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Universidad de Oviedo

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## Foreword

It is with great pleasure that we present the Proceedings of the 26$^{th}$ Congress of Differential Equations and Applications / 16$^{th}$ Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier "Pancho" Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: "a mathematician is a device for turning coffee into theorems". Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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# Convergence rates for Galerkin approximation for magnetohydrodynamic type equations

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**Abstract**

The motion of incompressible electrical conducting fluids can be modeled by magnetohydrodynamics equations, which consider the Navier-Stokes equations coupled with Maxwell's equations. For the classical Navier-Stokes system, there exists an extensively study of the convergence rate for the Galerkin approximations. Here, we extend the estimates rates of spectral Galerkin approximations for the magnetohydrodynamic equations. We prove optimal error estimates in the $L^2(\Omega)$ and $H^1(\Omega)$-norms, we obtain a result similar to the Rautmann for the $H^2(\Omega)$-norm, and we reach basically the same level of knowledge as in the case of the classical Navier-Stokes.

## 1. Introduction

The motion of incompressible electrical conducting fluids can be modeled by the so-called equations of magnetohydrodynamics, which can be described as the coupling of the Navier-Stokes equations and the Maxwell's equations. To describe these equations, we consider a bounded domain $\Omega \subset \mathbb{R}^3$, $T > 0$, denoted $Q_T \equiv \Omega \times (0,T)$ and $S_T \equiv \partial\Omega \times (0,T)$. In the case where there is free motion of heavy ions, not directly due to the electric field (see [11], [19], [20]), these equations can be reduced to the form:

$$\left\{\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{\eta}{\rho_m}\Delta\mathbf{u} + \frac{1}{\rho_m}\nabla\left(p^* + \frac{\mu}{2}\mathbf{h}^2\right) - \frac{\mu}{\rho_m}\mathbf{h}\cdot\nabla\mathbf{h} &= \mathbf{f}, && \text{in } Q_T,\\ \frac{\partial \mathbf{h}}{\partial t} - \frac{1}{\mu\sigma}\Delta\mathbf{h} + (\mathbf{u}\cdot\nabla)\mathbf{h} - (\mathbf{h}\cdot\nabla)\mathbf{u} + \nabla q &= 0, && \text{in } Q_T,\\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{h} &= 0, && \text{in } Q_T, \end{aligned}\right. \tag{1.1}$$

together with the following boundary and initial conditions:

$$\left\{\begin{aligned} \mathbf{u} &= \mathbf{0}, & \mathbf{h} &= \mathbf{0} && \text{on } S_T,\\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), & \mathbf{h}(x,0) &= \mathbf{h}_0(x) && \text{in } \Omega, \end{aligned}\right. \tag{1.2}$$

Here, $\mathbf{u}$ and $\mathbf{h}$ are unknown velocity and magnetic field, respectively, $p^*$ is an unknown hydrostatic pressure, $q$ is an unknown function related to the heavy ions (in such way that the density of electric current, $\mathbf{j}_0$, generated by this motion satisfies the relation $\operatorname{rot}\mathbf{j}_0 = -\sigma\nabla q$), $\rho_m$ is the density of mass of the fluid (assumed to be a positive constant), $\mu > 0$ is a constant magnetic permeability of the medium, $\sigma > 0$ is a constant electric conductivity, $\eta > 0$ is a constant viscosity of the fluid and $\mathbf{f}$ is a given external force field.

There are an extensive literature on the magnetohydrodynamic system (1.1)–(1.2): Lassner [9], by using the semigroup results of Kato and Fujita [7], proved the existence and uniqueness of strong solutions, local in time for any data and global in time for small data. Boldrini and Rojas-Medar [3] studied the existence of weak solutions and the reproductive property using the Galerkin method. The same authors improved this result to local and global strong solutions by using the spectral Galerkin method in [4, 5]. Damázio and Rojas-Medar [6] studied the regularity of weak solutions, and Notte-Cuello and Rojas-Medar [10] used an iterative approach to show the existence and uniqueness of the strong solutions. The initial value problem in time dependent domains was studied by Rojas-Medar and Beltrán-Barrios in [17], and by Berselli and Ferreira in [1]. The problem in unbounded domains with boundary uniformly of $C^3$-class was studied by Zhao in [22].

