

**Proceedings**  
**of the**  
**XXVI Congreso de Ecuaciones**  
**Diferenciales y Aplicaciones**  
**XVI Congreso de Matemática Aplicada**

**Gijón (Asturias), Spain**

**June 14-18, 2021**



**SēMA**  
Sociedad Española  
de Matemática Aplicada



Universidad de Oviedo

**Editors:**  
Rafael Gallego, Mariano Mateos

Esta obra está bajo una licencia Reconocimiento- No comercial- Sin Obra Derivada 3.0 España de Creative Commons. Para ver una copia de esta licencia, visite <http://creativecommons.org/licenses/by-nc-nd/3.0/es/> o envíe una carta a Creative Commons, 171 Second Street, Suite 300, San Francisco, California 94105, USA.



Reconocimiento- No Comercial- Sin Obra Derivada (by-nc-nd): No se permite un uso comercial de la obra original ni la generación de obras derivadas.



Usted es libre de copiar, distribuir y comunicar públicamente la obra, bajo las condiciones siguientes:



Reconocimiento – Debe reconocer los créditos de la obra de la manera especificada por el licenciador:

Coordinadores: Rafael Gallego, Mariano Mateos (2021), Proceedings of the XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones / XVI Congreso de Matemática Aplicada. Universidad de Oviedo.

La autoría de cualquier artículo o texto utilizado del libro deberá ser reconocida complementariamente.



No comercial – No puede utilizar esta obra para fines comerciales.



Sin obras derivadas – No se puede alterar, transformar o generar una obra derivada a partir de esta obra.

© 2021 Universidad de Oviedo

© Los autores

Universidad de Oviedo

Servicio de Publicaciones de la Universidad de Oviedo

Campus de Humanidades. Edificio de Servicios. 33011 Oviedo (Asturias)

Tel. 985 10 95 03 Fax 985 10 95 07

[http: www.uniovi.es/publicaciones](http://www.uniovi.es/publicaciones)

[servipub@uniovi.es](mailto:servipub@uniovi.es)

ISBN: 978-84-18482-21-2

Todos los derechos reservados. De conformidad con lo dispuesto en la legislación vigente, podrán ser castigados con penas de multa y privación de libertad quienes reproduzcan o plagien, en todo o en parte, una obra literaria, artística o científica, fijada en cualquier tipo de soporte, sin la preceptiva autorización.

## Foreword

It is with great pleasure that we present the Proceedings of the 26<sup>th</sup> Congress of Differential Equations and Applications / 16<sup>th</sup> Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics SĒMA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the SĒMA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier “Pancho” Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: “a mathematician is a device for turning coffee into theorems”. Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

The Local Organizing Committee from the Universidad de Oviedo

## **Scientific Committee**

- Juan Luis Vázquez, Universidad Autónoma de Madrid
- María Paz Calvo, Universidad de Valladolid
- Laura Grigori, INRIA Paris
- José Antonio Langa, Universidad de Sevilla
- Mikel Lezaun, Euskal Herriko Unibersitatea
- Peter Monk, University of Delaware
- Ira Neitzel, Universität Bonn
- José Ángel Rodríguez, Universidad de Oviedo
- Fernando de Terán, Universidad Carlos III de Madrid

## **Sponsors**

- Sociedad Española de Matemática Aplicada
- Departamento de Matemáticas de la Universidad de Oviedo
- Escuela Politécnica de Ingeniería de Gijón
- Gijón Convention Bureau
- Ayuntamiento de Gijón

## **Local Organizing Committee from the Universidad de Oviedo**

- Pedro Alonso Velázquez
- Rafael Gallego
- Mariano Mateos
- Omar Menéndez
- Virginia Selgas
- Marisa Serrano
- Jesús Suárez Pérez del Río

