



Undergraduates' conceptions of mathematics teaching and learning: an empirical study

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Accepted: 25 August 2021 / Published online: 21 September 2021
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Abstract

Previous research in mathematics education has explored teachers' conceptions of mathematics and its teaching and learning, and how their instructional tendencies (e.g., “traditional”, “technological”, “spontaneous” and “investigative”) relate to these conceptions. However, empirical evidence on this topic from large samples of pre-service teachers is limited. This study adapts and validates an instrument originally designed for in-service teachers to analyse the conceptions of mathematics and mathematics teaching and learning. This was done in a sample of undergraduate students in several different degree programmes (primary education, mathematics, and the education itinerary in psychology) in a Spanish university. Existing theory about instructional tendencies and conceptions of mathematics teaching and learning that was developed in the context of in-service teachers is then re-examined in the context of empirical evidence from this sample of individuals (all potential future teachers) without teaching experience. Results show that items from the instrument can be separated into four factors focussed on investigative stances, the role of textbooks, the role of teachers and lesson planning. Individual participants are not characterised by single tendencies; rather, they can be described in terms of several combinations of tendencies, grouped into four clusters. In line with the previous literature on in-service teachers, results suggest that conceptions of mathematics and its teaching and learning are not best captured by rigid, sharply delineated profiles. Rather, individuals configure their own conceptions in terms of combinations of different characteristics of prototypical tendencies.

Keywords Beliefs · Conceptions · Instructional tendency · Mathematics · Mathematics education

In Memoriam José Carrillo-Yáñez (1959–2021)

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1 Introduction

Teachers' conceptions of mathematics and its teaching and learning are powerful moderators of their mathematical teaching practices. The role of these conceptions and the nature of their relationship to knowledge and practice have been deeply analysed within the literature on mathematics education. Previous research also reports on the impact of beliefs, affects and conceptions on mathematics teachers' practices at all educational levels (Goldin et al., 2016; Philipp, 2007). Difficulties in measuring teachers' conceptions have also been pointed out (Ernest, 1989; Lerman, 1990; Lui & Boomer, 2016). Teachers' conceptions are not directly observed but inferred from observations, there are frequent inconsistencies between beliefs and practices, contradictory beliefs and other difficulties coming from external factors (school policies, lesson planning, etc.).

In many parts of the world, mathematics teaching practices remained somewhat static over many centuries, assuming an expositive, deductive approach in which the teacher's role was predominant (Abaté & Cantone, 2005). During the last decades of the twentieth century, calls to reform mathematics teaching and learning led to the introduction of constructivist approaches in which students have a leading role (Jaworski, 1994), but inertia in teachers' behaviours as well as teachers' beliefs about teaching approaches produced tensions between tradition and reform in mathematics (Boaler, 2002a; Stocks & Schofield, 1997).

Although several instruments for investigating mathematics teachers' conceptions have been designed (Contreras et al., 1999; Roelofs et al., 2003; Woolley et al., 2004), many of them have not been validated or have been used in empirical studies with small sample sizes. One of these instruments (Contreras et al., 1999) has been used to analyse how instructional tendencies manifest among several teachers working at different educational levels, but quantitative validation of this instrument is lacking, as the authors employed only qualitative studies with very small sample sizes (9 teachers).

Thus, our first goal is to adapt and validate this previously defined instrument to measure different instructional tendencies related to teachers' conceptions of mathematics and its teaching and learning, so that the new instrument can be used with individuals without teaching experience. A second goal is to use empirical evidence to consider previously described theory about instructional tendencies in the context of a sample of individuals without teaching experience (specifically, undergraduate students in Spain). The reason for considering undergraduate students and not teachers is to validate an instrument for that population, so that it can be used in a forthcoming study in which undergraduate students with different backgrounds will observe and rate mathematics teaching practices by analysing videotaped lessons and artefacts. Previous research supports the effectiveness of using videotaped lessons to develop future teachers' competences (Muñiz-Rodríguez et al., 2018). In this sense, it becomes relevant to analyse whether undergraduates' conceptions of mathematics and its teaching and learning introduce a bias when observing and rating teaching practices. These two goals lead us to the following research questions: (1) Is it possible to adapt and to validate an instrument designed for use with teachers to be used with undergraduate students? (2) Can undergraduate students be classified into different categories of instructional tendencies related to their conceptions of mathematics and its teaching and learning?

2 Theoretical framework

2.1 Beliefs and conceptions

Studies about beliefs constitute a common line of research in mathematics education, even when there is no “universal acceptance by mathematics education researchers of a definition of beliefs that can ground the various theories” (Goldin et al., 2009, p. 2). Teachers’ beliefs, conceptions, affects and their interactions with knowledge and teachers’ practices become relevant in any analysis of an instructional process (Pajares, 1992), since they “play a significant role in shaping teachers’ characteristic patterns of instructional behaviour” (Thompson, 1992, pp. 130–131). Beliefs are defined as “psychologically held understandings, premises, or propositions about the world” (Philipp, 2007, p. 259). The relevant literature does not offer a common answer to how the role of beliefs can be understood. For some authors “the relationship between belief and practice has been a philosophical enquiry with, inevitably, conjectural outcomes [...], whilst others have sought confirmatory evidence” (Andrews & Hatch, 2000, p. 31). Nevertheless, it seems to be the case that teachers’ beliefs influence their practice in some way (Clark et al., 2014; Cross, 2009), even though the interaction between beliefs and practice is not always clear (Schoenfeld, 2011; Wilhelm, 2014).

Beliefs are generally considered to be organised into belief systems (Philipp, 2007), which cluster around an idea or an object with different degrees of strength and nature of relationships amongst the constituent beliefs. Belief systems may sometimes link seemingly incompatible beliefs (Andrews & Hatch, 2000). Individuals can, however, construct sensible systems (Leatham, 2006) that reconcile such apparent incompatibility by organising their beliefs in ways that make sense to them personally.

Conceptions can be considered mental structures “encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences” (Philipp, 2007, p. 259). In Thompson’s words, conceptions include “what a teacher considers to be desirable goals of the mathematics programme, his or her own role in teaching, the students’ role, appropriate classroom activities, desirable instructional approaches and emphases, legitimate mathematical procedures, and acceptable outcomes of instruction” (Thompson, 1992, p. 135). In this work, we assume Philipp’s view, accepting the interrelatedness of beliefs and conceptions, and considering, as Flores and Carrillo (2014) and Zoitsakos et al. (2015), that, for our purposes, it makes no sense to push ourselves into finding distinctions between the two.

