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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Recent developments in modeling free-surface flows with vertically-resolved velocity profiles using moments

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Abstract

Shallow water moment models are non-linear PDEs in balance law form for free-surface flows that allow for vertical variations in the horizontal velocity. The models are extensions of the standard shallow water equations. However, the models in their original form lack global hyperbolicity. The loss of hyperbolicity already occurs for small vertical variations of the velocity and this leads to instabilities in numerical test cases. We review two recently developed hyperbolic shallow water moment models, which are based on two different linearizations during the derivation. Recently, the models have been extended to consider sediment transport and bottom topographies, for which new well-balanced numerical schemes based on analytical derivation of steady states can be constructed. We summarize the recent developments focusing on analytical properties of the models and their derivation.

1. Introduction

The well-known Shallow Water Equations (SWE), sometimes also called Saint-Venant equations, are a simplified model for free-surface flows and are commonly used to model different physical phenomena. However, the main deficiency of these models is that they assume a constant velocity profile of the horizontal velocity. In fact, the model only takes into account the mean velocity averaged along the vertical axis. This limits the applicability of the SWE model for complex flows and situations in which bottom friction plays an important role such as sediment transport.

One option to include vertical variations of the velocity is the use of multiple layers with piecewise constant velocities [2] leading to a system of equations that is coupled via the interfaces. However, the analysis of the model is difficult and no analytical eigenvalues can be obtained. Additionally, many layers are necessary to accurately describe varying profiles.

A polynomial velocity ansatz was used in [9] and the system of equations for the coefficients can be obtained by projection onto orthogonal test functions. This can be seen as an extension of the standard SWE model using an extended set of variables, so-called moments. These new Shallow Water Moment Equations (SWME) have been applied to several test cases which showed the accuracy and flexibility of the approach.

The main drawback of the SWME model in its original version is that the model looses hyperbolicity even for small variations of the velocity profile, as shown in [7]. This can lead to oscillations and a breakdown of the solution during simulations, which was exemplified using a dam-break test case.

Hyperbolicity was restored using two different linearization of the model in [7] and [6]. We will summarize the derivations of both models in this paper and outline the different analytical properties.

While hyperbolicity is a main ingredient for a stable numerical simulation, different physical phenomena need to be modeled by means of special friction terms or additional equations. We show a recently developed example of sediment transport [3].

2. Shallow Water Moment Models

The standard shallow water equations (SWE) for a Newtonian fluid in one horizontal direction x for water height h and mean velocity u_m using a flat bottom topography are given by

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{pmatrix} = -\frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m \end{pmatrix}, \tag{2.1}$$

where λ and ν denote the slip length and the kinematic viscosity, respectively.

While the SWE model is efficient to compute approximate solutions of simple flows in very shallow conditions, the model is inaccurate in case of horizontal variations of the vertical velocity. This is due to the fact that only the average velocity u_m is a variable of the model. In [9], the Shallow Water Moment Equations (SWME) were developed to overcome this problem. The derivation is based on two main ideas:

- The first idea is to scale vertical position variable $\zeta(t,x)$ as

$$\zeta(t,x) := \frac{z - h_b(t,x)}{h_s(t,x) - h_b(t,x)} = \frac{z - h_b(t,x)}{h(t,x)},$$

  with $h(t,x) = h_s(t,x) - h_b(t,x)$ the water height from the bottom h_b to the surface h_s. This transforms the vertical z-direction from a physical space to a projected space $\zeta : [0,T] \times \mathbb{R} \to [0,1]$, see [9].

- The second idea assumes a polynomial expansion of the velocity variable, in the transformed vertical direction. We thus expand $u : [0,T] \times \mathbb{R} \times [0,1] \to \mathbb{R}$ as

$$u(t,x,\zeta) = u_m(t,x) + \sum_{j=1}^{N} \alpha_j(t,x)\phi_j(\zeta), \tag{2.2}$$

  where $u_m : [0,T] \times \mathbb{R} \to \mathbb{R}$ is the mean velocity and $\phi_j : [0,1] \to \mathbb{R}$ are the *scaled Legendre polynomials* of degree j defined by

$$\phi_j(\zeta) = \frac{1}{j!}\frac{d^j}{d\zeta^j}(\zeta - \zeta^2)^j. \tag{2.3}$$

Note that the basis polynomials fulfill $\phi_j(0) = 1$ and they are orthogonal basis functions as

$$\int_0^1 \phi_m \phi_n d\zeta = \frac{1}{2n+1}\delta_{mn}, \tag{2.4}$$

with Kronecker delta δ_{mn} [9].