On the other hand, for the classical Navier–Stokes system there exists an extensively study of the convergence rate for the Galerkin approximations. The first work in this way was given by Rautmann in [12], where he proved the optimal convergence in the $H^1(\Omega)$-norm, but the optimal convergence in the $L^2(\Omega)$-norm was left as an open problem in [12] and was answered by Salvi in [18] (see also [2]). Applying the same method and assuming the uniform boundedness in time of the $L^2(\Omega)$-norm of the gradient of the velocity and the exponential stability in the $H^1(\Omega)$-norm of the solution, Heywood [8] was able to derive optimal uniform in time error estimates for the velocity in the $H^1(\Omega)$-norm. Also, without explicitly assuming $H^1(\Omega)$-exponential stability, Boldrini and Rojas–Medar [2] proved optimal uniform in time error estimates for the spectral Galerkin approximations in the $H^1(\Omega)$ and $L^2(\Omega)$-norms, assuming that the external force field has a mild form of decay.

The study of the convergence rate in the $\mathbf{H}^2(\Omega)$-norm is difficult, because estimates of higher order derivates spatial of solution are required, which needs a compatibility condition to be satisfied by the initial value of the solution. The work of Rautmann [15] give an answer to the question *"how smooth a Navier–Stokes solution can be at time $t = 0$ without any compatibility condition"*. Making use of this result, Rautmann [13], [14] proved the convergence rate in the $\mathbf{H}^2(\Omega)$-norm of the spectral Galerkin approximation to the solution without any compatibility condition.

The aim of this work is to extend the estimates rates of spectral Galerkin approximations for the the Navier-Stokes system to the magnetohydrodynamic equations (1.1)-(1.2). We prove optimal error estimates in the $L^2(\Omega)$ and $H^1(\Omega)$-norms and obtain a result similar to the Rautmann in [13], [14] for $H^2(\Omega)$-norms. In this way, we reach the same level of knowledge as in the case of the classical Navier-Stokes equations. The complete proofs of all the results contained in this manuscript can be consulted in [16].

## 2. Function Spaces and framework

Throughout this paper we will use the following notation: Vector functions will be written in bold letters. The $H^m$ norm is denoted by $\|\cdot\|_m$. Here $H^m = W^{m,2}(\Omega)$ $(m > 0)$ are the usual Sobolev spaces. $H_0^1$ denotes the closure of $C_0^\infty(\Omega)$ in the $H^1$–norm. Let

$$\mathbf{C}_{0,\sigma}^\infty(\Omega) := \{\mathbf{v} \in (C_0^\infty(\Omega))^3 : \operatorname{div}\mathbf{v} = 0 \quad \text{in} \quad \Omega\}, \quad \mathbf{V} = \left\{\text{closure of } \mathbf{C}_{0,\sigma}^\infty(\Omega) \text{ in } \mathbf{H}_0^1(\Omega)\right\},$$

$$\mathbf{H} = \left\{\text{closure of } \mathbf{C}_{0,\sigma}^\infty(\Omega) \text{ in } \mathbf{L}^2(\Omega)\right\} \quad \text{and} \quad \mathbf{V}^* = \{\text{topological dual of } \mathbf{V}\}.$$

In order to give an operator interpretation of problem (1.1)-(1.2), we shall introduce the well known Helmholtz and Weyl decomposition. The Hilbert space $\mathbf{L}^2(\Omega)$ admits the Helmholtz and Weyl decomposition (cf. [21]):

$$\mathbf{L}^2 = \mathbf{H} \oplus \mathbf{H}^\perp,$$

where $\oplus$ denotes direct sum and $\mathbf{H}^\perp = \{\nabla\pi : \pi \in H^{1,\cdot}(\Omega)\}$. Let $P$ be the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$. Then the operator $A : \mathbf{H} \to \mathbf{H}$ given by $A = -P\Delta$ with domain $D(A) = \mathbf{V} \cap \mathbf{H}^2(\Omega)$ is called the Stokes operator. It is well known that $A$ is a positive self-adjoint operator and is characterized by the following relation:

$$(A\mathbf{w}, \mathbf{v}) = (\nabla\mathbf{w}, \nabla\mathbf{v}) \ \text{ for all } \mathbf{w} \in D(A),\ \mathbf{v} \in \mathbf{V}.$$

