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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Contents

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada
Gijón, 14-18 junio 2021
(pp. 271–276)

Fractional evolution equations in dicrete sequences spaces

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Abstract

In this talk, we consider fractional differential equations (in the sense of Caputo) on the sequence Lebesgue spaces $\ell^p(\mathbb{Z})$ with $p \geq 1$. The associated operator to the Cauchy problem is defined by convolution with a sequence of compact support. We use techniques from Functional Analysis to calculate the solution of the problem. In the case of fractional powers of operators, we give explicitly the solution of the problem. As a consequence, we obtain new integral formulae for certain special functions.

1. Introduction

Of concern in this paper is the following semi discrete Cauchy problem

$$\begin{cases} \partial_t u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where B has the form of a convolution operator in the discrete variable, i.e.

$$Bu(n,t) = \sum_{j\in\mathbb{Z}} b(n-j)u(j,t), \tag{1.2}$$

and b belong to the Banach algebra $\ell^1(\mathbb{Z})$. A typical example is one dimensional discrete Laplacian, Δ_d, which can be obtained taking $b = \delta_{-1} - 2\delta_0 + \delta_1$, where $\delta_i(j)$ denotes the Kronecker delta (or discrete Dirac measure). In such case, equation (1.1) corresponds to the non-homogeneous semi discrete diffusion equation (also known as the semi discrete heat equation or the lattice diffusion equation). The analytical study of such kind of equations have received an increasing interest in the last decade, mainly due to their many applications in diverse areas of knowledge. For instance, in probability theory, the value $u(n,t)$ of (1.1) with $B = \Delta_d$, describes the probability that a continuous-time symmetric random walk on $\mathbb{Z}$ visits a point n at time t; cf. [6, Section 4]. In chemistry, (1.1) describes the flow of a chemical in an infinite system of tanks arranged in a row, where each two neighbors are connected by pipes [10, Section 3] and in transport theory, (1.1) describes the dynamics of an infinite chain of cars, each of them being coupled to its two neighbours. The value $u(n;t)$ is the displacement of car n at time t from its equilibrium position; cf. [5, Example 1]. From an analytical point of view, quite recently Slavik [11] studied the asymptotic behavior of solutions of (1.1) when $B = \Delta_d$, showing that a bounded solution approaches the average of the initial values if the average exists. Note that choosing $b = \delta_{-1} - \delta_0$ in (1.2) we obtain the forward difference operator $B = \Delta$ and hence (1.2) corresponds to the semi discrete transport equation, studied recently by Abadias et.al. [1].

Interestingly, the references [4] and [9] studied fundamental solutions of (1.1) and the second order semi discrete equation

$$\begin{cases} \partial_{tt} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), \quad u_t(n,0) = \phi(n), & n \in \mathbb{Z}, \end{cases} \tag{1.3}$$

when $B = -(-\Delta_d)^\alpha$ is the discrete fractional Laplacian. Particularly, in [9], the authors combine operator theory techniques with the properties of the Bessel functions to develop a theory of analytic semigroups and cosine operators generated by Δ_d and $-(-\Delta_d)^\alpha$. Also note that the fractional forward difference operator $B = -(-\Delta)^\alpha$ has been studied in [1] where maximum and comparison principles in the context of harmonic analysis are proved.

However, to the best of our knowledge, to date there is no attempt to investigate in an unified way fundamental solutions of the general equation (1.1).

Our key observation in this paper concerning this issue is that the discrete fractional Laplacian can be obtained from (1.2) by allowing fractional powers of b as element of the Banach algebra $\ell^1(\mathbb{Z})$. This original approach, that we provide in this paper, allow us to obtain new insights by introducing a completely new method to analyze both qualitative behavior and fundamental solutions of (1.1) in an unified way.

