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Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Well-balanced algorithms for relativistic fluids on a Schwarzschild background

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Abstract

A class of well-balanced finite volume methods with first and higher order of accuracy is designed for two spherical symmetric fluid models on a Schwarzschild curved background: the Burgers-Schwarzschild model and the Euler-Schwarzschild model. We take advantage of the explicit or implicit forms available for the stationary solutions of these models to design numerical methods that preserve them. These methods are then used to investigate the late time behaviour of the flows.

1. Introduction

We are interested in the numerical approximation and the long time behaviour of relativistic compressible fluid flows on a Schwarzschild black hole background. The flow is assumed to enjoy spherical symmetry and therefore we deal with nonlinear hyperbolic systems of partial differential equations (PDEs) in one space variable. The objective is two-fold: on the one hand, designing and testing numerically finite volume algorithms that are well-balanced; on the other hand, to perform a thorough investigation of the behavior of the solutions and numerically infer definite conclusions about the long-time behavior of such flows. Our study should provide first and useful insights for, on the one hand, further development concerning the mathematical analysis of the models and, on the other hand, further investigations to the same problem in higher dimensions without symmetry restriction.

We consider first the *relativistic Burgers-Schwarzschild model* (see [12, 13]):

$$v_t + F(v,r)_r = S(v,r), \qquad t \geq 0, \quad r > 2M, \tag{1.1a}$$

where $v = v(t,r) \in [-1,1]$ is the unknown function and the flux and source terms read

$$F(v,r) = \Big(1 - \frac{2M}{r}\Big)\frac{v^2-1}{2}, \qquad S(v,r) = \frac{2M}{r^2}\,(v^2-1), \tag{1.1b}$$

while the constant $M > 0$ represents the mass of the black hole. The speed of propagation for this scalar balance law reads

$$\partial_v F(v,r) = \Big(1 - \frac{2M}{r}\Big)v, \tag{1.2}$$

which vanishes at the boundary $r = 2M$, so that no boundary condition is required in order to pose the Cauchy problem.

Next, we consider the *relativistic Euler-Schwarzschild model* (as it is called in [12, 13]):

$$V_t + F(V,r)_r = S(V,r), \qquad t \geq 0, \quad r > 2M, \tag{1.3a}$$

whose unknowns are the fluid density $\rho = \rho(t,r) \geq 0$ and the normalized velocity $v = v(t,r) \in (-1,1)$. These functions are defined for all $r > 2M$ and the limiting values $v = \pm 1$ can be reached at the boundary $r = 2M$ only, and

$$V = \begin{pmatrix} V^0 \\ V^1 \end{pmatrix} = \begin{pmatrix} \dfrac{1+k^2v^2}{1-v^2}\rho \\ \dfrac{1+k^2}{1-v^2}\rho v \end{pmatrix}, \qquad F(V,r) = \begin{pmatrix} \Big(1-\dfrac{2M}{r}\Big)\dfrac{1+k^2}{1-v^2}\rho v \\ \Big(1-\dfrac{2M}{r}\Big)\dfrac{v^2+k^2}{1-v^2}\rho \end{pmatrix}, \tag{1.3b}$$

$$S(V,r) = \begin{pmatrix} -\dfrac{2}{r}\Big(1-\dfrac{2M}{r}\Big)\dfrac{1+k^2}{1-v^2}\rho v \\ \dfrac{-2r+5M}{r^2}\dfrac{v^2+k^2}{1-v^2}\rho - \dfrac{M}{r^2}\dfrac{1+k^2v^2}{1-v^2}\rho + 2\dfrac{r-2M}{r^2}k^2\rho \end{pmatrix}, \tag{1.3c}$$

with

$$v = \frac{1+k^2-\sqrt{(1+k^2)^2-4k^2\left(\frac{V^1}{V^0}\right)^2}}{2k^2\frac{V^1}{V^0}}, \qquad \rho = \frac{V^1(1-v^2)}{v(1+k^2)}. \tag{1.3d}$$

Here, $k \in (-1, 1)$ denotes the (constant) speed of sound. The eigenvalues of the Jacobian of the flux function are

$$\mu_\pm = \Big(1 - \frac{2M}{r}\Big)\frac{v \pm k}{1 \pm k^2 v}, \tag{1.4}$$

so that the system is strictly hyperbolic. As usual, a state (ρ, v), by definition, is said to be *sonic* if one of the eigenvalues vanishes, i.e. if $|v| = |k|$, *supersonic* if both eigenvalues have the same sign, i.e. if $|v| > |k|$, or *subsonic* if the eigenvalues have different signs, i.e. if $|v| < |k|$. Both eigenvalues $\mu_\pm$ vanish at the boundary $r = 2M$, so that no boundary condition is required in order to pose the Cauchy problem.

