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On a class of nonlocal evolution equations with the $p[u(x,t)]$-Laplace operator

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ABSTRACT

We study the homogeneous Dirichlet problem for a class of nonlocal singular parabolic equations

$$u_t - \operatorname{div}\left(|\nabla u|^{p[u]-2}\nabla u\right) = f \quad \text{in } \Omega\times(0,T),$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a smooth bounded domain, $p[u] = p(l(u))$ is a given function with values in the interval $[p^-, p^+] \subset (1,2)$, and $l(u) = \displaystyle\int_\Omega |u(x,t)|^\alpha\,dx$, $\alpha \in [1,2]$, is a functional of the unknown solution. We find sufficient conditions for global or local in time solvability of the problem, prove the uniqueness, and show that every solution gets extinct in a finite time.

1. Introduction

In the present work, we study the homogeneous Dirichlet problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p[u]-2}\nabla u\right) = f \quad \text{in } Q_T,\\ u = 0 \text{ on } \partial\Omega\times(0,T),\\ u(x,0) = u_0 \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $Q_T = \Omega\times(0,T]$, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain with the boundary $\partial\Omega$. The exponent of nonlinearity $p(s)$ is a given function, $p : \mathbb{R} \mapsto [p^-, p^+] \subset (1,2)$, $p^\pm = const$. In Eq. (1.1), the argument of

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$p(\cdot)$ is the functional

$$l(v) = \int_{\Omega} |v(x,t)|^{\alpha}\, dx : L^{\infty}(0,T; L^{\alpha}(\Omega)) \mapsto \mathbb{R}, \qquad \alpha \in [1,2]. \tag{1.2}$$

To indicate the nonlocal dependence of the exponent p on the function $u(x,t)$ we use the notation $p[u] \equiv p(l(u))$. We prove that problem (1.1) has a unique strong solution, global or local in time, and show that every strong solution vanishes in finite time.

The special form of the functional $l(u)$ in (1.2) is chosen for convenience of presentation, the results easily extend to a wider class of functionals. For example, all the results remain true if $l(u)$ is a functional over $L^{\alpha}(\Omega)$,

$$l(u) = \int_{\Omega} g(x)u(x,t)\, dx, \qquad \text{with some } g \in L^{\alpha'}(\Omega),\ \alpha \in [1,2].$$

1.1. Motivation and previous work

The mathematical modelling of many real-life processes leads to systems of nonlinear equations whose structure may depend on certain components of the sought solution. For example, the stationary thermo-convective flow of a non-Newtonian fluid is described by the following system for the velocity $v(x)$, the pressure $p(x)$ and the temperature $\theta(x)$, [1],

$$\begin{cases} (v\cdot\nabla)b(\theta) - \Delta\theta = g, \qquad \operatorname{div} v = 0 \quad \text{in } \Omega, \\ (v\cdot\nabla)v - \operatorname{div}(\mu(\theta) + \tau(\theta)|S(v)|^{q(\theta)-2}S(v)) + \nabla p = f, \end{cases}$$

endowed with the boundary conditions for v, p and θ. Here $\Omega \subset \mathbb{R}^d$ is a bounded domain, g, f are given functions, b, μ, τ and q are given functions of θ, and $S(v)$ is the deformation rate tensor. Another example is the model of the thermistor, [2,3],

$$-\operatorname{div}(|\nabla u|^{\sigma(\theta)-2}\nabla u) = g, \qquad -\Delta\theta = \lambda|\nabla u|^{\sigma(\theta)}, \tag{1.3}$$

where u is the electric potential and θ in the temperature in a conductor in the presence of Joule heating, or the models of electro-rheological fluids in which the character of nonlinearity in the governing Navier–Stokes equations varies according to the applied electromagnetic field [4].

Formally, each of these models can be regarded as a nonlinear equation, or a system of equations, whose nonlinearity depends on the sought solution. For instance, if the equation for θ in (1.3) has a solution for every given u, this dependence defines the nonlocal operator, $\theta \equiv \theta(u)$, and the first equation reads

$$-\operatorname{div}(|\nabla u|^{\sigma[u]-2}\nabla u) = g, \quad x \in \Omega, \tag{1.4}$$

where the exponent has the form $\sigma[u] \equiv \sigma(\theta(u))$.

Functionals with the growth condition depending on the solution or its gradient are successfully used for denoising of digital images — see, e.g., [5–7] for the models based on minimization of functionals with $p(|\nabla u|)$-growth and [8] for a discussion of the model of denoising of the image f based on the minimization of the functional

$$\lambda\|f-u\|^2_{L^2(\Omega)} + \int_{\Omega}\left(\alpha_1(u)|\nabla u\cdot\xi(u)|^{p_1(u)} + \alpha_2(u)|\nabla u\cdot\xi^{\perp}(u)|^{p_2(u)}\right) dx,$$

where $p_1(u), p_2(u) \in [1,2]$, $(\xi(u), \xi^{\perp}(u))$ is an orthonormal coordinate system such that $\xi(u)$ is approximately parallel to ∇u, wherever $\nabla u \neq 0$.

To the best of our knowledge, the equations involving the $p[u]$-Laplace operators were studied thus far only in papers [9,10]. Both papers address elliptic equations of the structure (1.4) and consider the cases of local and nonlocal dependence of σ on u, but their approach to the problem is different.

Paper [9] deals with the equation

$$b(u) - \operatorname{div} a(x,u,\nabla u) = f. \tag{1.5}$$

In this equation $b : \mathbb{R} \mapsto \mathbb{R}$ is a nondecreasing function, $b(0) = 0$, and $a(x,z,\xi)$ is a strictly monotone operator of Leray–Lions type which satisfies the growth and coercivity conditions with the variable exponent $p(x,u)$ such that $p(x,u) \in [p^-,p^+] \subset (1,\infty)$, $p^\pm = const$. This class of equations includes (1.4) as a partial case. A challenging feature of the equations that involve $p[u]$-Laplacian is that they cannot be interpreted as a duality relation in a fixed Banach space. For this reason, the authors of [9] reduce the study to the L^1-setting and use the Young measures to obtain a solution of the degenerate equation of the type (1.5) as the limit of a sequence $\{u_n\}$ of solutions of the regularized equations with $p_n(x)$ and p^+-Laplacian operators. The authors of [10] take another direction and overcome this difficulty adapting the idea of [11] about passing to the limit in a sequence $\left\{|\nabla v_k(x)|^{q_k(x)}\right\}$. The results of [9] and [10] are obtained under the assumption $p^- = \min p > d$, which yields compactness of the sequence of solutions of the regularized problems in a space of Hölder-continuous functions. Besides equations (1.4), (1.5) with the exponent $p[u] = (p \circ u)(x)$ defined as a composite function on Ω, the authors of [9,10] consider the case of nonlocal dependence of p on the solution u and discuss the question of uniqueness.

The nonlocal evolution equations are widely used in modelling of various processes in physics and biology and are intensively studied, see, e.g., [12–16] and references therein. Eq. (1.1) with $\alpha = 1$ can be regarded as the diffusion equation for the concentration $u(x,t)$, with the diffusion flux $|\nabla u|^{p[u]-2}\nabla u$ which depends on the total mass $m(t) = \int_\Omega u(x,t)\,dx$ at the instant t, or the inverse of the specific volume $m(t)/|\Omega|$, $|\Omega| = \operatorname{meas}\Omega$.

2. Assumption and results

2.1. The function spaces

For convenience of the reader, we collect here the basic facts on the Lebesgue and Sobolev with variable exponents. For a detailed presentation of the theory of these spaces we refer to the monograph [17], see also [18, Ch. 1].

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with the Lipschitz-continuous boundary $\partial\Omega$. Given a measurable function $p(x) : \Omega \mapsto [p^-,p^+] \subset (1,\infty)$, $p^\pm = const$, the set

$$L^{p(\cdot)}(\Omega) = \left\{ f : \Omega \mapsto \mathbb{R} : f \text{ is measurable on } \Omega,\ \int_\Omega |f|^{p(x)}\,dx < \infty \right\}$$

equipped with the Luxemburg norm

$$\|f\|_{p(\cdot),\Omega} := \inf\left\{ \alpha > 0 : \int_\Omega \left|\frac{f}{\alpha}\right|^{p(x)} dx \le 1 \right\}$$

becomes a Banach space. The relation between the modular $\int_\Omega |f|^{p(x)}\,dx$ and the norm follows from the definition:

$$\min\left(\|f\|^{p^-}_{p(\cdot)}, \|f\|^{p^+}_{p(\cdot)}\right) \le \int_\Omega |f|^{p(x)}\,dx \le \max\left(\|f\|^{p^-}_{p(\cdot)}, \|f\|^{p^+}_{p(\cdot)}\right). \tag{2.1}$$

In case of $p(\cdot) = const > 1$ these inequalities transform into equalities. For all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with

$$p(x) \in (1,\infty), \quad p'(x) = \frac{p(x)}{p(x)-1}$$

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the generalized Hölder inequality holds:

$$\int_{\Omega} |f\, g|\, dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \leq 2 \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}. \tag{2.2}$$

If $p(x)$ is measurable and $1 < p^- \leq p(x) \leq p^+ < \infty$ in Ω, then $L^{p(\cdot)}(\Omega)$ is a reflexive and separable Banach space, and $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.

Let $p_1(x), p_2(x)$ be measurable on Ω functions such that $p_i(x) \in [p_i^-, p_i^+] \subset (1, \infty)$ a.e. in Ω. If $p_1(x) \geq p_2(x)$ a.e. in Ω, then the inclusion $L^{p_1(\cdot)}(\Omega) \subset L^{p_2(\cdot)}(\Omega)$ is continuous and

$$\|u\|_{p_2(\cdot),\Omega} \leq C\|u\|_{p_1(\cdot),\Omega} \qquad \forall\, u \in L^{p_1(\cdot)}(\Omega) \tag{2.3}$$

with a constant $C = C(|\Omega|, p_1^{\pm}, p_2^{\pm})$.

The variable Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the collection of functions

$$W_0^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega) : |\nabla u|^{p(x)} \in L^1(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot),\Omega} + \|u\|_{p(\cdot),\Omega}. \tag{2.4}$$

By $C_{\log}(\overline{\Omega})$ we denote the set of functions continuous on $\overline{\Omega}$ with the logarithmic modulus of continuity:

$$|p(x_2) - p(x_1)| \leq \omega(|x_2 - x_1|) \tag{2.5}$$

where $\omega \geq 0$ satisfies the condition

$$\overline{\lim_{\tau \to 0^+}} \omega(\tau) \ln \frac{1}{\tau} = C < \infty, \qquad C = const.$$

It is known that for $p(x) \in C_{\log}(\overline{\Omega})$ the set $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and the space $W_0^{1,p(\cdot)}(\Omega)$ coincides with the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.4).

We will use the notation $p(z) \in C_{\log}(\overline{Q}_T)$ for the functions p of the argument $z = (x,t)$ which are continuous in the closure of the cylinder $Q_T = \Omega \times (0,T)$ with the logarithmic modulus of continuity, that is, satisfy condition (2.5) in the cylinder Q_T with x_i substituted by z_i.