More generally, and in order to provide simultaneously in our analysis the sub diffusive and super difussive cases associated to the equations (1.1) and (1.3), in this paper we include the representation of the fundamental solutions for the following semi discrete equations:

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases} \tag{1.4}$$

in case $0 < \beta \leq 1$ and

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n),\ \ u_t(n,0) = \phi(n), & n \in \mathbb{Z}, \end{cases} \tag{1.5}$$

in case $1 < \beta \leq 2$. In both cases, B is the convolution operator $Bf(n) := (b * f)(n)$ defined on $\ell^p(\mathbb{Z})$, $p \in [1, \infty]$, $b \in \ell^1(\mathbb{Z})$ and $\beta \in (0, 2]$ is a real number. The symbol $\mathbb{D}_t^{\beta}$ denotes the Caputo fractional derivative of order $\beta > 0$. These results have been jointly obtained with Jorge González-Camus and Carlos Lizama from the Universidad de Santiago de Chile to appear in Advances in Difference Equations (2021).

2. A Banach algebra framework

Given $1 \leq p \leq \infty$, we recall that the Banach spaces $(\ell^p(\mathbb{Z}), \|\ \ \|_p)$ are formed by bi-infinite sequences $f = (f(n))_{n\in\mathbb{Z}} \subset \mathbb{C}$ such that

$$\begin{aligned} \|f\|_p : &= \left(\sum_{n=-\infty}^{\infty} |f(n)|^p \right)^{\frac{1}{p}} < \infty, \qquad 1 \leq p < \infty; \\ \|f\|_\infty : &= \sup_{n\in\mathbb{Z}} |f(n)| < \infty. \end{aligned}$$

We remind the natural embeddings $\ell^1(\mathbb{Z}) \hookrightarrow \ell^p(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$, for $1 \leq p \leq \infty$ and that the dual of $\ell^p(\mathbb{Z})$ is identified with $\ell^{p'}(\mathbb{Z})$ where $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 < p < \infty$ and $p = 1$ if $p' = \infty$.

In the case that $f \in \ell^1(\mathbb{Z})$ and $g \in \ell^p(\mathbb{Z})$, we define

$$(f * g)(n) := \sum_{j=-\infty}^{\infty} f(n-j)g(j), \qquad n \in \mathbb{Z}.$$

From Young's Inequality, it follows that $f * g \in \ell^p(\mathbb{Z})$. Note that $(\ell^1(\mathbb{Z}), *)$ is a commutative Banach algebra with identity, that we denote $\delta_0 := \chi_{\{0\}}$. We observe that $\delta_1 * \delta_1 = \delta_2$ and, in general, $\delta_n * \delta_m = \delta_{n+m}$ for $n, m \in \mathbb{Z}$.

The Gelfand transform associated to $(\ell^1(\mathbb{Z}), *)$, is the discrete Fourier transform $\mathcal{F} : \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ (or Fourier series) where

$$\hat{f}(\theta) := \mathcal{F}(f)(e^{i\theta}) := \sum_{n\in\mathbb{Z}} f(n)e^{in\theta}, \quad \theta \in \mathbb{T}.$$

We recall that the spectrum of f, denoted as $\sigma_{\ell^1(\mathbb{Z})}(f)$, is defined by

$$\sigma_{\ell^1(\mathbb{Z})}(f) := \{\lambda \in \mathbb{C}\ :\ (\lambda\delta_0 - f)^{-1} \in \ell^1(\mathbb{Z})\}.$$

In what follows, we consider the general theory of commutative Banach algebra as framework. We collect the results that will be of our interest in the following theorem.

Theorem 2.1 *The following properties hold*:

(i) *The spectrum $Spec(\ell^1(\mathbb{Z}))$ is compact and, consequently, homeomorphic to the unit complex circle, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.*

(ii) $\sigma_{\ell^1(\mathbb{Z})}(f) \subset \{z \in \mathbb{C}\ ;\ |z| < \|f\|_1\}$ *and*

$$(\lambda\delta_0 - f)^{-1} = \sum_{n\geq 0} \lambda^{-n-1} f^n, \qquad \|f\|_1 < |\lambda|. \tag{2.1}$$

(iii) *The algebra $\ell^1(\mathbb{Z})$ is a semi simple regular Banach algebra and the discrete Fourier transform $\mathcal{F}$ is injective.*

