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Universidad de Oviedo

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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics SēMA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the SēMA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Contents

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada
Gijón, 14-18 junio 2021
(pp. 177–183)

On the Amplitudes of Spherical Harmonics of Gravitational Potencial and Generalised Products of Inertia

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Abstract

The vector field of the force of gravitational attraction due to an extended rigid body (of arbitrary irregular geometrical shape, and with an arbitrary internal mass distribution inside it) at any point outside the body can be derived from the gradient of a scalar field, its gravitational potential. In terms of spherical polar coordinates (distance from the origin, colatitude or latitude, and longitude) that potential can be expanded as an absolutely convergent series of spherical harmonics, involving Legendre polynomials and associated Legendre functions of the first kind depending on the colatitude (or the latitude) and circular functions depending on the longitude.

In the present contributed paper we establish, in terms of the so–called "integrals of inertia" (or "generalised products of inertia") of the body, general formulae for the amplitudes (i.e., for the coefficients) of the different zonal, tesseral, and sectorial harmonics of any degree and order in the said series expansion of the gravitational potential outside the body.

Key words and expressions: Celestial Mechanics, Potential Theory, extended rigid body, gravitational potential, Legendre functions, spherical harmonics, inertia integrals (generalised products of inertia).

Mathematics Subject Classification (MSC) 2020: 70 F 15, 33 C 55, 42 C 10, 86 A 20.

1. Introduction: Theoretical Context and Scope

We consider the *usual model of three–dimensional space* $\mathbf{R}^3$, endowed with the well–known algebraic, geometric and topological structures of a linear, affine and Euclidean space over the field $\mathbf{R}$ of the real numbers.

We also consider a *rigid body* of arbitrary geometrical shape containing a mass distribution inside its volume. A simple mathematical model for this situation is provided by a *bounded, connected open subset* $\mathcal{D}$ in ordinary space $\mathbf{R}^3$, delimited by a closed and sufficiently smooth surface $\mathcal{S} = \partial\,\mathcal{D}$ (the *boundary* of $\mathcal{D}$). We further assume that this distribution of matter is characterised by an arbitrary scalar function of position describing the *local density of mass* at each point of the body (say, in a neighbourhood of the point); although in principle this function can be supposed to be bounded and Riemann–integrable over the volume of the body, for certain purposes it should be assumed to be of the class $C^{(1}\left(\mathcal{D} \subseteq \mathbf{R}^3, \mathbf{R}\right)$ over the said volume of the body.

In particular, this model provides us with a first approach to the study of the force field of gravitational attraction created by many celestial bodies (and, more specifically, the Earth).

It is a well–known fact in some branches of Space Technology and Mathematical and Physical Sciences (e. g., Vector Analysis, Potential Theory, Celestial Mechanics, Astrodynamics, Physical Geodesy, Geophysics) that the vector field corresponding to the gravitational force of attraction created by a mass distribution confined inside an open, bounded and connected set contained in ordinary, three dimensional space can be expressed in terms of the gradient of a single scalar function of position, known as the *(scalar) potential* of that vector field.

Moreover, if spherical polar coordinates (r, θ, λ) are chosen to analyse this issue, that scalar potential can be expressed in the form of an *absolutely convergent series of spherical harmonics* in which products of associated Legendre functions of the first kind (depending on $\cos\theta$, the cosine of the colatitude θ) and elementary circular functions (namely, cosine and sine functions) of integer multiples of the longitude λ are involved.

The so–called *integrals of inertia* (also known as *inertial integrals* or *inertia integrals*) were introduced as a generalisation of the triple integrals (taken over the whole volume of the body) that define the position vector of the centre of mass of the body and its moments and products of inertia. For this reason they are also called *generalised products of inertia*. Accordingly, the volume integrals defining both the centre of mass and the moments and products of inertia of the body are viewed as particular instances of inertia integrals.

