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A methodology for phenomenological analysis of cumulative damage processes. Application to fatigue and fracture phenomena

Alfonso Fernández-Canteli [a,*], Enrique Castillo [b], Sergio Blasón [c]

[a] Dep. of Construction and Manufacturing Engineering, University of Oviedo, Spain
[b] Royal Academy of Engineering and Royal Academy of Sciences, Madrid, Spain
[c] Division 9.4, Bundesanstalt für Materialforschung und -prüfung (BAM), Berlin, Germany

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ABSTRACT

Sample functions, i.e., stochastic process realizations, are used to define cumulative damage phenomena which end into an observable terminal state or failure. The complexity inherent to such phenomena justifies the use of phenomenological models associated with the evolution of a physical magnitude feasible to be monitored during the test. Sample functions representing the damage evolution may be identified, once normalized to the interval [0,1], with cumulative distribution functions (cdfs), generally, of the generalized extreme value (GEV) family. Though usually only a fraction of the whole damage evolution, according to the specific problem handled, is available from the test record, the phenomenological models proposed allow the whole damage process to be recovered. In this way, down- and upwards extrapolations of the whole damage process beyond the scope of the experimental program are provided as a fundamental tool for failure prediction in the practical design. The proposed methodology is detailed and its utility and generality confirmed by its successive application to representative well-known problems in fatigue and fracture characterization. The excellent fittings, the physical interpretation of the model parameters and the good expectations to achieve a complete probabilistic analysis of these phenomena justify the interest of the proposed phenomenological approach with possible applications to other cumulative damage processes.

1. Introduction and motivation

Most families of k-parametric cumulative distribution functions (cdf) are such that the definition of k different points of any distribution of such family allows them to be uniquely identified within the family. However, if the correspondence is not accurate, the approach fails and more points would be needed to have a reliable parameter estimation. This means that, in particular, the cdf can be identified within the family only with the determination of a very reduced fraction of its range, because it contains enough information for the identification to be possible. However, to make the estimation sufficiently precise, an adequate selection of this range or a sufficiently large number of data points are required.

Consequently, if the family of cdfs is theoretically and practically justified to model reality, we can expect good estimations and predictions based on this basic idea of a partial range observation. In this work, we present and discuss in detail a methodology to make this estimation possible and practical. This becomes especially relevant in the case of experimental tests in which only a limited range of data can be observed.

The evolution of many cumulative damage phenomena (fatigue, fracture, wear, creep, ratcheting, etc.) often exhibits sigmoidal (or quasi-sigmoidal, as explained later) shaped curves. In particular, different physical magnitudes used to describe the evolution of fracture and fatigue phenomena adopts such evolution pattern, irrespective of the material involved, as corroborated by the following examples:

- log(da/dN) vs. log(ΔK), CGR curves for an aluminum alloy at different R-ratios [1];
- Total strain in % vs. fatigue life ratio in plain concrete under compressive loading [2];
- log(vacancy concentration) vs. log(N) of cycles for deformed copper [3,4];
- Deflection vs. fatigue life ratio (%) for prestressed concrete beams [5];
- Damage (measured as acoustic emission) vs. fatigue life ratio (%) in concrete for different stress conditions [6];

* Corresponding author.
E-mail address: afc@uniovi.es (A. Fernández-Canteli).

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Nomenclature

a	generic crack size
a_0	initial crack size predicted from the model
a_{up}	asymptotic fatigue crack size
cdf	cumulative distribution function
f ()	probability density function
pdf	probability density function
CGR	crack growth rate
CMOD	crack mouth opening displacement
da/dN	crack grow rate
DCPD	direct current potential drop
FCGR	fatigue crack growth rate (da/dN vs. ΔK)GEVgeneralized extreme value
GLM	generalized local model
K_{IC}	fracture toughness
K_{max}	maximum stress intensity factor
MCMC	Markov chain Monte Carlo
N	generic number of cycles
N*	normalized number of cycles
N_f	number of cycles to failure
N_{up}	predictive number of cycles for the crack to grow hypothetically from the initial crack size a_0 to the asymptotic crack size a_{up}
NSF	normalized sample function
P	load
R	stress or load ratio
SIF	stress intensity factor
S	stress
S-N field	Wöhler field (fatigue material characterization in terms of stress vs. number of cycles)
W_F	fracture energy
W_F^*	normalized fracture energy
$W_{F,M}$	measured fracture energy
$W_{F,t}$	total fracture energy
$W_{F,t}^*$	normalized total fracture energy
$W_{F,NM}$	non-measured fracture energy
ΔK	stress intensity factor range
ΔK_{th}	threshold value of the stress intensity factor range. It is referred to the long-crack regimen in this work
ΔK_{up}	asymptotic upper limit of the stress intensity factor range
ΔK^+	normalized stress intensity factor range
ΔK_{eff}	effective stress intensity factor range
λ	location parameter of either the GEV distributions
δ	scale parameter of either the GEV distributions
δ	deflection in the load–deflection curve
δ_u	mid-span deflection at test stop
β	shape parameter of the GEV distributions
γ	scale parameter of the GEV distributions
κ	shape parameter of the GEV distributions
Ω	normalizing scaling parameter
ε	strain
ε_t	fatigue total compressive strain
$\varepsilon_{t,0}$	initial fatigue total compressive strain
$\varepsilon_{t,f}$	strain at test failure
ε_f	specific limit value of the deformation

- Pulse velocity vs. fatigue life ratio (%) in concrete for a particular loading condition [7], etc.;
- Other functions in fracture, as the evolution of the total strain in concrete [8], and in ratcheting [9,10].

Sometimes the sigmoidal or pseudo-sigmoidal shape is not conveniently recognized because only a fraction of the function sought can be recorded, as it usually happens because of the inherent experimental limitations, see for instance the CGR curves for an axle steel at different R-ratios [11] or the a-N curves for 2024T-351 [12], 2024T-3 [13] and A1 7075-T6 aluminum alloys [14]. In other cases, the scale, linear or logarithmic, chosen for the representation may not be the most convenient one, as it can be observed in the two examples concerning the reduction (in %) of the secant modulus vs. fatigue life ratio in concrete for different stress levels shown in [2], where the sigmoidal shape of the curves vanishes when the cycle ratio is represented in a logarithmic scale.

Cumulative damage processes are stochastic phenomena in which physical degradation, i.e. deleterious evolution or loss of physical properties, occurs and ends as an observable and identifiable predetermined terminal level of damage state, not necessarily failure [15–18]. They represent sample functions, i.e. realizations of stochastic processes, associated with the evolution of a physical magnitude. This is the case of the CGR curves, cyclic creep curves and many other examples in which the data assessment aims at obtaining an analytical truthfully description of such sample functions. On the other hand, experimental test results can be classified as sample data, i.e. independent random results, when the data assessment pursues merely to determine the lifetime distribution to reach a certain terminal state. This is the case of the data involved in the S-N field assessment [19–23]. According to the above, the probabilistic assessment of descriptive and predictive approaches requires two different methodologies of analysis, which can be unified based on Bayesian techniques.

A fundamental contribution in this paper consists in pointing out that the sigmoidal shapes are observed to be ubiquitous in practically all manifestations of phenomena related to damage irrespective of the technique applied to its measurement (total deformation, crack growth, loss of stiffness, frequency variation, acoustic emission, pulsatile speed, potential drop, thermography, etc). The consistent evolution of the corresponding physical magnitudes evidences the similarity among these manifestations of the damage processes and the underlying common statistical background behind them. All the above, besides the difficulty of approaching such phenomena from a microstructural perspective justify the convenience of resorting to robust phenomenological models, see Freudenthal [24], Freudenthal-Gumbel [25], [Bolotin 26–28] and Bogdanoff-Kozin [15–18]. The application of a common phenomenological approach supported by a general methodology applicable to all these damage processes is envisaged to infer the whole damage evolution. In this way, a reliable extrapolation (often as asymptotic extrapolation) is achieved, which allows us to access scenarios not attainable in the scope of the experimental program, or to interrupt prematurely the test to spare time and costs without noticeable loss of reliability of the output results. In this work, a phenomenological methodology is proposed and its successive steps are explained and justified to allow for a satisfactorily descriptive and predictive analysis of different fatigue and fracture phenomena.

The methodology allows the followings objectives to be achieved:

- To present a comprehensive methodology for handling damage evolution cases as phenomenological models identified as “sample functions” in the sense of stochastic process realizations. The normalized sample functions are represented analytically as cdfs, generally of the GEV family.
- To discern this type of phenomenological sample function models from those used to fit a set of single independent random data related to failure lifetimes. In this case, several specimens of a sample are

tested and a cdf of the GEV family is proposed as the regression function representing the median percentile curve (see Section 4.3).
- To acknowledge the ubiquitous character of the sample function methodology, its high-quality fitting capacity and reliability of these models, proving their suitability when applied to a variety of phenomena characterization (fatigue, fracture, rheological properties) and to different materials (metallic, quasi-brittle, polymers) provided the adequate physical variable is selected for the test monitoring.
- To emphasize the observed property of recovering the whole sample function, representative of the complete damage evolution process, from its fraction monitored during the test.
- To identify the methodology as the start of a new way to establish fracture- and fatigue-related models in a simple but reliable way that can be integrated into the frame of a general scheme of the practical design. In this way, the generalized local model (GLM) [29–30] is feasible to be applied for the assessment of the experimental results to ensure transferability to the practical design.

