A note on spacelike submanifolds through light cones in Lorentzian space forms

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Abstract

We analyse the intrinsic and extrinsic geometry of spacelike submanifolds in light cones $\Lambda_c(p)$ of de Sitter and anti-de Sitter spacetimes. Every light cone $\Lambda_c(p)$ contains the lightlike geodesics starting from p and, essentially, it coincides with the horisms $E^+(p) \cup E^-(p)$. The analysis works by means of an explicit correspondence with the spacelike submanifolds through the light cone in the Lorentz-Minkowski spacetime. In particular, a characterization of totally umbilical compact surfaces through light cones in de Sitter and anti-de Sitter is shown and we obtain an estimation of the first eigenvalue of the Laplace operator on a compact spacelike surface in a light cone.

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1 Introduction

A null (or degenerate) hypersurface $\mathcal{L}$ into a spacetime is a codimension one embeded submanifold such that the pullback of the Lorentzian metric is degenerate at every point. Null hypersurfaces have not only an interesting geometry, but they also play an important role in General Relativity, where they arise as black hole event horizons and Cauchy horizons.

For the study of null hypersurfaces the theory of non-degenerate submanifolds fails. In fact, there is a non trivial intersection between the tangent and the normal bundles of null hypersurfaces. In order to avoid such difficulty, a distribution, transverse to the radical of the pullback of the metric, is usually introduced on $\mathcal{L}$ (see for instance [4] and references therein).

Although the induced metric is degenerate on $\mathcal{L}$, the family of (non-degenerate) spacelike submanifolds through $\mathcal{L}$ gives remarkable properties to the null hypersurface and conversely, under the assumption that a spacelike submanifold Σ factorizes through a fixed null hypersurface $\mathcal{L}$, the intrinsic geometry of Σ becomes limited. For example, recall the classical result by Brinkmann which states that an n-dimensional Riemannian manifold, with $n > 2$, is locally conformally flat if and only if it can be locally isometrically immersed in the light cone of the $(n+2)$-dimensional Lorentz-Minkowski spacetime $\mathbb{L}^{n+2}$ (see [3] for a modern proof). From the extrinsic point of view, the spacelike submanifolds through null hypersurfaces have also been considered. For example, codimension two spacelike submanifolds which factorize through a light cone of de Sitter spacetime have been recently studied in [1], where the compact marginally trapped ones have been characterized.

On the other hand, it has been pointed out that there is a strong relationship between the extrinsic and the intrinsic geometries of submanifolds through the light cone in the Lorentz-Minkowski spacetime [10], [11]. Inspired by this point of view, our main aim in this note is to show a natural correspondence between the light cone of the Lorentz-Minkowski spacetime and the null hypersurfaces, also called light cones, of de Sitter and anti-de Sitter spacetimes. By means of this correspondence, several results of [10] and [11] can be adapted to de Sitter and anti-de Sitter spacetimes. We denote here (anti)-de Sitter when we are talking about any of them.

Specifically, we consider codimension two spacelike submanifolds which factorize through a light cone of (anti)-de Sitter spacetime and we stablish a correspondence between these submanifolds and codimension two spacelike submanifolds which factorize through the light cone in the Lorentz-Minkowski spacetime $\mathbb{L}^n$. The intrinsic geometries of the corresponding submanifolds are actually the same but the extrinsic ones are different (see Section 3). Then, we focus on the case of spacelike surfaces which factorize through a light cone of the 4-dimensional (anti)-de Sitter spacetime, where we obtain our main results.

This note is organized as follows. Section 2 is devoted to recall the basic formulae of codimension two spacelike immersions in a Lorentzian manifold with constant sectional curvature. In Section 3 we define the light cones in (anti)-de Sitter spacetime, and we establish the cited correspondence between codimension two spacelike submanifolds through a light cone in $\mathbb{S}^n_1(c)$

and $\mathbb{H}_1^n(c)$ and codimension two spacelike submanifolds that factorize through the light cone of $\mathbb{L}^n$. This correspondence allows us to show, in Section 4, that the scalar curvature S of a spacelike immersion $\psi\colon \Sigma \to \mathbb{M}_1^n(c)$ through a light cone $\Lambda_c(p)$ is given by (see notation in Sections 2, 3)

$$S = n(n-1)\Big[\langle \mathbf{H}_\psi, \mathbf{H}_\psi\rangle + \frac{\varepsilon}{c^2}\Big].$$

This formula relates intrinsic data (the scalar curvature) and extrinsic data (the mean curvature vector field $\mathbf{H}_\psi$).

