


Article

An Intervention Based on Identifying Topics That Students Have Difficulties with

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Abstract: Students who start university coming from secondary school commit some common conceptual errors. This paper presents a study developed in the subject of Numerical Methods taught in the second term of the Industrial Engineering Degrees of Oviedo University (Spain). The study was comprised of several activities, with two main objectives. Firstly, detecting over time the main deficiencies in mathematical knowledge presented by students in the first year of their university degrees and secondly, overcoming these shortcomings. The activities carried out involved preparing teaching materials to be uploaded to a shared folder and made available to the students of the subject, the preparation of a questionnaire and the performance of a task group by the students and then giving them feedback about the shortcomings identified. These activities have been well-received by the students and the results obtained have been positive. Our intention, for later courses, is to continue developing this work with some supplementary activities.

Keywords: conceptual errors in mathematics; secondary education; first year university students

MSC: 97N10; 97N40

1. Introduction

Several authors note that certain errors in mathematics subjects are made by a high percentage of students. This circumstance seems to be widespread not only in Spain [1], but also in most other countries [2–6] despite their different education systems. The reasons why students make mistakes in mathematics can be very diverse. But when these errors are very similar among very different students, coming from different schools and without having had any previous contact, the reason or reasons for these errors may be reduced to a short list.

It should be noted that Mathematical learning is not a process of incorporating data, rules, etc. into a blank mind, but involves a dialogue (implicit or explicit) between the students' previous knowledge and the new knowledge that the teacher is trying to teach them [7]. Since learning mathematics requires a dialogue between prior knowledge and new knowledge, the hypothesis that poorly-grounded knowledge can lead to 'common' mistakes among students gains strength. What can lead to poor prior knowledge?

One of the factors may be textbooks that are not very exacting or that themselves contain conceptual errors. When analyzing textbooks from different publishers, it was observed there was indeed a lack of rigor and, sometimes even errors in the conclusions reached, something which can lead to conceptual errors in students [8].

From a historical point of view, an important milestone in the transmission of knowledge was the appearance of the school book, which can be considered as a cultural element that reflects the social manipulation that selects some contents over others, imposes a certain way of structuring them and proposes certain types of problems to the next generation with some semiotic tools and not others [9,10]. The explosive mixture between the lack of rigor of some textbooks, the use of these as a resource for mainly expository methodologies, the haste in preparing students for the entrance exam to the University with a format that has been unchanged for a longer time than desired, one which favors the idea that mathematics are hermetic compartments classified by types of exercises, all mean that our students have not acquired consistency in the development of their mathematical competence [11].

The systematization of methods is something that can lead to neglect in the understanding of what is done, why it is done and for what purpose. The affirmation to the student that there is an automatic method for establishing a family of results, even if it is true, tends to unload the fundamental responsibility of the control of the intellectual work and blocks the transmission of the problem, which frequently makes the activity fail [12].

In Spain, primary school has three levels and it finishes when students are 16 years old. After the primary and compulsory schools, students can enroll in any medium-grade training cycles. In general, all those students who want to attend university go to a bachelor degree with two academic years. It is also possible for students to apply for occupational training that prepares students for specific jobs.

This two-year bachelor degree has three different specialties: Humanities and Social Sciences, Science and Technology and Art. Those who want to enroll in an engineering degree at university are supposed to have attended the bachelor degree in Science and Technology. It is remarkable that, during the last high school year, the pressure to pass a compulsory exam required in order to get access to university (EBAU), the broadness of the curriculum and the reduced teaching time can all lead teachers to systematize methods so as to speed up the acquisition of basic knowledge that will allow students to pass this exam.

Errors due to language difficulties [13], learning symbols and mathematical vocabulary are for many students a problem like those encountered when learning a foreign language. A lack of semantic understanding of mathematical texts is a source of errors. During secondary education, the use of mathematical symbols is not uniform among schools and teachers. There are teachers who use them but do not oblige students to do so in exams.

The errors due to poor learning of facts and skills include all deficiencies regarding content and specific procedures for carrying out a mathematical task [14]. In the studies carried out during the academic year 2017–18 [15], the following conclusion was reached:

“The students who are in the first year of the Engineering Degree come from high school with some deficiencies in conceptual knowledge of mathematical analysis. We have seen that these deficiencies improve after taking the subject of Calculus. In addition, no significant differences have been detected by the type of centre of origin, although they do depend on the gender”.

The role of technology in mathematics education has already been researched [16]. The potential of digital technologies has been known for at least 40 years ago [17,18]. Nowadays, information and communication technologies (ICT) allow students to learn mathematics with the practice of skills and the development of concepts [19]. Practical cases of use of ICT in mathematics teaching at university have shown that the use of computers can help students to focus on the learning of concepts without their achieving less in skills learned by hand [20].

A remarkable case is the growth in the availability of online applications for mathematical education that cover all levels, including university degrees [21,22]. These online applications can be accessed by students from mobile phones. It has been highlighted in previous studies that not only is the use of technology important [16], but also the design of how it will be employed, the role that teachers play in the use of the ICT and the educational context in which these technologies are implemented.

The use of virtual teaching platforms as support for learning mathematics is no longer such a novel issue. In some studies, it was shown that through the virtual forum, collaborative work and

autonomous learning skills are promoted [23]. In the case of the present research, the virtual campus platform was employed for students to upload their tasks and receive feedback about their content.