For the elements of $W_0^{1,p(\cdot)}(\Omega)$ with $p(x) \in C^0(\overline{\Omega})$ the Poincaré inequality holds:

$$\|u\|_{p(\cdot),\Omega} \leq C(d,\Omega)\|\nabla u\|_{p(\cdot),\Omega}. \tag{2.6}$$

An immediate consequence of the Poincaré inequality is that an equivalent norm of $W_0^{1,p(\cdot)}(\Omega)$ can be defined by

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot),\Omega}.$$

Let $p(x), q(x) \in C^0(\overline{\Omega})$, $1 < p^- \leq p(x) \leq p^+ < \infty$, $d \geq 2$. If $q(x) < \frac{dp(x)}{d-p(x)}$ in $\overline{\Omega}$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$ is continuous, compact, and

$$\|v\|_{q(\cdot),\Omega} \leq C\|\nabla v\|_{p(\cdot),\Omega} \qquad \forall v \in W_0^{1,p(\cdot)}(\Omega).$$

According to (2.3) $W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p^-}(\Omega)$. If $p^- > \frac{2d}{d+2}$, then the embedding $W_0^{1,p^-}(\Omega) \subset L^2(\Omega)$ is compact.

Let us introduce the spaces of functions defined on the cylinder Q_T

$$\begin{gathered}\mathbf{V}_t(\Omega) = \{u : \Omega \mapsto \mathbb{R} | u \in L^2(\Omega) \cap W_0^{1,1}(\Omega),\, |\nabla u|^{p(x,t)} \in L^1(\Omega)\}, \quad t \in (0,T),\\ \mathbf{W}(Q_T) = \{u : (0,T) \mapsto \mathbf{V}_t(\Omega)|\, u \in L^2(Q_T),\, |\nabla u|^{p(x,t)} \in L^1(Q_T)\}\end{gathered}$$

with the norm

$$\|v\|_{\mathbf{W}(Q_T)} = \|v\|_{2,Q_T} + \|\nabla u\|_{p(\cdot),Q_T}.$$

Given a measurable in Q_T function u, a function p, and a functional $l(\cdot)$ defined by (1.2), we define the set

$$\mathbf{W}_u(Q_T) = \{v \in L^2(Q_T) : |\nabla v|^{p[u]} \in L^1(Q_T),\ v = 0 \text{ on } \partial\Omega \times (0,T) \text{ in the sense of traces}\}.$$

If we denote $\widetilde{p}(x,t) = p[u(x,t)]$, then $\mathbf{W}_u(Q_T)$ coincides with the space $\mathbf{W}(Q_T)$ with the given variable exponent $\widetilde{p}(x,t)$. The inclusion $u \in \mathbf{W}_u(Q_T)$ means that $u \in L^2(Q_T)$, $|\nabla u|^{\widetilde{p(x,t)}} \in L^1(Q_T)$ and $u = 0$ on $\partial\Omega \times (0,T)$. The norm of $\mathbf{W}_u(Q_T)$ is defined as the norm of $\mathbf{W}(Q_T)$ with the exponent $\widetilde{p}(x,t) = p[u(x,t)]$.

Notation. Throughout the text we use the notation

$$|v_{xx}|^q = \sum_{i,j=1}^{d} |D^2_{x_i x_j} v|^q$$

where the exponent q may depend on t. By C we denote the constants which can be computed or estimated through the data of the problem, but whose precise values are unimportant. The value of C may differ from line to line even in the same formula.

2.2. The main result and organization of the paper

Definition 2.1. A function u is called **strong solution** of problem (1.1) if

1. $u \in C^0([0,T]; L^2(\Omega))$, $|\nabla u|^{p[u]} \in L^\infty(0,T; L^1(\Omega))$, $u_t \in L^2(Q_T)$;
2. $\|u(\cdot,t) - u_0\|_{2,\Omega} \to 0$ as $t \to 0+$;
3. for every test-function $\phi \in L^2(Q_T)$ with $|\nabla \phi|^{p[u]} \in L^1(Q_T)$

$$\int_{Q_T} \left(u_t \phi + |\nabla u|^{p[u]-2} \nabla u \cdot \nabla \phi \right) dz = \int_{Q_T} f \phi \, dz. \tag{2.7}$$

The main result of this work is given in the following theorem.

Theorem 2.1. *Let $p : \mathbb{R} \mapsto [p^-, p^+]$, $p^\pm = const$, be a given function, $l(v)$ be the functional defined by (1.2), and $p[v] = p(l(v))$. Assume that*

$$\begin{aligned}
&\text{(a)} \quad \Omega \text{ is a bounded domain with the boundary } \partial\Omega \in C^2,\\
&\text{(b)} \quad u_0 \in W^{1,2}_0(\Omega), \quad f \in L^{(p^-)'}(Q_T),\\
&\text{(c)} \quad p(s) \text{ is differentiable in } \mathbb{R}, \quad \sup_{s\in\mathbb{R}} |p'(s)| \le C_*, \qquad C_* = const.,\\
&\text{(d)} \quad \frac{2d}{d+2} < p^- \le p^+ < 2.
\end{aligned} \tag{2.8}$$

Then problem (1.1) has a strong solution in the sense of Definition 2.1 and the following estimate holds:

$$\sup_{(0,T)} \|\nabla u(\cdot,t)\|^2_{2,\Omega} + \|u_t\|^2_{2,Q_T} + \int_{Q_T} |u_{xx}|^{p[u]} \, dz \le C \left(1 + \|\nabla u_0\|^2_{2,\Omega} + \int_{Q_T} |f|^{(p^-)'} \right) dz. \tag{2.9}$$

The paper is organized as follows. In Section 3 we consider the regularized non-singular problem (3.1). The solution of this problem is obtained as the limit of the sequence of Galerkin's approximations in the basis composed of the eigenfunctions of the Laplace operator. This section is almost entirely devoted to obtaining uniform a priori estimates for the approximate solutions.

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In Section 4 we justify first the passage to the limit in the sequence of Galerkin's approximations and obtain a solution of the regularized problem. We make use of monotonicity of the function $\gamma_\epsilon(q,\xi)\xi = (\epsilon^2+|\xi|^2)^{\frac{q-2}{2}}\xi$ in ξ with a fixed q, continuity of $\gamma_\epsilon(q,\xi)\xi$ with respect to q with a fixed ξ, and the fact that in the singular case, $p^+ < 2$, the solutions u_ϵ of the regularized problems and their approximations possess extra regularity: $\|\nabla u_\epsilon(t)\|_{2,\Omega}$ are uniformly bounded for all $t \in (0,T)$. It is worth mentioning here that if $p(x,t) \in C_{\log}(\overline{Q}_T)$ is a given function, then for every $u_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$ problem (1.1) admits a weak solution $u \in C([0,T]; L^2(\Omega)) \cap \mathbf{W}(Q_T)$, see, e.g. [18, Ch.4]. The time derivative of the weak solution is a distribution which may not belong to any Lebesgue space $L^s(Q_T)$ with $s > 1$. In case of Eq. (1.1) with $p = p[u]$, such a regularity is insufficient for the convergence of the sequence of the exponents p corresponding to the approximate solutions. To overcome this difficulty, we construct strong solutions with $u_t \in L^2(Q_T)$.

To pass to the limit as $\epsilon \to 0$ in the sequence $\{u_\epsilon\}$ of solutions to (3.1) we use the a priori estimates of Section 4, which remain true for the solutions of the regularized problem (3.1). The procedure of passing to the limit in ϵ requires an additional step because now the exponent $p_\epsilon = p[u_\epsilon]$ also depends on ϵ.

Uniqueness Theorem 5.1 is proven in Section 5. We show that the strong solution is unique in the class of functions the solution constructed in Theorem 2.1 belongs to.

In Section 6 we prove the local in time existence of a strong solution if condition (2.8) (d) on the range of $p[v]$ is omitted and substituted by the claim for the initial function $p[u_0] \in \left(\frac{2d}{d+2}, 2\right)$. It is shown next that every strong solution of problem (1.1) vanishes in finite time: if $f \equiv 0$ for all $t \geq t_f$, then there exists $t^* \geq t_f$ such that $\|u(t)\|_{2,\Omega} = 0$ for all $t \geq t^*$.

3. Regularized problem

We will obtain a solution of the singular problem (1.1) as the limit when $\epsilon \to 0$ of the family of solutions of the regularized problems

$$\begin{cases} u_{\epsilon t} = \operatorname{div}\left((\epsilon^2+|\nabla u_\epsilon|^2)^{\frac{p[u_\epsilon]-2}{2}}\nabla u_\epsilon\right) + f(z) \quad \text{in } Q_T, \\ u_\epsilon = 0 \text{ on } \partial\Omega \times (0,T), \\ u_\epsilon(x,0) = u_0(x) \text{ in } \Omega, \qquad \epsilon > 0. \end{cases} \tag{3.1}$$

3.1. Galerkin's approximations

The solution of problem (3.1) is understood in the sense of Definition 2.1. It is constructed as the limit of the sequence of finite-dimensional approximations

$$u_\epsilon \equiv u(x,t) = \lim_{m\to\infty} u^{(m)}, \qquad u^{(m)} = \sum_{i=1}^{\infty} u_{i,m}(t)\psi_i(x),$$

where $\{\psi_i\}$ is the orthonormal basis of $L^2(\Omega)$ composed of the eigenfunctions of the Dirichlet problem for the Laplace operator

$$(\nabla\psi_i, \nabla\phi)_{2,\Omega} = \lambda_i(\psi_i,\phi)_{2,\Omega} \quad \forall\phi \in W_0^{1,2}(\Omega), \quad i = 1,2,\dots. \tag{3.2}$$

The system $\left\{\frac{1}{\sqrt{\lambda_i}}\psi_i\right\}$ forms an orthogonal basis of $W_0^{1,2}(\Omega)$. Let us accept the notation

$$\begin{aligned} &\gamma_\epsilon(q,\mathbf{s}) = \left(\epsilon^2+|\mathbf{s}|^2\right)^{\frac{q-2}{2}}, \qquad \mathbf{s} \in \mathbb{R}^d,\ t \in (0,T),\ \epsilon > 0,\ q \in (1,2], \\ &p_m(t) = p[u^{(m)}], \qquad u^{(m)} = \sum_{i=1}^{m} u_{i,m}(t)\psi_i(x). \end{aligned} \tag{3.3}$$

The coefficients $u_{i,m}(t)$ are defined as the solutions of the Cauchy problem for the system of m ordinary nonlinear differential equations

$$\begin{aligned} & u'_{i,m}(t) = -\int_{\Omega} \gamma_\epsilon(p_m(t), \nabla u^{(m)}) \nabla u^{(m)} \cdot \nabla \psi_i \, dx + \int_{\Omega} f(z) \psi_i \, dx, \\ & u_{i,m}(0) = u_{0i}^{(m)}, \qquad i = 1, 2, \ldots, m, \end{aligned} \tag{3.4}$$

where the constants $v_i^{(m)}$ are the Fourier coefficients of u_0 in the basis $\{\psi_i\}$:

$$u_0^{(m)} = \sum_{i=1}^{m} u_{0i}^{(m)} \psi_i(x) \to u_0(x) \quad \text{in } L^2(\Omega).$$

By the Caratheodory theorem for every finite m system (3.4) has a continuous solution on an interval $(0, T_m)$. In the next subsection we derive the uniform estimates on $u^{(m)}$ and its derivatives, which show that the solutions of system (3.4) can be continued to the interval $(0, T)$.

3.2. Uniform a priori estimates

Lemma 3.1. *Under conditions (2.8)*

$$\sup_{(0,T)} \|u^{(m)}(t)\|_{2,\Omega}^2 + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)}) \left|\nabla u^{(m)}\right|^2 dz \leq C \left(\|u_0\|_{2,\Omega}^2 + \|f\|_{2,Q_T}^2 \right), \tag{3.5}$$

$$\int_{Q_T} \left|\nabla u^{(m)}\right|^{p_m(t)} dz \leq C \left(1 + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)}) \left|\nabla u^{(m)}\right|^2 dz \right) \tag{3.6}$$

with absolute constants C.

