A Problem-Based Learning Proposal to Teach Numerical and Analytical Nonlinear Root Searching Methods

Nonlinear equations are usually solved numerically. However, approximated analytical solutions of nonlinear equations are useful as an initial iteration point for numerical methods. Furthermore, these analytical approximations are sometimes quite close to the actual root. In order to teach at undergraduate level how to perform this kind of approximation, we propose problem-based learning with some nonlinear equations extracted from Applied Sciences. We also present a simple method to analytically solve nonlinear equations which are not solvable by standard computer algebra solvers.

Keywords: Roots of nonlinear equations; Fixed-point method; Problem-based learning

MSC: 65H05, 97D50

1. Numerical vs. analytical, a complementary view in teaching

Nonlinear equations are found in many branches of Applied Mathematics, Physics, and Engineering, but very few of them can be solved analytically. The impact of this fact on teaching at undergraduate level has been highly significant. For instance, in the calculation of planetary elliptic orbits, we need to evaluate the roots of Kepler's equation,

$$E - \varepsilon \sin E = M,$$

where E is the eccentric anomaly, ε is the eccentricity and M the mean anomaly. The solution of Kepler's equation can be formulated analytically as

$$E = M + \sum_{n=1}^{\infty} \frac{2J_n(n\varepsilon)}{n} \sin(nM), \tag{1}$$

where J_n denotes the Bessel function of the first kind, and the sum in (1) converges for $\varepsilon < 1$ like a geometric series with ratio (Colwell, 1993):

$$r(\varepsilon) = \frac{\varepsilon \exp\left(\sqrt{1+\varepsilon^2}\right)}{1+\sqrt{1+\varepsilon^2}}.$$

Nevertheless, (1) involves the calculation of high order Bessel functions and a numerical method to evaluate the sum. Therefore, it is worth considering a direct numerical approach. For example, the simple iterative method

$$E_{i+1} = M + \varepsilon \sin E_i,$$

with the initial iteration point $E_0 = 0$, works quite well. Therefore, from an educational point of view, the numerical approach is quite appealing because of its simplicity in comparison to the analytical one. Moreover, numerical solutions have become so popular that sometimes analytical solutions have not been considered in existing literature. This is the case of Freudenstein's equation, as we explain next.

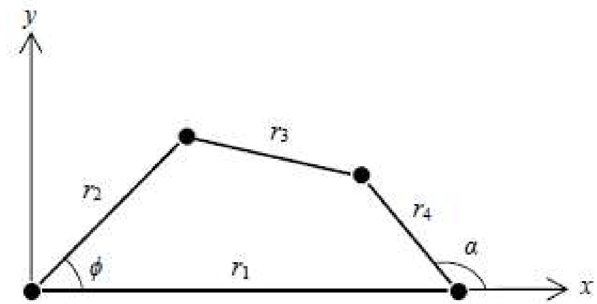

Figure 1: Four-bar linkage.

***Example 1**: Consider the four-bar linkage depicted in Figure 1, where the length of each bar is denoted as r_1, r_2, r_3, and r_4. The relationship between the angles α and ϕ of the linkage is given by Freudenstein's equation* (Hoffmann, 2001)*:*

$$R_1 \cos\alpha - R_2 \cos\phi + R_3 = \cos(\alpha - \phi), \tag{2}$$

where $R_1 = r_1/r_2$, $R_2 = r_1/r_4$, $R_3 = \left(r_1^2 + r_2^2 + r_3^2 + r_4^2\right)/\left(2r_2r_4\right)$. *We can rewrite* (2) *as follows*

$$A\cos\phi + C = B\sin\phi, \tag{3}$$

where $A = -R_2 - \cos\alpha$, $B = \sin\alpha$, *and* $C = R_3 + R_1\cos\alpha$. *Performing the change of variables* $x = \cos\phi$, (3) *becomes*

$$\left(A^2 + B^2\right)x^2 + 2AC\,x + C^2 - B^2 = 0.$$

Solving for x and undoing the change of variables performed, we arrive at

$$\begin{aligned}\phi(\alpha) = \cos^{-1}\Bigg(& \frac{(R_2 + \cos\alpha)(R_3 + R_1\cos\alpha)}{1 + R_2^2 + 2R_2\cos\alpha} \\ & \pm \frac{\sin\alpha\sqrt{1 + R_2^2 - R_3^2 + 2(R_2 - R_1R_3)\cos\alpha - R_1^2\cos^2\alpha}}{1 + R_2^2 + 2R_2\cos\alpha}\Bigg).\end{aligned} \tag{4}$$

