Proceedings

of the

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada

Gijón (Asturias), Spain

June 14-18, 2021

SēMA
Sociedad Española de Matemática Aplicada

Universidad de Oviedo

Editors:

Rafael Gallego, Mariano Mateos

Universidad de Oviedo
Servicio de Publicaciones de la Universidad de Oviedo
Campus de Humanidades. Edificio de Servicios. 33011 Oviedo (Asturias)
Tel. 985 10 95 03 Fax 985 10 95 07
http: www.uniovi.es/publicaciones
servipub@uniovi.es

ISBN: 978-84-18482-21-2

Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

The Local Organizing Committee from the Universidad de Oviedo

Scientific Committee

- Juan Luis Vázquez, Universidad Autónoma de Madrid
- María Paz Calvo, Universidad de Valladolid
- Laura Grigori, INRIA Paris
- José Antonio Langa, Universidad de Sevilla
- Mikel Lezaun, Euskal Herriko Unibersitatea
- Peter Monk, University of Delaware
- Ira Neitzel, Universität Bonn
- JoséÁngel Rodríguez, Universidad de Oviedo
- Fernando de Terán, Universidad Carlos III de Madrid

Sponsors

- Sociedad Española de Matemática Aplicada
- Departamento de Matemáticas de la Universidad de Oviedo
- Escuela Politécnica de Ingeniería de Gijón
- Gijón Convention Bureau
- Ayuntamiento de Gijón

Local Organizing Committee from the Universidad de Oviedo

- Pedro Alonso Velázquez
- Rafael Gallego
- Mariano Mateos
- Omar Menéndez
- Virginia Selgas
- Marisa Serrano
- Jesús Suárez Pérez del Río

Contents

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones
XVI Congreso de Matemática Aplicada
Gijón, 14-18 junio 2021
(pp. 277–283)

KPZ equation approximated by a nonlocal equation

Alexis Molino
Universidad de Almería, Spain

Abstract

Our main concern is the study of several aspects related with solutions of nonlocal problems whose prototype is

$$\begin{cases} u_t = \displaystyle\int_{\mathbb{R}^N} J(x-y)\big(u(y,t)-u(x,t)\big)\mathcal{G}\big(u(y,t)-u(x,t)\big)dy & \text{in } \Omega\times(0,T)\,, \\ u(x,0)=u_0(x) & \text{in } \Omega\,, \\ u(x,t)=h(x,t) & \text{in } \quad (\mathbb{R}^N\setminus\Omega)\times(0,T)\,. \end{cases}$$

where we take, as the most important instance, $\mathcal{G}(s)\sim 1+\frac{\mu}{2}\frac{s}{1+\mu^2 s^2}$ with $\mu\in\mathbb{R}$ as well as $u_0\in L^1(\Omega)$, J is a smooth symmetric function with compact support and Ω is a bounded smooth subset of $\mathbb{R}^N$, with nonlocal Dirichlet boundary condition $h(x,t)$.

The results deal with existence, uniqueness and comparison principle. The main motivation for dealing with these types of equations is that, under a kernel $\mathcal{G}$ rescaled in a suitable way, the unique solution of the above problem converges to a solution of the deterministic Kardar-Parisi-Zhang equation.

1. Introduction

We present some partial results from [15] concerning to the Dirichlet problem. Concretely, existence, uniqueness, comparison principle and rescaling kernel for the following nonlinear parabolic equation with nonlocal diffusion,

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)\big(u(y,t)-u(x,t)\big)\mathcal{G}\big(u(y,t)-u(x,t)\big)dy & \text{in } \Omega\times(0,T)\,, \\ u(x,0)=u_0(x) & \text{in } \Omega\,, \\ u(x,t)=h(x,t) & \text{in } \quad (\mathbb{R}^N\setminus\Omega)\times(0,T)\,, \end{cases} \tag{1.1}$$

for an appropriate functions J and $\mathcal{G}$ (see below (J) and ($\mathcal{G}$)), and its relationship with the deterministic KPZ equation

$$\begin{cases} u_t-\Delta u=\mu|\nabla u|^2 & \text{in } \Omega\times(0,T)\,, \\ u(x,0)=u_0(x) & \text{in } \Omega\,, \\ u(x,t)=h(x,t) & \text{on } \partial\Omega\times(0,T)\,, \end{cases} \tag{1.2}$$

where

1. Ω is a bounded smooth subset of $\mathbb{R}^N$ adding the boundary condition $u(x,t)=h(x,t)$ in $(\mathbb{R}^N\setminus\Omega)\times(0,T)$ for h sufficiently smooth;
2. $T>0$ (possibly infinite) and $\mu\in\mathbb{R}$;
3. u_0 is a smooth enough datum.