Section 5 is focussed on the case of surfaces which factorize through a light cone of the 4-dimensional (anti)-de Sitter spacetime. In this case, we obtain an explicit formula for the Gauss curvature in terms of a height function (see Cor. 5.2). Finally, in Section 5.1 we study the compact case. In this case, the surface Σ must be a topological sphere and the above formula for the scalar curvature can be integrated to give the integral formula

$$\int_\Sigma |\mathbf{H}_\psi|^2\, dA = 4\pi - \frac{\varepsilon}{c^2}\,\mathrm{Area}(\Sigma).$$

This integral formula looks very similar to the equality case of the generalized Wintgen inequality [12], [13]. However, in the Lorentzian setting, the generalized Wintgen inequality is not satisfied in general. Finally, we deal with the first eigenvalue of the Laplace operator of such kind of surfaces through light cones in (anti)-de Sitter spacetime. We obtain a Reilly type inequality (17), which is used to characterize the total umbilical round spheres in a light cone of (anti)-de Sitter spacetime (Th. 5.9). We also study the global geometry of the surfaces. In particular, we give a Liebmann-type result for these surfaces, i.e., we have that a compact spacelike immersion with constant Gauss curvature in a light cone of (anti)-de Sitter spacetime must be totally umbilical (Th. 5.5).

2 Preliminaries

Let $\mathbb{L}^{n+1}$ be the $(n+1)$-dimensional *Lorentz-Minkowski spacetime*. That is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric $\langle\ ,\ \rangle$ given by

$$\langle\ ,\ \rangle = -(dx_0)^2 + (dx_1)^2 + \cdots + (dx_n)^2, \tag{1}$$

where $(x_0, x_1, \cdots, x_n)$ are the canonical coordinates of $\mathbb{R}^{n+1}$. The n-dimensional *de Sitter spacetime* of radius $c > 0$ is defined as the hyperquadric

$$\mathbb{S}_1^n(c) = \{x \in \mathbb{L}^{n+1}\ :\ \langle x\,, x\rangle = c^2\}. \tag{2}$$

As it is well known, $\mathbb{S}_1^n(c)$ inherits from $\mathbb{L}^{n+1}$ a time orientable Lorentzian metric with constant sectional curvature equal to $1/c^2$.

On the other hand, we denote by $\mathbb{E}_2^{n+1}$ the $(n+1)$-dimensional real space $\mathbb{R}^{n+1}$ endowed with the indefinite metric of index 2

$$\langle\ ,\ \rangle' = -(dx_0)^2 - (dx_1)^2 + (dx_2)^2 + \cdots + (dx_n)^2, \tag{3}$$

Then, the n-dimensional *anti-de Sitter spacetime* of radius $c > 0$ is defined as

$$\mathbb{H}^n_1(c) = \{x \in \mathbb{E}^{n+1}_2 \;:\; \langle x\,, x\rangle' = -c^2\}. \tag{4}$$

$\mathbb{H}^n_1(c)$ is a time orientable Lorentzian manifold with constant sectional curvature equal to $-1/c^2$. From now on, we will write $\mathbb{M}^n_1(c)$ to refer to either $\mathbb{S}^n_1(c)$ or $\mathbb{H}^n_1(c)$, and, in order to simplify the notation, we will represent both metrics (1) and (3) by $\langle\ ,\ \rangle$. Let $\varepsilon = \pm 1$ be the sign of the sectional curvature of $\mathbb{M}^n_1(c)$. Then, $\mathbb{M}^n_1(c)$ is always defined by the equation $\langle x, x\rangle = \varepsilon c^2$.

Unless otherwise were started, from now on, we assume $n \geq 4$. Let Σ be an $(n-2)$-dimensional connected manifold and $\psi : \Sigma \to \mathbb{M}^n_1(c)$ a smooth immersion such that the induced metric on Σ is Riemannian. In this case, Σ is said to be a codimension two spacelike submanifold in $\mathbb{M}^n_1(c)$. The induced metric on Σ via ψ will be also denoted by $\langle\ ,\ \rangle$. Let us write $\overline{\nabla}$ and ∇ for the Levi-Civita connections of $\mathbb{M}^n_1(c)$ and Σ respectively, and we denote by $\nabla^\perp$ the normal connection of Σ in $\mathbb{M}^n_1(c)$. With this notation, the Gauss and Weingarten formulae of ψ are written respectively as

$$\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y) \qquad \text{and} \qquad \overline{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{5}$$

for any tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$ and any normal vector field $\xi \in \mathfrak{X}^\perp(\Sigma)$. Here, II denotes the vector valued second fundamental form of Σ,

$$\mathrm{II} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^\perp(\Sigma).$$

The shape (or Weingarten) operator A_ξ corresponding to ξ is related to the second fundamental form by

$$\langle A_\xi X, Y\rangle = \langle \mathrm{II}(X, Y), \xi\rangle. \tag{6}$$

As usual, we define the mean curvature vector field of the submanifold Σ by

$$\mathbf{H} = \frac{1}{n-2}\,\mathrm{tr}_{\langle\ ,\ \rangle}\mathrm{II} \in \mathfrak{X}^\perp(\Sigma),$$

where $\mathrm{tr}_{\langle\ ,\ \rangle}$ denotes the trace with respect to the induced metric $\langle\ ,\ \rangle$.