Conceptions become relevant when considering many different aspects of mathematical instruction. Amongst those aspects, some are more directly related to the aim of the present work, such as:

- The relationship between beliefs and instructional practices (Beswick, 2005; Mesa et al., 2014)
- Instructional decision-making (Beswick, 2012)
- The identification of different conceptions of mathematics and mathematics teaching and learning, and the relationship of such conceptions with knowledge for teaching (Andrews & Hatch, 2000; Ribeiro & Carrillo, 2011; Voss et al., 2013)
- The effectiveness of classroom practices (Opdenakker & Van Damme, 2006; Wilkins, 2008)
- The relationship with students’ attitudes towards mathematics and their approaches to mathematics learning (Crawford et al., 1994, 1998; Di Martino & Zan, 2011)
- The importance of cultural context (Chan & Wong, 2014; Xenofontos, 2018; Xie & Cai, 2018)

2.2 Analysing conceptions and their teaching implications

Instructional tendency can be understood as a teacher's prevailing disposition to teach towards specific conceptions. As such, they affect the way teachers interact in the teaching and learning process (Philipp, 2007). As Carrillo and Contreras (1995) and Porlán (1992), we use this term because it is easier for teachers to identify themselves with an instructional tendency rather than with a specific didactic model. Using this definition, Lui and Bonner (2016) describe two main instructional tendencies and their connections with teachers' conceptions. On the one hand, the constructivist tendency is more student-centred, fostering active learning and co-construction activities "where students engage in inquiry and discovery, construct their own mathematical knowledge, and develop mathematical creativity and independence" (Lui & Bonner, 2016, p. 2). Therefore, teachers whose conceptions are most closely aligned with the constructivist tendency organise their teaching to focus more on students' understanding than on content, and they promote inquiry-based processes (Goos, 2004). On the other hand, the often so-called traditional tendency is more transmission-based and focuses on algorithms and procedures. Thus, teachers "instruct by providing information in the form of facts, rules, and laws to students, whose subsequent responsibility is to absorb and process the information" (Lui & Bonner, 2016, p. 3), rather than encouraging students' reasoning and understanding. Teachers most closely aligned with the traditional tendency usually try to foster consolidation of mathematical knowledge by focusing on practicing procedures (Goldsmith & Schifter, 1997; Hewitt, 1996). These two main instructional tendencies as described in Lui and Bonner (2016) can be found in previous research though using different terminology. Sometimes, an intermediate tendency is also identified, as in Miller and Seller's (1985) ternary: transmission, *transaction* and transformation. According to these authors, the goal of teaching is along a spectrum from transmission of knowledge to problem solving and student-teacher interaction based on curriculum (i.e., transaction), to students' discovery within learning environments (i.e., transformation). A similar approach can be found in Askew et al. (1997), where three instructional tendencies were defined in terms of beliefs: transmission, connectionist and discovery. It is important to note that discussing these conceptions does not mean reducing teachers to a binary categorisation of teachers as traditional or constructivist (Clarke, 2006). On the contrary, teachers' conceptions arise in more complex combinations, depending on factors relating to context, topic, assessment, etc. The tendencies explained above try to approach the multifaceted process of teaching by describing archetypical conceptions as useful descriptors, though teachers cannot be simply classified as one or the other and often combine different approaches and teaching styles (Andrews & Sayers, 2013; Lepik et al., 2012).

How to analyse the role of teachers' conceptions has been an ongoing goal in the research agenda of mathematics education. Roelofs et al. (2003) developed a scale for measuring teachers' conceptions of learning, discovering some preference for transmission over negotiation or discovery, but these authors underlined that "one cannot speak of a pure transmission model" (p. 96). Woolley et al. (2004) introduced the Teacher Belief Survey (TBS), whose application suggested the existence of three tendencies: traditional management, traditional teaching and constructivist teaching, although different combinations can be found (e.g., a constructivist teaching approach can be supported by a traditional class management). Barkatsas and Malone (2005) classified teachers' conceptions according to two main tendencies, described as contemporary-constructivist and traditional-transmission-information processing. Their view of these tendencies as ways for characterising teachers' conceptions has been widely accepted, where teachers find their

own combinations of features from the two tendencies to configure their teaching style (Hewitt, 1999, 2001; Roelofs et al., 2003).

Keeping the idea of combination or balance between transmission/traditional and discovery/innovative tendencies, Climent (2005) developed the CEAM instrument (Spanish acronym of Conceptions about Teaching and Learning Mathematics) which was based on a previous instrument developed by Carrillo (1998), and Carrillo and Contreras (1995). While Carrillo's instrument was designed for measuring secondary mathematics teachers' conceptions, Climent's was designed for primary education teachers. Both versions were developed for in-service teachers. The CEAM instrument addresses the difficulty of analysing conceptions by defining indicators that refer to easy-to-observe teachers' actions and routines. Thus, this instrument provides researchers with an explicit description of teachers' conceptions, which justifies its adequacy for the purposes of this research study. Moreover, quantitative validation of this instrument is lacking, since the authors conducted only qualitative studies with very small sample sizes (9 teachers).

Based on Ernest (1989), Lerman (1990) and Porlán (1992), four instructional tendencies were defined in the CEAM: Traditional (TR), Technological (TE), Spontaneous (S) and Investigative (I). The following descriptions of each tendency are translated and paraphrased from Contreras et al. (1999, p. 54). The TR tendency is characterised by the adoption of a lecturing style as the sole teaching method and the textbook as the unique resource. The lesson is focused on an exclusively informative goal, and the learning is supposed to be accomplished using only memory, by adding information units. Assessment in the TR tendency is conceived as a final activity. In the TE tendency, teachers do not show contents in the final stage, but they perform a simulation of the process of construction of knowledge. The subject is taught from an informative and practical perspective, which allows its application. The learning is supposed to be accomplished by memory but with an internal organisation, a well-detailed planning of the activities that the teacher intends to develop in the classroom. The results of the assessment are considered to reflect about the learning process. The S tendency is characterised by teachers' proposals of manipulating activities, through which is eventually expected the production of unorganised knowledge. The essential nature of the subject is formative, assuming its utility for changing pupils' attitudes. Teachers think that pupils learn when the learning objective has a meaning for them, which emerges randomly from the context and the activities. Assessment in the S tendency is viewed as a permanent sensor of learning that gives the possibility of redirecting it. Finally, in the I tendency, teachers organise the process to acquire specific knowledge through investigation. The ultimate goal of the subject is providing the students instruments that make autonomous learning possible. Teachers conceive assessment as a permanent sensor of learning, giving the opportunity of redirecting and orienting it towards the foreseen learning through more appropriate contexts.