In order to be able of running reliable and accurate numerical simulations for these two models, we design shock-capturing, high-order, and well-balanced finite volume methods of first- and second-order of accuracy (and even third-order accurate for (1.1)). Specifically, we extend to the present problem the well-balanced methodology proposed recently by Castro and Parés [7] for nonlinear hyperbolic systems of balance laws. For earlier work on well-balanced schemes we also refer to [5, 16, 17] and, concerning the design of geometry-preserving schemes, we refer for instance to [1–3, 6, 8–10, 15, 19] and the references therein.

The properties of the stationary solutions play a fundamental role in the design of well-balanced schemes, as well as in the study of the long time behavior of solutions. We thus also built here upon earlier investigations by LeFloch and collaborators [11–13] on the theory and approximation of the relativistic Burgers- and Euler-Schwarzschild model (1.1) and (1.3). Remarkably, the stationary solutions to both models are available in explicit or implicit form.

2. Well-balanced methodology

Both problems of interest are of the form

$$V_t + F(V, r)_r = S(V, r), \qquad r > 2M, \tag{2.1}$$

with unknown $V = V(t, r) \in \mathbb{R}^N$ and $N = 1$ or 2. Systems of this form have non-trivial stationary solutions, which satisfy the ODE

$$F(V, r)_r = S(V, r). \tag{2.2}$$

Our goal is to introduce a family of numerical methods that are well-balanced, i.e. that preserve the stationary solutions in a sense to be specified. We follow the strategy in [7] to which we refer for further details and arguments of proof.

We consider semi-discrete finite volume numerical methods of the form

$$\frac{dV_i}{dt} = -\frac{1}{\Delta r}\Big(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} - \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} S(\mathbb{P}_i^t(r), r)\, dr\Big), \tag{2.3}$$

where the following notation is used.

- $I_i = [r_{i-\frac{1}{2}}, r_{i+\frac{1}{2}}]$ denote the computational cells, whose length Δr is assumed to be constant for simplicity.
- $V_i(t)$ denotes the approximate average of the exact solution in the ith cell at the time t, that is,

$$V_i(t) \cong \frac{1}{\Delta r}\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} V(r, t)\, dr. \tag{2.4}$$

- $\mathbb{P}_i^t(r)$ denotes the approximation of the solution in the ith cell given by a high-order reconstruction operator based on the cell averages $\{V_i(t)\}$, that is, $\mathbb{P}_i^t(r) = \mathbb{P}_i^t(r; \{V_j(t)\}_{j \in \mathcal{S}_i})$. Here, $\mathcal{S}_i$ denotes the set of cell indices associated with the stencil of the ith cell.
- The flux terms are denoted by $F_{i+\frac{1}{2}} = \mathbb{F}\left(V^{t,-}_{i+\frac{1}{2}}, V^{t,+}_{i+\frac{1}{2}}, r_{i+\frac{1}{2}}\right)$, where $V^{t,\pm}_{i+\frac{1}{2}}$ are the reconstructed states at the interfaces, i.e.

$$V^{t,-}_{i+\frac{1}{2}} = \mathbb{P}_i^t(r_{i+\frac{1}{2}}), \quad V^{t,+}_{i+\frac{1}{2}} = \mathbb{P}_{i+1}^t(r_{i+\frac{1}{2}}). \tag{2.5}$$

  Here, $\mathbb{F}$ is a consistent numerical flux, i.e. a continuous function $\mathbb{F} : \mathbb{R}^N \times \mathbb{R}^N \times (2M, +\infty) \to \mathbb{R}^N$ satisfying $\mathbb{F}(V, V, r) = F(V, r)$ for all V, r.

Furthermore, given a stationary solution V^* of (2.2), we use the following terminology.

- The numerical method (2.3) is said to be well-balanced for V^* if the vector of cell averages of V^* is an equilibrium of the ODE system (2.3).

- The reconstruction operator is said to be well-balanced for V^* if we have $\mathbb{P}_i(r) = V^*(r)$ for all $r \in [r_{i-\frac{1}{2}}, r_{i+\frac{1}{2}}]$, where $\mathbb{P}_i$ is the approximation of V^* obtained by applying the reconstruction operator to the vector of cell averages of V^*.