Proof. Multiplying the ith equation of (3.4) by $u_i^{(m)}$ and summing the results lead to the energy relation

$$\frac{1}{2} \frac{d}{dt} \left(\|u^{(m)}(t)\|_{2,\Omega}^2 \right) + \int_{\Omega} \gamma_\epsilon(p_m(t), \nabla u^{(m)}) \left|\nabla u^{(m)}\right|^2 dx = \int_{\Omega} u^{(m)} f \, dx. \tag{3.7}$$

Estimate (3.5) follows from (3.7) after integration in t. By Young's inequality, for every $\delta > 0$

$$\begin{aligned} \int_{Q_T} \left|\nabla u^{(m)}\right|^{p_m} dz &= \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)})^{-\frac{p_m}{2}} \left(\gamma_\epsilon(p_m(t), \nabla u^{(m)}) |\nabla u^{(m)}|^2 \right)^{\frac{p_m}{2}} dz \\ &\leq \delta \int_{Q_T} (\epsilon^2 + |\nabla u^{(m)}|^2)^{\frac{p_m}{2}} dz + C_\delta \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)}) |\nabla u^{(m)}|^2 \, dz \\ &\leq \delta \int_{Q_T} \left|\nabla u^{(m)}\right|^{p_m} dz + C_\delta \left(1 + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)}) |\nabla u^{(m)}|^2 \, dz \right). \end{aligned}$$

Estimate (3.6) follows if we take $\delta = 1/2$. □

Remark 3.1. Equality (3.7) yields the inequality

$$\|u^{(m)}(t)\|_{2,\Omega} \frac{d}{dt} \left(\|u^{(m)}(t)\|_{2,\Omega} \right) \leq \|u^{(m)}(t)\|_{2,\Omega} \|f(t)\|_{2,\Omega}.$$

Simplifying and integrating it in t we obtain the inequality

$$\|u^{(m)}(t)\|_{2,\Omega} \leq \|u_0\|_{2,\Omega} + \int_0^t \|f(t)\|_{2,\Omega} dt. \tag{3.8}$$

Corollary 3.1.

$$\int_{Q_T} (\gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|)^{(p_m(t))'}\, dz \le C$$

uniformly with respect to m and ϵ.

Proof. The estimate follows from (3.6) because for $1 < p^- \le p^+ \le 2$

$$\gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}| \le (\epsilon^2 + |\nabla u^{(m)}|^2)^{\frac{p_m(t)-1}{2}} \le C(p^\pm)\left(1 + |\nabla u^{(m)}|^{p_m(t)-1}\right). \quad \square$$

Lemma 3.2. *Let conditions (2.8) be fulfilled. Then the functions $u^{(m)}$ satisfy the estimate*

$$\begin{aligned}\sup_{(0,T)} \|\nabla u^{(m)}(\cdot,t)\|^2_{2,\Omega} &+ \int_{Q_T} \left((\epsilon^2 + |\nabla u^{(m)}|^2)^{\frac{p_m(t)-2}{2}} |u^{(m)}_{xx}|^2 + |u^{(m)}_{xx}|^{p_m(t)}\right) dz \\ &\le C\left(\|\nabla u_0\|^2_{2,\Omega} + \int_{Q_T} |f|^{(p^-)'}\, dz + 1\right)\end{aligned} \tag{3.9}$$

with a constant C independent of m and ϵ.

Proof. Multiplying ith equation in (3.4) by $\lambda_i u_i^{(m)}$, summing up for $i = 1, 2, \ldots, m$ and then following the proof of [19, Lemma 2.2] we arrive at the equality

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u^{(m)}\|^2_{2,\Omega}\right) + \int_\Omega \gamma_\epsilon(p_m(t), \nabla u^{(m)})|u^{(m)}_{xx}|^2\, dx = -I - I_{\partial\Omega} + I_f, \tag{3.10}$$

where

$$I = \int_\Omega (p-2)\left(\epsilon^2 + |\nabla u^{(m)}|^2\right)^{\frac{p_m(t)-2}{2}-1} \left(\sum_{k=1}^d \left(\nabla u^{(m)} \cdot \nabla(D_k u^{(m)})\right)^2\right) dx, \tag{3.11}$$

$$I_f = \int_\Omega f(z)\Delta u^{(m)}\, dx, \tag{3.12}$$

$$I_{\partial\Omega} = \int_{\partial\Omega} \gamma_\epsilon(p_m(t), \nabla u^{(m)})\left(\Delta u^{(m)}(\nabla u^{(m)} \cdot \mathbf{n}) - \nabla u^{(m)} \cdot \nabla(\nabla u^{(m)} \cdot \mathbf{n})\right) dS. \tag{3.13}$$

It is straightforward to check that

$$|I| \le (2 - p^-)\int_\Omega \gamma_\epsilon(p_m(t), \nabla u^{(m)})|u^{(m)}_{xx}|^2\, dx.$$

By Young's inequality

$$\begin{aligned}|I_f| &\le \int_\Omega |f||u^{(m)}_{xx}|\, dx \le \delta\int_\Omega |u^{(m)}_{xx}|^{p_m(t)}\, dx + C(\delta)\int_\Omega |f|^{p'_m(t)}\, dx \\ &\le \delta\int_\Omega \gamma_\epsilon(p_m(t), \nabla u^{(m)})^{-\frac{p_m(t)}{2}}\left(\gamma_\epsilon(p_m(t), \nabla u^{(m)})|u^{(m)}_{xx}|^2\right)^{\frac{p_m(t)}{2}} dx \\ &\quad + C(\delta, p^\pm)\left(1 + \int_\Omega |f|^{(p^-)'}\, dx\right) \\ &\le \delta\int_\Omega \gamma_\epsilon(p_m(t), \nabla u^{(m)})|u^{(m)}_{xx}|^2\, dx \\ &\quad + C'(\delta)\int_\Omega (\epsilon^2 + |\nabla u^{(m)}|^2)^{\frac{p_m(t)}{2}}\, dx + C''(\delta)\left(1 + \int_\Omega |f|^{(p^-)'}\, dx\right)\end{aligned}$$

with an arbitrary $\delta > 0$. Choosing δ appropriately small and using (3.6) we arrive at the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\nabla u^{(m)}(\cdot,t)\|^2_{2,\Omega}\right) &+ \int_\Omega \left(\gamma_\epsilon(p_m(t), \nabla u^{(m)})|u^{(m)}_{xx}|^2 + |u^{(m)}_{xx}|^{p_m(t)}\right) dx \\ &\le C\left(1 + |I_\Omega| + \int_\Omega |f|^{(p^-)'} + \int_\Omega |\nabla u^{(m)}|^{p_m}\, dx\right)\end{aligned} \tag{3.14}$$

with a constant C which does not depend on m and ϵ. It is known (see [20, Ch.1,Sec.1.5] for the case $d=2$ and [19, Lemma A.1] for the general case $d\geq 3$) that if $\partial\Omega\in C^2$, then there exist constants K, K', depending on $\partial\Omega$, such that

$$|I_{\partial\Omega}|\leq K\int_{\partial\Omega}\gamma_\epsilon(z,\nabla u^{(m)})\left(\nabla u^{(m)}\cdot\mathbf{n}\right)^2\,dS\leq K'\left(\int_{\partial\Omega}|\nabla u^{(m)}|^{p_m}\,dS+1\right).$$

Inequality (3.14) can be written in the form

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla u^{(m)}\|^2_{2,\Omega}\right)+\int_\Omega\left(\gamma_\epsilon(z,\nabla u^{(m)})|u^{(m)}_{xx}|^2+|u^{(m)}_{xx}|^{p_m}\right)\,dx\\&\qquad\leq C\left(\int_{\partial\Omega}|\nabla u^{(m)}|^{p_m}\,dS+\int_\Omega f^{(p^-)'}dx+\int_\Omega|\nabla u^{(m)}|^{p_m}\,dx+1\right).\end{aligned}\tag{3.15}$$

To estimate the integral over $\partial\Omega$ we use the following embedding inequality (see [21, Theorem 1.5.1.10]): there exists a constant $L=L(d,\Omega)$ such that for every $\delta\in(0,1)$

$$\int_{\partial\Omega}|\nabla u^{(m)}|^{p_m}dS\leq L\left(\delta^{1-\frac{1}{p_m}}\int_\Omega|u^{(m)}_{xx}|^{p_m}dx+\delta^{-\frac{1}{p_m}}\int_\Omega|\nabla u^{(m)}|^{p_m}dx\right)\tag{3.16}$$

It follows that for all $t\in[0,T]$ and every $\delta\in(0,1)$

$$\int_{\partial\Omega}|\nabla u^{(m)}|^{p_m}dS\leq L\left(\delta^{1-\frac{1}{p^+}}\int_\Omega|u^{(m)}_{xx}|^{p_m}dx+\delta^{-\frac{1}{p^-}}\int_\Omega|\nabla u^{(m)}|^{p_m}dx\right)\tag{3.17}$$

Combining (3.15) and (3.16) with $2\delta^{1-\frac{1}{p^+}}CK'\leq 1$, we arrive at the inequality

$$\frac{d}{dt}\|\nabla u^{(m)}\|^2_{2,\Omega}+\int_\Omega\left(\gamma_\epsilon(z,\nabla u^{(m)})|u^{(m)}_{xx}|^2+|u^{(m)}_{xx}|^{p_m}\right)\,dx\leq C\left(\int_\Omega|\nabla u^{(m)}|^{p_m}\,dx+\int_\Omega f^{(p^-)'}\,dx+1\right).$$

To complete the proof, we integrate this inequality with respect to t and plug in estimates (3.5), (3.6). □

Lemma 3.3. *Under conditions (2.8) the functions $u^{(m)}$ satisfy the estimates*

$$\|u^{(m)}_t\|^2_{2,Q_T}+\sup_{(0,T)}\int_\Omega(\epsilon^2+|\nabla u^{(m)}|^2)^{\frac{p_m(t)}{2}}\,dx\leq C\tag{3.18}$$

with a constant $C=C(\|\nabla u_0\|_{2,\Omega},\|f\|_{(p^-)'(\cdot),Q_T},p^\pm,C^*)$ independent of m and ϵ.