In what follows, an attempt is presented to provide such a systematic methodology to develop phenomenological models to describe fatigue and fracture properties analytically and to predict lifetimes in their application to practical design. The scheme followed in the paper is the following: in Section 2 some advantages when using phenomenological models are described and the background of the phenomenon is presented emphasizing the dual character of the damage processes. In Section 3, the proposed methodology is introduced and the steps to follow are described. In Section 4, the application of the approach to some examples of fatigue and fracture is illustrated followed by a discussion in Section 5. Finally, the conclusions are summarized in Section 6.

2. The dual descriptive and predictive analysis of the phenomenological probabilistic modeling of cumulative damage processes.

2.1. Phenomenological vs. empirical and micromechanical models for cumulative damage

The evolution of a reference physical variable in cumulative damage processes often follows a similar sigmoidal or quasi-sigmoidal shape irrespective of the monitoring technique applied. This points out the common features controlling a broad spectrum of different damage phenomena and suggests their possible modeling using a unique general phenomenological methodology.

The utility of phenomenological models was already envisaged by Freudenthal [24], which stated that ”*by applying the fundamental rules of the theory of probability many of the experimentally established relations between the principal variables can be theoretically deduced from the purely formal assumption of the existence of a statistical distribution function*”. Generally, fracture and fatigue models can be assumed to consist of a high number of primary resistant elements, to which a virtual statistical distribution of the mechanical resistance properties is associated, see Bolotin [26–28]. This justifies the possible statistical background arising from the equilibrium between the critical minimum resistant energy and maximum external loading energy. The maxima or minima character participating in the fatigue and fracture phenomena is determined by the mechanical behavior of the material at a macroscale.

The contribution of Bogdanoff and Kozin [15–18] is very meritorious. They establish, for the first time, a methodology for the analysis of cumulative damage problems using phenomenological models. They focus the attention on the analysis of the terminal distribution of fatigue lifetimes assuming Markov chain models, paying perhaps less attention to the modeling of sample functions from which, no particular solution of a predetermined type is proposed, which leads to less general models implying a high number of parameters.

2.2. Descriptive and predictive probabilistic analysis of cumulative damage phenomena

The probabilistic modeling of cumulative damage phenomena implies their descriptive and predictive analysis. This twofold scenario requires the definition of a sample function describing the stochastic progress of the cumulative damage and that of the cumulative distribution function providing the variability of the ultimate damage state predefined as an ultimate limit state.

When the crack size is assumed as the physical magnitude representing damage accumulation, the main objective of the crack growth modeling focuses on reproducing the physical reality of the crack growth phenomenon from the very beginning of the process, here denoted “back-extrapolation region”. This contrast with providing merely a good fit of the data recorded over the test. In this way, the crack growth behavior can be analytically inferred in phase I of the fatigue process, where the crack nucleates and evolve to a propagating one, i.e. in a region where, at least for the moment, the crack size cycles is not measurable as a function of the number of cycles.

In the first scenario, the particular evolution of the physical or main (dependent) variable (such as crack size, deflection, CMOD, crack growth rate, potential drop, acoustic emission, pulse velocity) is monitored as a function of a normalizing reference (independent) variable (such as load, fracture energy, SIF, lifetime) over the test, see [8,31–33], and identified as a stochastic sample function of the cumulative damage [15–18,26–28]. The sample functions are monotonic non-decreasing functions, such that, once conveniently normalized, grow from 0 to 1 so that they represent, by definition, cumulative distribution functions (cdfs). Though these cdfs cannot be immediately identified, they happen to be suitably recognized as generalized extreme value distribution (GEV) family, i.e. either Fréchet, Gumbel and Weibull distributions [34–35]. A possible explanation is not sufficiently cleared yet. The complex damage evolution can be understood as a succession of competitive states ruled by the maximum values of the load and the minimum resistance of the material. At a local basis, the local stable failures are then governed by a statistical law of extreme values under the assumption of the weakest link principle and independent statistical conditions applicable to a large number of primary elements, as suggested by Bolotin [26–28], that constitutes the material bulk in the test specimen or component. Due to the complexity implied in the analysis of such a process, a phenomenological model is envisaged as the suitable way to solve the problem according to Freudenthal and Freudenthal-Gumbel [24–25]. Similar reasoning is adduced by Bogdanoff and Kozin, when dealing with the assessment of crack growth data using a high-parametric model, see [15] page 4, which state “*Conceptually, there are at least an accountable infinity of sample functions associated with a given cumulative damage process forming a family of sample functions. Such a family, whose samples evolve in a probabilistic manner, constitutes a random (stochastic) process*”.

The second scenario is concerned with the inherent randomness exhibited by the fracture main variable, i.e. driving force, or the fatigue lifetime up to the terminal damage state, which must not be necessarily identified with physical failure. In all these cases, due to the assumed validity of the weakest link principle, the experimental results prove to be adequately fitted using a Weibull cumulative distribution function (cdf) as a particular case of the generalized extreme value family, see [34,35]. Note that in this second option no monitoring test is required. In the fatigue case, it corresponds to the conventional probabilistic fatigue analysis as given by the S-N field, when several stress levels are applied in the tests, see [19].

Under these assumptions and the features observed from the normalized sample functions, the statistical background of the cumulative damage process can be accepted. Both scenarios highlight by themselves the great significance of the extreme value statistical theory in the probabilistic analysis of cumulative damage phenomena (see [34,35]).

2.3. The Markov chain model of Bogdanoff-Kozin for damage accumulation description

Bogdanoff-Kozin are recognized as the pioneers of the stochastic analysis of fatigue using phenomenological models [15–18]. They identified shrewdly that a cumulative damage process is a succession of damage states, such that any damage state can be related back not only to the immediately precedent damage situation but to an initial state that can be attributed to those microstructural random characteristics of the material exhibited by the current specimen tested. This leads to the adoption of Markov-chain models to represent cumulative damage processes. Accordingly, an intermediate or even final state in the damage evolution can be inferred simply from a short register of the initial damage evolution, which provides the necessary information for reproducing the process.

The probabilistic sample function model proposed by Bogdanoff-Kozin [15–18] lacks recognition of the test register as a mere fraction of the whole sample function, which describes the cumulative damage process including its down- and upwards-extrapolation. In the particular case of crack growth analysis, this impedes:

- to reconstruct the whole crack growth sample functions under an analytical expression,
- to describe and predict in a probabilistic way the evolution of the crack size for initial crack sizes less than that used in the test program and
- to describe and predict the distribution of the total lifetimes to failure for initial crack sizes less than that used in the test program.

Despite these limitations, the Bogdanoff-Kozin contribution is fundamental in what concerns the sample function modeling and the theoretical fundamental principles of the Markov chains models. They present, presumably for the first time, a probabilistic treatment of the evolution of the cumulative damage processes pointing out the theoretical statistical background inherent to these phenomena. In fact, their insight into the probabilistic models of cumulative damage paves the way to confirm the denoted "micromechanical genetic heritage" that allows a surprisingly accurate description of the damage phenomenon evolution to be achieved using statistical functions even from a small fraction of the total process.

2.4. Extension to the damage cases represented by fracture

In addition to the typical applications of phenomenological models found in the fatigue field, the cumulative damage concept can be extended straightforwardly to the fracture case, as will be shown in the following. The fatigue lifetime is then replaced by the suitable magnitude controlling the test (such as displacement, CMOD, etc.).

3. The proposed methodology

The analysis of the proposed approach encompasses the following steps, see Fig. 1:

i) *Physical interpretation, identification of variables and bounds*

The first step consists of the physical interpretation of the sigmoidal or quasi-sigmoidal shaped phenomenon from the monitored test register, which has to be analyzed as a part of the stochastic sample function representing the complete cumulative damage process. This implies recognition of the model variables, i.e. the dependent or physical variable, associated with the damage state evolution along the test (fatigue crack growth, fatigue crack growth rate, fatigue total strain, deflection, CMOD, etc.), as a function of the independent or reference variable (number of cycles, driving force, fracture energy, etc.). This function is feasible of being normalized to the interval [0,1] after recognition of the lower and upper bounds that limit its existence field, which are model parameters. Usually, the test register represents only a fraction of the whole damage process in which the initial and censored final phases may not be able to be recorded. Nevertheless, the whole cumulative damage process can be reconstructed beyond the limits established by the test conditions with the adequate modelling proposed.

ii) *Definition of the normalized sample function (NSF)*

The independent or reference variable in the sample function is normalized according to its bounds. Usually, the bounds of the reference variable are wider than those set in the test. Actually, the proposed approach provides the whole sample function including its experimentally inaccessible domains. In the lower region, the inaccessibility usually arises from the incompatibility of measurement of the physical variable with the applied experimental technique, while in the upper