The preceding statements and comments, as well as most of the theoretical background concerning this paper, can be documented in detail and justified with the help of some pertinent bibliographical references. For example (just to mention but a few of them), Brouwer and Clemence, [2], Chapter III, pp. 115–133; Cid and Ferrer, [3], Chapter 7, pp. 185–216, and Appendix B, pp. 443-479; Fitzpatrick, [4], Chapter 12, pp. 265–309; Heiskanen and

Moritz, [5], Chapter 1, pp. 1–45, and Chapter 2, §2.5–§2.6, pp. 57–63; MacMillan, [6], Chapter II, pp. 24–95, and Chapter VII, pp. 325–406; Roy, [7], Chapter 7, §7.5, pp. 201–206, and Chapter 11, §11.7, p. 342.

In the present paper we derive *general expressions, in terms of inertia integrals, for the coefficients of the diverse (zonal, tesseral, and sectorial) spherical harmonics* of any degree n and order k occurring in the series expansion of the gravitational potential.

2. Some Basic Concepts and Notations

• We consider the usual affine and Euclidean three–dimensional space $\mathbf{R}^3$, and *a fixed, inertial Cartesian reference frame* $O x_1 x_2 x_3$, or $O x y z$, with its origin at a point O in $\mathbf{R}^3$. This rectangular coordinate system is determined by the choice of point O and an ordered basis $\{ \mathbf{i}_1 , \mathbf{i}_2 , \mathbf{i}_3 \}$ that we also suppose orthonormal and positively oriented (right–handed or dextrorse basis) in Euclidean vector space $\mathbf{R}^3$. This spatial coordinate frame is also denoted $\{ O , \{ \mathbf{i}_1 , \mathbf{i}_2 , \mathbf{i}_3 \} \}$.

• Given a point P in $\mathbf{R}^3$, it **position vector** *with respect to this Cartesian reference frame* $O x y z$ is

$$\overrightarrow{OP} \equiv \mathbf{r} \equiv \mathbf{x} = x \mathbf{i}_1 + y \mathbf{i}_2 + z \mathbf{i}_3 \equiv (x , y , z)_{(\mathbf{i}_1 , \mathbf{i}_2 , \mathbf{i}_3)} \equiv (x , y , z) . \tag{2.1}$$

• Let $\mathcal{D} \subseteq \mathbf{R}^3$ be a bounded *domain* or bounded *region* (a connected open subset) in $\mathbf{R}^3$, delimited by a closed and smooth surface $\mathcal{S} = \partial \mathcal{D}$ (the boundary of $\mathcal{D}$), and $\overline{\mathcal{D}} = \mathcal{D} \cup \partial \mathcal{D}$ its topological closure.

• Let $Q \in \mathcal{D}$ be an arbitrary point in this domain, located in space by its position vector (relative to the above Cartesian coordinate system), $\overrightarrow{OQ} = \xi \mathbf{i}_1 + \eta \mathbf{i}_2 + \zeta \mathbf{i}_3 \equiv (\xi , \eta , \zeta)$. The Euclidean distance between Q and P, that is, the Euclidean norm of the position vector of P relative to Q, is

$$\overline{QP} = || \overrightarrow{QP} || = \sqrt{ (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 } . \tag{2.2}$$

• In what follows the **notations** for the position variables of the orthogonal curvilinear system of *spherical polar coordinates* will be (r , θ , λ), where $r = || \overrightarrow{OP} || = || \mathbf{r} ||$ stands for the radius vector of P (**Euclidean distance** of point P from the origin O of the coordinate system), θ designates the **colatitude** of P (that is, the polar angle of the radius vector, measured from the positive part of the $O z \equiv O x_3$ coordinate axis), and λ is the **longitude** of P (azimuthal angle –measured from the positive part of the $O x \equiv O x_1$ coordinate axis– that locates the plane that contains point P and is orthogonal to the coordinate plane $O x y \equiv O x_1 x_2$.

Accordingly, $r \geq 0$, i.e., $r \in [0 , + \infty) = \mathbf{R}_+ \cup \{ 0 \}$; $0 \leq \theta \leq \pi$, that is, $\theta \in [0 , \pi]$; and $0 \leq \lambda < 2 \pi$, or $\lambda \in [0 , 2 \pi)$.