From now on, we also denote the inner product in $\mathbf{H}$ by the $\mathbf{L}^2(\Omega)$–inner product $(\cdot,\cdot)$. The general $L^p(\Omega)$-norm will be denoted by $\|\cdot\|_{L^p(\Omega)}$; to make easier the notation, in the case $p = 2$ we simply denote the $L^2$–norm by $\|\cdot\|$. We shall denote by $\mathbf{w}^k(x)$ and $\lambda_k$ the eigenfunctions and the eigenvalues of the Stokes operator. It is well known (see [21]) that $\mathbf{w}^k(x)$ are orthogonal in the inner products $(\cdot,\cdot)$, $(\nabla\cdot,\nabla\cdot)$ and $(A\cdot, A\cdot)$ and complete in the spaces $\mathbf{H}$, $\mathbf{V}$ and $\mathbf{V} \cap \mathbf{H}^2(\Omega)$, respectively. For each $k \in \mathbb{N}$, we denote by $P_k$ the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}_k = \operatorname{span}[\mathbf{w}^1(x), \ldots, \mathbf{w}^k(x)]$.

Throughout this work, we will deal with the following notion of strong solution for (1.1)-(1.2).

**Definition 2.1** Let $\mathbf{u}_0, \mathbf{h}_0 \in \mathbf{V}$ and $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$. By a strong solution of the problem (1.1)–(1.2), we mean a pair of vector-valued functions $(\mathbf{u}, \mathbf{h})$ such that $\mathbf{u}, \mathbf{h} \in L^\infty(0,T;\mathbf{V}) \cap \mathbf{L}^2(0,T;D(A))$ and that satisfies (1.1)–(1.2).

As a first step to set up and prove the main results of this work, and using the properties of the operator $P$, we can reformulate the problem (1.1)–(1.2), as follows: find $\mathbf{u}, \mathbf{h}$ in suitable spaces, satisfying:

$$\left\{\begin{array}{rcll} (\mathbf{u}_t, \mathbf{v}) + (\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) - ((\mathbf{h}\cdot\nabla)\mathbf{h}, \mathbf{v}) & = & (\mathbf{f}, \mathbf{v}), & \forall\, \mathbf{v} \in \mathbf{V}, \\ (\mathbf{h}_t, \mathbf{z}) + (\nabla\mathbf{h}, \nabla\mathbf{z}) + ((\mathbf{u}\cdot\nabla)\mathbf{h}, \mathbf{z}) - ((\mathbf{h}\cdot\nabla)\mathbf{u}, \mathbf{z}) & = & 0, & \forall \mathbf{z} \in \mathbf{V}, \\ \mathbf{u}(x,0) & = & \mathbf{u}_0(x), & x \in \Omega, \\ \mathbf{h}(x,0) & = & \mathbf{h}_0(x), & x \in \Omega. \end{array}\right. \tag{2.1}$$

Observe that, because we do not focus on the dependence of the error on the $\eta, \mu, \sigma$ or $\rho_m$, then we consider them all equal to 1.