1.1. Local problem

The equation $u_t-\Delta u=\mu|\nabla u|^2$, at least for $\mu>0$, is known in the literature as the deterministic Kardar-Parisi-Zhang (KPZ) equation. The KPZ equation was proposed in [13] in the physical theory of growth and roughening of surfaces. Further developments on physical applications of the KPZ equation can be found in [5] (for a survey on more recent aspects we refer to [19]). The deterministic case corresponds to the smoothing from an initially rough surface to a flat one.

The Kardar–Parisi–Zhang equation has given rise to a rich mathematical theory which has had a spectacular recent progress (see [10, 11]). From the point of view of Partial Differential Equations, equations having a gradient term with the so-called natural growth have been largely studied in the last decades by many mathematicians: in addition to the classical reference [14] let us just mention the pioneer paper by Aronson and Serrin [3] and also the result due to Boccardo, Murat and Puel [6].

1.2. Nonlocal problem

Nonlocal evolution equations have been extensively studied to model diffusion processes. The prototype example in this framework is the following one

$$u_t(x,t) = \int_{\mathbb{R}^N} K(x,y)(u(y,t) - u(x,t))dy, \tag{1.3}$$

where the kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative smooth function (not necessarily symmetric) satisfying $\int_{\mathbb{R}^N} K(x,y)dx = 1$ for any $y \in \mathbb{R}^N$ (or variations of it, see for instance [2]). If $u(y,t)$ is thought of as a density at location y at time t and $K(x,y)$ as the probability distribution of jumping from place y to place x, then the rate at which individuals from any other location go to the place x is given by $\int_{\mathbb{R}^N} K(x,y)u(y,t)dy$. On the other hand, the rate at which individuals leave the location x to travel to all other places is $-\int_{\mathbb{R}^N} K(y,x)u(x,t)dy = -u(x,t)$. In the absence of external sources this implies that the density must satisfy equation (1.3).

We are especially interested in symmetric kernels that have compact support; it means that the individuals can jump from a place to other, but they cannot go "too far away". On the contrary, for instance, nonlocal operators that allow "long jumps"correspond to a different choice of kernels. It is the case of the fractional Laplacian that involves a kernel that is singular and that does not have compact support (see, for instance [18] for a survey on this latter class of processes and [1] for the KPZ equation in fractional framework).

In particular, we consider $J : \mathbb{R}^N \to \mathbb{R}$ as a nonnegative radial symmetric function such that

$$J \in C_c(\mathbb{R}^n), \quad \text{with} \quad \int_{\mathbb{R}^N} J(z)\, dz = 1.$$

Choosing the kernel as $K(x,y) = J(x-y)$, equation (1.3) changes into a diffusion equation of convolution type, namely

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t), \qquad \text{in } \Omega \times (0,T) \tag{1.4}$$

(see for instance [4, 7, 9]).

1.3. Background

One of the most important features of nonlocal equations is that they can be rescaled to approximate local ones.

In [8] (see also [16] and [17] for the same type of result in a more general case) it has been proved that, under an appropriate rescaling kernel, solutions of (1.4) converge uniformly to solutions of heat equation. To be more specific, solutions of

$$u_t^\varepsilon(x,t) = \frac{C}{\varepsilon^2}\left[\int_{\mathbb{R}^N} J_\varepsilon(x-y)u^\varepsilon(y,t)dy - u^\varepsilon(x,t)\right] \qquad \text{in } \Omega \times (0,T) \tag{1.5}$$

converge uniformly (when $\epsilon \to 0$) to solutions of

$$v_t = \Delta v \qquad \text{in } \Omega \times (0,T)\,,$$

where $C^{-1} = \frac{1}{2}\int_{\mathbb{R}^N} J(z)z_N^2\, dz$ and $J_\varepsilon(s) = \frac{1}{\varepsilon^N} J\left(\frac{s}{\varepsilon}\right)$.

Let us mention that results in this direction, with the presence of a gradient term of convection type can be found, for instance, in [12]: in such a case the equation is the sum of two terms, one corresponding to the diffusion one, the other to the convection term.