# Contents

<b>On numerical approximations to diffuse-interface tumor growth models</b> Acosta-Soba D., Guillén-González F. and Rodríguez-Galván J.R. . . . . .	8
<b>An optimized sixth-order explicit RKN method to solve oscillating systems</b> Ahmed Demba M., Ramos H., Kumam P. and Watthayu W. . . . . .	15
<b>The propagation of smallness property and its utility in controllability problems</b> Apraiz J. . . . . .	23
<b>Theoretical and numerical results for some inverse problems for PDEs</b> Apraiz J., Doubova A., Fernández-Cara E. and Yamamoto M. . . . . .	31
<b>Pricing TARN options with a stochastic local volatility model</b> Arregui I. and Ráfales J. . . . . .	39
<b>XVA for American options with two stochastic factors: modelling, mathematical analysis and numerical methods</b> Arregui I., Salvador B., Ševčovič D. and Vázquez C. . . . . .	44
<b>A numerical method to solve Maxwell's equations in 3D singular geometry</b> Assous F. and Raichik I. . . . . .	51
<b>Analysis of a SEIRS metapopulation model with fast migration</b> Atienza P. and Sanz-Lorenzo L. . . . . .	58
<b>Goal-oriented adaptive finite element methods with optimal computational complexity</b> Becker R., Gantner G., Innerberger M. and Praetorius D. . . . . .	65
<b>On volume constraint problems related to the fractional Laplacian</b> Bellido J.C. and Ortega A. . . . . .	73
<b>A semi-implicit Lagrange-projection-type finite volume scheme exactly well-balanced for 1D shallow-water system</b> Caballero-Cárdenas C., Castro M.J., Morales de Luna T. and Muñoz-Ruiz M.L. . . . . .	82
<b>SEIRD model with nonlocal diffusion</b> Calvo Pereira A.N. . . . . .	90
<b>Two-sided methods for the nonlinear eigenvalue problem</b> Campos C. and Roman J.E. . . . . .	97
<b>Fractionary iterative methods for solving nonlinear problems</b> Candelario G., Cordero A., Torregrosa J.R. and Vassileva M.P. . . . . .	105
<b>Well posedness and numerical solution of kinetic models for angiogenesis</b> Carpio A., Cebrián E. and Duro G. . . . . .	109
<b>Variable time-step modal methods to integrate the time-dependent neutron diffusion equation</b> Carreño A., Vidal-Ferrándiz A., Ginestar D. and Verdú G. . . . . .	114

<b>Homoclinic bifurcations in the unfolding of the nilpotent singularity of codimension 4 in <math>R^4</math></b> Casas P.S., Drubi F. and Ibáñez S. . . . .	122
<b>Different approximations of the parameter for low-order iterative methods with memory</b> Chicharro F.I., Garrido N., Sarría I. and Orcos L. . . . .	130
<b>Designing new derivative-free memory methods to solve nonlinear scalar problems</b> Cordero A., Garrido N., Torregrosa J.R. and Triguero P. . . . .	135
<b>Iterative processes with arbitrary order of convergence for approximating generalized inverses</b> Cordero A., Soto-Quirós P. and Torregrosa J.R. . . . .	141
<b>FCF formulation of Einstein equations: local uniqueness and numerical accuracy and stability</b> Cordero-Carrión I., Santos-Pérez S. and Cerdá-Durán P. . . . .	148
<b>New Galilean spacetimes to model an expanding universe</b> De la Fuente D. . . . .	155
<b>Numerical approximation of dispersive shallow flows on spherical coordinates</b> Escalante C. and Castro M.J. . . . .	160
<b>New contributions to the control of PDEs and their applications</b> Fernández-Cara E. . . . .	167
<b>Saddle-node bifurcation of canard limit cycles in piecewise linear systems</b> Fernández-García S., Carmona V. and Teruel A.E. . . . .	172
<b>On the amplitudes of spherical harmonics of gravitational potencial and generalised products of inertia</b> Floría L. . . . .	177
<b>Turing instability analysis of a singular cross-diffusion problem</b> Galiano G. and González-Tabernero V. . . . .	184
<b>Weakly nonlinear analysis of a system with nonlocal diffusion</b> Galiano G. and Velasco J. . . . .	192
<b>What is the humanitarian aid required after tsunami?</b> González-Vida J.M., Ortega S., Macías J., Castro M.J., Michelini A. and Azzarone A. . . . .	197
<b>On Keller-Segel systems with fractional diffusion</b> Granero-Belinchón R. . . . .	201
<b>An arbitrary high order ADER Discontinuous Galerking (DG) numerical scheme for the multilayer shallow water model with variable density</b> Guerrero Fernández E., Castro Díaz M.J., Dumbser M. and Morales de Luna T. . . . .	208
<b>Picard-type iterations for solving Fredholm integral equations</b> Gutiérrez J.M. and Hernández-Verón M.A. . . . .	216
<b>High-order well-balanced methods for systems of balance laws based on collocation RK ODE solvers</b> Gómez-Bueno I., Castro M.J., Parés C. and Russo G. . . . .	220
<b>An algorithm to create conservative Galerkin projection between meshes</b> Gómez-Molina P., Sanz-Lorenzo L. and Carpio J. . . . .	228
<b>On iterative schemes for matrix equations</b> Hernández-Verón M.A. and Romero N. . . . .	236
<b>A predictor-corrector iterative scheme for improving the accessibility of the Steffensen-type methods</b> Hernández-Verón M.A., Magreñán A.A., Martínez E. and Sukhjit S. . . . .	242