With $\alpha_j : [0,T] \times \mathbb{R} \to \mathbb{R}$ for $j \in [1,2,\dots,N]$ we denote the corresponding *basis coefficients* at time t and position x. These coefficients are also called *moments*. Different values of the coefficients describe different horizontal velocity profiles, which allows for more complex flows and extends the standard SWE (2.1), where the horizontal velocity is constant. In the expansion, $N \in \mathbb{N}$ is the order of the velocity expansion and at the same time the maximum degree of the Legendre polynomials. A larger N typically enables the representation of more complex flows, whereas $N = 0$ corresponds to the constant velocity profile of the standard SWE (2.1).

To derive evolution equations for the basis coefficients, the expansion is inserted into the Navier-Stokes equations, which have been properly transformed to the new $\zeta(t,x)$ variable, see [9] for details. Then, the equations are projected onto the Legendre polynomials of degree $i = 1,\dots,N$, by multiplication with ϕ_j and integration over ζ, which gives one additional equation for each coefficient in the expansion. The arising integrals of the basis polynomials A_{ijk}, B_{ijk}, C_{ij} are denoted as follows

$$A_{ijk} = (2i+1)\int_0^1 \phi_i \phi_j \phi_k \, d\zeta, \tag{2.5}$$

$$B_{ijk} = (2i+1)\int_0^1 \partial_\zeta \phi_i \left(\int_0^\zeta \phi_j \, d\hat{\zeta}\right)\phi_k \, d\zeta, \tag{2.6}$$

$$C_{ij} = \int_0^1 \partial_\zeta \phi_i \, \partial_\zeta \phi_j \, d\zeta. \tag{2.7}$$

More details can be found in [6, 9].

The model with variables $U = (h, hu, h\alpha_1, \dots, h\alpha_N)^T \in \mathbb{R}^{N+2}$ can be written in compact form as

$$\partial_t U + \frac{\partial F}{\partial U}\partial_x U = Q\partial_x U + S, \tag{2.8}$$

where the conservative flux Jacobian $\frac{\partial F}{\partial U}$ is given by

$$\frac{\partial F}{\partial U} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ gh - u^2 - \sum_{i=1}^{N} \frac{\alpha_i}{2i+1} & 2u & \frac{2\alpha_1}{2\cdot 1+1} & \dots & \frac{2\alpha_N}{2N+1} \\ -2u\alpha_1 - \sum_{j,k=1}^{N} A_{1jk}\alpha_j\alpha_k & 2\alpha_1 & 2u\delta_{11} + 2\sum_{k=1}^{N} A_{11k}\alpha_k & \dots & 2u\delta_{1N} + 2\sum_{k=1}^{N} A_{1Nk}\alpha_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2u\alpha_N - \sum_{j,k=1}^{N} A_{Njk}\alpha_j\alpha_k & 2\alpha_N & 2u\delta_{N1} + 2\sum_{k=1}^{N} A_{N1k}\alpha_k & \dots & 2u\delta_{NN} + 2\sum_{k=1}^{N} A_{NNk}\alpha_k \end{pmatrix},$$

and the non-conservative matrix Q reads

$$Q = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & u\delta_{11} + \sum_{k=1}^{N} B_{11k}\alpha_k & \dots & u\delta_{1N} + \sum_{k=1}^{N} B_{1Nk}\alpha_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & u\delta_{N1} + \sum_{k=1}^{N} B_{N1k}\alpha_k & \dots & u\delta_{NN} + \sum_{k=1}^{N} B_{NNk}\alpha_k \end{pmatrix}.$$

The friction term on the right-hand side $S = (0, S_0, S_1, \dots, S_N)^T \in \mathbb{R}^{N+2}$ is defined in [9] as $S_0 = 0$ and

$$S_i = -(2i+1)\frac{\nu}{\lambda}\left(u + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h}C_{ij}\right)\alpha_j\right), i = 0, \dots, N. \tag{2.9}$$

The model (2.8) can also be written in the form of

$$\partial_t U + A(U)\partial_x U = S(U), \tag{2.10}$$

where the combined transport matrix $A = \frac{\partial F}{\partial U} - Q$ can easily be obtained from conservative flux Jacobian and the non-conservative terms.