• For any point $Q \in \mathcal{D}$. its position in space will be characterized by means of its Cartesian coordinates (ξ , η , ζ), related to its spherical polar coordinates $(\rho , \Theta , \Lambda)$ by means of the equations

$$\xi = \rho \sin \Theta \cos \Lambda , \quad \eta = \rho \sin \Theta \sin \Lambda , \quad \zeta = \rho \cos \Theta , \quad \text{with} \quad \xi^2 + \eta^2 + \zeta^2 = \rho^2 . \tag{2.3}$$

• In a similar way, let $P \in \mathbf{R}^3 \setminus \overline{\mathcal{D}}$ be an exterior point, with Cartesian and spherical polar coordinates (x , y , z) and (r , θ , λ), respectively,

$$x = r \sin \theta \cos \lambda , \quad y = r \sin \theta \sin \lambda , \quad z = r \cos \theta , \quad \text{and} \quad x^2 + y^2 + z^2 = r^2 . \tag{2.4}$$

• We consider a distribution of matter confined in the bounded domain $\mathcal{D}$. The Newtonian potential of the gravitational attraction created at point $P (x , y , z)$, at which a mass m_P is located outside of the body, by a systems of material points $Q (\xi , \eta , \zeta)$ contained in a domain $\mathcal{D}$, is given by ([2], Ch. III, §2, Eqs. (5)–(6), p. 117; [3], Ch. 7, §7.1, Eq. (7.1.2), p. 185, and §7.2, §§7.2.1, Eq. (7.2.6), p. 187; [4], Ch. 12, §12.4, Eq. (12.4.6), p. 290; [5], Ch. 1, §1.2, Eq. (1.11), pp. 3–4; [6], Ch. II, §20, Eq. (1), p. 24; [7], Ch. 7, §7.5, p. 202)

$$V (P) = \mathcal{G} m_P \int\int\int_{\mathcal{D}} \frac{ d m (Q) }{ || \overrightarrow{QP} || } = \mathcal{G} m_P \int\int\int_{\mathcal{D}} \frac{ \rho_{vol.} (Q) }{ || \overrightarrow{QP} || } \, d v (Q) , \tag{2.5}$$

where $\mathcal{G}$ is the *universal gravitational constant*, while $d m (Q)$ is the *differential element of mass* (or elementary mass) at point Q, and $|| \overrightarrow{QP} ||$ is the Euclidean *distance* between Q and P. In practice one takes $m_P = 1$, the unit mass. As for $\mathcal{G}$, its value in SI units is $\mathcal{G} \approx 6.67259 \times 10^{-11} \text{ N m}^2 / \text{kg}^2 = 6.67259 \times 10^{-11} \text{ m}^3 / \text{kg s}^2$. If $\rho_{vol.} (Q)$ is the *local density of mass* at point Q, and $d v (Q)$ the *differential element of volume* in the neighbourhood of Q, then the differential element of mass can be expressed as $d m (Q) = \rho_{vol.} (Q) \, d v (Q)$.

• This function V turns out to be *harmonic at points outside the domain* $\mathcal{D}$ (and consequently satisfies Laplace's equation $\Delta V = \nabla^2 V = 0$ outside $\mathcal{D}$), while it satisfies Poisson's equation $\Delta V = \nabla^2 V = - 4 \pi \mathcal{G} \rho$ in $\mathcal{D}$.

• Let n be a non–negative integer number, and k a non–negative integer between 0 and n, that is, $n \in \mathbf{N} \cup \{0\}$, and $k \in \{0, 1, 2, \cdots, n\}$. The *Legendre polynomial of degree* n in the independent variable t is denoted $P_n(t)$, while $P_n^k(t)$ will be the *associated Legendre function of the first kind of degree* n *and order* k. Note that for $k = 0$, $P_n^0(t) = P_n(t)$. For the purposes of the present paper, the scalar variable t will be taken as the *cosine function of the colatitude*.

• Within the framework of this theory of (surface) spherical harmonics, terms of the form $P_n(t)$ are called *zonal harmonics* of degree n. Terms $P_n^k(t)\cos(k\lambda)$ and $P_n^k(t)\sin(k\lambda)$ with $0 \neq k \neq n$ are *tesseral harmonics* of degree n and order k, while $P_n^n(t)\cos(n\lambda)$ and $P_n^n(t)\sin(n\lambda)$ are known as *sectorial harmonics* of degree n (and order n).