As can be deduced from the descriptions above, the authors of the CEAM instrument also assume, in line with Roelofs et al. (2003), Barkatsas and Malone (2005) or Lui and Bonner (2016), that I and TR tendencies could be respectively viewed as the most and the least constructivist conceptions, with TE and S in between. Also, one can easily establish certain correspondences with Askew and colleagues' (Askew et al., 1997) tendencies, particularly between TR and transmission, and between I and discovery, whereas TE shares features with transmission and connectionist, and S with connectionist and discovery.

The CEAM instrument consisted of a set of items organised in the following categories: mathematics, teaching methodology, learning processes, student's roles and teacher's roles. For each item, four indicators were defined according to the four tendencies (TR, TE, S

or I). These indicators included a description of expected conceptions for each tendency. Finally, the model was reviewed and adjusted using a sample of 9 teachers.

This design coexists with the assumption of teachers finding their own combination or balance between different tendencies and not fitting to any canonical description of a tendency. Hence, teachers are not expected to be consistently aligned with a TR or I tendency. Likewise, S and TE tendencies provide more nuanced points of view, characterising different combinations of conceptions, as in Miller and Seller (1985), Roelofs et al. (2003); Woolley et al. (2004) or Swan (2006), that is: “These categories are ‘ideal types’ and an individual teacher’s conception of mathematics, teaching and learning may combine elements of each of them, even where they appear to conflict” (Swan, 2006, p. 59).

3 Methodology

3.1 Population and sample

Our target population is slightly different from the original one the CEAM instrument was designed for, because the research interest lies in undergraduate students from different bachelor’s degrees: primary education, mathematics and the education itinerary in psychology. The rationale behind this target population involves analysing in the near future whether the conceptions of mathematics and its teaching and learning amongst undergraduates with different backgrounds are a source of bias when observing and rating mathematics teachers’ practices through videotaped lessons and artefacts. Thus, not only observers with strong mathematical backgrounds are needed, but also others with an understanding of social-emotional and psychological issues, because rating teachers includes considering both pedagogical and mathematical issues. Besides, the three sample groups each represent a potential set of future teachers. Some considerations about this population within the Spanish context must be stated. First, students within the primary education degree are pre-service teachers, since that degree qualifies them for the profession. Second, mathematics students do not receive any courses in education during the degree, so to become mathematics teachers (in secondary education), they must complete a master’s degree (Muñiz-Rodríguez et al., 2016). Third, psychology students within the education itinerary are mainly planning to become counsellors, in elementary and secondary schools. Obviously, there are likely to be differences among the mathematical backgrounds across these different degrees. Whereas students in the degree in mathematics have completed a scientific path during their highschool, students in the degree in primary education often come from social sciences paths in highschool and from vocational studies (López-Beltrán et al., 2020). Students in the degree in psychology are commonly mixed; some have scientific and some social sciences backgrounds. Students enrolled in the selected courses (i.e., the bachelor’s degrees: mathematics, primary education and the education itinerary within the psychology bachelors’ programme) at the University of Oviedo (Spain) were invited to participate in the study. The average response rate by degree programme was above 85% and, thus, the final (non-random) sample consisted of 247 students distributed as follows: 170 in the primary education degree (2nd and 3rd years, they had undertaken an initial internship, consisting only in observing), 43 in the mathematics degree (1st year) and 34 in the psychology degree (3rd year, they had not performed any internship). Since the purpose of this research is to validate an instrument, the heterogeneity of the sample

actually helps to underpin claims about a population of undergraduate students. Participants completed the questionnaire anonymously.

3.2 Instrument

For the purpose of this study, the original CEAM instrument was adapted in two ways. First, the original CEAM instrument was not administered as a questionnaire, but as a grid. For each item, the instrument comprised different indicators for each tendency, so that researchers assigned one of the tendencies based on teachers' answers (if interviewed) or behaviours (if observed). Second, the original CEAM instrument contained indicators related to previous teaching experience. Since one of the goals of the present study was to design a valid instrument for measuring the conceptions of undergraduate students who may have no previous teaching experience, and in order to reach a larger sample, we substituted the original indicators in the grid for items on a Likert-scale, from 1 (totally disagree) to 5 (totally agree) and removed some items about teaching experience. In a second step, the indicators classified either within the TR or I tendencies by the original authors were selected as item statements. Then, a content analysis of the original indicators was performed so that a higher level of agreement with each item statement meant respondents' conceptions were more aligned with the tendency in which the item was classified. We used a 5-point scale to allow participants to provide a neutral answer (3 = not agree, nor disagree), considering that as undergraduate students, it might be hard for them to take a stance on some of the items. Obviously, by substituting the indicators in the grid with Likert-scale items, we may have lost some nuances in terms of allowing participants to identify themselves with a comprehensive single tendency, but we gained ease of applicability to a larger, non-experienced sample.

Therefore, the final instrument consists of 35 items, displayed in Table 1, which also indicates the tendency towards which each item is classified. Regarding the aforementioned categories, items 1–7 and 10–15 refer to different methodological aspects (teacher's praxis, goals, information sources, lesson plans, class activities, etc.), items 8–9 are about conceptions of school mathematics, items 16–23 deal with different aspects of the learning process (the way it is produced, interactions, types of groups, students' interests, etc.), items 24–29 refer to students' roles, and items 30–35 to teachers' roles.

Content validation was carried out by the authors of the present manuscript, after discussions with the authors of the original CEAM instrument and other experts in mathematics education from the UK and Chile, who provided a more international view of the instrument. Considering the mother tongue of respondents, the instrument was originally written and administered in Spanish (being translated into English for publication). Therefore, no translation validation was needed.

3.3 Data analysis

Initially, a descriptive analysis of questionnaire responses was performed. Secondly, a Kruskal–Wallis test was conducted for each item to check whether there were significant differences in the responses between groups of students from different degree programmes.