Proof. Estimates (3.18) follow upon multiplication the ith equation of (3.4) by $u'_{i,m}(t)$ and summation of the results. Following the proof of [19, Lemma 2.4] we arrive at the relations

$$\begin{aligned}&\|u^{(m)}_t(t)\|^2_{2,\Omega}\,dx+\frac{d}{dt}\left(\int_\Omega\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}\,dx\right)\\&=-\int_\Omega\frac{dp_m(t)}{dt}\frac{\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}}{p_m^2(t)}\left(1-\frac{p_m(t)}{2}\ln\left(\epsilon^2+|\nabla u^{(m)}|^2\right)\right)\,dx+\int_\Omega fu^{(m)}_t\,dx\\&\leq C\left|\frac{dp_m(t)}{dt}\right|\left(1+\int_\Omega|\nabla u^{(m)}|^{p_m(t)}\,dx+\int_\Omega\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}\ln^2\left(\epsilon^2+|\nabla u^{(m)}|^2\right)\,dx\right)\\&\quad+\frac{1}{2}\|f(t)\|^2_{2,\Omega}+\frac{1}{2}\|u^{(m)}_t(t)\|^2_{2,\Omega}\qquad\text{for every } t\in[0,T]\end{aligned}\tag{3.19}$$

with $C=C(C^*,p^-)$. Using the formula

$$(|u|^\alpha)_t=\left((u^2)^{\frac{\alpha}{2}}\right)_t=\frac{\alpha}{2}(u^2)^{\frac{\alpha}{2}-1}2uu_t=\alpha u_t|u|^{\alpha-1}\operatorname{sign}u$$

and (3.5) we estimate

$$
\begin{aligned}
\left|\frac{dp_m(t)}{dt}\right| &= \alpha\left|p'\left(\|u(\cdot,t)\|^{\alpha}_{\alpha,\Omega}\right)\right|\left|\int_{\Omega}\left|u^{(m)}\right|^{\alpha-1}u^{(m)}_t\operatorname{sign}u\,dx\right| \\
&\leq \alpha C_*\|u^{(m)}_t\|_{2,\Omega}\left(\int_{\Omega}\left|u^{(m)}\right|^{2(\alpha-1)}dx\right)^{\frac{1}{2}} \\
&\leq \alpha C_*|\Omega|^{1-\dfrac{\alpha}{2}}\|u^{(m)}_t\|_{2,\Omega}\|u^{(m)}\|^{\frac{\alpha-1}{2}}_{2,\Omega} \\
&\leq C\|u^{(m)}_t\|_{2,\Omega}, \qquad C = C\left(\alpha,|\Omega|,C_*,u_0,f\right).
\end{aligned}
\tag{3.20}
$$

Then

$$
\begin{aligned}
\left|\frac{dp_m(t)}{dt}\right|\left|\int_{\Omega}\frac{\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}}{p^2_m(t)}dx\right| &\leq \frac{C}{(p^-)^2}\|u^{(m)}_t\|_{2,\Omega}\left(1+\int_{\Omega}|\nabla u^{(m)}|^{p_m(t)}dx\right) \\
&\leq C'\left(1+\sup_{(0,T)}\int_{\Omega}|\nabla u^{(m)}|^2dx\right)\|u^{(m)}_t\|_{2,\Omega},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\left|\frac{dp_m(t)}{dt}\right|\left|\int_{\Omega}\frac{1}{p_m(t)}\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}\ln^2\left(\epsilon^2+|\nabla u^{(m)}|^2\right)dx\right| \\
&\quad\leq C''\|u^{(m)}_t\|_{2,\Omega}\int_{\Omega}\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}\ln^2\left(\epsilon^2+|\nabla u^{(m)}|^2\right)dx = I.
\end{aligned}
$$

For every $0<\mu<\min\{p^-/2,(2-p^+)/2\}$,

$$
s^{\frac{p_m(t)}{2}}\ln^2 s \leq \begin{cases} s^{\frac{p_m(t)-\mu}{2}}\left(s^{\mu/2}\ln^2 s\right) & \text{if } s\in(0,1), \\ s^{\frac{p_m(t)+\mu}{2}}\left(s^{-\mu/2}\ln^2 s\right) & \text{if } s>1 \end{cases} \leq C(\mu,p^{\pm})(1+s^{\frac{p^++\mu}{2}}) \leq C(1+s).
\tag{3.21}
$$

Gathering (3.21) with (3.9) we obtain the estimate

$$
\begin{aligned}
&\int_{\Omega}\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}\ln^2\left(\epsilon^2+|\nabla u^{(m)}|^2\right)dx \leq C\int_{\Omega}(1+|\nabla u^{(m)}|^2)\,dx \\
&\quad\leq C\left(\|\nabla u_0\|^2_{2,\Omega}+\int_{Q_T}|f|^{(p^-)'}dz+1\right)
\end{aligned}
$$

for all $t\in(0,T)$. By Young's inequality

$$
\begin{aligned}
I &\leq C\|u_t(t)\|_{2,\Omega}\left(\|\nabla u_0\|^2_{2,\Omega}+\int_{Q_T}|f|^{(p^-)'}dz+1\right) \\
&\leq \frac{1}{4}\|u_t(t)\|^2_{2,\Omega}+C'\left(\|\nabla u_0\|^2_{2,\Omega}+\int_{Q_T}|f|^{(p^-)'}dz+1\right)^2.
\end{aligned}
\tag{3.22}
$$

Plugging (3.22), (3.5) and (3.6) into (3.19) we rewrite it in the form

$$
\begin{aligned}
&\frac{1}{4}\|u^{(m)}_t(t)\|^2_{2,\Omega}+\frac{d}{dt}\left(\int_{\Omega}\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}dx\right) \\
&\quad\leq C\left(\left(1+\|f\|^{(p^-)'}_{(p^-)',Q_T}+\|\nabla u_0\|^2_{2,\Omega}\right)^2+\|f(t)\|^2_{2,\Omega}\right)
\end{aligned}
$$

for every $t\in[0,T]$ with a constant depending on α, $p^{\pm}$, C^*, $|\Omega|$. Inequality (3.18) follows after integration in time. □

Corollary 3.2. *Under the conditions of Lemma 3.3* $p_m(t) \in C^{1/2}([0,T])$ *and*

$$\|p_m(t)\|_{C^{1/2}([0,T])} \leq C$$

with an independent of m *and* ϵ *constant* C.

Proof. By virtue of (3.18) and (3.20), for every $0 \leq \tau \leq t \leq T$

$$|p_m(t) - p_m(\tau)| = \left| \int_\tau^t \frac{dp_m(s)}{dt}\, ds \right| \leq C \int_\tau^t \|u_t^{(m)}\|_{2,\Omega}\, ds \leq C' \|u_t^{(m)}\|_{2,Q_T} |t-\tau|^{\frac{1}{2}}$$

with an independent of m and ϵ constant C. □

4. Passing to the limit

4.1. Strong solution of the regularized problem

Lemma 4.1. *If the data satisfy conditions (2.8), then problem (3.1) has a strong solution* $u_\epsilon = \lim u^{(m)}$ *as* $m \to \infty$. *The solution satisfies the estimate*

$$\|u_{\epsilon t}\|_{2,Q_T}^2 + \operatorname{ess\,sup}_{(0,T)} \|\nabla u_\epsilon(\cdot,t)\|_{2,\Omega}^2 + \int_{Q_T} |u_{\epsilon xx}|^{p[u_\epsilon]}\, dz \leq C \left(\|\nabla u_0\|_{2,\Omega}^2 + \int_{Q_T} |f|^{(p^-)'}\, dz + 1 \right). \tag{4.1}$$

For the sake of simplicity of notation, throughout this subsection we omit the subindex ϵ and denote by $u(z)$ the limit of the sequence $\{u^{(m)}\}$, which approximates the solution of the regularized problem (3.1). The uniform estimates (3.5), (3.6), (3.9), (3.18) allow one to extract from $\{u^{(m)}\}$ a subsequence (which we assume coinciding with the whole sequence) such that for some $u \in L^2(Q_T) \cap L^\infty(0,T;W_0^{1,2}(\Omega))$ and $\chi \in (L^{(p[u])'}(Q_T))^d$

$$\begin{aligned} &u_t^{(m)} \rightharpoonup u_t \text{ in } L^2(Q_T), \\ &\nabla u^{(m)} \rightharpoonup \nabla u \text{ in } (L^2(Q_T))^d, \\ &\gamma_\epsilon(p_m(t), \nabla u^{(m)}) \nabla u^{(m)} \rightharpoonup \chi \text{ in } (L^{(p[u])'}(Q_T))^d. \end{aligned} \tag{4.2}$$

The first two relations follow directly from (3.9) and (3.18). Let us prove the third relation of (4.2). According to [22, Th.5], the sequence $\{u^{(m)}\}$ is relatively compact in $C([0,T];L^2(\Omega))$:

$$u^{(m)} \to u \text{ in } C([0,T];L^2(\Omega)) \text{ and a.e. in } Q_T. \tag{4.3}$$

Due to (4.3), for every $t \in [0,T]$ there exists

$$\lim_{m\to\infty} \|u^{(m)}(\cdot,t)\|_{\alpha,\Omega}^\alpha = \|u(\cdot,t)\|_{\alpha,\Omega}^\alpha$$

whence, by continuity of $p(l)$,

$$p_m(t) = p\left(\|u^{(m)}(\cdot,t)\|_{\alpha,\Omega}^\alpha\right) \to p\left(\|u(\cdot,t)\|_{\alpha,\Omega}^\alpha\right) = p[u] \quad \forall t \in [0,T].$$

Fix some $\beta \in (0,1/2)$. By Corollary 3.2 the sequence $\{p_m(t)\}$ is equicontinuous in $C^{0,1/2}[0,T]$. It follows then that $\{p_m(t)\}$ is precompact in $C^{0,\beta}[0,T]$:

$$p_m(t) \to p(t) \equiv p[u] \text{ in } C^{0,\beta}[0,T] \subset C_{\log}[0,T]. \tag{4.4}$$

Notice that

$$\left(\gamma_\epsilon(p_m(t),\nabla u^{(m)})|\nabla u^{(m)}|\right)^{(p[u])'} \leq C\left(1+|\nabla u^{(m)}|^{p_m(t)-1}\right)^{\frac{p[u]}{p[u]-1}} \leq C\left(1+|\nabla u^{(m)}|^{\lambda_m(t)}\right)$$

with

$$\lambda_m(t)=(p_m(t)-1)\frac{p[u]}{p[u]-1}.$$

It is easy to see that

$$\lambda_m(t)<2 \quad\Leftrightarrow\quad p_m(t)+\frac{2}{p[u]}<3 \quad\Leftrightarrow\quad (p[u]-1)(p[u]-2)<p[u](p[u]-p_m(t)),$$

which is true for all sufficiently big m because for $1<p^-\leq p^+<2$ and $p_m(t)\to p[u]$ uniformly in $[0,T]$

$$(p[u]-1)(p[u]-2)\leq(p^--1)(p^+-2)<0 \quad\text{while}\quad p[u](p[u]-p_m(t))\to 0 \text{ as } m\to\infty.$$

Hence,

$$\left(\gamma_\epsilon(p_m(t),\nabla u^{(m)})|\nabla u^{(m)}|\right)^{(p[u])'} \leq C\left(1+|\nabla u^{(m)}|^2\right)$$

and

$$\int_{Q_T}(\gamma_\epsilon(p_m(t),\nabla u^{(m)})|\nabla u^{(m)}|)^{(p[u])'}\,dz\leq C$$

by virtue of (3.9). These arguments prove the following assertion.

Lemma 4.2. *If conditions (2.8) are fulfilled, then there exist $u\in L^2(Q_T)\cap L^\infty(0,T;W_0^{1,2}(\Omega))$ and $\chi\in(L^{(p[u])'}(Q_T))^d$ such that relations (4.2) are fulfilled and*

$$\int_{Q_T}(\gamma_\epsilon(p[u],\nabla u)|\nabla u|)^{(p[u])'}\,dz\leq C, \qquad (p[u])'=\frac{p[u]}{p[u]-1} \tag{4.5}$$

with a constant C depending only on the data.

