$

e) Whenever $O_3(d) = 0$ and $O_4(d) > 0$ (which implies that $O_1(d) < 0$, $O_2(d) < 0$), it is fulfilled that

$$T_X(d) \xrightarrow{\mathcal{L}} -z_1 + (2d-1)z_2 \quad \text{as } n \to \infty. \tag{18}$$

f) Whenever $O_3(d) > 0$ and $O_4(d) = 0$ (which implies that $O_1(d) < 0$, $O_2(d) < 0$), it is fulfilled that

$$T_X(d) \xrightarrow{\mathcal{L}} -z_1 - z_2 \quad \text{as } n \to \infty. \tag{19}$$

g) Whenever 3 values among $O_1(d)$, $O_2(d)$, $O_3(d)$ and $O_4(d)$ are lower than 0, it is fulfilled that

$$T_X(d) \xrightarrow{n\to\infty} -\infty \quad \text{as } n \to \infty. \tag{20}$$

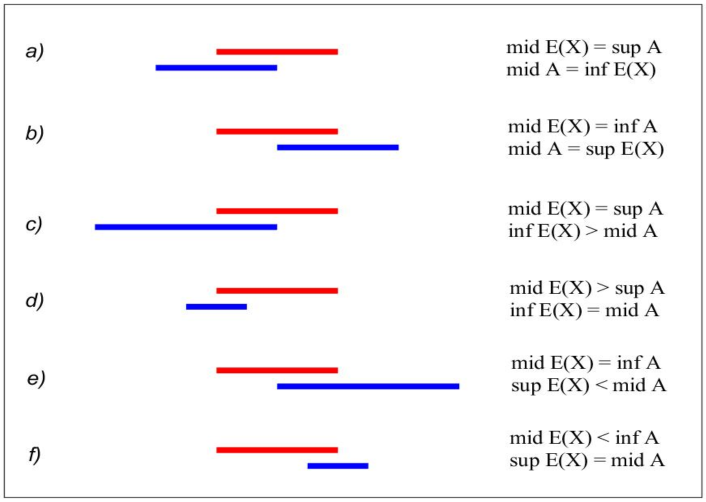

Figure 2: Situations a)-f) in Lemma 3 for a degree of overlap between $E(X)$ (in red) and A (in blue) equal to 1/2

A scheme of the different situations addressed in Lemma 3 for case $d = 1/2$ is provided in Figure 2.

Remark 1. *As it is observed in Lemma 3 the asymptotic distribution of $T_X(d)$ under the null hypothesis H_0 depends on X and, specifically, on $\operatorname{mid} E(X)$ and $\operatorname{spr} E(X)$. Thus, in order to develop the theoretical analysis of the testing procedure it is necessary to consider an X-dependent distribution to compare with. For this purpose, the following statistic will be taken into account:*

$$\begin{aligned}
T'_X(d) = \max\Big\{ &\min\Big\{ \sqrt{n}\left(\operatorname{mid}\overline{X_n} - \operatorname{mid}E(X) + (2d-1)(\operatorname{spr}\overline{X_n} - \operatorname{spr}E(X))\right) \\
&\qquad + \max\left(0, n^{1/4}(\operatorname{spr}\overline{X_n} - \operatorname{spr}A)\right), \\
&\qquad \sqrt{n}\left(\operatorname{mid}\overline{X_n} - \operatorname{mid}E(X) - \operatorname{spr}\overline{X_n} + \operatorname{spr}E(X)\right) \\
&\qquad + \max\left(0, n^{1/4}(\operatorname{spr}A - \operatorname{spr}\overline{X_n})\right) \Big\}, \\
&+ \min\left(0, n^{1/4}(\operatorname{mid}\overline{X_n} - \operatorname{mid}A + 2(d-1)\operatorname{spr}A)\right), \\
&\min\Big\{ \sqrt{n}\left(-\operatorname{mid}\overline{X_n} + \operatorname{mid}E(X) + (2d-1)(\operatorname{spr}\overline{X_n} - \operatorname{spr}E(X))\right) \\
&\qquad + \max\left(0, n^{1/4}(\operatorname{spr}\overline{X_n} - \operatorname{spr}A)\right) \\
&\qquad \sqrt{n}\left(-\operatorname{mid}\overline{X_n} + \operatorname{mid}E(X) - \operatorname{spr}\overline{X_n} + \operatorname{spr}E(X)\right) \\
&\qquad + \max\left(0, n^{1/4}(\operatorname{spr}A - \operatorname{spr}\overline{X_n})\right) \Big\} \\
&+ \min\left(0, n^{1/4}(\operatorname{mid}A - \operatorname{mid}\overline{X_n} + 2(d-1)\operatorname{spr}A)\right) \Big\}.
\end{aligned}$$

Mimicking the procedure in [22], the minima and maxima included in the expression above have been introduced to determine the component of the expression of $T'_X(d)$ which has more influence depending on each situation under H_0. The possible limit distributions of $T'_X(d)$ are established below:

- *If* $d = 1$ *and* $|\mathrm{mid}\, E(X) - \mathrm{mid}\, A| = |\mathrm{spr}\, E(X) - \mathrm{spr}\, A|$*, then* $T'_X(d)$ *converges in law to the same distribution as in (11).*
- *If* $\mathrm{mid}\, E(X) > \mathrm{mid}\, A + 2(d-1)\mathrm{spr}\, A$ *and* $\mathrm{spr}\, E(X) = \mathrm{spr}\, A$*, then* $T'_X(d)$ *converges in law to the same distribution as in (14).*
- *If* $\mathrm{mid}\, E(X) < \mathrm{mid}\, A + 2(d-1)\mathrm{spr}\, A$ *and* $\mathrm{spr}\, E(X) = \mathrm{spr}\, A$*, then* $T'_X(d)$ *converges in law to the same distribution as in (15).*
- *If* $\mathrm{mid}\, E(X) > \mathrm{mid}\, A + 2(d-1)\mathrm{spr}\, A$ *and* $\mathrm{spr}\, E(X) < \mathrm{spr}\, A$*, then* $T'_X(d)$ *converges in law to the same distribution as in (16).*
- *If* $\mathrm{mid}\, E(X) > \mathrm{mid}\, A + 2(d-1)\mathrm{spr}\, A$ *and* $\mathrm{spr}\, E(X) > \mathrm{spr}\, A$*, then* $T'_X(d)$ *converges in law to the same distribution as in (17).*
- *If* $\mathrm{mid}\, E(X) < \mathrm{mid}\, A + 2(d-1)\mathrm{spr}\, A$ *and* $\mathrm{spr}\, E(X) < \mathrm{spr}\, A$*, then* $T'_X(d)$ *converges in law to the same distribution as in (18).*
- *If* $\mathrm{mid}\, E(X) < \mathrm{mid}\, A + (2d-1)\mathrm{spr}\, A$ *and* $\mathrm{spr}\, E(X) > \mathrm{spr}\, A$*, then* $T'_X(d)$ *converges in law to the same distribution as in (19).*
- *If* $\mathrm{mid}\, E(X) = \mathrm{mid}\, A + (2d-1)\mathrm{spr}\, A$ *and* $\mathrm{spr}\, E(X) < \mathrm{spr}\, A$*, then* $T'_X(d)$ *converges in law to*

$$\max\{z_1 + (2d-1)z_2, -z_1 + (2d-1)z_2\}.$$

- *If* $\mathrm{mid}\, E(X) = \mathrm{mid}\, A + (2d-1)\mathrm{spr}\, A$ *and* $\mathrm{spr}\, E(X) > \mathrm{spr}\, A$*, then* $T'_X(d)$ *converges in law to*

$$\max\{z_1 - z_2, -z_1 - z_2\}.$$

- *If* $\mathrm{mid}\, E(X) = \mathrm{mid}\, A + (2d-1)\mathrm{spr}\, A$ *and* $\mathrm{spr}\, E(X) = \mathrm{spr}\, A$*, then* $T'_X(d)$ *converges in law to*

$$\max\{\min\{z_1 + (2d-1)z_2, z_1 - z_2\}, \min\{-z_1 + (2d-1)z_2, -z_1 - z_2\}\}.$$

It can be concluded that the asymptotic distribution of $T_X(d)$ is stochastically bounded by the one of $T'_X(d)$ in all the situations under H_0.

The theoretical analysis of the testing procedure in case $d \in (0,1)$ in terms of its consistency and its correctness is provided in the following lines. On one hand, if $\alpha \in [0,1]$ and $k_{1-\alpha}$ is the $(1-\alpha)$-quantile of the asymptotic distribution of $T'_X(d)$, and if H_0 in (29) is true, then it is easy to prove that

$$\limsup_{n\to\infty} P\left(T_X(d) > k_{1-\alpha}\right) \leq \alpha \ ,$$

taking into account Lemma 3 and that $T_X(d)$ is stochastically bounded by $T'_X(d)$ under H_0. In addition, the equality is achieved whenever conditions in *a)*, *b)*, *c)* and *d)* in Lemma 3 are fulfilled. On the other hand, if H_0 is not true, then it is also easy to verify that

$$\lim_{n\to\infty} P\left(T_X(d) > k_{1-\alpha}\right) = 1.$$

As an immediate consequence, the test rejecting H_0 in (29) at the significance level α whenever $T_X(d) > k_{1-\alpha}$ is asymptotically correct and consistent.