For internal consistency, as well as for the factor analysis, following psychometric guidelines, all items should be oriented in the same direction at the analysis stage. Thus, items classified within the TR tendency were reverse-coded. Cronbach's alpha (0.795) was sufficiently large to infer adequate internal consistency based on Nunnally's (1978)

Table 1 Descriptive analysis of responses to items

Item	Statement	Tendency	Mean	Median	Mode	SD
1	Mathematical activity in the classroom should include practice exercises	TR	3.77	4	4	1.047
2	The teacher should offer students the opportunity to investigate in order to acquire knowledge	I	4.4	4	5	0.672
3	The source of information in a classroom should be the teacher	TR	3.54	4	4	0.774
4	The source of information in a classroom should be the textbook	TR	2.65	3	3	0.889
5	There should be different levels of activities on which students have the opportunity to work individually	I	4.2	4	4	0.692
6	Once established, learning objectives can be modified by the teacher	I	3.81	4	4	0.909
7	The established curricular programming should be followed	TR	3.3	3	3	0.836
8	It is important that students acquire concepts, procedures and mathematical rules	TR	4.19	4	4	0.618
9	The purpose of mathematics is based on developing thinking that allows students to organise, interpret and understand the reality that surrounds them	I	4.27	4	4	0.694
10	Students should confront problems for which they do not have an answer	I	3.94	4	4	0.863
11	Following a textbook is an appropriate approach on the part of the teacher	TR	2.66	3	3	0.874
12	It is appropriate to use information sources such as the internet, everyday situations, experiences, etc., in the classroom	I	4.47	5	5	0.623
13	The teacher should set a level in the classroom and teach students according to this level	TR	2.99	3	4	1.11
14	The teacher should adapt the lesson plan if it is detected that there are different interests or levels amongst the students	I	4.32	4	4	0.722
15	Mathematical argumentation and the promotion of attitudes should have the same importance as the acquisition of mathematical concepts and rules	I	3.97	4	4	0.86
16	It is important to connect mathematics in the classroom to other contexts	I	4	4	4	0.752
17	Learning mathematics requires observing patterns that allow one to guess, check and try to establish a generalisation	I	3.8	4	4	0.708
18	It is important for teachers to transmit knowledge in the classroom	TR	4.23	4	4	0.658
19	The teacher should be the one who has the greatest importance in the classroom	TR	2.68	3	2	1.144
20	When students work individually, this improves learning	TR	3.34	3	3	0.854
21	An ideal teacher knows the interests of their students and takes them into consideration	I	4.51	5	5	0.63
22	The attitudes of students towards their learning can be modified	I	4.24	4	4	0.566

Table 1 (continued)

Item	Statement	Tendency	Mean	Median	Mode	SD
23	Learning mathematical techniques develops transferrable skills in students that can be used in their broader social environment	I	4.16	4	4	0.737
24	Students should be part of the design of teaching planning (e.g., for a lesson/unit)	I	3.71	4	4	0.804
25	Students learn because they have completed a teaching/learning process and are aware of it	I	3.89	4	4	0.729
26	It is necessary that students look for ways to answer certain questions	I	4.3	4	4	0.637
27	It is important to make students aware of what they do and why when they are learning	I	4.59	5	5	0.59
28	Students should maintain a critical attitude to the information that appears in the classroom	I	4.15	4	4	0.722
29	Students should look for alternative sources of information to those used in the classroom	I	3.92	4	4	0.805
30	The textbook reflects the contents by which the teacher should be guided	TR	2.74	3	3	0.915
31	Student curiosity should be provoked in the learning process	I	4.66	5	5	0.555
32	The ideal teacher should validate/justify the ideas that arise in the classroom	I	4.22	4	4	0.718
33	The teacher should always give the correct information	TR	3.81	4	4	0.967
34	The teacher should go to class with prior knowledge about the characteristics of the students	I	4.06	4	4	0.882
35	The ideal teacher raises questions for which there are no a priori answers	I	3.79	4	4	0.886

commonly used rule of thumb. While we recognise critiques of the use of Cronbach's alpha in this way (e.g., Taber, 2018), we report this statistic as it is expected to be familiar and interpretable for readers in the field.

For validation, given that the sample size was not large enough to perform both Exploratory and Confirmatory Factor Analysis (Mundfrom et al., 2005), we conducted an Exploratory Factor Analysis (EFA) using FACTOR 10.9 (Ferrando & Lorenzo-Seva, 2017). The number of factors was determined using the method of Optimal Implementation of Parallel Analysis proposed by Timmerman and Lorenzo-Seva (2011). The fit of the data to the model was checked using the Goodness of Fit Index (GFI) and Root Mean Squares of Residuals (RMSR). Polychoric correlation was used as an input matrix for the data. After trials of different combinations of methods, the best fit was achieved by using factorisation by unweighted least squares and normalised Varimax rotation (Ferrando & Lorenzo-Seva, 2017).

After validation, a further analysis was performed to classify individuals in the sample into different profiles. For this purpose, a cluster analysis was carried out (again using the original scores, that is, reversing the change of orientation made for factor analysis). As we were looking for a non-supervised classification, the *k*-means algorithm was used, by pre-fixing $k=2, 3, 4, 5$ clusters (carrying out trials with different initial points). Thus, we combined two types of analyses, which are not alternative but complementary. While with the EFA we obtained the structure of the items, with the cluster analysis we studied the similarities amongst individuals. Descriptive statistical analysis, Kruskal–Wallis tests and cluster analysis were undertaken using SPSS (IBM Corp., 2016), version 24.2.

4 Results

4.1 Descriptive results

Of the participants, 175 declared not having experience as a mathematics teacher at any level (including formal or informal education), whereas 72 said they had previous experience (all in the form of private mathematics tutoring) of some months (only one person said having 2 years of experience). This description supports the suitability of the sample for investigating individuals without mathematics teaching experience.

Table 1 shows mean, median, mode and standard deviation (SD) scores for each Likert-scale item. Levels of agreement with the statements in the items are quite high. Some items (2, 12, 21, 27 and 31) whose mode is in 5 points are classified within the I tendency and show a high level of agreement with considering students' interests and curiosity when teaching mathematics as well as making them aware of their learning process. All the mean values exceed 3, except for items 4, 11, 13, 19 and 30 (which are classified within the TR tendency). Medians are also quite high; even some of the items previously remarked (12, 21, 27 and 31) have medians as high as 5. There is similarity between means and medians for most items, though some items have more dispersed responses: 1, 13, 19 and 33. These are classified within the TR tendency, which underlines a greater disagreement with respect to the I tendency. Actually, items positively classified within the TR tendency have mean values lower than items within the I tendency.