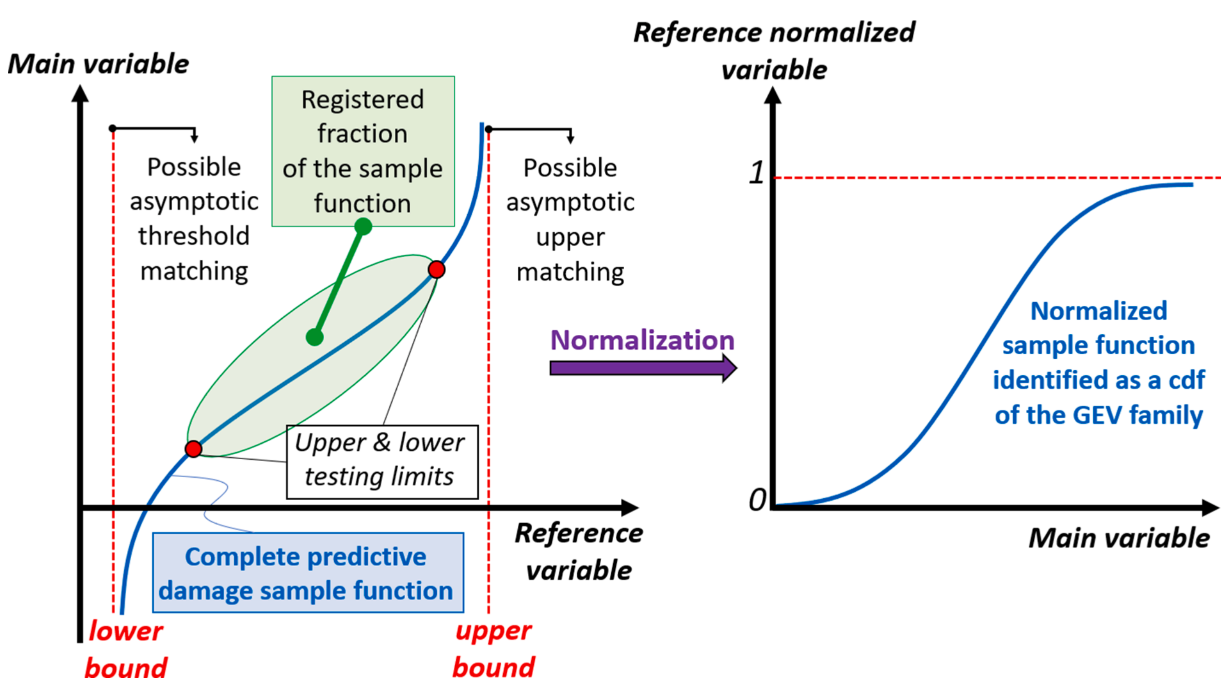


Fig. 1. Schematic representation of the recognition of the complete damage process from only a partial register of the test course.

region a fracture mechanism, not identifiable with the current damage process, may emerge as an interruption of the running damage course causing the abrupt failure. This is the case of fatigue crack growth.

Once the reference variable is normalized, the ensuing NSF represents a monotonically non-decreasing function ranging in the interval [0,1]. Accordingly, it can be identified as a cumulative distribution function (cdf), in principle from an unknown family. As a result, the normalizing reference variable, which extends beyond the test limits, is identified as an index of cumulative damage related to the probability of the corresponding cdf. This denotes the potential statistical background of the approach.

iii) *Identification of the normalized sample function (NSF) as cdf of the generalized extreme value (GEV) family*

As already pointed out by different researchers [3,4,15–18,24–28], there are sound reasons to admit the statistical background of the cumulative damage processes. A high number of "primary elements" whose specific resistance follows a random distribution are subject to a succession of load situations to each of which a progressive weakening or even the weakest link principle be applied. The microstructure of the material, as a homogeneous messo-scale media, determines the law of fracture or fatigue stable crack growth over the whole process and explains why the presupposed damage evolution previous and following the test can be inferred only from a partial register of the physical variable.

Furthermore, because the extreme value character of both the minimum resistance and relative maximum load are the governing factors in the succession of damage state progress representing the cumulative damage process, it is reasonable to assume that the normalized sample function identified as a cdf, belongs to the generalized extreme value family (GEV) [34,35]. Only three possible functions, namely, the Weibull, Gumbel and Fréchet ones, constitutes the GEV family where the Gumbel one can be contemplated as the boundary between the other two families. Note that the presence of both lower and upper bounds determines the existence field of the reference variable, while the particular way of approaching the bounds gives clues to recognize the eligible type of function identifiable with the normalized sample function. This means that the justification for identifying the problem as a case of maximum or minimum values may be only decided after performing a comprehensive analysis of the problem studied, not always clear. The selection of the suitable extreme distribution function can also follow as the optimal solution when the general expression of the GEV family is applied to the fitting process.

The location, scale and shape parameters of the cdf can be directly attributed to the particular physical and mechanical material characteristics such as microstructure features, grain size and shape distribution, grain boundaries and failure mechanism besides specimen geometry and test type that determine the phenomenon evolution. A feasible interpretation of the physical meaning of these parameters is generally achieved rather from the latter final version of the model when applied to experimental results.

iv) *Estimation of the model parameters*

Model completion includes identification of the dependent and independent variables, its bounds as related parameters and the GEV model parameters, three parameters in the Weibull or Fréchet distributions and two in the Gumbel case. Either the least-squares or the maximum likelihood procedures, indistinctly, can be applied to the parameter estimation. In this way, the analytical solution of the sample function is determined allowing the cumulative damage process to be wholly defined extending its reliable definition even beyond the limits of the test.

Note that up to this point, although a statistical cdf distribution is implied in the approach, the procedure represents simply a deterministic solution.

v) *Probabilistic analysis*

The probabilistic analysis of the phenomenological approach proposed can be performed in a simple way based on a dual viewpoint of the variability of the cumulative damage process: That concerning the normalized sample function, i.e. as that focused merely on the sample function shape when it is assumed as a GEV function, and that concerning the estimation of the statistical distribution of the terminal state (ultimate limit state) associated with the reference variable.

In the first case, each experimental sample function is fitted to a GEV function as defined by the two lower and upper bounds and the three (Weibull, Fréchet) or two (Gumbel) GEV parameters. Under the premises of the first viewpoint, only the variability of the latter is of concern. In the second case, a traditional analysis, considering the terminal data as independent random results, is performed assuming, according to the features of the problem handled, the suitable GEV function, such as a Weibull one. This corresponds to the conventional case of the probabilistic analysis of the fatigue S-N field, in which the damage evolution information is not available due to the absence of a test register, focusing the analysis merely on the terminal results [19].

vi) *Bayesian concept*

Unlike conventional methods, the Bayesian techniques [36–37] consider the model parameters as random variables. In the proposed methodology, this implies the consideration of the lower and upper bounds limits of the reference variable as model parameters along with those defining the normalized sample function, identified as a GEV function.

In this way, a more advanced and coherent assessment is achieved under a global probabilistic concept by unifying the dual assessment suggested above. Furthermore, the basis for a more complete and demanding approach to the cumulative damage concept is established providing a deeper comprehension of the physical background of the phenomenon studied.

Given the inherent stochastic character of the crack growth process for the specimens of a sample, the Bayesian theory based on MCMC (Markov chain Monte-Carlo) techniques, implemented in the OpenBUGS software, see Castillo et al. [38,39], can be applied to provide the univariate probability distributions of the intervening parameters on the a-N curve or directly their multivariate distribution, which goes further than the confidence bands of the parameter estimates. In this way, an important advance for the probabilistic assessment and prediction of cumulative damage phenomena in the practical design is achieved, particularly when the damage tolerance concept is applied. Due to the lack of space, the application of the Bayesian technique to the probabilistic assessment of a particular cumulative damage case is here omitted but announced as a future work.

4. Practical examples of application

The potentiality of the phenomenological procedure proposed in this work is illustrated when applied to the descriptive and predictive assessment of different damage cases of fatigue and fracture. In all the examples, very satisfactory fits are achieved by identifying the stochastic sample functions with statistical cdfs belonging to the generalized extreme value (GEV) family, i.e. Weibull, Gumbel and Fréchet functions, see [31–35].

4.1. *GEV sample functions for the analysis of fatigue phenomena*

In what follows two examples of fitting fatigue-related models are presented using the phenomenological procedure based on cdfs of the GEV family.

4.1.1. *The crack growth rate model of Castillo et al.*

In general, the experimental crack growth rate curves show a sigmoidal shape trend when the tests are conducted sufficiently offside of the Paris region though this trend is often unnoticeable in the upper region, see Fig. 2. This and the intuitive representation of fatigue as a succession of stable crack growth steps related to the weakest link principle and, as a result, to the statistical extreme value theory, induced

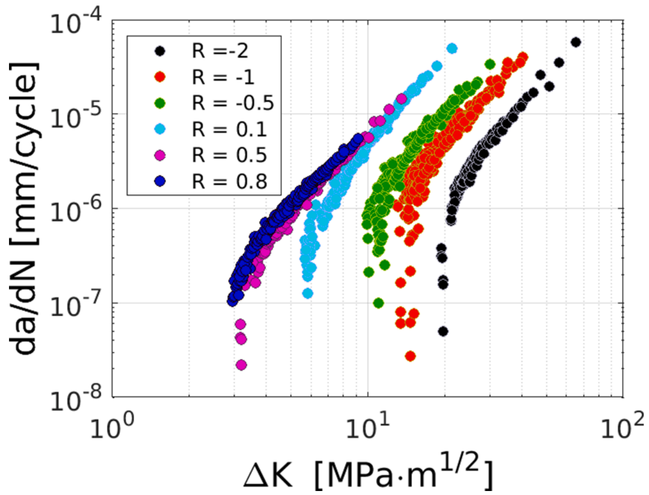


Fig. 2. Experimental results of fatigue crack growth rate from Pokorný et al. [11].

Castillo et al., see [32], to propose a phenomenological model based on the normalization of the sample function da/dN-ΔK and its identification as a cdf of the Gumbel family, as seen in Fig. 3.