The present research follows the line started with a study [1] published in 2019 in which the subject of analysis were, on the one hand, 52 students from our Calculus course who were studying the degree of Computer Software engineering in the 2017–18 academic year and had recently entered the University from high school and, on the other hand, 300 students (including the previous ones) who were taking the same subject in different engineering degrees at our university.

The former responded to a knowledge questionnaire consisting of eight multiple choice questions on two occasions, at the beginning and the end of the course. The latter responded, anonymously, to a survey with the aim of discovering their opinion of mathematics in the last year of high school, their perception of the mathematical knowledge they have at the end of high school, and the difficulty of mathematics in the first year of the University. We highlighted that approximately half of the surveyed students considered themselves well-prepared and when they arrived at university, their perception was that math subjects were much more difficult than they thought.

One of the main conclusions that was reached is that the mathematics syllabus for the last year of high school is too broad, which motivates teachers to look for shortcuts that allow them to solve a wide variety of typical problems, but at the cost of rigour. This causes the student to focus on procedures rather than concepts.

This previous work has mainly helped the authors to find out some of the causes of the high number of failures that occur in first-year mathematics subjects in the different engineering degrees. Therefore, we considered that the research presented in that paper is more research work than it is innovation in the classroom, as no activities for improving were performed with the students.

Taking into account the previous experiences of the authors, the aim of the present work was the reinforcement of the basic mathematical concepts in students of engineering degrees by means of the detection of their conceptual errors. It was developed by a group of four secondary school and two university teachers working at the University of Oviedo and different high schools, all of them located in the Principality of Asturias (Spain).

In the work that we are now presenting, the students involved were taking, during the 2018–19 academic year, the subject of Numerical Methods during the 2018–19 academic year. This subject belongs to the second semester of the first year of the degree. These students had already taken Calculus and Algebra subjects in the first semester.

In fact, this research is linked to a previously-presented one [15]. Please note that the students who were the objects of the innovation are the same and the questionnaire carried out on the first day of class is the one referred to in our previous work [15]. The difference is that in the one published before, neither the innovation carried out with the students nor the team work performed was analysed, and nor was the information obtained from the survey carried out at the end of the course was analysed. In short, the previous research was carried out only with the data provided by the questionnaire.

Nowadays, some authors accept that many problems that high school students have in learning mathematics come from their attitude to changes in their mathematical register [24]. From Duval's point of view [24], we must be aware that students must decipher symbols and shapes in an expected and logical manner to get a core of mathematical knowledge. He also highlights that:

“Understanding in mathematics that first consists of recognizing the mathematical object represented and the possible transformations of their semiotic representation depend on their semiosis. As long as they cannot achieve this cognitive activity, students will have a block in their learning of mathematics that is interpreted as ‘incomprehension’”.

In our own opinion and in that of other authors, [25] we should not forget that mathematical language is simply an environment in which to effectively explore the potential of some forms and registers with which mathematics expresses itself, and that changes in the register can affect the performance of the problem-solving process.

These ideas are shared by other authors [25–27] who have shown that the teaching and learning of mathematics are favorably affected by an approach that considers the student’s social context [28]. A further motivation is given when teachers are able to find some cultural elements that link with the student [25,29–31]. Also, other researchers have found the importance of the historical development of mathematics in how possible cultural elements have to be considered.

Other authors [32,33] evidenced that history of mathematics certainly takes on a relevant role among the possible cultural elements to consider. Some of them even suggest that the history of mathematics can provide examples that would help students to understand new mathematical concepts [34].

2. Materials and Methods

The subject of Numerical Methods involved in this work is taught in the second term of the first year of the engineering degrees in Electrical Engineering, Electronics, Mechanics, Industrial Chemistry and Industrial Technologies. All are taught at the Polytechnic School of Engineering of Oviedo University, located in Gijón, Spain. The students of those degrees that belong to the groups who took part in this study, a total of 115, had already studied the subjects of Calculus and Algebra in the first term of that academic year.

The research methodology was performed during the academic year 2018–2019. The methodology consisted of four coordinated parts whose main objective was to improve the skills of students in the subject of Numerical Methods and to have an impact on a higher percentage of students passing the course. Therefore, our research question could be formulated as follows: ‘Is it possible to increase the knowledge of students of Numerical Methods and as a consequence to improve their marks in this subject by means of different tasks focused on reducing their conceptual errors about certain mathematical topics?’.

At the beginning of this research, the conceptual errors that a high percentage of students made in the Calculus and Algebra subjects were detected with the help of a level test. Afterwards, what we did in the Numerical Methods subject was to form voluntary groups of three or four students and propose that they carry out different tasks in which directly or indirectly the competences related to the questions of the questionnaire were worked on. At the end of the research an opinion and satisfaction questionnaire was filled out by students. The number of students involved in innovation, with greater or lesser participation, was 81 out of a total of 115 enrolled.

In order to clarify the research sample, Figure 1 shows a flowchart that explains it. A total of 115 individuals were enrolled in the subject of Numerical Methods, and therefore, would potentially take part in the study. A total of 89 of them took the level test and 81 of those that took the test level decided to participate in the activity’s groups. Of these, only 27 answered the anonymous survey.

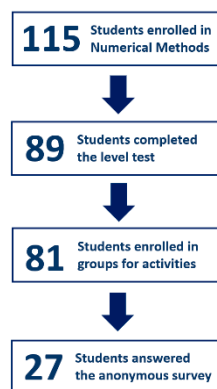


Figure 1. Flowchart sample composition.