Using Kruskal–Wallis tests, significant differences (p -value < 0.0001) were found in the median values of the responses depending on the bachelor's degree programme of the participants for items 1, 3, 4, 6, 7, 8, 11, 13, 14, 18, 19, 22, 30, 33 and 34. Items classified

within the I tendency constitute most of these differences. That means individuals' academic backgrounds are more easily differentiated by considering their answers in relation to the I tendency.

4.2 Exploratory factor analysis results

The Kaiser Meyer Olin index ($KMO=0.7935$) and Barlett's test ($p\text{-value}<0.00001$) show that the data matrix is appropriate for factorisation. The method of Optimal Implementation of Parallel Analysis suggests the extraction of four factors, explaining 45.6% of the common variance. Factor loadings are displayed in Table 2, whereas the full loadings are displayed in an Appendix Table 5. There are two loadings (for items 17 and 32) lower than 0.3, but we decided to include them in the model, with the precautions that we will discuss later.

The first factor is a mix of items related to *investigative stances*, which explains 23.59% of the common variance and includes items 2, 5, 8–10, 12, 15–17, 21–23, 25–29, 31 and 35. All these items were classified within the I tendency, except item 8 (which appears with a negative loading). The rationale behind this exception lies in the strongly extended influence of this traditional conception, particularly in mathematics education (Boaler, 2002b). This item will be discussed later.

The second factor, explaining 7.5% of the common variance, refers mainly to *the role of textbooks*, and includes items 4, 11, 24 and 30. Except item 24, the rest are classified within the TR tendency. It is interesting that item 24, which pertains to students' involvement in teaching planning, "hangs together" with the textbook-related items, as the link between these aspects is not immediately intuitive.

The third factor is made up of items 3, 18, 19 and 32–34, which are related to *the role of teachers* (i.e., the teacher as the most important and the source of knowledge, the teacher's knowledge transmission). This factor explains 7.46% of the common variance. All these items were classified within the TR tendency, except item 32, but this appears with a negative loading.

The fourth factor explains 7.05% of the common variance and consists of items 1, 6, 7, 13, 14, 20 and 34. These items are related to *lesson planning* (e.g., the possibility of changing the lesson plan, adapting it to students' interests, using group vs. individual work, knowledge of students' interests, ability levels and characteristics). These items are approximately evenly split in terms of their classification within I and TR tendencies.

Statistical indicators show a good fit of the data to the 4-factor model ($GFI=0.963$, with a good fit considered to be over 0.95, and $RMSR=0.0549$, with a good fit considered to be under 0.06). Items classified within the TR tendency are within the third (*the role of teachers*), second (*the role of students*) and fourth (*lesson planning*) factors, whereas the first factor (*investigative stances*) contains mostly items classified within the I tendency (except item 8, with a negative load) and explains almost one quarter of the common variance.

4.3 Cluster analysis results

Results from the different trials with cluster analysis by k -means showed that the best fitting model was achieved by using 4 clusters, obtaining full convergence in 16 steps and introducing considerable differences between cluster centres compared to $k=3$ (which provided a cluster with only 2 individuals), whereas with $k=5$, no significant increase of

Table 2 Factor loadings

Tendency	Item	Statement	F1	F2	F3	F4
I	2	The teacher should offer students the opportunity to investigate in order to acquire knowledge	0.666			
I	5	There should be different levels of activities on which students have the opportunity to work individually	0.497			
TR	8	It is important that students acquire concepts, procedures and mathematical rules	-0.581			
I	9	The purpose of mathematics is based on developing thinking that allows students to organise, interpret and understand the reality that surrounds them	0.544			
I	10	Students should confront problems for which they do not have an answer	0.523			
I	12	It is appropriate to use information sources such as the internet, everyday situations, experiences, etc. in the classroom	0.637			
I	15	Mathematical argumentation and the promotion of attitudes should have the same importance as the acquisition of mathematical concepts and rules	0.454			
I	16	It is important to connect mathematics in the classroom to other contexts	0.448			
I	17	Learning mathematics requires observing patterns that allow one to guess, check and try to establish a generalisation	0.286			
I	21	An ideal teacher knows the interests of their students and takes them into consideration	0.579			
I	22	The attitudes of students towards their learning can be modified	0.542			
I	23	Learning mathematical techniques develops transferable skills in students that can be used in their broader social environment	0.582			
I	25	Students learn because they have completed a teaching/learning process and are aware of it	0.347			
I	26	It is necessary that students look for ways to answer certain questions	0.673			
I	27	It is important to make students aware of what they do and why when they are learning	0.580			
I	28	Students should maintain a critical attitude to the information that appears in the classroom	0.621			
I	29	Students should look for alternative sources of information to those used in the classroom	0.589			
I	31	Student curiosity should be provoked in the learning process	0.807			
I	35	The ideal teacher raises questions for which there are no a priori answers	0.407			
TR	4	The source of information in a classroom should be the textbook		0.619		
TR	11	Following a textbook is an appropriate approach on the part of the teacher		0.796		
I	24	Students should be part of the design of teaching planning (e.g., for a lesson/unit)		0.341		
TR	30	The textbook reflects the contents by which the teacher should be guided		0.675		

Table 2 (continued)

Tendency	Item	Statement	F1	F2	F3	F4
TR	3	The source of information in a classroom should be the teacher			0.524	
TR	18	It is important for teachers to transmit knowledge in the classroom			0.554	
TR	19	The teacher should be the one who has the greatest importance in the classroom			0.411	
I	32	The ideal teacher should validate/justify the ideas that arise in the classroom			- 0.282	
TR	33	The teacher should always give the correct information			0.637	
TR	1	Mathematical activity in the classroom should include practice exercises				0.343
I	6	Once established, learning objectives can be modified by the teacher				0.397
TR	7	The established curricular programming should be followed				0.460
TR	13	The teacher should set a level in the classroom and teach students according to this level				0.374
I	14	The teacher should adapt the lesson plan if it is detected that there are different interests and levels amongst the students				0.342
TR	20	When students work individually, this improves learning				0.334
I	34	The teacher should go to class with prior knowledge about the characteristics of the students				0.394

Table 3 Final number of individuals in each cluster

Degree	Cluster			
	C1	C2	C3	C4
Primary education	53	40	55	22
Mathematics	7	10	2	24
Psychology	5	20	0	9
Total	65	70	57	55

information was obtained. Previously, hierarchical cluster analyses were carried out, with different linking methods, showing that optimal choices were between 3 and 5 clusters.

