Social Pooling of Beliefs and Values with Desirability

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Abstract

The problem of aggregating beliefs and values of rational subjects is treated with the formalism of sets of desirable gambles. This leads on the one hand to a new perspective of traditional results of social choice (in particular Arrow's theorem as well as sufficient conditions for the existence of an oligarchy and democracy) and on the other hand to use the same framework to create connections with opinion pooling. In particular, we show that weak Pareto can be derived as a coherence requirement and discuss the aggregation of state independent beliefs.

Introduction

This work is concerned with the question of aggregating beliefs (probabilities) and values (utilities) of a given number of rational subjects. The problem is of a foundational and philosophical nature; at the same time it has concrete statistical implications given that in applications we often want to aggregate information coming from different sources, or even predictions of different models.

Not surprisingly, the problem has a long history in the literature. A prominent example is *social choice theory* (Feldman and Serrano 2006), which aims at defining social functions that best represent the preferences of a group of rational voters. In this context, the celebrated *Arrow theorem* (1950) establishes limits to what is rationally possible to do while avoiding dictatorial solutions; these limits are severe in particular with complete preferences. Social choice theory is concerned with preferences over simple options (such as candidates to an election). As such it is not directly concerned with questions of probability. The related research field of *probabilistic opinion pooling* is instead concerned with finding a model that best 'summarises' a given number of probabilistic beliefs (Lindley, Tversky, and Brown 1979). Stewart and Quintana (2018) argue that imprecise probability has much potential in opinion pooling, in that precise probabilistic approaches incur problems that—we add—remind the Arrowian limitations. Similar considerations were made long ago by Walley (1982).

In this work we address the aggregation problem with the formalism of desirability. This has a few main advantages: the formalism is equivalent to that of preferences over horse lotteries and for this reason we can simultaneously deal with considerations of beliefs and values (Zaffalon and Miranda 2017); the framework is very general also because we can deal with any domain and possibility space (Zaffalon and Miranda 2018); moreover, it allows us to work in opinion pooling using preferences rather than probability, and this makes it much easier to carry over to pooling some of the ideas developed in social choice.

On this basis, after giving some preliminary notions from sets of desirable gambles and social choice theory, we study how some of the traditional results (Seidenfeld, Kadane, and Schervish 1989; Weymark 1984) in social choice transform in our setting: dictatorship, oligarchy and democracy. Next, we introduce the concept of coherence for social rules and deduce its intrinsic nature of linear pooling, that is, the idea of aggregating preferences (i.e., beliefs and values) via convex mixtures. This seems to indicate that the aggregation problem can be solved in a principled way, in particular when we use imprecise probability. Finally, we consider the setting of state-independent utility in the precise case and identify conditions that lead to a dictatorship in such a context. Due to space limitations, proofs have been omitted.

Preliminaries

We start by introducing the necessary notation and basic definitions. For additional comments, we refer to Walley's seminal work (1991).

Coherent sets of desirable gambles

The main modelling tool that we shall use in this paper is based on the concept of a gamble:

Definition 1 (Gamble) *Given a possibility space Ω, a gamble $f : \Omega \to \mathbb{R}$ is a bounded real-valued function on Ω.*

A gamble is interpreted as an uncertain reward in a linear utility scale. We might desire a gamble or not, depending on the information we have about the experiment whose possible outcomes are the elements of Ω. We denote the set of all gambles on Ω by $\mathscr{L}(\Omega)$, or more simply by $\mathscr{L}$ when there is no possible ambiguity. We also let $\mathscr{L}^+(\Omega) := \{f \in \mathscr{L}(\Omega) : f \geq 0, f \neq 0\}$, or simply $\mathscr{L}^+$, denote the subset of positive gambles. These are gambles that we always desire, since they may increase our wealth with no risk of decreasing it.

Definition 2 (Conic hull) *Given a set $\mathscr{K} \subseteq \mathscr{L}(\Omega)$, we let*

$$\text{posi}(\mathscr{K}) := \left\{ \sum_{j=1}^{r} \lambda_j f_j : f_j \in \mathscr{K}, \lambda_j > 0, r \geq 1 \right\}$$

denote the conic hull *of the original set.*

The conic hull operator is at the basis of the procedure of natural extension:

Definition 3 (Natural extension for gambles) *Given a set $\mathscr{K} \subseteq \mathscr{L}(\Omega)$, we call $\mathscr{D} := \text{posi}(\mathscr{K} \cup \mathscr{L}^+)$ its* natural extension.