When this proposal is contemplated as a particular application case of the methodology presented in Section 3, the procedure can be systematized according to the following steps for the assessment of crack growth rate results:

i) *Physical interpretation, identification of variables and bounds*: The sigmoidal shaped sample function relating fatigue crack growth rate to stress intensity factor (SIF) range ΔK is recognized. The sample function derived point by point from the *a-N* crack growth curve, represents a censored record, which is interrupted by a static fracture mechanism extraneous to the crack growth process, that impedes the crack growth progress to continue virtually beyond the real fracture towards infinity. The independent variable is the SIF ΔK with variability range [ΔK_{th}, ΔK_{up}], where ΔK_{th} and ΔK_{up} are the bounds, which, as model parameters, prescribe the existence interval of the variable ΔK. The dependent variable is the crack growth rate da/dN with a variability range [0,∞] i.e. [-∞,∞] on the logarithmic scale. The representation of the sample functions succeeds on log–log scale.

ii) *Definition of the NSF:* Once the reference variable ΔK^{+} is normalized to the interval [0,1] as Eq. (1):

$$\Delta K^{+} = \frac{log(\Delta K) - log(\Delta K_{th})}{log(\Delta K_{up}) - log(\Delta K_{th})} \tag{1}$$

The NSF is represented by a cdf of the dependent variable: $\Delta K^{+} = cdf\left[\log\left(\frac{da}{dN}\right)\right]$.

iii) *Identification of the NSF as a cdf of the GEV family*: A Gumbel cdf is proposed in [32] to fit the crack growth rate (CGR), as providing the best fitting in the particular cases evaluated in the variability range of the variable log (da/dN):

$$\Delta K^{+} = \frac{log(\Delta K) - log(\Delta K_{th})}{log(\Delta K_{up}) - log(\Delta K_{th})} = exp\left[-exp\left(\frac{\lambda - \log\frac{da}{dN}}{\delta}\right)\right] \tag{2}$$

where the model parameters are ΔK_{th}, ΔK_{up} and λ and δ as the Gumbel location and scale parameters, respectively. Alternatively, the non-dimensional parameters $\Delta K^{*} = \Delta K/K_{Ic}$, $K^{*}_{th} = \Delta K_{th}/K_{Ic}$ and $K^{*}_{up} = \Delta K_{up}/K_{Ic}$ can be used without modifying the definition of the normalizing variable ΔK^{+}.

iv) *Estimation of model parameters*: The least-squares technique is applied by minimizing the sum of squared errors measured in the ΔK scale, see [32]:

Minimize Q (λ, δ, ΔK_{th}, ΔK_{up}) given as, see Eq. (3):

$$Q = \sum_{i-1}^{n}\left\{log\Delta K_i - log\Delta K_{th} - \left(log\Delta K_{up} - log\Delta K_{th}\right)\cdot exp\left[-exp\left(\frac{\lambda - \log\frac{da}{dN}\Big]_i}{\delta}\right)\right]\right\}^2 \tag{3}$$

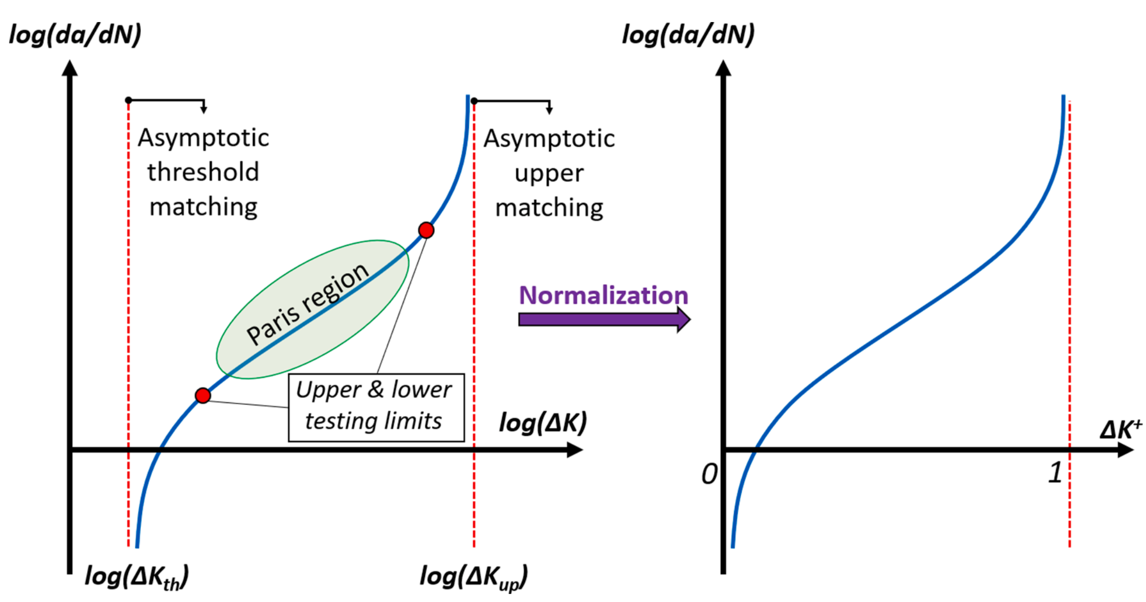


Fig. 3. Schematic explanation of the CGR model proposed by Castillo et al. [32].

where $\left.\frac{da}{dN}\right]_i$ and ΔK_i, i = 1,2,…, n are the data points.

A possible improvement of Eq. (2) to take into account crack closure effects (ΔK_{eff}) due to plasticity or others, could be envisaged by incorporating the corrections already arranged into other current models as NASGRO [40,41].

The model is applied to the fatigue CGR results from the experimental campaign carried out by Pokorny et al. on EA4T railway axle steel for different R stress ratios [11]. The use of K_{max} instead of ΔK as independent variable seems to be advantageous providing a unique ΔK_{th} irrespective of the R ratio, at least for $-1 < R < 0.5$, as shown in Fig. 4(a) and (b), where the assessed CGR curves have been represented in the normalized and natural modalities, respectively. The assessment of the CGR curve is facilitated taking into account the linearity of the Gumbel cdf when represented in log–log scale of the reference variable, i.e. ΔK or K_{max}, see Fig. 4(c). A detailed discussion about the interpretation of the R influence and the influence of the normalizing variable selection is given in [42].

v) *Probabilistic analysis:* To implement the methodology proposed by Castillo et al. for the assessment of experimental results from FCGR tests, the free software code ProPagation, to be downloaded from [43], is developed. It provides a user-friendly, intuitive environment using graphical user interfaces (GUI) in Matlab.

4.1.2. The total strain curves of plain and steel-fiber reinforced concrete

As a final fatigue modeling case using the proposed phenomenological methodology, the evolution of the total strain ε-N or cyclic creep curve under registered compressive fatigue of steel-fiber reinforced concrete is investigated as a stochastic sample function of cumulative damage, see Fig. 5 and [31,44]. It includes also the lifetime prediction from only an initial fraction of the registered strain evolution after the planned and monitored interruption of the test. Once the total strain curve is fitted as a sample function, an ultimate fatigue limit criterion for practical design is derived from the completed total strain ε-N curve.

i) *Physical interpretation, identification of variables and bounds*: The sigmoidal shaped sample function relating the fatigue total compressive strain ε_t and the applied number of cycles, N, is recognized from the test record. The sample function is derived point by point from the ε_t -*N* total strain curve. The total fracture can be imagined to continue virtually until the total strain is approaching asymptotically to infinity, see Fig. 6.

The independent variable is the number of cycles, N, with variability interval [0, N_{up}] where N_{up} represents the asymptotic number of cycles to achieve unlimited total strain (not reachable in the test). Without noticeable error, it can be approximated by N_f, i.e. the final number of cycles in the corresponding test. The dependent variable is the total compressive strain ε_t with a variability range: [$\varepsilon_{t,0}$,∞], where ε_{t0} is the initial total strain (see below). The latter may be discarded as a model parameter by assuming it to be, *a priori,* the total strain under the static load using the Young's modulus of the material. The representation of the sample functions succeeds on natural-natural scale.

