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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Stability of a one degree of freedom Hamiltonian system in a case of zero quadratic and cubic terms

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Abstract

We consider the stability of the equilibrium position of a periodic Hamiltonian system with one degree of freedom. It is supposed that the series expansion of the Hamiltonian function, in a small neighborhood of the equilibrium position, does not include terms of second and third degree. Moreover, we focus on a degenerate case, when fourth-degree terms in the Hamiltonian function are not enough to obtain rigorous conclusions on stability or instability. A complete study of the equilibrium stability in the above degenerate case is performed, giving sufficient conditions for instability and stability in the sense of Lyapunov. The above conditions are expressed in the form of inequalities with respect to the coefficients of the Hamiltonian function, normalized up to sixth-degree terms inclusive.

1. Introduction

Let us consider a one degree of freedom Hamiltonian system, periodically dependent on time, defined by the canonical differential equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \qquad \frac{dy}{dt} = -\frac{\partial H}{\partial x}. \tag{1.1}$$

We assume that the origin, $x = y = 0$, is an equilibrium position and that the Hamiltonian function $H = H(x, y, t)$ can be expanded in a convergent power series in a sufficiently small neighborhood of the origin. That is,

$$H(x, y, t) = \sum_{k=2}^{\infty} H_k(x, y, t), \qquad H_k(x, y, t) = \sum_{\nu+\mu=k} h_{\nu\mu} x^\nu y^\mu, \tag{1.2}$$

where ν and μ are nonnegative integers and the coefficients $h_{\nu\mu}$ are continuous 2π periodic functions of time, t. We also assume that a resonance of first or second order takes place in system (1.1). That is, the corresponding linear system has multiple characteristic multipliers. In particular, $\rho_{1,2} = 1$, for a first order resonance, and $\rho_{1,2} = -1$, for a second order resonance. In addition, the monodromy matrix is supposed to be diagonal. In the case it is nondiagonal, the problem of stability, in the sense of Lyapunov, has been completely solved [3, 8].

Under these assumptions, the origin is linearly stable, but nonlinear analysis is necessary to obtain a rigorous result about stability in the Lyapunov sense. Thus, terms of order three or higher in the Hamiltonian function $H(x, y, t)$ must be taken into account. It can be seen that, after a series of canonical change variables, the Hamiltonian function $H(x, y, t)$ can be brought to the following form [10, 12]

$$H(q, p, t) = \sum_{k=3}^{N} H_k(q, p) + \sum_{k=N+1}^{\infty} H_k(q, p, t), \qquad H_k = \sum_{\nu+\mu=k} h_{\nu\mu} q^\nu p^\mu, \tag{1.3}$$

where, for $3 \le k \le N$ (N can be set arbitrarily large), the coefficients $h_{\nu\mu}$ in H_k are real numbers, whereas, for $k > N$, they are T-periodic functions of time t.

The stability of the origin for the system (1.1) with the Hamiltonian (1.3), in the case $H_3 \ne 0$, has been studied in [10, 11] and we consider here the case $H_3 \equiv 0$, which appears in the presence of second order resonance. Now, the terms of fourth order in (1.3) play the most important role in the stability analysis of the equilibrium.

After a linear canonical change of variables [10], H_4 can be brought to one of the following nine simple forms:

$$\begin{array}{lll}
1)\ q^4 + aq^2p^2 + p^4,\ \ a > -2, & 5)\ q^2(q^2 + p^2), & 9)\ q^4. \\
2)\ q^4 + aq^2p^2 + p^4,\ \ a < -2 & 6)\ q^2p^2, & \\
3)\ q^4 + aq^2p^2 - p^4, a \in \mathbb{R}, & 7)\ q^3p, & \\
4)\ q^2(q^2 - p^2), & 8)\ -q^3p, &
\end{array} \tag{1.4}$$

In [10], it is also proved that in the case 1) the equilibrium is stable in the sense of Lyapunov, whereas it is unstable in the cases 2), 3), 4), 7). Cases 5) and 6) are considered in [12] and [9], respectively. In particular, considering terms up to six order, sufficient conditions for stability and instability in the Lyapunov sense are derived.

We concentrate our attention on the case 9), already considered in [7], where partial stability results are given. Our goal is to apply the results developed in [2] to derive complete and rigorous solution of the stability problem in this particular case.

2. Method of study and main result

To study the stability of the origin, it is convenient to introduce polar canonical variables by means of the canonical transformation

$$q = \sqrt{2r}\sin\varphi, \quad p = \sqrt{2r}\cos\varphi. \tag{2.1}$$

Now, the Hamiltonian function (1.3) is written as

$$H = r^2\Psi(\varphi, r) + O(r^{(N+1)/2}), \tag{2.2}$$

where

$$\Psi(\varphi, r) = \sum_{k=4}^{N} r^{\frac{k-4}{2}}\,\Psi_k(\varphi), \tag{2.3}$$

and $\Psi_k(\varphi)$ is a homogeneus function of order k with respect to $\sin\varphi$ and $\cos\varphi$.