Note that performing the substitution $R_1 \leftrightarrow -R_2$ *the angles α and* ϕ *are interchanged in* (2)*. Therefore, to obtain the inverse function α(*ϕ*) just exchange* $R_1 \leftrightarrow -R_2$ *and* $\alpha \leftrightarrow \phi$ *in* (4)*. Notice as well that we have obtained two solutions in* (4)*. This means that when we calculate the numerical solution, we may miss one of them, as occurs in* (Hoffmann, 2001). Δ

Analytical solutions are preferable to numerical ones because we can study the behaviour of the solution with respect to the parameters of the problem. For instance, in the above example, we can easily derive the condition for which the solution is unique equating in (4) the discriminant to zero. However, analytical solutions are not usually possible, so several numerical methods have been developed over time to solve nonlinear equations in a recursive way. Among the most popular methods we find the fixed-point method, Newton's method and bracketing methods. (Henrici, 1964), (Kincaid & Cheney, 1991). These numerical methods have been refined in existing literature (Suhadolnik, 2013), (Jain & Sethi, 2016). Nevertheless, these numerical methods have to be applied carefully in order to assure convergence to the proper root. For instance, Newton's method needs as its initial iteration a point near enough to the root we are looking for. Also, bracketing methods need to know the initial interval where the root lies. Moreover, complex roots cannot be found with bracketing methods or when the initial iteration is real in Newton's method. However, in teaching, we do not usually justify to students how to choose the initial iteration point in order to assure convergence to the proper root. We usually just say that a graphical plot suffices to select the initial iteration point. Nonetheless, if we have several parameters in the nonlinear equation, we have to graph the function for every particular case. In this sense, if we approximate a non-solvable nonlinear equation into a solvable one, the roots of the latter can be used as initial iteration points. Next, we present an example of the latter.

Example 2*: Consider quasi-one-dimensional isentropic flow of a perfect gas through a variable-area channel. The relationship between the Mach number M and the flow area A is given by* (Zucrow & Hoffman, 1976)*:*

$$\varepsilon = \frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}, \tag{5}$$

where A^* is the area for $M = 1$ and γ is the specific heat ratio of the flowing gas. For $0 < \varepsilon \neq 1$, there exist two values of M, i.e. $M < 1$ (subsonic flow) and $M > 1$ (supersonic flow). Taking $a = \frac{\gamma-1}{\gamma+1}$, $b = \frac{2}{\gamma+1}\left(\frac{\gamma-1}{2\varepsilon^2}\right)^a$, and the variable $x = \frac{\gamma-1}{2}M^2$, we can rewrite (5) *as*

$$f(x) = \frac{x^a}{1+x} = b > 0. \tag{6}$$

Note that the specific heat ratio γ of an ideal gas is $\gamma = 1 + \frac{2}{f}$ where f is the number of degrees of freedom. Therefore, we have $\gamma > 1$, hence $0 < a < 1$. Thereby, the function $f(x)$ has a unique local maximum at $x_{\max} = \frac{a}{1-a} > 0$, and (6) *has two roots: $0 < x_1 < x_{\max}$ (subsonic flow) and $x_2 > x_{\max}$ (supersonic flow), as shown in Figure* 2.

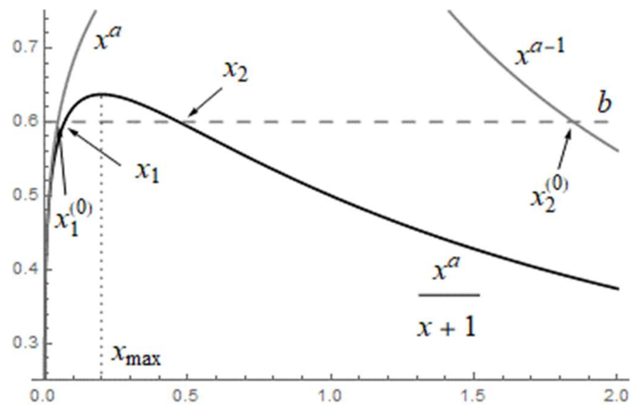

Figure 2. Graph of the initial guesses for the subsonic and supersonic flow.

