Proceedings

of the

XXVI Congreso de Ecuaciones Diferenciales y Aplicaciones XVI Congreso de Matemática Aplicada

Gijón (Asturias), Spain

June 14-18, 2021

CEDYA CMA 2020 GIJÓN

SeMA
Sociedad Española de Matemática Aplicada

Universidad de Oviedo

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Universidad de Oviedo
Servicio de Publicaciones de la Universidad de Oviedo
Campus de Humanidades. Edificio de Servicios. 33011 Oviedo (Asturias)
Tel. 985 10 95 03 Fax 985 10 95 07
http: www.uniovi.es/publicaciones
servipub@uniovi.es

ISBN: 978-84-18482-21-2

Foreword

It is with great pleasure that we present the Proceedings of the 26th Congress of Differential Equations and Applications / 16th Congress of Applied Mathematics (XXVI CEDYA / XVI CMA), the biennial congress of the Spanish Society of Applied Mathematics S$\vec{e}$MA, which is held in Gijón, Spain from June 14 to June 18, 2021.

In this volume we gather the short papers sent by some of the almost three hundred and twenty communications presented in the conference. Abstracts of all those communications can be found in the abstract book of the congress. Moreover, full papers by invited lecturers will shortly appear in a special issue of the S$\vec{e}$MA Journal.

The first CEDYA was celebrated in 1978 in Madrid, and the first joint CEDYA / CMA took place in Málaga in 1989. Our congress focuses on different fields of applied mathematics: Dynamical Systems and Ordinary Differential Equations, Partial Differential Equations, Numerical Analysis and Simulation, Numerical Linear Algebra, Optimal Control and Inverse Problems and Applications of Mathematics to Industry, Social Sciences, and Biology. Communications in other related topics such as Scientific Computation, Approximation Theory, Discrete Mathematics and Mathematical Education are also common.

For the last few editions, the congress has been structured in mini-symposia. In Gijón, we will have eighteen minis-symposia, proposed by different researchers and groups, and also five thematic sessions organized by the local organizing committee to distribute the individual contributions. We will also have a poster session and ten invited lectures. Among all the mini-symposia, we want to highlight the one dedicated to the memory of our colleague Francisco Javier “Pancho” Sayas, which gathers two plenary lectures, thirty-six talks, and more than forty invited people that have expressed their wish to pay tribute to his figure and work.

This edition has been deeply marked by the COVID-19 pandemic. First scheduled for June 2020, we had to postpone it one year, and move to a hybrid format. Roughly half of the participants attended the conference online, while the other half came to Gijón. Taking a normal conference and moving to a hybrid format in one year has meant a lot of efforts from all the parties involved. Not only did we, as organizing committee, see how much of the work already done had to be undone and redone in a different way, but also the administration staff, the scientific committee, the mini-symposia organizers, and many of the contributors had to work overtime for the change.

Just to name a few of the problems that all of us faced: some of the already accepted mini-symposia and contributed talks had to be withdrawn for different reasons (mainly because of the lack of flexibility of the funding agencies); it became quite clear since the very first moment that, no matter how well things evolved, it would be nearly impossible for most international participants to come to Gijón; reservations with the hotels and contracts with the suppliers had to be cancelled; and there was a lot of uncertainty, and even anxiety could be said, until we were able to confirm that the face-to-face part of the congress could take place as planned.

On the other hand, in the new open call for scientific proposals, we had a nice surprise: many people that would have not been able to participate in the original congress were sending new ideas for mini-symposia, individual contributions and posters. This meant that the total number of communications was about twenty percent greater than the original one, with most of the new contributions sent by students.

There were almost one hundred and twenty students registered for this CEDYA / CMA. The hybrid format allows students to participate at very low expense for their funding agencies, and this gives them the opportunity to attend different conferences and get more merits. But this, which can be seen as an advantage, makes it harder for them to obtain a full conference experience. Alfréd Rényi said: “a mathematician is a device for turning coffee into theorems”. Experience has taught us that a congress is the best place for a mathematician to have a lot of coffee. And coffee cannot be served online.