The final number of individuals in every cluster is displayed in Table 3, as well as the distribution of bachelor's degree programmes. Obviously, the distribution by degree programme is not homogeneous. Psychology students are mostly in the second cluster, and mathematics students are mostly in cluster 4. Primary education students are more homogeneously distributed, except in cluster 4 where there are considerably fewer primary education students.

4.4 Joint interpretation of factors and clusters

Given that the cluster analysis was conducted using the original item scores, for this joint interpretation, we reverse the transposing of item coding where this was performed for the factor analysis. Although the average item scores for each cluster centroid could be given, that is a 35×4 matrix, whose interpretation would be onerous. Thus, to increase the interpretability, we multiply the factor loadings (a 4×35 matrix) by the centroid average item scores (a 35×4 matrix), obtaining a 4×4 score matrix with the factorial scores of each cluster centroid (see Table 4). The reader must keep in mind that factorial scores are obtained by an aggregated weighting of Likert scores, so their magnitudes are not independently meaningful, in other words, a greater factorial score for a cluster indicates greater scores on the items constituting that factor from the individuals in that cluster. Table 4 shows that factor 1 (*investigative stances*) obtained the highest scores, followed by substantially lower scores for factor 3, related to *the role of teachers*. Factors 2 (*the role of textbooks*) and 4 (*lesson planning*) had much lower scores. Since factor 1 (*investigative stances*) gathered most of the items classified within the I tendency, from Table 4, we deduce that this tendency became the most relevant in explaining sample differences. Factor 3 (related to *the role of teachers*, with most of the items classified within the TR tendency) also helped in explaining differences.

Table 4 Factorial scores of clusters centroids

Factors	Clusters			
	C1	C2	C3	C4
F1	47.2	55.8	52.4	45.9
F2	8.1	7.8	4.3	8.9
F3	16.2	17.4	13.6	17.7
F4	2.6	4.0	5.3	0.4

From Table 4, we can see that individuals in cluster 1 scored the second lowest for factor 1 (*investigative stances*), which contained most of the items classified within the I tendency. On one hand, we can interpret this fact as if they were aligned more closely with the TE or S tendencies. On the other hand, we see that individuals in cluster 1 also scored the second lowest in factor 4 (*lesson planning*), that is, they do not tend to prioritise teachers' adapting lesson plans to students' interests, ability levels and characteristics. In this cluster, 81.5% of participants are students on the primary education degree programme.

In cluster 2, we observe a quite different pattern from that in cluster one. The predominant tendency is I, as participants in cluster 2 gave the highest scores on factor 1 (*investigative stances*). Participants in this cluster, however, still prioritise *the role of teachers* (factor 3). Here, more than half of the participants are students from the primary education degree programme (57.1%), whereas 28.6% are students on the psychology degree programme.

In cluster 3, the characterisation is closer to that of cluster 2. In this cluster, 96% of individuals are primary education students, and there are no psychology students. Participants in cluster 3 gave the second highest scores on factor 1 (*investigative stances*), but the lowest on factor 3 (*the role of teacher*). We interpret this to indicate a strong I tendency, but with an important representation of the S tendency as well. That is, members of cluster three are less I but also less TR than individuals in cluster 2.

Finally, cluster 4 is the only one in which the majority are mathematics students (43.6%), followed by students from the primary education degree (40%) and psychology (16.4%). Although this cluster shares similarities with cluster 1, there are also relevant differences. First, cluster 4 members gave the greatest scores on factor 3 about *the role of teachers*, and the lowest on factor 4 regarding *lesson planning*. That is, we can consider members of this cluster to be the most strongly TR-aligned respondents, but their scores on factor 1 (*investigative stances*) still indicate some strong influence of the I tendency. Thus, we are again close to the definition of the TE or S tendencies, but with a greater weight of tradition than in cluster one.

5 Discussion

The first goal of this work was to adapt and validate an instrument for analysing conceptions of mathematics and its teaching and learning in undergraduate students from different bachelor's degree programmes. Results demonstrate the validity of the instrument for analysing these conceptions. The internal consistency, as measured by Cronbach's alpha, is sufficiently high. Content validity has been previously discussed by researchers, and during the questionnaire administration process, no further controversial questions or issues about content were raised. Moreover, 70.9% of interviewees did not have previous teaching experience, and those that did (29.1%) had only informal short-term experience. Therefore, the contribution of this study consists of the validation of an instrument to measure undergraduate students' (including pre-service primary teachers as well as undergraduate students in mathematics and psychology) conceptions of mathematics and its teaching and learning. Additionally, although we plan to use the adapted instrument within a lesson observation project (with undergraduate students acting as observers), it can be used to measure students' conceptions of mathematics and its teaching and learning in a more general context, as observation is not directly mentioned within the adaptation.

Significant differences were found on several items depending on participants' bachelor's degree programmes. Nevertheless, there are also relevant similarities across degree

programmes; according to Table 1, the score spread was very small in items 2, 5, 8, 9, 10, 12, 22, 27 and 31. Additionally, the agreement level was very high on some items, for which 75% of the answers are greater than or equal to 4, and mode and median coincide at 5 (e.g., items 12, 21, 27 and 31). Differences would need further exploration to be completely explained, since, as stated in Petocz et al. (2007) and Wang et al. (2017), many variables could influence them, including students' backgrounds, their teachers' conceptions, students' expectations about their assessment and performance or the relevance of mathematics in their future plans.

The main result of the exploratory factor analysis concerns the validation of the structure of the adaptation of the CEAM instrument. It was originally established in the instrument that, theoretically, answers were classified within the TR or I tendencies, and the theoretical framework also hypothesised that respondents do not fit within an archetype but have a combination of tendencies instead. What the current study proves is the existence of four factors: *investigative stances*, *the role of textbooks*, *the role of teachers*, and *lesson planning*. Thus, the factor analysis reveals that the structure is not unidimensional, that is, even when assuming the construction of the instrument, reality is more complex than the dichotomy between constructivism and transmission.