By the method of construction of $u^{(m)}$, for every finite m and $\phi\in\mathcal{P}_k\equiv\operatorname{span}\{\psi_1,\dots,\psi_k\}$, $k\leq m$,

$$\int_{Q_T}\left(u_t^{(m)}\phi+\gamma_\epsilon(p_m(t),\nabla u^{(m)})\nabla u^{(m)}\cdot\nabla\phi-f\phi\right)\,dz=0. \tag{4.6}$$

Relations (4.2) and (4.5) allow one to pass in (4.6) to the limit as $m\to\infty$, which leads to the equality

$$\int_{Q_T}(u_t\phi+\chi\cdot\nabla\phi-f\phi)\,dz=0 \qquad \forall\phi\in\mathcal{P}_k. \tag{4.7}$$

Lemma 4.3. *For every $u\in\mathbf{W}_u(Q_T)$ there exists a sequence $\{\phi_N\}$, $\phi_N\in\mathcal{P}_N\cap\mathbf{W}_u(Q_T)$ such that $\phi_N\to u$ in $\mathbf{W}_u(Q_T)$.*

Proof. Recall that $\mathcal{P}_N\subset C([0,T];C^2(\overline{\Omega}))$. Because of the inclusions $p_m(t),p[u]\in C^{0,\beta}[0,T]$, the space $C^\infty(\overline{Q}_T)$ in dense in $\mathbf{W}_u(Q_T)$ and $\mathbf{W}_{u^{(m)}}(Q_T)$ with any $m\in\mathbb{N}$. For every $\epsilon>0$ there exists $\phi_\epsilon\subset C^\infty(\overline{Q}_T)$ such that $\|\phi_\epsilon-u\|_{\mathbf{W}_u(Q_T)}<\epsilon$. Since the systems $\{\psi_i\}$ and $\left\{\frac{1}{\sqrt{\lambda_i}}\psi_i\right\}$ form orthonormal bases of $L^2(\Omega)$ and $W_0^{1,2}(\Omega)$, then

$$\phi_\epsilon=\sum_{i=1}^\infty\phi_{\epsilon i}(t)\psi_i(x), \quad \|\phi_\epsilon(t)\|^2_{W_0^{1,2}(\Omega)}=\sum_{i=1}^\infty\lambda_i\phi_{\epsilon i}^2(t), \quad \|\phi_\epsilon\|^2_{L^2(0,T;W_0^{1,2}(\Omega))}=\sum_{i=1}^\infty\lambda_i\|\phi_{\epsilon i}\|^2_{2,(0,T)}<\infty,$$

$$\phi_\epsilon^{(N)} = \sum_{i=1}^{N} \phi_{\epsilon i}(t)\psi_i(x) \in \mathcal{P}_N, \qquad \|\phi_\epsilon^{(N)} - \phi_\epsilon\|^2_{L^2(0,T;W_0^{1,2}(\Omega))} = \sum_{i=N+1}^{\infty} \lambda_i \|\phi_{\epsilon i}\|^2_{2,(0,T)} \to 0 \quad \text{as } N \to \infty.$$

There exists $N = N(\epsilon)$ such that for every $m > N(\epsilon)$

$$\|\phi_\epsilon^{(m)} - \phi_\epsilon\|_{\mathbf{W}_u(Q_T)} \le C\|\phi_\epsilon^{(m)} - \phi_\epsilon\|_{L^2(0,T;W_0^{1,2}(\Omega))} < \epsilon$$

and

$$\|u - \phi_\epsilon^{(m)}\|_{\mathbf{W}_u(Q_T)} \le \|u - \phi_\epsilon\|_{\mathbf{W}_u(Q_T)} + \|\phi_\epsilon - \phi_\epsilon^{(m)}\|_{\mathbf{W}_u(Q_T)} \le \epsilon + C\|\phi_\epsilon^{(m)} - \phi_\epsilon\|_{L^2(0,T;W_0^{1,2}(\Omega))} < 2\epsilon.$$

□

Taking ϕ_N for the test-function in (4.7) and letting $N \to \infty$ we obtain the equality

$$\int_{Q_T} u_t u \, dz + \int_{Q_T} \chi \cdot \nabla u \, dz = \int_{Q_T} f u \, dz. \tag{4.8}$$

Let us return to (4.6) and take for the test-function $\phi = u^{(m)}$:

$$\begin{aligned} 0 &= \int_{Q_T} u_t^{(m)} u^{(m)} \, dz + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)}) |\nabla u^{(m)}|^2 \, dz - \int_{Q_T} f u^{(m)} \, dz \\ &= \int_{Q_T} u_t^{(m)} u^{(m)} \, dz + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)}) \nabla u^{(m)} \cdot \nabla(u^{(m)} - \psi) \, dz - \int_{Q_T} f u^{(m)} \, dz \\ &\quad + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)}) \nabla u^{(m)} \cdot \nabla \psi \, dz. \end{aligned} \tag{4.9}$$

We will use the following well-known inequality: if $q \in (1, 2]$, then for all $\xi, \zeta \in \mathbb{R}^d$, $\xi \neq \zeta$ and $\epsilon > 0$

$$(\gamma_\epsilon(q, \xi)\xi - \gamma_\epsilon(q, \zeta)\zeta) \cdot (\xi - \zeta) \ge (q-1)(1 + |\xi|^2 + |\zeta|^2)^{\frac{q-2}{2}} |\xi - \zeta|^2. \tag{4.10}$$

By virtue of (4.10) for every $\psi \in \mathcal{P}_k$ with $k \le m$

$$\begin{aligned} &\int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)}) \nabla u^{(m)} \cdot \nabla(u^{(m)} - \psi) \, dz \\ &\quad = \int_{Q_T} (\gamma_\epsilon(p_m(t), \nabla u^{(m)}) \nabla u^{(m)} - \gamma_\epsilon(p_m(t), \nabla \psi) \nabla \psi) \cdot \nabla(u^{(m)} - \psi) \, dz \\ &\qquad + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla \psi) \nabla \psi \cdot \nabla(u^{(m)} - \psi) \, dz \\ &\quad \ge \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla \psi) \nabla \psi \cdot \nabla(u^{(m)} - \psi) \, dz. \end{aligned} \tag{4.11}$$

Because of (4.4)

$$\sigma_m(\nabla \psi) \equiv \gamma_\epsilon(p_m(t), \nabla \psi) \nabla \psi - \gamma_\epsilon(p[u], \nabla \psi) \nabla \psi \to 0 \text{ as } m \to \infty \text{ uniformly in } Q_T. \tag{4.12}$$

It follows from (3.9), (4.12) and (4.2) that

$$\begin{aligned} &\int_{Q_T} \gamma_\epsilon(p_m(t), \nabla \psi) \nabla \psi \cdot \nabla(u_m - \psi) \, dz \\ &\quad = \int_{Q_T} \sigma_m(\nabla \psi) \cdot \nabla(u_m - \psi) \, dz + \int_{Q_T} \gamma_\epsilon(p[u], \nabla \psi) \nabla \psi \cdot \nabla(u_m - \psi) \, dz \\ &\quad \equiv J_1 + J_2 \to \int_{Q_T} \gamma_\epsilon(p[u], \nabla \psi) \nabla \psi \cdot \nabla(u - \psi) \, dz \quad \text{as } m \to \infty \end{aligned}$$

because

$$J_1 \le \|\sigma_m(\nabla\psi)\|_{\infty,Q_T}\|\nabla(u_m-\psi)\|_{1,Q_T} \le C\|\sigma_m(\nabla\psi)\|_{\infty,Q_T} \to 0,$$
$$J_2 \to \int_{Q_T} \gamma_\epsilon(p[u],\nabla\psi)\nabla\psi\cdot\nabla(u-\psi)\,dz.$$

Using (4.11) in (4.9) and then letting $m\to\infty$ we find that for $\psi\in\mathcal{P}_k$ with any $k\in\mathbb{N}$

$$0 \ge \int_{Q_T} u_t u\,dz + \int_{Q_T} \gamma_\epsilon(p[u],\nabla\psi)\nabla\psi\cdot\nabla(u-\psi)\,dz - \int_{Q_T} fu\,dz + \int_{Q_T} \chi\cdot\nabla\psi\,dz.$$

By Lemma 4.3 we may take $\psi=\psi^{(k)}\in\mathcal{P}_k\cap\mathbf{W}_u(Q_T)$ and then let $k\to\infty$. Plugging (4.8) we arrive at the inequality

$$0 \ge \int_{Q_T} (\gamma_\epsilon(p[u],\nabla\psi)\nabla\psi-\chi)\cdot\nabla(u-\psi)\,dz \qquad \forall\psi\in\mathbf{W}_u(Q_T).$$

Take $\psi=u+\lambda\zeta$ with an arbitrary $\zeta\in\mathbf{W}_u(Q_T)$ and $\lambda>0$. Simplifying and then letting $\lambda\downarrow 0$ we obtain the inequality

$$I(u,\chi,\zeta) \equiv \int_{Q_T} (\gamma_\epsilon(p[u],\nabla u)\nabla u-\chi)\cdot\nabla\zeta\,dz \le 0 \qquad \forall\zeta\in\mathbf{W}_u(Q_T).$$

Since ζ is arbitrary, it is necessary that $I(u,\chi,\zeta)=0$ for all $\zeta\in\mathbf{W}_u(Q_T)$, whence

$$\int_{Q_T} (u_t\zeta+\gamma_\epsilon(p[u],\nabla u)\nabla u\cdot\nabla\zeta-f\zeta)\,dz = 0 \qquad \forall\zeta\in\mathbf{W}_u(Q_T). \tag{4.13}$$

Estimate (4.1) follows from the uniform in m and ϵ estimates (3.5), (3.6), (3.9), (3.18). It follows from (3.9) that $D^2_{ij}u^{(m)}\rightharpoonup D^2_{ij}u$ in $L^{p^-}(Q_T)$ (up to a subsequence). Because of (4.4) and the uniform estimates (3.9) on $|u^{(m)}_{xx}|^{p_m(t)}$, the estimate on $\||u_{xx}|^{p[u]}\|_{1,Q_T}$ follows from [10, Lemma 3.1].

4.2. Strong solution of the singular problem

Let u_ϵ be the strong solution of problem (3.1) with $\epsilon>0$ obtained as the limit of the sequence of Galerkin's approximations (see Lemma 4.1). The functions u_ϵ satisfy the independent of ϵ estimates (4.1). Therefore, there exist functions u and χ such that, up to a subsequence,

$$\begin{aligned} &u_{\epsilon t}\rightharpoonup u_t \text{ in } L^2(Q_T),\\ &\nabla u_\epsilon\rightharpoonup\nabla u \text{ in } (L^2(Q_T))^d,\\ &\gamma_\epsilon(p[u_\epsilon],\nabla u_\epsilon)\nabla u_\epsilon\rightharpoonup\chi \text{ in } (L^{p'[u]}(Q_T))^d. \end{aligned} \tag{4.14}$$

Moreover, $u\in C^0([0,T];L^2(\Omega))$. For every $\epsilon>0$ the function u_ϵ satisfies equality (4.13):

$$\int_{Q_T} (u_{\epsilon t}\phi+\gamma_\epsilon(p[u_\epsilon],\nabla u_\epsilon)\nabla u_\epsilon\cdot\nabla\phi-f\phi)\,dz = 0 \qquad \forall\phi\in\mathbf{W}_{u_\epsilon}(Q_T). \tag{4.15}$$