In Gijón, June 4, 2021

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Overdetermined elliptic problems in onduloid-type domains with general nonlinearities

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Abstract

In this paper, we prove the existence of solutions to a general semilinear elliptic problem with overdetermined boundary conditions. The proof uses a local bifurcation argument from the straight cylinder, in analogy with the onduloids and the theory of Constant Mean Curvature surfaces. Such examples have been found already for linear problems or with nonlinearity $f(u) = 1$. In this work we are able to extend this phenomenon for a large class of functions $f(u)$.

Remark: This manuscript, especially the whole proof, is a work in progress in collaboration with David Ruiz and Pieralberto Sicbaldi.

1. Introduction

This paper is devoted to the existence of new solutions of a semilinear overdetermined elliptic problem in the form

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = \text{constant} & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a domain of $\mathbb{R}^{n+1}$, $n \geq 1$, $f : [0, +\infty) \to \mathbb{R}$ is a $C^{1,\alpha}$ function and ν stands for the exterior normal unit vector about $\partial\Omega$.

A classical result by Serrin [16,23] states that the existence of a positive solution to the overdetermined problem (1.1) yields that the smooth bounded domain Ω must be a ball. This result has applications in various mathematical and physical problems, such as isoperimetric inequalities, spectral geometry and hydrodynamics (see [4,26,27] for the details).

The case when the domain Ω is supposed to be unbounded is also very interesting. Indeed, overdetermined boundary conditions appear in free boundary problems if the variational structure imposes suitable conditions on the separation interface (see [2,6]). In this process, several methods applied to study the regularity of free boundary problems are based on blow-up techniques that lead to the study of an elliptic problem in an unbounded domain. In this framework, Berestycki, Caffarelli and Nirenberg [5] were concerned with the problem (1.1) in unbounded domains and concluded the following conjecture:

BCN Conjecture. Assume that Ω is a smooth domain with $\mathbb{R}^n \backslash \bar{\Omega}$ connected, then the existence of a bounded positive solution to problem (1.1) for some Lipschitz function f implies that Ω is either a ball, a half-space, a generalized cylinder $B^k \times \mathbb{R}^{n-k}$ (B^k is a ball in $\mathbb{R}^k$), or the complement of one of them.

Such conjecture, in the case of exterior domains, is motivated by the works of Reichel [17], Aftalion and Busca [1]. BCN Conjecture actually has motivated various interesting works. For example, Farina and Valdinoci [11] obtained some natural assumptions to conclude that Ω must be a half-space and u is a function only depending on one variable, when Ω is an epigraph for which the problem (1.1) has a solution. Furthermore, in [18] the BCN conjecture is proved for some classes of nonlinearities f; the work [28] gives a complete classification of solutions to harmonic overdetermined problems in the plane; Ros, Ruiz and Sicbaldi in [19] proved that if $\partial\Omega$ is connected and unbounded in dimension 2, then Ω is a half-plane.

The conjecture has been answered with a counterexample for $n \geq 3$ in [25], where the second author constructed a domain by a periodic perturbation of the straight cylinder $B^n \times \mathbb{R}$ for which there exists a periodic solution to the problem (1.1) for $f(u) = \lambda u, \lambda > 0$. More precisely, such domains, as shown in [22], belong to a 1-parameter family $\{\Omega_s\}_{s\in(-\epsilon,\epsilon)}$ and are given by

$$\Omega_s = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \; : \; |x| < 1 + s\cos\left(\frac{2\pi}{T_s} t\right) + O(s^2) \right\}$$

where ϵ is a small constant, $T_s = T_0 + O(s)$ and T_0 depends only on the dimension n. In [10], Fall, Minlend and Weth provided the same kind of work for $f(u) = 1$. In [8] similar solutions are found for the Allen-Cahn nonlinearity $f(u) = u - u^3$, but in domains that are perturbations of a dilated straight cylinder, i.e. perturbations of $(\epsilon^{-1} B^n) \times \mathbb{R}$ for ϵ small. In addition, Ros, Ruiz and Sicbaldi [20] found a perturbation of the complement of a ball B_R that supports a bounded solution to the problem (1.1), when f is a nonlinear function $f(u) = u^p - u$.

