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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier “Pancho” Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: “a mathematician is a device for turning coffee into theorems”. Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Nonlinear Analysis in Lorentzian Geometry: The maximal hypersurface equation in a Generalized Robertson-Walker spacetime

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Abstract

In this article we obtain a uniqueness result for the maximal hypersurface equation in a spatially open Generalized Robertson-Walker spacetime by means of Bochner's technique and a generalized maximum principle.

1. Introduction

Maximal hypersurfaces have played a key role in the study of General Relativity since they describe the physical space that can be measured in the transition from an expanding to a contracting phase of the universe. Maximal hypersurfaces constitute a useful initial set for the Cauchy problem in General Relativity [20]. Namely, Lichnerowicz proved that a Cauchy problem with initial conditions on a maximal hypersurface is reduced to a second order nonlinear elliptic differential equation and a first order linear differential system [10]. Moreover, the existence of constant mean curvature spacelike hypersurfaces (and in particular maximal) is necessary for the study of the structure of singularities in the space of solutions of Einstein's equations [2].

From a mathematical standpoint, maximal hypersurfaces enable us to understand the structure of the spacetime. Indeed, for some asymptotically flat spacetimes the existence of a foliation by maximal hypersurfaces was proved in [4]. As a matter of fact, maximal hypersurfaces appear as critical points of the area functional (see for instance [3]).

The study of maximal hypersurfaces from a mathematical perspective was boosted by the discovery of new nonlinear elliptic problems associated to these geometric objects. Indeed, the function defining a maximal graph in the $(n+1)$-dimensional Lorentz-Minkowski spacetime satisfies a second order PDE known as the maximal hypersurface equation in $\mathbb{L}^{n+1}$. Furthermore, the well-known Calabi-Bernstein theorem states that the only entire solutions to the maximal hypersurface equation in $\mathbb{L}^{n+1}$ are the affine functions. This result was proved by Calabi [5] for $n \leq 4$ and later extended to arbitrary dimension by Cheng and Yau [6].

These Calabi-Bernstein type results for the maximal hypersurface equation have been a subject of study in recent years, being extended to several ambient spacetimes such as standard static spacetimes [18, 19], pp-waves [16], doubly warped product spacetimes [8], among others. In this article we will focus on the models known as Generalized Robertson-Walker (GRW) spacetimes. These spacetimes were introduced in [1] to extend the classical notion of Robertson-Walker spacetime to the case where the fiber does not necessarily have constant sectional curvature. In particular, we will deal with the spatially open case, i.e., the case where the fiber is a complete non-compact Riemannian manifold. This is due to the fact that some experimental observations and theoretical arguments suggest that spatially open models provide a more accurate description of our current universe [7]. Furthermore, spatially closed universes lead to a violation of the holographic principle, making spatially open spacetimes compatible with a possible theory that unifies gravity and quantum mechanics [13].

Consequently, our aim in this article will be to particularize some of the results obtained in [14] to the maximal case, which will enable us to obtain uniqueness results for the maximal hypersurface equation in spatially open GRW spacetimes. The technique that will be used is based on combining Bochner formula with a generalized maximum principle (see [9, 15] for different ways of using these ideas to obtain parametric uniqueness results).

2. Preliminaries

Let (F, g_F) be an $n(\geq 2)$-dimensional (connected) Riemannian manifold, I an open interval in $\mathbb{R}$ and f a positive smooth function defined on I. Consider now the product manifold $\overline{M} = I \times F$ endowed with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\, \pi_F^*(g_F), \tag{2.1}$$

where π_I and π_F denote the projections onto I and F, respectively. The Lorentzian manifold $(\overline{M}, \overline{g})$ is a warped product (in the sense of [12, Chap. 7]) with base $(I, -dt^2)$, fiber (F, g_F) and warping function f. Endowing

$(\overline{M}, \overline{g})$ with the time orientation induced by $\partial_t := \partial/\partial t$ we can call it, following the terminology introduced in [1], an $(n+1)$-dimensional Generalized Robertson-Walker (GRW) spacetime.

In any GRW spacetime there is a distinguished timelike and future pointing vector field, $K := f(\pi_I)\partial_t$ that satisfies

$$\overline{\nabla}_X K = f'(\pi_I)\, X \tag{2.2}$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of the Lorentzian metric (2.1). Thus, K is conformal and its metrically equivalent 1-form is closed.

