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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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A predictor-corrector iterative scheme for improving the accessibility of the Steffensen-type methods

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Abstract

Solving equations of the form $H(x) = 0$ is usually done by applying iterative methods. The Steffensen-type methods, defined by means divided differences and derivative free, are usually considered to solve these problems when H is a non-differentiable operator due its accuracy and efficiency. But, in general, the accessibility of iterative methods that use divided differences in their algorithms is reduced. The main interest of this paper is to improve the accessibility, domain of starting points, for Setffensen-type methods. So, by using a predictor-corrector iterative process we can improve this accessibility. For this, we use a predictor iterative process with a good accessibility and after we consider a Steffensen-type iterative method for a good accuracy, since this type of iterative process has quadratic convergence. Thus we will obtain a predictor-corrector iterative process with good accessibility, given by the predictor iterative process, and an accuracy like the Steffensen-type methods. Moreover, we analyze the semilocal convergence of the predictor-corrector iterative process proposed in two cases: when H is differentiable and H is non-differentiable. So, we present a good alternative for the non-applicability of Newton's method to non-differentiable operators. The theoretical results are illustrated with numerical experiments. **CEDYA/CMA 2020**.

1. Introduction

One of the most studied problems in numerical mathematics is the solution of nonlinear systems of equations

$$H(x) = 0, \tag{1.1}$$

where $H : \Omega \subset \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a nonlinear operator, $H \equiv (H_1, H_2, \ldots, H_m)$ with $H_i : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}$, $1 \leq i \leq m$, and Ω is a non-empty open convex domain. Iterative methods are a powerful tool for solving these equations.

In this paper, we consider iterative processes free of derivatives. But these methods have a serious shortcoming: they have a region of reduced accessibility. In [4], the accessibility of an iterative process is increased by means of an analytical procedure, that is, by modifying the convergence conditions. However, in our work, we will increase accessibility by building an iterative predictor-corrector process. This iterative process has a first prediction phase and a second accurate approximation phase.

Kung and Traub presented in [10] a class of iterative processes without derivatives. These iterative processes considered by Kung and Traub contain Steffensen-type methods as a special case. From the biparametric family of iterative processes given in [2],

$$\begin{cases} x_0 \in \Omega, \quad \alpha, \ \beta \geq 0 \\ y_n = x_n - \alpha H(x_n), \\ z_n = x_n + \beta H(x_n), , \\ x_{n+1} = x_n - [y_n, z_n; H]^{-1} H(x_n), n \geqslant 0. \end{cases} \tag{1.2}$$

The three best-known Steffensen-type methods appear. So, for $\alpha = 0$ and $\beta = 1$ we have the original Steffensen method, the Backward-Steffensen method for $\alpha = 1$ and $\beta = 0$ and the Central-Steffensen method for $\alpha = 1$ and $\beta = 1$.

Notice that, if we consider the Newton's method,

$$x_{n+1} = x_n - [H'(x_n)]^{-1} H(x_n), \quad n \geq 0; \quad x_0 \in \Omega \text{ is given}, \tag{1.3}$$

which is one of the most used iterative methods to approximate a solution x^* of $H(x) = 0$, the Steffensen-type methods are obtained as a special case of this method, where the evaluation of $H'(x)$ in each step of Newton's method is approximated by the divided difference of first order $[x - \alpha H(x), x + \beta H(x); H]$. The Stetffensen-type methods have been widely studied ([1, 3, 6]).

Symmetric divided differences generally perform better. Moreover, maintain the quadratic convergence of Newton's method, by approximating the derivative through symmetric divided differences with respect to the x_n, and the Center-Steffensen method also has the same computational efficiency as Newton's method. But, in order to achieve the second order in practice, we need an iteration close enough to the solution to have a good approximation of the first derivative of H used in Newton's method. In other case, some extra iterations in comparison with Newton's method are required. Basically, when the norm of $H(x)$ is big, the approximation of the divided difference to the first derivative of H is bad. So, in general, the set of starting points of the Steffensen-type methods is poor. This justify that Steffensen-type methods are less used than Newton's method to approximate solutions of equations for differentiable operators.