The aim of this paper is to perform such a construction under somewhat minimal assumptions on the nonlinearity $f(u)$. For technical reasons, we need the following assumptions:

Assumption 1: There exists a positive radially symmetric solution $\phi_1 \in C^{2,\alpha}(B)$ of the problem

$$\begin{cases} \Delta\phi_1 + f(\phi_1) = 0 & \text{in } B \\ \phi_1 = 0 & \text{on } \partial B \end{cases} \tag{1.2}$$

with $\partial_\nu(x) \neq 0$ for $x \in \partial B$.

Assumption 2: Define the linearized operator $L_D : C^{2,\alpha}_{0,r}(B) \to C^{0,\alpha}_r(B)$ by

$$L_D(\phi) = \Delta\phi + f'(\phi_1)\phi\,, \tag{1.3}$$

where $C^{2,\alpha}_{0,r}(B)$ and $C^{0,\alpha}_r(B)$ denote the spaces of radial functions in $C^{2,\alpha}_0(B)$ and $C^{0,\alpha}(B)$ respectively. We assume that the linearized operator L_D is non-degenerate; in other words, if $L_D(\phi) = 0$ then $\phi = 0$.

Observe that by [12], any solution ϕ_1 of (1.2) needs to be a radially symmetric function.

We are now in position to state our main result:

Theorem 1.1 *If $n \geq 1$, $f : [0, +\infty) \to \mathbb{R}$ is $C^{1,\alpha}$ and assumptions 1 and 2 hold, then there exists a positive number T_* and a smooth map*

$$\begin{array}{ccc} (-\epsilon, \epsilon) & \to & C^{2,\alpha}(\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \\ s & \mapsto & (v_s, T_s) \end{array}$$

with $v_0 = 0$ and $T_0 = T_*$ such that the overdetermined problem (1.1) *has a solution in the domain*

$$\Omega_s = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \,:\, |x| < 1 + v_s\left(\frac{t}{T_s}\right) \right\}.$$

The solution $u = u_s$ of problem (1.1) *is T_s-periodic in the variable t and hence bounded. Moreover*

$$\int_0^1 v_s(t)\, dt = 0$$

and

$$v_s(t) = s\, \cos(2\pi\, t) + O(s^2)\,.$$

As a consequence, for all functions f satisfying assumptions 1 and 2 we produce a counterexample to the BCN conjecture diffeomorphic to a cylinder. Assumptions 1 and 2 hold for example in the following cases among many others:

(1) If $f(0) > 0$ and $f'(s) < \lambda_1$ for any $s \in (0, +\infty)$, where λ_1 is the first eigenvalue of the Dirichlet Laplacian in the unit ball of $\mathbb{R}^n$.

(2) If $f(u) = u^p - u$, $1 < p < \frac{n+2}{n-2}$ if $n > 2$, see [15].

(3) If $f(u) = \lambda e^u$ and $\lambda \in (0, \lambda^*)$, $\lambda^* > 0$ receives the name of extremal value, see for instance [9].

Obviously, our theorem covers the result in [10] and is complementary to the results in [22, 25].

2. Some details

The operator L_D defined in Assumption 2 has a diverging sequence of eigenvalues γ_{D_j}, hence there are only a finite number l of them which are negative, i.e.

$$\gamma_{D_1} < \gamma_{D_2} < \cdots < \gamma_{D_l} < 0,\ \gamma_{D_{l+1}} > 0.$$

Actually, these eigenvalues γ_{D_j} are all simple.

Let $z_j \in C^{2,\alpha}_{0,r}(B)$ (normalized by $\|z_j\|_{L^2} = 1$) be the eigenfunctions corresponding to the eigenvalues γ_{D_j}, i.e.

$$\begin{cases} \Delta z_j + f'(\phi_1) z_j + \gamma_{D_j} z_j = 0 & \text{in } B \\ z_j = 0 & \text{on } \partial B \end{cases}. \tag{2.1}$$

As is well known, the operator L_D is related to the quadratic form

$$Q_D : H^1_{0,r}(B) \to \mathbb{R}, \;\; Q_D(\phi) := \int_B \left(|\nabla\phi|^2 - f'(\phi_1)\phi^2\right).$$

The first eigenvalue of L_D is given by

$$\gamma_{D_1} = \inf\left\{Q_D(\phi) : \|\phi\|_{L^2(B)} = 1\right\}.$$

We also define the quadratic form

$$Q : H^1_r(B) \to \mathbb{R}, \;\; Q(\psi) := \int_B \left(|\nabla\psi|^2 - f'(\phi_1)\psi^2\right) + c\,\omega_n \psi(1)^2,$$

where ω_n is the area of $\mathbb{S}^{n-1}$ and $c = -\phi_1''(1) = n - 1 + \frac{f(0)}{\phi_1'(1)}$.