3. On the Series Expansion of the Gravitational Potential in Terms of Spherical Harmonics

• Taking $m_P = 1$, the gravitational potential given in Eq. (2.5) can be recast in the form ([3], Ch. 7, §7.6, §§7.6.1, p. 206, Eq. (7.6.4); [4], Chapter 12, §12.1, p. 275, Eqs. (12.1.23)–(12.1.24), and §12.2, p. 279, Eq. (12.2.5); [5] Ch. 2, §2.5, Eqs. (2.37)–(2.38) and (2.39)–(2.40), pp. 59–60, with notations as in Ch. 1, §1.13, Eqs. (1.67), p. 29; [7] Ch. 11 §11.7 p. 342)

$$\begin{aligned} V = \frac{\mathcal{G}M}{r}\Bigg\{ 1 + \frac{1}{M}\sum_{n=1}^{\infty} \int\!\!\int\!\!\int_{\mathcal{D}} \left(\frac{\rho}{r}\right)^n \Bigg[P_n(\cos\theta)\, P_n(\cos\Theta) \\ + 2\sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} P_n^k(\cos\theta) P_n^k(\cos\Theta) \cos\{k(\Lambda - \lambda)\}\Bigg] dm \Bigg\}, \end{aligned} \tag{3.1}$$

where M stands for the total mass of the distribution of matter contained in the domain $\mathcal{D}$.

• Introducing an auxiliary quantity $R = \sup\left\{\rho = \|\overrightarrow{OQ}\| = \text{distance}(O, Q) \,/\, Q \in \mathcal{D}\right\}$ ([3], Ch. 7, §7.6, §§7.6.1, p. 206; [4], Ch. 12, §12.1, p. 275), and defining the following *coefficientes* ([3], pp. 206–207, Eqs. (7.6.5); [5] Ch. 2, §2.5, Eqs. (2.38), p. 59, and Eqs. (2.40), p. 60),

$$J_n = -\frac{1}{M}\int\!\!\int\!\!\int_{\mathcal{D}} \left(\frac{\rho}{R}\right)^n P_n(\cos\Theta)\, dm , \tag{3.2}$$

$$C_n^k = -\frac{2}{M}\frac{(n-k)!}{(n+k)!}\int\!\!\int\!\!\int_{\mathcal{D}} \left(\frac{\rho}{R}\right)^n P_n^k(\cos\Theta)\cos k\Lambda\, dm , \tag{3.3}$$

$$S_n^k = -\frac{2}{M}\frac{(n-k)!}{(n+k)!}\int\!\!\int\!\!\int_{\mathcal{D}} \left(\frac{\rho}{R}\right)^n P_n^k(\cos\Theta)\sin k\Lambda\, dm , \tag{3.4}$$

the preceding potential (3.1) takes on the form ([3], Eq. (7.6.6)), p. 207; [5], Eqs. (2.39)–(2.40), pp. 59–60)

$$\begin{aligned} V &= \frac{\mathcal{G}M}{r}\left\{1 - \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n \left[J_n P_n(\cos\theta) + \sum_{k=1}^{n} P_n^k(\cos\theta)\left(C_n^k\cos k\lambda + S_n^k \sin k\lambda\right)\right]\right\} \\ &= \frac{\mathcal{G}M}{r}\left\{1 - \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n \left[J_n P_n(\cos\theta) + \sum_{k=1}^{n}\left(C_n^k P_n^k(\cos\theta)\cos k\lambda + S_n^k P_n^k(\cos\theta)\sin k\lambda\right)\right]\right\} \\ &= \frac{\mathcal{G}M}{r}\Bigg\{1 - \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n \Bigg[J_n P_n(\cos\theta) \\ &\qquad + \sum_{k=1}^{n}\left(C_n^k\left\{P_n^k(\cos\theta)\cos k\lambda\right\} + S_n^k\left\{P_n^k(\cos\theta)\sin k\lambda\right\}\right)\Bigg]\Bigg\} \\ &= \frac{\mathcal{G}M}{r}\left\{1 - \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n \left[J_n P_n(\cos\theta) + \sum_{k=1}^{n}\left(C_n^k\, \mathcal{C}_n^k(\theta,\lambda) + S_n^k\, \mathcal{S}_n^k(\theta,\lambda)\right)\right]\right\}, \end{aligned} \tag{3.5}$$