In order to establish the results concerning estimates for spectral Galerkin approximation, we need to fix some problems. The *spectral Galerkin approximations* for $(\mathbf{u},\mathbf{h})$ are defined for each $k\in\mathbb{N}$ as the solution $(\mathbf{u}^k,\mathbf{h}^k)\in C^2([0,T];\mathbf{V}_k)\times C^2([0,T];\mathbf{V}_k)$ of:

$$\left\{\begin{array}{rcll} (\mathbf{u}^k_t,\mathbf{v})+(\nabla\mathbf{u}^k,\nabla\mathbf{v})+((\mathbf{u}^k\cdot\nabla)\mathbf{u}^k,\mathbf{v})-((\mathbf{h}^k\cdot\nabla)\mathbf{h}^k,\mathbf{v}) &=& (\mathbf{f},\mathbf{v}), & \forall\,\mathbf{v}\in\mathbf{V}_k,\\ (\mathbf{h}^k_t,\mathbf{z})+(\nabla\mathbf{h}^k,\nabla\mathbf{z})+((\mathbf{u}^k\cdot\nabla)\mathbf{h}^k,\mathbf{z})-((\mathbf{h}^k\cdot\nabla)\mathbf{u}^k,\mathbf{z}) &=& 0, & \forall\mathbf{z}\in\mathbf{V}_k,\\ \mathbf{u}(x,0) &=& P_k\mathbf{u}_0(x), & x\in\Omega,\\ \mathbf{h}(x,0) &=& P_k\mathbf{h}_0(x), & x\in\Omega. \end{array}\right. \tag{2.2}$$

Recall that the eigenfunctions expansion of $\mathbf{u}$ and $\mathbf{h}$ can be written, respectively, as:

$$\mathbf{u}(x,t)=\sum_{i=1}^{\infty}a_i(t)\mathbf{w}^i(x)\quad\text{and}\quad\mathbf{h}(x,t)=\sum_{i=1}^{\infty}c_i(t)\mathbf{w}^i(x), \tag{2.3}$$

where $\mathbf{w}^i$ are the eigenfunctions of the Stokes operator. The partial sums of the series for $\mathbf{u}$ and $\mathbf{h}$ will also appear in our study, whose expression are given, respectively, by:

$$\mathbf{v}^k(t)=P_k\mathbf{u}(t)=\sum_{i=1}^{k}a_i(t)\mathbf{w}^i(x)\quad\text{and}\quad\mathbf{b}^k(t)=P_k\mathbf{h}(t)=\sum_{i=1}^{k}c_i(t)\mathbf{w}^i(x). \tag{2.4}$$

## 3. Known results

By using the spectral Galerkin approximations (2.2), Rojas-Medar and Boldrini ( [4], [5]) proved the following results:

**Theorem 3.1** *Assume the following condition for the initial data* $\mathbf{u}_0$, $\mathbf{h}_0$, *and the external force* $\mathbf{f}$ *of* (1.1)-(1.2):

$$\mathbf{u}_0,\ \mathbf{h}_0\in\mathbf{V},\quad \mathbf{f}\in L^2(0,T;\mathbf{L}^2(\Omega)) \tag{3.1}$$

*Then, on a (possibly small) time interval* $[0,T_1]$, $0<T_1\le T$, *problem* (1.1)-(1.2) *has a unique strong solution* $(\mathbf{u},\mathbf{h})$. *This solution belongs* $C([0,T_1];\mathbf{V})\times C([0,T_1];\mathbf{V})$. *Moreover, there exist* $C^1$*-functions* $F(t)$ *and* $G(t)$ *such that for any* $t\in[0,T_1]$, *there hold*:

$$\begin{array}{rcl} \|\nabla\mathbf{u}(t)\|^2+\|\nabla\mathbf{h}(t)\|^2+\displaystyle\int_0^t(\|A\mathbf{u}(s)\|^2+\|A\mathbf{h}(s)\|^2)ds &\le& F(t),\\ \displaystyle\int_0^t(\|\mathbf{u}_t(s)\|^2+\|\mathbf{h}_t(s)\|^2)ds &\le& G(t). \end{array}$$

*Moreover, the same kind of estimates holds uniformly in* $n\in\mathbb{N}$ *for the Galerkin approximations* $(\mathbf{u}^n,\mathbf{h}^n)$.