The bootstrap statistic proposed in case $d \in (0,1)$ is based on the expression of $T'_X(d)$ and it is defined as

$$\begin{aligned}
T^*_X(d) = \max\Big\{ & \min\Big\{ \sqrt{n}\left(\operatorname{mid}\overline{X^*_n} - \operatorname{mid}\overline{X_n} + (2d-1)(\operatorname{spr}\overline{X^*_n} - \operatorname{spr}\overline{X_n})\right) \\
& \qquad + \max\left(0, n^{1/4}(\operatorname{spr}\overline{X_n} - \operatorname{spr}A)\right), \\
& \qquad \sqrt{n}\left(\operatorname{mid}\overline{X^*_n} - \operatorname{mid}\overline{X_n} - \operatorname{spr}\overline{X^*_n} + \operatorname{spr}\overline{X_n}\right) \\
& \qquad + \max\left(0, n^{1/4}(\operatorname{spr}A - \operatorname{spr}\overline{X_n})\right) \Big\}, \\
& + \min\left(0, n^{1/4}(\operatorname{mid}\overline{X_n} - \operatorname{mid}A + 2(d-1)\operatorname{spr}A)\right), \\
& \min\Big\{ \sqrt{n}\left(-\operatorname{mid}\overline{X^*_n} + \operatorname{mid}\overline{X_n} + (2d-1)(\operatorname{spr}\overline{X^*_n} - \operatorname{spr}\overline{X_n})\right) \\
& \qquad + \max\left(0, n^{1/4}(\operatorname{spr}\overline{X_n} - \operatorname{spr}A)\right) \\
& \qquad \sqrt{n}\left(-\operatorname{mid}\overline{X^*_n} + \operatorname{mid}\overline{X_n} - \operatorname{spr}\overline{X^*_n} + \operatorname{spr}\overline{X_n}\right) \\
& \qquad + \max\left(0, n^{1/4}(\operatorname{spr}A - \operatorname{spr}\overline{X_n})\right) \Big\} \\
& + \min\left(0, n^{1/4}(\operatorname{mid}A - \operatorname{mid}\overline{X_n} + 2(d-1)\operatorname{spr}A)\right) \Big\}.
\end{aligned} \tag{21}$$

The different possibilities for the asymptotic distribution of $T^*_X(d)$ are provided in Lemma 4.

Lemma 4. *Let X in $\mathcal{P}$. Then:*

a) Whenever $O_1(d) = O_2(d) = 0$, it is fulfilled that

$$T^*_X(d) \xrightarrow{\mathcal{L}} \min\{z_1 + (2d-1)z_2, z_1 - z_2\} \quad as - [P]. \tag{22}$$

b) Whenever $O_3(d) = O_4(d) = 0$, it is fulfilled that

$$T^*_X(d) \xrightarrow{\mathcal{L}} \min\{-z_1 + (2d-1)z_2, -z_1 - z_2\} \quad as - [P]. \tag{23}$$

c) Whenever $O_1(d) = 0$ and $O_2(d) > 0$, it is fulfilled that

$$T_X^*(d) \xrightarrow{\mathcal{L}} z_1 + (2d-1)z_2 \quad as-[P]. \tag{24}$$

d) Whenever $O_1(d) > 0$ and $O_2(d) = 0$, it is fulfilled that

$$T_X^*(d) \xrightarrow{\mathcal{L}} z_1 - z_2 \quad as-[P]. \tag{25}$$

e) Whenever $O_3(d) = 0$ and $O_4(d) > 0$, it is fulfilled that

$$T_X^*(d) \xrightarrow{\mathcal{L}} -z_1 + (2d-1)z_2 \quad as-[P]. \tag{26}$$

f) Whenever $O_3(d) > 0$ and $O_4(d) = 0$, it is fulfilled that

$$T_X^*(d) \xrightarrow{\mathcal{L}} -z_1 - z_2 \quad as-[P]. \tag{27}$$

Other situations under H_0 leads to other limit distributions of the bootstrap statistic different from the ones provided in Lemma 4.

In practice, the distribution of $T_X^*(d)$ is approximated by means of the Monte Carlo method as in the case $d = 1$.

4. Two-sample test for the degree of overlap between the expected value of two RIs

The theoretical developments for the two-sample test are analogous to the ones of Section 3, so the proofs of the corresponding lemmas are very similar and are not included.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and $X, Y : \Omega \longrightarrow \mathcal{K}_c(\mathbb{R})$ be two independent RIs such that $\operatorname{spr} E(X) > 0$ and $\operatorname{spr} E(Y) > 0$ and belonging to the class $\mathcal{P}$ defined in (5).

In this case, given a degree $d \in (0,1)$, the aim is to test

$$H_0 : O(E(X), E(Y)) \geq d \quad \text{vs.} \quad H_1 : O(E(X), E(Y)) < d, \tag{28}$$

that can be equivalently expressed as

$$\begin{aligned} H_0 &: \max\Big\{ \min\big\{\Theta_1(d), \Theta_2(d)\big\}, \min\big\{\Theta_3(d), \Theta_4(d)\big\}\Big\} \leq 0; \\ H_1 &: \max\Big\{ \min\big\{\Theta_1(d), \Theta_2(d)\big\}, \min\big\{\Theta_3(d), \Theta_4(d)\big\}\Big\} > 0, \end{aligned} \tag{29}$$

where

$$\begin{aligned}
\Theta_1(d) &= \operatorname{mid} E(X) - \operatorname{mid} E(Y) + (2d-1)\operatorname{spr} E(X) - \operatorname{spr} E(Y),\\
\Theta_2(d) &= \operatorname{mid} E(X) - \operatorname{mid} E(Y) - \operatorname{spr} E(X) + (2d-1)\operatorname{spr} E(Y),\\
\Theta_3(d) &= -\operatorname{mid} E(X) + \operatorname{mid} E(Y) + (2d-1)\operatorname{spr} E(X) - \operatorname{spr} E(Y),\\
\Theta_4(d) &= -\operatorname{mid} E(X) + \operatorname{mid} E(Y) - \operatorname{spr} E(X) + (2d-1)\operatorname{spr} E(Y).
\end{aligned} \tag{30}$$

Let $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^m$ be simple random samples drawn from X and Y, respectively, so that $n/(n+m) \to p_1 \in (0,1)$ and $m/(n+m) \to p_2 \in (0,1)$. The test statistic in this case is defined below.

$$T_{(X,Y)}(d) = \sqrt{n}\max\Big\{\min\big\{\widehat{\Theta}_1(d), \widehat{\Theta}_2(d)\big\}, \min\big\{\widehat{\Theta}_3(d), \widehat{\Theta}_4(d)\big\}\Big\}, \tag{31}$$

where

$$\begin{aligned}
\widehat{\Theta}_1(d) &= \operatorname{mid}\overline{X_n} - \operatorname{mid}\overline{Y_m} + (2d-1)\operatorname{spr}\overline{X_n} - \operatorname{spr}\overline{Y_m},\\
\widehat{\Theta}_2(d) &= \operatorname{mid}\overline{X_n} - \operatorname{mid}\overline{Y_m} - \operatorname{spr}\overline{X_n} + (2d-1)\operatorname{spr}\overline{Y_m},\\
\widehat{\Theta}_3(d) &= -\operatorname{mid}\overline{X_n} + \operatorname{mid}\overline{Y_m} + (2d-1)\operatorname{spr}\overline{X_n} - \operatorname{spr}\overline{Y_m},\\
\widehat{\Theta}_4(d) &= -\operatorname{mid}\overline{X_n} + \operatorname{mid}\overline{Y_m} - \operatorname{spr}\overline{X_n} + (2d-1)\operatorname{spr}\overline{Y_m}.
\end{aligned} \tag{32}$$

Consider the bivariate normal distributions $U = (u_1, u_2)^T \equiv \mathcal{N}_2\Big(\vec{0}, \Sigma_1\Big)$ and $V = (v_1, v_2)^T \equiv \mathcal{N}_2\Big(\vec{0}, \Sigma_2\Big)$, where Σ_1 and Σ_2 are the covariance matrices for the random vectors $(\operatorname{mid} X, \operatorname{spr} X)$ and $(\operatorname{mid} Y, \operatorname{spr} Y)$, respectively. The asymptotic distribution of the statistic $T_{(X,Y)}(d)$ in all the different situations under H_0 is studied in Lemma 5.