ii) *Definition of the NSF:* Once the normalizing variable: N* is defined

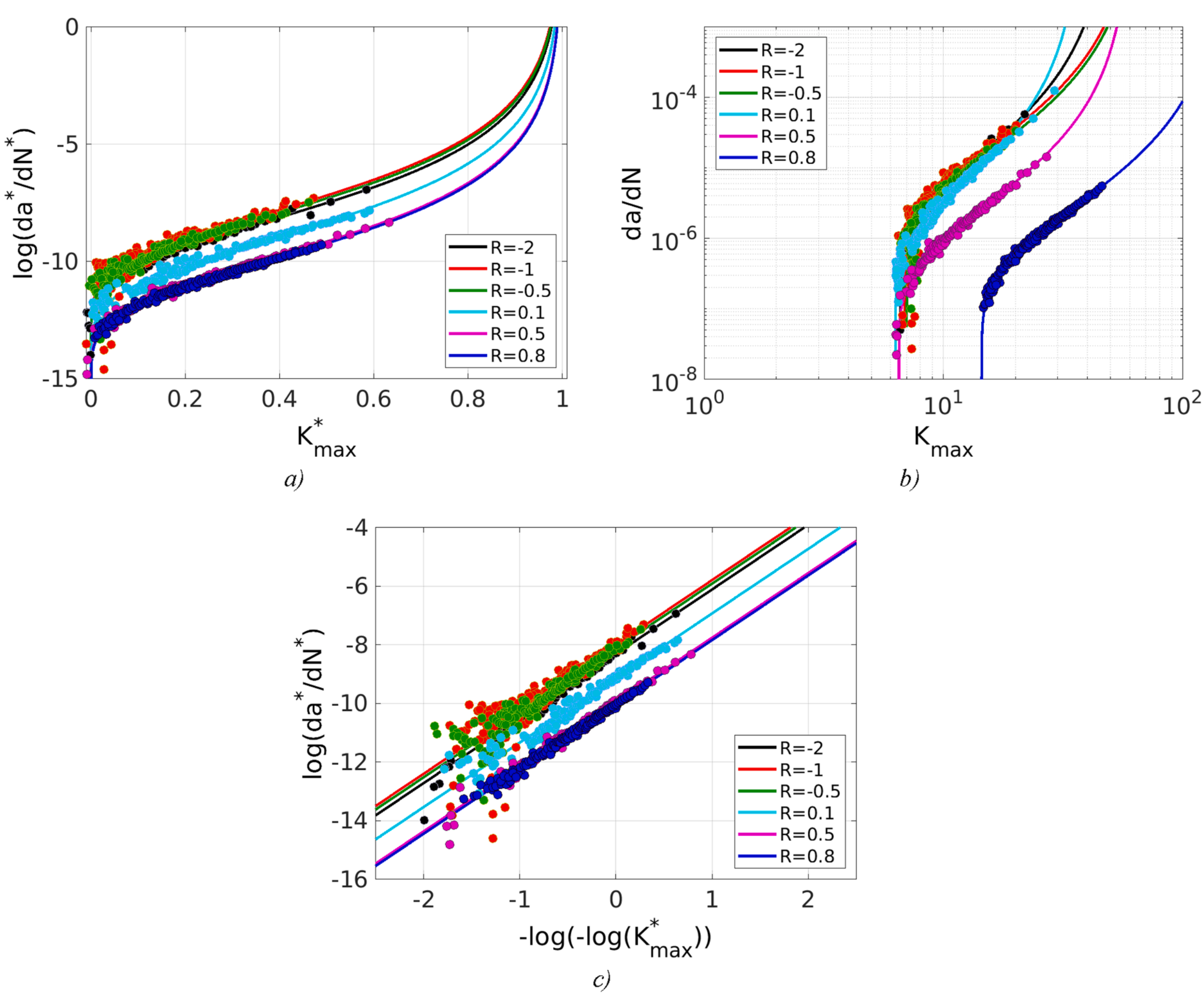


Fig. 4. Assessment of crack growth rate results from Pokorný et al [11] using the phenomenological approach proposed: (a) normalized CGR curves, (b) CGR curves in log–log scale (c) representation on Gumbel paper (log-(log–log) scale). From [42].

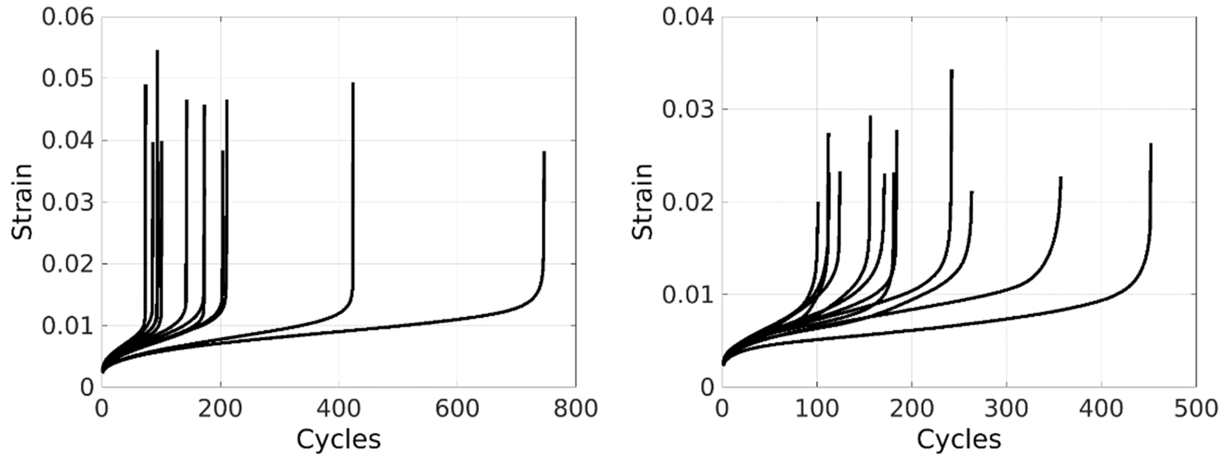


Fig. 5. Typical ε - N curves for steel-fiber reinforced concretes with different fiber content (plain concrete on the left; H15 on the right). From [44].

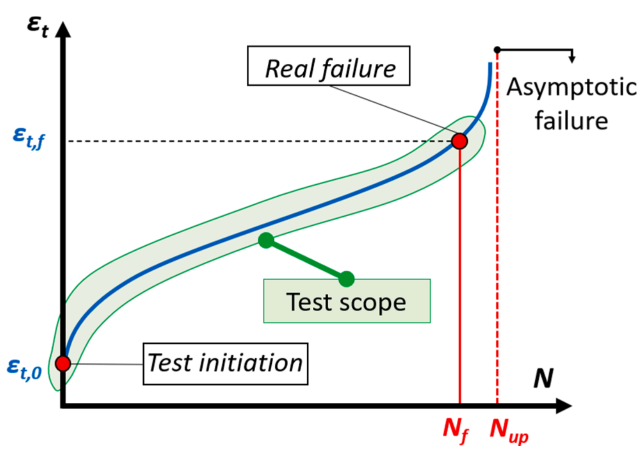


Fig. 6. Schematic explanation of the Weibull model for fatigue compressive curves in concrete.

as Eq. (4):

$$N^* = \frac{N}{N_{up}} \tag{4}$$

in the interval [0,1], the NSF is represented by a cdf of the dependent or physical variable: $N^* = cdf[\varepsilon_t]$.

iii) *Identification of the NSF as cdf of the GEV family:* A three-parameter Weibull cdf is proposed to fit the total strain in accordance with the variability range of the variable [ε_{t0},∞]. Following former definitions, the sample function equation becomes, see Eq, (5):

$$N^* = \frac{N}{N_{up}} = 1 - exp\left\{-\left[\frac{\varepsilon_t - \varepsilon_{t,0}}{\delta}\right]^\beta\right\};\ \varepsilon_t \geq \varepsilon_{t,0} \tag{5}$$

where the model parameters are N_{up} and $\varepsilon_{t,0}$, δ and β as the Weibull location, scale and shape parameters, respectively.

iv) *Estimation of the model parameters*: The least-square technique is used by minimizing the sum of squared errors measured in the N scale, see [31]:

Minimize Q($\varepsilon_{t,0}$, δ, β, N_{up}) given as Eq. (6):

$$Q = \sum_{i=1}^{n}\left\{\beta\left[\log\left(\varepsilon_{t,i} - \varepsilon_{t,0}\right) - \log(\delta)\right] - \log\left[-\log\left(1 - \frac{N}{N_{up}}\right)\right]\right\}^2 \tag{6}$$

where $\varepsilon_{t,i}$ and N_i, $i = 1,2,\ldots,n$ are the data points

v) *Probabilistic analysis*: The proposed phenomenological methodology is applied to provide the analytical fitting of the ε_t - N curves from the experimental campaign reported in [44]. In Fig. 7, two representative cases of the complete assessment performed in [31] for the fatigue strain evolution of steel-fiber reinforced concretes with different fiber content (plain concrete and H15, i.e. 15% fiber content) are shown. On the left column, the individual fits of the normalized ε_t - N curves using Weibull cdfs are represented for either particular fiber content case, while on the right column, a comparison is visible for all the specimens pertaining to the sample of the corresponding concrete type. In the latter case, the scatter among the normalized ε_t - N curves is noticeable evidencing the differences in the damage evolution occurring for the different specimens of the same sample.

Besides the interest in defining the cumulative damage evolution as a first step for a complete probabilistic definition of phenomena, the variability assessment of the terminal lifetimes, N_{up}, in a sample is of paramount importance in the practical design. The lifetime prediction can be performed only from the final lifetimes of the sample without the strain evolution record, i.e. without sample function assessment, to be required. This is the case when the probabilistic S-N field of the material is determined when only this information is considered of utility in the design. Nevertheless, higher reliability of the lifetime prediction is achieved when, additionally, the registered strain evolution is available. Fig. 8 (left) shows the Weibull cdfs obtained for the two samples (plain concrete and H15) reported in [44], whose normalized ε - N curves are represented in Fig. 7. No test results are available on further stress ranges in this experimental program could be extended straightforwardly to derive the whole probabilistic S-N field analysis by applying the model proposed by Castillo and Canteli [19].

The proposed phenomenological approach contributes to some advantageous innovations of the test methodology in testing and practical design. Among them:

a) The fatigue ultimate limit state criterion of the material can be determined as a specific limit value of the deformation, ε_f, defined as the intersection of a tangent to the cdf in the middle region (for instance at $N^* = 0.632$ that represents a characteristic value of the Weibull distribution, as the sum of the location plus scale parameters, i.e. $\lambda + \delta$, from Eq. (8), and $N^*=1$). In this way, the ultimate fatigue limit state criterion, i.e. ε_f, can be obtained directly from the estimated Weibull parameters in the experimental program, see Fig. 8 (right), to be applied in the practical design as a representative damage value preceding an unacceptable degree of the deterioration in the material

b) It deserves to mention the possibility of premature interruption of the fatigue tests to achieve a favorable shortening of test duration and cost, see Fig. 9. In this way, within certain limits, a reliable prediction of the remaining analytical sample function till failure is achieved from the registered test data, which represents only a fraction of the whole fatigue cumulative damage process. This includes, obviously, the estimation of