It is easily checked that, if the reconstruction operator is well-balanced for a continuous stationary solution V^* of (2.2) then the numerical method is also well-balanced for V^*. The following strategy to design a well-balanced reconstruction operator $\mathbb{P}_i$ on the basis of a standard operator $\mathbb{Q}_i$ was introduced in [5]:

Given a family of cell values $\{V_i\}$, in every cell $I_i = [r_{i-\frac{1}{2}}, r_{i+\frac{1}{2}}]$ we proceed as follows.

1. Seek, (whenever possible), a stationary solution $V_i^*(x)$ defined in the stencil of cell I_i $(\cup_{j\in\mathcal{S}_i} I_j)$ such that

$$\frac{1}{\Delta r}\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} V_i^*(r)\,dr = V_i. \tag{2.6}$$

   If such a solution does not exist, take $V_i^* \equiv 0$.

2. Apply the reconstruction operator to the cell values $\{W_j\}_{j\in\mathcal{S}_i}$ given by

$$W_j = V_j - \frac{1}{\Delta r}\int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} V_i^*(r)\,dr, \qquad j \in \mathcal{S}_i, \tag{2.7}$$

   in order to obtain $\mathbb{Q}_i(r) = \mathbb{Q}_i(r; \{W_j\}_{j\in\mathcal{S}_i})$. We consider the MUSCL reconstruction operator (see [18]) in the second-order case and the CWENO3 (see [14]) in the third-order case.

3. Define finally

$$\mathbb{P}_i(r) = V_i^*(r) + \mathbb{Q}_i(r). \tag{2.8}$$

It can be then easily shown that the reconstruction operator $\mathbb{P}_i$ in (2.8) is well-balanced for every stationary solution provided that the reconstruction operator $\mathbb{Q}_i$ is exact for the zero function. Moreover, if $\mathbb{Q}_i$ is conservative then $\mathbb{P}_i$ is conservative, in the sense that

$$\frac{1}{\Delta r}\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \mathbb{P}_i(r)\,dr = V_i, \tag{2.9}$$

and $\mathbb{P}_i$ has the same accuracy as $\mathbb{Q}_i$ if the stationary solutions are sufficiently regular.

If a quadrature formula (whose order of accuracy must be greater or equal to the one of the reconstruction operator)

$$\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} f(x)\,dx \approx \Delta r \sum_{l=0}^{q} \alpha_l f(r_{i,l})$$

where $\alpha_0, \dots, \alpha_q, r_{i,0}, \dots, r_{i,q}$ represent the weights and the nodes of the formula, is used to compute the averages of the initial condition, namely $V_{i,0} = \sum_{l=0}^{q} \alpha_l V_0(r_{i,l})$, the reconstruction procedure has to be modified to preserve the well-balanced property: Steps 1 and 2 have to be replaced by the following ones

1. Seek, if possible, the stationary solution $V_i^*(x)$ defined in the stencil of cell I_i $(\cup_{j\in\mathcal{S}_i} I_j)$ such that

$$\sum_{l=0}^{q} \alpha_l V_i^*(r_{i,l}) = V_i. \tag{2.10}$$

   If this solution does not exist, take $V_i^* \equiv 0$.

2. Apply the reconstruction operator to the cell values $\{W_j\}_{j\in\mathcal{S}_i}$ given by

$$W_j = V_j - \sum_{l=0}^{q} \alpha_l V_i^*(r_{j,l}), \qquad j \in \mathcal{S}_i.$$

For first- or second-order methods, if the midpoint rule is selected to compute the initial averages, i.e. $V_{i,0} = V_0(r_i)$, then at the first step of the reconstruction procedure, the problem (2.10) reduces to finding the stationary solution satisfying

$$V_i^*(r_i) = V_i. \tag{2.11}$$

The well-balanced property of the method can be lost if the quadrature formula is used to compute the integral appearing at the right-hand side of (2.3). In order to circumvent this difficulty, in [7] it is proposed to rewrite the methods as follows:

$$\begin{aligned}\frac{dV_i}{dt} = &-\frac{1}{\Delta r}\left(F_{i+\frac{1}{2}} - F\left(V_i^{t,*}(r_{i+\frac{1}{2}}), r_{i+\frac{1}{2}}\right) - F_{i-\frac{1}{2}} + F\left(V_i^{t,*}(r_{i-\frac{1}{2}}), r_{i-\frac{1}{2}}\right)\right)\\ &+ \frac{1}{\Delta r}\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \left(S(\mathbb{P}_i^t(r), r) - S(V_i^{t,*}(r), r)\right)\, dr,\end{aligned} \tag{2.12}$$