where the **notations** $\mathcal{C}_n^k(\theta, \lambda)$ and $\mathcal{S}_n^k(\theta, \lambda)$ represent the *surface spherical harmonics*, namely

$$\mathcal{C}_n^k(\theta,\lambda) = P_n^k(\cos\theta)\cos k\lambda , \qquad \mathcal{S}_n^k(\theta,\lambda) = P_n^k(\cos\theta)\sin k\lambda . \tag{3.6}$$

• The above constants J_n, C_n^k, and S_n^k, introduced in Eqs. (3.2)–(3.4), are *measures of the amplitudes of the various harmonics* $\mathcal{C}_n^0(\theta,\lambda) = P_n(\cos\theta)$, $\mathcal{C}_n^k(\theta,\lambda)$, and $\mathcal{S}_n^k(\theta,\lambda)$, respectively ([7], §11.7, p. 342).

♣ In what follows we will propose general expressions for the adimensional coefficients (3.2), (3.3), and (3.4), in terms of inertia integrals (4.5) of the body (see below).

4. Mathematical Formulae and Results Invoked in the Derivation of General Expressions for the Amplitudes

In this Section we collect some formulae and results to which we resort in our considerations and developments leading to the construction of general expression for the coefficients of the spherical harmonics of the gravitational potential.

4.1. Legendre Functions of the First Kind

• To calculate those coefficients, we will start from the following algebraic expression of the *associated Legendre function of the first kind of degree n and order k* ([3], App. B, §B.2, §§B.2.1, p. 453; [5], Ch. 1, §1.11, Eq. (1.62), p. 24; [6], Ch. VII, §197, Eq. (8), p. 370),

$$P_n^k(t) = \frac{\left(1 - t^2\right)^{k/2}}{2^n} \sum_{j=0}^{s} (-1)^j \frac{(2n - 2j)!}{j!\,(n-j)!\,(n-k-2j)!}\; t^{n-k-2j} , \tag{4.1}$$

where s is the greatest integer number $\leq (n-k)/2$; i.e., $s = (n-k)/2$ or $s = (n-k-1)/2$, whichever is an integer. That is, $s = (n-k)/2$ or $s = (n-k-1)/2$ according as $n-k$ is even or odd. In other words, number s is the *integer part* of $(n-k)/2$.

• Taking $t = \cos\Theta$ allows us to rewrite the preceding algebraic expression (4.1) in the *trigonometric form*

$$P_n^k(\cos\Theta) = \frac{\sin^k\Theta}{2^n} \sum_{j=0}^{s} (-1)^j \frac{(2n-2j)!}{j!\,(n-j)!\,(n-k-2j)!}\; (\cos\Theta)^{n-k-2j}\,\Theta , \tag{4.2}$$

• In particular, when $k = 0$, the *Legendre polynomial of degree n* ([3], App. B, §B.2, §§B.2.1 Eq. (B.2.6), p. 452;) reads

$$P_n(t) = \frac{1}{2^n} \sum_{\ell=0}^{r} (-1)^\ell \frac{(2n-2\ell)!}{\ell!\,(n-\ell)!\,(n-2\ell)!}\; t^{n-2\ell} , \tag{4.3}$$

$$P_n(\cos\Theta) = \frac{1}{2^n} \sum_{\ell=0}^{r} (-1)^\ell \frac{(2n-2\ell)!}{\ell!\,(n-\ell)!\,(n-2\ell)!}\; (\cos\Theta)^{n-2\ell} , \tag{4.4}$$

where r is the integer part of $n/2$.

