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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Weakly nonlinear analysis of a system with nonlocal diffusion

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Abstract

We study, through a weakly nonlinear analysis, the pattern formation for a system of partial differential equations of the Shigesada-Kawasaki-Teramoto type with nonlocal diffusion in the one-space dimensional case with periodic boundary conditions. We obtain the pattern of the solutions for values of the bifurcation parameter in the proximity of the onset of instabilities. Finally, we compare the results of the nonlocal model with those of the usual local diffusion model.

1. Introduction

Let $T > 0$. We consider the following nonlocal diffusion problem. Find $u_i : [0,T]\times\mathbb{R} \to \mathbb{R}_+$, for $i = 1,2$, such that

$$\partial_t u_i(t,x) = \int_{\mathbb{R}} J(x-y)\big(p_i(\mathbf{u}(t,y)) - p_i(\mathbf{u}(t,x))\big)dy + \gamma f_i(\mathbf{u}(t,x)), \tag{1.1}$$

$$u_i(0,x) = u_{0i}(x), \tag{1.2}$$

for $(t,x) \in Q_T = [0,T]\times\mathbb{R}$, and for some $u_{0i} : \mathbb{R} \to \mathbb{R}_+$, periodic functions of period L. The diffusion kernel, $J : \mathbb{R} \to \mathbb{R}_+$, is an even function with compact support included in $(-L/2, L/2)$. We assume $J \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, and $u_{0i} \in L^\infty(0,L) \cap BV(0,L)$. Here, $\mathbb{R}_+ = [0,\infty)$, $\mathbf{u} = (u_1, u_2)$, and, for $i, j = 1,2$, $i \neq j$, the diffusion and reaction functions are given by

$$p_i(\mathbf{u}) = u_i(c_i + a_i u_i + u_j), \quad f_i(\mathbf{u}) = u_i\big(\alpha_i - (\beta_{i1}u_1 + \beta_{i2}u_2)\big), \tag{1.3}$$

for some non-negative constant coefficients $\gamma,\ c_i,\ a_i,\ \alpha_i,\ \beta_{ij}$.

Problem (1.1)-(2.1) is a nonlocal version of the clasical Shigesada-Kawasaki-Teramoto (SKT) population model, see [4]. For the relationship between local and nonlocal diffusion models, see the monograph by Andreu et al. [1]

2. Existence and uniqueness of solution

Theorem 2.1 *Assume the above conditions and $a_i + \beta_{ii} > 0$ for $i = 1,2$. Then, there exists a unique strong solution (u_1, u_2) of problem (1.1)-(1.2) with $u_i \geq 0$ a. e. in Q_T and such that, for $i = 1,2$ and $t \in [0,T]$,*

$$u_i \in W^{1,\infty}(0,T; L^\infty(\mathbb{R})) \cap C([0,T]; L^\infty(\mathbb{R})) \cap BV_{loc}(\mathbb{R}))$$

Proof If u_i are periodic funtions, then the right hand side of equation (1.1) is a periodic funtion with period L and, for fixed x, the integrand is nonzero outside the interval $(x - L/2, x + L/2)$. Defining a periodic extension of J of period L, that we will denote by J_p, the integral in (1.1) may be computed from the corresponding integral in any interval of legth L. Thus, we reformulate the problem (1.1)-(2.1) as defined in the bounded domain $[0,L]$:

$$\partial_t u_i(t,x) = \int_0^L J_p(x-y)\big(p_i(\mathbf{u}(t,y)) - p_i(\mathbf{u}(t,x))\big)dy + \gamma f_i(\mathbf{u}(t,x)), \tag{2.1}$$

$$u_i(0,x) = u_{0i}(x), \tag{2.2}$$

for $(t,x) \in [0,T]\times[0,L]$. Theorem 1 in [2] implies the existence of a unique solution to this problem. The periodic extension of this solution to $\mathbb{R}$ is a solution to our original problem (1.1)–(1.2). The uniqueness follows from the uniqueness result in [2]. □

In the following, we will use the notation J instead of J_p.