Observe that,

$$Q|_{H^1_{0,r}(B)} = Q_D.$$

Analogously, we can define

$$\gamma_1 = \inf\left\{Q(\psi) : \|\psi\|_{L^2(B)} = 1\right\}. \tag{2.2}$$

It is rather standard to show that γ_1 is achieved by the minimizer ψ_1, and that γ_1 is simple, so ψ_1 is uniquely determined up to a sign. In addition, there holds: $\gamma_1 < \min\{0, \gamma_{D_1}\}$. In fact, it is evident that $\gamma_1 \le \gamma_{D_1}$ from the variational characterization of the eigenvalues. The strict inequality follows because of the uniqueness of solutions of Initial Vale Problems for ODEs (see [21] for details).

Next, we will consider the Dirichlet problem for the linearized equation in a straight cylinder for periodic functions, namely,

$$\begin{cases} \Delta\psi + f'(\phi_1)\psi = 0 & \text{in } B \times \mathbb{R} \\ \psi(x) = 0 & \text{on } (\partial B) \times \mathbb{R} \end{cases} \tag{2.3}$$

where $\psi(x,t)$ is T-periodic in the variable t.

Define:

$$C_1^T = B \times \mathbb{R}/T\mathbb{Z}.$$

Hence (2.3) is just the linearization of the problem:

$$\begin{cases} \Delta\phi + f(\phi) = 0 & \text{in } C_1^T \\ \phi = 0 & \text{on } \partial C_1^T \end{cases}. \tag{2.4}$$

If ϕ_1 is the solution of Problem (1.2), then the function $\phi_1(x,t) = \phi_1(x)$ (we use a natural abuse of notation) solves (2.4). Define the linearized operator $L_D^T : C^{2,\alpha}_{0,r}(C_1^T) \to C^\alpha_r(C_1^T)$ (associated to Problem (2.4)) by

$$L_D^T(\phi) = \Delta\phi + f'(\phi_1)\phi,$$

and consider the eigenvalue problem

$$L_D^T(\phi) + \tau\phi = 0.$$

Then the functions $z_j(x,t) = z_j(x)$ from (2.1) solve the problem

$$\begin{cases} \Delta z_j + f'(\phi_1) z_j + \tau_j z_j = 0 & \text{in } C_1^T \\ z_j = 0 & \text{on } \partial C_1^T \end{cases}.$$

Let us define the quadratic form $Q_D^T : H^1_{0,r}(C_1^T) \to \mathbb{R}$ related to L_D^T,

$$Q_D^T(\psi) := \int_{C_1^T} \left(|\nabla\psi|^2 - f'(\phi_1)\psi^2\right).$$

We will also need to define the quadratic form $Q^T : H^1_r(C^T_1) \to \mathbb{R}$,

$$Q^T(\psi) := \int_{C^T_1} \left(|\nabla\psi|^2 - f'(\phi_1)\psi^2\right) + c\int_{\partial C^T_1} \psi^2.$$

In next proposition we study the behavior of these quadratic forms:

Proposition 2.1 *Define*:

$$\alpha = \inf\left\{Q^T_D(\psi) : \psi \in H^1_{0,r}(C^T_1),\ \|\psi\|_{L^2} = 1,\ \int_{C^T_1} \psi\, z_j = 0,\ j = 1,\dots l.\right\},$$

$$\beta = \inf\left\{Q^T(\psi) : \psi \in H^1_r(C^T_1),\ \|\psi\|_{L^2} = 1,\ \int_{\partial C^T_1} \psi = 0,\ \int_{C^T_1} \psi\, z_j = 0,\ j = 1,\dots l.\right\},$$

then

$$\alpha = \min\left\{\gamma_{D_{l+1}}, \gamma_{D_1} + \frac{4\pi^2}{T^2}\right\}, \quad \beta = \min\left\{\gamma_{D_{l+1}}, \gamma_1 + \frac{4\pi^2}{T^2}\right\}.$$