The new SWME model was used for simulation of smooth waves and dam-break scenarios in [9]. The model was more accurate than the standard SWE model and converged towards a reference solution with increasing number of moments/coefficients N.

3. Hyperbolic Regularization

As already noted in [9], the SWME model is not hyperbolic for values $N > 1$. Loosing hyperbolicity may or may not lead to instabilities and non-physical values during numerical simulations. In [7], the hyperbolicity was studied in more detail and a breakdown of hyperbolicity inducing instable oscillations in time could be found for standard simulations.

Two hyperbolic models have recently been developed. The first one called the Hyperbolic Shallow Water Moment Equations (HSWME) from [7] is based on insights from moment models for rarefied gases [1, 5, 8]. The second one called the Shallow Water Linearized Moment Equations (SWLME) from [6] is based on the assumption of small deviations from equilibrium and neglecting small terms in the derivation. Both models are hyperbolic. We will outline the main ideas and state the model equations following both approaches in the next two subsections.

3.1. Hyperbolic Shallow Water Moment Equations

The Hyperbolic Shallow Water Moment Equations (HSWME) [7] overcome the loss of hyperbolicity using a linearization of the SWME model around linear velocity deviations, denoted by the case $N = 1$. This leads to setting all other coefficients $\alpha_i = 0$ for $i = 2, \dots, N$ in the SWME model matrices. Note that the model is still non-linear and includes the dependencies on all other variables h, u_m, α_1 on the dynamics of the other coefficients.

The HSWME system from [7] is written in the same non-conservative form as (2.10) as

$$\partial_t U + A_H(U)\partial_x U = S(U), \tag{3.1}$$

with regularized hyperbolic system matrix $A_H(U) \in \mathbb{R}^{(N+2)\times(N+2)}$ given by

$$A_H(U) = \begin{pmatrix} 0 & 1 & & & & \\ gh - u_m^2 - \frac{1}{3}\alpha_1^2 & 2u_m & \frac{2}{3}\alpha_1 & & & \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & \\ -\frac{2}{3}\alpha_1^2 & 0 & \frac{1}{3}\alpha_1 & u_m & \ddots & \\ & & & \ddots & \ddots & \frac{N+1}{2N+1}\alpha_1 \\ & & & & \frac{N-1}{2N-1}\alpha_1 & u_m \end{pmatrix}. \tag{3.2}$$

In [7], it was shown up to a certain N, that the model has real eigenvalues and is therefore hyperbolic, for all variable states. The proof was recently extended to arbitrary order in [4] which yields the following theorem

Theorem 3.1 *The HSWME model* (3.1) *of arbitrary order N is globally hyperbolic and the eigenvalues are*

$$\lambda_{1,2} = u_m \pm \sqrt{gh + \alpha_1^2},$$
$$\lambda_{i+2} = u_m + r_{i,N}\alpha_1, \quad i = 1, 2, \ldots, N,$$

where $r_{i,N} \in \mathbb{R}$ *is the i-th root of the real polynomial* $p_N(z)$ *of degree N, defined by the recursion* $p_k(z) = zp_{k-1}(z) - b_k p_{k-2}(z)$, *for* $2 \le k \le N$, $p_1(z) = 1$, $b_k = \frac{(k-1)(k+1)}{(2k-1)(2k+1)}$.

3.2. Shallow Water Linearized Moment Equations

The second hyperbolic model called Shallow Water Linearized Moment Equations (SWLME) derived in [6] is based on a careful investigation of non-linear terms in the underlying model equations. One example is the term

$$\int_0^1 \phi_i u^2 \, d\zeta.$$

Using the polynomial velocity expansion (2.2), this terms can be computed according to [6] as

$$\int_0^1 \phi_i u^2 \, d\zeta = \int_0^1 \phi_i \left(u_m + \sum_{j=1}^N \alpha_j \phi_j \right)^2 d\zeta \tag{3.3}$$

$$= u_m^2 \int_0^1 \phi_i \, d\zeta + \sum_{j=1}^N 2u_m\alpha_j \int_0^1 \phi_i\phi_j \, d\zeta + \sum_{j,k=1}^N 2\alpha_j\alpha_k \int_0^1 \phi_i\phi_j\phi_k \, d\zeta \tag{3.4}$$

$$= 0 + \frac{2}{2i+1} u_m \alpha_i + \frac{1}{2i+1} \sum_{j,k}^N A_{ijk}\alpha_j\alpha_k. \tag{3.5}$$