(iv) $\mathcal{F}(f*g)=\mathcal{F}(f)\mathcal{F}(g)$ *and*

$$\sigma_{\ell^1(\mathbb{Z})}(f)=\mathcal{F}(f)(\mathbb{T}),\qquad f\in\ell^1(\mathbb{Z}). \tag{2.2}$$

We observe that the range of the Gelfand transform is the Wiener algebra $\mathcal{A}(\mathbb{T})$, the pointwise algebra of absolutely convergent Fourier series, i.e., $F(e^{i\theta})=\sum_{n\in\mathbb{Z}}f(n)e^{i\theta n}$, ($\theta\in\mathbb{T}$) with $f\in\ell^1(\mathbb{Z})$. For $F\in\mathcal{A}(\mathbb{T})$, we also write $F(z)=\sum_{n\in\mathbb{Z}}f(n)z^n$, for $|z|\le 1$.

The inverse discrete Fourier transform is given by the following expressions

$$\mathcal{F}^{-1}(F)(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}F(e^{i\theta})e^{-in\theta}\,d\theta=\frac{1}{2\pi i}\int_{|z|=1}F(z)\frac{dz}{z^{n+1}},\quad n\in\mathbb{Z},$$

for $F\in\mathcal{A}(\mathbb{T})$ (and for other functions in larger sets).

The classical formulation of Wiener's lemma characterizes functions $F\in\mathcal{A}(\mathbb{T})$ which are invertible in $\mathcal{A}(\mathbb{T})$ as follows:

Given $F\in\mathcal{A}(\mathbb{T})$ where $F(e^{i\theta})=\sum_{n\in\mathbb{Z}}f(n)e^{i\theta n}$ for $\theta\in\mathbb{T}$. Then $F(e^{i\theta})\neq 0$ for all $\theta\in\mathbb{T}$ if and only if $1/F\in\mathcal{A}(\mathbb{T})$, i.e., $(1/F)(e^{i\theta})=\sum_{n\in\mathbb{Z}}g(n)e^{i\theta n}$ with $(g(n))_{n\in\mathbb{Z}}\in\ell^1(\mathbb{Z})$; in this case $f*g=\delta_0$. ([7, Theorem 5.5]).

Definition 2.2 Given $\alpha,\beta>0$, we define the vector-valued Mittag-Leffler function, $E_{\alpha,\beta}:\ell^1(\mathbb{Z})\to\ell^1(\mathbb{Z})$, by

$$E_{\alpha,\beta}(a):=\sum_{j=0}^{\infty}\frac{a^j}{\Gamma(\alpha j+\beta)},\qquad a\in\ell^1(\mathbb{Z}).$$

Note that

$$E_{1,1}(a)=\sum_{j=0}^{\infty}\frac{a^j}{j!}=e^a;\qquad E_{2,1}(a)=\sum_{j=0}^{\infty}\frac{a^j}{(2j)!}.$$

The set $\exp(\ell^1(\mathbb{Z})):=\{e^a\ ;a\in\ell^1(\mathbb{Z})\}$ is the connected component of δ_0 in the set of regular elements in $\ell^1(\mathbb{Z})$ ([8, Theorem 6.4.1]).

We follow the usual terminology in semigroup theory: the element a is called the generator of the entire group $(e^{za})_{z\in\mathbb{C}}$; a cosine function, $\mathrm{Cos}(z,a):=E_{2,1}(z^2a)$, and a sine function, $\mathrm{Sin}(z,a):=zE_{2,2}(z^2a)$. We have

$$\mathrm{Sin}(z,a)=\int_{[0,z]}\mathrm{Cos}(s,a)ds,\qquad z\in\mathbb{C},$$

for $a\in\ell^1(\mathbb{Z})$, see [2, Sections 3.1 and 3.14]. Moreover, the Laplace transform of a entire group or a cosine function is connected with the resolvent of its generator as follows:

$$\begin{aligned}(\lambda-a)^{-1}&=\int_0^\infty e^{-\lambda s}e^{as}ds,\qquad \lambda>\|a\|_1,\\ \lambda(\lambda^2-a)^{-1}&=\int_0^\infty e^{-\lambda s}\mathrm{Cos}(s,a)ds,\qquad \lambda>\sqrt{\|a\|_1},\end{aligned} \tag{2.3}$$

see, for example, [2, p. 213].

