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Universidad de Oviedo

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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Symmetry analysis and conservation laws of a family of non-linear viscoelastic wave equations

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Abstract

This work considers a non-linear viscoelastic wave equation with non-linear damping and source terms. We analyze the partial differential equation from the point of view of Lie symmetries. Firstly, we apply Lie's method to obtain new symmetries. Hence, we transform the partial differential equation into an ordinary differential equation, by using the symmetries. Moreover, new solutions are derived from the ordinary differential equation. Finally, by using the direct method of multipliers, we construct low-order conservation laws depending on the form of the damping and source terms.

1. Introduction

Recently, several viscoelastic wave equations have been studied. The single viscoelastic wave equation of the form

$$u_{tt} - \Delta u + \int_0^1 h(t-s)\Delta u(x,s)ds + f(u_t) = g(u)$$

in $\Omega \times (0, \infty)$, where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$, with initial and boundary conditions, has been extensively studied. Many results concerning non-existence and blow-up in finite time have been proved [3–7, 10].

Furthermore, the non-linear viscoelastic wave equation with damping and source terms

$$u_{tt} - u_{xx} + f(u_t) = g(u), \qquad x \in \Omega,\ t > 0, \tag{1.1}$$

has also been very studied obtaining similar results. As in the single viscoelastic wave equation, in the absence of the source term ($g = 0$), it is well-known that the damping term $f(u_t)$ assures global existence and decay of the solution energy for arbitrary initial data. In the same way, in the absence of the damping term, the source term causes finite time blow-up of solutions with a large initial data (negative initial energy). Here, the interaction between the damping term and the source term makes the problem more interesting.

The aim of this work is to obtain the Lie point symmetries of equation (1.1). Afterwards, we present the reductions obtained from the symmetries, transforming the PDE into an ODE. Moreover, we obtain traveling wave solutions by the comparison between equation (1.1) and similar equations studied previously [1, 2, 8]. Finally, we give a complete classification of the conservation laws admitted by equation (1.1).

2. Lie point symmetries and reductions

It is considered a one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \end{aligned} \tag{2.1}$$

where ϵ is the group parameter. These transformations leave invariant the set of solutions of equation (1.1). The associated Lie algebra of infinitesimal symmetries is given by the infinitesimal generator

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \tag{2.2}$$

Each infinitesimal generator (2.2) generates a transformation obtained by solving the system of ODEs

$$\frac{\partial \hat{x}}{\partial \epsilon} = \xi(\hat{x}, \hat{t}, \hat{u}), \qquad \frac{\partial \hat{t}}{\partial \epsilon} = \tau(\hat{x}, \hat{t}, \hat{u}), \qquad \frac{\partial \hat{u}}{\partial \epsilon} = \eta(\hat{x}, \hat{t}, \hat{u}),$$

satisfying the initial conditions

$$\hat{x}|_{\epsilon=0} = x, \qquad \hat{t}|_{\epsilon=0} = t, \qquad \hat{u}|_{\epsilon=0} = u,$$

with ϵ the group parameter.

The symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi(x,t,u)u_x + \tau(x,t,u)u_t - \eta(x,t,u) = 0.$$

For equation (1.1), a PDE with two independent variables, a single group reduction transforms the PDE into ODEs, easier to solve than the original equation.

We require that the transformation (2.1) leaves invariant the set of solutions of equation (1.1). This leads to an overdetermined linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$, generated by requiring that

$$\mathrm{pr}^{(2)}X\,(u_{tt} - u_{xx} + f(u_t) - g(u)) = 0,$$

where $\mathrm{pr}^{(2)}X$ is the 2-th order prolongation of the vector field X defined by

$$\mathrm{pr}^{(2)}X = X + \eta_x\frac{\partial}{\partial u_x} + \eta_t\frac{\partial}{\partial u_t} + \eta_{xx}\frac{\partial}{\partial u_{xx}} + \eta_{xt}\frac{\partial}{\partial u_{xt}} + \eta_{tt}\frac{\partial}{\partial u_{tt}},$$

with the coefficients

$$\begin{aligned}
\eta_x &= D_x\eta - u_t D_x\tau - u_x D_x\xi,\\
\eta_t &= D_t\eta - u_t D_t\tau - u_x D_t\xi,\\
\eta_{xx} &= D_x(\eta_x) - u_{xt} D_x\tau - u_{xx} D_x\xi,\\
\eta_{xt} &= D_t(\eta_x) - u_{xt} D_x\tau - u_{xx} D_t\xi,\\
\eta_{tt} &= D_t(\eta_t) - u_{tt} D_t\tau - u_{xt} D_t\xi,
\end{aligned}$$

where D_x and D_t are the total derivatives of x and t, respectively.