3 Light cones in (anti)-de Sitter spacetime

In this section we start introducing the notion of light cone in $\mathbb{M}^n_1(c)$. Let $p \in \mathbb{M}^n_1(c)$ be a fixed point, then the *light cone of (anti)-de Sitter spacetime with vertex at* p is the hypersurface

$$\Lambda_c(p) := \big\{q \in \mathbb{M}^n_1(c) \;:\; \langle q - p\,, q - p\rangle = 0, \quad q \neq p\big\}, \tag{7}$$

or in an equivalent way, $\Lambda_c(p) = \big\{q \in \mathbb{M}^n_1(c) \;:\; \langle q, p\rangle = \varepsilon c^2, \quad q \neq p\big\}$. For each $q \in \Lambda_c(p)$, the tangent space at q is expressed as

$$T_q\Lambda_c(p) = \big\{v \in T_q\mathbb{M}^n_1(c) \;:\; \langle v\,, p\rangle = 0\big\} = \big\{v \in \mathbb{R}^{n+1} \;:\; \langle v\,, q\rangle = \langle v\,, p\rangle = 0\big\}. \tag{8}$$

In this setting, it is easy to check that $T_q\Lambda_c(p) \cap \big(T_q\Lambda_c(p)\big)^\perp = \mathrm{Span}\{q - p\}$, and therefore $\Lambda_c(p)$ is a degenerate hypersurface in $\mathbb{M}^n_1(c)$.

Remark 3.1. For a general Lorentzian manifold (M, g), the light cone $\Lambda_M(p)$ with vertex at $p \in M$ is the image by means of the exponential map of the light cone at the tangent space T_pM. Thus, the light cone $\Lambda_M(p)$ collects the lightlike geodesics starting at p. Until the first conjugate point, the light cone gives a smooth degenerate immersed hypersurface. This is the situation in our setting, since every Lorentzian manifold of constant sectional curvature has no conjugate points along its lightlike geodesics. On the other hand, recall that the future and the past horisms of $p \in M$ are defined by $E^{\pm}(p) = J^{\pm}(p) \setminus I^{\pm}(p)$, where $J^+(p)$ and $I^+(p)$ are the causal future and chronological future of p, respectively. The causal and chronological past are denoted by replacing $+$ by $-$. For a geodesically complete Lorentzian manifold (M, g) of constant sectional curvature, $E^+(p) \cup E^-(p) = \Lambda_M(p)$ holds. In fact, the inclusion $E^+(p) \cup E^-(p) \subset \Lambda_M(p)$ is a direct consequence of [9, Cor. 14.5] and the converse is obvious.

Let us fix a point $p \in \mathbb{M}^n_1(c)$, we say that a codimension two spacelike immersion $\psi : \Sigma^{n-2} \to \mathbb{M}^n_1(c)$ factorizes through the light cone at $p \in \mathbb{M}^n_1(c)$ when $\psi(\Sigma) \subset \Lambda_c(p)$. From now on, $\mathbb{E}^{n+1}_s$ will denote the $(n+1)$-dimensional semi-Euclidean space of signature s. Thus, we have $\mathbb{S}^n_1(c) \subset \mathbb{E}^{n+1}_1$ and $\mathbb{H}^n_1(c) \subset \mathbb{E}^{n+1}_2$. From every immersion $\psi : \Sigma^{n-2} \to \mathbb{M}^n_1(c)$ through the light cone at $p \in \mathbb{M}^n_1(c)$, we can consider $p^\perp \subset \mathbb{E}^{n+1}_s$. It is clear that $p^\perp$ is isometric to the n-dimensional Lorentz-Minkowski spacetime. The light cone in $p^\perp \simeq \mathbb{L}^n$ with vertex at the origin is the set

$$\Lambda = \{x \in p^\perp \ : \ \langle x\, , x\rangle = 0, \ x \neq 0\}.$$

The translation $T(x) = x - p$ in $\mathbb{E}^{n+1}_s$ induces an isometry from $\Lambda_c(p)$ to Λ. By means of this isometry, we have a one-to one correspondence between codimension two spacelike immersions $\psi : \Sigma^{n-2} \to \mathbb{M}^n_1(c)$ through $\Lambda_c(p)$ and spacelike immersions $\bar{\psi} := T \circ \psi \colon \Sigma^{n-2} \to \Lambda \subset p^\perp$. This correspondence can be summarized in the following commutative square.