## CONTENTS

<b>Recent developments in modeling free-surface flows with vertically-resolved velocity profiles using moments</b> Koellermeier J. . . . .	247
<b>Stability of a one degree of freedom Hamiltonian system in a case of zero quadratic and cubic terms</b> Lanchares V. and Bardin B. . . . .	253
<b>Minimal complexity of subharmonics in a class of planar periodic predator-prey models</b> López-Gómez J., Muñoz-Hernández E. and Zanolin F. . . . .	258
<b>On a non-linear system of PDEs with application to tumor identification</b> Maestre F. and Pedregal P. . . . .	265
<b>Fractional evolution equations in discrete sequences spaces</b> Miana P.J. . . . .	271
<b>KPZ equation approximated by a nonlocal equation</b> Molino A. . . . .	277
<b>Symmetry analysis and conservation laws of a family of non-linear viscoelastic wave equations</b> Márquez A. and Bruzón M. . . . .	284
<b>Flux-corrected methods for chemotaxis equations</b> Navarro Izquierdo A.M., Redondo Neble M.V. and Rodríguez Galván J.R. . . . .	289
<b>Ejection-collision orbits in two degrees of freedom problems</b> Ollé M., Álvarez-Ramírez M., Barrabés E. and Medina M. . . . .	295
<b>Teaching experience in the Differential Equations Semi-Virtual Method course of the Tecnológico de Costa Rica</b> Oviedo N.G. . . . .	300
<b>Nonlinear analysis in lorentzian geometry: the maximal hypersurface equation in a generalized Robertson-Walker spacetime</b> Pelegrín J.A.S. . . . .	307
<b>Well-balanced algorithms for relativistic fluids on a Schwarzschild background</b> Pimentel-García E., Parés C. and LeFloch P.G. . . . .	313
<b>Asymptotic analysis of the behavior of a viscous fluid between two very close mobile surfaces</b> Rodríguez J.M. and Taboada-Vázquez R. . . . .	321
<b>Convergence rates for Galerkin approximation for magnetohydrodynamic type equations</b> Rodríguez-Bellido M.A., Rojas-Medar M.A. and Sepúlveda-Cerda A. . . . .	325
<b>Asymptotic aspects of the logistic equation under diffusion</b> Sabina de Lis J.C. and Segura de León S. . . . .	332
<b>Analysis of turbulence models for flow simulation in the aorta</b> Santos S., Rojas J.M., Romero P., Lozano M., Conejero J.A. and García-Fernández I. . . . .	339
<b>Overdetermined elliptic problems in unduloid-type domains with general nonlinearities</b> Wu J. . . . .	344