The natural extension is the set of all gambles that we should regard desirable once we state that $\mathscr{K}$ is a set of gambles we desire. This is a consequence of the linearity of our utility scale and of the fact that $\mathscr{L}^+$ is always desirable.

Definition 4 (Coherent set of desirable gambles) *We say that a subset $\mathscr{D}$ of $\mathscr{L}(\Omega)$ is a coherent set of desirable gambles if and only if $\mathscr{D}$ satisfies the following properties:*

D1. $\mathscr{L}^+ \subseteq \mathscr{D}$ [Accepting Partial Gains];

D2. $0 \notin \mathscr{D}$ [Avoiding Null Gain];

D3. $f, g \in \mathscr{D} \Rightarrow f + g \in \mathscr{D}$ [Additivity];

D4. $f \in \mathscr{D}, \lambda > 0 \Rightarrow \lambda f \in \mathscr{D}$ [Positive Homogeneity].

Note that the natural extension is coherent if and only if it avoids null gain.

Definition 5 (Measurable gambles) *Given a partition $\mathscr{B}$ of Ω, we say that a gamble f on Ω is $\mathscr{B}$-*measurable *if and only if it is actually a function on $\mathscr{B}$:*

$$(\forall B \in \mathscr{B})(\forall \omega, \omega' \in B)\ f(\omega) = f(\omega').$$

We shall denote by $\mathscr{L}_{\mathscr{B}}(\Omega)$ the subset of $\mathscr{L}(\Omega)$ given by the $\mathscr{B}$-measurable gambles. Note that there is a one-to-one correspondence between $\mathscr{L}_{\mathscr{B}}(\Omega)$ and $\mathscr{L}(\mathscr{B})$.

Definition 6 (Conditional gambles) *Given a non-empty set $B \subseteq \Omega$, we say that a gamble f on Ω is* conditional on B *if and only if it is zero outside B: $f = Bf$.*

We shall denote by $\mathscr{L}(\Omega)|B$ the subset of $\mathscr{L}(\Omega)$ made of gambles that are conditional on $B \subseteq \Omega$. Note that there is a one-to-one correspondence between $\mathscr{L}(\Omega)|B$ and $\mathscr{L}(B)$.

Definition 7 (Marginal set of gambles) *Let $\mathscr{D} \subseteq \mathscr{L}(\Omega)$ be a coherent set of desirable gambles and consider a partition $\mathscr{B}$ of Ω. The $\mathscr{B}$-marginal of $\mathscr{D}$ is the set $\mathscr{D}_{\mathscr{B}} := \mathscr{D} \cap \mathscr{L}_{\mathscr{B}}(\Omega)$.*

Definition 8 (Conditional set of gambles) *Let $\mathscr{D} \subseteq \mathscr{L}(\Omega)$ be a coherent set of desirable gambles and consider a non-empty set $B \subseteq \Omega$. The B-conditional of $\mathscr{D}$ is the set $\mathscr{D}|B := \mathscr{D} \cap \mathscr{L}(\Omega)|B$.*

A coherent set of desirable gambles encompasses a probabilistic model for Ω made of lower and upper expectations:

Definition 9 (Coherent lower and upper prevision) *Let $\mathscr{D}$ be a coherent set of desirable gambles in $\mathscr{L}$. For all $f \in \mathscr{L}$, let:*

$$\underline{P}(f) := \sup\{\mu \in \mathbb{R} : f - \mu \in \mathscr{D}\}. \tag{1}$$

It is called the lower prevision *of f. The conjugate value given by $\overline{P}(f) := -\underline{P}(-f)$ is called the* upper prevision *of f. The functionals $\underline{P}, \overline{P} : \mathscr{L} \to \mathbb{R}$ are respectively called a* coherent lower prevision *and a* coherent upper prevision. *If $\underline{P} = \overline{P}$ for some $f \in \mathscr{L}$, then we call the common value the* prevision *of f and we denote it by $P(f)$. If this happens for all $f \in \mathscr{L}$ then we call the functional P a* linear prevision.

There is a one-to-one correspondence between coherent lower previsions and a special type of coherent desirable sets:

Definition 10 (Strict desirability) *A coherent set of gambles $\mathscr{D}$ is said to be* strictly desirable *if and only if it satisfies $f \in \mathscr{D} \setminus \mathscr{L}^+ \Rightarrow (\exists \delta > 0)\ f - \delta \in \mathscr{D}$.*

In fact, from $\underline{P}$ we can induce set:

$$\mathscr{D}_{\triangleright} := \mathscr{L}^+ \cup \{f \in \mathscr{L} : \underline{P}(f) > 0\}, \tag{2}$$

which is coherent and strictly desirable and moreover induces $\underline{P}$ through Eq. (1).