Given an n-dimensional manifold M, an immersion $\psi : M \to \overline{M}$ is called spacelike if the Lorentzian metric (2.1) induces a Riemannian metric g on M through ψ. In this codimension one case, M is called a spacelike hypersurface. Along this article, we will denote the restriction of π_I along ψ by τ. It can be easily seen that its gradient is given by $\nabla\tau = -\partial_t^\top$, where $\partial_t^\top$ is the tnagential component of ∂_t along ψ. In addition, we also have $\sinh^2\varphi = |\nabla\tau|^2$.

Furthermore, the time-orientation of $\overline{M}$ allows to globally define on each spacelike hypersurface M in $\overline{M}$ a unique unitary timelike vector field $N \in \mathfrak{X}^\perp(M)$ with the same time-orientation as ∂_t .

Denoting by A the shape operator associated to N, the mean curvature function associated to N is $H := -(1/n)\mathrm{trace}(A)$. A spacelike hypersurface with identically zero constant mean curvature is called maximal hypersurface.

Among the family of spacelike hypersurfaces in a GRW spacetime we should highlight the subfamily of spacelike graphs. Given an $n(\geq 2)$-dimensional Riemannian manifold (F, g_F) and a smooth function $f : I \longrightarrow \mathbb{R}^+$ we can consider in the GRW spacetime $\overline{M} = I \times_f F$ the graph

$$\Sigma_u = \{(u(p), p) : p \in \Omega\},$$

where $\Omega \subseteq F$, $u \in C^\infty(\Omega)$ and $u(\Omega) \subseteq I$. The induced metric on Ω from the Lorentzian metric on $\overline{M}$, via the graph Σ_u is given by

$$g_u = -du^2 + f(u)^2 g_F.$$

Note that g_u is positive definite (i.e., Σ_u is spacelike) if and only if u satisfies

$$|Du| < f(u).$$

In this case,

$$N = \frac{1}{f(u)\sqrt{f(u)^2 - |Du|^2}}\left(f(u)^2\partial_t + Du\right)$$

is a future pointing unit normal vector field on Σ_u and when $\Omega = F$ the spacelike graph is said to be entire. From [12, Prop. 7.35] we obtain that the mean curvature function of a spacelike graph associated to N is

$$H = \mathrm{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 - |Du|^2}}\right) + \frac{f'(u)}{n\sqrt{f(u)^2 - |Du|^2}}\left(n + \frac{|Du|^2}{f(u)^2}\right), \tag{2.3}$$

where div represents the divergence operator in (F, g_F).

Our aim in this article will be to obtain a uniqueness result for the solutions of the maximal hypersurface equation in a spatially open GRW spacetime. Namely, we are interested in the solutions on (F, g_F) of the following second order nonlinear elliptic PDE:

$$\mathrm{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 - |Du|^2}}\right) + \frac{f'(u)}{n\sqrt{f(u)^2 - |Du|^2}}\left(n + \frac{|Du|^2}{f(u)^2}\right) = 0, \tag{E.1}$$

$$|Du| < \lambda f(u), \;\; 0 < \lambda < 1, \tag{E.2}$$

3. Main results

In order to obtain our uniqueness result for equation (E) we will first deal with the parametric version of the problem, considering maximal hypersurfaces in a spatially open GRW spacetime which are not necessarily graphs. To prove our main uniqueness results we will need the following lemma, which bounds the Laplacian of the hyperbolic angle of these hypersurfaces.

Lemma 3.1 *Let $\psi : M \to \overline{M}$ be a complete maximal hypersurface in a GRW spacetime $\overline{M} = I \times_f F$. Then, the hyperbolic angle of M satisfies*

$$\begin{aligned} \frac{1}{2}\Delta \sinh^2 \varphi &= \cosh^2 \varphi \left(\mathrm{Ric}^F(N^F, N^F) - n(\log f)''(\tau)\sinh^2 \varphi\right) + |\mathrm{Hess}(\tau)|^2 \\ &\quad + \frac{f'(\tau)^2}{f(\tau)^2}\sinh^2 \varphi (n + \sinh^2 \varphi) + |\nabla \cosh \varphi|^2. \end{aligned} \tag{3.1}$$

where Ric^F *denotes the Ricci tensor of the fiber F and N^F is the projection of N on F.*