Notice that asymptotically the subsonic and supersonic flows occur when $x \gg 1$ and $x \ll 1$ respectively. Therefore, for the subsonic flow we have $x+1 \approx 1$, and the solution of (6) *is $x_1 \approx b^{1/a} = x_1^{(0)}$. For the supersonic flow we have $x+1 \approx x$, thus $x_2 \approx b^{1/(a-1)} = x_2^{(0)}$. On the one hand, note that $0 < x_1^{(0)} < x_1$ since $x^a > f(x)$ $(x > 0)$ is a monotonically-increasing function going from 0 to ∞. On the other hand, $x_2 < x_2^{(0)}$ since $x^{a-1} > f(x)$ $(x > 0)$ is a monotonically-decreasing function going from ∞ to 0. Since $f(x)$ is a monotonically-increasing function for $x < x_{\max}$, if we apply Newton's method to* (6) *with $x_1^{(0)}$ as initial guess, we will obtain x_1. Likewise, Newton's method will converge to x_2, if the initial guess is $x_2^{(0)}$. Consequently, we have two possibilities for the initial guess of the iteration of* (5),

$$M_0 = \begin{cases} \sqrt{2b^{1/a}/(\gamma-1)}, \\ \sqrt{2b^{1/(a-1)}/(\gamma-1)}. \end{cases} \tag{7}$$

Considering $\varepsilon = 10$ *and air as the flowing gas, i.e.* $\gamma = 1.4$, *the approximation given in* (7) *for the subsonic flow is* $M_0 = 0.0578704$, *whereas the numerical solution of* ((5) *applying Newton's method is very close to it*, $M = 0.0579872$. *For the supersonic flow, we have* $M_0 = 4.64398$, *and* $M = 3.92255$. Δ

The above example shows that an analytical approach may be quite helpful for the numerical root searching of nonlinear equations in a systematic way without a graphical pre-search. Also, the initial guesses for the iteration scheme given in (7) indicate that the original equation has two different solutions, something which is not apparent from a numerical point of view. In addition, the analytical approximation for the subsonic flow is quite accurate. Notice that the key idea in obtaining the initial guesses for the iteration scheme is how to approximate the nonlinear equation (6). Next, we propose another example to be solved by the students in which we have to consider the value range of the variables involved.

Example 3: *In the design of open rectilinear channels of rectangular section, the flow Q is given by the following equation* (White, 1999)*:*

$$Q = \frac{\sqrt{S}\,(BH)^{5/3}}{n(B+2H)^{2/3}}, \tag{8}$$

where n is Manning roughness coefficient, S = tan α denotes the slope of the channel, B its width and H its depth (which is unknown). Generally, the width of the channel is much greater than its depth $B \gg H$ *, thus taking into account this approximation and solving for H, we obtain the following starting iteration point,*

$$H_0 = \left(\frac{nQ}{B\sqrt{S}}\right)^{3/5}.$$

Taking $Q = 5$ *m*^{3}*/s*, $B = 20$ *m*, $n = 0.03$ *and* $S = 2\times10^{-4}$, *we obtain* $H_0 = 0.683483$. *Solving* (8) *by Newton's method we obtain* $H = 0.702293$*, thus the initial iteration point* H_0 *is quite near to the actual root.* Δ

In light of the previous example, a possible objection to these analytical approximations might be that they are not very accurate. In the next example, we discuss this issue with a nonlinear equation arising in the theory of diffraction of light.

Example 4: *To calculate the magnitude of the successive maximums in the Fraunhofer diffraction diagram of a slit, we have to solve the following nonlinear equation* (Alonso & Finn, 1967)

$$\tan x = x. \tag{9}$$

A trivial solution of (9) *is $x = 0$. Notice that if $x = x^*$ is a root of (9), then $x = -x^*$ is also a root because both* $\tan x$ *and x are odd functions. Notice as well that in each period,* $\tan x$ *is a continuous and monotonically-increasing function whose image is $\mathbb{R}$. Therefore,* (9) *has got a unique root in each period, i.e.* $\exists!\, x^* \in ((k - 1/2)\pi, (k + 1/2)\pi)$, $k \in \mathbb{Z}$, *such that* $\tan x^* = x^*$. *In order to find the positive non-trivial solutions, i.e.* $k \in \mathbb{Z}^+$ *let us rewrite* (9) *as follows:*

$$x - \pi k = \tan^{-1} x. \tag{10}$$

Expanding the $\tan^{-1} x$ *function at* $x = \pi k$ *up to fourth order, and defining the variable* $y = \frac{x-u}{1+u^2}$ *with* $u = \pi k$, *we approximate* (10) *as,*

$$y^4 + a y^3 + b y^2 + c y + d \approx 0, \tag{11}$$

where

$$a = \frac{3u^2 - 1}{3u\left(1 - u^2\right)}, \quad b = \frac{-1}{1 - u^2}, \quad c = \frac{-u}{1 - u^2}, \quad d = \frac{\tan^{-1} u}{u\left(1 - u^2\right)}.$$