Finally, we consider the most informative cases of coherent sets of gambles:

Definition 11 (Maximal coherent set of gambles) *A coherent set of desirable gambles $\mathscr{D}$ is called* maximal *if and only if*

$$(\forall f \in \mathscr{L} \setminus \{0\})\ f \notin \mathscr{D} \Rightarrow -f \in \mathscr{D}.$$

A maximal set of desirable gambles has no coherent superset, and, conversely, any coherent set of desirable gambles is the intersection of its coherent maximal supersets. Moreover, it induces a linear prevision by means of Eq. (1).

More generally speaking, a linear prevision P may be induced by several different coherent sets of desirable gambles. It may be interesting then to consider the least informative one:

Definition 12 (Maximal strict desirability) *A set of strictly desirable gambles $\mathscr{D}_{\triangleright}$ is called* maximally strictly desirable *if and only if there is a linear prevision P that determines $\mathscr{D}_{\triangleright}$ by means of Eq.* (2).

Social rules

Having introduced sets of gambles allows us to reformulate the main concepts in social choice theory with them. This is what we set out to do in the present section.

We start with a set of n 'voters' $\mathscr{H}$, each of them with beliefs about a possibility space $\mathscr{S}$. In our setting it is straightforward to deal also with utility considerations, by assuming that voters have in addition values on a set of prizes $\mathscr{X}$ and by taking the new 'possibility' space to be $\mathscr{Z} := \mathscr{S} \times \mathscr{X}$ (Zaffalon and Miranda 2017; 2018). In fact this work lives at such a level of generality; however the reader need not be concerned with it, given that technically $\mathscr{Z}$ is treated just as a possibility space.

Let us next define voters' profiles.

Definition 13 (Profiles) *Let $\mathbb{D}$ be the set of coherent sets of desirable gambles in $\mathscr{L}$. $\mathbb{D}^n$, the n-times Cartesian product of $\mathbb{D}$, is the set of logically possible profiles of individual sets of desirable gambles.*

A profile of coherent sets of desirable gambles is thus a vector $[\mathscr{D}_i]_{i\in\mathscr{H}} \in \mathbb{D}^n$. To keep the notation simple, we shall often denote it simply by $[\mathscr{D}_i]$.

With this formalism we can re-define also the concept of social welfare function, which we rename in this context as *social rule*, as well as the properties that it may satisfy (for the original concepts and definitions see Weymark 1984).

Definition 14 (Social rule) *A* social rule Γ *is a function from the admissible set of profiles $\mathscr{A} \subseteq \mathbb{D}^n$ to a coherent set of desirable gambles (social coherent set of desirable gambles).*

Social coherent sets of desirable gambles remain unsubscripted while subscripts distinguish individuals' coherent sets of desirable gambles. Note that our social rules are in particular applicable when the beliefs are modelled by probability measures, coherent lower previsions, or other uncertainty models, such as belief functions; it suffices to make the correspondence with coherent sets of desirable gambles in Eq. (1).

Example 1 *As a simple running example in order to clarify some of the notions that follow, consider a set of $n = 2$ voters and a possibility space $\mathscr{X}$ such that $|\mathscr{X}| \geq 3$. The following are three instances of social rules:*

- $\Gamma_1(\mathscr{D}_1, \mathscr{D}_2) = \mathscr{D}_1$.
- $\Gamma_2(\mathscr{D}_1, \mathscr{D}_2) = \mathscr{D}_1 \cap \mathscr{D}_2$.
- $\Gamma_3(\mathscr{D}_1, \mathscr{D}_2) = \mathscr{M}_2$, *where $\mathscr{M}_2$ is a maximally coherent set of gambles that includes $\mathscr{D}_2$.*

Next we consider a number of additional properties that a social rule may satisfy. From our formulation, such a rule turns the, possibly imprecise, assessments of a number of voters (such as those that lead to a coherent lower prevision by means of Eq. (1)) into an aggregated profile, which may be imprecise too. As particular cases of interest, we may consider the case when the aggregated set represents precise assessments:

Definition 15 (Completeness) *A social rule Γ satisfies* completeness *if and only if $\Gamma([\mathscr{D}_i])$ is a maximal set of gambles for every profile $[\mathscr{D}_i]$, and it satisfies* strict completeness *if and only if $\Gamma([\mathscr{D}_i])$ is a maximally strictly desirable set of gambles for every profile $[\mathscr{D}_i]$.*

If we consider the social rules in Example 1, we see that only Γ_3 is complete, because both Γ_1, Γ_2 will not return a maximal set of gambles if for instance $\mathscr{D}_1 = \mathscr{L}^+$.