Proposition 3.2. *Let $\psi : \Sigma^{n-2} \to \mathbb{M}^n_1(c)$ be a codimension two spacelike submanifold which factorizes through the light cone $\Lambda_c(p)$. Then, there exists a unique spacelike immersion $\bar{\psi} \colon \Sigma^{n-2} \to \Lambda \subset p^\perp$ such that makes commutative the following diagram,*

$$\begin{array}{ccccc} \Lambda \subset p^\perp & & \xrightarrow{\quad j \quad} & & \mathbb{E}^{n+1}_s \\ \uparrow \bar{\psi} & & & & \uparrow T \\ \Sigma & \longrightarrow & \Lambda_c(p) & \longrightarrow & \mathbb{M}^n_1(c) \\ & & \underset{\psi}{\frown} & & \end{array} \tag{9}$$

where j is the inclusion. Moreover, the intrinsic geometries on Σ induced from ψ and $\bar{\psi}$ are the same.

This correspondence $\psi \leftrightarrow \bar{\psi}$ allows us to obtain geometrical properties for ψ from the ones of $\bar{\psi}$. The rest of this note will develop several aspects of this point of view.

Remark 3.3. As consequence of [5, Cor. 7.6] (see also [3]) and the previous paragraph, we can deduce that any Riemannian manifold M^n, $n \geq 3$, is locally conformally flat if and only if it can be locally isometrically immersed in the light cone $\Lambda_c(p)$.

If we fix a timelike vector $W \in p^{\perp}$, we can consider the future light cone $\Lambda^{+} \subset p^{\perp}$ and the past lightcone Λ^{-} of Λ with respect to W. Observe that, using the isometry T, we are able to define the future and the past components of the light cone $\Lambda_c(p)$ of $\mathbb{M}_1^n(c)$ with respect to W in a natural way.

Let $\psi\colon \Sigma^{n-2} \to \mathbb{M}_1^n(c)$ be a codimension two spacelike submanifold through the light cone at $p \in \mathbb{M}_1^n(c)$. Then $\bar{\psi}$ is a spacelike immersion through $\Lambda \subset p^{\perp}$ and for every unit timelike vector $W \in p^{\perp}$, we introduce the height function on Σ as

$$\begin{aligned} h_W(x) : \Sigma^{n-2} &\to \mathbb{R} \\ x &\to -\langle \bar{\psi}(x), W \rangle = -\langle \psi(x), W \rangle. \end{aligned} \tag{10}$$

Note that $h_W(x) \neq 0$ for every $x \in \Sigma$ and $h_W > 0$ when $\bar{\psi}$ factorizes through the future light cone $\Lambda^{+} \subset p^{\perp}$ corresponding to W in $p^{\perp}$.

4 Codimension two submanifolds through a light cone of (anti)-de Sitter spacetime

Let $\psi\colon \Sigma \to \Lambda_c(p) \subset \mathbb{M}_1^n(c)$ be a codimension two spacelike submanifold through a light cone. As was stated in Proposition 3.2, the intrinsic geometries corresponding to ψ and $\bar{\psi}$ are the same. At this point we wonder what we can say about the extrinsic geometries of ψ and $\bar{\psi}$. Let us denote here with a subscript on $\mathbf{H}$ the mean curvature vector field corresponding to every given immersion. With this notation we can state the next result.

Proposition 4.1. *Let $\psi\colon \Sigma^{n-2} \to \mathbb{M}_1^n(c)$ be a codimension two spacelike submanifold which factorizes through the light cone at $p \in \mathbb{M}_1^n(c)$. Then*

$$\langle \mathbf{H}_{\bar{\psi}}, \mathbf{H}_{\bar{\psi}} \rangle = \langle \mathbf{H}_{\psi}, \mathbf{H}_{\psi} \rangle + \frac{\varepsilon}{c^2}, \tag{11}$$

where $\bar{\psi}$ is the corresponding immersion given in Proposition 3.2.

Proof. By the commutative diagram (9) we have $\mathbf{H}_{j\circ\bar{\psi}} = \mathbf{H}_{T\circ\psi}$ and, since $T\colon \mathbb{M}_1^n(c) \to \mathbb{E}_s^{n+1}$ is a totally umbilical immersion, it follows $\mathbf{H}_{T\circ\psi} = T_*(\mathbf{H}_{\psi}) - \frac{\varepsilon}{c}N$ where N is the outward unit normal vector field to $T\colon \mathbb{M}_1^n(c) \to \mathbb{E}_s^{n+1}$ with $\langle N, N \rangle = \varepsilon$. Finally, we obtain $\langle \mathbf{H}_{\bar{\psi}}, \mathbf{H}_{\bar{\psi}} \rangle = \langle \mathbf{H}_{\psi}, \mathbf{H}_{\psi} \rangle + \frac{\varepsilon}{c^2}$, as we wanted to prove. □

The next corollary is a direct consequence of [10, Cor. 4.5] and previous Proposition 4.1.