Lemma 5. *For $n, m \in \mathbb{N}$, let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be simple random samples drawn from X and Y, respectively, so that $n/(n+m) \to p_1 \in (0,1)$ and $m/(n+m) \to p_2 \in (0,1)$. Let $T_{(X,Y)}(d)$ be defined as in (31). If $X, Y \in \mathcal{P}$, then:*

a) Whenever $\Theta_1(1) = \Theta_2(1) = \Theta_3(1) = \Theta_4(1) = 0$ (which only arises when $d = 1$ and $|\operatorname{mid} E(X) - \operatorname{mid} E(Y)| = |\operatorname{spr} E(X) - \operatorname{spr} E(Y)|$), it is fulfilled that

$$\begin{aligned}
T_{(X,Y)}(1) \xrightarrow{\mathcal{L}} \max\Big\{&\min\{u_1 - v_1 + u_2 - v_2, u_1 - v_1 - u_2 + v_2\},\\
&\min\{-u_1 + v_1 + u_2 - v_2, -u_1 + v_1 - u_2 + v_2\}\Big\}
\end{aligned} \tag{33}$$

as $n \to \infty$.

b) Whenever $\Theta_1(d) = \Theta_2(d) = 0$ (which means that $\operatorname{mid} E(X) = \operatorname{mid} E(Y) + 2(1-d)\operatorname{spr} E(Y)$ and $\operatorname{spr} E(X) = sprE(Y)$), it is fulfilled that

$$\begin{aligned}
T_{(X,Y)}(d) \xrightarrow{\mathcal{L}} \min\{\, &u_1 - v_1 + (2d-1)u_2 - v_2,\\
&u_1 - v_1 - u_2 + (2d-1)v_2\} \quad \text{as } n \to \infty.
\end{aligned} \tag{34}$$

c) Whenever $\Theta_3(d) = \Theta_4(d) = 0$ *(which means that* $\mathrm{mid}\, E(X) = \mathrm{mid}\, E(Y) + 2(d-1)\mathrm{spr}\, E(Y)$ *and* $\mathrm{spr}\, E(X) = spr E(Y)$*), it is fulfilled that*

$$T_{(X,Y)}(d) \xrightarrow{\mathcal{L}} \min\{-u_1 + v_1 + (2d-1)u_2 - v_2, \\ -u_1 + v_1 - u_2 + (2d-1)v_2\} \quad \text{as } n \to \infty. \tag{35}$$

d) Whenever $\Theta_1(d) = 0$ *and* $\Theta_2(d) > 0$ *(which means that* $\Theta_3(d) < 0$, $\Theta_4(d) < 0$*), it is fulfilled that*

$$T_{(X,Y)}(d) \xrightarrow{\mathcal{L}} u_1 - v_1 + (2d-1)u_2 - v_2 \quad \text{as } n \to \infty. \tag{36}$$

e) Whenever $\Theta_1(d) > 0$ *and* $\Theta_2(d) = 0$ *(which means that* $\Theta_3(d) < 0$, $\Theta_4(d) < 0$*), it is fulfilled that*

$$T_{(X,Y)}(d) \xrightarrow{\mathcal{L}} u_1 - v_1 - u_2 + (2d-1)v_2 \quad \text{as } n \to \infty. \tag{37}$$

f) Whenever $\Theta_3(d) = 0$ *and* $\Theta_4(d) > 0$ *(which means that* $\Theta_1(d) < 0$, $\Theta_2(d) < 0$, $\mathrm{mid}\, E(X) < \mathrm{mid}\, E(Y) + 2(d-1)\mathrm{spr}\, E(Y)$ *and* $\mathrm{spr}\, E(X) < \mathrm{spr}\, E(Y)$*), it is fulfilled that*

$$T_{(X,Y)}(d) \xrightarrow{\mathcal{L}} -u_1 + v_1 + (2d-1)u_2 - v_2 \quad \text{as } n \to \infty. \tag{38}$$

g) Whenever $\Theta_3(d) > 0$ *and* $\Theta_4(d) = 0$ *(which means that* $\Theta_1(d) < 0$, $\Theta_2(d) < 0$, $\mathrm{mid}\, E(X) < \mathrm{mid}\, E(Y) + 2(d-1)\mathrm{spr}\, E(Y)$ *and* $\mathrm{spr}\, E(X) > \mathrm{spr}\, E(Y)$*), it is fulfilled that*

$$T_{(X,Y)}(d) \xrightarrow{\mathcal{L}} -u_1 + v_1 - u_2 + (2d-1)v_2 \quad \text{as } n \to \infty. \tag{39}$$

h) Whenever 3 values among $\Theta_1(d)$, $\Theta_2(d)$, $\Theta_3(d)$ *and* $\Theta_4(d)$ *are lower than 0, it is fulfilled that*

$$T_{(X,Y)}(d) \overset{n\to\infty}{\longrightarrow} -\infty \quad \text{as } n \to \infty. \tag{40}$$

As in the one-sample test, in the case $d \in (0,1)$ we can define a statistic $T'_{(X,Y)}$ which stochastically bounds $T_{(X,Y)}$ in all situations under H_0. In addition, it can be easily proved that the test rejecting H_0 in (28) at the significance level α whenever $T_{(X,Y)} > k_{1-\alpha}$ is asymptotically correct and consistent.

Finally, the corresponding bootstrap approach is proposed. Given X and Y RIs such that $\mathrm{spr}\, E(X) > 0$ and $\mathrm{spr}\, E(Y) > 0$, $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^m$ two simple random samples drawn from X and Y, and $\{X_i^*\}_{i=1}^n$ and $\{Y_i^*\}_{i=1}^m$ two resamplings from

$\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^m$, respectively. Then, the corresponding bootstrap statistics for cases $d = 1$ and $d \in (0, 1)$ can be defined as follows:

$$
\begin{aligned}
T^*_{(X,Y)}(1) = \max\Big\{ \min\Big\{ &\sqrt{n}\big(\mathrm{mid}\,\overline{X^*_n} - \mathrm{mid}\,\overline{X_n} - \mathrm{mid}\,\overline{Y^*_m} + \mathrm{mid}\,\overline{Y_m} \\
&\quad +\mathrm{spr}\,\overline{X^*_n} - \mathrm{spr}\,\overline{X_n} - \mathrm{spr}\,\overline{Y^*_m} + \mathrm{spr}\,\overline{Y_m}\big) \\
&\sqrt{n}\big(\mathrm{mid}\,\overline{X^*_n} - \mathrm{mid}\,\overline{X_n} - \mathrm{mid}\,\overline{Y^*_m} + \mathrm{mid}\,\overline{Y_m} \\
&\quad +\mathrm{spr}\; - \overline{X^*_n} + \mathrm{spr}\,\overline{X_n} + \mathrm{spr}\,\overline{Y^*_m} - \mathrm{spr}\,\overline{Y_m}\big)\Big\}, \\
\min\Big\{ &\sqrt{n}\big(- \mathrm{mid}\,\overline{X^*_n} + \mathrm{mid}\,\overline{X_n} + \mathrm{mid}\,\overline{Y^*_m} - \mathrm{mid}\,\overline{Y_m} \\
&\quad +\mathrm{spr}\,\overline{X^*_n} - \mathrm{spr}\,\overline{X_n} - \mathrm{spr}\,\overline{Y^*_m} + \mathrm{spr}\,\overline{Y_m}\big) \\
&\sqrt{n}\big(- \mathrm{mid}\,\overline{X^*_n} + \mathrm{mid}\,\overline{X_n} + \mathrm{mid}\,\overline{Y^*_m} - \mathrm{mid}\,\overline{Y_m} \\
&\quad +\mathrm{spr}\; - \overline{X^*_n} + \mathrm{spr}\,\overline{X_n} + \mathrm{spr}\,\overline{Y^*_m} - \mathrm{spr}\,\overline{Y_m}\big)\Big\};
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
T^*_{(X,Y)}(d) = \max\Big\{ \min\Big\{ &\sqrt{n}\Big(\mathrm{mid}\,\overline{X^*_n} - \mathrm{mid}\,\overline{X_n} - \mathrm{mid}\,\overline{Y^*_m} + \mathrm{mid}\,\overline{Y_m} \\
&+(2d-1)(\mathrm{spr}\,\overline{X^*_n} - \mathrm{spr}\,\overline{X_n}) - \mathrm{spr}\,\overline{Y^*_m} + \mathrm{spr}\,\overline{Y_m}\Big) \\
&+\max\big(0, n^{1/4}(\mathrm{spr}\,\overline{X_n} - \mathrm{spr}\,\overline{Y_m})\big), \\
&\sqrt{n}\Big(\mathrm{mid}\,\overline{X^*_n} - \mathrm{mid}\,\overline{X_n} - \mathrm{mid}\,\overline{Y^*_m} + \mathrm{mid}\,\overline{Y_m} \\
&-\mathrm{spr}\,\overline{X^*_n} + \mathrm{spr}\,\overline{X_n} + (2d-1)(\mathrm{spr}\,\overline{Y^*_m} - \mathrm{spr}\,\overline{Y_m})\Big) \\
&+\max\big(0, n^{1/4}(\mathrm{spr}\,\overline{Y_m} - \mathrm{spr}\,\overline{X_n})\big)\Big\} \\
+\min(0, n^{1/4})&(\mathrm{mid}\,\overline{X_n} - \mathrm{mid}\,\overline{Y_m} + 2(d-1)\mathrm{spr}\,\overline{Y_m}), \\
\min\Big\{ &\sqrt{n}\Big(- \mathrm{mid}\,\overline{X^*_n} + \mathrm{mid}\,\overline{X_n} + \mathrm{mid}\,\overline{Y^*_m} - \mathrm{mid}\,\overline{Y_m} \\
&+(2d-1)(\mathrm{spr}\,\overline{X^*_n} - \mathrm{spr}\,\overline{X_n}) - \mathrm{spr}\,\overline{Y^*_m} + \mathrm{spr}\,\overline{Y_m}\Big) \\
&+\max\big(0, n^{1/4}(\mathrm{spr}\,\overline{X_n} - \mathrm{spr}\,\overline{Y_m})\big), \\
&\sqrt{n}\Big(- \mathrm{mid}\,\overline{X^*_n} + \mathrm{mid}\,\overline{X_n} + \mathrm{mid}\,\overline{Y^*_m} - \mathrm{mid}\,\overline{Y_m} \\
&-\mathrm{spr}\,\overline{X^*_n} + \mathrm{spr}\,\overline{X_n} + (2d-1)(\mathrm{spr}\,\overline{Y^*_m} - \mathrm{spr}\,\overline{Y_m})\Big) \\
&+\max\big(0, n^{1/4}(\mathrm{spr}\,\overline{Y_m} - \mathrm{spr}\,\overline{X_n})\big)\Big\} \\
+\min(0, n^{1/4})&(-\mathrm{mid}\,\overline{X_n} + \mathrm{mid}\,\overline{Y_m} + 2(d-1)\mathrm{spr}\,\overline{Y_m})\Big\}.
\end{aligned}
\tag{42}
$$