The work plan developed was divided into four parts that are detailed below:

2.1. Part 1. Conceptual Errors Determination

In the first phase, in order to find out what conceptual errors the students have, the main task performed was the classification of the conceptual errors most likely to be made by students. According to the teachers' experience, they were related to the following topics:

- Logarithms and matrices operations.
- Relative and absolute extremes.
- Asymptotes of a real variable function.
- Application of functional analysis to the study of polynomial zeros.
- Curricular factors affecting the resolution of systems of linear equations.
- Types of discontinuities.
- Concerning the definite integral.
- On the concept of the limit of a function at a point.
- Concerning the number of zeros in a function.
- Solving inequalities with absolute value.
- On the growth or decline of a function.

In order to create this detailed list, a number of meetings were held and in addition some internal group documents were created. The work done by each member of the team was uploaded to a repository available in the Microsoft Teams (Redmond, WA, USA) platform with editing permissions for the whole group so that they could work collaboratively.

2.2. Part 2. Level Test Preparation

The team prepared a level test with 8 multiple-choice questions on basic mathematical concepts, which were answered by the students on the first day of class of the subject Numerical Methods (January 2019). Table 1 shows the questions proposed in this questionnaire. These questions do not cover all the topics listed in the previous section, but only some of them. Questions 1 and 7 are about logarithms and matrices operations. Question 2 concerns the growth or decline of a function, while question 3 covers the topic related to the number of zeros in a function. Question 4 is about relative and absolute extremes. Questions 5 and 6 are related to the definite integral and, finally, question 8 is about linear systems resolution. Please note that four of these questions, numbered as 1, 3, 4 and 5 were employed by authors in a previous study [1].

The results from a total of 89 students are available. These results allowed for an analysis of the degree of assimilation of the subjects of Calculus and Algebra by these students. Once the test was completed, the students received the material prepared by the team and took advantage of this to review the mathematical concepts that they will need most in the development of the subject

Table 1. Questions proposed in the multiple-choice level test.

Name and Surname:	
High School of origin in the academic year 2017–18:	
Have you passed the subject of Calculus?	Have you passed the subject of Algebra?
(1) $\log_2(1/4) =$	
(a) 2 (b) -2 (c) 1/2 (d) does not exist	
(2) The function $f(x) = 1/x$ as defined in $D = \{x \in R / x \neq 0\}$	
(a) is not strictly increasing or decreasing in D (b) is not injective in D	
(c) is strictly decreasing in D (d) is strictly increasing in D	
(3) The number of real roots in the equation $x^3 + x - 5 = 0$ is	
(a) 0 (b) 1 simple (c) 1 simple and 1 double (d) 3 simple	

Table 1. Cont.

-
- (4) Be $f(x) = x^2$ defined in the closed interval $[-2, 1]$. The absolute maximum of $f(x)$ is
 (a) 4 (b) -2 (c) 1 (d) there is no absolute maximum
- (5) If f is continuous in $[a, b]$, the area of the flat region bounded by the $y = f(x)$, the vertical lines $x = a$, $x = b$ and the abscissa axis, is given by:
 (a) $\int_a^b f(x) dx$ (b) $\left| \int_a^b f(x) dx \right|$ (c) $\int_a^b |f(x)| dx$ (d) none of the above
- (6) If f is continuous in $[a, b]$ and $\int_a^b f(x) dx > 0$ then $\forall x \in [a, b]$ it is verified:
 (a) $f(x) \geq 0$ (b) $f(x) > 0$ (c) $f(x) \neq 0$ (d) none of the above
- (7) There are A, B, C , three matrices such that $AB = AC$ When can it be said that $B = C$?
 (a) always (b) if all three matrices are square
 (c) if it is square with non-zero determinant (d) if B and C are square and reversible
- (8) There are $Ax = b$ a Cramer system, that is, a system of linear equations where the coefficients A is square and reversible Which of the following methods is generally more efficient at solving the system?
 (a) Gauss (b) Gauss-Jordan (c) Cramer (d) Obtain A^{-1}
- Note. The most efficient corresponds to the one that requires the least elementary operations.
-

2.3. Part 3. Activities Offered to Students and Teachers Feedback

Our Numerical Methods students that formed groups A and B of expository classes were divided into four groups for classroom practice (PA1 to PA4). Each student received seven one-hour sessions of this type of practice. It was proposed to form groups of three or four students belonging to the same classroom practice group, of whom one was the coordinator. Taking into account the previous requirements, groups were formed by students on their own. It was designed so that they would work as a team, both in the classroom practices and in the performance of tasks outside the classroom to be uploaded to the Virtual Campus. They were told that the formation of these groups was voluntary and that the tasks to be carried out should help them to be better-prepared for the exams and to get better marks at the end of the course. There were five topics addressed in the classroom practices and for each of them the students received theoretical material, solved exercises, and proposed exercises. Each group created was assigned a task for each of the five topics, selecting one or two exercises from those proposed. All students received feedback about all the tasks uploaded before the exam. This means that they had an accurate assessment regarding their knowledge in the subject of Numerical Methods. Please note that this feedback is the main innovation of the present research, as it has never before been done in this subject. The topics selected are described below.