Example 2.3 For $\alpha,\beta>0$, we have that

$$E_{\alpha,\beta}(z\delta_0)=E_{\alpha,\beta}(z)\delta_0;\qquad E_{\alpha,\beta}(z\delta_1)=\sum_{j=0}^{\infty}\frac{z^j\delta_j}{\Gamma(\alpha j+\beta)}.$$

In particular, $e^{z\delta_1}=\sum_{j=0}^{\infty}\frac{z^j\delta_j}{j!}$ and $\mathrm{Cos}(z,\delta_1)=\sum_{j=0}^{\infty}\frac{z^{2j}\delta_j}{(2j)!}$ are generated by δ_1.

In the next proposition, we collect some basic properties of these vector-valued Mittag-Leffler functions. As usual, we consider Bochner vector-valued integration in the Banach space $\ell^1(\mathbb{Z})$, see for example [12, Section 1.2].

Proposition 2.4 *For $\alpha, \beta > 0$ and $a \in \ell^1(\mathbb{Z})$, we have that*

(i) $\|E_{\alpha,\beta}(a)\|_1 \leq E_{\alpha,\beta}(\|a\|_1)$.

(ii) $\mathcal{F}(E_{\alpha,\beta}(a)) = E_{\alpha,\beta}(\mathcal{F}(a))$; *in particular* $\mathcal{F}(e^{az}) = e^{z\mathcal{F}(a)}$ *and* $\mathcal{F}(Cos(z,a)) = Cos(\mathcal{F}(z), a)$ *for* $z \in \mathbb{C}$.

(iii) $\sigma_{\ell^1(\mathbb{Z})}(E_{\alpha,\beta}(a)) = E_{\alpha,\beta}(\sigma_{\ell^1(\mathbb{Z})}(a))$.

(iv) *The following Laplace transform formula holds*

$$\int_0^\infty e^{-\lambda t} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(t^\alpha a)dt = k!\lambda^{\alpha-\beta}\left((\lambda^\alpha - a)^{-1}\right)^{(k+1)}, \quad \Re(\lambda) > \|a\|_1^{1/\alpha}, \tag{2.4}$$

for $k \in \mathbb{N} \cup \{0\}$.

(v) *For* $0 < \gamma < 1$, $E_{\gamma,1}(a) = \int_0^\infty \Phi_\gamma(t)e^{ta}dt$.

A nice application of the classical Wiener's lemma is the invariance of spectrum for convolution operators defined on $\ell^p(\mathbb{Z})$ for $1 \leq p \leq \infty$. This issue is contained in the following theorem that is the key abstract result in this paper.

Theorem 2.5 *Given* $a \in \ell^1(\mathbb{Z})$, *we define*

$$A(b)(n) := (a * b)(n), \quad n \in \mathbb{Z}, \quad b \in \ell^p(\mathbb{Z}), \tag{2.5}$$

then $A \in \mathcal{B}(\ell^p(\mathbb{Z}))$ *for all* $1 \leq p \leq \infty$. *Moreover,* $\|A\| = \|a\|_1$ *and, for all* $1 \leq p \leq \infty$, *the following identities hold*:

$$\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(A) = \sigma_{\ell^1(\mathbb{Z})}(a) = \mathcal{F}(a)(\mathbb{T}). \tag{2.6}$$

For all $a \in \ell^1(\mathbb{Z})$, *we have that* e^{za} *is an entire group in* $\ell^p(\mathbb{Z})$ *with generator* a *and for all* $1 \leq p \leq \infty$, *the following identities hold*:

$$\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(e^{za}) = \sigma_{\ell^1(\mathbb{Z})}(e^{za}) = e^{z\mathcal{F}(a)(\mathbb{T})}, \qquad z \in \mathbb{C}. \tag{2.7}$$

The element a in the above theorem is also called the *symbol* of the operator A.