4.2. Integrals of Inertia of a Body

• The general definition of this concept (also known as *inertial integrals*, *integrals of inertia*, or *generalised products of inertia* of the body) obeys the formula ([4], Ch. 12, §12.4, p. 293; [6], Ch. II, §50, p. 89)

$$I_{i,j,k} = I_{ijk} = \int\!\!\int\!\!\int_{\mathcal{D}} \xi^i \eta^j \zeta^k \, dm , \quad \text{where } i, j, k\ i \text{ are non–negative integers.} \tag{4.5}$$

For a given non–negative integer number n, integrals $I_{i,j,k}$, with i, j, k non–negative integers such that $i + j + k = n$, are also called *moments of orden n*.

• When dealing with spherical harmonics of degree n we will furthermore consider that $i + j + k = n =$ degree of the harmonic.

• Some authors use these integrals only in the case of harmonics of low degree ([2], Ch. III, §6, p. 126 for $n = 3$, and §7, p 126, for $n = 4$; [7], Ch. 7, §7.5, p. 204 for $n = 3$).

• Obviously ([4], p. 293), $I_{0,0,0} = M =$ total mass of the body. Fitzpatrick ([4], Ch. 12, §12.7, Eqs. (12.7.1)–(12.7.4), pp. 306–307) gives explicit expressions for the terms of the gravitational potential up to degree 3, in terms of spherical coordinates, with the amplitudes of the spherical harmonics, namely coefficients

$$J_1,\ C_1^1,\ S_1^1,\ J_2,\ C_2^1,\ S_2^1,\ C_2^2,\ S_2^2,\ J_3,\ C_3^1,\ S_3^1,\ C_3^2,\ S_3^2,\ C_3^3,\ S_3^3,$$

represented in terms of inertia integrals as linear combinations of the said integrals up to order three.

♣ Here we shall establish that the coeficientes J_n, C_n^k, and S_n^k of the harmonics of degree n depend on linear combinations of integrals $I_{i,j,k}$ with $i + j + k = n =$ degree of the harmonic.

4.3. Multiple–angle Formulae

According to Weisstein [9], for a positive integer k,

$$\sin(k\,\alpha) \;=\; \sum_{p=0}^{h} (-1)^p \binom{k}{2p+1} \sin^{2p+1}\alpha\, \cos^{k-2p-1}\alpha\ , \qquad (4.6)$$

$$\cos(k\,\alpha) \;=\; \sum_{p=0}^{H} (-1)^p \binom{k}{2p} \sin^{2p}\alpha\, \cos^{k-2p}\alpha\ , \qquad (4.7)$$

where h is the integer part of $(k-1)/2$, and H denotes the integer part of $k/2$.

4.4. The Multinomial Theorem

• The *Multinomial Theorem* (attributed to Johann Bernoulli and Leibniz) is a generalisation of Newton´s Binomial Theorem that provides us with a formula for the non–negative entire powers of a polynomial (say, multinomial) expression. Let m be a positive integer number, and n a non–negative integer.

• Consider a multinomial expression $(a_1 + a_2 + \cdots + a_m)$ with m terms (m monomials). Then, from Abramowitz and Stegun ([1], Ch. 24, §24.1, §§24.1.2, §§§24.1.2.I, p. 823), and Weisstein [8],

$$(a_1 + a_2 + \cdots + a_m)^n = \left(\sum_{i=1}^{m} a_i\right)^n$$

$$= \sum_{n_1+n_2+\cdots+n_m=n} \frac{n!}{n_1!\,n_2!\cdots n_m!}\, a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m}$$

$$= \sum_{n_1+n_2+\cdots+n_m=n} \binom{n}{n_1, n_2, \cdots, n_m} \prod_{i=1}^{m} a_i^{n_i}\ , \qquad (4.8)$$

where the sum of the (non–negative) exponents $n_i \in \mathbf{N} \cup \{0\}$ is n: $\sum_{i=1}^{m} n_i = n$. Note that the sum is taken over all combinations of non–negative integers $n_1, n_2, \cdots, n_m$ such that $n_1 + n_2 + \cdots + n_m = n$.