## Recent developments in modeling free-surface flows with vertically-resolved velocity profiles using moments

Julian Koellermeier<sup>1</sup>

*julian.koellermeier@kuleuven.be KU Leuven, Belgium*

### Abstract

Shallow water moment models are non-linear PDEs in balance law form for free-surface flows that allow for vertical variations in the horizontal velocity. The models are extensions of the standard shallow water equations. However, the models in their original form lack global hyperbolicity. The loss of hyperbolicity already occurs for small vertical variations of the velocity and this leads to instabilities in numerical test cases. We review two recently developed hyperbolic shallow water moment models, which are based on two different linearizations during the derivation. Recently, the models have been extended to consider sediment transport and bottom topographies, for which new well-balanced numerical schemes based on analytical derivation of steady states can be constructed. We summarize the recent developments focusing on analytical properties of the models and their derivation.

### 1. Introduction

The well-known Shallow Water Equations (SWE), sometimes also called Saint-Venant equations, are a simplified model for free-surface flows and are commonly used to model different physical phenomena. However, the main deficiency of these models is that they assume a constant velocity profile of the horizontal velocity. In fact, the model only takes into account the mean velocity averaged along the vertical axis. This limits the applicability of the SWE model for complex flows and situations in which bottom friction plays an important role such as sediment transport.

One option to include vertical variations of the velocity is the use of multiple layers with piecewise constant velocities [2] leading to a system of equations that is coupled via the interfaces. However, the analysis of the model is difficult and no analytical eigenvalues can be obtained. Additionally, many layers are necessary to accurately describe varying profiles.

A polynomial velocity ansatz was used in [9] and the system of equations for the coefficients can be obtained by projection onto orthogonal test functions. This can be seen as an extension of the standard SWE model using an extended set of variables, so-called moments. These new Shallow Water Moment Equations (SWME) have been applied to several test cases which showed the accuracy and flexibility of the approach.

The main drawback of the SWME model in its original version is that the model loses hyperbolicity even for small variations of the velocity profile, as shown in [7]. This can lead to oscillations and a breakdown of the solution during simulations, which was exemplified using a dam-break test case.

Hyperbolicity was restored using two different linearizations of the model in [7] and [6]. We will summarize the derivations of both models in this paper and outline the different analytical properties.

While hyperbolicity is a main ingredient for a stable numerical simulation, different physical phenomena need to be modeled by means of special friction terms or additional equations. We show a recently developed example of sediment transport [3].

### 2. Shallow Water Moment Models

The standard shallow water equations (SWE) for a Newtonian fluid in one horizontal direction  $x$  for water height  $h$  and mean velocity  $u_m$  using a flat bottom topography are given by

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{pmatrix} = -\frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m \end{pmatrix}, \quad (2.1)$$

where  $\lambda$  and  $\nu$  denote the slip length and the kinematic viscosity, respectively.

While the SWE model is efficient to compute approximate solutions of simple flows in very shallow conditions, the model is inaccurate in case of horizontal variations of the vertical velocity. This is due to the fact that only the average velocity  $u_m$  is a variable of the model. In [9], the Shallow Water Moment Equations (SWME) were developed to overcome this problem. The derivation is based on two main ideas:



- The first idea is to scale vertical position variable  $\zeta(t, x)$  as

$$\zeta(t, x) := \frac{z - h_b(t, x)}{h_s(t, x) - h_b(t, x)} = \frac{z - h_b(t, x)}{h(t, x)},$$

with  $h(t, x) = h_s(t, x) - h_b(t, x)$  the water height from the bottom  $h_b$  to the surface  $h_s$ . This transforms the vertical  $z$ -direction from a physical space to a projected space  $\zeta : [0, T] \times \mathbb{R} \rightarrow [0, 1]$ , see [9].