Corollary 4.2. *Let $\psi\colon \Sigma^{n-2} \to \Lambda_c(p) \subset \mathbb{M}_1^n(c)$ be a codimension two spacelike submanifold through the light cone $\Lambda_c(p)$. Then, the scalar curvature S of Σ is given by*

$$S = n(n-1)\Big(\langle \mathbf{H}_{\psi}, \mathbf{H}_{\psi} \rangle + \frac{\varepsilon}{c^2}\Big). \tag{12}$$

□

In this instance, an immediate consequence of [10, Prop. 5.1] and the correspondence between immersions through the light cone $\Lambda_c(p) \subset \mathbb{M}^n_1(c)$ and $p^\perp$ is the following.

Corollary 4.3. *Every codimension two compact spacelike submanifold in $\mathbb{S}^n_1(c)$ or $\mathbb{H}^n_1(c)$ that factorizes through a light cone is a topological $(n-2)$-sphere $\mathbb{S}^{n-2}$.*

Remark 4.4. Even more, Corollary 4.3 implies that every compact spacelike submanifold $\psi\colon \Sigma^{n-2} \to \Lambda_c(p) \subset \mathbb{M}^n_1(c)$ must be embedded. In fact, let us consider the spacelike immersion $\bar{\psi} : \Sigma^{n-2} \to \Lambda \subset p^\perp$ as in Proposition 3.2. Without loss of generality, we can assume that $\bar{\psi}$ factorizes through the future light cone $\Lambda^+ \subset p^\perp$. The proof of [10, Cor. 4.5] provides a diffeomorphism $F\colon \Sigma^{n-2} \to \mathbb{S}^{n-2}$ which is given by the composition $F = \pi \circ \alpha \circ \bar{\psi}$ where $\alpha\colon \Lambda^+ \to (0, +\infty) \times \mathbb{S}^{n-2}$ is the diffeomorphism

$$\alpha(v) = (v_0, 1/v_0(v_1, \cdots v_n)) \,,$$

and π is the projection on the second factor $\mathbb{S}^2$. The continuous map $\bar{\psi}\colon \Sigma^{n-2} \to \psi(\Sigma^{n-2})$ is one-to-one from a compact manifold on a Hausdorff space and then $\bar{\psi}$ is a homeomorphism.

Remark 4.5. There are compact spacelike submanifolds of codimension two in $\mathbb{M}^n_1(c)$ which are not topological spheres. For example in the case of $\mathbb{S}^n_1(c)$, we can consider the Euclidean sphere $\mathbb{S}^{n-1}(c)$ embedded as a totally geodesic spacelike hypersurface in $\mathbb{S}^n_1(c)$ and then, every hypersurface in $\mathbb{S}^{n-1}(c)$ can be seen as codimension two spacelike submanifold in $\mathbb{S}^n_1(c)$. In the case of $\mathbb{H}^n_1(1)$, we consider its universal cover $\widetilde{\mathbb{H}}^n_1(1)$. A model for $\widetilde{\mathbb{H}}^n_1(1)$ is given by $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ endowed with the Lorentzian metric obtained by the pull-back of the covering map, [9, Exam. 8.27],

$$k\colon \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{H}^n_1(1), \quad k(t,x) = (\sqrt{1 + \|x\|^2}\cos t, \sqrt{1 + \|x\|^2}\sin t, x),$$

where $\|x\|^2$ is the square of the Euclidean norm. In this picture, the slices at t constant are totally geodesic spacelike hypersurfaces isometric to the hyperbolic Riemannian space $\mathbb{H}^{n-1}(1)$. Now, every compact hypersurface in $\mathbb{H}^{n-1}(1)$ projects by means of k on a compact spacelike submanifold in $\mathbb{H}^n_1(1)$. In particular, when such a hypersurface is non-embedded, the projected one is also non-embedded.

Remark 4.6. We would like to point out that Corollary 4.3 remains true when we replace $\mathbb{H}^n_1(c)$ by $\widetilde{\mathbb{H}}^n_1(c)$. Namely, for every $(t,x) \in \widetilde{\mathbb{H}}^n_1(c)$ the lightlike cone at (t,x) is the image by means of the exponential map of the light cone at the tangent space $T_{(t,x)}\widetilde{\mathbb{H}}^n_1(c)$. Recall that any Lorentzian manifold of constant sectional curvature has no conjugate points along its lightike geodesics and then, the lightlike cone at $(t,x) \in \widetilde{\mathbb{H}}^n_1(c)$ is a smooth degenerate embedded hypersurface. Then, every compact codimension two spacelike submanifold in $\widetilde{\mathbb{H}}^n_1(c)$ through a light like cone is an embedded topological sphere. In fact, first we claim that every light cone in $\widetilde{\mathbb{H}}^n_1(c)$ is topologically two copies of $I \times \mathbb{S}^{n-2}$ where $I \subset \mathbb{R}$ is an interval of the real line. Indeed, without loss of generality, we assume $c = 1$. Now, taking into account the homogeneity of $\widetilde{\mathbb{H}}^n_1(1)$, it suffices to show the above assertion at a single point. Let us fix $(0,0) \in \widetilde{\mathbb{H}}^n_1(1)$, then $k(0,0) = (1, 0, \cdots, 0) = p_0 \in \mathbb{H}^n_1(1)$ and a direct computation shows that