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

An important case of finite difference operators are given by sequences in the set

$$c_c(\mathbb{Z}) := \{a \in \ell^1(\mathbb{Z}) : \exists\, m \in \mathbb{Z}_+ : a(n) = 0, \forall\, |n| > m)\}.$$

In such case, the discrete Fourier Transform of $a \in c_c(\mathbb{Z})$ is a trigonometric polynomial

$$\mathcal{F}(a)(e^{i\theta}) = \sum_{j=-m}^{m} a(j)e^{ij\theta}. \tag{3.1}$$

It is interesting to observe that if $\sum_{j=-m}^{m} a(j) = 0$ then $0 \in \sigma_{\ell^1(\mathbb{Z})}(a)$. This follows immediately from (2.6).

Definition 3.1 For $f \in \ell^p(\mathbb{Z})$, with $1 \leq p \leq \infty$, we define the following operators

1. $-\Delta f(n) := f(n) - f(n+1) = ((\delta_0 - \delta_{-1}) * f)(n)$;
2. $\nabla f(n) := f(n) - f(n-1) = ((\delta_0 - \delta_1) * f)(n)$;
3. $\Delta_d f(n) := f(n+1) - 2f(n) + f(n-1) = ((\delta_{-1} - 2\delta_0 + \delta_1) * f)(n)$;
4. $\Delta_{dd} f(n) := f(n+2) - 2f(n) + f(n-2) = ((\delta_{-2} - 2\delta_0 + \delta_2) * f)(n)$;

for $n \in \mathbb{Z}$.

We remark that when considering the above defined operators in the context of numerical analysis, the operators $-\Delta$ and ∇ are related to Euler scheme of approximation, and the operator Δ_d corresponds to the second-order central difference approximation for the second order derivative. The operator Δ_{dd} appears in Bateman's paper [3, Page 506] in connection with the equations of Born and Karman on crystal lattices in vibration.

3.1. The operator $-\Delta$

The forward difference operator $\Delta f(n) := f(n+1) - f(n)$ is a classical operator used in approximation theory and in the theory of difference equations. Considered as an operator from $\ell^p(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$, our main result read as follows.

Theorem 3.2 *The operator $-\Delta f = a * f$ where $a := \delta_0 - \delta_{-1}$ enjoys the following properties*

1. *The norm is given by $\|\Delta\| = 2$;*
2. *The Fourier transform is $\mathcal{F}(a)(z) = 1 - z,\ |z| = 1$;*
3. *For all $1 \le p \le \infty$ the spectrum is given by $\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(-\Delta) = \{z \in \mathbb{T}\ :|z-1| = 1\}$;*
4. *For $|\lambda + 1| > 1$,*
$$(\lambda\delta_0 + a)^{-1} = \sum_{j\ge 0} \frac{\delta_{-j}}{(1+\lambda)^{j+1}}.$$
5. *The associated group is $e^{-za}(n) = e^{-z}\dfrac{z^{-n}}{(-n)!}\chi_{-\mathbb{N}_0}(n),\ z \in \mathbb{C},\ n \in \mathbb{Z}$ and its generator is $-a$.*
6. *The norm of the group is given by $\|e^{-ta}\|_1 = 1, \quad t > 0$;*
7. *The associated cosine function is $Cos(z, -a)(n) = \frac{\sqrt{\pi}}{(-n)!}\left(\frac{z}{2}\right)^{-n+\frac{1}{2}} J_{-n-\frac{1}{2}}(z)\chi_{-\mathbb{N}_0}(n)$ where $z \in \mathbb{C},\ n \in \mathbb{Z}$.*

Similar results are proved for operator ∇ , Δ_d and Δ_{dd}.