It is shown in [5, 10, 14] that, if the function $\Psi_4(\varphi)$ does not have real roots, then the origin is a stable equilibrium point. This is what happens in the case 1), listed in (1.4). On the other hand, if $\Psi_4(\varphi)$ has a simple real root φ_0, such that $\frac{d\Psi_4}{d\varphi}(\varphi_0) < 0$, there is instability. This situation takes place in cases 2), 3), 4) and 7).

In the cases 5), 6) and 9) the function $\Psi_4(\varphi)$ has only multiple real roots and we say that a degeneracy takes place. To solve now the stability problem, it is necessary to consider the terms of order higher than r^2. To this end, we will use a technique for degenerate cases developed in [2]. The key idea is that simple roots of the function (2.3), coming from a multiple root of $\Psi_4(\varphi)$, play an important role for the stability problem. Thus, it is necessary to determine whether multiple roots of $\Psi_4(\varphi)$ give rise to simple distinct roots, when terms of order higher than r^2 in the Hamiltonian function (2.2) are considered. Even more, we have to ensure that additional terms of higher order cannot destroy the simple real roots of function (2.3). In this way, we introduce the concepts of main part and simple main part of a root (see [2]).

Let φ_0 be a root of multiplicity $M > 1$ of the function $\Psi_4(\varphi)$. Thus, according to the implicit function theorem [6], $\Psi(\varphi, r) = 0$ has exactly M roots approaching φ_0 with $r \to 0$. Lut us denote by $\varphi_*(r)$ one of these roots, which can be represented as a series expansion in fractional powers of r

$$\varphi_*(r) = \varphi_0 + \sum_{j=1}^{\infty} a_j r^{\frac{j}{m}}, \tag{2.4}$$

where m is an even integer ($2 \le m \le 2M$) and a_j are obtained by equating to zero the coefficients of powers of r, after substituting (2.4) into (2.3).

Definition 2.1 Let us consider the finite series

$$\varphi_q(r) = \varphi_0 + \sum_{j=1}^{q} a_j r^{\frac{j}{m}}, \tag{2.5}$$

which is obtained by omitting terms of order higher than q/m in (2.4); q is the maximal integer number such that the equation for a_q is obtained by substituting (2.5) in (2.3) and equating to zero the coefficient of $r^{\frac{\nu}{m}}$, where $\frac{\nu}{m} < \frac{N-3}{2}$. We call finite series (2.3) *main part* of root (2.4).

Definition 2.2 We say that root (2.4) has a *simple main part* if among roots of the equation $\Psi(\varphi, r) = 0$ there is not another root with the same main part.

Taking these two definitions in mind, general conditions for instability in the case of a degeneracy are given by the following theorem [2].

Theorem 2.3 *Let us consider the canonical system defined by Hamiltonian* (2.2). *Suppose that all real roots of the function $\Psi_4(\varphi)$ are multiple and the function $\Psi(\varphi, r)$ has a real root φ_* of form* (2.4). *If the root φ_* has a simple main part φ_q and, for sufficiently small r, the inequality $\frac{\partial\Psi}{\partial\varphi}(\varphi_*, r) < 0$ is satisfied, then the equilibrium $r = 0$ is unstable.*

As it was said previously, in the case 9), all real roots of the function $\Psi_4(\varphi) = 4\sin^4\varphi$ have multiplicity four and the use of Theorem 2.3 will be our main tool to obtain sufficient conditions for instability. To begin our analysis, we perform a series of near identity canonical transformations, in order to simplify the Hamiltonian function. This procedure has been previously introduced by Markeev to study other degenarete cases [9, 11, 12] and applied by Gutiérrez and Vidal [7] in the case we are dealing with.

Let us take $N = 6$ in (1.3). Thus, the Hamiltonian function reads as

$$H = q^4 + H_5(q,p) + H_6(q,p) + H^{(7)}(q,p,t), \tag{2.6}$$

where $H^{(7)}(q,p,t)$ is a convergent series in powers of q and p, starting from terms of degree seven or higher, whose coefficients are T-periodic functions of t.