Now the model assumes small deviations from a constant profile, i.e., $\alpha_i = \mathcal{O}(\epsilon)$, such that the last term containing the coefficient coupling $\alpha_j\alpha_k = \mathcal{O}(\epsilon^2)$ can be neglected in comparison to the first term. The result is the simpler expression

$$\int_0^1 \phi_i u^2 \, d\zeta \approx \frac{2}{2i+1} u_m \alpha_i.$$

Based on this strategy, the SWLME model includes fewer terms than the original (2.10) and reads

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \ldots + \frac{1}{2N+1}h\alpha_N^2 \\ 2hu_m\alpha_1 \\ \vdots \\ 2hu_m\alpha_N \end{pmatrix} = Q\partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix} + P, \tag{3.6}$$

where the non-conservative term simplifies to

$$Q = (0, 0, \mathrm{u_m}, \ldots, \mathrm{u_m}),$$

and the combined transport system matrix of the new SWLME can be written as

$$A_N = \begin{pmatrix} 0 & 1 & 0 & \vdots & 0 \\ gh - u_m^2 - \frac{\alpha_1^2}{3} - \ldots - \frac{\alpha_N^2}{2N+1} & 2u_m & \frac{2\alpha_1}{3} & \ldots & \frac{2\alpha_N}{2N+1} \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & & \\ \vdots & \vdots & & \ddots & \\ -2u_m\alpha_N & 2\alpha_N & & & u_m \end{pmatrix} \in \mathbb{R}^{(N+2)\times(N+2)}. \tag{3.7}$$

It was shown in the following theorem from [6] that the eigenvalues of the SWLME model are indeed real such that the model is hyperbolic

Theorem 3.2 *The SWLME system matrix* $A_N \in \mathbb{R}^{(N+2)\times(N+2)}$ (3.7) *has the following characteristic polynomial*

$$\chi_{A_N}(\lambda) = (u_m - \lambda)\left[(\lambda - u_m)^2 - gh - \sum_{i=1}^N \frac{3\alpha_i^2}{2i+1} \right]$$

and the eigenvalues are given by

$$\lambda_{1,2} = u_m \pm \sqrt{gh + \sum_{i=1}^{N} \frac{3\alpha_i^2}{2i+1}} \quad \textit{and} \quad \lambda_{i+2} = u, \;\; \textit{for}\;\; i = 1, \dots, N. \tag{3.8}$$

The system is thus hyperbolic.

3.3. Steady states of SWLME

Another main benefit of the SWLME model is the possibility of obtaining analytical steady states that generalize the standard SWE Rankine-Hugoniot conditions. According to [6] the steady states can be derived as follows for flat bottom $\partial_x b = 0$ and zero friction:

$$\partial_x \left(h u_m\right) = 0 \tag{3.9}$$

$$\partial_x \left(h u_m^2 + \frac{1}{2} g h^2 + \frac{1}{3} h \alpha_1^2 + \ldots + \frac{1}{2N+1} h \alpha_N^2\right) = 0 \tag{3.10}$$

$$\partial_x \left(2 h u_m \alpha_1\right) = u_m \partial_x \left(h \alpha_1\right) \tag{3.11}$$

$$\vdots \tag{3.12}$$

$$\partial_x \left(2 h u_m \alpha_N\right) = u_m \partial_x \left(h \alpha_N\right), \tag{3.13}$$

which first leads to

$$h u_m = const, \tag{3.14}$$

$$u_m = 0 \text{ or } \frac{\alpha_i}{h} = const, \text{ for } i = 1, \dots, N. \tag{3.15}$$