Proof The crucial idea of this proof is to compute the Laplacian of the function $\cosh \varphi$, which is defined by

$$\cosh \varphi = -\overline{g}(N, \partial_t).$$

Using (2.2) we can compute this function's gradient, obtaining

$$\nabla \cosh \varphi = A\partial_t^\top + \frac{f'(\tau)}{f(\tau)}\cosh \varphi\, \partial_t^\top. \tag{3.2}$$

Choosing a local orthonormal reference frame $\{E_1, \dots, E_n\}$ on TM we can obtain the Laplacian of $\cosh \varphi$ using (3.2) as follows

$$\Delta \cosh \varphi = \sum_{i=1}^n g(\nabla_{E_i}(A\partial_t^\top), E_i) + \sum_{i=1}^n g\left(\nabla_{E_i}\left(\frac{f'(\tau)}{f(\tau)}\cosh \varphi\, \partial_t^\top\right), E_i\right). \tag{3.3}$$

In fact, we can rewrite (3.3) as

$$\begin{aligned} \Delta \cosh \varphi &= \sum_{i=1}^n g((\nabla_{E_i}A)\partial_t^\top, E_i) + \sum_{i=1}^n g(\nabla_{E_i}\partial_t^\top, AE_i) - \frac{f''(\tau)}{f(\tau)}\cosh \varphi \sinh^2 \varphi \\ &\quad + 2\frac{f'(\tau)^2}{f(\tau)^2}\cosh \varphi \sinh^2 \varphi + \frac{f'(\tau)}{f(\tau)} g(A\partial_t^\top, \partial_t^\top) \\ &\quad + \frac{f'(\tau)}{f(\tau)}\cosh \varphi \sum_{i=1}^n g(\nabla_{E_i}\partial_t^\top, E_i). \end{aligned} \tag{3.4}$$

where we have used that $(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y)$ for all $X, Y \in \mathfrak{X}(M)$. On the other hand, using Codazzi equation $\overline{g}(\overline{\mathrm{R}}(X,Y)N, Z) = \overline{g}((\nabla_Y A)X, Z) - \overline{g}((\nabla_X A)Y, Z)$ (where $\overline{\mathrm{R}}$ denotes the curvature tensor of $\overline{M}$) and choosing our local frame in T_pM satisfying $\left(\nabla_{E_j}E_i\right)_p = 0$ we deduce from (3.4)

$$\begin{aligned} \Delta \cosh \varphi &= -\overline{\mathrm{Ric}}(\partial_t^\top, N) + 2\frac{f'(\tau)}{f(\tau)} g(A\partial_t^\top, \partial_t^\top) + \cosh \varphi \,\mathrm{trace}(A^2) \\ &\quad - \frac{f''(\tau)}{f(\tau)}\cosh \varphi \sinh^2 \varphi + 3\frac{f'(\tau)^2}{f(\tau)^2}\cosh \varphi \sinh^2 \varphi + n\frac{f'(\tau)^2}{f(\tau)^2}\cosh \varphi, \end{aligned} \tag{3.5}$$

where $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\overline{M}$. Decomposing N as $N = N^F - \overline{g}(N, \partial_t)\partial_t$, being N^F the projection of N on the fiber F, we can use [12, Cor. 7.43] to write

$$\overline{\mathrm{Ric}}(\partial_t^\top, N) = -\cosh \varphi \left(\mathrm{Ric}^F(N^F, N^F) + (n-1)(\log f)''(\tau)\sinh^2 \varphi\right). \tag{3.6}$$

Now, (3.6) can be used in (3.5) to obtain

$$\begin{aligned}\Delta\cosh\varphi &= \cosh\varphi\left(\mathrm{Ric}^F(N^F,N^F)-(n-1)(\log f)''(\tau)\sinh^2\varphi\right) \qquad (3.7)\\ &+2\,\frac{f'(\tau)}{f(\tau)}g(A\partial_t^\top,\partial_t^\top)+\cosh\varphi\,\mathrm{trace}(A^2)-\frac{f''(\tau)}{f(\tau)}\cosh\varphi\sinh^2\varphi\\ &+3\,\frac{f'(\tau)^2}{f(\tau)^2}\cosh\varphi\sinh^2\varphi+n\,\frac{f'(\tau)^2}{f(\tau)^2}\cosh\varphi.\end{aligned}$$