Another important assumption we shall consider is that Γ can be applied to any profile of voters:

Definition 16 (Unlimited domain) *A social rule Γ satisfies* unlimited domain *if and only if its set $\mathscr{A}$ of admissible set of profiles is $\mathscr{A} = \mathbb{D}^n$. It is said to satisfy* unlimited maximal domain *when $\mathscr{A} = \hat{\mathbb{D}}^n$, the set of profiles constituted by maximal coherent sets of desirable gambles, and it satisfies* unlimited maximal strict domain *when $\mathscr{A} = \hat{\mathbb{D}}^n_{\triangleright}$, the set of profiles constituted by maximally strictly coherent sets of desirable gambles.*

Next we consider a property called independence of irrelevant alternatives, which shall be instrumental in characterising a number of social rules.

Definition 17 (Independence of irrelevant alternatives) *A social rule is* independent of irrelevant alternatives *if and only if*

$$(\forall f \in \mathscr{L})(\forall [\mathscr{D}_i], [\mathscr{D}'_i] \in \mathscr{A})((\forall i \in \mathscr{H})(f \in \mathscr{D}_i \Leftrightarrow f \in \mathscr{D}'_i)) \quad f \in \Gamma([\mathscr{D}_i]) \Leftrightarrow f \in \Gamma([\mathscr{D}'_i]).$$

The interpretation of this property is that whether a gamble f belongs to the aggregated decision set depends only on which voters are endorsing f (but note that this may depend on the gamble f we consider, in the sense that two gambles f, g may belong to the same elements in the profile and one may belong to the aggregated set while the other one is not).

If we consider the social rules in the running example, we observe that both Γ_1 and Γ_2 satisfy independence of irrelevant alternatives, because their definition depends only on which sets in the profile include the gamble; but Γ_3 does not when it also satisfies the assumption of unlimited domain. Indeed we may have a profile $[\mathscr{D}_1, \mathscr{D}_2]$ and take $f \in \Gamma_3([\mathscr{D}_1, \mathscr{D}_2]) \setminus \mathscr{D}_2$, then given $\mathscr{D}'_1 = \Gamma_3(\mathscr{D}_1, \mathscr{D}_2)$ and $\mathscr{D}'_2 = \mathscr{D}_2$, it holds that $f \in \mathscr{D}'_1 \setminus \mathscr{D}'_2$ and $f \in \Gamma_3(\mathscr{D}'_1, \mathscr{D}'_2)$, while given $\mathscr{D}''_1 := \text{posi}(\{f\} \cup \mathscr{L}^+)$ and $\mathscr{D}''_2 := \text{posi}(\{-f\} \cup \mathscr{L}^+)$, then $f \in \mathscr{D}''_1 \setminus \mathscr{D}''_2$ and $f \notin \Gamma_3(\mathscr{D}''_1, \mathscr{D}''_2)$.

The next condition states that if a gamble is deemed desirable by all the voters, then it should belong to the aggregated set:

Definition 18 (Weak Pareto) *A social rule Γ satisfies* weak Pareto *if and only if*

$$(\forall [\mathscr{D}_i] \in \mathscr{A}) \ \cap_{i\in\mathscr{H}} \mathscr{D}_i \subseteq \Gamma([\mathscr{D}_i]).$$

Note that since the intersection of a family of coherent sets of desirable gambles is again a coherent set of desirable gambles, this definition is consistent. We can also see that the three social rules in our running example satisfy weak Pareto.

We also have the following:

Theorem 1 *Let Γ be a social rule that satisfies weak Pareto, and let $[\mathscr{D}_i]$ be a profile. Then $\Gamma([\mathscr{D}_i])$ is an intersection of a family of maximal sets that include $\cap_i \mathscr{D}_i$.*

In this paper, we shall analyse situations where the overall decision is determined or influenced by the behaviour of a number of particular voters. We shall consider a number of cases.