In general, we consider nonlocal problems of the type

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)\big(u(y,t) - u(x,t)\big)\, \mathcal{G}\big(u(y,t) - u(x,t)\big)\, dy, \tag{1.6}$$

where $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ is a suitable continuous function. For instance, if $\mathcal{G} \equiv 1$, then we recover problem (1.4). Let us mention that the case $\mathcal{G}(s) = |s|^{p-2}$, with $p \geq 2$ has been treated in [2] where it is proved that solutions to the rescaled nonlocal problem converge to solutions of the Dirichlet problem for the p–Laplacian evolution equation.

On the contrary, the kind of kernels $\mathcal{G}$ we consider does not have the same structure of the previous ones, since they are bounded and do not satisfy any symmetry assumptions (neither odd nor even).

With this background, it is not surprising that problem (1.2) can be approximated by nonlocal equations. The question is to identify what kind of nonlocal equation approximates, under rescaling, problem (1.2).

1.4. Main results

To conclude this introduction we want to state the most relevant results of this work. In order to not enter in technicalities, let us fix a family of kernels $\mathcal{G}_\mu$ that are the easiest (not trivial) example we can consider: for $\mu \in \mathbb{R}$ let

$$\mathcal{G}_\mu(s) = 1 + \frac{\mu s}{2(1+\mu^2 s^2)}, \qquad s \in \mathbb{R}, \quad \mu \in \mathbb{R},$$

and the corresponding family of nonlocal Dirichlet problems

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)\big(u(y,t)-u(x,t)\big)\,\mathcal{G}_\mu\big(u(y,t)-u(x,t)\big)\,dy & \text{in } \Omega\times(0,T)\,, \\ u(x,0) = u_0(x) & \text{in } \Omega\,, \\ u(x,t) = h(x,t) & \text{in } (\mathbb{R}^N\setminus\Omega)\times(0,T)\,. \end{cases} \tag{1.7}$$

with Ω a bounded domain and u_0 and h smooth enough (see Definition 2.1 and Definition 2.4 for more precise hypotheses).

After have proved the existence, uniqueness (see Theorem 2.3) and a Comparison Principle (see Theorem 2.5) for solutions of (1.7), we face the problem of *rescaled kernels*.

The result we prove, in this model case, reads like this.

Let u be the unique smooth solution to (1.2)*, with suitable initial data u_0 and smooth enough boundary condition $u(x,t) = h(x,t)$ on $\partial\Omega\times(0,T)$. Then there exists a family of functions $\{u^\varepsilon\}$, $\varepsilon>0$, such that u^ε solves the approximating nonlocal problem*

$$\begin{cases} u^\varepsilon_t(x,t) = \displaystyle\frac{C}{\varepsilon^2}\int_{\Omega_{J_\varepsilon}} J_\varepsilon(x-y)\left[\big(u^\varepsilon(y,t)-u^\varepsilon(x,t)\big) + \frac{\mu}{2}\frac{\big(u^\varepsilon(y,t)-u^\varepsilon(x,t)\big)^2}{1+\mu^2\big(u^\varepsilon(y,t)-u^\varepsilon(x,t)\big)^2}\right]dy & \textit{in } \Omega\times(0,T), \\ u^\varepsilon(x,0) = u_0(x) & \textit{in } \Omega, \\ u^\varepsilon(x,t) = h(x,t) & \textit{in } (\Omega_{J_\varepsilon}\setminus\Omega)\times(0,T), \end{cases}$$

with C a suitable constant, $\Omega_{J_\varepsilon} = \Omega + \operatorname{supp} J_\varepsilon$ and the family $\{u^\varepsilon\}$ satisfies

$$\lim_{\varepsilon\to 0}\ \sup_{t\in[0,T]}\left\|u^\varepsilon(x,t)-u(x,t)\right\|_{L^\infty(\Omega)} = 0\,.$$

2. Statement of the results

Let us consider the following equation:

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y;x,t)\,\mathcal{G}(x,u(y;x,t))\,dy, \tag{2.1}$$

where $J:\mathbb{R}^N\to\mathbb{R}$ is a nonnegative radial symmetric function such that

$$J\in C_c(\mathbb{R}^n), \quad \text{with} \quad \int_{\mathbb{R}^N} J(z)\,dz = 1, \tag{J}$$

and where, here and throughout, we denote by $u(y;x,t) := u(y,t)-u(x,t)$ and by

$$C(J) := \int_{\mathbb{R}^N} J(z)z_N^2\,dz < \infty, \quad z=(z_1,z_2,\dots,z_N).$$