where $V_i^{t,*}$ is the function selected in Step 1 for the ith cell at time t. In this equivalent form, a quadrature formula can be applied to the integral without losing the well-balanced property, and this leads to a numerical method of the form:

$$\begin{aligned}\frac{dV_i}{dt} = &-\frac{1}{\Delta r}\left(F_{i+\frac{1}{2}} - F\left(V_i^{t,*}(r_{i+\frac{1}{2}}), r_{i+\frac{1}{2}}\right) - F_{i-\frac{1}{2}} + F\left(V_i^{t,*}(r_{i-\frac{1}{2}}), r_{i-\frac{1}{2}}\right)\right)\\ &+ \sum_{l=0}^{q} \alpha_l \left(S(\mathbb{P}_i^t(r_{i,l}), r_{i,l}) - S(V_i^{t,*}(r_{i,l}), r_{i,l})\right).\end{aligned} \tag{2.13}$$

First-order well-balanced methods are obtained by selecting the trivial constant piecewise reconstruction operator as the standard one, i.e.

$$\mathbb{Q}_i(r, V_i) = V_i, \qquad r \in [r_{i-\frac{1}{2}}, r_{i+\frac{1}{2}}]. \tag{2.14}$$

It can be easily checked that the numerical method then reduces to

$$\frac{dV_i}{dt} = -\frac{1}{\Delta r}\left(F_{i+\frac{1}{2}} - F\left(V_i^{t,*}(r_{i+\frac{1}{2}}), r_{i+\frac{1}{2}}\right) - F_{i-\frac{1}{2}} + F\left(V_i^{t,*}(r_{i-\frac{1}{2}}), r_{i-\frac{1}{2}}\right)\right), \tag{2.15}$$

where $F_{i+\frac{1}{2}} = \mathbb{F}\left(V_i^*(r_{i+\frac{1}{2}}), V_{i+1}^*(r_{i+\frac{1}{2}}), r_{i+\frac{1}{2}}\right)$.

Notice that the implementation of these methods requires to find a stationary solution with prescribed average at Step 1 of the reconstruction procedure. In the case of the Burgers-Schwarzschild model, the explicit expression of the stationary solutions is available

$$v^*(r) = \pm\sqrt{1 - K^2\Big(1 - \frac{2M}{r}\Big)}, \quad K > 0. \tag{2.16}$$

and it can be easily checked that (2.10) and (2.11) have always a unique solution. In the case of the Euler-Schwarzschild model, the following implicit form of the stationary solutions is available

$$\frac{\operatorname{sgn}(v)(1 - v^2)|v|^{\frac{2k^2}{1-k^2}} r^{\frac{4k^2}{1-k^2}}}{\left(1 - \frac{2M}{r}\right)} = C_1, \qquad r(r - 2M)\rho\frac{v}{1 - v^2} = C_2, \tag{2.17}$$

where C_1, C_2 are constants. Once the constants are fixed by imposing (2.11), a nonlinear system has to be solved to evaluate the stationary solution at a point of the stencil. This system can have 0, 1, or 2 solutions. If there is no solution the standard reconstruction is used. When there are two solutions, one of them is supersonic and the other is subsonic: the one whose regime is equal to that of V_i is selected.

3. Numerical tests

First-, second- and third-order methods for the Burgers-Schwarzschild and first- and second-order methods for Euler-Schwarzschild have been implemented. Several numerical test are presented here to show the relevance of the well-balanced property for the investigation of the asymptotic behaviour of the flows.

3.1. Burgers-Schwarzschild

We consider the spatial interval $[2M, L]$ with $M = 1$ and $L = 4$, a 256-point uniform mesh, and the CFL number equal to 0.5. At $r = 2M$, $F_{-\frac{1}{2}} = 0$ is imposed. At $r = L$, a transmissive boundary condition based on the use ghost-cells is used. The following numerical flux is considered:

$$F_{i+\frac{1}{2}} = \mathbb{F}(v_i, v_{i+1}, r_{i+\frac{1}{2}}) = \left(1 - \frac{2M}{r_{i+\frac{1}{2}}}\right)\frac{q^2(0; v_i, v_{i+1}) - 1}{2},$$

where $q(\cdot; v_L, v_R)$ is the self-similar solution of the Riemann problem for the standard Burgers equation with the initial condition

$$v_0(r) = \begin{cases} v_L, & r < 0,\\ v_R, & r > 0.\end{cases}$$

In order to check the relevance of the well-balanced property, the well-balanced methods will be compared with standard ones based on the same numerical fluxes and the standard first-, second-, or third-order reconstructions.