Definition 19 (Almost decisiveness) *A set of individuals $\mathscr{G} \subseteq \mathscr{H}$ is* almost decisive for a gamble f *if and only if*

$$(\forall [\mathscr{D}_i] \in \mathscr{A}) \ f \in \cap_{i\in\mathscr{G}} \mathscr{D}_i \text{ and } f \notin \cup_{i\notin\mathscr{G}} \mathscr{D}_i \Rightarrow f \in \Gamma([\mathscr{D}_i]).$$

It is called almost decisive *when it is almost decisive for every gamble f.*

With respect to the social rules in the running example, for Γ_1 the first voter is almost decisive, while the second is not; for Γ_2, none of the two voters is almost decisive; and for Γ_3, the second voter is almost decisive, while the first one is not.

Note that when $\mathscr{G} = \mathscr{H}$, almost decisiveness reduces to Γ satisfying weak Pareto and then, if $\mathscr{G}$ is a proper subset of $\mathscr{H}$, we may assume without loss of generality that $f \notin \mathscr{L}^+ \cup \mathscr{L}^- \cup \{0\}$, considering that any $f \in \mathscr{L}^+$ must belong to $\cap_{i \notin \mathscr{G}} \mathscr{D}_i$ and that any $f \in \mathscr{L}^- \cup \{0\}$ cannot belong to $\mathscr{D}_i$ for any $i \in \mathscr{G}$.

A slightly stronger notion is the following:

Definition 20 (Decisiveness) *Given a social rule Γ, a set of individuals $\mathscr{G} \subseteq \mathscr{H}$ is* decisive *for a gamble f if and only if*

$$(\forall [\mathscr{D}_i] \in \mathscr{A})\ f \in \cap_{i \in \mathscr{G}} \mathscr{D}_i \Rightarrow f \in \Gamma([\mathscr{D}_i]),$$

and it is decisive *when it is decisive for every gamble f.*

In our running example, the first voter is decisive for Γ_1 and the second is decisive for Γ_3. Since in this example we have only two voters, they correspond to what we shall call next a dictator.

Definition 21 (Dictatorship) *An individual $i \in \mathscr{H}$ is a* dictator *if and only if $\{i\}$ is decisive.*

This means that $\mathscr{D}_i \subseteq \Gamma([\mathscr{D}_j])$ for any profile $[\mathscr{D}_j]$. Note that the two sets need not coincide: dictatorship means that those gambles that are considered desirable by voter i must also be considered desirable in the overall assessment, but the latter may include others, meaning that $\Gamma([\mathscr{D}_j])$ may be a strict superset. When they coincide, we may say that $\{i\}$ is a *strong dictator* (Pini et al. 2009). One particular case where the two sets will necessarily coincide is when $\mathscr{D}_i$ is a maximal set of gambles, because these cannot be strictly included in any other coherent set of gambles.

The above reasoning also means that, if the assumption of unlimited domain is satisfied, there can be at most one dictator: for given two different dictators $j_1 \neq j_2$, it should be $\mathscr{D}_{j_1} \cup \mathscr{D}_{j_2} \subseteq \Gamma([\mathscr{D}_i])$, and for any two different maximal sets of gambles $\mathscr{D}_{j_1}, \mathscr{D}_{j_2}$, their union has no coherent superset.

Dictatorship

We shall now establish a version of Arrow's theorem for the case where the voters' preferences are assessed by means of coherent sets of desirable gambles (for a version of Arrow's theorem using preference profiles see Feldman and Serrano 2006). In order to do this, we establish first a couple of lemmas that lie in the core of our version of Arrow's theorem under desirability. The first one shows, somewhat surprisingly, that under some assumptions the conditions of being almost decisive and decisive are equivalent:

Lemma 1 *Assume that $|\mathscr{X}| \geq 3$. If a social rule satisfies the following properties:*

- *unlimited (maximal) domain,*
- *independence of irrelevant alternatives,*
- *weak Pareto,*

and it admits a group $\mathscr{G}$ that is almost decisive for a gamble f, then it is decisive.

Next we show that, if in addition to the previous assumptions we have completeness, then any decisive group of voters can be contracted:

Lemma 2 *Assume that $|\mathscr{X}| \geq 3$. If a social rule satisfies the following properties:*

- *completeness,*
- *unlimited maximal domain,*
- *independence of irrelevant alternatives,*
- *weak Pareto,*

then if a group $\mathscr{G}$ containing at least two individuals is decisive, then it contains a proper subset of individuals that are also decisive.