- The second idea assumes a polynomial expansion of the velocity variable, in the transformed vertical direction. We thus expand  $u : [0, T] \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$  as

$$u(t, x, \zeta) = u_m(t, x) + \sum_{j=1}^N \alpha_j(t, x) \phi_j(\zeta), \quad (2.2)$$

where  $u_m : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is the mean velocity and  $\phi_j : [0, 1] \rightarrow \mathbb{R}$  are the *scaled Legendre polynomials* of degree  $j$  defined by

$$\phi_j(\zeta) = \frac{1}{j!} \frac{d^j}{d\zeta^j} (\zeta - \zeta^2)^j. \quad (2.3)$$

Note that the basis polynomials fulfill  $\phi_j(0) = 1$  and they are orthogonal basis functions as

$$\int_0^1 \phi_m \phi_n d\zeta = \frac{1}{2n+1} \delta_{mn}, \quad (2.4)$$

with Kronecker delta  $\delta_{mn}$  [9].

With  $\alpha_j : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  for  $j \in [1, 2, \dots, N]$  we denote the corresponding *basis coefficients* at time  $t$  and position  $x$ . These coefficients are also called *moments*. Different values of the coefficients describe different horizontal velocity profiles, which allows for more complex flows and extends the standard SWE (2.1), where the horizontal velocity is constant. In the expansion,  $N \in \mathbb{N}$  is the order of the velocity expansion and at the same time the maximum degree of the Legendre polynomials. A larger  $N$  typically enables the representation of more complex flows, whereas  $N = 0$  corresponds to the constant velocity profile of the standard SWE (2.1).

To derive evolution equations for the basis coefficients, the expansion is inserted into the Navier-Stokes equations, which have been properly transformed to the new  $\zeta(t, x)$  variable, see [9] for details. Then, the equations are projected onto the Legendre polynomials of degree  $i = 1, \dots, N$ , by multiplication with  $\phi_j$  and integration over  $\zeta$ , which gives one additional equation for each coefficient in the expansion. The arising integrals of the basis polynomials  $A_{ijk}, B_{ijk}, C_{ij}$  are denoted as follows

$$A_{ijk} = (2i+1) \int_0^1 \phi_i \phi_j \phi_k d\zeta, \quad (2.5)$$

$$B_{ijk} = (2i+1) \int_0^1 \partial_\zeta \phi_i \left( \int_0^\zeta \phi_j d\hat{\zeta} \right) \phi_k d\zeta, \quad (2.6)$$

$$C_{ij} = \int_0^1 \partial_\zeta \phi_i \partial_\zeta \phi_j d\zeta. \quad (2.7)$$

More details can be found in [6, 9].

The model with variables  $U = (h, hu, h\alpha_1, \dots, h\alpha_N)^T \in \mathbb{R}^{N+2}$  can be written in compact form as

$$\partial_t U + \frac{\partial F}{\partial U} \partial_x U = Q \partial_x U + S, \quad (2.8)$$

where the conservative flux Jacobian  $\frac{\partial F}{\partial U}$  is given by

$$\frac{\partial F}{\partial U} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ gh - u^2 - \sum_{i=1}^N \frac{\alpha_i}{2i+1} & 2u & \frac{2\alpha_1}{2 \cdot 1 + 1} & \dots & \frac{2\alpha_N}{2N+1} \\ -2u\alpha_1 - \sum_{j,k=1}^N A_{1jk} \alpha_j \alpha_k & 2\alpha_1 & 2u\delta_{11} + 2 \sum_{k=1}^N A_{11k} \alpha_k & \dots & 2u\delta_{1N} + 2 \sum_{k=1}^N A_{1Nk} \alpha_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2u\alpha_N - \sum_{j,k=1}^N A_{Njk} \alpha_j \alpha_k & 2\alpha_N & 2u\delta_{N1} + 2 \sum_{k=1}^N A_{N1k} \alpha_k & \dots & 2u\delta_{NN} + 2 \sum_{k=1}^N A_{NNk} \alpha_k \end{pmatrix},$$



**Theorem 3.1** *The HSWME model (3.1) of arbitrary order  $N$  is globally hyperbolic and the eigenvalues are*

$$\begin{aligned}\lambda_{1,2} &= u_m \pm \sqrt{gh + \alpha_1^2}, \\ \lambda_{i+2} &= u_m + r_{i,N} \alpha_1, \quad i = 1, 2, \dots, N,\end{aligned}$$

where  $r_{i,N} \in \mathbb{R}$  is the  $i$ -th root of the real polynomial  $p_N(z)$  of degree  $N$ , defined by the recursion  $p_k(z) = zp_{k-1}(z) - b_k p_{k-2}(z)$ , for  $2 \leq k \leq N$ ,  $p_1(z) = 1$ ,  $b_k = \frac{(k-1)(k+1)}{(2k-1)(2k+1)}$ .