Moreover, those infima are achieved. If $\beta = \gamma_1 + \frac{4\pi^2}{T^2}$, the minimizer is equal to

$$\psi_1(x)\cos\left(\frac{2\pi}{T}(t+\delta)\right),$$

where ψ_1 is the minimizer for (2.2) *and $\delta \in [0,1]$.*

Proof Just by defining $\bar{\psi}(x) = \int_0^T \psi(x,t)dt$ and the Poincaré-Wirtinger inequality, see [21] for details. □

Corollary 2.2 *Define $\overline{T}$ as*:

$$\overline{T} = \begin{cases} \frac{2\pi}{\sqrt{-\gamma_{D_1}}} & \text{if } \gamma_{D_1} < 0, \\ +\infty & \text{if } \gamma_{D_1} > 0. \end{cases} \tag{2.5}$$

Then, for $T \in (0,\overline{T})$, we have that $Q^T_D(\psi) > 0$ for any $\psi \in H^1_{0,r}(C^T_1)$ such that $\int_{C^T_1} \psi z_j = 0$, $j = 1,2,\cdots,l$. As a consequence, L^T_D is nondegenerate.

Defining the cylinder-type domain

$$C^T_{1+v} = \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R}/\mathbb{Z} \ :\ 0 \le |x| < 1 + v\left(\frac{t}{T}\right)\right\},$$

we start with the following result, that allows us to obtain a solution for the Dirichlet problem in the domain C^T_{1+v} and its smooth dependence on T and v.

Proposition 2.3 *Assume that $T < \overline{T}$, where $\overline{T}$ is given by* (2.5). *Then, for all $v \in C^{2,\alpha}_e(\mathbb{R}/\mathbb{Z})$ whose norm is sufficiently small, the problem*

$$\begin{cases} \Delta\phi + f(\phi) = 0 & \text{in } C^T_{1+v} \\ \phi = 0 & \text{on } \partial C^T_{1+v} \end{cases} \tag{2.6}$$

has a unique positive solution $\phi = \phi_{1+v,T} \in C^{2,\alpha}(C^T_{1+v})$. Moreover, ϕ depends smoothly on the function v, and $\phi = \phi_1$ when $v \equiv 0$.

Proof Following the nondegeneracy of the Dirichlet problem, please refer to [21] for details. □

For any $T < \overline{T}$, there exists a neighborhood $\mathcal{U}$ of 0 in $C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$ where the following Dirichlet-to-Neumann operator is well defined and C^1:

$$G : \mathcal{U} \times (0,\overline{T}) \to C^{1,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z}),$$

$$G(v,T)(t) = \left.\frac{\partial\phi_{1+v,T}}{\partial\nu}\right|_{\partial C^T_{1+v}}(T\,t) - \frac{1}{\mathrm{Vol}(\partial C^T_{1+v})}\int_{\partial C^T_{1+v}} \frac{\partial\phi_{1+v,T}}{\partial\nu}, \tag{2.7}$$

where $\phi(v,T)$ is the solution of (2.6) verified by Proposition 2.3.

We will next compute the Fréchet derivative of the operator G. For so, we will need the following lemmas.

Lemma 2.4 *Assume that $T < \overline{T}$, where $\overline{T}$ is given by* (2.5)*. Then for all $v \in C^{2,\alpha}_e(\mathbb{R}/\mathbb{Z})$, there exists a unique solution $\psi_{v,T}$ to the problem*

$$\begin{cases} \Delta\psi_{v,T} + f'(\phi_1)\psi_{v,T} = 0 & \text{in } C_1^T \\ \psi_{v,T} = v(\cdot/T) & \text{on } \partial C_1^T \end{cases}. \tag{2.8}$$