Defining the parameters:

$$r = 2b^3 - 9abc + 27c^2 + 27a^2 d - 72bd,$$

$$t = b^2 - 3ac + 12d,$$

$$R = \sqrt[3]{r + \sqrt{r^2 - 4t^3}},$$

$$A = \frac{\sqrt[3]{2}t}{3R} + \frac{R}{3\sqrt[3]{2}}, \quad B = \sqrt{\frac{a^2}{4} - \frac{2b}{3} + A},$$

the unique positive root of the quartic equation given in (11) *is* (Oldham, Myland, & Spanier, 2010)*:*

$$y_k \approx -\frac{a}{4} - \frac{B}{2} + \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - A - \frac{4ab - 8c - a^3}{4B}}.$$

Therefore, an approximate solution of (10) *for $k \in \mathbb{Z}^+$ is given by,*

$$x_k \approx \left(1 + \pi^2 k^2\right) y_k + \pi k.$$

It is worth noting that if we expand the $\tan^{-1} x$ *function at* $x = \pi k$ *up to second order in* (10)*, we will obtain a more compact analytical expression, which reads as*

$$x_k \approx \frac{1+\pi^2 k^2}{2}\sqrt{\pi^2 k^2 + 4\frac{\tan^{-1}(\pi k)}{\pi k}} + \pi k,$$

but its accuracy is not as good as the fourth order approximation obtained above. Nevertheless, the relative error of both approximations with respect to the exact solution solved numerically is extremely good, as shows Table 1. *Also,* Table 1 *shows that the numerical computation of the relative error seems to decrease as k increases. However, for* $k \geq 45$ *this tendency changes abruptly, probably due to numerical precision.* Δ

The above example shows that an analytical approach for solving nonlinear equations may be numerically worthwhile in practice. Further, we have obtained an approximate formula for all the roots of (9).

k	Fourth order	Second order
1	4.03×10^{-4}	3.05×10^{-3}
2	9.19×10^{-6}	1.90×10^{-4}
4	1.18×10^{-7}	8.46×10^{-6}
8	1.18×10^{-9}	3.18×10^{-7}
16	1.04×10^{-11}	1.09×10^{-8}
32	8.92×10^{-14}	3.59×10^{-10}

Table 1: Relative error between the analytical approximated solution and the numerical one.

Since the applications are widespread, the more nonlinear equations we know how to solve analytically, the better. The goal of this article is two-fold. On the one hand, we propose to the students problem-based learning about possible analytical approximations of nonlinear equations, studying the nature of the problem from which the nonlinear equation is posed. On the other hand, we provide a simple method based on the fixed-point theory in order to increase the number of nonlinear equations that can be solved analytically.

This paper is organized as follows. In Section 2 we present a simple fixed-point theorem to increase the number of the nonlinear equations with an analytical solution. As a corollary, we see how to use this theorem to solve a special kind of quartic equation. Section 3 is devoted to applying the fixed-point theorem to take nonlinear equations which are analytically solvable and transform them into other nonlinear equations which are not solvable by computer algebra, but which have the same roots as the original equation. For this purpose, we first introduce the Lambert W function. In Section 4, we consider the proper change of variables of the inverse problem, i.e. the change to obtain the solvable nonlinear equation from the transformed one. In Section 5 we present our conclusions.

2. A fixed-point theorem for root searching

Theorem 1: *If x^* is a root of $f(x) = x$, then x^* is also a root of $(f \circ \cdots \circ f)(x) = x$.*

Proof: Applying f to both sides of $f(x) = x$ we obtain $f[f(x)] = f(x) = x$. Therefore, successively applying function f to both sides of $f(x) = x$, we obtain the desired result. □

Notice that Theorem 1 can be used to simplify the nonlinear equation to be solved numerically, i.e. from $(f \circ \cdots \circ f)(x) = x.$ to $f(x) = x$.