**Theorem 3.2** *Assume* (3.1) *and*

$$\mathbf{u}_0,\ \mathbf{h}_0\in D(A),\quad \mathbf{f}_t\in L^2(0,T;\mathbf{L}^2(\Omega)). \tag{3.2}$$

*Then*:

$$\begin{array}{rcl} \|\mathbf{u}_t(t)\|^2+\|\mathbf{h}_t(t)\|^2+\displaystyle\int_0^t(\|\nabla\mathbf{u}_t(s)\|^2+\|\nabla\mathbf{h}_t(s)\|^2)ds &\le& H_0(t),\\ \|A\mathbf{u}(t)\|^2+\|A\mathbf{h}(t)\|^2 &\le& H_1(t),\\ \displaystyle\int_0^t(\|\mathbf{u}_{tt}(s)\|^2_{V^\star}+\|\mathbf{h}_{tt}(s)\|^2_{V^\star})ds &\le& H_2(t), \end{array}$$

*for any* $t\in[0,T_1]$, *where* $H_i(t)$, $i=0,1,2$ *are continuous functions* $t\in[0,T_1]$. *Therefore*:

$$\mathbf{u}(t),\mathbf{h}(t)\in C^1([0,T_1];\mathbf{V})\cap C([0,T_1];D(A)).$$

*Moreover, the same kind of estimates holds uniformly in* $n$ *for the Galerkin approximations* $(\mathbf{u}^n,\mathbf{h}^n)$.

Referring to the Navier–Stokes equations, the following lemma can be found in the Rautmann's paper [12].

**Lemma 3.3**
*If* $\mathbf{u} \in \mathbf{V}$*, then there holds*:

$$\|\mathbf{u} - P_k\mathbf{u}\|^2 \le \frac{1}{\lambda_{k+1}}\|\nabla\mathbf{u}\|^2.$$

*Also, if* $\mathbf{u} \in \mathbf{V} \cap \mathbf{H}^2(\Omega)$*, we have*:

$$\|\mathbf{u} - P_k\mathbf{u}\|^2 \le \frac{1}{\lambda_{k+1}^2}\|A\mathbf{u}\|^2, \qquad \|\nabla\mathbf{u} - \nabla P_k\mathbf{u}\|^2 \le \frac{1}{\lambda_{k+1}}\|A\mathbf{u}\|^2.$$

Some of the classical Sobolev interpolation inequalities, considered in this manuscript, can be found in the following result:

**Lemma 3.4** *The following estimates are true*:

- $\|\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)} \le C\|A\mathbf{v}\|, \quad \forall\mathbf{v} \in \mathbf{V} \cap \mathbf{H}^2(\Omega),$
- $\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)} \le C\|\nabla\mathbf{v}\|, \quad \forall\mathbf{v} \in \mathbf{V},$
- $\|\mathbf{v}\|_{\mathbf{L}^3(\Omega)} \le C\|\mathbf{v}\|^{1/2}\|\nabla\mathbf{v}\|^{1/2}, \quad \forall\mathbf{v} \in \mathbf{V}$
- $\|\mathbf{v}\|_{\mathbf{L}^4(\Omega)} \le C\|\mathbf{v}\|^{1/4}\|\nabla\mathbf{v}\|^{3/4}, \quad \forall\mathbf{v} \in \mathbf{V}.$

## 4. Estimates for the solution in $\mathbf{H}^1(\Omega)$

Our first result on error estimates read as follows:

**Theorem 4.1** *Assume hypothesis* (3.1) *for the data. Then, the approximations* $(\mathbf{u}^k, \mathbf{h}^k)$ *satisfy*:

$$\|\mathbf{u}(t) - \mathbf{u}^k(t)\|^2 + \|\mathbf{h}(t) - \mathbf{h}^k(t)\|^2 + \int_0^t (\|\nabla\mathbf{u}(s) - \nabla\mathbf{u}^k(s)\|^2 + \|\nabla\mathbf{h}(s) - \nabla\mathbf{h}^k(s)\|^2)\, ds \le \frac{C}{\lambda_{k+1}}.$$