Positive stationary solution We consider the initial condition

$$v_0(r) = \sqrt{\frac{3}{4} + \frac{1}{2r}} \tag{3.1}$$

corresponding to a positive stationary solution. Table 1 shows the error in L^1 norm between the initial condition and the numerical solution at time $t = 50$. Figure 1 compares the numerical solutions obtained with the well-balanced and the non-well-balanced methods: it can be seen how the latter are unable to capture the stationary solution. After a time that decreases with the order, the numerical solutions depart from the steady state.

Scheme (256 cells)	Error (1st)	Error (2nd)	Error (3rd)
Well-balanced	1.13E-14	8.72Ee-17	7.22E-14
Non well-balanced	1.89	1.61	8.78E-02

Tab. 1 Well-balanced versus non-well-balanced schemes: L^1 errors at $t = 50$ for the Burgers model with the initial condition (3.1).

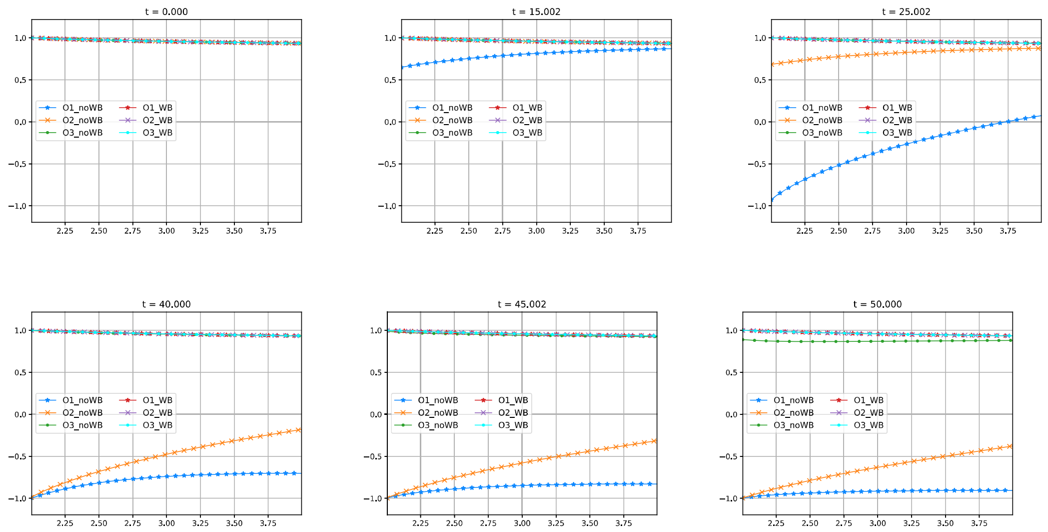


Fig. 1 Burgers-Schwarzschild model with the initial condition (3.1): first-, second-, and third-order well-balanced and not-well-balanced methods at various times.

Perturbation of a steady shock solution In this test case we consider the initial condition:

$$\tilde{v}_0(r) = v_0(r) + p_L(r), \tag{3.2}$$

where v_0 is the steady shock solution given by

$$v_0(r) = \begin{cases} \sqrt{\frac{3}{4} + \frac{1}{2r}}, & 2 < r < 3, \\ -\sqrt{\frac{3}{4} + \frac{1}{2r}}, & \text{otherwise}, \end{cases} \tag{3.3}$$

and

$$p_L(r) = \begin{cases} -\frac{1}{5} e^{-200(r-2.5)^2}, & 2.2 < r < 2.8, \\ 0, & \text{otherwise}. \end{cases} \tag{3.4}$$

The first-, second-, and third-order well-balanced methods have been applied to this problem. In Figure 2 it can be observed that, after the wave generated by the initial perturbation leaves the computational domain, the stationary solution (3.3) is not recovered: a different stationary solution is obtained whose shock is placed at a different location. Observe that all the three methods capture the same stationary solution.