The asymptotic distributions of $T^*_{(X,Y)}(1)$ and $T^*_{(X,Y)}(d)$ are (almost sure) the ones provided in Lemma 5 for $T_{(X,Y)}$, under the same conditions. In addition, the consistency of the bootstrap procedure can be straightforwardly derived. Finally, the distributions of both $T^*_{(X,Y)}(1)$ and $T^*_{(X,Y)}(d)$ are approximated in practice again by means of the Monte Carlo method.

5. Power analysis of the tests

The capability of the tests for the degree of overlap defined in the previous sections is analyzed. A suitable way to carry out this analysis is through the study of the power function under a sequence of alternatives which converges to the null one as the sample size increases, i.e., by using the so-called local alternatives. They allow us to measure how sensitive is a test under small deviations from the null hypothesis.

Withouth loss of generality we will check the power under the different situations established in Lemma 5 for the two-sample case. The power analysis of the corresponding one-sample test can be addressed analogously.

Suppose that X and Y are two RIs so that $O(E(X), E(Y)) = d$. Let $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^m$ be two simple random samples drawn from X and Y, and let $\{Y_i^{'}\}_{i=1}^m$ $\{Y_i^{''}\}_{i=1}^m$ be two 'corrections' of $\{Y_i\}_{i=1}^m$ defined as

$$Y_i^{'} = Y_i + \frac{\delta_m}{\sqrt{m}} \quad \text{and} \quad Y_i^{''} = Y_i - \frac{\delta_m}{\sqrt{m}} \tag{43}$$

whose sample means are $\overline{Y_m^{'}} = \overline{Y_m} + \frac{\delta_m}{\sqrt{m}}$ and $\overline{Y_m^{''}} = \overline{Y_m} - \frac{\delta_m}{\sqrt{m}}$, respectively. If $\delta_m \nearrow \infty$ and $\delta_m/\sqrt{m} \to 0$ as $m \to \infty$, then both sequences of sample means $\{\overline{Y_m^{'}}\}_m$ and $\{\overline{Y_m^{''}}\}_m$ converges to $\overline{Y_m}$ as the sample size m tends to ∞. Thus, it is easy to check that the null hypothesis is not satisfied but it is approached as $m \to \infty$.

Theorem 2 shows that the power of the test under different situations converges to 1. Its proof is provided in the Appendix.

Theorem 2. *Let X and Y be two RIs, let $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^m$ be two simple random samples drawn from them and let $\{Y_i^{'}\}_{i=1}^m$ $\{Y_i^{''}\}_{i=1}^m$ be the 'corrections' of $\{Y_i\}_{i=1}^m$ defined above. Let $T_{(X,Y^{'})}(d)$ and $T_{(X,Y^{''})}(d)$ be defined as in (31) and let $k_{1-\alpha}$ and $k_{1-\beta}$ be the $(1-\alpha)$ and $(1-\beta)$-quantiles of the asymptotic distributions of $T_{(X,Y^{'})}(d)$ and $T_{(X,Y^{''})}(d)$, respectively.*

i) If the asymptotic testing procedures in cases a), b), d) and e) of Lemma 5 are applied to $\{Y_i^{'}\}_{i=1}^m$, then

$$\lim_{n\to\infty} P(\, T_{(X,Y^{'})} > k_{1-\alpha}\,) = 1.$$

ii) If the asymptotic testing procedures in cases c), f) and g) of Lemma 5 are applied to $\{Y_i^{''}\}_{i=1}^m$, then

$$\lim_{n\to\infty} P(\, T_{(X,Y^{''})} > t_{1-\alpha}\,) = 1.$$

6. Simulation studies

In this section, some simulation studies are developed in order to show the empirical behavior of the bootstrap approach. For this purpose, several models are taken into account for the one-sample and two-sample tests proposed in Sections 3 and 4. Two different situations are tackled in both scenarios: in the first one the mid-point and spread components of the RI or RIs considered are independently generated; in the second one, it is allowed that those components have certain level of dependence each other. In addition, the consideration of equal or unequal sample sizes has been examined in the two-sample case.

6.1. Scenario 1: one-sample test

As a first study, the one-sample test is analyzed. Some different models are proposed according to the independency or dependency of the components. In all the cases, the fixed interval $A = [1, 3]$ is used to compare it with the expected value of X. The bootstrap test introduced in Section 3 has been run in every case for 10000 simulations with 1000 bootstrap replications each, by considering the usual significance level $\alpha = 0.05$ and different sample sizes. The different null hypothesis investigated as well as the distributions for the mid-point and spread components are gathered in Table 1.

Table 1: Distributions for the mid-point and spread of X considered in 10 different scenarios of the one-sample test for the degree of overlap between $E(X)$ and A

Model	H_0	$\text{mid}\, X$	$\text{spr}\, X$	$E(X)$
1.1.a)	$O(E(X), A) = 1$	$\mathcal{N}(2, 1)$	χ^2_1	$[1, 3]$
1.1.b)	$O(E(X), A) \geq 1/2$	$\mathcal{N}(3, 1)$	χ^2_1	$[2, 4]$
1.1.c)	$O(E(X), A) \geq 1/2$	$\mathcal{N}(1, 1)$	χ^2_1	$[0, 2]$
1.1.d)	$O(E(X), A) \geq 1/2$	$U(0, 6)$	$\chi^2_{0.5}$	$[2.5, 3.5]$
1.1.e)	$O(E(X), A) \geq 1/2$	$U(-2, 4)$	$\chi^2_{0.5}$	$[0.5, 1.5]$
1.2.a)	$O(E(X), A) = 1$	$\text{spr}\, X + \mathcal{N}(1, 1)$	χ^2_1	$[1, 3]$
1.2.b)	$O(E(X), A) \geq 1/2$	$\text{spr}\, X + \mathcal{N}(2, 1)$	χ^2_1	$[2, 4]$
1.2.c)	$O(E(X), A) \geq 1/2$	$\text{spr}\, X + \mathcal{N}(0, 1)$	$U(0, 2)$	$[0, 2]$
1.2.d)	$O(E(X), A) \geq 1/2$	$\mathcal{N}(3.5, 1) - \text{spr}\, X$	$U(0, 1)$	$[2.5, 3.5]$
1.2.e)	$O(E(X), A) \geq 1/2$	$\mathcal{N}(0.5, 1) + \text{spr}\, X$	$U(0, 1)$	$[2.5, 3.5]$

It is easy to check that models 1.a) and 2.a) satisfy the conditions of Lemma 1, whereas models 1.1.b)-1.2.b), 1.1.c)-1.2.c), 1.1.d)-1.2.d) and 1.1.e)-1.2.e) satisfy conditions of parts a), b), c) and e) of Lemma 3, respectively. Note that cases fulfilling conditions of parts d) and f) of the same lemma can be analogously considered leading to similar conclusions. The results obtained in the situations included in Table 1 are collected in Table 2.