As far as the function $\mathcal{G}$ is concerned, we assume that $\mathcal{G}:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a nonnegative Carathéodory function (namely, $\mathcal{G}(\cdot,s)$ is measurable for every $s\in\mathbb{R}$ and $\mathcal{G}(x,\cdot)$ is continuous for almost every $x\in\mathbb{R}^N$) satisfying

$$\exists\,\alpha_2\ge\alpha_1>0\ : \qquad \alpha_1 \le \frac{\mathcal{G}(x,s)s-\mathcal{G}(x,\sigma)\sigma}{s-\sigma} \le \alpha_2, \qquad \forall s,\sigma\in\mathbb{R}\ s\ne\sigma\,,\ \text{and for a.e. } x\in\mathbb{R}^N. \tag{$\mathcal{G}$}$$

Let us first point out that the above condition implies that $\mathcal{G}$ is a positive bounded function, since taking $\sigma=0$ in ($\mathcal{G}$), we get

$$0<\alpha_1\le\mathcal{G}(x,s)\le\alpha_2, \qquad \text{for any } s\in\mathbb{R} \text{ and for a.e. } x\in\mathbb{R}^N.$$

Moreover observe that the above condition relies to be a sort of uniform ellipticity for the operator, while ($\mathcal{G}$) corresponds to a strong monotonicity.

Anyway, let us stress again that, in contrast with all the known results about nonlocal equations of the above type, in our case we do not require any symmetry (neither odd nor even) assumption on $\mathcal{G}$.

The prototype of $\mathcal{G}$ we have in mind (we will come back on this example later) is the following one:

$$\mathcal{G}_\mu(x,s) = 1 + \frac{\mu(x)\, s}{2(1+\mu(x)^2 s^2)}, \quad x \in \Omega, \quad s \in \mathbb{R},$$

where $\mu : \Omega \to \mathbb{R}$ stands for a measurable function. Notice that this function satisfies $\mathcal{G}_\mu(x,0) = 0$ and $\frac{d}{ds}\mathcal{G}_\mu(x,0) = \mu(x)$.

The first kind of results we want to stablish deals with the existence and uniqueness of solutions. More precisely, consider the following problem in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$.

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\,\mathcal{G}(x,u(y;x,t))\,dy, & \text{in } \Omega \times (0,T) \\ u(x,t) = h(x,t), & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

with $h \in L^1\left((\mathbb{R}^N \setminus \Omega) \times (0,\infty)\right)$ and $u_0 \in L^1(\Omega)$.

Let us first observe that the integral expression vanishes outside of $\Omega_J = \Omega + \mathrm{supp}(J)$. In this way, h has only to be prescribed, in fact, in $\Omega_J \setminus \Omega$ and we can rewrite the above problem as

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\Omega_J} J(x-y)u(y;x,t)\,\mathcal{G}(x,u(y;x,t))\,dy, & \text{in } \Omega \times (0,T), \\ u(x,t) = h(x,t), & \text{in } (\Omega_J \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{P}$$

where $T > 0$ may be finite or $+\infty$.

We give now two definitions of solution.

Definition 2.1 Assume that J and $\mathcal{G}$ satisfy (J) and ($\mathcal{G}$), respectively.
For $h(x,t) \in L^1((\Omega_J \setminus \Omega) \times (0,T))$ and $u_0(x) \in L^1(\Omega)$, we define a weak solution of problem (P) as a function $u \in C([0,T); L^1(\Omega))$ such that:

$$u(x,t) = \int_0^t \int_{\Omega_J} J(x-y)u(y;x,\tau)\mathcal{G}(x,u(y;x,\tau))\,dy\,d\tau + u_0(x), \qquad \text{for a.e. x } \in \Omega,\ t \in (0,T), \tag{2.2}$$

$$u(y,t) = h(y,t) \qquad \text{for a.e. } y \in \Omega_J \setminus \Omega \text{ and } t \in (0,T)$$

$$\lim_{t \to 0^+} \|u(x,t) - u_0(x)\|_{L^1(\Omega)} = 0\,.$$

Moreover, if $h(x,t) \in C((\Omega_J \setminus \overline{\Omega}) \times (0,T))$ and $u_0(x) \in C(\overline{\Omega})$, we define a regular solution of problem (P) as a function $u \in C([0,\infty); C(\overline{\Omega}))$ such that:

$$u(x,t) = \int_0^t \int_{\Omega_J} J(x-y)u(y;x,\tau)\mathcal{G}(x,u(y;x,\tau))\,dy\,d\tau + u_0(x), \qquad \text{for any x } \in \overline{\Omega},\ t \in (0,T),$$

$$u(y,t) = h(y,t) \qquad \text{for any } y \in \Omega_J \setminus \overline{\Omega} \text{ and } t \in (0,T)$$

$$\lim_{t \to 0^+} \|u(x,t) - u_0(x)\|_{C(\overline{\Omega})} = 0\,.$$

Some more remarks about the meaning of weak and regular solutions are now in order.

Remark 2.2

i) Observe that, in addition to the different smoothness of the boundary condition and/or the initial datum, the main difference lies on the prescription of data on $\partial\Omega$. Indeed, for weak solutions, h is prescribed in $(\Omega_J \setminus \Omega) \times (0,T)$ and u_0 in Ω, while for regular solutions, h is prescribed in $(\Omega_J \setminus \overline{\Omega}) \times (0,T)$ and u_0 in $\overline{\Omega}$.

ii) As already noticed in [7] (in a different context) the boundary conditions are not understood in a classical way, i.e. it is not true that the solutions of problem (P) pointwise coincide with the prescribed boundary data $h(x,t)$. This is due to the fact that the value at any point $(x,t) \in \partial\Omega \times (0,T)$ depends both on the values of u inside $\overline{\Omega} \times [0,T]$ and on the boundary datum $h(x,t)$, since

$$u(x,t) = \int_0^t \int_{\Omega \cap \mathrm{supp} J} J(x-y)u(y;x,\tau)\, \mathcal{G}\big(x, u(y,\tau) - u(x,\tau)\big)\, dy\, d\tau$$
$$+ \int_0^t \int_{\Omega^c \cap \mathrm{supp} J} J(x-y)\big(h(y,\tau) - u(x,\tau)\big)\, \mathcal{G}\big(x, h(y,\tau) - u(x,\tau)\big)\, dy\, d\tau + u_0(x)\,.$$

Consequently, in contrast with the local case, the equation is solved up to the boundary, depending, near $\partial\Omega$, also of the prescribed boundary condition.

iii) Let us stress that the regularity required in the definition of weak solutions is the less restrictive in order to give sense to the formulation, and to the boundary and initial conditions. Anyway from (2.2) we deduce that the time derivative $u_t(x,t)$ of u also belongs to $C((0,\infty); L^1(\Omega))$.

Let us also point out that the weak solution framework is the more natural one in order to prove the existence of a solution. Indeed we only require an L^1 regularity to prove the existence of a solution.

Finally we want to underline that the nonlocal operator involved in such equation does not have the regularizing effect that is typical of the Laplacian, but leaves unchanged the regularity of the initial and boundary data.

In this framework, the existence result is the following:

Theorem 2.3 [*Existence*] *Consider problem (P) and suppose that (J) and $(\mathcal{G})$ are in force. Then*:

i) For any $u_0 \in L^1(\Omega)$ and $h \in L^1((\Omega_J \setminus \Omega) \times (0,T))$ there exists a unique weak solution;

ii) For any $u_0 \in C(\overline{\Omega})$ and $h \in C((\Omega_J \setminus \overline{\Omega}) \times [0,T))$ there exists a unique regular solution and moreover its time derivative belongs to $C(\overline{\Omega} \times (0,T))$.

Once we have deduced the existence of a solution, one important tool is to compare two solutions, or, more generally a sub and a supersolution. Here we recall what we mean by those concepts in our setting.

Definition 2.4 A function $u \in C(\overline{\Omega} \times [0,T])$ is a regular subsolution to problem (P) if it satisfies $u_t \in C(\overline{\Omega} \times (0,T))$ and

$$\begin{cases} u_t(x,t) \le \displaystyle\int_{\Omega_J} J(x-y)u(y;x,t)\, \mathcal{G}(x, u(y;x,t))\, dy, & \text{in } \overline{\Omega} \times (0,T), \\ u(x,t) \le h(x,t), & \text{in } (\Omega_J \setminus \overline{\Omega}) \times (0,T), \\ u(x,0) \le u_0(x), & \text{in } \overline{\Omega}, \end{cases} \tag{2.3}$$

with $u_0(x) \in C(\overline{\Omega})$ and $h(x,t) \in C((\Omega_J \setminus \overline{\Omega}) \times (0,T))$.
As usual, a regular supersolution is defined analogously by replacing "$\le$" with "$\ge$". Clearly, a regular solution is both a regular subsolution and a regular supersolution.