Proof Let $\psi_0(x,t) \in C^{2,\alpha}(C_1^T)$ such that $\psi_0|_{\partial C_1^T} = v(\cdot/T)$. If we set $\omega = \psi_{v,T} - \psi_0$, the problem (2.8) is equivalent to the problem

$$\begin{cases} \Delta\omega + f'(\phi_1)\omega = -\Big(\Delta\psi_0 + f'(\phi_1)\psi_0\Big) & \text{in } C_1^T \\ \omega = 0 & \text{on } \partial C_1^T \end{cases}.$$

Observe that the right hand side of the above equation is in $C_r^\alpha(C_1^T)$. Recall the Corollary 2.2, L_D^T is nondegenerate. Hence it is a bijection and the result follows. □

Lemma 2.5 *Let $v \in C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$ and $\psi_v = \psi_{v,T} \in C_r^{2,\alpha}(C_1^T)$ be the solution of* (2.8)*. Then*

$$\int_{C_1^T} \psi_v z_j = 0, \qquad \int_{\partial C_1^T} \frac{\partial\psi_v}{\partial\nu} = 0\,, \qquad j = 1, 2, \cdots, l\,.$$

Proof We can get these results by the straight computation, refer to [21]. □

For $T < \overline{T}$ we can define the linear and continuous operator $H_T : C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z}) \to C^{1,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$ by

$$H_T(v)(t) = \partial_\nu\psi_v(Tt) + c\,v,$$

and $\psi_v = \psi_{v,T}$ as in Lemma 2.4. We present some properties of H_T.

Lemma 2.6 *For any $T < \overline{T}$, the operator*

$$H_T : C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z}) \to C^{1,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$$

is a linear essentially self-adjoint operator and has closed range. Moreover, it is also a Fredholm operator of index zero.

Proof By the straight computation, we can get that the operator H_T is a linear essentially self-adjoint operator. And the rest results follow from [3, 14]. More details refer to [21]. □

We show now that the linearization of the operator G with respect to v at $v = 0$ is given by H_T, up to a constant.

Proposition 2.7 *The map G is C^1, and $D_v(G)|_{v=0} = -\phi_1'(1)\,H_T$.*

Proof By the Proposition 2.3 (the function $\phi(v,T)$ depends smoothly on v), the operator G is C^1. The linear operator obtained by the directional derivative of linearizing G with respect to v, computed at (v,T), is given by

$$G'(w) = \lim_{s\to 0} \frac{G(sw,T) - G(0,T)}{s} = \lim_{s\to 0} \frac{G(sw,T)}{s}.$$

Let $v = sw$, for $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we consider the parameterization of C_{1+v}^T given by

$$Y(y,t) := \left(\left(1 + v\left(\frac{t}{T}\right)\right)y, t\right).$$

Let g be the induced metric such that $\hat{\phi} = Y^*\phi$ (smoothly depending on the real parameter s) solves the problem

$$\begin{cases} \Delta_g\hat{\phi} + f(\hat{\phi}) = 0 & \text{in } C_1^T \\ \hat{\phi} = 0 & \text{on } \partial C_1^T \end{cases}.$$

We remark that $\hat{\phi}_1 = Y^*\phi_1$ is the solution of

$$\Delta_g\hat{\phi}_1 + f(\hat{\phi}_1) = 0$$

in C_1^T, and

$$\hat{\phi}_1(y,t) = \phi_1\big((1+sw)y, t\big)$$

on ∂C_1^T. Let $\hat{\phi} = \hat{\phi}_1 + \hat{\psi}$, we can get that

$$\begin{cases} \Delta_g \hat{\psi} + f(\hat{\phi}_1 + \hat{\psi}) - f(\hat{\phi}_1) = 0 & \text{in } C_1^T \\ \hat{\psi} = -\hat{\phi}_1 & \text{on } \partial C_1^T \end{cases}. \tag{2.9}$$