2.3.1. Error Analysis

In this task students work with symbolic calculation, numerical calculation and measuring errors, including the concepts of absolute relative error. Principal sources of error, including truncation and rounding errors, are also included.

In the numerical analysis of a problem, it is fundamental to study the error between the approximate calculated solution and the real solution. It is desirable to be able to carry out an error analysis, that is, a determination of quantitative estimates or dimensions for the error of the solution. The three main sources of error, apart from human error, are: data, truncation and rounding errors. Each of these errors is related to a class concept of numerical calculation: conditioning, convergence, and stability. In this task, exercises that aim to obtain a higher, as fine as possible, level for absolute error and relative error due to rounding are proposed. Also, students analyze situations where it is possible to avoid committing a very frequent type of rounding error, the so-called cancellation error.

Please note that in the case of this activity, the following logarithm property is employed:

$$\log(x + 1) - \log(x) = \log\left(\frac{x + 1}{x}\right) \quad (1)$$

The property employed in this activity is also useful for solving question number one of the multiple-choice level test, although just knowing how the logarithm in a certain base a ($a > 0$, $a \neq 1$) is defined should be enough.

2.3.2. Numerical Solutions of Non-Linear Equations

In this task, students must find the number of zeros of $f(x) = \log(x) - \frac{x^2}{2} + 2$, whilst also finding intervals that contain only one of them. Afterwards they are required to apply the secant method to find the value of zeros, expressing the result in terms of $\log(2)$ and taking as seeds $x_0 = 2$ and $x_1 = 4$.

For this task, the following concepts are applied: root separation, bisection method, secant method, convergence order, iterative fixed-point method with global and local convergence and Newton-Raphson's method.

It is also mandatory to review the theorems of Rolle and Bolzano. As an application of Rolle, we will try to obtain intervals separated from the real line, in such a way that in each of them the equation $f(x) = 0$ has, at most, a real root. For each of these intervals, it will be checked whether or not the hypotheses of the Bolzano theorem are fulfilled. If in one interval (of monotony) they are not fulfilled, we may deduce that it does not contain any roots. In each interval where these hypotheses are fulfilled, we have one and only one root and we would try to find a smaller interval that still contains it. Please also note that in this task the main iterative methods for obtaining solutions to a non-linear equation are addressed. The fixed-point iteration method is employed and the concept of order of convergence is introduced. Please note that Newton's method is a case of the fixed-point iteration methods to ensure at least quadratic convergence.

From an operational point of view, in this task, it is also necessary to use the following logarithm property: $\log(4) = \log(2^2) = 2\log(2)$, which is related to the first question of the multiple-choice questionnaire.

Question number 3 of the multiple-choice questionnaire is also related to this activity as it can be solved taking into account Rolle's corollary.

2.3.3. Numerical Methods for Linear Systems

The problems posed make use of direct methods like the Gaussian method, the LU factorization, the Gaussian method with partial pivot and the Cholesky method. The concept of condition number is also introduced.

It has been estimated [35] that 75% of scientific problems involve solving a system of linear equations. Typical cases are the interpolation and approximation with linear families of functions and also the resolution of differential equations by methods in differences.

In this task, students first deal with the so-called direct methods, i.e., the methods which, except for rounding errors, lead us to the exact solution of the system by means of a finite number of elementary arithmetic operations. The fundamental method for direct solutions is that of Gauss and its variants. Pivoting is necessary in some situations, to make the Gaussian algorithm possible, but it is also advisable for the numerical stability of the method, that is, to minimize the inevitable rounding errors that occur when dividing. When the matrix of coefficients is symmetrical and defined positive, the number of operations of the Cholesky method is a little more than half that of the Gauss method, which makes it very efficient for this type of matrix.

Iterative methods: Convergence Jacobi's method and Gauss-Seidel method. Characterization of convergence using the eigenvalues of the iteration matrix. Iterative methods are generally better suited to "large" systems of equations. By using them, from an initial seed or vector a succession of vectors is generated that under certain conditions converges to the solution of the system. For the

study of convergence, it is fundamental to locate the values of the iteration matrix of the method, that is, the zeros of the so-called characteristic polynomial of this matrix.

2.3.4. Interpolation. Minimum-Square Adjustment

In this task, students are required to build the Lagrange interpolation polynomial of the function $f(x) = \frac{1}{x-1}$ for the nodes $x_0 = -1$, $x_1 = 0$ and $x_2 = 2$. Students are also required to write the divided differences table and express the polynomial in Newton's form. Finally, the interpolation error for $x = 0.9$ must be bounded.

This means that students deal with the following methods: Lagrange interpolation, Lagrange base polynomials, divided differences and error in the interpolation, Chebyshev nodes, split interpolation and least squares adjustment. In many problems of Physics and Mathematics, there are complicated functions whose manipulation and calculation can be difficult. Therefore, it seems natural to substitute these functions for others that are easier to use and that are "close" to the previous ones. There are very different criteria for approximating functions. Sometimes the new function is required to coincide with the given one in a certain set of values (interpolation); at other times the sum of the squares of the "deviations" is required to be as small as possible. In this task, different ways to obtain the Lagrange interpolation polynomial by solving a system of linear equations are proposed, using the Lagrange base polynomials, and also employing Newton's form using divided differences.