Corollary: *Let $p(x)$ be a polynomial of order $n \geq 2$. Then*

$$\frac{p[p(x)] - x}{p(x) - x} = q(x),$$

where $q(x)$ is a polynomial of order $n\,(n - 1)$.

Proof: According to Theorem 1, all the roots of the polynomial $p(x) - x$ (of order $n \geq 2$) are also roots of the polynomial $p[p(x)] - x$ (of order n^2). Therefore, $p(x) - x$ divides $p[p(x)] - x$ and the quotient $q(x)$ is of order $n^2 - n = n\,(n - 1)$, as we wanted to prove. □

The above theorem can be applied to solve special cases of quartics, as is explained in the next example.

Example 7: *Find the roots of the following quartic equation:*

$$q(x) = x^4 + 2x^3 - 4x^2 - 6x + 3 = 0.$$

Consider the quadratic polynomial,

$$p(x) = a\,x^2 + b\,x + c,$$

thus

$$p[p(x)] - x$$
$$= a^3x^4 + 2a^2bx^3 + a\left(b + b^2 + 2ac\right)x^2 + \left(2abc + b^2 - 1\right)x + c\left(1 + b + ac\right).$$

Identifying the coefficients of $q(x)$ and $p[p(x)] - x$, we readily arrive at $a = 1$, $b = 1$, $c = -3$. Thereby, the roots of $p(x) - x = x^2 - 3$, i.e. $x = \pm\sqrt{3}$, are roots of $q(x)$ as well. Also, the polynomial

$$r(x)=\frac{p[p(x)]-x}{p(x)-x}=\frac{x^4+2x^3-4x^2-6x+3}{x^2-3}=x^2+2x-1,$$

has roots $x=-1\pm\sqrt{2}$. *In summary, the roots of* $q(x)$ *are* $\pm\sqrt{3}$ *and* $-1\pm\sqrt{2}$. Δ

Since the general solution of a quartic is cumbersome to calculate, the above example provides a shortcut. However, the main drawback is that this method only works for a very special kind of quartic, because we have 3 free parameters, i.e. a, b, c, to be set by the 5 coefficients of the given quartic. Nonetheless, it is a heuristic method for calculating irrational roots of a quartic. The latter is interesting form an educational point of view, since students usually only try integer or rational roots of a polynomial by Ruffini's method.

3. Applications

3.1 The Lambert W function

Many nonlinear equations found in scientific contexts can be solved using the Lambert W function, which is defined as the inverse function of xe^x. See (Olver, et al.) for branches, properties, and expansions of this function. Nowadays, the most common software packages of computer algebra include the $W(x)$ function and operate with it both numerically and symbolically. In (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996), we find a large collection of examples extracted from Physics, Engineering and Mathematics where the Lambert W function is applied. Next, we propose to the students that they apply the Lambert W function to solve the following nonlinear equation arising in Fluid Mechanics, which does not seem to appear in any existing literature.

Example 8: *When an incompressible fluid flows steadily through a round pipe, the pressure drop due to the effects of wall friction is given by the empirical formula:*

$$\Delta P=-\frac{1}{2}f\rho V^2\frac{L}{D},$$

where ΔP *is the pressure drop, r is the density, V is the velocity, L is the pipe length, D is the pipe diameter, and f is the D'Arcy friction coefficient. For flow in the turbulent regime between completely smooth pipe surfaces and wholly rough pipe surfaces, we found the following empirical equation for the friction coefficient f in* (Colebrook, et al., 1939)*,*

$$\frac{1}{\sqrt{f}}=-2\log_{10}\left(\frac{\varepsilon/D}{3.7}+\frac{2.51}{\mathrm{Re}\sqrt{f}}\right), \tag{12}$$

where ε is the pipe surface roughness, Re = *DVρ/μ is the dimensionless Reynolds number, and μ the viscosity. Defining the parameters* $a = \varepsilon/(3.7\,D)$, $b = 2.51/\mathrm{Re}$, $c = -1/2 \log 10$, *and the variable* $x = f^{-1/2}$, *we can rewrite* (12) *as*

$$a + bx = e^{cx},$$

Performing now the change of variables $y = a + bx$*, and reordering terms, we obtain,*

$$y\,e^{-cy/b} = e^{-ac/b} \rightarrow y = -\frac{b}{c} W\left(-\frac{c\,e^{-ac/b}}{b}\right).$$

Undoing the change of variables performed, we finally arrive at

$$f = \left(\frac{bc}{ac + b\,W\left(-e^{-ac/b}\,c/b\right)}\right)^2.$$

Δ

The above example shows that the Lambert *W* function significantly increases the number of nonlinear equations that can be solved analytically. Moreover, we can approximate quite accurately the actual root of a given nonlinear equation with the aid of the Lambert *W* function, as is explained in the following example proposed to the students.