Obviously, $\hat{\psi}$ is a smooth function of s. When $s = 0$, we have $\phi = \phi_1$. Then, $\hat{\psi} = 0$ and $\hat{\phi}_1 = \phi_1$ as $s = 0$. We set

$$\dot{\psi} = \partial_s \hat{\psi}|_{s=0}.$$

Differentiating (2.9) with respect of s and evaluating the result at $s = 0$, we have

$$\begin{cases} \Delta \dot{\psi} + f'(\phi_1)\dot{\psi} = 0 & \text{in } C_1^T \\ \dot{\psi} = -\phi_1'(1) w & \text{on } \partial C_1^T \end{cases}$$

where $r := |y|$. Then $\dot{\psi} = -\phi_1'(1)\, \psi_w$ where ψ_w is as given by Lemma 2.4 (with $\tilde{v} = w$). Then, we can write

$$\hat{\phi}(x,t) = \hat{\phi}_1(x,t) + s\dot{\psi}(x,t) + O(s^2).$$

In particular, in a neighborhood of ∂C_1^T we have

$$\begin{aligned} \hat{\phi}(y,t) &= \phi_1\big((1+sw)y, t\big) + s\dot{\psi}(y,t) + O(s^2) \\ &= \phi_1(y,t) + s\big(wr\partial_r \phi_1 + \dot{\psi}(y,t)\big) + O(s^2). \end{aligned}$$

In order to complete the proof of the result, it is enough to calculate the normal derivation of the function $\hat{\phi}$ when the normal is calculated with respect to the metric g. By using cylindrical coordinates $(y,t) = (rz,t)$ where $r > 0$ and $z \in \mathbb{S}^{n-1}$, then the metric g can be expanded in C_1^T as

$$g = (1+sw)^2 dr^2 + 2srw'(1+sw)drdt + \big(1 + s^2r^2(w')^2\big)dt^2 + r^2(1+sw)^2 \mathring{h}$$

where $\mathring{h}$ is the metric on $\mathbb{S}^{n-1}$ induced by the Euclidean metric. It follows from this expression that the unit normal vector fields to ∂C_1^T for the metric g is given by

$$\hat{\nu} = \big((1+sw)^{-1} + O(s^2)\big)\partial_r + O(s)\partial_t.$$

By this, we conclude that

$$g(\nabla\hat{\phi}, \hat{\nu}) = \partial_r \phi_1 + s\big(w\partial_r^2 \phi_1 + \partial_r \dot{\psi}\big) + O(s^2)$$

on ∂C_1^T. From the fact that $\partial_r \phi_1$ is constant and the fact that the term $w\partial_r^2\phi_1 + \partial_r \dot{\psi}$ has mean 0 on ∂C_1^T we obtain

$$G'(w) = \partial_r \dot{\psi} + \phi_1''(1)\, w = -\phi_1'(1)\, \partial_r \psi_w + \phi_1''(1)\, w = -\phi_1'(1)\, H_T(w).$$

This concludes the proof of the result. □

We now define the first eigenvalue of the operator H_T as

$$\sigma(T) = \inf\left\{ \int_0^1 H_T(v)v : v \in C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z}),\ \int_0^1 v^2 = 1 \right\}.$$

By the Divergence formula, we have

$$Q^T(\psi_v) = T\omega_n \int_0^1 H_T(v)v.$$

Next lemma characterizes the eigenvalue $\sigma(T)$ in terms of the quadratic form Q^T.

Lemma 2.8 *For any $T < \overline{T}$, we have*

$$\sigma(T) = \min\left\{ \frac{1}{T} Q^T(\psi) : \psi \in E,\ \int_{\partial C_1^T} \psi^2 = 1 \right\},$$

where

$$E = \left\{ \psi \in H_r^1(C_1^T) : \int_{\partial C_1^T} \psi = 0, \int_{C_1^T} \psi z_j = 0, j = 1, \dots l \right\}. \tag{2.10}$$

Moreover, the infimum is attained.

Proof Define $\mu_1 := \inf\left\{Q_D^T(\psi) : \psi \in E, \int_{\partial C_1^T} \psi^2 = 1\right\} \in [-\infty, +\infty)$. We show that μ_1 is achieved by contradiction. Then we can get $\int_0^1 v^2 = \frac{1}{T\omega_n}$, $J_T(v) = \frac{1}{T\omega_n} Q^T(\psi) = \frac{1}{T\omega_n}\mu_1$, refer to [21]. □

We are now in position to prove the following useful result:

Proposition 2.9 *There exists a real positive number $T_* = \frac{2\pi}{\sqrt{-\gamma_1}} < \overline{T}$, then*

(*i*) *if $T < T_*$, then $\sigma(T) > 0$;*

(*ii*) *if $T = T_*$, then $\sigma(T) = 0$;*

(*iii*) *if $T > T_*$, then $\sigma(T) < 0$.*

Moreover, $\mathrm{Ker}(H_{T_*}) = \mathbb{R}\cos(2\pi t)$. *In particular,* $\dim \mathrm{Ker}(H_{T_*}) = 1$.