Next, we state the comparison principle:

Theorem 2.5 [*Comparison Principle*] *Let u and v be a regular subsolution and a regular supersolution of problem (P), respectively, with boundary data $h_1(x,t)$ and $h_2(x,t)$ and initial data $u_0(x)$ and $v_0(x)$, respectively. If $h_1(x,t) \le h_2(x,t)$ in $\Omega_J \setminus \overline{\Omega}$ and $u_0(x) \le v_0(x)$ in $\overline{\Omega}$, then $u \le v$ in $\overline{\Omega} \times [0,T]$.*

Remark 2.6 The existence, uniqueness and comparison principle are also true relaxing the hypotheses on the kernel $J(x-y)$ by considering a more general one of the form $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^+$ with compact support in $\Omega \times B(0,\rho)$, with $\rho > 0$ such that

$$0 < \sup_{y \in B(0,\rho)} K(x,y) = R(x) \in L^\infty(\Omega).$$

The next result we want to state relates solutions of local and nonlocal equations. In order to do it, let us fix a Hölder continuous function $\mu : \overline{\Omega} \to \mathbb{R}$ with exponent $\alpha \in (0, 1)$, and consider

$$\mathcal{G}_\mu(x, s) = 1 + \frac{\mu(x)\, s}{2(1 + \mu(x)^2 s^2)}, \qquad (x, s) \in \overline{\Omega} \times \mathbb{R}. \tag{2.4}$$

The local problem we are interested in is the following

$$\begin{cases} v_t(x,t) = \Delta v(x,t) + \mu(x)|\nabla v(x,t)|^2 & \text{in } \Omega \times (0,T), \\ v(x,t) = h_0(x,t) & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = v_0(x) & \text{in } \Omega\,. \end{cases} \tag{2.5}$$

Observe that if, for the same $0 < \alpha < 1$, we have $\partial\Omega \in C^{2+\alpha}$, $v_0 \in C^{1+\alpha}(\overline{\Omega})$, $h \in C^{1+\alpha,1+\alpha/2}(\partial\Omega \times [0,T])$ with v_0 and h compatible (namely, they are globally $C^{1+\alpha,1+\alpha/2}$ functions of the parabolic boundary of the cylinder) and the equation holds up to the boundary, then Theorem 6.1 of Chapter V in [14] provides a solution $v \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times (0,T])$.

Such a result becomes trivial if we assume $\mu(x) = \mu \in \mathbb{R}$, after the Hopf–Cole transformation, since solutions of the heat equation satisfy the required regularity.

We set here the definition of *classical solution* and then we state our convergence result.

Definition 2.7 We say that $v \in C(\overline{\Omega} \times [0,T]) \cap C^{2+\alpha,1+\alpha/2}(\Omega \times (0,T))$ is a *classical solution* to the Dirichlet problem (2.5) if it satisfies both the equations and the boundary and initial conditions in a pointwise sense.

Finally, the main result of this work establishes that solutions of the deterministic equation KPZ can be uniformly approximated by solutions of nonlocal problems by means of a suitable kernel rescaled.

Theorem 2.8 *Let Ω be a $C^{2+\alpha}$, with $\alpha \in (0,1)$, bounded domain of $\mathbb{R}^N$, $N \geq 1$, and let v be a classical solution of the quasilinear problem* (2.5) *with $h \in C^{1+\alpha}\left(\Omega_{J_\varepsilon} \backslash \Omega \times (0,T]\right)$ such that $h\big|_{\partial\Omega\times(0,T)} = h_0(x,t)$ and $v_0 \in C^{1+\alpha}(\overline{\Omega})$. Assume that J satisfies (J) and that for a.e. x in Ω, $\mathcal{G}(x,s)$ is a $C^{1+\alpha}$ function with respect to the s variable such that that $(\mathcal{G})$ holds true. For any $\varepsilon > 0$, let u^ε denote the unique solution to*