• The *multinomial coefficients* (or *multinomial numbers*) are

$$\binom{n}{n_1, n_2, \cdots, n_m} = \frac{n!}{n_1!\,n_2!\cdots n_m!}\ . \qquad (4.9)$$

• The *number of monomials in the above sums* is

$$\frac{(n+m-1)!}{n!\,(m-1)!}\ . \qquad (4.10)$$

• In particular we are interested in the special case of the multinomial formula for $m = 3$. More specifically, the *trinomial expansion* of $\rho^2 = \xi^2 + \eta^2 + \zeta^2$ (MacMillan [6], Ch. VII, §204, p. 383), namely

$$\rho^{2\ell} = \left(\rho^2\right)^{\ell} = \left(\xi^2 + \eta^2 + \zeta^2\right)^{\ell} = \sum_{\ell_1+\ell_2+\ell_3=\ell} \frac{\ell!}{\ell_1!\,\ell_2!\,\ell_3!}\, \xi^{2\ell_1} \eta^{2\ell_2} \zeta^{2\ell_3}\ , \qquad (4.11)$$

$\ell_j \geq 0$ integer numbers. The number of terms of an expanded trinomial is

$$\frac{(\ell+3-1)!}{\ell!\,(3-1)!} = \frac{(\ell+2)!}{\ell!\,2!} = \frac{(\ell+2)\,(\ell+1)}{2}\ , \qquad (4.12)$$

where ℓ is the exponent to which the trinomial is raised.

5. Formulae for the Amplitudes of Spherical Harmonics in terms of Inertia Integrals

Theorem 5.1 *Coefficients of zonal harmonics. Let r be the integer part of $n/2$. Then*

$$J_n = -\frac{1}{2^n M R^n} \sum_{\ell=0}^{r} (-1)^{\ell} \frac{(2n-2\ell)!}{(n-\ell)!\,(n-2\ell)!} \left(\sum_{\ell_1+\ell_2+\ell_3=\ell} \frac{1}{\ell_1!\,\ell_2!\,\ell_3!}\, I_{2\ell_1, 2\ell_2, n-2\ell_1-2\ell_2} \right). \qquad (5.1)$$

Proof From the definition (3.2) and the trigonometric form (4.4) of the Legendre polynomial of degree n,

$$\begin{aligned} J_n &= -\frac{1}{M}\int\int\int_{\mathcal{D}}\left(\frac{\rho}{R}\right)^n P_n(\cos\Theta)\,dm \\ &= -\frac{1}{MR^n}\frac{1}{2^n}\int\int\int_{\mathcal{D}}\rho^n\sum_{\ell=0}^{r}\frac{(-1)^\ell(2n-2\ell)!}{\ell!(n-\ell)!(n-2\ell)!}(\cos\Theta)^{n-2\ell}\,dm \\ &= -\frac{1}{2^nMR^n}\int\int\int_{\mathcal{D}}\sum_{\ell=0}^{r}\frac{(-1)^\ell(2n-2\ell)!}{\ell!(n-\ell)!(n-2\ell)!}\rho^n\cos^{n-2\ell}\Theta\,dm \\ &= -\frac{1}{2^nMR^n}\int\int\int_{\mathcal{D}}\mathcal{I}_{J_n}\,dm = -\frac{1}{2^nMR^n}\mathbf{I}_{J_n}. \end{aligned}$$

The integrand $\mathcal{I}_{J_n}$ of $\mathbf{I}_{J_n}$ will be treated in the following way,