### 3.2. Shallow Water Linearized Moment Equations

The second hyperbolic model called Shallow Water Linearized Moment Equations (SWLME) derived in [6] is based on a careful investigation of non-linear terms in the underlying model equations. One example is the term

$$\int_0^1 \phi_i u^2 d\zeta.$$

Using the polynomial velocity expansion (2.2), this terms can be computed according to [6] as

$$\int_0^1 \phi_i u^2 d\zeta = \int_0^1 \phi_i \left( u_m + \sum_{j=1}^N \alpha_j \phi_j \right)^2 d\zeta \quad (3.3)$$

$$= u_m^2 \int_0^1 \phi_i d\zeta + \sum_{j=1}^N 2u_m \alpha_j \int_0^1 \phi_i \phi_j d\zeta + \sum_{j,k=1}^N 2\alpha_j \alpha_k \int_0^1 \phi_i \phi_j \phi_k d\zeta \quad (3.4)$$

$$= 0 + \frac{2}{2i+1} u_m \alpha_i + \frac{1}{2i+1} \sum_{j,k} A_{ijk} \alpha_j \alpha_k. \quad (3.5)$$

Now the model assumes small deviations from a constant profile, i.e.,  $\alpha_i = \mathcal{O}(\epsilon)$ , such that the last term containing the coefficient coupling  $\alpha_j \alpha_k = \mathcal{O}(\epsilon^2)$  can be neglected in comparison to the first term. The result is the simpler expression

$$\int_0^1 \phi_i u^2 d\zeta \approx \frac{2}{2i+1} u_m \alpha_i.$$

Based on this strategy, the SWLME model includes fewer terms than the original (2.10) and reads

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix} + \partial_x \begin{pmatrix} hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \dots + \frac{1}{2N+1}h\alpha_N^2 \\ 2hu_m\alpha_1 \\ \vdots \\ 2hu_m\alpha_N \end{pmatrix} = Q \partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix} + P, \quad (3.6)$$

where the non-conservative term simplifies to

$$Q = (0, 0, u_m, \dots, u_m),$$

and the combined transport system matrix of the new SWLME can be written as

$$A_N = \begin{pmatrix} 0 & 1 & 0 & \vdots & 0 \\ gh - u_m^2 - \frac{\alpha_1^2}{3} - \dots - \frac{\alpha_N^2}{2N+1} & 2u_m & \frac{2\alpha_1}{3} & \dots & \frac{2\alpha_N}{2N+1} \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & & \\ \vdots & \vdots & & \ddots & \\ -2u_m\alpha_N & 2\alpha_N & & & u_m \end{pmatrix} \in \mathbb{R}^{(N+2) \times (N+2)}. \quad (3.7)$$

It was shown in the following theorem from [6] that the eigenvalues of the SWLME model are indeed real such that the model is hyperbolic

**Theorem 3.2** *The SWLME system matrix  $A_N \in \mathbb{R}^{(N+2) \times (N+2)}$  (3.7) has the following characteristic polynomial*

$$\chi_{A_N}(\lambda) = (u_m - \lambda) \left[ (\lambda - u_m)^2 - gh - \sum_{i=1}^N \frac{3\alpha_i^2}{2i+1} \right]$$

and the eigenvalues are given by

$$\lambda_{1,2} = u_m \pm \sqrt{gh + \sum_{i=1}^N \frac{3\alpha_i^2}{2i+1}} \quad \text{and} \quad \lambda_{i+2} = u, \quad \text{for } i = 1, \dots, N. \quad (3.8)$$

The system is thus hyperbolic.