3. Linear stability and bifurcation parameter

Under suitable assumptions on the coefficients α_i and β_{ij}, the coexistence equilibrium

$$\tilde{\mathbf{u}} = \Big(\frac{\beta_{22}\alpha_1 - \beta_{12}\alpha_2}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}}, \frac{\beta_{11}\alpha_2 - \beta_{21}\alpha_1}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}}\Big) \tag{3.1}$$

is a constant stationary solution of equation (1.1). In what follows, and for simplicity, we will consider $L = 2\pi$. We also introduce the notation $\Delta_{nl}u = J * u - u$.

The linearization of problem (1.1)-(1.2) around $\tilde{\mathbf{u}}$ suggests to look for a solution of the form: $\mathbf{u} = \boldsymbol{\rho} e^{\lambda t}\cos(nx - \Phi_n)$, with $\boldsymbol{\rho} \in \mathbb{R}^2$. The resulting matrix eigenvalue problem is

$$A_n\boldsymbol{\rho} = \lambda\boldsymbol{\rho}, \quad \text{where } A_n = \gamma K - \kappa(n)D,$$

with $\kappa(n) = \int_{-\pi}^{\pi} J(x)\big(1 - \cos(nx)\big)dx \geq 0$, and

$$K := D\mathbf{f}(\tilde{\mathbf{u}}) = \begin{pmatrix} -\beta_{11}\tilde{u}_1 & -\beta_{12}\tilde{u}_1 \\ -\beta_{21}\tilde{u}_2 & -\beta_{22}\tilde{u}_2 \end{pmatrix}, \qquad D = \begin{pmatrix} d_1 + 2a_{11}\tilde{u}_1 + a_{12}\tilde{u}_2 & a_{12}\tilde{u}_1 \\ a_{21}\tilde{u}_2 & d_2 + a_{21}\tilde{u}_1 + 2a_{22}\tilde{u}_2 \end{pmatrix}. \tag{3.2}$$

We are going to assume that the uniform problem in space is stable, so K has two real eigenvalues and his determinant is positive.

Matrix A_n has two different real eigenvalues for any n and one of the eigenvalues is negative.

We want to find a bifurcation parameter such that the problem is stable on one side of the threshold and unestable on the other side, that is, the matrix has zero as eigenvalue for certain value n for the threshold value.

The weakly nonlinear analysis follows the local case in [3].

In the study of the linear stability of the local problem in a bounded domain we are concerned in the arrays $A_k = \gamma K - k^2 D$ with k a natural number.

We want to obtain a condition than implies that the expression $det(A_k)$ attains a minimum value 0 for some value $k > 0$. We oberserve that $Det(A_k)$ is a polynomial of degree 2 in the variable k^2:

$$h(k^2) = det(\gamma K - k^2 D) = \gamma^2 det(K) + \gamma q k^2 + k^4 det(D)$$

where

$$q = \beta_{11}\tilde{u}_1(2a_{22}\tilde{u}_2 + d_2) + \beta_{22}\tilde{u}_2(2a_{11}\tilde{u}_1 + d_1) + a_{12}\tilde{u}_2(\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1) + a_{21}\tilde{u}_1(\beta_{11}\tilde{u}_1 - \beta_{12}\tilde{u}_2)$$

If we want the parabolyc expression to have a positive root, we need $q < 0$.

The only possible negative terms in q are $\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1$ and $\beta_{11}\tilde{u}_1 - \beta_{12}\tilde{u}_2$.

It can be proved that these two terms have different sign. In this work we will assume $\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1 < 0$ and then, we will consider as bifurcation parameter $b := a_{12}$. We will use the superscript b to highlight the dependence on b of the quantities.

In [3] it is proved that there exists a thershold value b^c and a unique value $k_c > 0$ such that $h^{b^c}(k_c^2) = 0$, and for any $b > b^c$ there exist intervals (x_1, x_2) such that $h^b(k) < 0$ for $k^2 \in (k_1^2, k_2^2)$. Furthermore the size of the interval increases in γ

In our nonlocal problem in a bounded domain, instead of k^2 we have to consider $\kappa(n)$ for $n \in \mathbb{N}$. So, for γ big enough, there exist a natural number n and a threshold $b_n^* > b^c$ such that $det(\gamma K - \kappa(n)D^{b_n^*}) = 0$.