Let us introduce new canonical variables Q, P by using a generating function $S(q,P)$ of the form

$$S(q,P) = qP + S_3(q,P) + S_4(q,P), \quad S_k(q,P) = \sum_{\nu+\mu=k} s_{\nu\mu} q^\nu P^\mu, \tag{2.7}$$

being $s_{\nu\mu}$ constant coefficients properly chosen in order to simplify the expression of the new Hamiltonian function. Taking into account the relations

$$p = \frac{\partial S}{\partial q}, \qquad Q = \frac{\partial S}{\partial P}, \tag{2.8}$$

we can express the old variables in a power series expansion of the new ones in such a way that the new Hamiltonian function, K, becomes [7]

$$\begin{aligned} K &= Q^4 + K_5(Q,P) + K_6(Q,P) + K^{(7)}(Q,P,t), \\ K_5(Q,P) &= \gamma_{23}Q^2P^3 + \gamma_{14}QP^4 + \gamma_{05}P^5, \\ K_6(Q,P) &= \gamma_{24}Q^2P^4 + \gamma_{15}QP^5 + \gamma_{06}P^6. \end{aligned} \tag{2.9}$$

The coefficients γ_{ij} in (2.9) are related to the coefficients of Hamiltonian (1.3) through the following identities [7]

$$\begin{aligned} &\gamma_{23} = h_{23}, \quad \gamma_{14} = h_{14}, \quad \gamma_{05} = h_{05}, \\ &\gamma_{24} = h_{24} - \tfrac{3}{7}h_{32}^2 + \tfrac{7}{4}h_{50}h_{14} - \tfrac{1}{8}h_{23}h_{41}, \\ &\gamma_{15} = h_{15} - \tfrac{1}{2}h_{32}h_{23} + \tfrac{1}{4}h_{41}h_{14} + \tfrac{5}{2}h_{50}h_{05}, \\ &\gamma_{06} = h_{06} - \tfrac{1}{4}h_{32}h_{14} + \tfrac{5}{8}h_{41}h_{05}. \end{aligned} \tag{2.10}$$

The main result of our stability study can be formulated in terms of the coefficients of the Hamiltonian (2.9) and it is collected in the following Theorem [4].

Theorem 2.4 *Let us consider the Hamiltonian system defined by* (2.9)*, then*

1. *If at least one of the inequalities* $\gamma_{05} \neq 0$, $\gamma_{14} \neq 0$ *or* $\gamma_{23}^2 - 4\gamma_{06} > 0$ *is fulfilled, then the origin is an unstable equilibrium.*
2. *If* $\gamma_{05} = \gamma_{14} = 0$ *and* $\gamma_{23}^2 - 4\gamma_{06} < 0$ *, then the origin is stable in the sense of Lyapunov.*
3. *In the case* $\gamma_{05} = \gamma_{14} = 0$ *and* $\gamma_{23}^2 - 4\gamma_{06} = 0$ *and* $\gamma_{15} \neq 0$ *the origin is unstable.*

3. Sketch of the proof

A complete proof of Theorem 2.4 is given in [4]. Here we outline the main ideas. To begin with, we introduce a scaling of the variables that will help us to see which are the relevant terms contributing to the proper splitting of the multiple roots. In this way, we introduce the following canonical transformation

$$Q = \varepsilon\bar{Q}, \qquad P = \varepsilon^\kappa \bar{P}, \tag{3.1}$$

and the Hamiltonian (2.9) reads as

$$\begin{aligned} K &= Q^4 + K_5(Q,P) + K_6(Q,P) + K^{(7)}(Q,P,t), \\ K_5(Q,P) &= \varepsilon^{3\kappa-2}\gamma_{23}Q^2P^3 + \varepsilon^{4\kappa-3}\gamma_{14}QP^4 + \varepsilon^{5\kappa-4}\gamma_{05}P^5, \\ K_6(Q,P) &= \varepsilon^{4\kappa-2}\gamma_{24}Q^2P^4 + \varepsilon^{5\kappa-3}\gamma_{15}QP^5 + \varepsilon^{6\kappa-4}\gamma_{06}P^6, \end{aligned} \tag{3.2}$$

where bars have been suppressed. The scaling introduces an ordering which depends on the exponent κ. Indeed, a different exponent κ gives rise to a different ordering and, to solve the degeneracy, we look for a proper choice of κ. To this end, we introduce the concept of leading exponent of a monomial.

Definition 3.1 We say that the leading exponent of a monomial $P^\alpha Q^\beta$ is $\lambda(\alpha,\beta)$ if the scaling (3.1), with $\kappa = \lambda(\alpha,\beta)$, places this monomial at the same order than Q^4. That is $\lambda(\alpha,\beta) = \frac{4-\alpha}{\beta}$.

monomial	P^5	QP^4	Q^2P^3	P^6	QP^5	Q^2P^4	P^7	QP^6	Q^2P^5	P^8
$\kappa(\alpha,\beta)$	4/5	3/4	2/3	2/3	3/5	1/2	4/7	1/2	2/5	1/2

Tab. 1 Leading exponent $\lambda(\alpha,\beta)$ for different monomials.