Example 9: *If we have a mortgage loan P at an annual interest rate i (in percent) to be repaid in n months, the monthly instalment c is calculated as* (Kohn, 1990)*,*

$$c = \frac{P\,i/1200}{1-\left(1+i/1200\right)^{-n}}. \tag{13}$$

Performing the substitution $r = i/1200$*, we can rewrite* (13) *as*

$$\frac{c}{P} = \left[1-(1+r)^{-n}\right] = r. \tag{14}$$

Knowing c and P, we can numerically evaluate r with a fixed-point method. However, we can analytically solve (14) *in an approximate way, taking into account the following asymptotic formula:*

$$(1+r)^{-n} \approx e^{-nr}, \quad n \rightarrow \infty, r \rightarrow 0. \tag{15}$$

Thereby, taking $x = -nr$ *and* $a = -nc/P$*, we have*

$$a\left(1-e^{x}\right) \approx x. \tag{16}$$

Fortunately, (16) *can be solved with the aid of the Lambert W function as*

$$x \approx a - W\left(a\,e^{a}\right),$$

thus, undoing the changes of variables performed, eventually we arrive at,

$$i \approx 1200\left[\frac{c}{P}+\frac{1}{n}W\left(-\frac{nc}{P}e^{-nc/P}\right)\right]. \tag{17}$$

It turns out that (17) *provides a very good approximation in practice. For instance, considering a standard loan* P = 120000 € *to be paid in* 25 *years (i.e.* n = 300 months*) with an instalment* c = 450 €, *the relative error of* (17) *is* ≈ 0.3%. Δ

In the above example, the key idea is the approximation given in (15). This approximation is usually possible because n is large, and r is low in a mortgage loan. A recent example of this type applied to the relaxation time in wet grinding can be found in (González-Santander & Monreal, 2019).

We can further expand the number of nonlinear equations that the Lambert W function can solve by using Theorem 1. For instance, consider the function

$$f(x)=\frac{1}{a}\log^{1/b}x.$$

On the one hand, the solution to the fixed-point equation $f(x) = x$ is given in terms of the Lambert W function as:

$$x=\left(-\frac{W\left(-b\,a^{b}\right)}{b\,a^{b}}\right)^{1/b}. \tag{18}$$

On the other hand, the equation $f[f(x)] = x$ can be expressed as

$$b\left[\log a+(ax)^{b}\right]=\log(\log x). \tag{19}$$

According to Theorem 1, a solution to the nonlinear equation (19) is given by (18). As specific examples, we provide the solution of the following equations, which cannot be solved by symbolic computer algebra.

Example 10: *Setting* a = 1 *and* b = –1/2, *we have*

$$\frac{1}{2\sqrt{x}}+\log(\log x)=0 \;\rightarrow\; x=\frac{1}{4W^{2}(1/2)}\approx 2.0207.$$

Δ

Example 11: *Also, setting* $a = e$ *and* b = 1, *we have*

$$1+e\,x=\log(\log x) \;\rightarrow\; x=-\frac{W(-e)}{e}\approx -0.1453-0.6578\,i.$$

Notice that in this case the solution is complex. This means that although the graphs of $1 + e\,x$ *and* $\log(\log x)$ *do not cross on the real plane, there is a complex solution, since all the solutions (real or complex) of* $f(x) = x$ *are solutions of* $f[f(x)] = x$.

Δ

This last example prevents students from only graphically pre-searching potential roots of nonlinear equations, since complex roots are possible. It is worth noting that these kinds of complex roots may arise in problems of Applied Sciences. For instance, in electrical circuits, the impedance Z has a representation in the complex plane, i.e. $Z = R + i X$, where R denotes the resistance and X is the reactance.