On the other hand, approximately half of the explained variance is provided by factor 1 (*investigative stances*), whereas the rest can be explained by jointly considering participants' views relevant to factors 2 (*the role of textbooks*), 3 (*the role of teachers*) and 4 (*lesson planning*). On average, individuals scoring higher on the *investigative stances* gave lower scores on the three factors referring to *the role of teachers*, *the role of textbooks* and *lesson planning*. This is quite clear with individuals in cluster 2 (mostly future primary teachers). However, this lower scoring is not such a simple and homogeneous behaviour. There are differences coming from nuances relevant to conceptions of mathematics and its teaching and learning, leading participants to locate themselves somewhere in between the I and TR tendencies. Therefore, our results support the theoretical conceptualisation of the existence of two main tendencies, formulated in the model by Carrillo's group, but also in Askew et al. (1997), Roelofs et al. (2003) or Woolley et al. (2004); and, also, that these tendencies and the intermediate ones do not correspond unequivocally or in a one-to-one manner to personal conceptions, which are complex combinations of these tendencies. Nevertheless, what can be deduced from the results is that our adapted instrument allows for a better identification of the two tendencies the items are oriented to (I and TR), while it seems to be more difficult to identify the other two (TE and S). Perhaps we could infer these tendencies only from the lack of correspondence to either of the other two, but finding a clear relationship between TE and S and the middle scores in the Likert scale remains as an unsolved problem.

Regarding the critical interpretation of the factor analysis, we must highlight the behaviour of item 8 ("It is important that students acquire concepts, procedures and mathematical rules"), which is the only one classified within the TR tendency within the first factor (related to *investigative stances*, with almost all I-classified items). This fact illustrates that most of the students still identify acquiring concepts, procedures and rules as a relevant mathematical learning process, even when coinciding with more investigative stances. There are several possible reasons for this. One lies in the phrasing of the item, which combines three different notions (concepts, procedures, and rules). It is also possible that the original classification of this item in the original CEAM instrument within the TR tendency could be wrong, since concepts are also important for the I tendency. Besides, this statement, which might be in nature closer to the TR tendency, can be related with an extended conception based on a purely procedural notion of mathematical learning (Boaler, 2002b; Crawford et al., 1994). Another issue concerning

the EFA relates to the lower loadings of items 17 and 32. Item 17 has a low score in all factors. We assigned it to factor 1 (*investigative stances*), since its loading (0.286) is almost double of the rest (see the Appendix Table 5). The loadings of item 32 are more distributed. We assigned it to factor 3 (*the role of teachers*) since it appears with a negative loading, so that it represents a teacher who validates and discusses the ideas arising during the classroom, instead of a teacher being the undisputed source of knowledge. Nevertheless, these and other items with some smaller cross-loadings (as 1, 8, 13 or 19) endorse the idea of multi-dimensionality and combination amongst tendencies. Thus, the behaviour of these items needs to be further studied using a Confirmatory Factor Analysis (CFA) as a potential extension of this work.

The contribution of the second part of this work also merits discussion, because it provides empirical evidence of the combination of the different theoretical positionings that we have demonstrated by profiling the sample using cluster analysis. This procedure allows us to characterise four profiles of undergraduate students regarding their conceptions of mathematics and its teaching and learning. The cluster analysis results further support the existence of different patterns by academic background, as the distribution of students across clusters varies depending on their bachelor's degrees. The first remarkable emerging idea is that there are no sharp delineations of clusters of conceptions amongst this sample. That is, respondents cannot be described exclusively by one of the four tendencies; instead, each cluster represents a combination of the four tendencies, with different weights of each of the tendencies in each of the clusters. If we had obtained a one-to-one correspondence between factor and clusters, these would endorse the existence of archetypical groups similar to what is described in the tendencies, but the overlapping between factor scores and clusters reinforces the idea of conceptions as much more complex systems, with a stronger representation of the I (in two clusters) and the S (in the other two clusters) tendencies, and a lesser presence of the TR tendency. This finding is consistent with Carrillo and Contreras' theoretical approach, supporting the idea that: "The existence of direct relationships between such epistemological beliefs seem coherent; that is, a certain conception model corresponds to a certain didactic tendency. Nevertheless, [...] a case study [...] revealed that these relationships do not hold true in general" (Carrillo & Contreras, 1995, p. 91; original in Spanish). In our work, this is supported by results from a much larger sample, and we show how conception models correspond to different combinations of instructional tendencies. These results allow us to classify the sample into four different groups, each of them having a different combination of the four instructional tendencies. This is not an isolated view. Other authors have supported similar ideas regarding such combinations (Roelofs et al., 2003; Swan, 2006) and the complexity of the relevant conceptual mental framework (Cross, 2009).

Our results endorse Lui and Bonner's (2016) claim, after their quantitative analysis of 47 in-service and pre-service teachers' beliefs: "although teachers may hold constructivist orientations for mathematics teaching and learning, they may hold traditional beliefs about what mathematics is, leading to less utilisation of conceptual analyses. Further research is warranted to better understand these relationships" (Lui & Bonner, 2016, p. 8). Similarly, Xenofontos states that: "a teacher may hold beliefs pertaining to more than one sub-theme" (2018, p. 52). Although we are not here directly analysing teachers' practice, our results allow us to extend previous authors' claims about the combination of instructional tendencies to the target population of undergraduate students.

Our findings also illustrate that the existence of different combinations of conceptions of mathematics and its teaching and learning is connected to differences in the academic backgrounds of (prospective) teachers and other respondents. Xenofontos (2018) pointed out this possibility, wondering whether teachers' beliefs are similar between those holding

a degree in elementary education and those holding a mathematics degree for secondary education in the same socio-cultural context. Additionally, Muñiz-Rodríguez et al. (2020) found differences amongst in-service secondary teachers' perceptions of professional competence, depending on their previous background (mathematics teachers in secondary education can hold other degrees apart from those related to mathematics, such as engineering or physics), so the present study reinforces the existence of differences according to academic background (which could be seen to correspond to differences between possible future primary and secondary school teachers).