The curve of $f(x)$ is an equilateral hyperbola as the one in the question 2 of the multiple-choice test. Both functions present a lack of continuity in a point that does not belong to the function domain range. Therefore, this function cannot be derivate in the interval $(-1, 2)$ and the result employed to obtain an error estimation is not applicable. This activity should be enough for students to realize that function $g(x) = \frac{1}{x}$ defined in $\mathbb{R} - \{0\}$ is neither strictly increasing or decreasing in all its domain range, although it is decreasing in $(-\infty, 0)$ and $(0, +\infty)$.

2.3.5. Numerical Integration

Interpolation-type formulas. Newton-Côtes formulas. Trapezium formulas. Simpson's formula. Formula of the midpoint. Error calculation. Compound formulas.

The problem of numerically evaluating a defined integral occurs naturally when it is not possible to obtain it analytically, or when the function to be integrated is only known in a discrete set of points.

The methods we study in task 5 consist of interpolating the function by integrating in $n + 1$ points by means of a polynomial function, calculating the integral of this function. In this way we obtain the quadrature formulas of interpolation type. One of the functions we propose be integrated is $f(x) = \frac{x-1}{x^2+1}$ in the interval $[0, 3]$ using different formulae. This function goes from negative to positive values in this interval and therefore it is necessary to make use of the integral absolute value which links this activity with question number 5 of the multiple-choice test.

It is not advisable to use a large number of interpolation points for numerical integration. It is more practical to divide the integration interval into smaller intervals and apply a simple integration formula to each of them. This gives rise to the composite quadrature formulas.

In this activity students are also required to integrate $I = \int_{-1}^3 \frac{x}{x^2+1} dx$ using some properties of the definite integral to check that $I > 0$. This activity is linked to question 6, as $\frac{x}{x^2+1}$ is an odd function and the result of $\int_{-1}^1 \frac{x}{x^2+1} dx$ I equals zero. Please note that the function that is integrated takes positive and negative values in the integration interval and is equal to zero in one point inside the interval.

2.4. Part 4. Opinions and Satisfaction Survey

To discover how students perceive the subject and the work performed by the teachers and also what they think about their own mathematical knowledge, an opinions and satisfaction survey was prepared. The questions therein are presented in Table 2.

Table 2. Questions proposed in the opinions and satisfaction survey.

1. This is the first time I've been enrolled in the subject a. <input type="checkbox"/> yes b. <input type="checkbox"/> no, but I didn't prepare it well before c. <input type="checkbox"/> no						
2. I am receiving support classes ("tutoring") for this Numerical Methods course a. <input type="checkbox"/> never b. <input type="checkbox"/> rarely c. <input type="checkbox"/> occasionally d. <input type="checkbox"/> Often e. <input type="checkbox"/> very often						
3. I attend theory classes a. <input type="checkbox"/> never b. <input type="checkbox"/> rarely c. <input type="checkbox"/> occasionally d. <input type="checkbox"/> Often e. <input type="checkbox"/> very often						
4. I attend classroom practices a. <input type="checkbox"/> never b. <input type="checkbox"/> rarely c. <input type="checkbox"/> occasionally d. <input type="checkbox"/> Often e. <input type="checkbox"/> very often						
5. I have difficulty following the teacher's explanations in lectures and classroom practice a. <input type="checkbox"/> never b. <input type="checkbox"/> rarely c. <input type="checkbox"/> occasionally d. <input type="checkbox"/> Often e. <input type="checkbox"/> very often						
6. I think I have enough knowledge in math to take this subject a. <input type="checkbox"/> disagree b. <input type="checkbox"/> I'm starting to have my doubts c. <input type="checkbox"/> agree						
7. I have taken part in the Virtual Campus forums by actively participating in the task assigned to my group a. <input type="checkbox"/> never b. <input type="checkbox"/> rarely c. <input type="checkbox"/> occasionally d. <input type="checkbox"/> Often e. <input type="checkbox"/> very often						
8. The members of the group meet to carry out the task together and/or communicate by mail. a. <input type="checkbox"/> never b. <input type="checkbox"/> rarely c. <input type="checkbox"/> occasionally d. <input type="checkbox"/> Often e. <input type="checkbox"/> very often						
9. I followed the answers given by the teacher to the interventions of other colleagues in the forums a. <input type="checkbox"/> never b. <input type="checkbox"/> rarely c. <input type="checkbox"/> occasionally d. <input type="checkbox"/> Often e. <input type="checkbox"/> very often						
10. The interventions of my colleagues and the answers of the teacher in the forums have helped me to clarify doubts and to be better prepared for the exam. a. <input type="checkbox"/> disagree b. <input type="checkbox"/> I'm not entirely sure c. <input type="checkbox"/> agree						
		very unsatisfied	unsatisfied	satisfied	quite happy	very satisfied
11.	The degree of satisfaction with the work done by the teacher in the forums is	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
12.	The degree of satisfaction with the work done by the teacher in the lectures and classroom practices is	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
13.	The degree of satisfaction with the work done by the teacher in the laboratory practices is	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
14.	The general level of satisfaction with the work done by the teachers is	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

3. Results

As was previously mentioned, 89 students out of a total of 115 who enrolled in the subject completed the level test. 16 (18%) of the students interviewed were women and the other 73 (82%) men. As to the origin of these students, 29.2% had already been enrolled in university the previous year, 15.7% came from private or subsidised centres, and the other 55.1% from a public centre.

Of the total number of students surveyed, 37 (41.57%) had passed the subject of Calculus at that time, while 52 (58.43%) had failed it. In the subject of Algebra, 44 (49.44%) of the students had passed, while 45 (50.56%) had failed.