$$k^{-1}(\Lambda_{-1}(p_0)) = \Big\{(t,x) \in \mathbb{R} \times \mathbb{R}^{n-1} : \|x\|^2 = \tan^2 t, \; x \neq 0\Big\}.$$

Therefore, the light cone at $(0,0) \in \widetilde{\mathbb{H}}^n_1(1)$ is topologically two copies of $(0, \pi/2) \times \mathbb{S}^{n-2}$. Now, the same proof of [10, Prop. 5.1] remains valid in this case and thus, every compact codimension two spacelike submanifold in $\widetilde{\mathbb{H}}^n_1(c)$ through a light like cone is a topological sphere $\mathbb{S}^{n-2}$ which is also embedded by the same method as in Remark 4.4.

From [2, Prop. 5.2], we know that, under some assumptions on the height function, every complete codimension two spacelike submanifold factorizing through the light cone in the Lorentz-Minkowski spacetime is compact and conformally diffeomorphic to the Euclidean sphere. By means of the correspondence given in Proposition 3.2, we have the following consequence.

Proposition 4.7. *Let $\psi\colon \Sigma \to \Lambda_c(p) \subset \mathbb{M}^n_1(c)$ be a codimension two complete spacelike submanifold factorizing through the light cone $\Lambda_c(p)$. If the height function h_W defining in* (10) *is bounded from above, then Σ is compact and conformally diffeomorphic to the Euclidean sphere $\mathbb{S}^{n-2}$.*

Remark 4.8. Actually, from [2, Lem. 5.1] in Proposition 4.7 it is enough to assume that h_W satisfies condition

$$h_W(p) \leq C\, r(p) \log(r(p)), \quad r(p) \gg 1,$$

where C is a positive constant and r denotes the Riemannian distance function from a fixed origin $o \in \Sigma$.

It is directly deduced from commutative diagram (9) that a codimension two spacelike submanifold $\psi\colon \Sigma \to \Lambda_c(p) \subset \mathbb{M}^n_1(c)$ is totally umbilical if and only if, $\bar{\psi} : \Sigma \to \Lambda \subset p^{\perp}$ is totally umbilical. Then, the following result is a direct consequence of [2, Th. 5.1].

Proposition 4.9. *Let $\psi\colon \Sigma^{n-2} \to \Lambda_c(p) \subset \mathbb{M}^n_1(c)$ be a codimension two spacelike submanifold which factorizes through the light cone $\Lambda_c(p)$. If ψ is totally umbilical, then there exist $\mathbf{v} \in \mathbb{E}^{n+1}$ and $\tau > 0$ such that*

$$\psi(\Sigma) \subset \Sigma(\mathbf{v}, \tau) = \{x \in \Lambda_c(p) : \langle x - p, \mathbf{v}\rangle = \tau\}.$$

5 Spacelike surfaces through a light cone of (anti)-de Sitter 4-dimensional spacetime

In this section we focus on the case $n = 4$, that is, we consider Σ a spacelike surface immersed in $\mathbb{M}^4_1(c)$. As a direct consequence of Corollary 4.2 or from [11, Cor. 3.7] we are now able to give the following identity.

Corollary 5.1. *Let $\psi\colon \Sigma \to \mathbb{M}^4_1(c)$ be a spacelike surface through a light cone $\Lambda_c(p)$. Then, the Gauss curvature of Σ may be expressed as*

$$K = \langle \mathbf{H}_\psi, \mathbf{H}_\psi \rangle + \frac{\varepsilon}{c^2}. \tag{13}$$

□

On the other hand, taking into account the definition of the height function h_W in (10) and [11, Cor. 3.7], we obtain the following expression for the Gauss curvature of Σ.