In the weakly nonlinear approximation we must consider b_n^* as the threshold value of the bifurcation parameter instead of b^c. However, we will use, for simplicity, b^c in the equations in the following section.

Remark 3.1 We observe that the temporal evolution of the amplitudes associated to a same wavelength (the cos(nx) term and the sin(nx) term) satisfafy the same equation, so we can consider Φ_n is independent in time.

4. Weakly nonlinear analysis

We follow [3].

Let b^c be the bifurcation threshold. We will consider the expansions

$$b = b^c + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 b_3 + O(\varepsilon^4)$$
$$\mathbf{w} = \varepsilon \mathbf{w}_1 + \varepsilon^2 \mathbf{w}_2 + \varepsilon^3 \mathbf{w}_3 + O(\varepsilon^4)$$
$$\partial_t = \varepsilon \partial_{T_1} + \varepsilon^2 \partial_{T_2} + \varepsilon^3 \partial_{T_3} + O(\varepsilon^4)$$

then

$$\partial_t \mathbf{w} = \varepsilon^2 \partial_{T_1} \mathbf{w}_1 + \varepsilon^3 (\partial_{T_1} \mathbf{w}_2 + \partial_{T_2} \mathbf{w}_1) + O(\varepsilon^4)$$

We will use the following notation for the nonlocal diffusion terms:

$$\mathcal{L}^b = \mathcal{L}^{b^c} + \sum_{j=1}^{3} \varepsilon^j \begin{pmatrix} b_j \tilde{u}_2 & b_j \tilde{u}_1 \\ 0 & 0 \end{pmatrix} \Delta_{nl}$$

where

$$\mathcal{L}^{b^c} = \gamma K + D^{b^c} \Delta_{nl}$$

Substituting the above expansion into our nonlinear problem and collecting at each order in ε, we obtain the following succesion of linear problems:

Order ε :

The linear system $\mathcal{L}^{b^c} \mathbf{w}_1 = 0$ has solutions:

$$\mathbf{w}_1 = A(T_1, T_2) \rho cos(k_c x - \Phi)$$

with $\rho \in ker\big(\gamma K - \kappa(k_c) D^{b^c}\big)$.

We can select

$$\boldsymbol{\rho} = (1, M)^t, \quad \text{with } M = \frac{-\gamma K_{21} + D_{21}^{b^c} \kappa(k_c)}{\gamma K_{22} - D_{22}^{b^c} \kappa(k_c)} < 0$$

Order ε^2 :

We have the system:

$$\mathcal{L}^{b^c} \mathbf{w}_2 = \partial_{T_1} \mathbf{w}_1 - \frac{1}{2}\big(Q_K(\mathbf{w}_1, \mathbf{w}_1) + \Delta_{nl} Q_D^{b^c}(\mathbf{w}_1, \mathbf{w}_1)\big) - b_1 \begin{pmatrix} \tilde{u}_2 & \tilde{u}_1 \\ 0 & 0 \end{pmatrix} \Delta_{nl} \mathbf{w}_1 =: \mathbf{F}$$

where

$$\partial_{T_1} \mathbf{w}_1 = \partial_{T_1} A(T_1, T_2) \rho cos(k_c x - \Phi)$$

$$Q_K(\mathbf{w}_1, \mathbf{w}_1) = A(T_1, T_2)^2 Q_K(\rho, \rho) cos^2(k_c x - \Phi) = \frac{1}{2} A(T_1, T_2)^2 Q_K(\rho, \rho)\big(1 + cos(2k_c x - 2\Phi)\big)$$

$$Q_D^{b^c}(\mathbf{w}_1, \mathbf{w}_1) = A(T_1, T_2)^2 Q_D^{b^c}(\rho, \rho) cos^2(k_c x - \Phi)$$