$$\begin{cases} u_t^\varepsilon(x,t) = \dfrac{C(x)}{\varepsilon^2} \displaystyle\int_{\Omega_{J_\varepsilon}} J_\varepsilon(x-y) u^\varepsilon(y;x,t) \mathcal{G}(x, u^\varepsilon(y;x,t))dy & \text{in } \overline{\Omega} \times (0,T), \\ u^\varepsilon(x,t) = h(x,t) & \text{in } (\Omega_{J_\varepsilon} \setminus \overline{\Omega}) \times (0,T), \\ u^\varepsilon(x,0) = v_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{2.6}$$

with $C(x)^{-1} = \frac{1}{2} C(J) \mathcal{G}(x,0)$ and $\mu(x) = \dfrac{2\mathcal{G}_s'(x,0)}{\mathcal{G}(x,0)}$ for any a.e. $x \in \Omega$.

Then we have:

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \left\| u^\varepsilon(x,t) - v(x,t) \right\|_{L^\infty(\Omega)} = 0\,.$$

Acknowledgements

The author is supported under project PGC2018–096422–B–I00 (MCIU/AEI/FEDER, UE) and by Junta de Andalucía FQM-116 (Spain).

References

[1] B. Abdellaoui, I. Peral and A. Primo. On the KPZ equation with fractional diffusion. *preprint. arXiv:1904.04593.*

[2] F. Andreu, J.M. Mazón, J.D. Rossi and J. Toledo. A nonlocal p-Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions. *SIAM J. Math. Anal.*, 40:1815–1851 (2009).

[3] D.G. Aronson, J. Serrin. Local behavior of solutions of quasilinear parabolic equations. *Arch. Rat. Mech. Anal.*, 25:81–122 (1967).

[4] P. Bates, P. Fife, X. Ren, X. Wang. Travelling waves in a convolution model for phase transitions. Arch. Ration. Mech. Anal., 138:105–136 (1997).

[5] A.L. Barabási and H.E. Stanley. Fractal Concepts in Surface Growth. *Cambridge University Press* (1995).

[6] L. Boccardo, F. Murat, J.P. Puel. Existence results for some quasilinear parabolic equations. *Nonlinear Analysis T.M.A.*, 13:373–392 (1989).

[7] E. Chasseigne, M. Chaves, J.D. Rossi. Asymptotic behavior for nonlocal diffusion equations. *J. Math. Pures Appl.*, 86:271–291 (2006).

[8] C. Cortázar, M. Elgueta and J.D. Rossi. Nonlocal diffusion problems that approximate the heat equation with Dirichlet boundary conditions. *Israel Journal of Mathematics*, 170:53–60 (2009).

[9] C. Cortázar, M. Elgueta, J.D. Rossi, N. Wolanski. Boundary fluxes for non-local diffusion. *J. Differential Equations*, 234:360–390 (2007).

[10] I. Corwin. The Kardar-Parisi-Zhang equation and universality class. *Random Matrices Theory Appl.*, 1:1130001 (2012).

[11] M. Hairer. Solving the KPZ equation. *Annals of Mathematics*, 178:559–664 (2013).

[12] L. Ignat and J.D. Rossi. A nonlocal convection-diffusion equation. *J. Funct. Anal.*, 251:399–437 (2007).

[13] M. Kardar, G. Parisi, Y.C. Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56:889–892 (1986).

[14] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva. Linear and quasi-linear equations of parabolic type. *Translations of Math. Monographs*, vol 23, Providence, 1968.

[15] T. Leonori, A. Molino and S. Segura de León. Parabolic equations with natural growth approximated by nonlocal equations. *Communications in Contemporary Mathematics*, 23(1): 32 pp. (2021).

[16] A. Molino and J.D. Rossi. Nonlocal diffusion problems that approximate a parabolic equation with spatial dependence. *Z. Angew. Math. Phys.*, 67:41 (2016).

[17] A. Molino and J.D. Rossi. Nonlocal Approximations to Fokker-Planck Equations. *Funkcialaj Ekvacioj*, 62:35–60 (2019).

[18] E. Valdinoci. From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SeMA*, 49:33–44 (2009).

[19] H. S. Wio, C. Escudero, J. A. Revelli, R. R. Deza, M. S. de La Lama. Recent developments on the Kardar-Parisi-Zhang surface-growth equation. *Phil. Trans. R. Soc. A* 369:396 (2011).