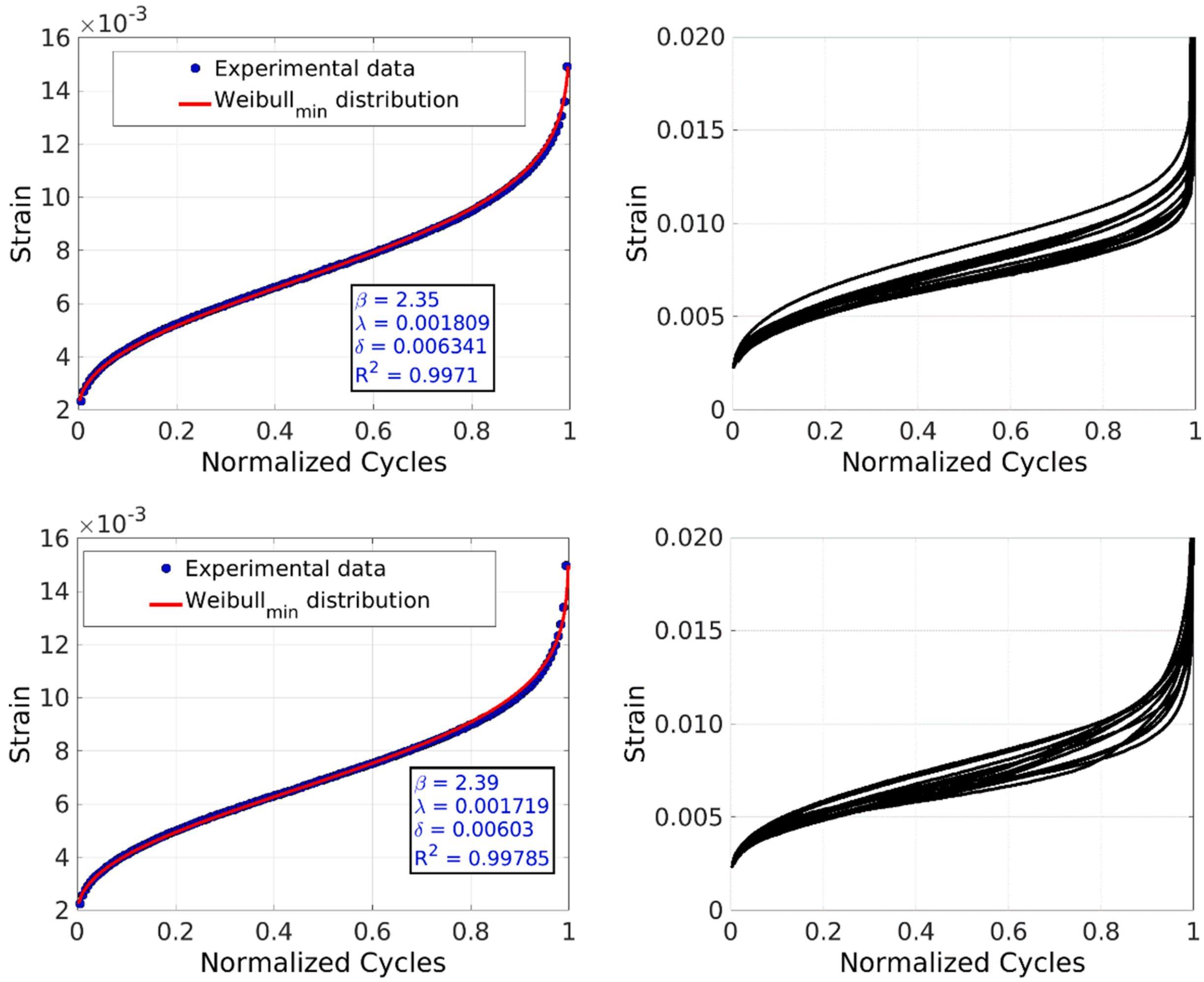


Fig. 7. Normalized ε - N curves for steel-fiber reinforced concretes with different fiber content (top: plain concrete, bottom: H15) fitted as Weibull cdfs. Left) Typical individual fitting; right) Variability for the sample tested. From [31]

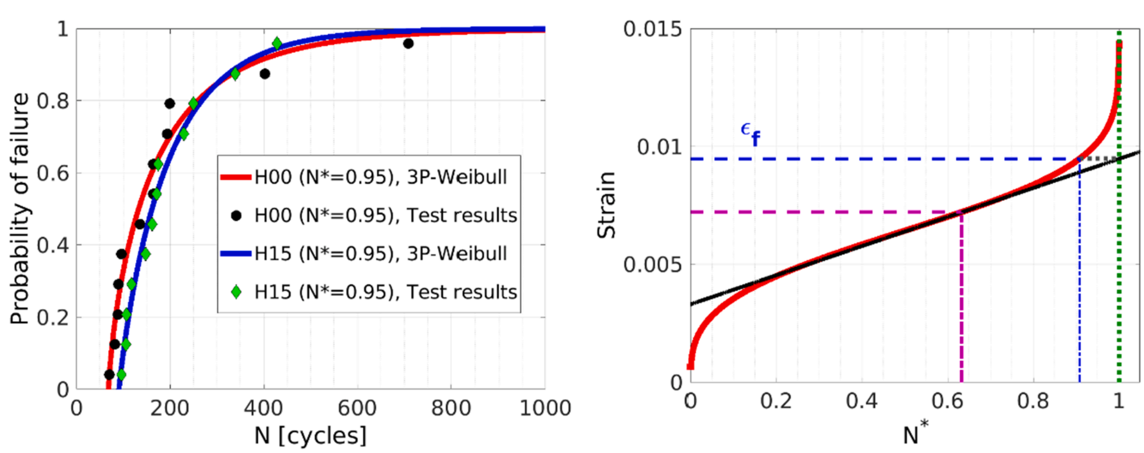


Fig. 8. Fatigue terminal lifetime distribution for H0 (plain concrete) and H15 (steel fiber reinforced concrete with 15% fiber content) and definition of the limit state strain in ε_t -N curves according to the model proposed. From [31].

the predictive lifetime. The utility of this technique is already confirmed when applied to the experimental results of concrete under fatigue compressive loading, see [31], where it represents a complete alternative to the Sparks-Menzies equation since it provides the explanation to the suitability of this law and overcomes its meaning giving an analytical solution to the prediction.

Note that such a reliable extrapolation, or rather reconstruction, of the damage process is only sustainable by the existence of the “microstructural genetic heritage”, i.e. the predetermination of the shape of the damage evolution from the underlying microstructure of the material, that support the suitability of the proposed type and shape of the sample function. The comparison of the reconstructed damage sample from the

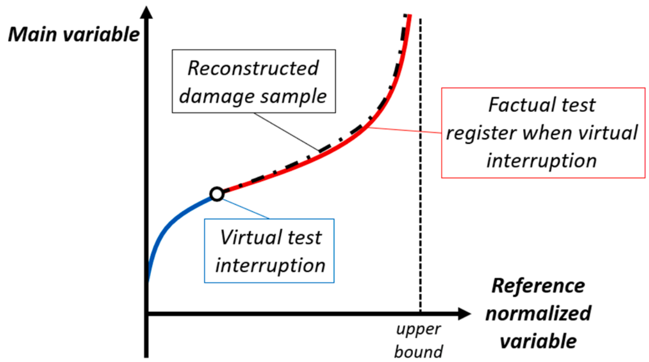


Fig. 9. Reconstruction of the damage process after virtual interruption of the test register and comparison with the real registered damage evolution for compressive fatigue of concrete [31].

registered test data till the virtual interruption and the real damage evolution confirms the reliability of the extrapolation, see Fig. 9, [31].

4.2. GEV sample functions for the analysis of a fracture phenomena

In what follows one example of fitting fracture-related models are presented using the phenomenological procedure based on cdfs of the GEV family.

4.2.1. The fracture energy (P-δ) model in concrete

The behavior of concrete in a 3-point bending (3 PB) test under monotonic load is investigated. It implies the experimental record of the load–deflection P- δ curve over the tests as a representative measurement of the damage evolution and, subsequently, the analytical fitting using the above phenomenological methodology, see Fig. 10. Instead of the deflection δ values, the CMOD results are recommended for future results assessment.

Since the test must be interrupted at a certain deflection, δ_u, of the loading process due to the limited affordable specimen deflection [45], the missing record of the fraction of the fracture energy remaining till the virtual end of the monotonic fracture process must be determined by asymptotic matching, see Fig. 11.

The recorded load–deflection P_i - δ_i experimental data are transformed into the fracture energy results of the W_i - δ_i curve according to Eq. (7):

$$W_{F,i} = W_{F,i-1} + \frac{P_i - P_{i-1}}{2}\Delta\delta_i \cong \sum_{i=1}^{n} P_i\delta_i \tag{7}$$

The scheme of the whole evaluation process implies the following steps:

i) *Physical interpretation, identification of parameters and bounds*: The cumulative area under the experimentally registered load–deflection P-δ curve is recognized as representing in a linear-linear scale the current supplied fracture energy, W_F, over the test as a function of the registered mid-deflection, δ, of the 3 PB specimen. The reference normalizing variable is the supplied fracture energy, W_F, bounded in the interval [0, $W_{F,t}$], where $W_{F,t}$ is the total energy to be supplied when the test could be continued virtually up to failure. The dependent physical variable is the mid-deflection δ of the 3 PB specimen with asymptotic range variability [0,∞].

ii) *Definition of the NSF:* The normalized fracture energy is defined as Eq. (8):

$$W_F^* = \frac{W_F}{W_{F,t}} \tag{8}$$

as a function of the dependent or physical variable, δ, in the interval [0,1].

iii) *Identification of the NSF as a cdf of the GEV family*: The normalized $W_F^* - \delta$ curve represents a monotonically non-decreasing function, whose sigmoidal shape tends asymptotically to unity. Consequently, it is, by definition, a cdf, which is supposed to pertain to the GEV family. Because the non-negligible contribution of the upper tail of the W-δ curve to the total fracture energy and the initial energy value, $W_F = 0$, implying a lower bound in the distribution, only a Fréchet distribution, as a particular case of heavy tail functions, emerges as the uniquely eligible GEV sample function candidate. This is confirmed, see below, using objective criteria based on optimization of the function fitting when resorting to the general expression of the GEV family in the fitting process. The negative sign of the κ-parameter excludes the alternative Gumbel and Weibull distributions, see [8,34,35]