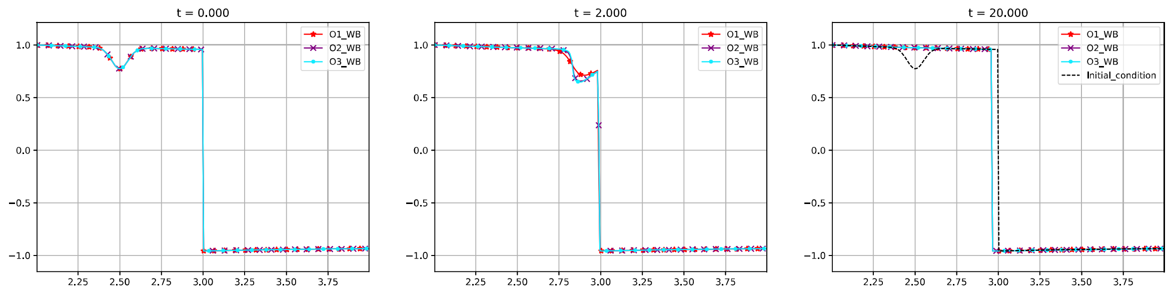


Fig. 2 Burgers-Schwarzschild model with the initial condition (3.2)-(3.3)-(3.4): first-, second-, and third-order well-balanced methods at selected times.

3.2. Euler-Schwarzschild

We consider the spatial interval $[2M, L]$ with $M = 1$ and $L = 10$, a 500-point uniform mesh, $k = 0.3$, and the CFL number equal to 0.5. At $r = 2M$ we impose $F_{-\frac{1}{2}} = 0$ as boundary condition since $\left(1 - \frac{2M}{r}\right) = 0$. The boundary conditions are the same as in the previous test case. A HLL-like numerical flux in PVM form (see [4]) will be used:

$$F_{i+\frac{1}{2}} = \frac{1}{2}(F(V_i) + F(V_{i+1})) - \frac{1}{2}\left(\alpha_0(V_{i+1} - V_i) + \alpha_1(F(V_{i+1}) - F(V_i))\right), \tag{3.5}$$

with

$$\alpha_0 = \frac{\overline{\lambda_2}|\overline{\lambda_1}| - \overline{\lambda_1}|\overline{\lambda_2}|}{\overline{\lambda_2} - \overline{\lambda_1}}, \qquad \alpha_1 = \frac{|\overline{\lambda_2}| - |\overline{\lambda_1}|}{\overline{\lambda_2} - \overline{\lambda_1}}, \tag{3.6}$$

where $\overline{\lambda_1}$ and $\overline{\lambda_2}$ are the eigenvalues of some intermediate matrix $J_{i+\frac{1}{2}}$ of the form

$$J_{i+\frac{1}{2}} = \left(1 - \frac{2M}{r_{i+\frac{1}{2}}}\right)\begin{bmatrix} 0 & 1 \\ \dfrac{k^2 - v_m^2}{1 - k^2 v_m^2} & \dfrac{2(1-k^2)v_m}{1 - k^2 v_m^2} \end{bmatrix} \tag{3.7}$$

where v_m is some intermediate value between v_i^n and v_{i+1}^n.

Discontinuous stationary entropy weak solution We consider the initial condition

$$V_0(r) = \begin{cases} V_-^*(r), & r \le 6, \\ V_+^*(r), & \text{otherwise}, \end{cases} \tag{3.8}$$

where $V_-^*(r)$ is the supersonic stationary solution such that

$$\rho_-^*(6) = 4, \quad v_-^*(6) = 0.6 \tag{3.9}$$

and $V_+^*(r)$ is the subsonic one such that

$$\rho_+^*(6) = \frac{\rho_-^*(6)(v_-^*(6)^2 - k^4)}{k^2(1 - v_-^*(6)^2)}, \quad v_+^*(6) = \frac{k^2}{v_-^*(6)}. \tag{3.10}$$

V_0 is an entropy weak stationary solution of the system: see [12,13]. Table 2 shows the error in L^1 norm between the numerical solution at time $t = 50$ and Figure 3 shows the comparison of the numerical results obtained with well-balanced and non-well-balanced methods at selected times. The numerical results of this section put on evidence, as for the Burgers-Schwarzschild system, the relevance of using well-balanced methods for the Euler-Schwarzschild model.