Applying the previous condition to equation (1.1), we get a system of equations for the infinitesimals. Then, by solving the system, we can make a Lie symmetries classification.

Theorem 2.1 *The Lie point symmetries of the non-linear viscoelastic wave equation* (1.1)*, with $f(u_t)$ and $g(u)$ arbitrary functions, are generated by the operators*

$$X_1 = \partial_x, \quad X_2 = \partial_t.$$

The symmetries of Theorem 2.1 yield to the one-parameter symmetry transformation groups

$$\begin{aligned}
(\hat{x},\hat{t},\hat{u})_1 &= (x+\epsilon, t, u), \quad \text{space translation,}\\
(\hat{x},\hat{t},\hat{u})_2 &= (x, t+\epsilon, u), \quad \text{time translation.}
\end{aligned}$$

From the generator $\lambda X_1 + X_2$, we obtain the traveling wave reductions

$$z = x - \lambda t, \qquad u(x,t) = h(z), \tag{2.3}$$

where $h(z)$ satisfies

$$(\lambda^2 - 1)h'' + f(-\lambda\, h') - g(h) = 0. \tag{2.4}$$

3. Traveling wave solutions

Let us consider the second-order equation (2.4)

$$h'' = \frac{1}{1-\lambda^2} f(-\lambda h') + \frac{1}{1-\lambda^2} g(h). \tag{3.1}$$

We can find equation (2.4) studying other mathematical models. For instance, the general solution of a second-order ODE of the form

$$h'' = \frac{1}{\lambda}\left(\mu h' + \frac{1}{2}h^2 - \omega h - c_0\right), \tag{3.2}$$

with c_0 an arbitrary constant and λ, μ, ω satisfying $\omega = \frac{6\mu^2}{25\lambda}$, was obtained by Kudryashov [8]. The general solution is given in terms of the Weierstrass elliptic function, with invariants $g_2 = 0$ and $g_3 = c_1$,

$$h(z) = \omega_k + \frac{6\alpha^2}{25\beta} - \exp\left\{\frac{2z\alpha}{5\beta}\right\}\mathcal{P}\left(c_2 - \frac{5\beta}{\alpha\sqrt{12\beta}}\exp\left\{\frac{z\alpha}{5\beta}\right\},\ 0,\ c_1\right),$$

where c_1 and c_2 are arbitrary constants.

The comparison between equation (3.1) and equation (3.2) shows that these equations are the same if

$$\begin{aligned} f(-\lambda h') &= \tfrac{1-\lambda^2}{\lambda}\mu h', \\ g(h) &= \tfrac{1}{\lambda}\left(\tfrac{1}{2}h^2 - \omega h - c_0\right). \end{aligned}$$

Hence, the solutions of equation (3.1) and equation (3.2) are equal with the previous condition. Finally, by undoing the change of variables (2.3), a exact solution of the non-linear viscoelastic wave equation (1.1) is

$$u(x,t) = \omega + \frac{6\alpha^2}{25\beta} - \exp\left\{\frac{2(x-\lambda t)\alpha}{5\beta}\right\}\mathcal{P}\left(c_2 - \frac{5\beta}{\alpha\sqrt{12\beta}}\exp\left\{\frac{(x-\lambda t)\alpha}{5\beta}\right\}, 0, c_1\right). \tag{3.3}$$