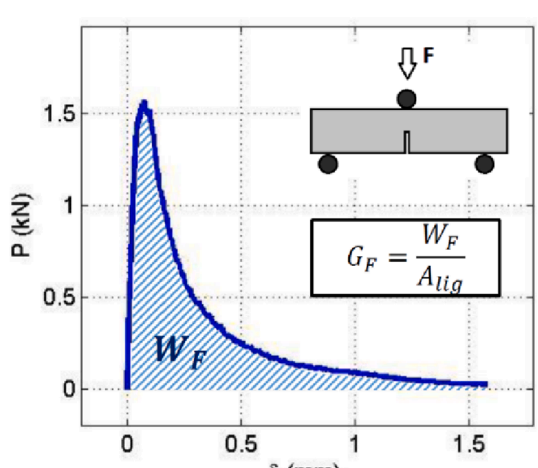
iv) *Estimation of the model parameters*: As illustrated in Eq. (9), the least-square technique is used by minimizing the sum of squared errors measured in the W_F scale, see [8]:

Minimize $Q(W_{F,T}, \lambda, \gamma, \beta)$ given as

$$Q = \sum_{i-1}^{n} \left\{\beta[\log(\gamma) - \log(\delta_i - \lambda)] - \log\left[\log\left(W_{F,T}\right) - \log\left(W_{F,i}\right)\right]\right\}^2 \tag{9}$$

where $W_{F,i}$ and δ_i, $i = 1,2,\ldots, n$ are the data points

Since the normalized fracture energy curve $W_F^* - \delta$ is identified as a Fréchet cdf, the normalized P-δ curve, as its derivative, represents a probability density function (pdf) of the same family. This allows an alternative procedure to be applied using the registered data of the P-δ curve in the model assessment without any transformation, see [8]. In



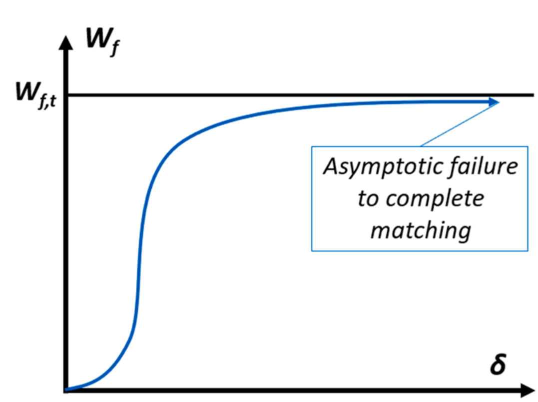


Fig. 10. Schematic representation of the fracture curve of a concrete sample from a 3-PB test and fracture energy curve derived as the integral of the former curve.

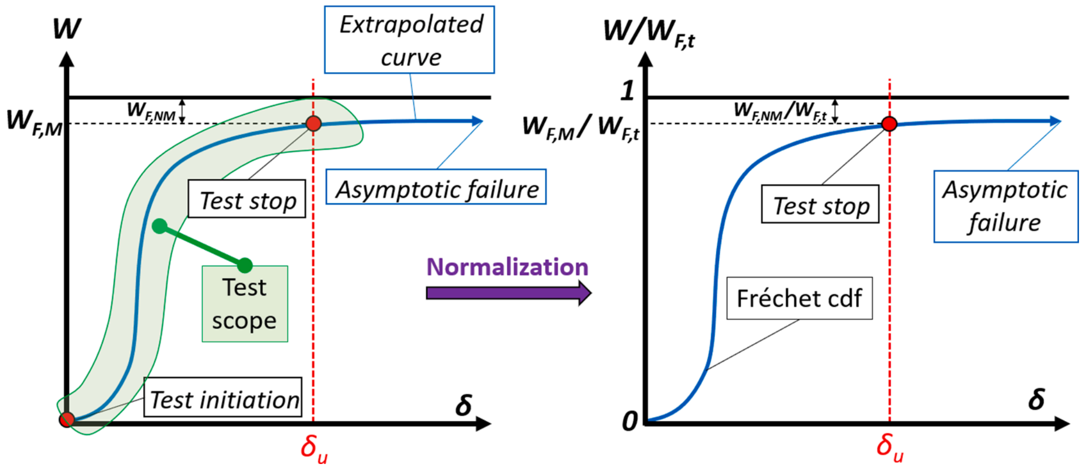


Fig. 11. Schematic explanation of the model applied to the assessment of P-δ curves and fracture energy from 3 PB tests on plain and fiber-reinforced concretes.

any case, the conceptual background remains unaltered. The probability density function of the GEV family for maxima is applied to the assessment process, without presupposing a Fréchet distribution, see [34,35]:

$$f(\delta;\lambda,\gamma,\kappa) = exp\left[-\left(1-\kappa\left(\frac{\delta-\lambda}{\gamma}\right)\right)^{\frac{1}{\kappa}}\right]\cdot\left(1-\kappa\frac{\delta-\lambda}{\gamma}\right)^{\frac{1}{\kappa}-1}\cdot\frac{1}{\gamma};\ \delta \geq \lambda \tag{10}$$

where δ is the mid-deflection at each test time and λ, γ and κ are, respectively, the location, scale and shape parameters of the GEV function. The value of the κ parameter determines the particular distribution type of Eq. (10), i.e. Weibull, Gumbel and Fréchet pdfs for $\kappa > 0$, $= 0$ and < 0, respectively, see [34,35]. These parameters are estimated under the condition of optimal fitting to the experimental data, i.e. those minimizing the difference between the pdf of the corresponding GEV distribution and the sample function, once normalized to the scaling parameter, Ω, that happens to be the total fracture energy, $W_{F,t}$. Accordingly:

$$W_{F,T} = \int_{-\infty}^{+\infty} P(\delta)d\delta = \int_{\lambda}^{\infty} \Omega\cdot f_{GEVD}(\delta;\lambda,\gamma,\kappa)d\delta = \Omega\cdot\int_{\lambda}^{\infty} f_{GEVD}(\delta;\lambda,\gamma,\kappa)d\delta = \Omega \tag{11}$$

noting that the area under the whole pdf happens to be one.

In the present study, κ is found to be negative confirming that the optimal fitting is provided by a Fréchet pdf. In this case, $f_{GEV} = f_{Fréchet}$ which leads finally, to the following expressions of the P- δ, load–deflection, curve and the W_F-δ, fracture energy-deflection, curve, see Eqs. (12) and (13):

$$P(\delta) = \Omega\cdot f_{Frechet}(\delta;\lambda,\gamma,\beta) = W_{F,t}\frac{\beta\gamma^{\beta}}{(\delta-\lambda)^{\beta+1}}exp\left[-\left(\frac{\gamma}{\delta-\lambda}\right)^{\beta}\right];\ \delta \geq \lambda, \tag{12}$$

$$W_{F,T}(\delta) = \Omega\cdot F_{Frechet}(\delta;\lambda,\gamma,\beta) = W_{F,t}\cdot\left(exp\left[-\left(\frac{\gamma}{\delta-\lambda}\right)^{\beta}\right]\right);\ \delta \geq \lambda. \tag{13}$$

being

$$\lambda = \lambda^{*} + \frac{\gamma^{*}}{\kappa^{*}};\gamma = -\frac{\gamma^{*}}{\kappa^{*}};\beta = -\frac{1}{\kappa^{*}} \tag{14}$$

where the parameters labelled with stern in Eq. (14) are understood as the ones from Eq. (10).

Note that the definition of the model, as a Fréchet cdf allows both lower and upper tails of the distribution to be analytically defined. The upper tail is not provided from the test record, though the reliable definition of the final region of the sample function is crucial for the correct estimation of the fracture energy value. In this way, the proposed approach represents a comprehensive alternative to other empirical solutions as those based on power or exponential fitting laws used in

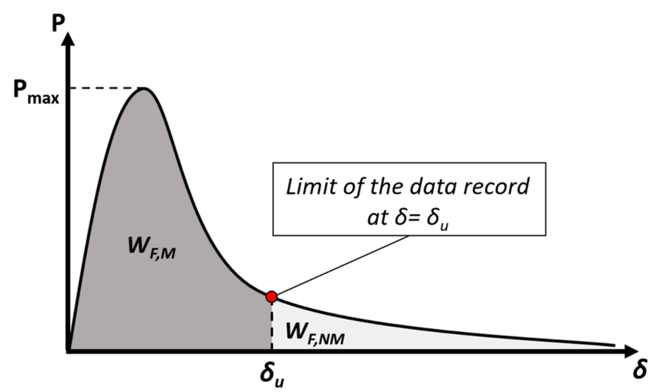


Fig. 12. Illustration of the measured ($W_{F,M}$) and non-measured ($W_{F,NM}$) work of fracture in a registered P-δ curve for a three-point bending test on concrete, censored at a certain level δ_u of the deflection. From [8].

[45] for the extrapolation of the P- δ curve, see Fig. 12. This particularity makes the present analysis to differ from other phenomenological models, like those studied in Sections 4.1. and 4.2, where the definition of the final phase of the phenomenon has little influence on the practical design, though it is essential to capture the real dimension of the phenomenon studied.