### 3.3. Steady states of SWLME

Another main benefit of the SWLME model is the possibility of obtaining analytical steady states that generalize the standard SWE Rankine-Hugoniot conditions. According to [6] the steady states can be derived as follows for flat bottom  $\partial_x b = 0$  and zero friction:

$$\partial_x (hu_m) = 0 \quad (3.9)$$

$$\partial_x \left( hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 + \dots + \frac{1}{2N+1}h\alpha_N^2 \right) = 0 \quad (3.10)$$

$$\partial_x (2hu_m\alpha_1) = u_m\partial_x (h\alpha_1) \quad (3.11)$$

$$\vdots \quad (3.12)$$

$$\partial_x (2hu_m\alpha_N) = u_m\partial_x (h\alpha_N), \quad (3.13)$$

which first leads to

$$hu_m = \text{const}, \quad (3.14)$$

$$u_m = 0 \text{ or } \frac{\alpha_i}{h} = \text{const}, \quad \text{for } i = 1, \dots, N. \quad (3.15)$$

The Rankine-Hugoniot conditions for a shock from a given state  $(h_0, h_0u_{m,0}, h_0\alpha_{1,0}, \dots, h_0\alpha_{N,0})$  to a state  $(h, hu_m, h\alpha_1, \dots, h\alpha_N)$  then read

$$(h - h_0) \left[ -\frac{u_{m,0}^2}{gh_0} + \frac{1}{2} \left( \left( \frac{h}{h_0} \right)^2 + \left( \frac{h}{h_0} \right) \right) + \sum_{i=1}^N \frac{1}{2i+1} \frac{\alpha_{i,0}^2}{gh_0} \left( \left( \frac{h}{h_0} \right)^3 + \left( \frac{h}{h_0} \right)^2 + \left( \frac{h}{h_0} \right) \right) \right] = 0. \quad (3.16)$$

Introducing the dimensionless flow numbers

$$Fr = \frac{u_{m,0}}{\sqrt{gh_0}}, \quad (3.17)$$

$$(M\alpha)_i = \frac{\alpha_{i,0}}{u_{m,0}}, \quad \text{for } i = 1, \dots, N \quad (3.18)$$

and writing  $y = \frac{h}{h_0}$ , leads to the non-dimensional solutions

$$h = h_0 \vee -Fr^2 + \frac{1}{2}(y^2 + y) + \sum_{i=1}^N \frac{1}{2i+1} (M\alpha)_i^2 Fr^2 (y^3 + y^2 + y) = 0. \quad (3.19)$$

This gives rise to a new dimensionless number  $M\alpha^2 := \sum_{i=1}^N \frac{1}{2i+1} (M\alpha)_i^2$ . According to [6],  $M\alpha$  measures the total deviation from equilibrium. Note, that there is at least one non-trivial solution for non-zero  $Fr$  and  $M\alpha$ .

It is also possible to derive steady states for smooth and frictionless flows including bottom topographies that can later be used to derive well-balanced schemes. In the momentum equation, this requires

$$\partial_x \left( \frac{1}{2}u_m^2 + g(h+b) + \frac{3}{2} \sum_{i=1}^N \frac{1}{2i+1} \alpha_i^2 \right) = 0, \quad (3.20)$$

where  $b(x)$  is the bottom topography term.

The full non-trivial steady state solution is then computed by solving

$$hu_m = \text{const}, \quad (3.21)$$

$$\frac{1}{2}u_m^2 + g(h+b) + \frac{3}{2} \sum_{i=1}^N \frac{1}{2i+1} \alpha_i^2 = \text{const}, \quad (3.22)$$

$$\frac{\alpha_i}{h} = \text{const}, \quad \text{for } i = 1, \dots, N. \quad (3.23)$$

The analytically computed equations to determine steady-states are used within a well-balanced numerical scheme to conserve certain steady-states numerically. We refer to [6] for detailed examples.