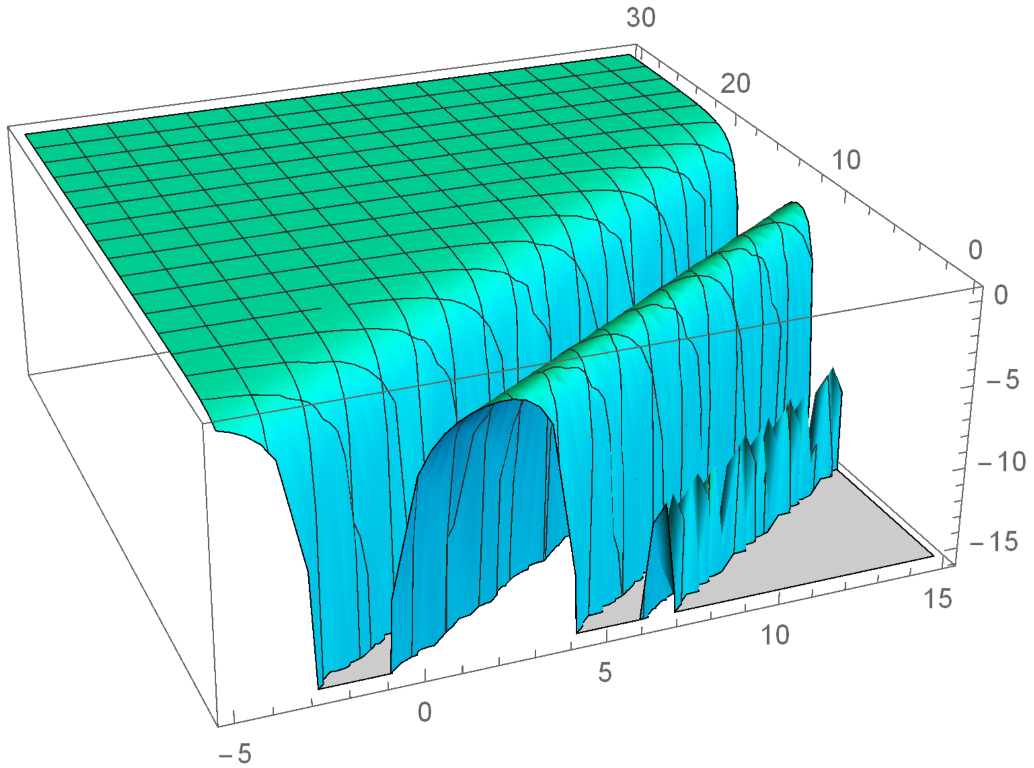

Fig. 1 Solution (3.3) for $\lambda = \alpha = \beta = c_1 = c_2 = 1$.

Solution (3.3) is a soliton (see Fig. 1).

4. Conservation laws

A conservation law admitted by equation (1.1) satisfies the divergence identity

$$D_tT + D_xX = (u_{tt} - u_{xx} + f(u_t) - g(u))Q,$$

called the characteristic equation for the conserved density T and the conserved flux X.

However, the general form for low-order multipliers Q in terms of u and derivatives of u is given by those variables that can be derived to obtain a leading derivative of the equation. Clearly, u_{tt} can be obtained by the derivative of u_t with respect to t, and u_{xx} can be obtained by the derivative of u_x with respect to x.

This determines

$$Q(t,x,u,u_t,u_x)$$

as the general form for a low-order multiplier for equation (1.1).

All low-order multipliers can be found by solving the determining equation

$$E_u((u_{tt} - u_{xx} + f(u_t) - g(u))Q) = 0, \tag{4.1}$$

where E_u represents the Euler operator with respect to u [9], that is

$$E_u = \partial_u - D_x\partial_{u_x} - D_t\partial_{u_t} + D_xD_t\partial_{u_{xt}} + D_x^2\partial_{u_{xx}} + \cdots .$$

Hence, we write and split the determining equation (4.1) with respect to u_{xx}, u_{tt}, u_{tx}, yielding an overdetermined system in Q, $f(u_t)$, $g(u)$. The multipliers are found by solving the system with the same algorithmic method used for the determining equation for infinitesimal symmetries. Thus, we obtain a complete classification of multipliers and conservation laws.

Tab. 1 Multipliers admitted by equation (1.1), with $f(u_t) \neq 0$.

$f(u_t)$	$g(u)$	Q
$f_0u_t + f_1$	arbitrary	$u_x\, e^{f_0t}$
f_0	arbitrary	$u_x,\, u_t$
f_0	$g_1\, e^{g_0u} - f_0$	$u_t,\, u_x,\, tu_t + xu_x + \frac{2}{g_0}$
$-g_0 - \frac{1}{f_0u_t+f_1}$	g_0	$f_0u_tu_x + f_1u_x$
$-\frac{4f_0}{u_t+f_1} + f_2$	$\frac{4f_0}{f_1} - f_2$	$u_t + f_1$

The multipliers with $f(u_t) \neq 0$ are shown in Tab. 1.