If we now compute $|\mathrm{Hess}(\tau)|^2$ we have

$$\begin{aligned}|\mathrm{Hess}(\tau)|^2 &= \sum_{i=1}^n g(\nabla_{E_i}\partial_t^\top,\nabla_{E_i}\partial_t^\top)=\frac{f'(\tau)^2}{f(\tau)^2}\left(n-1+\cosh^4\varphi\right) \qquad (3.8)\\ &+\cosh^2\varphi\,\mathrm{trace}(A^2)+2\,\frac{f'(\tau)}{f(\tau)}\cosh\varphi\,g(A\partial_t^\top,\partial_t^\top).\end{aligned}$$

Combining (3.7) and (3.8) leads to

$$\begin{aligned}\cosh\varphi\,\Delta\cosh\varphi &= \cosh^2\varphi\left(\mathrm{Ric}^F(N^F,N^F)-n(\log f)''(\tau)\sinh^2\varphi\right) \qquad (3.9)\\ &+|\mathrm{Hess}(\tau)|^2+\frac{f'(\tau)^2}{f(\tau)^2}\sinh^2\varphi(n+\sinh^2\varphi).\end{aligned}$$

We conclude the proof noticing that

$$\frac{1}{2}\Delta\sinh^2\varphi=\cosh\varphi\,\Delta\cosh\varphi+|\nabla\cosh\varphi|^2,$$

and using (3.9) to obtain (3.1). □

To prove our main results we also need the following lemma that extends [17, Lemma 3] and gives a bound for the Ricci curvature of constant mean curvature spacelike hypersurfaces in GRW spacetimes.

Lemma 3.2 *Let $\psi: M \to \overline{M}$ be a maximal hypersurface in a GRW spacetime $\overline{M}=I\times_f F$ whose warping function satisfies $(\log f)''(\tau)\le 0$. If either the fiber F has non-negative sectional curvature, M has bounded hyperbolic angle and the sectional curvature of the fiber F is bounded from below, then the Ricci curvature of M is bounded from below.*

Proof Given a point $p\in M$, let us choose a local orthonormal reference frame $\{E_1,\dots,E_n\}$ around p. From the Gauss equation we have that the Ricci curvature of M, Ric, satisfies

$$\mathrm{Ric}(X,X)\ge\sum_{i=1}^n\overline{g}(\overline{\mathrm{R}}(X,E_i)E_i,X)$$

for all $X\in\mathfrak{X}(M)$. Using [12, Prop. 7.42] we obtain

$$\begin{aligned}\sum_{i=1}^n\overline{g}(\overline{\mathrm{R}}(X,E_i)E_i,X) &= \sum_{i=1}^n g_F(\mathrm{R^F}(X^F,E_i^F)E_i^F,X^F)+(n-1)\frac{f'(\tau)^2}{f(\tau)^2}|X|^2\\ &-(n-2)(\log f)''(\tau)\,g(X,\nabla\tau)^2-(\log f)''(\tau)|\nabla\tau|^2|X|^2,\end{aligned}$$

being $\mathrm{R^F}$ the curvature tensor of F and X^F and E_i^F the projections of X and E_i on the fiber. If $(\log f)''(\tau)\le 0$ and F has non-negative sectional curvature, we see that the Ricci curvature of M is bounded from below. On the other hand, if φ is bounded and the sectional curvature of F is bounded from below we can consider $X\in\mathfrak{X}(M)$ such that $|X|^2=1$ and decompose it as

$$X=-\overline{g}(X,\partial_t)\partial_t+X^F.$$

Moreover, we can also see that

$$|X^F|^2|E_i^F|^2 = \left(1 + g(X,\nabla\tau)^2\right)\left(1 + g(E_i,\nabla\tau)^2\right),$$

as well as

$$\overline{g}(X^F, E_i^F)^2 = g(X,E_i)^2 + g(X,\nabla\tau)^2 g(E_i,\nabla\tau)^2 + 2g(X,E_i)g(X,\nabla\tau)g(E_i,\nabla\tau).$$

Thus, if the sectional curvature of F is bounded from below by a constant C the above expressions yield

$$\sum_{i=1}^{n} g_F(\mathrm{R^F}(X^F, E_i^F)E_i^F, X^F) \geq C\left(n - 1 + \sinh^2\varphi + (n-2)g(X,\nabla\tau)^2\right). \tag{3.10}$$

Thus, if the hyperbolic angle of M is bounded the classical Schwarz inequality guarantees that the left hand side of (3.10) is bounded from below by a constant. Therefore, we conclude again that if $(\log f)''(\tau) \leq 0$, then the Ricci curvature of M is bounded from below. □

In addition, we will make use of the following consequence of the Omori-Yau maximum principle obtained by Cheng and Yau in [6].