#### 4. Sediment transport and friction models

In [3], the HSWME model was coupled to an Exner equation [11], modeling sediment transport at the bottom. This means that the bottom topography  $b(t, x)$  is also a function of time and evolves according to

$$\partial_t b + \partial_x Q_b = 0, \quad (4.1)$$

where  $Q_b$  is the solid transport discharge that can be modeled by the Meyer-Peter & Müller formula [10].

It was shown in [3] that the eigenvalues of the coupled model are a generalization of the eigenvalues of the standard SWE model coupled to the Exner equation. The additional eigenvalues are real such that the model is again hyperbolic. The model leads to a much more realistic sediment transport. Unlike as for the SWE model, the velocity at the bottom is not the same as the average velocity  $u_m$ , which means that the coupled sediment equation (4.1) is correctly transported with the bottom velocity according to the polynomial expansion (2.2).

#### 5. Summary and future work

In this paper, recent developments in modeling free-surface flows with vertically resolved velocity profiles were summarized and compared. Based on a polynomial expansion of the velocity profile, the derivation of the Shallow Water Moment Equations was outlined. Two hyperbolic regularizations based on different linearizations of the model are described and the results for the eigenvalues and steady states are given. As one application, a sediment transport model that builds up on the previously discussed models is described.

The recently developed models are a major step forward for the simulation of complex free-surface flows. The models open up a lot of possibilities for future work. Firstly, the inclusion of a coriolis force term and the analytical investigation of wave properties is necessary for applications and to understand the structure of the models. Additional efforts should focus on the numerical simulation of the model equation, e.g., regarding the implementation of wet-dry fronts or asymptotic-preserving schemes for the limits of large friction terms. Lastly, the inclusion of more realistic friction terms of Savage-Hutter type [11] to model granular flows, e.g., for avalanches, land slides, or mud flows would be beneficial for real-world applications.

#### Acknowledgements

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement no. 888596. The first author is a postdoctoral fellow in fundamental research of the Research Foundation - Flanders (FWO), funded by FWO grant no. 0880.212.840.

#### References

- [1] Y. Fan, J. Koellermeier, J. Li, R. Li, and M. Torrilhon. Model reduction of kinetic equations by operator projection. *Journal of Statistical Physics*, 162(2):457–486, 2016.
- [2] E. D. Fernández-Nieto, J. Garres-Díaz, A. Mangeney, and G. Narbona-Reina. A multilayer shallow model for dry granular flows with the  $\mu$  ( $\dot{\gamma}$ )-rheology: Application to granular collapse on erodible beds. *Journal of Fluid Mechanics*, 798:643–681, 2016.
- [3] J. Garres-Díaz, M. J. C. Díaz, J. Koellermeier, and T. M. de Luna. Shallow water moment models for bedload transport problems. *accepted by Adv. Appl. Math. Mech.*, 2021.
- [4] Q. Huang, J. Koellermeier, and W.-A. Yong. Equilibrium stability analysis of hyperbolic shallow water moment equations. *submitted*.
- [5] J. Koellermeier. *Derivation and numerical solution of hyperbolic moment equations for rarefied gas flows*. PhD thesis, 2017.
- [6] J. Koellermeier and E. Pimentel-Garcia. Steady states and well-balanced schemes for shallow water moment equations with topography. *submitted*.
- [7] J. Koellermeier and M. Rominger. Analysis and numerical simulation of hyperbolic shallow water moment equations. *Communications in Computational Physics*, 28(3):1038–1084, 2020.
- [8] J. Koellermeier, R. P. Schaefer, and M. Torrilhon. A framework for hyperbolic approximation of kinetic equations using quadrature-based projection methods. *Kinetic and Related Models*, 7(3):531–549, 2014.
- [9] J. Kowalski and M. Torrilhon. Moment approximations and model cascades for shallow flow. *Communications in Computational Physics*, 25(3):669–702, 2019.
- [10] E. Meyer-Peter and R. Müller. Formulas for bed-load transport. Technical report, 1948.
- [11] S. B. Savage and K. Hutter. The motion of a finite mass of granular material down a rough incline. *Journal of Fluid Mechanics*, 199(2697):177–215, 1989.