Note that in this second lemma we are not requiring unlimited domain because, perhaps surprisingly, it can be checked to be incompatible with the assumptions of weak Pareto and independence of the irrelevant alternatives under completeness.

From this result it is just a small step to immediately derive our version for Arrow's theorem in terms of sets of desirable gambles:

Theorem 2 *Assume that $|\mathscr{X}| \geq 3$. Any social rule that satisfies:*

- *completeness,*
- *unlimited maximal domain,*
- *independence of irrelevant alternatives,*
- *weak Pareto,*

makes one (unique) person a dictator.

It is not difficult to show that, under the assumption of unlimited maximal domain, independence of the irrelevant alternatives is also necessary for the existence of a dictator.

Regarding the social rules in the running example, it is possible to notice that restricting the set of admissible profiles of Γ_3 to $\hat{\mathbb{D}}^n$, makes it satisfy also independence of irrelevant alternatives. Hence it satisfies all the hypothesis of Theorem 2, giving rise to the dictatorship of the second voter.

Oligarchy

Maintaining the axioms of Arrow's theorem but modifying the requirement to work only with maximal sets, it is possible to obtain a general oligarchy theorem.

Definition 22 (Oligarchy) *A set of individuals $\mathscr{G} \subseteq \mathscr{H}$ is an oligarchy if and only if:*

- *$\mathscr{G}$ is decisive;*
- $(\forall f \in \mathscr{L}(\mathscr{X}))(\forall [\mathscr{D}_i] \in \mathscr{A})$ $((\exists i \in \mathscr{G})(f \in \mathscr{D}_i) \Rightarrow -f \notin \Gamma([\mathscr{D}_i])$.

In other words, $\mathscr{G}$ is an oligarchy if and only if for any profile $[\mathscr{D}_i]$ it holds that

$$\cap_{i \in \mathscr{G}} \mathscr{D}_i \subseteq \Gamma([\mathscr{D}_i]) \subseteq \cap_{i \in \mathscr{G}} (-\mathscr{D}_i)^c. \quad (3)$$

The reason why we are not imposing completeness in this section is that, if $\Gamma([\mathscr{D}_i])$ was a complete set, then for any oligarchy $\mathscr{G}$ we should have $\cup_{i \in \mathscr{G}} \mathscr{D}_i \subseteq \Gamma([\mathscr{D}_i])$, and this is only compatible with the assumption of unlimited domain in case the oligarchy consists of only one voter, that is, in case of a dictatorship. Note also that in the case of a dictatorship the second condition is a consequence of the first.

It is not difficult to show that, under mild conditions, there can be at most one oligarchy:

Lemma 3 *For any social rule satisfying unlimited (maximal) domain, there can be at most one oligarchy.*

Next we show that if we remove the hypothesis of completeness from Theorem 2 we can deduce that the collective rule is an oligarchy.

Theorem 3 *Assume that* $|\mathscr{X}| \geq 3$. *For any social rule* Γ *that satisfies:*

- *unlimited (maximal) domain,*
- *independence of irrelevant alternatives,*
- *weak Pareto,*

there exists a unique oligarchy.

Regarding the social rules in the running example, we notice that Γ_2 gives rise to an oligarchy that in this case corresponds to the whole society (what we shall call democracy in our next section).

Democracy

By adding the property of anonymity to the axioms of Theorem 3 we obtain the polar case to Arrow's dictator, the oligarchy must consist of the whole society.

Definition 23 (Anonymity) *A social rule* Γ *satisfies* anonymity *if and only if for every permutation* σ *of* $\mathscr{H}$ *and for every profile* $[\mathscr{D}_i]$, *it holds that*

$$\Gamma([\mathscr{D}_i]) = \Gamma([\mathscr{D}_{\sigma(i)}]).$$

Thus, anonymity requires the social rule to treat all individuals equally. In our running example, it is only satisfied by Γ_2, since we do not have in general that $\Gamma_1(\mathscr{D}_1, \mathscr{D}_2) = \Gamma_1(\mathscr{D}_2, \mathscr{D}_1)$ nor $\Gamma_3(\mathscr{D}_1, \mathscr{D}_2) = \Gamma_3(\mathscr{D}_2, \mathscr{D}_1)$.

From our previous results, we easily deduce the following:

Theorem 4 *Assume that* $|\mathscr{X}| \geq 3$. *For any social rule* Γ *that satisfies:*

- *unlimited (maximal) domain,*
- *independence of irrelevant alternatives,*
- *weak Pareto,*
- *anonymity,*

there exists a unique oligarchy, which is the whole society.