Lemma 3.3 [6, 11] *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. If $u \in C^2(M)$ is a non-negative function that satisfies $\Delta u \geq Cu^2$ for a positive constant C, then u vanishes identically on M.*

Taking these three lemmas into account, we are now in a position to prove our main parametric uniqueness result.

Theorem 3.4 *Let $\psi : M \to \overline{M}$ be a complete maximal hypersurface in a GRW spacetime $\overline{M} = I \times_f F$ whose fiber F has non-negative sectional curvature. If the warping function satisfies*

$$\sup_M\, (\log f)''(\tau) < 0, \tag{A}$$

then M is a totally geodesic spacelike slice.

Proof Under these assumptions, we deduce from Lemma 3.1 that φ satisfies

$$\frac{1}{2}\Delta \sinh^2\varphi \geq -n\,(\log f)''(\tau)(1 + \sinh^2\varphi)\sinh^2\varphi.$$

Moreover, (A) allows us to use Lemma 3.2 to guarantee that the Ricci curvature of M is bounded from below as well as ensures the existence of a positive constant C such that

$$\Delta\sinh^2\varphi \geq C\sinh^4\varphi.$$

Finally, we can use Lemma 3.3 to conclude that M is a totally geodesic spacelike slice. □

Remark 3.5 Note that assumption (A) cannot be omitted in order to obtain these results in spatially open GRW spacetimes. For instance, in the Lorentz-Minkowski spacetime of arbitrary dimension $\mathbb{L}^{n+1}$ this assumption does not hold and there is no analogous uniqueness result for complete maximal spacelike hypersurfaces.

As a consequence of Theorem 3.4 we can obtain our main non parametric result for the maximal hypersurface equation in a GRW spacetime whose fiber has non-negative sectional curvature.

Corollary 3.6 *Let $f : I \longrightarrow \mathbb{R}^+$ be a smooth function such that* $\inf f > 0$ *and* $\sup\,(\log f)'' < 0$. *Then, the only entire solutions to the equation*

$$\operatorname{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 - |Du|^2}}\right) + \frac{f'(u)}{n\sqrt{f(u)^2 - |Du|^2}}\left(n + \frac{|Du|^2}{f(u)^2}\right) = 0, \tag{E.1}$$

$$|Du| < \lambda f(u), \quad 0 < \lambda < 1, \tag{E.2}$$

on a complete Riemannian manifold F with non-negative sectional curvature are the constant functions $u = t_0$, with $t_0 \in I$ such that $f'(t_0) = 0$.

Proof Note that constraint (E.2) implies that the hyperbolic angle of the graph Σ_u satisfies

$$\cosh\varphi < \frac{1}{\sqrt{1-\lambda^2}}. \tag{3.11}$$

Furthermore, using the classical Schwarz inequality we deduce

$$g_u(v,v) \geq |Du|^2 g_u(v,v) + f(u)^2 g_F(d\pi_F(v), d\pi_F(v)), \text{ for all } v \in T\Sigma_u. \tag{3.12}$$

Hence, from (3.12) we obtain

$$g_u(v,v) \geq \frac{f(u)^2}{\cosh^2\varphi} g_F(d\pi_F(v), d\pi_F(v)). \tag{3.13}$$

Denoting by $L_F(\gamma)$ and $L_u(\gamma)$ the length of a smooth curve γ on F with respect to the metrics g_F and g_u, respectively, from (3.13) and (3.11) we have

$$L_u(\gamma) \geq (1-\lambda^2)(\inf f(u)^2) L_F(\gamma). \tag{3.14}$$

Thus, since (F, g_F) is complete and $\inf f > 0$ we obtain that the metric g_u is also complete. This fact and the rest of our assumptions enable us to apply Theorem 3.4 to end the proof. □

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