Since $u_\epsilon\to u$ in $C^0([0,T];L^2(\Omega))$, then $\|u_\epsilon(\cdot,t)\|^\alpha_{\alpha,\Omega}\to\|u(\cdot,t)\|^\alpha_{\alpha,\Omega}$ for every $t\in[0,T]$ and

$$p\left(\|u_\epsilon(\cdot,t)\|^\alpha_{\alpha,\Omega}\right)\to p\left(\|u(\cdot,t)\|^\alpha_{\alpha,\Omega}\right) \quad \text{as } \epsilon\to 0$$

by continuity. As in Corollary 3.2, one may check that the functions $p_\epsilon(t):=p[u_\epsilon]$ are equicontinuous in $C^{0,1/2}[0,T]$: by Lemma 4.1

$$\begin{aligned} |p_\epsilon(t)-p_\epsilon(\tau)| &= \left|\int_\tau^t \frac{dp_\epsilon(s)}{ds}\,ds\right| \le \alpha\sup_{\mathbb{R}}|p'|\int_\tau^t\int_\Omega |u_{\epsilon t}||u|^{\alpha-1}\,dxds\\ &\le C(\alpha,C_*)\|u_{\epsilon t}\|_{2,Q_T}\left(\int_\tau^t\int_\Omega |u_\epsilon|^{2(\alpha-1)}\,dz\right)^{\frac{1}{2}}\\ &\le C\sup_{(0,T)}\|u_\epsilon(t)\|^{\alpha-1}_{2,\Omega}|t-\tau|^{1/2} \le C'|t-\tau|^{1/2} \end{aligned} \tag{4.16}$$

with an independent of ϵ constant C'. Hence,

$$p[u_\epsilon] \to p[u] \text{ in } C^{0,\beta}[0,T] \text{ with some } \beta \in (0,1/2). \tag{4.17}$$

It follows that $C^\infty(\overline{Q}_T)$ is dense in $\mathbf{W}_u(Q_T)$ and $\mathbf{W}_{u_\epsilon}(Q_T)$ with every ϵ. Let $\phi_\delta \in C^\infty(\overline{Q}_T)$ and $\phi_\delta \to u$ in $\mathbf{W}_u(Q_T)$ as $\delta \to 0$. Repeating the proof of Lemma 4.2 we find that $\chi \in (L^{(p[u])'}(Q_T))^d$, and by (4.14)

$$\int_{Q_T} \gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla \phi_\delta \, dz \to \int_{Q_T} \chi \cdot \nabla \phi_\delta \, dz \quad \text{as } \epsilon \to 0.$$

Taking ϕ_δ for the test-function in (4.15) and letting $\epsilon \to 0$ we obtain

$$\int_{Q_T} (u_t \phi_\delta + \chi \cdot \nabla \phi_\delta - f\phi_\delta) \, dz = 0.$$

Letting now $\delta \to 0$ we arrive at the equality

$$\int_{Q_T} (u_t u + \chi \cdot \nabla u - fu) \, dz = 0. \tag{4.18}$$

Choosing $u_\epsilon \in \mathbf{W}_{u_\epsilon}(Q_T)$ for the test-function in (4.15) we obtain

$$\int_{Q_T} u_{\epsilon t} u_\epsilon \, dz + \int_{Q_T} (\gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla u_\epsilon - fu_\epsilon) \, dz = 0. \tag{4.19}$$

Let us take $\psi \in C^\infty([0,T]; C_0^\infty(\Omega)) \subset \mathbf{W}_{u_\epsilon}(Q_T)$ with any $\epsilon > 0$. By (4.10)

$$\begin{aligned}
&\int_{Q_T} \gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla u_\epsilon \, dz \\
&\quad = \int_{Q_T} \gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla(u_\epsilon - \psi) \, dz + \int_{Q_T} \gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla \psi \, dz \\
&\quad = \int_{Q_T} (\gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon - \gamma_\epsilon(p[u_\epsilon], \nabla \psi)\nabla \psi) \cdot \nabla(u_\epsilon - \psi) \, dz \\
&\qquad + \int_{Q_T} \gamma_\epsilon(p[u_\epsilon], \nabla \psi)\nabla \psi \cdot \nabla(u_\epsilon - \psi) \, dz + \int_{Q_T} \gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla \psi \, dz \\
&\quad \geq \int_{Q_T} \gamma_\epsilon(p[u_\epsilon], \nabla \psi)\nabla \psi \cdot \nabla(u_\epsilon - \psi) \, dz + \int_{Q_T} \gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla \psi \, dz \\
&\quad = \int_{Q_T} \left(\gamma_\epsilon(p[u_\epsilon], \nabla \psi)\nabla \psi - |\nabla \psi|^{p[u_\epsilon]-2}\nabla \psi\right) \cdot \nabla(u_\epsilon - \psi) \, dz \\
&\qquad + \int_{Q_T} |\nabla \psi|^{p[u_\epsilon]-2}\nabla \psi \cdot \nabla(u_\epsilon - \psi) \, dz + \int_{Q_T} \gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla \psi \, dz \equiv I_1 + I_2 + I_3,
\end{aligned}$$

where

$$I_3 \to \int_{Q_T} \chi \cdot \nabla \psi \, dz \quad \text{as } \epsilon \to 0$$

by virtue of (4.14). Let us denote

$$\Phi_\epsilon \equiv (\epsilon^2 + |\nabla \psi|^2)^{\frac{p[u_\epsilon]-2}{2}}\nabla \psi - |\nabla \psi|^{p[u_\epsilon]-2}\nabla \psi.$$

For every $\psi \in C^\infty(0,T; C_0^\infty(\Omega))$, $\Phi_\epsilon \to 0$ as $\epsilon \to 0$ uniformly in Q_T. Since $\|u_\epsilon - \psi\|_{\mathbf{W}_{u_\epsilon}} \leq C$, it follows that

$$\begin{aligned}
|I_1| = \left| \int_{Q_T} \Phi_\epsilon \nabla(u_\epsilon - \psi) \, dz \right| &\leq 2\|\Phi_\epsilon\|_{p'_\epsilon(\cdot),Q_T} \|\nabla(u_\epsilon - \psi)\|_{p_\epsilon(\cdot),Q_T} \\
&\leq 2\|\Phi_\epsilon\|_{p'_\epsilon(\cdot),Q_T} \|u_\epsilon - \psi\|_{\mathbf{W}_{u_\epsilon}} \to 0 \quad \text{when } \epsilon \to 0.
\end{aligned}$$

Finally, set

$$\Psi_\epsilon \equiv |\nabla\psi|^{p[u_\epsilon]-2}\nabla\psi - |\nabla\psi|^{p[u]-2}\nabla\psi$$

and represent

$$I_2 = \int_{Q_T} \Psi_\epsilon \nabla(u_\epsilon - \psi)\,dz + \int_{Q_T} |\nabla\psi|^{p[u]-2}\nabla\psi\cdot\nabla(u_\epsilon-\psi)\,dz \equiv K_1 + K_2.$$

By (4.17), $\Psi_\epsilon \to 0$ as $\epsilon\to 0$ uniformly in Q_T. Since $|\nabla\psi|^{p[u]-2}\nabla\psi \in (C^0(\overline{Q_T}))^d \subset (L^2(Q_T))^d$, $\nabla u_\epsilon \rightharpoonup \nabla u$ in $(L^2(Q_T))^d$, and $\|\nabla(u_\epsilon-\psi)\|_{2,Q_T} \leq C$ with an independent of ϵ constant C, then

$$\begin{aligned}
&|K_1| \leq \|\Psi_\epsilon\|_{2,Q_T}\|\nabla(u_\epsilon-\psi)\|_{2,Q_T} \to 0,\\
&K_2 = \int_{Q_T} |\nabla\psi|^{p[u]-2}\nabla\psi\cdot\nabla(u_\epsilon-\psi)\,dz \to \int_{Q_T} |\nabla\psi|^{p[u]-2}\nabla\psi\cdot\nabla(u-\psi)\,dz \quad \text{as } \epsilon\to 0,
\end{aligned}$$

whence

$$I_2 \to \int_{Q_T} |\nabla\psi|^{p[u]-2}\nabla\psi\cdot\nabla(u-\psi)\,dz \quad \text{as } \epsilon\to 0.$$

Thus,

$$I_1 + I_2 + I_3 \to \int_{Q_T} \chi\cdot\nabla\psi\,dz + \int_{Q_T} |\nabla\psi|^{p[u]-2}\nabla\psi\cdot\nabla(u-\psi)\,dz \quad \text{as } \epsilon\to 0.$$

It follows that for every $\psi \in C^\infty([0,T];C_0^\infty(\Omega))$

$$\lim_{\epsilon\to 0}\int_{Q_T} \gamma_\epsilon(p[u_\epsilon],\nabla u_\epsilon)\nabla u_\epsilon\cdot\nabla u_\epsilon\,dz \geq \int_{Q_T}\chi\cdot\nabla\psi\,dz + \int_{Q_T}|\nabla\psi|^{p[u]-2}\nabla\psi\cdot\nabla(u-\psi)\,dz.$$

By (4.18), (4.19) as $\epsilon\to 0$, for every $\psi\in C^\infty(0,T;C_0^\infty(\Omega))$

$$\begin{aligned}
0 &= \int_{Q_T} u_t u\,dz - \int_{Q_T} fu\,dz + \lim_{\epsilon\to 0}\int_{Q_T}\gamma_\epsilon(p[u_\epsilon],\nabla u_\epsilon)\nabla u_\epsilon\cdot\nabla u_\epsilon\,dz\\
&\geq -\int_{Q_T}\chi\cdot\nabla u\,dz + \int_{Q_T}\chi\cdot\nabla\psi\,dz + \int_{Q_T}|\nabla\psi|^{p[u]-2}\nabla\psi\cdot\nabla(u-\psi)\,dz\\
&= \int_{Q_T}\left(|\nabla\psi|^{p[u]-2}\nabla\psi - \chi\right)\cdot\nabla(u-\psi)\,dz.
\end{aligned} \tag{4.20}$$

Let us take $\psi\equiv\psi_\delta + \lambda\zeta$ where $\lambda = const > 0$,

$$\zeta,\psi_\delta \in C^\infty([0,T];C_0^\infty(\Omega)) \text{ and } \psi_\delta \to u \text{ in } \mathbf{W}_u(Q_T) \text{ as } \delta\to 0.$$

Inequality (4.20) takes the form

$$\begin{aligned}
J_1 + J_2 \equiv &\int_{Q_T}\left(|\nabla(\psi_\delta+\lambda\zeta)|^{p[u]-2}\nabla(\psi_\delta+\lambda\zeta) - \chi\right)\cdot\nabla(u-\psi_\delta)\,dz\\
&- \lambda\int_{Q_T}\left(|\nabla(\psi_\delta+\lambda\zeta)|^{p[u]-2}\nabla(\psi_\delta+\lambda\zeta)-\chi\right)\cdot\nabla\zeta\,dz \leq 0.
\end{aligned}$$

By the generalized Hölder inequality (2.2)

$$\begin{aligned}
|J_1| &\leq 2\|u-\psi_\delta\|_{\mathbf{W}_u(Q_T)}\left\||\nabla(\psi_\delta+\lambda\zeta)|^{p[u]-2}\nabla(\psi_\delta+\lambda\zeta)-\chi\right\|_{p'[u],Q_T}\\
&\leq 2\|u-\psi_\delta\|_{\mathbf{W}_u(Q_T)}\left(\left\||\nabla(\psi_\delta+\lambda\zeta)|^{p[u]-1}\right\|_{p'[u],Q_T} + \|\chi\|_{p'[u],Q_T}\right)\\
&\leq C\|u-\psi_\delta\|_{\mathbf{W}_u(Q_T)}\left(1 + \|\chi\|_{p'[u],Q_T} + \int_{Q_T}|\nabla\psi_\delta|^{p[u]}\,dz + \int_{Q_T}|\lambda\nabla\zeta|^{p[u]}\,dz\right)\\
&\leq C\|u-\psi_\delta\|_{\mathbf{W}_u(Q_T)} \to 0 \quad \text{as } \delta\to 0,
\end{aligned}$$

while

$$J_2 \to -\lambda \int_{Q_T} \left(|\nabla(u+\lambda\zeta)|^{p[u]-2}\nabla(u+\lambda\zeta) - \chi \right) \cdot \nabla\zeta \, dz.$$

Hence,

$$\lambda \int_{Q_T} \left(|\nabla(u+\lambda\zeta)|^{p[u]-2}\nabla(u+\lambda\zeta) - \chi \right) \cdot \nabla\zeta \, dz \geq 0.$$

Simplifying and letting $\lambda \to 0^+$ we obtain the inequality

$$\int_{Q_T} \left(|\nabla u|^{p[u]-2}\nabla u - \chi \right) \cdot \nabla\zeta \, dz \geq 0 \qquad \forall \zeta \in C^\infty([0,T]; C_0^\infty(\Omega)).$$

Because of the density of smooth functions in $\mathbf{W}_u(Q_T)$, this inequality is possible only if

$$\int_{Q_T} \left(|\nabla u|^{p[u]-2}\nabla u - \chi \right) \cdot \nabla\phi \, dz = 0 \qquad \forall \phi \in \mathbf{W}_u(Q_T).$$

Returning to (4.15) and passing to the limit as $\epsilon \to 0$ we find that for every test-function $\phi \in \mathbf{W}_u(Q_T)$

$$\int_{Q_T} \left(u_t\phi + |\nabla u|^{p[u]-2}\nabla u \cdot \nabla\phi - f\phi \right) dz = 0.$$

Estimate (2.9) follows from (4.17) and the uniform estimates of Lemma 4.1.