since $\Delta_{nl} cos(jk_c x - j\Phi) = -\kappa(jk_c) cos(jk_c x - j\Phi)$, we have

$$\Delta_{nl} Q_D^{b^c}(\mathbf{w}_1, \mathbf{w}_1) = -\frac{1}{2} \kappa(2k_c) A(T_1, T_2)^2 Q_D^{b^c}(\rho, \rho) cos(2k_c x - 2\Phi)$$

$$\Delta_{nl} \mathbf{w}_1 = -\kappa(k_c) A(T_1, T_2) \rho cos(k_c x - \Phi)$$

So, we have the following source term in the system:

$$\mathbf{F} = -\frac{1}{4} A(T_1, T_2)^2 \sum_{j=0,2} \mathcal{M}_j(\rho, \rho) cos(jk_c x) + \big(\partial_{T_1} A \rho + b_1 \kappa(k_c) A (\tilde{u}_2 + \tilde{u}_1 M, 0)^t\big) cos(k_c x - \Phi)$$

with

$$\mathcal{M}_j = Q_K - \kappa(jk_c) Q_D^{b^c}$$

The functions $cos(k_c x - \Phi)$ and $cos(2k_c x - 2\Phi)$ are linearly independent, so the solvability Fredholm condition is satisfied if the source term has no component in $cos(k_c x - \Phi)$.

We deduce, that we need $T_1 = b_1 = 0$.

If we assume this condition we can solve the linear problems and we obtain:

$$\mathcal{L}^{b^c} \mathbf{w}_2 = A(T_2)^2 \sum_{j=0,2} (\gamma K - \kappa(jk_c) D^{b^c}) \mathbf{w}_{2j} cos(jk_c x - j\Phi)$$

and so, $\mathbf{w}_{2j}$ verify the systems:

$$L_j \mathbf{w}_{2j} = -\frac{1}{4} \mathcal{M}_j(\rho, \rho), \text{ para } j = 0, 2$$

with $L_j = \gamma K - \kappa(jk_c)D^{b^c}$, and finally

$$\mathbf{w}_2 = A(T_2)^2(\mathbf{w}_{20} + \mathbf{w}_{22}cos(2k_cx - 2\Phi))$$

Order ε^3:
In this case, the system reads:

$$\mathcal{L}^{b^c}\mathbf{w}_3 = \partial_{T_2}\mathbf{w}_1 - Q_K(\mathbf{w}_1,\mathbf{w}_2) - \Delta_{nl}Q_D^{b^c}(\mathbf{w}_1,\mathbf{w}_2) - \begin{pmatrix}\tilde{u}_2 & \tilde{u}_1\\ 0 & 0\end{pmatrix} b_2\Delta_{nl}\mathbf{w}_1 =: \mathbf{G}, \tag{4.1}$$

where

$$\partial_{T_2}\mathbf{w}_1 = \partial_{T_2}A(T_2)\rho cos(k_cx - \Phi)$$

$$\begin{aligned} Q_K(\mathbf{w}_1,\mathbf{w}_2) &= A(T_2)^3Q_K(\rho,\mathbf{w}_{20})cos(k_cx-\Phi) + A(T_2)^3Q_K(\rho,\mathbf{w}_{22})cos(k_cx-\Phi)cos(2k_cx-2\Phi)\\ &= A(T_2)^3\Big(\big(Q_K(\rho,\mathbf{w}_{20}) + \frac{1}{2}Q_K(\rho,\mathbf{w}_{22})\big)cos(k_cx-\Phi) + \frac{1}{2}Q_K(\rho,\mathbf{w}_{22})cos(3k_cx-3\Phi)\Big)\end{aligned}$$

and

$$\begin{aligned}\Delta_{nl}Q_D^{b^c}(\mathbf{w}_1,\mathbf{w}_2) = -A(T_2)^3\Big(&\big(Q_D^{b^c}(\rho,\mathbf{w}_{20}) + \frac{1}{2}Q_D^{b^c}(\rho,\mathbf{w}_{22})\big)\kappa(k_c)cos(k_cx-\Phi)\\ &+\frac{1}{2}Q_D^{b^c}(\rho,\mathbf{w}_{22})\kappa(3k_c)cos(3k_cx-3\Phi)\Big)\end{aligned}$$