The Rankine-Hugoniot conditions for a shock from a given state $\left(h_0, h_0 u_{m,0}, h_0 \alpha_{1,0}, \dots, h_0 \alpha_{N,0}\right)$ to a state $\left(h, h u_m, h\alpha_1, \dots, h\alpha_N\right)$ then read

$$(h - h_0)\left[-\frac{u_{m,0}^2}{g h_0} + \frac{1}{2}\left(\left(\frac{h}{h_0}\right)^2 + \left(\frac{h}{h_0}\right)\right) + \sum_{i=1}^{N} \frac{1}{2i+1} \frac{\alpha_{i,0}^2}{g h_0} \left(\left(\frac{h}{h_0}\right)^3 + \left(\frac{h}{h_0}\right)^2 + \left(\frac{h}{h_0}\right)\right)\right] = 0. \tag{3.16}$$

Introducing the dimensionless flow numbers

$$Fr = \frac{u_{m,0}}{\sqrt{g h_0}}, \tag{3.17}$$

$$(M\alpha)_i = \frac{\alpha_{i,0}}{u_{m,0}}, \quad \text{for } i = 1, \dots, N \tag{3.18}$$

and writing $y = \frac{h}{h_0}$, leads to the non-dimensional solutions

$$h = h_0 \;\vee\; -Fr^2 + \frac{1}{2}\left(y^2 + y\right) + \sum_{i=1}^{N} \frac{1}{2i+1} \left(M\alpha\right)_i^2 Fr^2 \left(y^3 + y^2 + y\right) = 0. \tag{3.19}$$

This gives rise to a new dimensionless number $M\alpha^2 := \sum_{i=1}^{N} \frac{1}{2i+1} \left(M\alpha\right)_i^2$. According to [6], $M\alpha$ measures the total deviation from equilibrium. Note, that there is at least one non-trivial solution for non-zero Fr and $M\alpha$.

It is also possible to derive steady states for smooth and frictionless flows including bottom topographies that can later be used to derive well-balanced schemes. In the momentum equation, this requires

$$\partial_x \left(\frac{1}{2} u_m^2 + g(h + b) + \frac{3}{2} \sum_{i=1}^{N} \frac{1}{2i+1} \alpha_i^2\right) = 0, \tag{3.20}$$

where $b(x)$ is the bottom topography term.

The full non-trivial steady state solution is then computed by solving

$$h u_m = const, \tag{3.21}$$

$$\frac{1}{2} u_m^2 + g(h + b) + \frac{3}{2} \sum_{i=1}^{N} \frac{1}{2i+1} \alpha_i^2 = const, \tag{3.22}$$

$$\frac{\alpha_i}{h} = const, \text{ for } i = 1, \dots, N. \tag{3.23}$$

The analytically computed equations to determine steady-states are used within a well-balanced numerical scheme to conserve certain steady-states numerically. We refer to [6] for detailed examples.

4. Sediment transport and friction models

In [3], the HSWME model was coupled to an Exner equation [11], modeling sediment transport at the bottom. This means that the bottom topography $b(t,x)$ is also a function of time and evolves according to

$$\partial_t b + \partial_x Q_b = 0, \tag{4.1}$$

where Q_b is the solid transport discharge that can be modeled by the Meyer-Peter & Müller formula [10].

It was shown in [3] that the eigenvalues of the coupled model are a generalization of the eigenvalues of the standard SWE model coupled to the Exner equation. The additional eigenvalues are real such that the model is again hyperbolic. The model leads to a much more realistic sediment transport. Unlike as for the SWE model, the velocity at the bottom is not the same as the average velocity u_m, which means that the coupled sediment equation (4.1) is correctly transported with the bottom velocity according to the polynomial expansion (2.2).

5. Summary and future work

In this paper, recent developments in modeling free-surface flows with vertically resolved velocity profiles were summarized and compared. Based on a polynomial expansion of the velocity profile, the derivation of the Shallow Water Moment Equations was outlined. Two hyperbolic regularizations based on different linearizations of the model are described and the results for the eigenvalues and steady states are given. As one application, a sediment transport model that builds up on the previously discussed models is described.

The recently developed models are a major step forward for the simulation of complex free-surface flows. The models open up a lot of possibilities for future work. Firstly, the inclusion of a coriolis force term and the analytical investigation of wave properties is necessary for applications and to understand the structure of the models. Additional efforts should focus on the numerical simulation of the model equation, e.g., regarding the implementation of wet-dry fronts or asymptotic-preserving schemes for the limits of large friction terms. Lastly, the inclusion of more realistic friction terms of Savage-Hutter type [11] to model granular flows, e.g., for avalanches, land slides, or mud flows would be beneficial for real-world applications.

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