It can be seen that the first term that can solve the degeneracy is the one with the largest leading exponent [4]. Table 1 shows the leading exponent for those monomials appearing in the Hamiltonian function up to six order and also monomials of order seven and eight. We can see that the monomial P^5 has the maximum leading exponent and it is the first term to be taken into account to proper split the multiple root. If this term fails, then the next term to be considered is QP^4 and so on.

Now, we move to polar coordinates (2.1) in order to apply Theorem 2.3. The Hamiltonian function in the form (2.2) is given by

$$\begin{aligned} K &= 4r^2(\sin^4\varphi + r^{1/2}\Psi_5(\varphi) + r\Psi_6(\varphi)) + \tilde{K}(\varphi,r,t),\\ \Psi_5(\varphi) &= \sqrt{2}(\gamma_{23}\sin^2\varphi\cos^3\varphi + \gamma_{14}\sin\varphi\cos^4\varphi + \gamma_{05}\cos^5\varphi),\\ \Psi_6(\varphi) &= 2(\gamma_{24}\sin^2\varphi\cos^4\varphi + \gamma_{15}\sin\varphi\cos^5\varphi + \gamma_{06}\cos^6\varphi). \end{aligned} \tag{3.3}$$

Our goal is to analyze the real roots of the equation

$$\Psi(\varphi,r) \equiv \sin^4\varphi + r^{1/2}\Psi_5(\varphi) + r\Psi_6(\varphi) = 0, \tag{3.4}$$

emanating from multiple roots $\varphi = 0$ and $\varphi = \pi$ of the function $\sin^4\varphi$.

To determine the main part of the roots, we introduce a fractional power series of the form (2.4), where the fractional exponents are chosen according to the leading exponent. In this way, if $\lambda(\alpha,\beta)$ is the maximum leading exponent, the first fractional exponent with nonzero coefficient in (2.4) is given by

$$\frac{j}{m} = \frac{1-\lambda(\alpha,\beta)}{2\lambda(\alpha,\beta)}. \tag{3.5}$$

For instance, if $\gamma_{05} \neq 0$ we consider the series

$$\varphi_1 = a_1 r^{1/8} + a_2 r^{2/8} + \cdots, \qquad \varphi_2 = \pi + b_1 r^{1/8} + b_2 r^{2/8} + \cdots.$$

It is almost straightforward to check that, in all the cases of instability of Theorem 2.4, the conditions of Theorem 2.3 are satisfied and we are done.

To prove item 2. of Theorem 2.4, we introduce proper action-angle variables. In this sense, we rewrite the Hamiltonian function as

$$H = H_0(Q,P) + \hat{H}(Q,P,t)\,, \tag{3.6}$$

where

$$H_0(Q,P) = (Q^2 + \alpha P^3)^2 + \beta P^6, \quad \hat{H}(Q,P,t) = \gamma_{15}QP^5 + \gamma_{24}Q^2P^4 + H^{(7)}(Q,P,t)\,. \tag{3.7}$$

The coefficients α and β read

$$\alpha = \frac{1}{2}\gamma_{23}, \quad \beta = \gamma_{06} - \frac{1}{4}\gamma_{23}^2\,.$$

We note that H_0 is a positive definite function, provided that, under the conditions of item 2., $\beta > 0$. Thus, the origin of the truncted system with Hamiltonian H_0 is stable and it is encircled by a family of closed curves, describing periodic motion in a sufficient small neighborhood. Let be the action variable

$$I = \frac{1}{2\pi}\oint P(Q,h)\,dQ, \tag{3.8}$$

where the integral is calculated along a closed phase trajectory. Every trajectory is completely defined by h, where h is a constant such that $H_0 = h$. A direct calculation shows that

$$I = h^{\frac{5}{12}} J_0\,, \tag{3.9}$$

being J_0 a constant. Then, it follows that H_0 reduces to

$$h(I) = \left(\frac{I}{J_0}\right)^{\frac{12}{5}}. \tag{3.10}$$

Moreover, it can be proved that, in action-angle variables, the Hamiltonian takes the form

$$\Gamma = h(I) + h_1(I, w, t), \tag{3.11}$$

where $h_1(I, w, t) = o(h(I))$. However, the nondegeneracy condition

$$\frac{d^2h}{dI^2} = \frac{84 I^{\frac{2}{5}}}{25 J_0^{\frac{12}{5}}} \neq 0 \tag{3.12}$$

is fulfilled for $0 < I \ll 1$. Thus, the Arnold-Moser theorem [1, 13] guarantees the stability of the equilibrium position of the original canonical system.

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