The source term is, in this case

$$\mathbf{G} = \Big(\partial_{T_2}A\rho + A\mathbf{G}_1^{(1)} + A^3\mathbf{G}_1^{(3)}\Big)cos(k_cx-\Phi) + \mathbf{G}_3A^3cos(3k_cx-3\Phi)$$

with

$$\mathbf{G}_1^{(1)} = (\tilde{u}_2 + \tilde{u}_1M)\kappa(kc)b_2(1,0)^t$$

$$\mathbf{G}_1^{(3)} = -\big(\mathcal{M}_1(\rho,\mathbf{w}_{20}) + \frac{1}{2}\mathcal{M}_1(\rho,\mathbf{w}_{22})\big)$$

$$\mathbf{G}_3 = -\frac{1}{2}\mathcal{M}_3(\rho,\mathbf{w}_{22})$$

The Fredholm condition is satisfied if $\mathbf{G} \in Ker\big((\mathcal{L}^{b^c})^*\big)^{\perp}$.

The vectorial space $Ker((\mathcal{L}^{b^c})^*)$ has dimension 1.The elements of this space are multples of $\Psi = \boldsymbol{\eta}cos(k_cx-\alpha)$ with $\alpha \in \mathbb{R}$, $\boldsymbol{\eta} = (1, M^*) \in Ker\big(\gamma K^t - \kappa(k_c)(D^{b^c})^t\big)$.

If we define

$$\hat{\sigma} = -\frac{\mathbf{G}_1^{(1)}\cdot\boldsymbol{\eta}}{\boldsymbol{\rho}\cdot\boldsymbol{\eta}} > 0, \qquad \hat{L} = \frac{\mathbf{G}_1^{(3)}\cdot\boldsymbol{\eta}}{\boldsymbol{\rho}\cdot\boldsymbol{\eta}},$$

We obtain the Stuart-Landau equation in the supercritical case $\hat{L} > 0$:

$$\partial_{T_2}A = \hat{\sigma}A - \hat{L}A^3$$

and the weakly nonlinear approximation of the limit in time of the solution:

$$\mathbf{w} = \varepsilon\boldsymbol{\rho}\sqrt{\frac{\hat{\sigma}}{\hat{L}}}cos(k_cx-\Phi) + \varepsilon^2\frac{\hat{\sigma}}{\hat{L}}(\mathbf{w}_{20} + \mathbf{w}_{22}cos(2k_cx-2\Phi)) + O(\varepsilon^3).$$

5. Numerical simulations.

We consider a finite difference discretization in space and an explicit discretization in time for problem (1.1)-(1.2). We must use a discrete space length h small enough to have an adequate number or discrete points in the domain of the nonlocal kernel. In Fig. 1, we show the the numerical approximation of the asymptotic solution in time (for a set of parameters) for the nonlinear problem (1.1)-(1.2) and the weakly nonlinear approximation. We took $h = \frac{2\pi}{100}$ and a triangular kernel with support in $[-\frac{\pi}{10}, \frac{\pi}{10}]$ and variance 2.