Corollary 5.2. *Let $\psi\colon \Sigma \to \mathbb{M}^4_1(c)$ be a spacelike surface that factorizes through the light cone at $p \in \mathbb{M}^4_1(c)$. Then, for every unit timelike vector $W \in p^\perp$, the Gauss curvature of Σ is given by*

$$K = \frac{1 + |\nabla h_W|^2}{h_W^2} - \frac{\Delta h_W}{h_W}. \tag{14}$$

In particular, when $\bar{\psi}$ factorizes through the future light cone $\Lambda^+ \subset p^\perp$ cooresponding to W, we have

$$K = \frac{1}{h_W^2} - \Delta \log(h_W).$$

□

Remark 5.3. *For example, the vector $P = -\varepsilon c^2 \partial_0 + \langle \partial_0, p\rangle p$ satisfies $P \in p^\perp$ and it is not difficult to show that, for $\varepsilon = 1$, P is timelike. Then, the height function h_W for $W := \frac{1}{\sqrt{-\langle P,P\rangle}} P$ is given by,*

$$h_W(x) = \frac{c}{\sqrt{c^2 + \varepsilon \langle \partial_0, p\rangle^2}} \langle \psi(x) - p, \partial_0\rangle, \quad x \in \Sigma.$$

Now we can relate the sign of the Gauss curvature K with the existence of local extreme points of the function h_W by mean of [11, Prop. 3.11] as follows.

Proposition 5.4. *Let $\psi : \Sigma \to \mathbb{M}^4_1(c)$ be a spacelike surface that factorizes through a future (resp. past) light cone $\Lambda_c(p)$ corresponding to $W \in p^\perp$ and with Gauss curvature $K \leq 0$. Then, the function h_W does not attain a local maximum (resp. minimum) value.* □

5.1 Some results for the compact case

We focus now on the case of a compact surface $\psi : \Sigma \to \mathbb{M}^n_1(c)$ that factorizes through the light cone $\Lambda_c(p)$.

The correspondence expressed in the square (9) gives that a spacelike surface $\psi\colon \Sigma \to \mathbb{M}^4_1(c)$ that factorizes through the light cone $\Lambda_c(p) \subset \mathbb{M}^4_1(c)$ is totally umbilical if and only if $\bar{\psi}\colon \Sigma \to p^\perp$ is totally umbilical. This fact can be used to characterize the totally umbilical surfaces in $\mathbb{M}^4_1(c)$ through light cones from [11, Theor. 5.4].

Theorem 5.5. *Let $\psi : \Sigma \to \mathbb{M}^4_1(c)$ be a compact spacelike surface that factorizes through the light cone at a point $p \in \mathbb{M}^4_1(c)$. If K is constant, then Σ is a totally umbilical round sphere.*

□

The totally umbilical compact spacelike surfaces that factorizes through the light cone $\Lambda \subset p^\perp$ are given by the two-parameter family

$$\mathbb{S}^2(W, \tau) = \{x \in \Lambda : \langle x, W\rangle = \tau\}$$

where $\tau > 0$ and $W \in p^{\perp}$ with $\langle W, W\rangle = -1$, [11]. As a direct application of Theorem 5.5, we have the following.

Corollary 5.6. *Let $\psi : \Sigma \to \mathbb{M}^4_1(c)$ be a compact spacelike surface that factorizes through the light cone at a point $p \in \mathbb{M}^4_1(c)$. The following assertions are equivalent*

1. *K is constant.*
2. *$\psi : \Sigma \to \mathbb{M}^4_1(c)$ is totally umbilical.*
3. *There exist $\tau > 0$ and $W \in p^{\perp}$ with $\langle W, W\rangle = -1$ such that*

$$\psi(\Sigma) = \Lambda_c(p) \cap \{x : \langle x, W\rangle = \tau\}.$$

Remark 5.7. Recall that a spacelike surface is called *marginally trapped* if its mean curvature vector field $\mathbf{H}$ is null, i.e., $\langle \mathbf{H}, \mathbf{H}\rangle = 0$ and $\mathbf{H}$ vanishes nowhere. For $\psi : \Sigma \to \mathbb{M}^4_1(c)$ a marginally trapped spacelike surface that factorizes through the light cone $\Lambda_c(p)$, we have from (13) that $K = \varepsilon/c^2$. If in addition Σ is compact, then Σ is a topological sphere $\mathbb{S}^2$. Therefore, the Gauss-Bonnet formula implies $\varepsilon = +1$. Therefore there are no closed marginally trapped surfaces that factorizes through a light cone in the anti-de Sitter spacetimes $\mathbb{H}^4_1(c)$ (already proved in a more general setting in [8]). It is known that there exist examples of closed marginally trapped surfaces in the 4-dimensional de Sitter spacetime that factorizes through a light cone (see for instance [6]).