3.1. Results of the Students in the Level Test

Table 3 shows the results of the students in the proposed test, while Table 4 shows the option chosen for each one of these questions. The right answer for question 1 is given by 47.19% of students. This means that over half are unable to operate with logarithms, despite the fact that this concept is

studied in Spain in the 4th year of secondary school, which is equivalent to Year 10 in England or sophomore in United States. In question 2, too many students consider that the function $f(x) = 1/x$ is strictly decreasing in its domain. Note that f is strictly decreasing in each of the intervals $(-\infty, 0)$ and $(0, \infty)$. The error is to deduce that it is also strictly decreasing at the junction of both intervals; it is sufficient to verify that $f(-1)$ is less than $f(1)$ to realize that such statement is false.

Table 3. Results of the student in the level test.

	Right Answer		Wrong Answer		Not Answered	
	N	(%)	N	(%)	N	(%)
P1	42	47.19%	46	51.69%	1	1.12%
P2	38	42.70%	50	56.18%	1	1.12%
P3	12	13.48%	71	79.78%	6	6.74%
P4	37	41.57%	52	58.43%	0	0.00%
P5	30	33.71%	59	66.29%	0	0.00%
P6	18	20.22%	67	75.28%	4	4.49%
P7	27	30.34%	59	66.29%	3	3.37%
P8	23	25.84%	62	69.66%	4	4.49%

Table 4. Answers chosen to each one of the questions of the level test proposed. The right answer is marked in bold.

	a		b		c		d		Not Answered	
	N	(%)	N	(%)	N	(%)	N	(%)	N	(%)
P1	8	8.99%	42	47.19%	28	31.46%	10	11.24%	1	1.12%
P2	38	42.70%	8	8.99%	39	43.82%	3	3.37%	1	1.12%
P3	20	22.47%	12	13.48%	24	26.97%	27	30.34%	6	6.74%
P4	37	41.57%	27	30.34%	2	2.25%	23	25.84%	0	0.00%
P5	13	14.61%	36	40.45%	30	33.71%	10	11.24%	0	0.00%
P6	15	16.85%	31	34.83%	21	23.60%	18	20.22%	4	4.49%
P7	19	21.35%	18	20.22%	27	30.34%	22	24.72%	3	3.37%
P8	23	25.84%	32	35.96%	18	20.22%	12	13.48%	4	4.49%

In question 3, only 13.5% of the students chose the correct answer. In our opinion, students have problems in understanding the difference between real and complex irrational roots. Answering question 3 correctly supposes that the Bolzano and Rolle theorems must be understood, although it would be just enough to know the derivative of the function, and to take into account that every odd degree polynomial equation has, at least, a real root. Despite this, and considering the experience of the secondary school teachers consulted, many students consider this question to be a purely algebraic problem.

Question 4 with 41.57% obtained the second-highest rate of right answers. It deals with the topic of relative and absolute extremes. Most of the students that give the wrong answer (30.34%) do so because they give as the right answer the value of the point instead of the value reached by the function at such a point. In question 5, regarding the calculation of the area of a flat region, 40% of the students proposed that the area coincides with the absolute value of the integral of the function. The reason for this error is two-fold. On the one hand are the difficulties that students have in understanding the mathematical language, and on the other, the fact that the absolute value of a number is always a quantity greater than or equal to zero. It is not difficult to graph functions that change sign in an interval $[a, b]$ in such a way that the absolute value of the defined integral does not coincide with the area enclosed by the graph of the function and the abscissa axis.

In question 6, only 20% of the students chose the correct answer. One of the properties of the defined integral states that if the integrand function is positive in the interval $[a, b]$ then the

corresponding integral defines a positive real number. The majority of students' errors lie in their thinking that the opposite implication is also verified.

In the case of question 7, linked to the topic of matrices operations, 30.34% of students gave the right answer, while the percentages of the rest of options were all around 20%.

In question 8, the majority option is to choose the Gauss-Jordan method as the most efficient one to solve a Cramer system; however, the one that requires the fewest operations is the Gauss method. Students have worked on systems resolution using Cramer's rule and the Gauss method, although the latter is not usually explained in a systematic way at high school. The so-called inverse matrix method, which we believe should be ruled out from the beginning, and the Gauss-Jordan method are also mentioned in the textbooks.

The average number of right answers given by each of the students that took the test was 2.55. The average number of answers was 3 and the standard deviation was 1.51. None of the students was able to correctly answer all eight questions. Only two (2.25%) gave seven right answers.

It was found that there were no statistically-significant differences in the median of the total number of correct answers given by those who answered the test, taking into account whether they had either passed or failed the subject of Calculus ($p = 0.114$) when compared with the help of a Chi-squared test. In this case, both groups averaged 3 right answers in the test proposed. Statistically-significant differences were found in the median of the number of questions of those that had passed Algebra when compared with those that had failed ($p = 0.029$). Please note that those who had passed averaged 3 right answers and those that had not averaged 2. Finally, non-statistically significant differences were found neither by gender ($p = 0.503$) nor by the kind of centre ($p = 0.342$) where they had studied before arriving at university.