Consider now the function,

$$f(x) = \frac{\log x}{b} - a.$$

On the one hand, the solution of the fixed-point equation $f(x) = x$ is given in terms of the Lambert W function as

$$x = -\frac{W\left(-b\, e^{ab}\right)}{b}. \tag{20}$$

On the other hand, the equation $f[f(x)] = x$ can be expressed as

$$\log b + \log\left(a + e^{b(x+a)}\right) = \log\left(\log x\right). \tag{21}$$

Next, we present some particular cases of (20) and (21), which cannot be solved by symbolic computer algebra.

Example 12: *Set* $a = b = 1$*, thereby*

$$1 + e^{x+1} = \log x \;\rightarrow\; x = -W(-e) \approx -0.3950 - 1.7882\, i.$$

Δ

Example 13: *Also, setting* $a = -1/e$ *and* $b = e$*, we have*

$$e^{ex} - 1 = \log x \;\rightarrow\; x = -\frac{W(-1)}{e} \approx -0.11703 - 0.49194\, i.$$

Δ

3.2 Some more cases

Now consider the function

$$f(x) = \frac{g(x)}{a} + x - b, \tag{22}$$

where $g(x)$ is an invertible function. Then, on the one hand,

$$f(x) = x \;\rightarrow\; x = g^{-1}(ab), \tag{23}$$

and on the other hand,

$$f\left[f(x)\right] = x \;\rightarrow\; \frac{g(x)}{a} + x - b = g^{-1}\left(2ab - g(x)\right). \tag{24}$$

To see the wide variety of nonlinear equations that (24) contains, we present two different examples, which cannot be solved by using symbolic computer algebra.

Example 14: *Consider* $g(x) = \log x$, *with* $a = 1$ *and* $b = 1/2$, *to obtain*

$$x\ \log x = e + \frac{x}{2} - x^2 \ \rightarrow\ x = \sqrt{e}. \tag{25}$$

Δ

Example 15: *Take* $g(x) = \sin^{-1}x$, *with* $a = 1$ *and* $b = \pi/4$, *thereby*

$$\sin^{-1} x + x = \frac{\pi}{4} + \sqrt{1-x^2} \ \rightarrow\ x = \frac{1}{\sqrt{2}}. \tag{26}$$

Δ

Yet consider another case with the following function,

$$f(x) = \frac{x}{a}\left[g(x) - b\right],$$

where $g(x)$ is an invertible function. Then, on the one hand,

$$f(x) = x \ \rightarrow\ x = g^{-1}(a+b), \tag{27}$$

and on the other hand,

$$f\left[f(x)\right] = x \ \rightarrow\ \frac{x}{a}\left[g(x)-b\right] = g^{-1}\left(\frac{a^2}{g(x)-b} + b\right). \tag{28}$$

Next, we present some specific examples of (27) and (28), which are not solvable by using symbolic computer algebra.

Example 16: *Set* $g(x) = \sin x$ *with* $a = 1$ *and* $b = 0$, *to obtain*

$$\sin(x \sin x) = 1 \ \rightarrow\ x = \frac{\pi}{2}. \tag{29}$$

Δ

Example 17: *Also, taking* $g(x) = \log x$ *with* $a = 1/e$ *and* $b = 0$, *we get*

$$\left[1 + \log x + \log(\log x)\right]\log x = \frac{1}{e^2} \ \rightarrow\ x = e^{1/e}.$$

Δ

Example 18: *Finally, take* $g(x) = \cos x$, *with* $a = 1/2$ *and* $b = 0$,

$$2\cos^2(x\cos x) = 1 + \frac{1}{4}\sec x \ \rightarrow\ x = \cos^{-1}(1/2).$$

Δ

We could propose that students experiment with different types of functions in order to obtain cases similar to those presented in this section. However, it is more interesting to provide a general method for solving nonlinear equations that are solvable using Theorem 1. This is the purpose of the next section.

4. The change of variables method

So far, we have used Theorem 1 to derive new nonlinear equations with the same roots as solvable nonlinear equations. Next, we provide a method for tackling the inverse problem. Consider that we have a nonlinear equation

$$g(x) = 0. \tag{30}$$

If we find a change of variables $y = f(x)$ for which the nonlinear equation (30) is written as $x = f(y)$, then $f[f(x)] = x$, and by virtue of Theorem 1, $f(x) = x$. Therefore, the roots of the latter equation are also roots of the original equation (30).

Our proposal to the students is that they solve the nonlinear equations presented in the previous Section by using this method. Next, we present some of the solutions.