Note that if the whole society $\mathscr{H}$ is an oligarchy, Eq. (3) becomes

$$\cap_i \mathscr{D}_i \subseteq \Gamma([\mathscr{D}_i]) \subseteq \cap_i (-\mathscr{D}_i)^c,$$

or, in other words, that Γ satisfies weak Pareto and that $f \in \cup_i \mathscr{D}_i$, then $-f$ cannot belong to $\Gamma([\mathscr{D}_i])$. Thus, the social rule cannot contradict any of the individual choices of the voters, and must incorporate those options where all of them agree. Note also that, considering the comments in the previous section, democracy is incompatible with the assumption of completeness.

Coherent social rules

Now we investigate more in detail what is the relation between the notion of coherence of sets of desirable gambles from Walley's theory and social rules. The following lemma makes a first step: it shows that if all we do is to *literally* translate the voters' profile into a set of desirable gambles $\mathscr{E}$, then this set is coherent and its marginal is $\cap_{i \in \mathscr{H}} \mathscr{D}_i$.

Lemma 4 *Consider a set of voters* $\mathscr{H}$ *and a profile* $[\mathscr{D}_i]$. *For each* $i \in \mathscr{H}$, *let us define* $\mathscr{D}|i$ *on* $\mathscr{H} \times \mathscr{X}$ *by:*

$$\mathscr{D}|i := \{f \in \mathscr{L}(\mathscr{H} \times \mathscr{X}) : f = \mathbb{I}_i \otimes f_i, f_i \in \mathscr{D}_i\}, \quad (4)$$

where $\mathbb{I}_i \otimes f_i$ *is the gamble given by*

$$\mathbb{I}_i \otimes f_i(j,z) = \begin{cases} f_i(z) & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Then $\cap_{i \in \mathscr{H}} \mathscr{D}_i = \mathrm{Marg}_{\mathscr{X}}(\mathscr{E})$, *where* $\mathscr{E}$ *is the natural extension of* $\cup_{i \in \mathscr{H}} \mathscr{D}|i$ *and* $\mathrm{Marg}_{\mathscr{X}}(\mathscr{E})$ *is given by Definition 7.*

The previous lemma implicitly hints at the fact that weak Pareto is a consequence of coherence and the bare fact that voters have a profile. In point 1 of the next theorem we make this claim precise.

Theorem 5 *Let* Γ *be a social rule. Consider a set of voters* $\mathscr{H}$ *with profile* $[\mathscr{D}_i]$, *and let*

$$\mathscr{E} := \left\{ \sum_{i \in \mathscr{H}} \mathbb{I}_i \otimes f_i : (\forall i \in \mathscr{H}) f_i \in \mathscr{D}_i \cup \{0\} \right\} \setminus \{0\}. \quad (5)$$

0. *$\mathscr{E}$ is equal to the natural extension of $\cup_{i \in \mathscr{H}} \mathscr{D}|i$.*
1. *$\Gamma([\mathscr{D}_i]) \supseteq \mathrm{Marg}_{\mathscr{X}}(\mathscr{E})$ is equivalent to the existence of a coherent $\mathscr{E}' \supseteq \mathscr{E}$ such that $\Gamma([\mathscr{D}_i]) = \mathrm{Marg}_{\mathscr{X}}(\mathscr{E}')$.*
2. *The smallest such set is*

$$\mathscr{E}' := \{f_0 \otimes \mathscr{H} + \sum_{i \in \mathscr{H}} \mathbb{I}_i \otimes f_i : f_0 \in \Gamma([\mathscr{D}_i]) \cup \{0\}, (\forall i \in \mathscr{H}) f_i \in \mathscr{D}_i \cup \{0\}\} \setminus \{0\}.$$

3. *For all $i \in \mathscr{H}$, $\mathscr{E}'|i$ coincides with the set $\mathscr{D}|i$ given by Eq.* (4).

Taking this theorem into account, we put forward the following definition:

Definition 24 (Coherent social rule) *A* coherent social rule *is a social rule such that for every profile* $[\mathscr{D}_i] \in \mathscr{A}$ *it holds that* $\Gamma([\mathscr{D}_i]) \supseteq \mathrm{Marg}_{\mathscr{X}}(\mathscr{E})$, *where* $\mathscr{E}$ *is defined as in Eq.* (5).