5. Uniqueness of strong solutions

Theorem 5.1. *Problem (1.1) has at most one strong solution in the class of functions*

$$\mathcal{S} = \left\{ v : \; v \in C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; W_0^{1,2}(\Omega)), \; v_t \in L^2(Q_T) \right\}.$$

Proof. Let $u_i \in \mathcal{S}$ be two different strong solutions of problem (1.1). Notice that this set in not empty: according to Theorem 2.1 for every $u_0 \in W_0^{1,2}(\Omega)$ and $f \in L^{(p^-)'}(Q_T)$ problem (1.1) has at least one strong solution $u \in \mathcal{S}$. Let us denote

$$p_1 = p[u_1], \qquad p_2 = p[u_2].$$

The inclusions $u_i \in \mathcal{S}$ yield

$$u_i \in \mathbf{W}_{u_1}(Q_T) \cap \mathbf{W}_{u_2}(Q_T),$$

which allows one to take the function $u = u_1 - u_2$ for the test-function in the integral identities (4.8) for u_i. Combining these identities we arrive at the equality

$$\frac{1}{2}\|u(t)\|_{2,\Omega}^2 + \int_{Q_t} \left(|\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla u_2|^{p_2-2}\nabla u_2 \right) \cdot \nabla u \, dz = 0. \tag{5.1}$$

We will prove first that the strong solution is unique on a time interval $[0,T^*]$ with some T^* depending only on the data. Writing

$$\begin{aligned}(|\nabla u_1|^{p_1-2}\,\nabla u_1 - |\nabla u_2|^{p_2-2}\,\nabla u_2)\cdot\nabla u &= (|\nabla u_1|^{p_2-2}\,\nabla u_1 - |\nabla u_2|^{p_2-2}\,\nabla u_2)\cdot\nabla u \\ &\quad + (|\nabla u_1|^{p_1-2}\,\nabla u_1 - |\nabla u_1|^{p_2-2}\,\nabla u_1)\cdot\nabla u\end{aligned}$$

and using inequality (4.10) we transform (5.1) into the form

$$\frac{1}{2}\|u(t)\|_{2,\Omega}^2 + (p_2^- - 1)\int_{Q_t} \Lambda\,|\nabla u|^2 \, dz \leq I(t), \tag{5.2}$$

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where

$$\Lambda = (1 + |\nabla u_1|^{p_2} + |\nabla u_2|^{p_2})^{\frac{p_2-2}{p_2}},$$
$$I(t) = \int_{Q_t} (|\nabla u_1|^{p_2-2}\nabla u_1 - |\nabla u_1|^{p_1-2}\nabla u_1) \cdot \nabla u \, dz.$$

By Young's inequality

$$|I(t)| \le \delta \int_{Q_t} \Lambda |\nabla u|^2 \, dz + C(\delta)J(t) \tag{5.3}$$

with

$$J(t) = \int_{Q_t} \left| |\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla u_1|^{p_2-2}\nabla u_1 \right|^2 \Lambda^{-1} \, dz$$

and any $\delta > 0$. Plugging (5.3) into (5.2) and choosing δ appropriately small, we rewrite (5.3) in the form

$$\frac{1}{2}\|u(t)\|^2_{2,\Omega} + (p^- - 1 - \delta)\int_{Q_t} |\nabla u|^2 \Lambda \, dz \le C(\delta)J(t). \tag{5.4}$$

For every $q, r > 1$ and $\xi \in \mathbb{R}^d$, $|\xi| \neq 0$,

$$\left||\xi|^{q-2}\xi - |\xi|^{r-2}\xi\right| = \left|\left(|\xi|^{q-1} - |\xi|^{r-1}\right)\frac{\xi}{|\xi|}\right| \le \left||\xi|^{q-1} - |\xi|^{r-1}\right|\left|\frac{\xi}{|\xi|}\right| = \left||\xi|^{q-1} - |\xi|^{r-1}\right|.$$

By the Lagrange theorem there exists $\theta \in (0, 1)$ such that

$$\left||\xi|^{q-1} - |\xi|^{r-1}\right| = |\xi|^{\theta q + (1-\theta)r - 1}|\ln|\xi|||q - r|.$$

It follows that at every point $z \in Q_T$ either $|\nabla u_1| = 0$ and

$$\left||\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla u_1|^{p_2-2}\nabla u_1\right| = 0,$$

or $|\nabla u_1| \neq 0$ and

$$\left||\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla u_1|^{p_2-2}\nabla u_1\right| \le |\nabla u_1|^{p-1}|\ln|\nabla u_1|| \, |p_1 - p_2| \tag{5.5}$$

with $p = \theta p_1 + (1-\theta)p_2$, $\theta \in (0, 1)$. Recall that the exponents p_1, p_2 are independent of x. By Young's inequality, for a.e. $t \in (0, T)$

$$\|\Lambda^{-1}\|^{\frac{2}{2-p_2}}_{\frac{2}{2-p_2},\Omega} = \int_\Omega (1 + |\nabla u_1|^{p_2} + |\nabla u_2|^{p_2})^{\frac{2}{p_2}} \, dx \le C\int_\Omega (1 + |\nabla u_1|^2 + |\nabla u_2|^2) \, dx \le C'$$

with a constant C' depending on d, $p^\pm$ and the constant in (3.5). Using the classical Hölder's inequality and then (5.5) we obtain

$$\begin{aligned} J(t) &\le \int_0^t \left\|\left||\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla u_1|^{p_2-2}\nabla u_1\right|^2\right\|_{\frac{2}{p_2(t)},\Omega} \|\Lambda^{-1}\|_{\frac{2}{2-p_2(t)},\Omega} \, dt \\ &\le C\int_0^t |p_1 - p_2|^2 \left(\int_\Omega \left(|\nabla u_1|^{p-1}|\ln|\nabla u_1||\right)^{\frac{4}{p_2}} dx\right)^{\frac{p_2}{2}} dt \end{aligned} \tag{5.6}$$

with a constant $C = C(C', p^\pm)$ and the exponent $p = \theta p_1 + (1-\theta)p_2$ where $\theta = \theta(t) \in (0, 1)$. Set

$$\kappa = \frac{4(p-1)}{p_2} = (\theta p_1 + (1-\theta)p_2 - 1)\frac{4}{p_2}.$$

The assumption $p_i \le p^+ < 2$ yields the inequality

$$\kappa \le \frac{4(p_2-1)}{p_2} < 2 \quad \text{if } p_2 \ge p_1.$$

Let us also claim that

$$\kappa < 2 \text{ if } p_1 \ge p_2,$$

that is,

$$p_1 - p_2 < 2 - p^+ \le 2 - p_1 \quad \text{if } p_1 \ge p_2. \tag{5.7}$$

Condition (5.7) is surely fulfilled on a sufficiently small time interval $(0, T^*)$ with T^* defined through the data. Indeed: repeating the derivation of (4.16) we obtain the inequalities

$$|p_i(t) - p_i(\tau)| \le C'|t-\tau|^{\frac{1}{2}} \quad \forall\, t,\tau \in [0,T]$$

with a constant C' depending only on u_0, f and d. It follows that

$$|p_1(t) - p_2(t)| \le |p_1(t) - p[u_0]| + |p_2 - p[u_0]| \le 2C't^{\frac{1}{2}} < 2 - p^+ \quad \text{for } t < T^* = \left(\frac{2-p^+}{2C'}\right)^2.$$

We will use inequality (3.21) in the following form: if $\mu \in (0,1)$ is so small that $\kappa(1+\mu) \le 2$, then for every $\xi > 0$

$$(\xi|\ln\xi|)^\kappa = \begin{cases} \xi^{\kappa(1+\mu)}\left(\xi^{-\mu}\ln\xi\right)^\kappa & \text{if } \xi > 1, \\ \xi^{\kappa(1-\mu)}\left(\xi^{\mu}|\ln\xi|\right)^\kappa & \text{if } \xi \in (0,1] \end{cases} \le C(1+\xi^2)$$

with a constant $C = C(\mu)$. This inequality together with (3.9) implies that for a.e. $t \in (0,T^*)$

$$\left(\int_\Omega \left(|\nabla u_1|\left|\ln|\nabla u_1|^{\frac{1}{p-1}}\right|\right)^{\frac{4(p-1)}{p_2}} dx\right)^{\frac{p_2}{2}} \le C\left(1 + \int_\Omega |\nabla u_1|^2 dx\right)^{\frac{p^+}{2}} \le C,$$

whence

$$J(t) \le C\int_0^t |p_1 - p_2|^2\, dt, \quad t < T^*. \tag{5.8}$$

By Hölder's inequality and due to the assumption $\alpha \in [1,2]$

$$\begin{aligned} |p_2 - p_1| &\le C\|u_2 - u_1\|_{2,\Omega}\left(\int_\Omega \left(|u_2|^{2(\alpha-1)} + |u_1|^{2(\alpha-1)}\right) dx\right)^{\frac{1}{2}} \\ &\le C\|u_2 - u_1\|_{2,\Omega}\left(1 + \|u_2\|_{2,\Omega}^{2(\alpha-1)} + \|u_1\|_{2,\Omega}^{2(\alpha-1)}\right)^{\frac{1}{2}} \\ &\le C\|u_2 - u_1\|_{2,\Omega}, \qquad C = C(\alpha, u_0, f). \end{aligned}$$

It follows now from (5.4) and (5.8) that $u = u_2 - u_1$ satisfies the inequality

$$\|u(t)\|_{2,\Omega}^2 \le C\int_0^t |p_1 - p_2|^2\, dt \le C\int_0^t \|u(\tau)\|_{2,\Omega}^2\, d\tau, \quad t \in (0,T^*). \tag{5.9}$$