3.2. Results of Those Students That Took Part in the Working Groups

In view of the poor results obtained in the test and the fact that more than half of the students surveyed had failed the subject of Calculus in the first term, the first action was to clarify some concepts, fundamentally of Calculus, that the students had distorted. Afterwards, groups of three or four students were formed. A total of 81 students took part in this initiative. They were divided into 24 groups of 3 or 4 students each. Eight groups performed the 5 activities proposed. This group included 29 students (35.8% of the total), two groups with 7 students (8.64% of the total) performed 4 activities, five groups with a total of 16 students (19.75%) performed 3 activities, five groups including 15 students (18.52%) performed two activities, three groups with a total of 10 students (12.35%) performed one activity and only one group of 4 students (4.94%) did not perform any of the proposed activities. All the tasks were sent by the students through the Virtual Campus.

Table 5 shows the results obtained for those students that enrolled in any of the groups formed for activities according to the number of activities completed. As can be observed, the best results were obtained for those students who completed the five proposed activities, 63.33% of whom passed the subject and obtained the highest average mark (5.51), those who completed 4 activities had an average mark of 4.08 and so on. This trend is only broken for those students who completed only one activity and obtained an average mark of 4.92. Please note that this subset is formed by only two students and would be considered as an exception to the rule. Also, in the same table, it can be said that generally speaking, the lower the number of activities completed, the lower the percentage of students that took the exam. For example, those who made five activities took the exam in 89.66% of cases, while those who made four took the exam in a 57.14%, those who completed three activities took the exam in the 75% of cases. It is also noticeable that the number of activities completed are in line with the average marks of individuals.

In summary, it can be said that a total of 56 of the 81 (69.14%) students who participate in the activities took the exam. While from the other 34 that did not enrol in the groups for activities only 5 (14.71%) took the exam and only one (20%) passed, while in the case of those that took the activities 27 of 56 (48.21%) passed.

Table 5. Results obtained for those students that enrolled in any of the groups formed for activities according to the number of activities completed.

Activities Completed	Students		Took the Exam		Average Mark	Passed	
	(N)	(N)	(N)	(%)		(N)	(%)
5	30	26	26	86.67%	5.51	19	63.33%
4	7	4	4	57.14%	4.08	2	28.57%
3	16	12	12	75.00%	3.54	2	12.50%
2	15	9	9	60.00%	3.52	2	13.33%
1	10	5	5	50.00%	4.92	2	20.00%
0	3	0	0	0.00%	NA	0	0.00%
Total	81	56	56	69.14%	4.61	27	33.33%

The most important way to measure the results obtained by the students in the subject of Numerical Methods is by their performance. In this research, the results are assessed by means of two indicators. The first and most important of these is the percentage of students that passed the subject. In this case, the percentage was 33.33%, of those who enrolled in the groups for activities while in the previous academic year it was about 25%.

3.3. Results of the Opinions and Satisfaction Survey

The second indicator measures the degree of the students’ satisfaction with the work and methodology performed by the teachers. From our point of view, it was also of interest to find out what perception students had of the work performed. For this purpose, we carried out an anonymous survey with 14 questions, which is presented in Table 2. It was answered by only 27 students; the results are presented in Table 6. While the first 10 questions of the table shows the answers of students to those questions about the subject and their own knowledge, the answers to questions 11 to 14 show that the degree of satisfaction that they have with the work of teachers in various forms of education. Please note that results are mostly between 4 and 5 (quite satisfied and very satisfied).

Table 6. Questions proposed in the opinions and satisfaction survey about the students’ knowledge and the subject.

Question	Answers									
	a		b		c		d		e	
	N	%	N	%	N	%	N	%	N	%
1	23	85.2	3	11.1	1	3.7	-	-	-	-
2	21	77.8	2	7.4	0	0	4	14.8	0	0
3	0	0	0	0	0	0	3	11.1	24	88.9
4	0	0	0	0	1	3.7	1	3.7	25	92.6
5	4	14.8	14	51.9	8	29.6	1	3.7	0	0
6	0	0	9	33.3	18	66.7	-	-	-	-
7	1	3.7	1	3.7	4	14.8	12	44.4	9	33.3
8	1	3.7	1	3.7	2	7.4	4	14.8	19	70.4
9	2	7.4	3	11.1	3	11.1	9	33.3	10	37
10	1	3.7	5	18.5	21	77.8	-	-	-	-
11	0	0	1	3.7	5	18.5	9	33.4	12	44.5
12	0	0	1	3.7	4	14.8	14	51.8	8	29.7
13	2	7.4	3	11.1	5	18.5	5	18.5	12	44.5
14	0	0	0	0	9	33.4	14	51.8	4	14.8

4. Discussion

According to the results obtained, the percentage of students that passed the subject in the academic year 2018–19 increased by 32% over the previous year, rising from 25% to 33.33%. Please note that the students that passed the subject in the academic year 2015–2016 was 16%, 26% in 2016–2017 and 25% in 2017–2018. This data could indicate that the methodology described in this report has been satisfactory as the results of previous years improved. In any case, however, the teachers cannot be satisfied with the percentage of passes we attained in the subject.

Regarding the multiple-choice level test that is analysed in Section 3.1, it must be considered that in any multiple-choice exam with four options, students guessing at random will score 25%. Also, in many cases, and with very little knowledge, it is possible to disregard at least one of the options. This means that results under 33% are close to random answers. In the results in Table 4 the students have only scored more than 33% in 3 of the 8 questions, so the knowledge of the class on this test is close to zero which is quite a discouraging result, as three of the answers (numbers 1, 4 and 7) should be known when students finish secondary education while the others are taught in the subjects of Calculus and Algebra.