Next, we use Theorem 5 to establish a preliminary result that gives conditions, when the individual preferences are all linear previsions, for the social rule to be a convex combination of them.

Theorem 6 *Let* Γ *be a coherent social rule, with unlimited maximal strict domain. If* Γ *satisfies strict completeness, then for all* $[\hat{\mathscr{D}}_i] \in \mathscr{A}$ *there is a probability mass function* π *on* $\mathscr{H}$ *such that the linear prevision induced by* $\Gamma([\hat{\mathscr{D}}_i])$ *on* $\mathscr{X}$ *can be written as* $\sum_{i \in \mathscr{H}} \pi(i) P_i$, *where, for every* $i \in \mathscr{H}$, P_i *is the linear prevision induced by* $\hat{\mathscr{D}}_i$ *on* $\mathscr{X}$.

State-independent utility and dictatorship

We shall next show that eliminating the hypothesis of independence of irrelevant alternatives and adding some other conditions, we can find again a dictatorship (for a similar version of the following theorems using preference profiles see Seidenfeld, Kadane, and Schervish 1989). Our result shall be related to the decomposition of the assessments.

Definition 25 (State independence) *A probability measure P on $\mathscr{S} \times \mathscr{X}$ is* state independent *when possibilities and prizes are stochastically independent. A complete social rule* Γ satisfies state independence *if and only if for each profile $[\mathscr{D}_i]$ the probability measure induced by $\Gamma([\mathscr{D}_i])$ satisfies state independence. It is said to have* state independent domain *when for each profile $[\mathscr{D}_i]$ in its domain, $\mathscr{D}_i$ is a maximal set of gambles that induces a state independent probability measure for every i.*

We have seen that a social rule that satisfies the hypotheses of Theorem 6 is always a linear pooling with weights that depend on the profile. In the particular case in which weights are constant with respect to the profile, we have the following:

Theorem 7 *Let Γ be a social rule. If Γ satisfies:*

- *State independent domain*
- *weak Pareto,*
- *completeness,*
- *state independence,*

and if then there is a probability mass function π on $\mathscr{H}$ such that for all $[\hat{\mathscr{D}}_i] \in \mathscr{A}$ the linear prevision induced by $\Gamma([\mathscr{D}_i])$ on $\mathscr{X}$ can be written as $\sum_{i \in \mathscr{H}} \pi(i) P_i$, then there exists $j \in \mathscr{H}$ such that $\pi(j) = 1$.

In the proof we deduce something different with respect to Seidenfeld, Kadane, and Schervish (1989): that π (independent of the profile) is a degenerate distribution, meaning that it is always the same individual that determines the collective choices. In other words, this result gives a sufficient condition for the social rule to be a dictatorship, if we focus on the linear previsions associated with the coherent sets of desirable gambles in the profile.

Note that the theorem cannot be extended to the general case in which weights depend on the profile, even if we ask only for state independence. Denote by (P_i, U_i) the state dependent preferences of voter i, and consider the social rule $\Gamma([(P_1, U_1), \ldots, (P_n, U_n)])$ given by

$$\begin{cases} (P_1, (U_1 + \cdots + U_n)/n) & \text{if } P_1 = \cdots = P_n \\ ((P_1 + \cdots + P_n)/n, U_1) & \text{if } U_1 = \cdots = U_n \\ (P_1, U_1) & \text{otherwise;} \end{cases}$$

this rule satisfies independence even if it is not a dictatorship. A detailed study of this problem was made by Goodman (1988).

Conclusions

In this paper, we have considered a problem of opinion pooling where the voters' joint states of beliefs and values (probabilities and utilities) are represented as coherent sets of desirable gambles. In addition to establishing a version of Arrow's theorem in this context, we have also shown that other forms of collective choice can be characterised in terms of rationality axioms. In particular, we have shown that the property of weak Pareto can be obtained as a consequence of coherence, and that, roughly speaking, democracy has incompleteness of the beliefs as an inherent feature. In addition, we have characterised the coherent rules and shown that these have a tight relation with linear pooling in the precise case.

As future lines of research, we would like to characterise other forms of government; in addition, we would also like to better characterise the extent to which our work and Walley's (1982) are related.

Acknowledgements

We would like to thank Seamus Bradley and Teddy Seidenfeld for stimulating discussion, and for providing some relevant references, as well as the financial support of projects PGC2018-098623-B-I00 and GRUPIN/IDI/2018/000176.

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