Relation between the perturbation and the displacement of the shock In order to study the relationship between the amplitude of the perturbation and the distance between the initial and the final shock locations, we consider the family of initial conditions:

$$\widetilde{V}_0(r) = \widetilde{V}^*(r) + \delta(\alpha, r), \tag{3.11}$$

Scheme (500 cells)	Error v (1st)	Error ρ (1st)	Error v (2nd)	Error ρ (2nd)
Well-balanced	2.20E-13	1.25E-11	1.92E-13	1.03E-11
Non well-balance	0.89	3.94	0.89	3.92

Tab. 2 Well-balanced versus non-well-balanced schemes: L^1 errors at time $t = 50$ for the Burgers-Schwarzschild model with the initial condition (3.8)

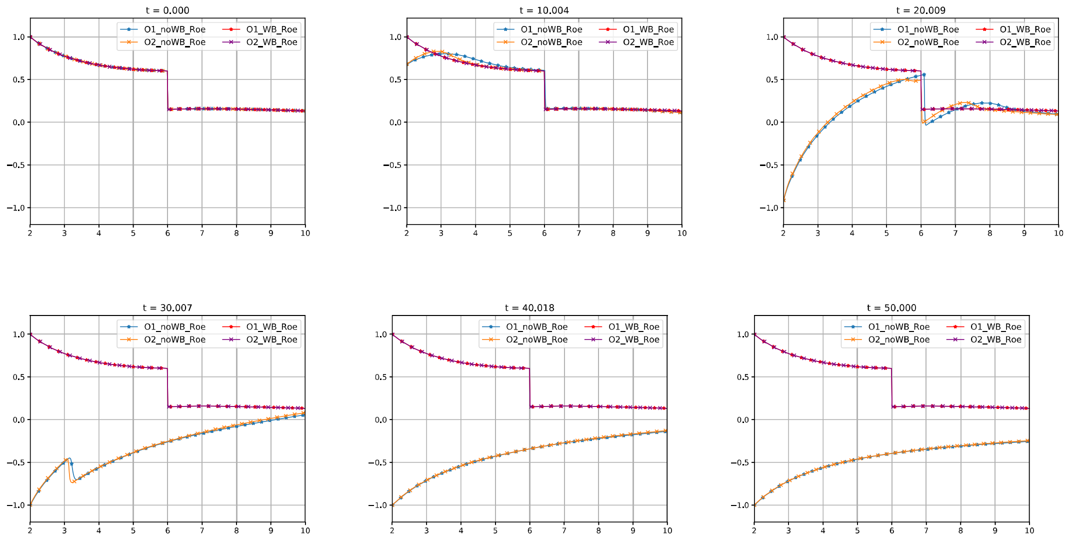
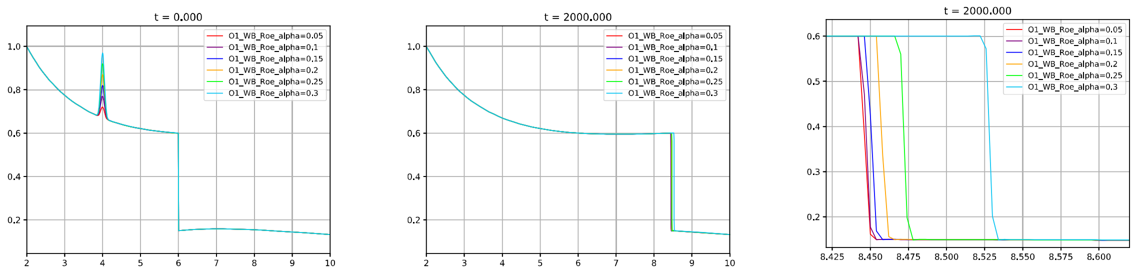


Fig. 3 Euler-Schwarzschild model with the initial condition (3.8): first- and second-order well-balanced and non-well-balanced methods at selected times for the variable v.

where $\widetilde{V}^*$ is the steady shock solution given by (3.8)-(3.10) and

$$\delta(\alpha, r) = [\delta_v(\alpha, r), \delta_\rho(\alpha, r)]^T = \begin{cases} [\alpha e^{-200(r-4)^2}, 0]^T, & 3 < r < 5, \\ [0, 0]^T, & \text{otherwise}, \end{cases} \tag{3.12}$$

with $\alpha > 0$. In this case we will also use a 2000-point uniform mesh. Figure 4 shows the numerical solution for different values of α and we observe that depending on the amplitude of the perturbation the numerical solutions converge in time to different steady shock solutions.



(a) Zoom.

Fig. 4 Euler-Schwarzschild model with the initial condition (3.11): first-order well-balanced method taking different values of α for variable v.