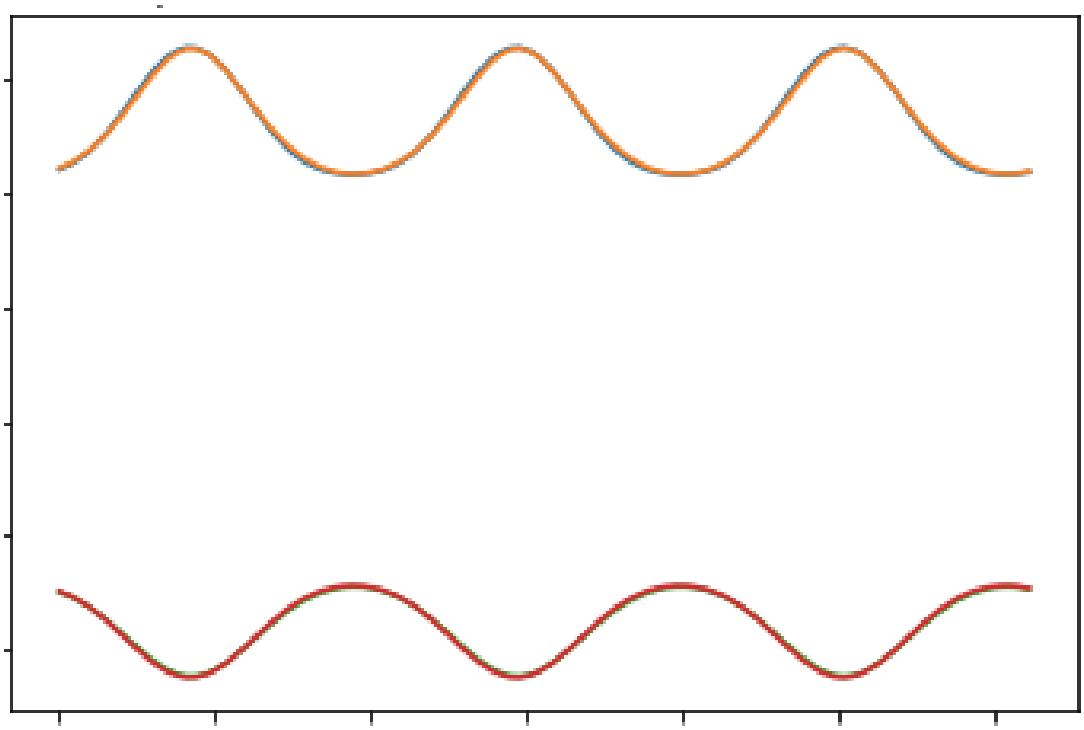

Fig. 1 Nonlocal solution and the weakly nonlinear approximation. The parameters are $d_1 = d_2 = 0.1$, $a_{11} = 1.e-4$, $a_{21} = 0.3$, $a_{22} = 0.1$, $\alpha_1 = 1.2$, $\alpha_2 = 1.0$, $\beta_{11} = 0.5$, $\beta_{12} = 0.4$, $\beta_{21} = 0.38$, $\beta_{22} = 0.41$, $\gamma = 49.75$. With these data, we have $b^c = 5.297$, $n = 3$, $b_3^* = 5.328$. Thus we took $b := a_{12} = 5.5$

Remark 5.1 (Rescaling of the nonlocal kernel) Let u be a 2π−periodic function twice continuously differentiable in $[0, 2\pi]$, and assume that the support of J is contained in [-1,1]. Then, we have:

$$\lim_{\varepsilon\to 0} \frac{c_1}{\varepsilon^3} \int_{x_0-\pi}^{x_0+\pi} J\Big(\frac{y-x}{\varepsilon}\Big)(u(y)-u(x))dy = u_{xx}(x_0)$$

where $c_1^{-1} = \frac{1}{2}\int_{-1}^{1} J(x)x^2 dx$. We consider the rescaling $J_\varepsilon(x) = \frac{c_1}{\varepsilon^3} J(\frac{x}{\varepsilon})$, so that, for fixed n,

$$\kappa_\varepsilon(n) = \int_{-1}^{1} J_\varepsilon(x)\big(1-\cos(nx)\big)dx = \frac{c_1}{\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} J\Big(\frac{x}{\varepsilon}\Big)\big(1-\cos(nx)\big)dx = c_1 \int_{-1}^{1} J(t)\frac{1-\cos(n\varepsilon t)}{\varepsilon^2}dt \to n^2$$

as $\varepsilon \to 0$. Thus, the weakly nonlinear approximation of the rescaled nonlocal diffusion problem converges, when $\varepsilon \to 0$, to the weakly nonlinear approximation of the corresponding local diffusion problem.

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