Thus, two are our main objectives in this work. On the one hand, in the case of differentiable operators, to which Newton's method can also be applied, to construct a predictor-corrector iterative process that has accessibility and efficiency like Newton's method, but using symmetric divided differences. And, secondly, that this predictor-corrector iterative process considered can have a behavior like Newton's method has in the differentiable case, but considering the case of non-differentiable operators where Newton's method cannot be applied.

Following this idea, in this paper we consider the derivative-free point-to-point iterative process given by

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n - \mathbf{Tol}, x_n + \mathbf{Tol}; H]^{-1} H(x_n), \quad n \geq 0, \end{cases} \tag{1.4}$$

where $\mathbf{Tol} = (tol, tol, \ldots, tol) \in \mathbb{R}^m$ for a real number $tol > 0$. Thus, we take a symmetric divided difference to approximate the derivative in Newton's method. Furthermore, by varying the parameter tol, we can approach the value $F'(x_n)$. Notice that, in the differentiable case, for $tol = 0$ we obtain the Newton's method.

However, although reducing the value of tol we can reach a speed of convergence similar to Newton's method, its order of convergence is linear. That is why we will consider this method as a predictor, due to its good accessibility, and we will consider the Center-Steffensen method as a corrector, whose order of convergence is quadratic.

So, we consider the predictor-corrector method given by

$$\begin{cases} \begin{cases} \text{Given an initial guess } u_0 \in \Omega, \\ u_{j+1} = u_j - [u_j - \mathbf{Tol}, u_j + \mathbf{Tol}; H]^{-1} H(u_j), \quad j = 0, \ldots, N_0 - 1, \end{cases} \\ \begin{cases} x_0 = u_{N_0}, \\ y_n = x_n - H(x_n), \ n \geqslant 0, \\ z_n = x_n + H(x_n), \ n \geqslant 0, \\ x_{n+1} = x_n - [y_n, z_n; H]^{-1} H(x_n), \ n \geqslant 0, \end{cases} \end{cases} \tag{1.5}$$

where $\mathbf{Tol} = (tol, tol, \ldots, tol) \in \mathbb{R}^m$ for a real number $tol > 0$. Thus, this predictor-corrector method will be a Steffensen-type method with good accessibility and quadratic convergence from an iteration to be determined.

The paper is organized as follows. First, we introduce the motivation of the paper. Next, we establish a semilocal convergence analysis of the new method when operator H is both differentiable and non-differentiable cases.

2. Semilocal convergence

The semilocal study of the convergence is based on demanding conditions to the initial approximation u_0, from certain conditions on the operator H , and provide conditions required to the initial approximation that guarantee the convergence of sequence (1.5) to the solution x^*. To analyze the semilocal convergence of iterative processes that do not use derivatives in their algorithms, the conditions usually are required for the operator divided difference. Although, in the case that the operator H is Fréchet differentiable, the divided difference operator can be defined from the Fréchet derivative of the operator H.

2.1. Differentiable operators

Now, we establish the semilocal convergence of iterative process given in (1.5) for differentiable operators. So, we consider $H : \Omega \subset \mathbb{R}^m \longrightarrow \mathbb{R}^m$ a Fréchet differentiable operator and there exists

$$[v, w; H] = \int_0^1 H'(tv + (1-t)w) \, dt, \tag{2.1}$$

for each pair of distinct points $v, w \in \Omega$. Notice that, as H is Fréchet differentiable $[x, x; H] = H'(x)$.

Now, we suppose the following initial conditions:

(D1) Let $u_0 \in \Omega$ such that exists $\Gamma_0 = [H'(u_0)]^{-1}$ with $\|\Gamma_0\| \leq \beta$ and $\|H(u_0)\| \leq \delta_0$.

(D2) $\|H'(x) - H'(y)\| \leq K\|x - y\|$, $x, y \in \Omega$, $K \in \mathbb{R}^+$.