*In addition, if we assume that* (3.2)*, then the approximations* $(\mathbf{u}^k, \mathbf{h}^k)$ *satisfy*:

$$\|\mathbf{u}(t) - \mathbf{u}^k(t)\|^2 + \|\mathbf{h}(t) - \mathbf{h}^k(t)\|^2 \le \frac{C}{\lambda_{k+1}^2}.$$

**Theorem 4.2** *If in addition to* (3.1) *we assume* (3.2)*, then we have that there exists a constant* $C > 0$ *such that*:

$$\|\nabla\mathbf{u}(t) - \nabla\mathbf{u}^k(t)\|^2 + \|\nabla\mathbf{h}(t) - \nabla\mathbf{h}^k(t)\|^2 \le \frac{C}{\lambda_{k+1}}.$$

**Corollary 4.3** *Under the hypothesis of Theorem* 4.2*, there exists a positive constant* $C > 0$ *such that*:

$$\int_0^t (\|\mathbf{u}_t(s) - \mathbf{u}_t^k(s)\|^2 + \|\mathbf{h}_t(s) - \mathbf{h}_t^k(s)\|^2)\, ds \le \frac{C}{\lambda_{k+1}}$$

*and, if* $\mathbf{f} \in L^2(0,T;\mathbf{H}^1(\Omega))$ *then*:

$$\int_0^t (\|A\mathbf{u}(s) - A\mathbf{u}^k(s)\|^2 + \|A\mathbf{h}(s) - A\mathbf{h}^k(s)\|^2)\, ds \le \frac{C}{\lambda_{k+1}}.$$

Note that these estimates are made in the Sobolev spaces related to the strong regularity of the solution (see Definition 2.1).

In the search of a proof for these theorems (and corollary), we have to use some preliminary results whose proof needs to define the following auxiliary variables and problems:
Using (2.3) and (2.4), we define:

$$\begin{aligned} \mathbf{e}^k(t) &= \mathbf{u}(t) - \mathbf{v}^k(t), \quad & \tilde{\mathbf{e}}^k(t) &= \mathbf{h}(t) - \mathbf{b}^k(t), \\ \mathbf{E}^k(t) &= \mathbf{v}^k(t) - \mathbf{u}^k(t), \quad & \tilde{\mathbf{E}}^k(t) &= \mathbf{b}^k(t) - \mathbf{h}^k(t), \end{aligned} \tag{4.1}$$

where $\mathbf{u}^k$ and $\mathbf{h}^k$ are the $k^{th}$ Galerkin approximations of $\mathbf{u}$ and $\mathbf{h}$ solutions of (2.2), respectively. One of our aim is to "mesure" the distance between the solutions of (2.1) and (2.2), that we split as:

$$\mathbf{u}(t) - \mathbf{u}^k(t) = \mathbf{e}^k(t) + \mathbf{E}^k(t), \quad \text{and} \quad \mathbf{h}(t) - \mathbf{h}^k(t) = \tilde{\mathbf{e}}^k(t) + \tilde{\mathbf{E}}^k(t). \tag{4.2}$$

These variables satisfy the following problem:

$$\begin{cases} (\mathbf{E}^k_t, \mathbf{v}) + (\nabla \mathbf{E}^k, \nabla \mathbf{v}) + ((\mathbf{e}^k \cdot \nabla)\mathbf{u}, \mathbf{v}) + ((\mathbf{E}^k \cdot \nabla)\mathbf{u}, \mathbf{v}) + ((\mathbf{u}^k \cdot \nabla)\mathbf{e}^k, \mathbf{v}) + ((\mathbf{u}^k \cdot \nabla)\mathbf{E}^k, \mathbf{v}) \\ -((\tilde{\mathbf{e}}^k \cdot \nabla)\mathbf{h}, \mathbf{v}) - ((\tilde{\mathbf{E}}^k \cdot \nabla)\mathbf{h}, \mathbf{v}) - ((\mathbf{h}^k \cdot \nabla)\tilde{\mathbf{e}}^k, \mathbf{v}) - ((\mathbf{h}^k \cdot \nabla)\tilde{\mathbf{E}}^k, \mathbf{v}) = 0, \qquad \forall\, \mathbf{v} \in \mathbf{V}_k, \\ (\tilde{\mathbf{E}}^k_t, \mathbf{z}) + (\nabla \tilde{\mathbf{E}}^k, \nabla \mathbf{z}) + ((\mathbf{e}^k \cdot \nabla)\mathbf{h}, \mathbf{z}) + ((\mathbf{E}^k \cdot \nabla)\mathbf{h}, \mathbf{z}) + ((\mathbf{u}^k \cdot \nabla)\tilde{\mathbf{e}}^k, \mathbf{z}) + ((\mathbf{u}^k \cdot \nabla)\tilde{\mathbf{E}}^k, \mathbf{z}) \\ -((\tilde{\mathbf{e}}^k \cdot \nabla)\mathbf{u}, \mathbf{z}) - ((\tilde{\mathbf{E}}^k \cdot \nabla)\mathbf{u}, \mathbf{z}) - ((\mathbf{h}^k \cdot \nabla)\mathbf{e}^k, \mathbf{z}) - ((\mathbf{h}^k \cdot \nabla)\mathbf{E}^k, \mathbf{z}) = 0, \qquad \forall\, \mathbf{z} \in \mathbf{V}_k, \\ \mathbf{E}^k(x,0) = \tilde{\mathbf{E}}^k(x,0) = 0, \qquad x \in \Omega. \end{cases}$$

Using adequate estimates (see [16] for more details), the following results can be proved:

**Lemma 4.4** *Assume hypothesis* (3.1) *for the data. Then*:

$$\|\mathbf{E}^k(t)\|^2 + \|\tilde{\mathbf{E}}^k(t)\|^2 \le \frac{C}{\lambda_{k+1}}.$$

*In addition, if we assume* (3.2)*, then*:

$$\|\mathbf{E}^k(t)\|^2 + \|\tilde{\mathbf{E}}^k(t)\|^2 \le \frac{C}{\lambda_{k+1}^2}.$$

**Corollary 4.5** *Assume hypothesis* (3.1) *for the data. Then*:

$$\int_0^t (\|\nabla \mathbf{E}^k(s)\|^2 + \|\nabla \tilde{\mathbf{E}}^k(s)\|^2 ds \le \frac{C}{\lambda_{k+1}}.$$

*In addition, if we assume* (3.2)*, then*:

$$\int_0^t (\|\nabla \mathbf{E}^k(s)\|^2 + \|\nabla \tilde{\mathbf{E}}^k(s)\|^2) ds \le \frac{C}{\lambda_{k+1}^2}.$$

**Lemma 4.6** *Assuming* (3.1) *and* (3.2) *for the data, we have that there exists a constant $C > 0$ such that*:

$$\|\nabla \mathbf{E}^k(t)\|^2 + \|\nabla \tilde{\mathbf{E}}^k(t)\|^2 \qquad \le \frac{C}{\lambda_{k+1}}.$$

**Corollary 4.7** *Under the hypotheses of Lemma* 4.6*, here exists a positive constant $C > 0$ such that*:

$$\int_0^t (\|\mathbf{E}^k_t(s)\|^2 + \|\tilde{\mathbf{E}}^k_t(s)\|^2) ds \le \frac{C}{\lambda_{k+1}}.$$

## 5. $H^2(\Omega)$-error estimates for the velocity and the magnetic field

The objective of this section is to state and sketch the estimates in the $\mathbf{H}^2(\Omega)$-norm for the solutions of (1.1)-(1.2) that we have obtained. Concretely, our result reads as follows:

**Theorem 5.1** *Assume* (3.1)-(3.2)*. If moreover $\mathbf{f} \in C([0,T], \mathbf{H}^1(\Omega))$ and $\mathbf{u}_0, \mathbf{h}_0 \in D(A^{1+\epsilon})$, with $\epsilon \in (0, \frac{1}{4})$, then*

$$\begin{aligned} \|A\mathbf{u}(t) - A\mathbf{u}^k(t)\| + \|\mathbf{u}_t(t) - \mathbf{u}^k_t(t)\| &\le C\left[\frac{C(\alpha+\epsilon)}{\lambda_{k+1}^{\epsilon}} + \frac{1}{\lambda_{k+1}}\right], \\ \|A\mathbf{h}(t) - A\mathbf{h}^k(t)\| + \|\mathbf{h}_t(t) - \mathbf{h}^k_t(t)\| &\le C\left[\frac{C(\alpha+\epsilon)}{\lambda_{k+1}^{\epsilon}} + \frac{1}{\lambda_{k+1}}\right]. \end{aligned}$$

For the proof of this theorem we will use the writing of $\mathbf{u}(t) - \mathbf{u}^k(t)$ and $\mathbf{h}(t) - \mathbf{h}^k(t)$ in terms of $\mathbf{e}^k(t)$ and $\mathbf{E}^k(t)$ and $\tilde{\mathbf{e}}^k(t)$ and $\tilde{\mathbf{E}}^k(t)$, respectively, given in (4.2). Therefore, if we want to estimate $A\mathbf{u} - A\mathbf{u}^k$ and $A\mathbf{h} - A\mathbf{h}^k$, then we need to estimate $A\mathbf{u} - A\mathbf{v}^k$ and $A\mathbf{E}^k$ and $A\mathbf{h} - A\mathbf{b}^k$ and $A\tilde{\mathbf{E}}^k$. With this objective, we precise, in first time, to estimate $A^\alpha\mathbf{u} - A^\alpha\mathbf{v}^k$ and $A^\alpha\mathbf{E}^k$ and $A^\alpha\mathbf{h} - A^\alpha\mathbf{b}^k$ and $A^\alpha\tilde{\mathbf{E}}^k$ for $\alpha \in [0,1)$ and then obtain the desired result.

The regularity results for the solution obtained in the Theorems 3.1 and 3.2 will be also necessary in order to obtain our results. Firstly, observe that we can write the following representation of the solution obtained in Theorem 3.1:

$$\begin{aligned}
\mathbf{u}(t) &= e^{-At}\mathbf{u}_0 + \int_0^t e^{-(t-s)A}P(\mathbf{f} - (\mathbf{u}(s)\cdot\nabla)\mathbf{u}(s) + (\mathbf{h}(s)\cdot\nabla)\mathbf{h}(s))ds,\\
\mathbf{h}(t) &= e^{-At}\mathbf{h}_0 + \int_0^t e^{-(t-s)A}(-(\mathbf{u}(s)\cdot\nabla)\mathbf{h}(s) + (\mathbf{h}(s)\cdot\nabla)\mathbf{u}(s))ds.
\end{aligned} \tag{5.1}$$

**Theorem 5.2** *Suppose that* $\mathbf{f} \in C([0,T], \mathbf{H}^1(\Omega))$ *and* $\mathbf{u}_0, \mathbf{h}_0 \in D(A^{1+\epsilon})$, *then the solution* $(\mathbf{u}, \mathbf{h})$ *of* (1.1)-(1.2) *satisfies for* $0 \le \epsilon < 1/4$,

$$\mathbf{u}, \mathbf{h} \in C([0,T]; D(A^{1+\epsilon})) \cap C^1([0,T]; D(A^\epsilon)).$$

The proof of Theorem 5.2 is based the properties of $D(A^\alpha)$, the Stokes operator properties and the use of (4.1), (4.2) and (5.1). In particular, the fractional powers $A^\alpha$ with domain of definition $D(A^\alpha) \subset \mathbf{H}$ are defined for any real $\alpha$ by means of the spectral representation of $A$. For $\alpha < \beta$ the imbedding $D(A^\beta) \subset D(A^\alpha)$ is compact and $D(A^\beta)$ is dense in $D(A^\alpha)$, therefore $A$ is a sectorial operator and $A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}$. On $D(A^\alpha)$, the operator $A^\alpha$ commute, with $e^{-tA}$, and satisfies several properties (see [7]).

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