4. Conclusions

The procedure introduced in [5] and recalled in [7] is extended to the relativistic fluid flows in the Schwarzschild background. More precisely, we develop first and higher order well-balanced schemes for the relativistic Burgers and Euler systems. Several numerical tests are used to validate the schemes and to highlight the relevance of the well-balanced property when dealing with these relativistic flows. We also use these schemes to perform a systematic numerical study of these two PDE systems in order to be able to extract general conclusions about the long time behavior of the flow. Such a study is expected to be a useful tool to direct the mathematical analysis of the models and the study of more complex relativistic models.

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References

[1] A. Beljadid, P.G. LeFloch, and M. Mohamadian, Late-time asymptotic behavior of solutions to hyperbolic conservation laws on the sphere, *Computer Methods in Applied Mechanics and Engineering*, 349:285–311 2019.

[2] F. Bouchut, Nonlinear stability of finite Volume Methods for hyperbolic conservation laws: And Well-Balanced schemes for sources, *Springer Science & Business Media*, 2004.

[3] A. Beljadid and P.G. LeFloch, A central-upwind geometry-preserving method for hyperbolic conservation laws on the sphere, Commun. Appl. Math. Comput. Sci. 12 (2017), 81–107.

[4] M.J. Castro and E. Fernández-Nieto, A class of computationally fast first order finite volume solvers: PVM methods. *SIAM Journal on Scientific Computing*, 34(4):A2173–A2196, 2012.

[5] M.J. Castro, J.M Gallardo, J.A López-García, and C. Parés, Well-balanced high order extensions of Godunov method for semilinear balance laws. *SIAM Journal on Numerical Analysis*, 46(2):1012–1039, 2008.

[6] M.J. Castro, T. Morales de Luna and C. Parés, Well-balanced schemes and path-conservative numerical methods. In *Handbook of Numerical Analysis*, 18:131–175, 2017.

[7] M.J. Castro and C. Parés, Well-balanced high-order finite volume methods for systems of balance laws. *Journal of Scientific Computing*, 82(2):1–48, 2020.

[8] S. Dong and P.G. LeFloch, Convergence of the finite volume method on a Schwarzschild background. *ESAIM: Mathematical Modelling and Numerical Analysis*, 53(5):1459–1476, 2019.

[9] D. Dziuk, D. Kröner, and T. Müller, Scalar conservation laws on moving hypersurfaces. *Interfaces and Free Boundaries*, 15(2):203–236, 2013.

[10] J. Giesselman and P.G. LeFloch, Formulation and convergence of the finite volume method for conservation laws on spacetimes with boundary. *Numerische Mathematik*, 144:751–785, 2020.

[11] P.G. LeFloch and H. Makhlof, A geometry-preserving finite volume method for compressible fluids on Schwarzschild spacetime. *Communications in Computational Physics*, 15(3):827–852, 2014.

[12] P.G. LeFloch and S. Xiang, A numerical study of the relativistic Burgers and Euler equations on a Schwarzschild black hole exterior. *Applied Mathematics and Computational Science*, 13(2):271–301, 2018.

[13] P.G. LeFloch and S. Xiang, Weakly regular fluid flows with bounded variation on the domain of outer communication of a Schwarzschild black hole spacetime. II, J. Math. Pure Appl. 122 (2019), 272–317.

[14] D. Levy, G. Puppo, and G. Russo. Compact central WENO schemes for multidimensional conservation laws. *SIAM Journal on Scientific Computing*, 22(2):656–672, 2000.

[15] A. Rossmanith, D.S. Bale, and R.J. LeVeque, A wave propagation algorithm for hyperbolic systems on curved manifolds. *Journal of Computational Physics*, 199(2):631–662, 2004.

[16] G. Russo, Central schemes for conservation laws with application to shallow water equations. *Trends and Applications of Mathematics to Mechanics*, Springer, p. 225–246, 2005.

[17] G. Russo, High-order shock-capturing schemes for balance laws.*Numerical solutions of partial differential equations, Advanced Courses in Mathematics — CRM Barcelona Centre de Recerca Matemàtica, Birkhäuser, Basel*, p. 59–147, 2009.

[18] B. Van Leer. Towards the ultimate conservative difference scheme. ii. monotonicity and conservation combined in a second-order scheme. *Journal of computational physics*, 14(4):361–370, 1974.

[19] M.-E. Vázquez-Cendón, Improved treatment of source terms in upwind schemes for the shallow water equations in channels with irregular geometry. *Journal of computational physics*, 148(2): 497–526, 1999.