They show that the empirical sizes of the test are in all cases quite close to the expected nominal significance level even for moderate sample sizes. Specifically, it should be noticed that even when that approximation is relatively good for all models, it is more conservative in the case of models 1.1.b), 1.2.b), 1.1.c) and 1.2.c) (which are those corresponding to parts a) and b) of Lemma 3) than in the rest of scenarios. Thus, in view of the previous results it can be concluded that models fulfilling conditions a) and b) of Lemma 3) for solving Test (29) whenever $d = 1/2$ behave a little bit better than those fulfilling conditions c) and e) of the same lemma, although all of them present a good behavior in terms of approximation to the expected nominal significance level.

Table 2: Empirical size of the bootstrap test for the degree of overlap for models included in Table 1

n \ Model	**1.1.a)**	**1.1.b)**	**1.1.c)**	**1.1.d)**	**1.1.e)**
$n = 10$	0.0616	0.0648	0.0629	0.0830	0.0856
$n = 30$	0.0555	0.0529	0.0497	0.0642	0.0636
$n = 50$	0.0489	0.0497	0.0482	0.0606	0.0592
$n = 100$	0.0503	0.0489	0.0489	0.0534	0.0531
$n = 200$	0.0499	0.0510	0.05	0.0509	0.0507
n \ Model	**1.2.a)**	**1.2.b)**	**1.2.c)**	**1.2.d)**	**1.2.e)**
$n = 10$	0.0913	0.0605	0.0641	0.0808	0.0757
$n = 30$	0.0603	0.0479	0.0482	0.0580	0.0606
$n = 50$	0.0575	0.0491	0.0489	0.0537	0.0555
$n = 100$	0.0532	0.0486	0.0496	0.0517	0.0534
$n = 200$	0.0507	0.05	0.0492	0.0511	0.0514

In addition, there are not remarkable differences among the results when considerating either independent or dependent distributions for the mid-point and spread components. Finally, some slight differences appreciated when comparing all the situations may also be due to the diverse nature of the distributions.

6.2. Scenario 2: two-sample test

Secondly, the behavior of the two-sample test introduced in Section 4 is investigated. Again, different models are considered for solving the two-sided test for the degree of overlap between $E(X)$ and $E(Y)$ in case $d = 1$ and the corresponding one-sided test ($\geq$) in case $d = 1/2$. Dependent and independent distributions for the mid-point and spread components of the RIs X and Y have been taken into account as well as equal and different type of distributions for X and Y. The distribution of the components of X and Y for each proposed test is provided in Tables 3 (case $d = 1$) and 4 (case $d = 1/2$).

Table 3: Distributions for the mid-point and spread of X and Y in 4 different situations of the two-sample test for the degree of overlap between $E(X)$ and $E(Y)$ in case $d = 1$ ($E(X) = E(Y) = [1,3]$)

Model	mid X	spr X	mid Y	spr Y
2.1.a)	$\mathcal{N}(2,1)$	χ_1^2	$\mathcal{N}(2,1)$	χ_1^2
2.1.b)	$U(1,3)$	χ_1^2	$\mathcal{N}(2,1)$	$U(0,2)$
2.1.c)	$\operatorname{spr} X + \mathcal{N}(1,1)$	χ_1^2	$\operatorname{spr} X + \mathcal{N}(1,1)$	χ_1^2
2.1.d)	$\operatorname{spr} X + U(0,2)$	χ_1^2	$\operatorname{spr} X + \mathcal{N}(1,1)$	$U(0,2)$

Table 4: Distributions for the mid-point and spread of X and Y in 8 different situations of the two-sample test for the degree of overlap between $E(X)$ and $E(Y)$ in case $d = 1/2$ ($E(Y) = [1,3]$ in all cases)

Model	mid X	spr X	mid Y	spr Y	$E(X)$
2.2.a)	$\mathcal{N}(3,1)$	χ_1^2	$\mathcal{N}(2,1)$	χ_1^2	$[2,4]$
2.2.b)	$U(1,5)$	χ_1^2	$\mathcal{N}(2,1)$	$U(0,2)$	$[2,4]$
2.2.c)	$\mathcal{N}(1,1)$	χ_1^2	$\mathcal{N}(2,1)$	χ_1^2	$[0,2]$
2.2.d)	$U(-2,4)$	χ_1^2	$\mathcal{N}(2,1)$	$U(0,2)$	$[0,2]$
2.2.e)	$\operatorname{spr} X + \mathcal{N}(2,1)$	$U(0,2)$	$\operatorname{spr} X + \mathcal{N}(1,1)$	$U(0,2)$	$[2,4]$
2.2.f)	$\operatorname{spr} X + U(0,4)$	χ_1^2	$\operatorname{spr} X + \mathcal{N}(1,1)$	$U(0,2)$	$[2,4]$
2.2.g)	$\mathcal{N}(2,1) - \operatorname{spr} X$	$U(0,2)$	$\operatorname{spr} X + \mathcal{N}(1,1)$	$U(0,2)$	$[0,2]$
2.2.h)	$U(0,4) - \operatorname{spr} X$	χ_1^2	$\operatorname{spr} X + \mathcal{N}(1,1)$	$U(0,2)$	$[2.5,3.5]$

Regarding the distributions included in Table 3, it should be remarked that models 2.1.a) and 2.1.c) provides equal distribution types for X and Y whereas models 2.1.b) and 2.1.d) present different distribution types for both RIs. On the other hand, the distributions of the components of models 2.1.a) and 2.1.b) are independently generated in contrast to the ones of models 2.1.c) and 2.1.d) which are dependently generated. Besides, the four models satisfy conditions of the part a) of Lemma 5.

With respect to the situations included in Table 4, models 2.2.a), 2.2.c), 2.2.e) and 2.2.g) show equal distribution types for X and Y while the others are determined by different distribution types for X and Y. In the same way, the distribution of the mid-point and spread in case of models from 2.2.a) to 2.2.d) has been independently generated whereas the rest of models present dependency in the generation of the distribution of both components. Finally, models 2.2.a), 2.2.b), 2.2.e) and 2.2.f) satisfy conditions of part b) of Lemma 5 and the other models satisfy conditions of part c). For the sake of readability of the manuscript, simulations considering other scenarios included in Lemma 5 have been ommited, although similar conclusions can be drawn in those cases.

In order to solve the tests gathered in Tables 3 and 4, 10000 simulations and 1000 bootstrap replications of the bootstrap approach proposed in Section 4 have been carried out at a significance level $\alpha = 0.05$ for equal and unequal sample sizes. The results obtained are displayed in Tables 5 and 6.

Table 5: Empirical size of the bootstrap test for the degree of overlap for models included in Table 3

(n,m)\ Model	**2.1.a)**	**2.1.b)**	**2.1.c)**	**2.1.d)**
$n = m = 10$	0.0597	0.0644	0.0737	0.077
$n = m = 30$	0.0578	0.054	0.0652	0.0604
$n = m = 50$	0.0538	0.0541	0.0533	0.0538
$n = m = 100$	0.0513	0.0520	0.0524	0.0522
$n = m = 200$	0.0504	0.0507	0.0512	0.0512
$n = 10$, $m = 30$	0.0546	0.0572	0.0682	0.0667
$n = 30$, $m = 50$	0.0626	0.0528	0.0504	0.0574
$n = 80$, $m = 100$	0.052	0.0534	0.0518	0.0546
$n = 100$, $m = 150$	0.0496	0.0524	0.0486	0.05
$n = 200$, $m = 300$	0.052	0.0492	0.0532	0.05

From the results shown in Table 5 it can be concluded that the corresponding empirical sizes of the test in the four cases are quite close to the nominal significance level whenever the sample sizes n and m are greater than 100, i.e., the behaviour of the test in this four situations is quite good for moderate/large sample sizes. However, for small sample sizes the empirical size of the test is around 0.06 when

Table 6: Empirical size of the bootstrap test for the degree of overlap for models included in Table 4

(n,m) \ Model	**2.2.a)**	**2.2.b)**	**2.2.c)**	**2.2.d)**
$n = m = 10$	0.0675	0.0604	0.0566	0.0586
$n = m = 30$	0.0505	0.0494	0.0529	0.0540
$n = m = 50$	0.0520	0.0512	0.0486	0.0489
$n = m = 100$	0.0484	0.0482	0.0497	0.0511
$n = m = 200$	0.0504	0.0494	0.0511	0.0490
$n = 10$, $m = 30$	0.0477	0.0571	0.0583	0.0586
$n = 30$, $m = 50$	0.0469	0.0509	0.0491	0.0489
$n = 80$, $m = 100$	0.0520	0.0489	0.0471	0.0484
$n = 100$, $m = 150$	0.0511	0.0487	0.0497	0.0496
$n = 200$, $m = 300$	0.0494	0.0497	0.0499	0.0505
(n,m) \ Model	**2.2.e)**	**2.2.f)**	**2.2.g)**	**2.2.h)**
$n = m = 10$	0.0608	0.0634	0.0604	0.0576
$n = m = 30$	0.0532	0.0518	0.0548	0.0494
$n = m = 50$	0.0516	0.0507	0.0472	0.0481
$n = m = 100$	0.0504	0.0488	0.0524	0.0489
$n = m = 200$	0.0493	0.0492	0.0487	0.505
$n = 10$, $m = 30$	0.0616	0.0632	0.0604	0.0568
$n = 30$, $m = 50$	0.05	0.0480	0.0484	0.0476
$n = 80$, $m = 100$	0.0496	0.0488	0.0525	0.0485
$n = 100$, $m = 150$	0.0485	0.0508	0.05	0.0492
$n = 200$, $m = 300$	0.0490	0.0504	0.0514	0.0497

the components have independent distributions and it is close to 0.07 when those distributions are dependent, which implies that the general behavior of the test is quite good.