4. Fundamental solution for semidiscrete evolution equations

Given $0 < \beta \le 1$, we first consider the equation

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}. \end{cases} \tag{4.1}$$

We recall that function $E_{\alpha,\beta}(b)$ (with $b \in \ell^1(\mathbb{Z})$) is the vector-valued Mittag-Leffler function given in Definition 2.2. The main result is the following Theorem.

Theorem 4.1 *Let $\varphi, \phi \in \ell^p(\mathbb{Z})$, and $g : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{C}$ be such that, for each $t \in \mathbb{R}_+$, $g(\cdot, t) \in \ell^p(\mathbb{Z})$ and $\sup_{s\in[0,t]} ||g(\cdot, s)||_p < \infty$ with $1 \le p \le \infty$.*

(i) *For $0 < \beta < 1$, the function*

$$\begin{aligned} u(n,t) =& (E_{\beta,1}(t^{\beta}b) * \varphi)(n) \\ &+ \int_0^t (t-s)^{\beta-1} \left(E_{\beta,\beta}((t-s)^{\beta}b) * g(\cdot, s)\right)(n)ds,\ n \in \mathbb{Z}, \end{aligned}$$

is the unique solution of the initial value problem (4.1). *Moreover, $u(\cdot, t)$ belong to $\ell^p(\mathbb{Z})$ for $t > 0$.*

(ii) *For $1 < \beta < 2$, the function*

$$\begin{aligned} u(n,t) =& (E_{\beta,1}(t^{\beta}b) * \varphi)(n) + t(E_{\beta,2}(t^{\beta}b) * \phi)(n) \\ &+ \int_0^t (t-s)^{\beta-1} \left(E_{\beta,\beta}((t-s)^{\beta}b) * g(\cdot, s)\right)(n)ds,\ n \in \mathbb{Z}, \end{aligned}$$

is the unique solution of the initial value problem (1.5). *Moreover, $u(\cdot, t)$ belong to $\ell^p(\mathbb{Z})$ for $t > 0$.*

5. Applications to special functions

Take $a = \delta_{-1} - \delta_0$ or $a = \delta_1 - \delta_0$.

(i) For $0 < \beta < 1$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$E^{(n)}_{\beta,1}(t) = \sum_{j=0}^{\infty} \frac{(j+n)!}{j!} \frac{t^j}{\Gamma(\beta(j+n)+1)} = \int_0^\infty \Phi_\beta(\tau) e^{\tau t} \tau^n d\tau.$$

(ii) For $1 < \beta < 2$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$(2t)^{n-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j (j+n)!}{j!} \frac{t^{2j}}{\Gamma(\beta(j+n)+1)} = \frac{\sqrt{\pi}}{2} \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) \tau^{n+\frac{1}{2}} J_{n-\frac{1}{2}}(\tau t)\, d\tau. \tag{5.1}$$

Now take $a = \delta_{-1} - 2\delta_0 + \delta_1$ or $a = \delta_{-2} - 2\delta_0 + \delta_2$.

(i) For $0 < \beta < 1$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$\sum_{j=0}^{\infty} (-1)^j \binom{2(j+n)}{j} \frac{t^{j+n}}{\Gamma(\beta(j+n)+1)} = \int_0^\infty \Phi_\beta(\tau) e^{-2\tau t} I_n(2\tau t) d\tau. \tag{5.2}$$

In particular, when $\beta = \frac{1}{3}$, we get the integral formula for Airy function,

$$\sum_{j=0}^{\infty} (-1)^j \binom{2(j+n)}{j} \frac{t^{j+n}}{\Gamma(\frac{j+n}{3}+1)} = \int_0^\infty 3^{\frac{2}{3}} Ai\left(\frac{\tau}{3^{\frac{1}{3}}}\right) e^{-2\tau t} I_n(2\tau t) d\tau,$$

for $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$.

(ii) For $1 < \beta < 2$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$\sum_{j=0}^{\infty} (-1)^j \binom{2(j+n)}{j} \frac{t^{2(j+n)}}{\Gamma(\beta(j+n)+1)} = \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) J_{2n}(2\tau t) d\tau. \tag{5.3}$$

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