In first place, we obtain some technical results.

Lemma 2.1 *The following items are verified.*

(i) *Let* $R > 0$ *with* $B(u_0, R + \|\mathbf{Tol}\|) \subseteq \Omega$. *If* $\beta K(R + \|\mathbf{Tol}\|) < 1$ *then, for each pair of distinct points* $y, z \in B(u_0, R + \|\mathbf{Tol}\|)$, *there exists* $[y, z; H]^{-1}$ *such that*

$$\|[y, z; H]^{-1}\| \leq \frac{\beta}{1 - \beta K(R + \|\mathbf{Tol}\|)}. \tag{2.2}$$

(ii) *If* $u_j, u_{j-1} \in \Omega$, *for* $j = 0, 1, \ldots, N_0$, *then*

$$\|H(u_j)\| \leq \frac{K}{2}(\|\mathbf{Tol}\| + \|u_j - u_{j-1}\|)\|u_j - u_{j-1}\|. \tag{2.3}$$

(iii) *If* $x_j, x_{j-1} \in \Omega$, , *for* $j \geqslant 1$, *then*

$$\|H(x_j)\| \leq \frac{K}{2}(\|H(x_{j-1}\| + \|x_j - x_{j-1}\|)\|x_j - x_{j-1}\|. \tag{2.4}$$

To simplify the notation, from now on, we denote

$$A_j = [u_j - \mathbf{Tol}, u_j + \mathbf{Tol}; H], \quad B_j = [x_j - H(x_j), x_j + H(x_j); H],$$

and the parameters $a_0 = \beta^2 K \delta_0$ and $b_0 = \beta K \mathbf{Tol}$. Other parameters that we are going to use are the following:

$$M = \frac{L}{2}(b_0 + La_0), \text{ where } L = \frac{1}{1 - b_0 - \beta K R}.$$

Moreover, notice that the polynomial equation $p(t) = 0$, with

$$p(t) = 2a_0(1 - b_0) - (2 + a_0 - 5b_0 + 3b_0^2)\beta K t + (4 - 5b_0)\beta^2 K^2 t^2 - 2\beta^3 K^3 t^3,$$

has at least a positive real root since that $p(0) > 0$ and $p(t) \to -\infty$ as $t \to \infty$. Then, we denote by R the smallest positive real root of the polynomial equation $p(t) = 0$.
Finally, we denote by $[x]$ the integer part of the real number x.

Theorem 2.2 *Let* $H : \Omega \subset \mathbb{R}^m \longrightarrow \mathbb{R}^m$ *a Fréchet differentiable operator defined on a nonempty open convex domain* Ω. *Suppose that conditions* $(D1)$ *and* $(D2)$ *are satisfied and there exists* $tol > 0$ *such that* $M < 1$, $R < \dfrac{1 - b_0}{\beta K}$ *and* $B(u_0, R + \|\mathbf{Tol}\|) \subseteq \Omega$. *If we consider*

$$N_0 \geqslant \begin{cases} 1 + \left[\dfrac{log(\|\mathbf{Tol}\|/\delta_0)}{log(M)}\right] & \text{if } \|\mathbf{Tol}\| < \delta_0, \\ 1 & \text{if } \|\mathbf{Tol}\| > \delta_0, \end{cases} \tag{2.5}$$

then the iterative process predictor-corrector (1.5), *starting at* u_0, *converges to* x^* *a solution of* $H(x) = 0$. *Moreover,* $u_j, x_n, x^* \in \overline{B(u_0, R)}$ *for* $j = 1, \ldots, N_0$ *and* $n \geqslant 0$.

Next, we get an uniqueness result for the iterative process predictor-corrector (1.5).

Theorem 2.3 *Under conditions of the previous Theorem, the solution* x^* *of the equation* $H(x) = 0$ *is unique in* $B(u_0, R)$.