In case $d = 1/2$ we get from Table 6 similar conclusions that in the one-sample case: the approximation of the empirical size of the test to the level 0.05 is relatively good in all the cases and it shows in general a conservative behavior. It should be also noticed that empirical size is around 0.06 for all the models when the corresponding sample sizes are small which implies again that the two-sample test behaves in a good way. Finally, there are no appreciable differences in the dependence or independence of the components as well as on the similarity or difference of the distributions of the two RIs. Again, the slight differences appreciated when comparing all the situations may be due to the nature of the distributions considered.

7. Real-life application

In this section, a real-life application is provided to show the applicability of the procedure presented in the manuscript. Thus, a questionnaire of some statements regarding students' mathematics related beliefs has been proposed to four groups of students attending the second course of the Degree in Primary Education of the University of Cantabria (Spain). Two of the questions of the questionnaire were the following ones:

M_1. "*I think it's interesting what I learn in math class.*"

M_2. "*I like to do math stuff.*"

Each one of the four groups, namely, G_1, G_2, G_3 and G_4, receives class from a different professor. The number of students of each group is 28, 31, 31 and 24, respectively, and the total sample size is 114. They are asked to reflect on the statements M_1 and M_2 by using intervals representing the set of values that the student considers compatible with his/her opinion at some extent (that is, the student considers that his/her opinion cannot be outside of this set).

Let us consider the following random intervals:

- $X \equiv$ answer of a student to the statement M_1.
- $Y \equiv$ answer of a student to the statement M_2.
- $X_k \equiv$ answer of a student of the Group k to the statement M_1, $k \in \{1, 2, 3, 4\}$.
- $Y_k \equiv$ answer of a student of the Group k to the statement M_2, $k \in \{1, 2, 3, 4\}$.

As a first study we are interested in analyzing if the mean of the responses to question M_1 has in common at least the 50% or at least the 75% of the information with the intervals $A = [5, 10]$ and $B = [6, 8]$, respectively. In addition, we will also analyze if the mean of the responses to question M_2 has in common at least the 50% or at least the 80% of the information with the intervals $C = [5, 6.5]$ and

$D = [0, 5]$, respectively. The corresponding sample means are $\overline{X} = [4.7807, 6.8553]$ and $\overline{Y} = [3.8553, 5.8333]$ i.e. we will consider the tests described in Table 7.

The bootstrap procedure developed in Section 3 has been applied with $B = 100000$ replications. The obtained p-values as well as the decision for each test are also shown in Table 7.

Table 7: Results corresponding to eight one-sample tests for the degree of overlap regarding questions M_1 and M_2

Test	H_0	H_1	p-value
Test 1.1	$O(E(X), A) \geq 0.5$	$O(E(X), A) < 0.5$	1
Test 1.2	$O(E(X), B) \geq 0.5$	$O(E(X), B) < 0.5$	0.2282
Test 1.3	$O(E(X), A) \geq 0.75$	$O(E(X), A) < 0.75$	0.9329
Test 1.4	$O(E(X), B) \geq 0.75$	$O(E(X), B) < 0.75$	$7 \cdot 10^{-4}$
Test 1.5	$O(E(Y), C) \geq 0.5$	$O(E(Y), A) < 0.5$	0.8321
Test 1.6	$O(E(Y), D) \geq 0.5$	$O(E(Y), B) < 0.5$	0.7404
Test 1.7	$O(E(Y), C) \geq 0.8$	$O(E(Y), A) < 0.8$	0.0841
Test 1.8	$O(E(Y), D) \geq 0.8$	$O(E(Y), B) < 0.8$	0.0554

Thus, taking into account the results in Table 7, only the null hypothesis in Test 1.4 is rejected at the usual significance levels, which implies that we can assure that the mean responses of the students to the question "I think it's interesting what I learn in math class" has in common with the interval $[6, 8]$ less than the 75% of the information. Besides, the null hypothesis of Tests 1.7 and 1.8 are also rejected at the level $\alpha = 0.1$ which implies that we can assure that the degree of overlap between the mean responses for question "I like to do math stuff" and the intervals $[5, 6.5]$ and $[0.5]$ is lower than 0.8 at the significance level $\alpha = 0.1$. Finally, the other null hypothesis are not rejected at the usual significance levels so neither H_0 nor H_1 can be assured in those cases.

Secondly, we are interested in compare the answers given to question M_1 by the students of different groups in order to check if their interest in Mathematics depends on the professor they have or not. Thus, we are going to check if the expected values of the responses given by the students from each pair of groups have in common at least the 90% of the information. To do that, the tests gathered in Table 8 are considered.

The sample means for question M_1 in each one of the groups are $\overline{X_1} = [5.6071, 7.9464]$, $\overline{X_2} = [4.3871, 6.3871]$, $\overline{X_3} = [4.0645, 6.0323]$ and $\overline{X_4} = [5.25, 7.25]$, respectively. In this case, the bootstrap procedure proposed in Section 4 has been

Table 8: Results corresponding to 6 two-sample tests for the degree of overlap in the case of question M_1

Test	H_0	H_1	p-value
Test 2.1	$O(E(X_1), E(X_2)) \geq 0.9$	$O(E(X_1), E(X_2)) < 0.9$	0.0619
Test 2.2	$O(E(X_1), E(X_3)) \geq 0.9$	$O(E(X_1), E(X_3)) < 0.9$	0.0094
Test 2.3	$O(E(X_1), E(X_4)) \geq 0.9$	$O(E(X_1), E(X_4)) < 0.9$	0.8447
Test 2.4	$O(E(X_2), E(X_3)) \geq 0.9$	$O(E(X_2), E(X_3)) < 0.9$	0.597
Test 2.5	$O(E(X_2), E(X_4)) \geq 0.9$	$O(E(X_2), E(X_4)) < 0.9$	0.3859
Test 2.6	$O(E(X_3), E(X_4)) \geq 0.9$	$O(E(X_3), E(X_4)) < 0.9$	0.1353

applied, again with $B = 100000$ replications, and the p-values are included in the last column of Table 8.

From the results in Table 8 we can derive that groups 1 and 3 are the ones differing the most since the corresponding p-value is lower than the usual significance levels. This means that the responses regarding the interest in what the students of group 1 learn in a math class has in mean less than a 90% of the information in common with respect to the responses of students of group 2 to the same statement. Besides, at a significance level $\alpha = 0.1$ we can also assume that groups 1 and 2 has in common less than the 90% of the information in mean. Paying attention to the values of the sample means we can conclude that the students of group 1 presents more interesest in Mathematics than the ones of groups 2 or 3.

Finally, we are going to mimick the last study above with respect to question M_2, to check if the Mathematics liking of the students can be considered to be similar in each pair of groups (again using and overlap degree of 0.9). The sample means of the responses to M_2 in each one of the groups are $\overline{Y_1} = [3.7321, 6.1071]$, $\overline{Y_2} = [3.7097, 5.5000]$, $\overline{Y_3} = [4, 5.7258]$ and $\overline{Y_4} = [4, 6.0833]$. Table 9 gathers the corresponding tests as well as the p-values obtained in each one on the situations.

In this case the Mathematics liking can be consider to be very similar in each one of the four groups since all the two-sample tests proposed are not rejected at the usual significance levels; what is more, all the p-values obtained are very high.

Therefore, by considering the studies developed in this section, it can be inferred the professor has not a lot of influence in students Mathematics liking whereas it seems they may influence their interest in Mathematics.