Example 19: *Rewrite the equation given in* (22) *as* $x + 1 = \log(\log x - 1)$ *and then take the change the variables* $y = \log x - 1$*, thus we have* $x + 1 = \log y$*. Thereby, we obtain the reduced equation* $x = \log x - 1$*, that can be solved using the Lambert W function as* $x = -W(-e)$*, as mentioned before.* Δ

Example 20: *Multiply by x both sides of equation* (29) *and take the change of variables* $y = x \sin x$*, thus the original equation is written as* $y \sin y = x$*. Therefore, the reduced equation is* $x \sin x = x$*, i.e.* $x = \sin^{-1} 1 = \pi/2$. Δ

Example 21: *Rewrite* (25) *as*

$$\log x + x - \frac{1}{2} = \frac{e}{x} = y,$$

where y is the new variable. According to this change of variables, calculate

$$\log y + y - \frac{1}{2} = \log\left(\frac{e}{x}\right) + \frac{e}{x} - \frac{1}{2}$$
$$= -\log x + \frac{e}{x} + \frac{1}{2} = x,$$

Therefore, $\log x + x - 1/2 = x$ *and* $x = \sqrt{e}$*, as aforementioned.* Δ

Example 22: *Perform in* (26) *the following change of variables*

$$y = \sin^{-1} x + x - \frac{\pi}{4} = \sqrt{1 - x^2}, \tag{31}$$

and calculate

$$\sin^{-1} y + y - \frac{\pi}{4} = \sin^{-1}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2} - \frac{\pi}{4}$$

$$= \cos^{-1} x + \sqrt{1-x^2} - \frac{\pi}{4}.$$

Applying the trigonometric identity $\sin^{-1} x + \cos^{-1} x = \pi/2$ *and* (31)*, we arrive at*

$$\sin^{-1} y + y - \frac{\pi}{4} = \frac{\pi}{4} - \sin^{-1} x + \sqrt{1-x^2} = x.$$

Therefore, $\sin^{-1} x + x - \pi/4 = x$*, and* $x = \sin(\pi/4) = 1/\sqrt{2}$. Δ

It is important to explain to the students that we can apply other changes of variables to reduce a given nonlinear equation to a solvable nonlinear equation. For instance, let us solve the last example with another change of variables.

Example 23: *Take the change of variables* $y = \sin^{-1} x$ *in* (26)*, thus we have*

$$y - \frac{\pi}{4} = \sqrt{1-x^2} - x. \tag{32}$$

Since $x = \sin y$*, we can rewrite the right hand side of* (32) *as*

$$y - \frac{\pi}{4} = \cos y - \sin y = \sqrt{2}\sin\left(\frac{\pi}{4} - y\right). \tag{33}$$

An apparent solution of (33) *is just* $y = \pi/4$*, hence* $x = \sin(\pi/4) = 1/\sqrt{2}$. Δ

The solutions to the remaining examples are left to the reader.

5. Conclusions

In the first section, we discussed the lack of an analytical approach in the teaching of nonlinear equation-solving at undergraduate level. Because of this, we have proposed a Problem-Based Learning methodology with several examples drawn from Applied Sciences to offer balanced learning between analytical and numerical methods. Analytical solutions are preferable when they are possible. However, when this is not the case, analytical approximations may be worthwhile in providing the initial iteration points for numerical methods, above all, when the nonlinear equation presents complex roots or multiple real roots. Also, they can provide a numerical approximation good enough for practical purposes. In addition, analytical approximations are relevant in order to study the behaviour of the solution with respect to the parameters involved in the problem.

Since analytical solutions to nonlinear equations are of considerable interest, we have seen how to increase the number of nonlinear equations that are analytically solvable by means of Theorem 1. This theorem states that if a root of the fixed-point equation $f(x) = x$ is known, then that root is also a solution of the equation $f[f(x)] = x$. Applying Theorem 1, we have derived a heuristic method to solve quartics. Further, we have given some examples of nonlinear equations for which the symbolic computer algebra does not yield an analytical solution. In addition, we have derived a change of variables for the inverse problem, i.e. to recover the original nonlinear equation $f(x) = x$ from $f[f(x)] = x$. Also, we have presented a great variety of examples to be proposed to the students as Problem-Based Learning.

However, it is worth noting that Theorem 1 may only be applied to a very specific kind of nonlinear equations. Therefore, most of the time numerical methods are unavoidable as root searching methods of nonlinear equations arising in Applied Sciences.

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