Proof It follows from Lemma 2.8 and Proposition 2.1, taking into account that $C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$ contains only even functions. □

Now, we are ready to prove that the operator G satisfies the hypotheses of the Crandall-Rabinowitz bifurcation theorem (see [7, 13, 24]). And then, Theorem 1.1 follows immediately from the following proposition and the Crandall-Rabinowitz theorem.

Proposition 2.10 *There exists a real number T_* such that the linearized operator $D_v G(0, T_*)$ has 1-dimensional kernel and can be spanned by the function $v_0 = \cos(2\pi t)$,*

$$\mathrm{Ker}\, D_v G(0, T_*) = \mathbb{R} v_0.$$

The cokernel of $D_v G(0, T_*)$ is also 1-dimensional, and

$$D_T D_v G(0, T_*)(v_0) \notin \mathrm{Im}\, D_v G(0, T_*).$$

Proof Recall from the Proposition 2.7, we know that $D_v G(0, T_*) = -\phi_1'(1)\phi_1 H_{T_*}$. Then we have

$$\mathrm{Im}\, D_v G(0, T_*) = \mathrm{Im}\, H_{T_*}.$$

By the Proposition 2.9, we have that the kernel of the linearized operator $D_v G(0, T_*)$ has dimension 1 and can be spanned by the function $v_0 = \cos(2\pi t)$,

$$\mathrm{Ker}\, D_v G(0, T_*) = \mathbb{R}\, v_0.$$

Then, codim Im $(H_{T_*}) = 1$ follows from the fact that H_T is a Fredholm operator of index zero by Lemma 2.6.

Here, we are ready to prove $D_T D_v G(0, T_*)(v_0) \notin \mathrm{Im}\, D_v G(0, T_*)$. Taking $\xi \in \mathrm{Im}\, D_v G(0, T_*) = \mathrm{Im}\,(H_{T_*})$, $\xi = H_{T_*}(v)$, then we have

$$\int_0^1 \xi v_0 = \int_0^1 H_{T_*}(v) v_0 = \int_0^1 H_{T_*}(v_0) v = 0,$$

because of the fact $H_{T_*}(v_0) = 0$. We have

$$\mathrm{Im}\,(H_{T_*}) = \left\{\xi : \int_0^1 \xi v_0 = 0\right\}.$$

Notice that $D_T D_v G(0, T_*)(v_0) = -\phi_1'(1) D_T|_{T=T_*} H_T(v_0)$, then, in order to prove $D_T D_v G(0, T_*)(v_0) \notin \mathrm{Im}\, D_v G(0, T_*)$, we just need to prove that

$$\int_0^1 \left(D_T|_{T=T_*} H_T(v_0)\right) v_0 \neq 0.$$

Actually,

$$\begin{aligned}
\int_0^1 \left(D_T|_{T=T_*} H_T(v_0) v_0\right) &= \frac{d}{dT}\Big|_{T=T_*} \int_0^1 H_T(v_0) v_0 = \frac{1}{\omega_n} \frac{d}{dT}\Big|_{T=T_*} \left(\frac{1}{T} Q^T(\psi_{v_0}, \psi_{v_0})\right) \\
&= \frac{1}{\omega_n} \frac{d}{dT}\Big|_{T=T_*} \left(\frac{1}{2} Q(\psi_1, \psi_1) + \frac{2\pi^2}{T^2} \int_B \psi_1^2\right) = -\frac{4\pi^2}{\omega_n T_*^3} \int_B \psi_1^2 \neq 0,
\end{aligned}$$

where the third equality is given by the straight computation of $Q^T(\psi, \psi)$ with the function $\psi_{v_0}(x, t) = \psi_1(x)\cos(\frac{2\pi t}{T})$. □

Acknowledgements

The author was supported by Junta de Andalucía Grant FQM116.

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