Theorem 4.1 *All non-trivial low-order conservation laws admitted by the non-linear viscoelastic wave equation* (1.1)*, with* $f(u_t) \neq 0$*, are given below.*

1. *For* $f(u_t) = f_0$*,* $g(u)$ *arbitrary function and* $Q = u_t$*, we obtain the conservation law*

$$T = \frac{1}{2}{u_x}^2 + \frac{1}{2}{u_t}^2 + \int g(u) + f_0\,\mathrm{d}u,$$
$$X = -u_tu_x.$$

2. *For* $f(u_t) = f_0$*,* $g(u)$ *arbitrary function and* $Q = u_x$*, we obtain the conservation law*

$$T = u_tu_x,$$
$$X = -\frac{1}{2}{u_x}^2 - \frac{1}{2}{u_t}^2 + \int g(u) + f_0\,\mathrm{d}u.$$

3. *For* $f(u_t) = f_0$*,* $g(u) = g_1\, e^{g_0u} - f_0$ *and* $Q = tu_t + xu_x + \frac{2}{g_0}$*, we obtain the conservation law*

$$T = \frac{1}{2g_0}\, 2\,te^{ug_0}g_1 + \left(t{u_t}^2 + t{u_x}^2 + 2\,u_xxu_t\right)g_0 + 4\,u_t,$$
$$X = \frac{1}{2g_0}\, 2\,xe^{ug_0}g_1 + \left(-2\,tu_tu_x - x{u_t}^2 - {u_x}^2x\right)g_0 - 4\,u_x.$$

4. *For* $f(u_t) = f_0u_t + f_1$*,* $g(u)$ *an arbitrary function and* $Q = u_x\, e^{f_0t}$*, we obtain the conservation law*

$$T = u_x\mathrm{e}^{f_0\,t}u_t,$$
$$X = \int \mathrm{e}^{f_0\,t}\left(g(u) + f_1\right)\mathrm{d}u + \frac{1}{2}\left(-{u_t}^2 - {u_x}^2\right)\mathrm{e}^{f_0\,t}.$$

5. *For* $f(u_t) = -g_0 - \frac{1}{u_t\,f_0+f_1}$*,* $g(u) = g_0$ *and* $Q = f_0u_tu_x + f_1u_x$*, we obtain the conservation law*

$$T = \frac{1}{6}\,f_0\,{u_x}^3 + \frac{1}{2}\,f_0\,{u_t}^2u_x + f_1\,u_xu_t,$$
$$X = -\frac{1}{2}\,f_0\,u_t{u_x}^2 - \frac{1}{2}\,{u_x}^2f_1 - \frac{1}{6}\,f_0\,{u_t}^3 - u - \frac{1}{2}\,{u_t}^2f_1.$$

6. *For* $f(u_t) = -\frac{4f_0}{u_t+f_1} + f_2$*,* $g(u) = \frac{4f_0}{f_1} - f_2$ *and* $Q = u_t + f_1$*, we obtain the conservation law*

$$T = \frac{1}{2}{u_x}^2 + \frac{1}{2}{u_t}^2 + f_1\,u_t + 4\,\frac{f_0\,u}{f_1},$$
$$X = (-u_t - f_1)\,u_x.$$

5. Conclusions

In this work, we have obtained some Lie point symmetries for the viscoelastic wave equation (1.1). However, for a future paper we will study the Lie point symmetries complete classification of equation (1.1), in the presence of damping and source terms, for different expressions of the functions f and g. Then, we have constructed the corresponding reduced equation. This reduction makes easier the resolution of the viscoelastic wave equation (1.1), in order to obtain solutions of physical interest such as solitons. Moreover, we have obtained a traveling wave solution from the reduced equation by the comparison between equation (1.1) and comparable equations studied before by other authors. Furthermore, classical Lie symmetries are not the only ones that can be studied. Another symmetries, such as non-classical or potential symmetries, can also be studied in the future. Finally, we have derived the non-trivial low-order conservation laws by using the direct multiplier method developed by Anco and Bluman.

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