Table 9: Results corresponding to 6 two-sample tests for the degree of overlap in the case of question M_2

Test	H_0	H_1	p-value
Test 3.1	$O(E(Y_1), E(Y_2)) \geq 0.9$	$O(E(Y_1), E(Y_2)) < 0.9$	0.9703
Test 3.2	$O(E(Y_1), E(Y_3)) \geq 0.9$	$O(E(Y_1), E(Y_3)) < 0.9$	0.9997
Test 3.3	$O(E(Y_1), E(Y_4)) \geq 0.9$	$O(E(Y_1), E(Y_4)) < 0.9$	0.9038
Test 3.4	$O(E(Y_2), E(Y_3)) \geq 0.9$	$O(E(Y_2), E(Y_3)) < 0.9$	0.6640
Test 3.5	$O(E(Y_2), E(Y_4)) \geq 0.9$	$O(E(Y_2), E(Y_4)) < 0.9$	0.9205
Test 3.6	$O(E(Y_3), E(Y_4)) \geq 0.9$	$O(E(Y_3), E(Y_4)) < 0.9$	0.9772

8. Concluding remarks and open problems

Some hypothesis tests for analyzing the degree of overlap between the expected value of an RI an a fixed interval and between the expectations of two RIs have been proposed. For this purpose an index for measuring the degree of overlap between two intervals was introduced based on the Szymkiewicz-Simpson coefficient. Asymptotic and bootstrap approaches have been developed to approximate the distribution of the corresponding test statistics. Besides, some simulation studies were carried out and they have shown a good empirical behavior of the tests for moderate and even for small sample sizes in some situations.

To improve the results obtained the introduction of the sample variability in the test statistics may be studied as well as the estimation of the covariance operator involved in their limit distributions. The problem of defining a confidence interval for the degree of overlap can be also explored. In addition, the extension of the proposed approaches to the case of having sets in $\mathbb{R}^p$ or to the functional framework could be addressed.

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Appendix: proofs

Proof of Lemma 1

The statistic $T_X(1)$ can be equivalently expressed as:

$$\begin{aligned}T_X(1) = \sqrt{n}\max\Big\{ \min\Big\{ & \text{mid}\,\overline{X_n} - \text{mid}\,E(X) + \text{mid}\,E(X) - \text{mid}\,A \\ & +\text{spr}\,\overline{X_n} - \text{spr}\,E(X) + \text{spr}\,E(X) - \text{spr}\,A, \\ & \text{mid}\,\overline{X_n} - \text{mid}\,E(X) + \text{mid}\,E(X) - \text{mid}\,A \\ & -\text{spr}\,\overline{X_n} + \text{spr}\,E(X) - \text{spr}\,E(X) + \text{spr}\,A\Big\}, \\ \min\Big\{ & -\text{mid}\,\overline{X_n} + \text{mid}\,E(X) - \text{mid}\,E(X) + \text{mid}\,A \\ & +\text{spr}\,\overline{X_n} - \text{spr}\,E(X) + \text{spr}\,E(X) - \text{spr}\,A, \\ & -\text{mid}\,\overline{X_n} + \text{mid}\,E(X) - \text{mid}\,E(X) + \text{mid}\,A \\ & -\text{spr}\,\overline{X_n} + \text{spr}\,E(X) - \text{spr}\,E(X) + \text{spr}\,A\Big\}\Big\}.\end{aligned}$$

The terms of the first minimum can be expressed as

$$\sqrt{n}\Big(\text{mid}\,\overline{X_n} - \text{mid}\,E(X) + \text{spr}\,\overline{X_n} - \text{spr}\,E(X)\Big) + \sqrt{n}O_1(1), \quad (44)$$

$$\sqrt{n}\Big(\text{mid}\,\overline{X_n} - \text{mid}\,E(X) - \text{spr}\,\overline{X_n} - \text{spr}\,E(X)\Big) + \sqrt{n}O_2(1), \quad (45)$$

whereas the terms of the second minimum are equal to

$$\sqrt{n}\Big(-\text{mid}\,\overline{X_n} + \text{mid}\,E(X) + \text{spr}\,\overline{X_n} - \text{spr}\,E(X)\Big) + \sqrt{n}O_3(1), \quad (46)$$

$$\sqrt{n}\Big(-\text{mid}\,\overline{X_n} + \text{mid}\,E(X) - \text{spr}\,\overline{X_n} - \text{spr}\,E(X)\Big) + \sqrt{n}O_4(1). \quad (47)$$

As $O_1(1) = O_2(1) = O_3(1) = O_4(1) = 0$, it is possible to apply the continuous mapping and the central limit theorems for real variables to assure that the asymptotic convergence of $T_X(1)$ in this situation is equal to the one in (11).

□

Proof of Lemma 3

The statistic $T_X(d)$ can be equivalently written as:

$$\begin{aligned}T_X(d) = \sqrt{n}\max\Big\{ \min\Big\{ & \text{mid}\,\overline{X_n} - \text{mid}\,E(X) + \text{mid}\,E(X) - \text{mid}\,A \\ & +(2d-1)(\text{spr}\,\overline{X_n} - \text{spr}\,E(X)) + (2d-1)\text{spr}\,E(X) - \text{spr}\,A, \\ & \text{mid}\,\overline{X_n} - \text{mid}\,E(X) + \text{mid}\,E(X) - \text{mid}\,A \\ & -\text{spr}\,\overline{X_n} + \text{spr}\,E(X) - \text{spr}\,E(X) + (2d-1)\text{spr}\,A\Big\}, \\ \min\Big\{ & -\text{mid}\,\overline{X_n} + \text{mid}\,E(X) - \text{mid}\,E(X) + \text{mid}\,A \\ & +(2d-1)(\text{spr}\,\overline{X_n} - \text{spr}\,E(X)) + (2d-1)\text{spr}\,E(X) - \text{spr}\,A, \\ & -\text{mid}\,\overline{X_n} + \text{mid}\,E(X) - \text{mid}\,E(X) + \text{mid}\,A \\ & -\text{spr}\,\overline{X_n} + \text{spr}\,E(X) - \text{spr}\,E(X) + (2d-1)\text{spr}\,A\Big\}\Big\}.\end{aligned}$$

a) The first term of the second minimum is equal to

$$\sqrt{n}\Big(- \operatorname{mid} \overline{X_n} + \operatorname{mid} E(X) + (2d-1)(\operatorname{spr} \overline{X_n} - \operatorname{spr} E(X))\Big) \\ -\sqrt{n}O_1(d) + 2\sqrt{n}((2d-1)\operatorname{spr} E(X) - \operatorname{spr} A). \tag{48}$$

As $\operatorname{mid} E(X) = \operatorname{mid} A + 2(1-d)\operatorname{spr} A$ and $\operatorname{spr} E(X) = \operatorname{spr} A$, then

$$2\sqrt{n}((2d-1)\operatorname{spr} E(X) - \operatorname{spr} A) = 4\sqrt{n}(d-1)\operatorname{spr} E(X) < 0$$

and diverges in probability to $-\infty$ as $n \to \infty$. Therefore, by the central limit and the Slutsky's theorems, the term (48) diverges in probability to $-\infty$ as $n \to \infty$. For the same reason, the second term of the second minimum can be written as

$$\sqrt{n}\Big(- \operatorname{mid} \overline{X_n} + \operatorname{mid} E(X) - (\operatorname{spr} \overline{X_n} - \operatorname{spr} E(X))\Big) \\ -\sqrt{n}O_2(d) - 2\sqrt{n}(\operatorname{spr} E(X) - (2d-1)\operatorname{spr} A). \tag{49}$$

Again, the last part can be expressed as

$$-2\sqrt{n}(\operatorname{spr} E(X) - (2d-1)\operatorname{spr} A) = -4\sqrt{n}(1-d)\operatorname{spr} E(X) < 0$$

and diverges in probability to $-\infty$ as $n \to \infty$. Thus, by the central limit and the Slutsky's theorems, also (49) diverges in probability to $-\infty$ as $n \to \infty$.

Finally, the terms of the first minimum can be written as

$$\sqrt{n}\Big(\operatorname{mid} \overline{X_n} - \operatorname{mid} E(X) + (2d-1)(\operatorname{spr} \overline{X_n} - \operatorname{spr} E(X))\Big) + \sqrt{n}O_1(d), \tag{50}$$

$$\sqrt{n}\Big(\operatorname{mid} \overline{X_n} - \operatorname{mid} E(X) - (\operatorname{spr} \overline{X_n} - \operatorname{spr} E(X))\Big) + \sqrt{n}O_2(d), \tag{51}$$

therefore, by the continuous mapping and the central limit theorems for real variables, the asymptotic convergence of $T_X(d)$ as $n \to \infty$ is the one in (14).

b) The proof of this part is analogous to the one of the part *a*). Following similar reasonings, as $\operatorname{mid} E(X) = \operatorname{mid} A + 2(d-1)\operatorname{spr} A$ and $\operatorname{spr} E(X) = \operatorname{spr} A$, then the first and the second terms of the first minimum (multiplied by $\sqrt{n}$) in T_X diverge in probability to $-\infty$ as $n \to \infty$. Additionally, the terms of the second minimum can be expressed as it was pointed out in (46) and (47).