2.2. Non-differentiable operators

To obtain a result of semilocal convergence for iterative process (1.5) when H is a non-differentiable operator, we must suppose that for each pair of distinct points $x, y \in \Omega$, there exists a first-order divided difference of H at these points. As we consider Ω an open convex domain of $\mathbb{R}^m$, this condition is satisfied ([5], [7]). Moreover, it is also necessary to impose a condition for the first-order divided difference of the operator H. As it appears in [11] and [9], a Lipschitz-continuous condition or a Hölder-continuous can be considered, but in the above cases, it is known [8], that the Fréchet derivative of H exists in Ω. Therefore, these conditions cannot be verified if the operator H is non-differentiable. Then, to establish the semilocal convergence of iterative process given in (1.5) for non-differentiable operator H, we suppose the following conditions:

(ND1) Let $u_0 \in \Omega$ such that A_0^{-1} exists with $||A_0^{-1}|| \leq \beta_0$ and $||H(u_0)|| \leq \delta_0$.

(ND2) $||[x, y; H] - [u, v; H]|| \leq P + K\left(||x - u|| + ||y - v||\right), \quad P, K \geq 0$, with $x, y, u, v \in \Omega, x \neq y, u \neq v$.

To simplify the notation, from now on, we denote

$$\widetilde{M} = \beta_0(P + K(\beta_0\delta_0 + 2\|\mathbf{Tol}\|)) \qquad \text{and} \qquad S = \frac{\widetilde{M}}{1 - \beta_0(P + 2K(R + \|\mathbf{Tol}\|))}$$

In this conditions, we start our study obtaining a technical result, the proof of which is evident from algorithm given in (1.5).

Lemma 2.4 *The following items can be easily verified.*

(i) *If $u_j, u_{j-1} \in \Omega$, for $j = 0, 1, \ldots, N_0$, then*

$$H(u_j) = \big([u_j, u_{j-1}; H] - A_{j-1})(u_j - u_{j-1}\big). \tag{2.6}$$

(ii) *If $x_j, x_{j-1} \in \Omega$, for $j \geqslant 1$, then*

$$H(x_j) = \big([x_j, x_{j-1}; H] - B_{j-1}\big)(x_j - x_{j-1}). \tag{2.7}$$

Theorem 2.5 *Under the conditions $(ND1)$-$(ND2)$, if the real equation*

$$t = \frac{\beta_0\delta_0(1 - \beta_0(P + 2K(t - \|\mathbf{Tol}\|)))}{1 - \beta_0(P + 2K(t + \|\mathbf{Tol}\|)) - \widetilde{M}}, \tag{2.8}$$

has at least one positive real root, the smallest positive root is denoted by R, and there exists $tol > 0$ such that satisfies

$$\widetilde{M} + \beta_0(P + 2K(R + \|\mathbf{Tol}\|)) < 1, \tag{2.9}$$

and $B(u_0, R + \|\mathbf{Tol}\|) \subset \Omega$. If we consider

$$N_0 \geqslant \begin{cases} 2 + \left[\dfrac{log(\|\mathbf{Tol}\|/\widetilde{M}\delta_0)}{log(M)}\right] & \text{if } \|\mathbf{Tol}\| < \dfrac{\beta_0\delta_0(P + \beta_0\delta_0K)}{1 - 2\beta_0\delta_0}, \\ 1 & \text{if } \|\mathbf{Tol}\| > \dfrac{\beta_0\delta_0(P + \beta_0\delta_0K)}{1 - 2\beta_0\delta_0}, \end{cases} \tag{2.10}$$

then the iterative process predictor-corrector (1.5), starting at u_0, converges to x^* a solution of $H(x) = 0$. Moreover, $u_j, x_n, x^* \in \overline{B(u_0, R)}$ for $j = 1, \ldots, N_0$ and $n \geqslant 0$.

Moreover, x^* is unique in $B(u_0, R) \subset \Omega$.

Theorem 2.6 *Under conditions of the previous Theorem, the solution x^* of the equation $H(x) = 0$ is unique in $B(u_0, R)$.*

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