As a direct consequence of (13) and the Gauss-Bonnet formula, the total mean curvature of compact spacelike surfaces in a light cone $\Lambda_c(p) \subset \mathbb{M}^4_1(c)$ may be expressed as

$$\int_{\Sigma} |\mathbf{H}_{\psi}|^2 \, \mathrm{d}A = 4\pi - \frac{\varepsilon}{c^2} \operatorname{Area}(\Sigma). \tag{15}$$

Remark 5.8. We recall that given an isometric immersion of a compact oriented surface S into an orientable Riemannian 4-dimensional manifold with constant sectional curvature ε/c^2, the following inequality holds

$$\int_S |\mathbf{H}|^2 \, dV \geq 2\pi\, \chi(S) + \Big| \int_S K_N \, dV \Big| - \frac{\varepsilon}{c^2} \operatorname{Area}(S), \tag{16}$$

where $\mathbf{H}$, K_N, and $\chi(S)$ are, respectively, the mean curvature vector field, the normal curvature and the Euler characteristic of S. This integral inequality is known as the generalized Wintgen inequality [12], [13]. A similar proof as in [10, Rem. 4.2] shows that K_N vanishes identically in our setting. Hence Formula (15) gives the equality in (16) for compact spacelike surfaces that factorize through a light cone in $\mathbb{M}^4_1(c)$. However, in the Lorentzian setting, the generalized Wintgen inequality is not satisfied, in general. We can see this even considering a compact spacelike surface of the Lorentz-Minkowski space. For instance, we define the isometric embedding

$$\psi : \mathbb{S}^2 \subset \mathbb{R}^3 \to \mathbb{L}^4, \quad \psi(x, y, z) = \big(\cosh(x), \operatorname{senh}(x), y, z\big).$$

Then, a straightforward computation (see details in [10]) shows

$$\int_{\mathbb{S}^2} |\mathbf{H}|^2 \, dV < \operatorname{Area}(\mathbb{S}^2) = 4\pi,$$

and hence, the generalized Wintgen inequality is not met for ψ.

In order to analyse the spectrum of the Laplace operator of $(\Sigma, \langle\ ,\ \rangle)$, formula (15) is very useful. First, let us recall that for an arbitrary Riemannian metric g on $\mathbb{S}^2$, the minimum non zero eigenvalue of the Laplace operator λ_1 of g satisfies the Hersch inequality [7] which states

$$\lambda_1 \leq \frac{8\pi}{\operatorname{Area}(\mathbb{S}^2, g)},$$

and the equality holds if and only if $(\mathbb{S}^2, g)$ has constant Gauss curvature. Therefore, taking into account (15), Hersch inequality may be written for a compact spacelike surface Σ in $\Lambda_c(p)$ as

$$\lambda_1 \leq \frac{2\int_\Sigma |\mathbf{H}_\psi|^2 \, dA}{\operatorname{Area}(\Sigma)} + \frac{2\varepsilon}{c^2}, \tag{17}$$

and from Corollary 5.6, the equality holds if and only if $\psi\colon \Sigma \to \mathbb{M}^4_1(c)$ is totally umbilical. This formula gives an extrinsic bound of the first non trivial eigenvalue of the Laplace operator of $(\Sigma, \langle\ ,\ \rangle)$, and formally is the same expression of the well-known Reilly inequality in the Euclidean space. However, Reilly equality is not true in general in a Lorentzian ambient (see for instance [11]).

In the compact case, Corollary 5.2 gives the following integral formula for a compact spacelike surface Σ through $\Lambda_c(p)$

$$\int_\Sigma \frac{1}{h^2_W} \, \mathrm{d}A = 4\pi, \tag{18}$$

for every unit timelike vector $W \in p^\perp$. Now, from Schwarz inequality and Theorem 5.5 we come to the following result.

Proposition 5.9. *Let $\psi : \Sigma \to \mathbb{M}^4_1(c)$ be a compact spacelike surface that factorizes through the future light cone $\Lambda_c(p)$ corresponding to the unit timelike vector $W \in p^\perp$. Then, we have the following upper bound for the area of Σ,*

$$\operatorname{Area}(\Sigma) \leq 2\sqrt{\pi}\, \|\langle \psi, W\rangle\|,$$

where $\|\cdot\|$ is the usual L^2 norm. Moreover, the equality holds for some W if and only if Σ is the totally umbilical round sphere $\Lambda_c(p) \cap \{x : \langle x, W\rangle = r\}$ with $r = -1/\langle \psi, W\rangle$.

Finally, from (15), the Hersch inequality and Corollary 5.6 we get the next theorem.

Theorem 5.10. *Let $\psi : \Sigma \to \mathbb{M}^4_1(c)$ be a compact spacelike surface that factorizes through the future light cone $\Lambda_c(p)$ corresponding to the unit timelike vector $W \in p^\perp$. Then, for every unit timelike vector $w \in p^\perp$ with $\langle w, W\rangle < 0$, we have*

$$\lambda_1 \leq \frac{2}{\min_\Sigma \left(h^2_w\right)},$$

and the equality holds for some w if and only if Σ is immersed as a totally umbilical round sphere in $\mathbb{M}^4_1(c)$. □

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