Again, by considering the continuous mapping and the central limit theorems for real variables, the asymptotic convergence of $T_X(d)$ as $n \to \infty$ is in this case the one in (15).

c) In this case it is easy to show that both terms of the second minimum (multiplied by $\sqrt{n}$) diverge in probability to $-\infty$ whereas the second component of the first minimum (multiplied by $\sqrt{n}$) diverges to ∞, as $n \to \infty$. In addition, the first component of the first minimum is equal to the one in (44), and converges in law to $z_1 + (2d-1)z_2$ by the CLT.

d) This case is analogous to the one in *c*) taking into account that the remaining term is the one in (45), which converges in law to $z_1 - z_2$ by the CLT.

e) In this it is easy to show that both components of the first minimum (multiplied by $\sqrt{n}$) diverge in probability to $-\infty$ whereas the second term of the second minimum (multiplied by $\sqrt{n}$) diverges to ∞, as $n \to \infty$. Moreover, the first term of the second minimum is equal to the one in (46), and converges in law to $-z_1 + (2d-1)z_2$ by the CLT.

f) This case is analogous to the one in *e*) taking into account that the remaining component is the one in (47), which converges in law to $-z_1 - z_2$ by the CLT.

g) Finally, in this case three terms of the expression diverge to $-\infty$ as $n \to \infty$, so the whole expression of $T_X(d)$ does.

□

Proof of Theorem 2

i) Consider the asymptotic testing procedure in case *a*) of Lemma 5. The corresponding cases *b*), *d*) and *e*) can be solved analogously. The test statistic in this case can be written as

$$
\begin{aligned}
T_{(X,Y')}(1) = \sqrt{n}\max\Big\{ \min\Big\{ & \operatorname{mid}\overline{X_n} - \operatorname{mid}E(X) - \operatorname{mid}\overline{Y'_m} + \operatorname{mid}E(Y') \\
& +\operatorname{spr}\overline{X_n} - \operatorname{spr}E(X) - \operatorname{spr}\overline{Y'_m} + \operatorname{spr}E(Y') \\
& +\operatorname{mid}E(X) - \operatorname{mid}E(Y') + \operatorname{spr}E(X) - \operatorname{spr}E(Y'), \\
& \operatorname{mid}\overline{X_n} - \operatorname{mid}E(X) - \operatorname{mid}\overline{Y'_m} + \operatorname{mid}E(Y') \\
& -\operatorname{spr}\overline{X_n} + \operatorname{spr}E(X) + \operatorname{spr}\overline{Y'_m} - \operatorname{spr}E(Y') \\
& +\operatorname{mid}E(X) - \operatorname{mid}E(Y') - \operatorname{spr}E(X) + \operatorname{spr}E(Y')\Big\}, \\
\min\Big\{ & -\operatorname{mid}\overline{X_n} + \operatorname{mid}E(X) - \operatorname{mid}\overline{Y'_m} + \operatorname{mid}E(Y') \\
& +\operatorname{spr}\overline{X_n} - \operatorname{spr}E(X) - \operatorname{spr}\overline{Y'_m} + \operatorname{spr}E(Y') \\
& -\operatorname{mid}E(X) + \operatorname{mid}E(Y') + \operatorname{spr}E(X) - \operatorname{spr}E(Y'), \\
& -\operatorname{mid}\overline{X_n} + \operatorname{mid}E(X) - \operatorname{mid}\overline{Y'_m} + \operatorname{mid}E(Y') \\
& -\operatorname{spr}\overline{X_n} + \operatorname{spr}E(X) + \operatorname{spr}\overline{Y'_m} - \operatorname{spr}E(Y') \\
& -\operatorname{mid}E(X) + \operatorname{mid}E(Y') - \operatorname{spr}E(X) + \operatorname{spr}E(Y')\Big\}\Big\}.
\end{aligned}
$$

The first term of the first minimum is equal to

$$\sqrt{n}\Big(\text{mid}\,\overline{X_n} - \text{mid}\,E(X) - \text{mid}\,\overline{Y'_m} + \text{mid}\,E(Y')$$
$$+\text{spr}\,\overline{X_n} - \text{spr}\,E(X) - \text{spr}\,\overline{Y'_m} + \text{spr}\,E(Y)\Big) + \sqrt{n}\Big(\Theta_1(1)\Big) - \delta,$$

where $\Theta_1(1)$ is the one defined in (30). As in the case $a)$ of Lemma 5 it is fulfilled that $\Theta_1(1)$, by the CLT and the Slutsky's theorem the term above diverges in probability to $-\infty$. In the same way, the second term of the first minimum also diverges in probability to $-\infty$.

On the other hand, the first term of the second minimum is equal to

$$\sqrt{n}\Big(- \text{mid}\,\overline{X_n} + \text{mid}\,E(X) + \text{mid}\,\overline{Y'_m} - \text{mid}\,E(Y')$$
$$+\text{spr}\,\overline{X_n} - \text{spr}\,E(X) - \text{spr}\,\overline{Y'_m} + \text{spr}\,E(Y')\Big) + \sqrt{n}\Big(\Theta_3(d)\Big) + \delta,$$

where $\Theta_3(1)$ is the one defined in (30). As $\Theta_3(1) = 0$, again by the CLT and the Slutsky's theorem it can be concluded that the term above diverges in probability to $+\infty$. Finally, as the minimum and the maximum are continuous functions, the statistic $T_{(X,Y')}(1)$ diverges in probability to ∞ and, therefore,

$$\lim_{n\to\infty} P(\,T_{(X,Y')} > t_{1-\alpha}\,) = 1.$$

ii) Consider the asymptotic testing procedure in case $b)$ of Lemma 5. The corresponding cases $c)$, $f)$ and $g)$ can be solved analogusly. The test statistic $T_{(X,Y'')}(d)$ in this case can be written as follows:

$$\begin{aligned}
\sqrt{n}\max\Big\{ \min\Big\{ & \text{mid}\,\overline{X_n} - \text{mid}\,E(X) - \text{mid}\,\overline{Y'_m} + \text{mid}\,E(Y') \\
& +(2d-1)(\text{spr}\,\overline{X_n} - \text{spr}\,E(X)) - \text{spr}\,\overline{Y'_m} + \text{spr}\,E(Y') \\
& +\text{mid}\,E(X) - \text{mid}\,E(Y') + (2d-1)\text{spr}\,E(X) - \text{spr}\,E(Y'), \\
& \text{mid}\,\overline{X_n} - \text{mid}\,E(X) - \text{mid}\,\overline{Y'_m} + \text{mid}\,E(Y') \\
& -(2d-1)(\text{spr}\,\overline{X_n} - \text{spr}\,E(X)) + \text{spr}\,\overline{Y'_m} - \text{spr}\,E(Y') \\
& +\text{mid}\,E(X) - \text{mid}\,E(Y') - (2d-1)\text{spr}\,E(X) + \text{spr}\,E(Y')\Big\}, \\
\min\Big\{ & - \text{mid}\,\overline{X_n} + \text{mid}\,E(X) - \text{mid}\,\overline{Y'_m} + \text{mid}\,E(Y') \\
& +(2d-1)(\text{spr}\,\overline{X_n} - \text{spr}\,E(X)) - \text{spr}\,\overline{Y'_m} + \text{spr}\,E(Y') \\
& -\text{mid}\,E(X) + \text{mid}\,E(Y') + (2d-1)\text{spr}\,E(X) - \text{spr}\,E(Y'), \\
& -\text{mid}\,\overline{X_n} + \text{mid}\,E(X) - \text{mid}\,\overline{Y'_m} + \text{mid}\,E(Y') \\
& -(2d-1)(\text{spr}\,\overline{X_n} - \text{spr}\,E(X)) + \text{spr}\,\overline{Y'_m} - \text{spr}\,E(Y') \\
& -\text{mid}\,E(X) + \text{mid}\,E(Y') - (2d-1)\text{spr}\,E(X) + \text{spr}\,E(Y')\Big\}\Big\}.
\end{aligned}$$

Using a similar reasoning that in part i), as in case b) of Lemma 5 it is fulfilled that $\Theta_1(d) = \Theta_2(d) = 0$, and then the two components of the first minimum diverge in probability to ∞. In addition, the first term of the second minimum can be written as

$$\sqrt{n}\Big(- \operatorname{mid} \overline{X_n} + \operatorname{mid} E(X) + \operatorname{mid} \overline{Y'_m} - \operatorname{mid} E(Y') \\ + \operatorname{spr} \overline{X_n} - \operatorname{spr} E(X) - \operatorname{spr} \overline{Y'_m} + \operatorname{spr} E(Y')\Big) + \sqrt{n}\Big(\Theta_3(d)\Big) + \delta.$$

As $\Theta_2(d) = 0$, it is satisfied that $\Theta_3(d) = 2(d-1)(\operatorname{spr} E(X) + \operatorname{spr} E(Y)) < 0$, and the term above diverges in probability to $-\infty$. In the same way, the nullity of $\Theta_1(d)$ implies that $\Theta_4(d)$ is also lower than 0 and it can be proven that the second term of the second minimum also diverges in probability to $-\infty$. Therefore, due to the continuity of the minimum and maximum functions, $T(X, Y'')(d)$ diverges in probability to ∞, and

$$\lim_{n\to\infty} P(\, T_{(X,Y'')} > t_{1-\alpha}\,) = 1.$$

□

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