In a certain way, this work can be considered as a continuation of the one started by the authors in the academic year 2017–2018, although in that research only conceptual errors were detected and not innovation as is the case for the working groups whose results are detailed in 3.2 proposed. Some of the problems found by the team are common to those highlighted in other studies [36]. The concern about mathematical errors is not new and can be traced back at least to the beginning of the 20th century [3]. The approach to this problem proposed in this study, with the help of a test, is not new and has already been employed by other authors in countries as varying as Malaysia [2] and Germany [5]. In the case of the German study, it was also carried out for engineering studies at the beginning of their degrees. The author proposed 31 different items covering algebra, trigonometry, elementary functions, and analytical geometry in the plain. The percentage of right answers was about 50%.

With regard to the anonymous survey that was performed by only 27 students, and is analysed in Section 3.3 the following should be noticed:

- One third of the students surveyed doubt that they have sufficient level in mathematics to take the subject. This leads us to reaffirm the belief that the mathematics in the second year of high school is not profound enough to prepare students for a university degree.
- Almost a quarter of the students surveyed said that they had participated little or very little in the tasks assigned to their groups. This is not surprising, as we suspected that in some groups the level of dedication to the assigned task had been very uneven.
- About 30% of the students surveyed followed little or very little the responses given by the teacher to the contributions of other colleagues on the forums. This fact makes us think that the amount of information provided to students on the subject forums has been too great.
- 77% of the students surveyed said that the contributions of their classmates and the answers given by the teacher on the forums helped them to clear up their doubts and to be better prepared for the exams. In contrast, 3.7% think the opposite and 18.5% are not sure about it. We think that these answers are extremely relevant.
- Only one of the 27 students surveyed expressed little satisfaction with the work done by the teacher on the subject's forums and in the face-to-face classes.

We believe that the above results are an indication that the specific objectives sought were achieved to a high degree.

From the point of view of the authors, the results of this study must be analysed with caution, as only 27 students filled in the survey. This makes it a self-selecting sample which would be not be representative of the views of the class as a whole.

Finally, we would like to mention a highly controversial issue that from our point of view would help students to improve their mathematical abilities. It is the requirement of a bridging course before starting

the first semester at university. This proposal was already mentioned in previous research [1] and it was considered convenient for more than 50% of students. Currently, most universities in Spain do not offer a formal bridging course that covers either one or two semesters like in other countries, but we agree with a previous study [6] that the current heterogeneity of the student cohort entering third-level education makes it necessary to implement different entry pathways that in some cases would require a bridging course. Also, taking into account the good performance of those groups of students that performed all the proposed activities, from the point of view of authors it would be of interest to make that part the global mark of the subject. This idea will be considered for future academic years.

5. Conclusions

From the point of view of the authors, the main strengths of this work are that it has encouraged high school and university teachers to meet regularly to discuss mathematical concepts and how they are perceived by students. Please note that not only has it been possible to share impressions among teachers of both academic levels and with different degrees of experience, but the working group also analysed objective information collected in the tests proposed to the students.

Another strength of this research is that through the virtual forums on the Virtual Campus, the students' collaboration skills have been promoted, a transversal competence which is very interesting. This research has facilitated communication between the teacher and the students, as well as with each other.

Also, during a four-month period, 91 homework assignments were sent by students with the corresponding feedback messages from the teacher. This means that a continuous assessment of the students was performed, giving them accurate information about their knowledge of the subject.

After this first experience of collaborative work, the Project is likely to be continued in future courses. For future academic years the weak points found will be taken into account in order to be improved. One of the first weaknesses detected was that in the middle of the quarter, coinciding with the partial examination of the subject, a decrease in activity on the forums began to be noted. This was due to the heavy workload that students have from all the subjects, therefore activities and workload will be re-scheduled to prevent this.

It is also remarkable how few hours are assigned to classroom practice. This means that we have little time for face-to-face activities related to the project. This weakness cannot be remedied by the subject teachers as this is the official number of hours assigned in the Teaching Guide and approved by The National Agency for Quality Assessment and Accreditation of Spain (ANECA).

Furthermore, it was noticed that in some groups, there are students who take little or no part in carrying out the tasks. We feel that it would be of interest to introduce some motivational methodologies into the project. Finally, in our opinion, due to the large amount of information that students receive through the forums, they are not always able to assimilate everything they are told. The amount of information given will be reduced for the next academic years.

As a summary, the fundamental results of this research can be summarized in that 81 out of a total of 115 students enrolled in this subject (70.4%) performed the group activities. In general, it can be said that those students who performed a higher number of activities obtained a better mark. Although the percentage of students that fulfilled the opinions and satisfaction survey was low, 27 out of 81 (33.33%) students, from our point of view their opinions were highly satisfactory.

Finally, we would like to highlight that from our point of view, the study worked in a satisfactory way. In spite of this, in our opinion it would be of interest to perform certain improvements. First of all, we consider that it is possible to create a larger multiple-choice questionnaire that covers all the topics of the Calculus subject that should be known by students at the beginning of their university studies. A test of this kind would allow us to propose some personalized activities that would help the students to understand those concepts they have problems with. If we proceed in this way, it would be possible for students to start their second semester at university with better mathematical knowledge, which would mean obtaining better results in the subject of Numerical Methods.

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