$$\begin{aligned} \mathcal{I}_{J_n} &= \sum_{\ell=0}^{r}\frac{(-1)^\ell(2n-2\ell)!}{\ell!(n-\ell)!(n-2\ell)!}(\rho\cos\Theta)^{n-2\ell}\left(\rho^2\right)^\ell \\ &= \sum_{\ell=0}^{r}\frac{(-1)^\ell(2n-2\ell)!}{\ell!(n-\ell)!(n-2\ell)!}\zeta^{n-2\ell}\left(\xi^2+\eta^2+\zeta^2\right)^\ell \\ &= \sum_{\ell=0}^{r}\frac{(-1)^\ell(2n-2\ell)!}{\ell!(n-\ell)!(n-2\ell)!}\zeta^{n-2\ell}\left(\sum_{\ell_1+\ell_2+\ell_3=\ell}\frac{\ell!}{\ell_1!\ell_2!\ell_3!}\xi^{2\ell_1}\eta^{2\ell_2}\zeta^{2\ell_3}\right) \\ &= \sum_{\ell=0}^{r}\frac{(-1)^\ell(2n-2\ell)!}{(n-\ell)!(n-2\ell)!}\left(\sum_{\ell_1+\ell_2+\ell_3=\ell}\frac{1}{\ell_1!\ell_2!\ell_3!}\xi^{2\ell_1}\eta^{2\ell_2}\zeta^{n-2\ell+2\ell_3}\right) \\ &= \sum_{\ell=0}^{r}\frac{(-1)^\ell(2n-2\ell)!}{(n-\ell)!(n-2\ell)!}\left(\sum_{\ell_1+\ell_2+\ell_3=\ell}\frac{1}{\ell_1!\ell_2!\ell_3!}\xi^{2\ell_1}\eta^{2\ell_2}\zeta^{n-2\ell_1-2\ell_2}\right). \end{aligned}$$

Note that $n-2\ell+2\ell_3 = n-2\ell_1-2\ell_2-2\ell_3+2\ell_3 = n-2\ell_1-2\ell_2$. And integral $\mathbf{I}_{J_n}$ reads

$$\begin{aligned} \mathbf{I}_{J_n} &= \int\int\int_{\mathcal{D}}\sum_{\ell=0}^{r}\frac{(-1)^\ell(2n-2\ell)!}{\ell!(n-\ell)!(n-2\ell)!}\rho^n\cos^{n-2\ell}\Theta\,dm \\ &= \sum_{\ell=0}^{r}\frac{(-1)^\ell(2n-2\ell)!}{(n-\ell)!(n-2\ell)!}\left(\sum_{\ell_1+\ell_2+\ell_3=\ell}\frac{1}{\ell_1!\ell_2!\ell_3!}I_{2\ell_1,2\ell_2,n-2\ell_1-2\ell_2}\right), \end{aligned}$$

from which (5.1) follows.

□

Theorem 5.2 *Coefficients of tesseral and sectorial harmonics of the* $C_n^k(\theta,\lambda)$ *type*:

$$\begin{aligned} C_n^k = -\frac{2}{2^nMR^n}\frac{(n-k)!}{(n+k)!}&\left(\sum_{j=0}^{s}(-1)^j\frac{(2n-2j)!}{(n-j)!(n-k-2j)!}\right. \\ &\left.\times\left[\sum_{\ell=0}^{p}(-1)^\ell\binom{k}{2\ell}\left\{\sum_{j_1+j_2+j_3=j}\frac{1}{j_1!j_2!j_3!}I_{k-2\ell+2j_1,2\ell+2j_2,n-k-2j_1-2j_2}\right\}\right]\right), \end{aligned} \tag{5.2}$$

with s = *integer part of* $(n-k)/2$, *and* p = *integer part of* $k/2$.

Theorem 5.3 *Coefficients of tesseral and sectorial harmonics of the* $S_n^k(\theta, \lambda)$ *type*:

$$S_n^k = -\frac{2}{2^n M R^n} \frac{(n-k)!}{(n+k)!} \left(\sum_{j=0}^{s} (-1)^j \frac{(2n-2j)!}{(n-j)!\,(n-k-2j)!} \left[\sum_{\ell=0}^{q} (-1)^\ell \binom{k}{2\ell+1} \left\{ \sum_{j_1+j_2+j_3=j} \frac{1}{j_1!\, j_2!\, j_3!} I_{k-2\ell+2j_1-1,\, 2\ell+2j_2+1,\, n-k-2j_1-2j_2} \right\} \right] \right), \tag{5.3}$$

where s = *integer part of* $(n-k)/2$, *and* q = *integer part of* $(k-1)/2$.

Remark 5.4 The proof of these last theorems follows the approach and treatment of the case of the coefficients of zonal harmonics.

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