In turn, the initial region in the loading process, though apparently inconsequential from the practical viewpoint, points out that the application of the load and the material immediate response is not so simple as one could think. In fact, the model reveals that the initial load increase in the test requires a previous almost gratuitous deformation. As a result, the location parameter, though almost negligible, is not supposed to be zero "a priori" but let as a free parameter. In this way, fitting the recorded data as a sample function turns out to be extremely accurate, as shown in the examples of Fig. 13 representing the experimental *P-δ* curves for different concretes fitted with the analytical Fréchet pdfs using the proposed phenomenological model

v) *Probabilistic analysis*: No probabilistic analysis has been undertaken on the results of this model though a comparison is performed between the estimated total energy obtained with the proposed methodology and that used by Elices et al. in [45], see [8].

4.3. GEV functions used as regression functions to model random independent failures

Some fatigue models are proposed, such as those of Kohout-Vechĕt [20] Ravi-Chandran [21–22] and d'Antuono [23], in which the fitting

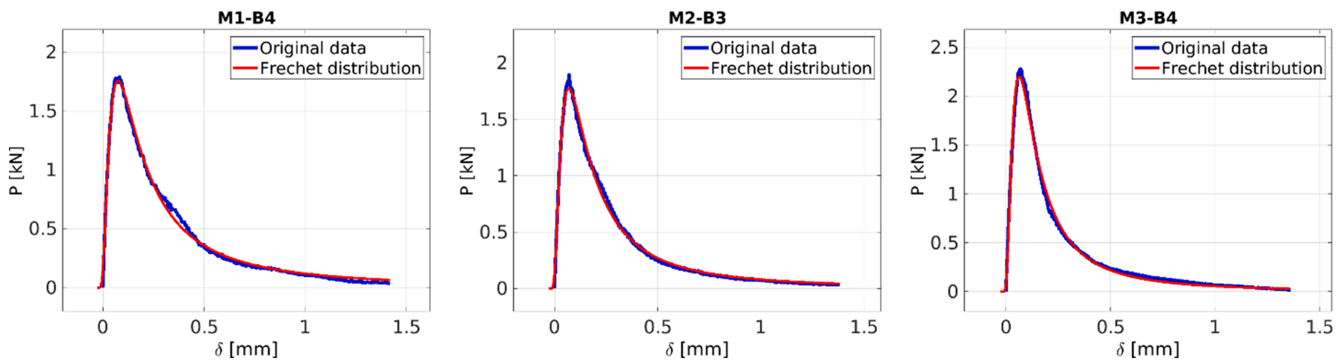


Fig. 13. Representative examples of P-δ fits for concretes of different resistances achieved with the proposed phenomenological methodology (from [8]).

procedure to describe the fatigue S-N median curve is based on analytical functions, which may pertain, or not, to the GEV family. In those cases, contrary to the former cases presented in Sections 4.1 and 4.2, the analysis is referred to test data from different specimens (or imaginary, from a virtual replication of the same specimen at different conditions) rather than to a record of a unique sample function describing the cumulative damage related to a single test. In this way, the aforementioned models should be classified as empirical models rather than phenomenological models and, as a consequence, the methodology proposed is not applicable to the assessment of such fatigue models.

5. Discussion

Besides the pioneer contributions of Freudenthal, Gumbel and Bolotin to substantiate the statistical background of phenomenological models related to fracture and fatigue phenomena, other microstructural derivations, such as the model of Polák [3,4] relating resistivity with strain, dislocations and vacancy concentrations of cooper under fatigue deformation, provide analytical arguments that demonstrate the statistical basis of the cumulative damage evolution. Polák establishes a differential equation that leads to an exponential sigmoidal shaped solution that can be interpreted as a particular case of Weibull cdf for the shape parameter $\beta=1$. In fact, microstructural models also imply unavoidable simplified assumptions in their derivation so that three-parameter Weibull models with shape parameter $\beta>1$ can be envisaged as possibly more complex or general analytical derivations from the above model.

The assumption of Markov chain models in the analysis of cumulative damage, as proposed by Bogdanoff-Kozin [15–18], and the high agreement found between the experimental registered damage evolution and that predicted in the prematurely interrupted test, justifies the concept denoted "micromechanical genetic heritage" as representing a predeterministic evolution of the cumulative damage. In fact, a surprisingly faithful description of the evolution of the phenomenon from a fragment of the total process is achieved when cdfs of the GEV family are applied in the fitting of sample functions in fatigue. This can be extended to other damage phenomena, to be studied in detail, such as wear, creep and ratcheting. In all these phenomenological models the evolution of the damage process can be referred to the initial state according to the Markov's chain theory, which is determined by the particular microstructural features shown by the material tested (such as random distribution of grains sizes and orientations, dislocations, flaws, etc., that constitute the initial distribution of the resistant property implied by the big number of primary elements).

So that the damage model is not only valid within the test scope (descriptive analysis) but also applies for the following damage states that virtually could be extrapolated when maintaining the test conditions. As a matter of fact, the fraction of the damage process as registered in the test provides the information to be applied beyond the test conditions. This is why the cdfs of the GEV family can be applied in all the cases.

This statement is supported by the high quality fits of the sample functions with extremely high correlation coefficients, as confirmed by the fracture and fatigue cases handled in Section 4, but more important, by the possible reconstruction of the whole curve from a relatively small recorded fraction of the whole damage process, as confirmed for fatigue lifetime tests both for metals, concrete and polymers, see [33], [31] and [46], respectively.

In any case, despite the satisfactory fittings observed when applying the proposed phenomenological methodology, the probabilistic definition of cumulative damage models is so far not yet achieved since the sample functions happen to be only a deterministic model of the damage evolution even if, paradoxically, their analytical expression corresponds to statistical distributions. If the attention of the data assessment is only focused on the lifetime prediction or, in general, on that of the terminal state, their distribution can be standardly performed assuming a three-parameter Weibull distribution, as shown in [19,31]. Otherwise, the single and multivariate distributions of the intervening parameters in the phenomenological model are required. The Bayesian technique, as already mentioned above provides the complete probabilistic definition of the stochastic sample functions representing a complementary but fundamental component of the general methodology proposed

Some final thoughts to finish are added:

(a) The sigmoidal shape of the cumulative damage sample functions may be not general because it depends on the shape parameter of the GEV function, but what is general is the normalized sample function as pertaining to the GEV family.
(b) The use of cdfs can be extended to the analysis of phenomenological models when deformation, rather than failure, becomes the reference magnitude justifying the physical phenomenon, such as in the case of viscoelastic characterization when the normal family is the natural solution to be proposed, see [47].
(c) The phenomenological methodology proposed in this work based on the applications of cdfs of the GEV family to fit sample functions related to cumulative damage does not simply pursue the search of quality fittings with high correlation coefficients as is proved to be reached in the examples handled. Such a goal could be simply achieved using empirical models.
(d) The variability of the estimated values of the model parameters deserves special attention in future studies, particularly that of parameters representing directly transcendental information of the phenomenon, such as the initial crack position in the retro-extrapolation model [33], the threshold crack growth rate in the CGR model [32,43], or even the lifetime as normalizing variable [31].
(e) The systematization in the methodology is intended, but its achievement is not the end of a stage in which fit excellence is fulfilled by a certain function as representing a happy idea, but rather the opposite, i.e. the beginning of a new stage of analysis in

which a basis for a new statistical interpretation of the phenomena is sought, relying in turn on the identification of the microstructural approach that is behind it, which incidentally can be formulated in statistical terms. This is the meaning of this phenomenological modeling, hopefully in the spirit of Freudenthal, Bolotin and Bogdanoff-Kozin, and therefore of the stochastic sample function interpreted as "microstructural genetic heritage" which profound meaning is not yet sufficiently understood.

6. Conclusions

The main conclusions drawn from this model are the following:

- A broad variety of experimental techniques, such as direct current potential drop (DCPD), acoustic emission, frequency drop and total strain, are successfully applied to measure the cumulative damage evolution associated with fracture and fatigue, which in many cases exhibits apparent sigmoidal shaped evolution.
- A phenomenological methodology based on statistical functions is applied to provide the analytical solution of some sigmoidal or quasi-sigmoidal sample functions representing cumulative damage phenomena in the fracture and fatigue domains.
- The general applicability and satisfactory suitability of the proposed methodology is confirmed when applied to the descriptive and predictive analyses of different experimental fracture and fatigue cases, such as monotonic load-displacement, CGR and a-N curves, and total fatigue strain evolution for different materials.
- The proposed approach promotes the comprehensive analysis of fracture and fatigue phenomena encompassing the possible interpretation and justification of the resulting statistical functions, their physical meaning and the identification of the model parameters.
- The fatigue and fracture damage evolution is envisaged as stochastic sample functions, which once adequately normalized, represent, by definition, cumulative distribution functions as monotonic non-decreasing functions ranging between 0 and 1.
- The use of cdfs of the generalized extreme value (GEV) family (Fréchet, Gumbel, Weibull) to fit the fatigue sample functions is, in some way, justified as the consequence of the continuous succession of equilibrium states over the fatigue or fracture processes. The maximum and minimum character of the variables intervening in the fatigue process and the high number of primary elements subject to the weakest link conditions are also arguments for this assignment as GEV functions.
- The statistical background of the phenomenon and the asserted "microstructural genetic heritage" are confirmed as the sample function representing a Markov chain process predetermined by the initial microstructural features of the specimen. Only in this way, the surprisingly accurate description of the damage phenomenon evolution can be explained, even from a registered fragment of the whole fatigue process.
- In this way, a comprehensive phenomenological methodology for descriptive and predictive probabilistic analysis of fracture and fatigue phenomena as cumulative damage processes is achieved.
- The application of Bayesian techniques provides the necessary complement allowing the probabilistic assessment of this methodology to be accomplished.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This work is devoted to the memory of J.L. Bogdanoff and F. Kozin, who notably and early contributed to the phenomenological comprehension and modeling of the stochastic cumulative damage phenomena.

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