By the Gronwall lemma $\|u(t)\|_{2,\Omega}^2 = 0$ for $t \in [0,T^*)$, which means that $u_2(x,T^*/2) = u_1(x,T^*/2)$ in Ω. Let us take $T^*/2$ for the initial instant and consider problem (1.1) in the cylinder $\Omega \times (T^*/2, T)$. As is already shown, the condition $u_2(x,T^*/2) - u_1(x,T^*/2) = 0$ in Ω yields the equality $u_2 = u_1$ in $\Omega \times (T^*/2, 3T^*/2)$. Repeating these arguments, in a finite number of steps of the length T^* we will exhaust the interval $(0,T)$. The proof of Theorem 5.1 is completed. □

6. Final remarks

6.1. Local in time existence without assumptions on the range of $p[u]$

The arguments used in the proof of Theorem 2.1 allow one to prove local in time solvability of problem (1.1) in the case when condition (2.8) (d) is removed and substituted by an assumption on $\|u_0\|_{2,\Omega}$. Fix a small δ and consider problem (1.1) with the exponent

$$p_\delta[s] = \begin{cases} \frac{2d}{d+2} + \delta & \text{if } p[s] \leq \frac{2d}{d+2} + \delta, \\ p[s] & \text{if } \frac{2d}{d+2} + 2\delta < p[s] < 2 - 2\delta, \\ 2 - \delta & \text{if } p[s] \geq 2 - \delta. \end{cases}$$

One may choose $p_\delta \in C^1(\mathbb{R})$ with $\sup_{\mathbb{R}} |p'_\delta| = C(C_*, \delta)$ with C_* from condition (2.8) (c). By Theorems 2.1, 5.1 problem (1.1) with the nonlocal exponent $p_\delta[u]$ has a unique global in time strong solution which satisfies the analog of estimate (4.16):

$$|p[u(t)] - p[u(\tau)]| \leq C|t-\tau|^{\frac{1}{2}}, \quad 0 \leq \tau < t \leq T.$$

In particular,

$$p[u_0] - C\sqrt{t} \leq p[u(t)] \leq p[u_0] + C\sqrt{t}$$

with a constant C depending only on u_0, f and d. Let us assume that u_0 satisfies the inequality

$$\frac{2d}{d+2} + 3\delta < p[u_0] < 2 - 3\delta.$$

Then there exists an interval $(0, T_\delta)$ wherein for the constructed solution

$$\frac{2d}{d+2} + 2\delta < p_\delta[u(t)] < 2 - 2\delta.$$

It follows that in the cylinder $\Omega \times (0, T_\delta)$ the function u solves problem (1.1) with the exponent $p_\delta[u] \equiv p[u]$. An example of such a situation is furnished by the functional $p[u] = \frac{2d}{2+d} + \|u(t)\|^2_{2,\Omega}$. Assume that $p[u_0] < 2$. On the one hand, $p[u] > \frac{2d}{d+2}$, on the other hand, for the sufficiently small t

$$p[u(t)] = \frac{2d}{d+2} + \left(\|u_0\|^2_{2,\Omega} + \int_0^t \|f(\tau)\|_{2,\Omega}\right)^2 < 2$$

due to estimate (3.8).

6.2. Vanishing in a finite time

In this section, we study the property of extinction in finite of the strong solutions of problem (1.1). We use the energy method developed in [23, Ch.2]. Let us assume that $f \in L^\infty(0, T; L^{(p^-)'}(\Omega))$ and u is a strong solution corresponding to the initial function $u_0 \in W_0^{1,2}(\Omega)$. Since the strong solution can be taken for the test-function in (2.7), the following energy equality holds: for every $t, t+h \in (0, T)$

$$\frac{1}{h}\int_t^{t+h} \frac{1}{2}\frac{d}{dt}\left(\|u(s)\|^2_{2,\Omega}\right) ds + \frac{1}{h}\int_t^{t+h}\int_\Omega |\nabla u|^{p[u]} \, dz = \frac{1}{h}\int_t^{t+h}\int_\Omega uf \, dz.$$

By the Lebesgue differentiation theorem for a.e. $t \in (0, T)$

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2_{2,\Omega}\right) + \int_\Omega |\nabla u|^{p[u]} \, dx = \int_\Omega uf \, dx. \tag{6.1}$$

Let us assume first that $f \equiv 0$. In this case (6.1) yields the inequality

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2_{2,\Omega}\right) + \int_\Omega |\nabla u|^{p[u]} dx \le 0 \tag{6.2}$$

and it follows that $\|u(t)\|^2_{2,\Omega} \le \|u_0\|^2_{2,\Omega}$. Let us introduce the function

$$Y(t) = \frac{\|u(t)\|^2_{2,\Omega}}{\|u_0\|^2_{2,\Omega}} \le 1.$$

Let $\widetilde{C}$ be the constant from the embedding inequality

$$\|v\|^2_{2,\Omega} \le \widetilde{C}\|\nabla v\|^2_{p^-,\Omega}, \qquad v \in W_0^{1,p^-}(\Omega).$$

By the generalized Hölder inequality (2.1), for every $t \in (0,T)$

$$\begin{aligned} Y(t) = \frac{\|u(t)\|^2_{2,\Omega}}{\|u_0\|^2_{2,\Omega}} &\le \widetilde{C}\frac{\|\nabla u\|^2_{p^-,\Omega}}{\|u_0\|^2_{2,\Omega}} \\ &\le 4\widetilde{C}\max\left\{|\Omega|^{\frac{2}{(p^-)'}}, |\Omega|^{\frac{2}{(p^+)'}}\right\}\frac{\|\nabla u\|^2_{p[u],\Omega}}{\|u_0\|^2_{2,\Omega}} \\ &= \widehat{C}\frac{\|\nabla u\|^2_{p[u],\Omega}}{\|u_0\|^2_{2,\Omega}}, \qquad \widehat{C} = \widehat{C}(\widetilde{C}, |\Omega|, p^-). \end{aligned} \tag{6.3}$$

Inequality (6.2) can be written in the form

$$\|u_0\|^2_{2,\Omega} Y'(t) + a(t) Y^{\frac{p[u]}{2}}(t) \le 0, \quad Y(0) = 1, \tag{6.4}$$

with the coefficient

$$a(t) = 2\left(\frac{1}{\widehat{C}}\|u_0\|^2_{2,\Omega}\right)^{\frac{p[u]}{2}} \ge 2\min\left\{\left(\frac{1}{\widehat{C}}\|u_0\|^2_{2,\Omega}\right)^{\frac{p^+}{2}}, \left(\frac{1}{\widehat{C}}\|u_0\|^2_{2,\Omega}\right)^{\frac{p^-}{2}}\right\}.$$

Let us denote

$$\beta = \frac{2}{\|u_0\|^2_{2,\Omega}}\min\left\{\left(\frac{1}{\widehat{C}}\|u_0\|^2_{2,\Omega}\right)^{\frac{p^+}{2}}, \left(\frac{1}{\widehat{C}}\|u_0\|^2_{2,\Omega}\right)^{\frac{p^-}{2}}\right\}. \tag{6.5}$$

Since $Y \le 1$, from (6.4) we obtain

$$Y'(t) + \beta Y^{\frac{p^+}{2}}(t) \le 0, \quad Y(0) = 1.$$

The straightforward integration of the previous inequality over the interval $(0,t) \subset (0,t^*)$ gives

$$Y^{\frac{2-p^+}{2}}(t) \le 1 - \beta\frac{2-p^+}{2}t.$$

Since $Y(t) \ge 0$, it is necessary that

$$Y(t) \equiv 0 \quad \text{for all } t \ge t^* = \frac{2}{\beta(2-p^+)}. \tag{6.6}$$

These arguments prove the following assertion.

Lemma 6.1. *Let in the conditions of Theorem 2.1 $f \equiv 0$ in Q_T. Then every strong solution of problem (1.1) vanishes at a finite moment:*

$$u(x,t) = 0 \text{ in } Q_T \cap \{t \ge t^*\}, \qquad t^* = \frac{2}{\beta(2-p^+)}$$

with the constant β from (6.5).

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6.3. Vanishing at a prescribed moment

Let us assume now that $f \not\equiv 0$ in Q_T and there is $\epsilon > 0$ such that

$$\|f(t)\|_{2,\Omega}^{\frac{p^+}{p^+-1}} \leq \epsilon \left(1 - \frac{t}{t_f}\right)_+^{\frac{p^+}{2-p^+}}, \qquad v_+ = \max\{v, 0\}, \tag{6.7}$$

where $t_f > t^*$ and t^* is the constant defined in (6.6). Consider the function

$$Z(t) = \frac{\|u(t)\|_{2,\Omega}^2}{M^2}, \quad M^2 = \|u_0\|_{2,\Omega}^2 + \|f\|_{L^1(0,T;L^2(\Omega))}.$$

By virtue of (3.8) $Z(t) \leq 1$ in $[0,T]$ and $Z(0) = 1$. It follows from (6.1) that $Z(t)$ satisfies the inequality

$$M^2 Z'(t) + b(t) Z^{\frac{p^+}{2}}(t) \leq 2M \int_\Omega \left|\frac{u}{M}\right| |f| \, dx \leq 2M Z^{\frac{1}{2}}(t) \|f(t)\|_{2,\Omega}$$

with the coefficient

$$b(t) = 2\left(\frac{M^2}{\widehat{C}}\right)^{\frac{p[u]}{2}} \geq 2 \max\left\{ (\widehat{C}^{-1} M^2)^{\frac{p^+}{2}}, (\widehat{C}^{-1} M^2)^{\frac{p^-}{2}} \right\} =: \gamma.$$

By Young's inequality

$$Z'(t) + \frac{\gamma}{M^2} Z^{\frac{p^+}{2}}(t) \leq \frac{\gamma}{2M^2} Z^{\frac{p^+}{2}}(t) + L \|f(t)\|_{2,\Omega}^{\frac{p^+}{p^+-1}}, \qquad L = L(M, \gamma, p^+)$$

and, due to assumption (6.7),

$$Z'(t) + \frac{\gamma}{2M^2} Z^{\frac{p^+}{2}}(t) \leq \epsilon L \left(1 - \frac{t}{t_f}\right)_+^{\frac{p^+}{2-p^+}}. \tag{6.8}$$

Let us consider the function

$$Y(t) = \left(1 - \frac{t}{t_f}\right)_+^{\frac{2}{2-p^+}}.$$

It is straightforward to check that $Y(t)$ solves the problem

$$\begin{cases} Y'(t) + \frac{\gamma}{2M^2} Y^{\frac{p^+}{2}}(t) = \epsilon L \left(1 - \frac{t}{t_f}\right)_+^{\frac{p^+}{2-p^+}}, & t \in (0, t_f), \\ Y(0) = Z(0) = 1, \end{cases}$$

provided that

$$-\frac{2}{t_f(2-p^+)} + \frac{\gamma}{2M^2} = \epsilon L. \tag{6.9}$$

It is easy to see that $Y(t)$ is a majorant for $Z(t)$, i.e.,

$$M^2 \|u(t)\|_{2,\Omega}^2 \equiv Z(t) \leq \left(1 - \frac{t}{t_f}\right)_+^{\frac{2}{2-p^+}}. \tag{6.10}$$

Lemma 6.2. *Let us assume that $f \not\equiv 0$ in Q_T and conditions (6.7), (6.9) are fulfilled. Then every strong solution of problem (1.1) satisfies (6.10), i.e., vanishes at the moment $t_f > t^*$.*

Remark 6.1. The effect of vanishing at the prescribed moment t_f takes place if the data satisfy condition (6.9). This condition involves three parameters: ϵ — the "intensity" of the source f in Eq. (1.1), t_f — the moment when the source vanishes, and the L^2-norm of the initial datum. The assertion of Lemma 6.2 remains true if two of the three parameters are given, while the third one is chosen according to condition (6.9).

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