The results also allow us to consider the sample not only as a potential set of future teachers but also as students who may have been taught mathematics in different ways. Thus, we follow Ball (1988) in assuming that teachers tend to reproduce, especially at the beginning of their professional careers, the models with which they have been taught. In this case, we do not have in-service teachers, but we can make an informed speculation that students' conceptions of mathematics, and its teaching and learning may also be strongly influenced by the type of mathematical instruction they received. This helps us to explain differences amongst clusters depending on students' academic backgrounds. In particular, most mathematics students notably cluster in the fourth group, which is the one with the highest presence of the TR tendency. They do not receive any instruction about didactics of mathematics or about mathematics education; thus, the mathematical instruction they know comes only from their mathematics teachers at high school and university level. Mathematics instruction at these levels is still greatly influenced by formalism (see López-Beltrán et al., 2020). However, the most represented tendencies are I and S across the sample as a whole. Students in the primary education degree are intensively trained (and not only in mathematics and mathematics education courses) in constructivist theories, and they constitute most of the sample. Therefore, this group influences the I and S tendencies in every cluster. This also applies to the psychology students, who receive quite technological/instrumental mathematical (mainly statistical) instruction but, at the same time, are hardly educated in theories underlying the importance and effectiveness of learner-centred teaching styles (Fernández, 2013).

As a corollary, and following Ball (1988) again, we can infer that in recent years, mathematics education has changed. The case study by Carrillo and Contreras (1995) established a predominance of the TR and TE tendencies, but in our larger-scale and more recent study, these are the two less strongly represented, while I and S are the two more strongly represented. We are convinced that this is not only the effect of the sample size, but of recent curricular reforms based on constructivism. This change of perspective in younger students has also been found in different cultural contexts, as in Lin et al. (2020). Additionally, we remark that we are not examining teaching styles, as this study did not involve direct observation; we are only analysing undergraduate students' conceptions. If they ever become mathematics teachers, the extant literature suggests that their beliefs and practices may not always agree (Mesa et al., 2014; Swan, 2006).

Before concluding, we must point out the limitations of this work. The first is obvious since, even with quite a large sample, participants come from the same university. Therefore, the influence of the context cannot be isolated. Secondly, we could not use a random sampling, so the generalisation of our conclusions must be very cautiously interpreted. Thirdly, the use of Likert-scale items provides only quantitative information and hampers us from obtaining more detailed interpretations of students' thoughts and therefore of possible richer frameworks for describing their conceptions. Finally, the instrument does not differentiate mathematical domains, and this could hide particular conceptions of certain topics such as statistics (Groth & Meletiou-Mavrotheris, 2018). Similarly, it does not distinguish different types of mathematical instruction, which could be an issue when

considering students from different degrees: students in the mathematics degree receive a mathematically oriented instruction, but students in the psychology degree are instructed into much more instrumental mathematics.

6 Conclusions

The need for empirical evidence within this field has been asserted by Adler et al. (2005), but also echoed more recently (Xie & Cai, 2018). Our work contributes substantially to increased knowledge and understanding of the conceptions of mathematics and its teaching and learning as well as the relationships amongst four instructional tendencies previously identified in the literature, by empirically testing hypotheses about the aggregation of different tendencies to configure individual profiles. Thus, this study provides evidence of the impossibility of encapsulating conceptions of mathematics and its teaching and learning into rigid, sharply delineated profiles. Instead, individuals configure their own conceptions by combining different characteristics of these prototypical profiles (Mura, 1993). Moreover, we have found that more constructivist-oriented tendencies (here, I or S), are predominant amongst this sample, which seems to be a difference from the main trends found in previous studies. Nevertheless, we would like to underscore that, in this research, we are only measuring self-reported conceptions and not measuring practices, so our conclusions cannot be extrapolated to changes in enacted teaching practices in the same context.

To further this research, we are already developing more quantitative and qualitative analyses of these data for determining other possible relationships or structures, including alternative approaches to the factor analysis. We also plan to obtain a different sample in order to conduct a CFA for the model proposed here, with students from different universities and different countries.

The current findings support the need for future research regarding the impact of beliefs and conceptions on, among other things, the application of mathematics classroom observation codes in lesson observations. There are important practical implications to be considered from an analysis of how observers' conceptions of mathematics and its teaching and learning and their relationship with the four instructional tendencies can influence their scores for observed lessons. Additionally, mapping teachers' conceptions onto teachers' knowledge or to their instructional practices is an interesting and important area for further research. Hence, another intended future line of investigation is to analyse the tendencies identified here together with teachers' knowledge. In order to do this, it will be particularly interesting to use models of mathematics teachers' knowledge in which conceptions (including beliefs) play a central role, as is the case in the Mathematics Teachers' Specialised Knowledge (MTSK) model (Aguilar-González et al., 2019; Carrillo et al., 2018), where conceptions constitute a central domain that permeates the mathematical and pedagogical-mathematical knowledge domains (Aguilar-González et al., 2018).

Finally, another interesting line of future investigation is to consider other analytical models that can capture in a much more interpretative way the nature of the relationships among the four instructional tendencies, with particular attention to better characterising the S and TE tendencies which seem to be partially hidden by the TR and I tendencies, and by the construction of the questionnaire. Additionally, further research is needed about the interpretation of middle scores in the Likert scale and its possible relationship with TR and I tendencies.

Appendix

Table 5 Factor loadings (bold) and cross-loadings

Item	Statement	F1	F2	F3	F4
2	The teacher should offer students the opportunity to investigate in order to acquire knowledge	0.666	0.077	0.225	0.100
5	There should be different levels of activities on which the students have the opportunity to work individually	0.497	-0.032	-0.078	0.018
8	It is important that students acquire concepts, procedures and mathematical rules	-0.581	0.081	0.250	0.391
9	The purpose of mathematics is based on developing thinking that allows students to organise, interpret and understand the reality that surrounds them	0.544	-0.003	0.040	-0.068
10	Students should confront problems for which they do not have an answer	0.523	-0.028	0.049	-0.050
12	It is appropriate to use information sources such as the internet, everyday situations, experiences, etc. in the classroom	0.637	-0.012	0.013	0.387
15	Mathematical argumentation and the promotion of attitudes should have the same importance as the acquisition of mathematical concepts and rules	0.454	-0.080	-0.117	-0.012
16	It is important to connect mathematics in the classroom to other contexts	0.448	-0.040	0.160	0.292
17	Learning mathematics requires observing patterns that allow one to guess, check and try to establish generalisations	0.286	-0.069	-0.155	-0.135
21	An ideal teacher knows the interests of their students and takes them into consideration	0.579	0.141	-0.307	0.330
22	The attitudes of students towards their learning can be modified	0.542	0.056	-0.131	0.055
23	Learning mathematical techniques develops transferable skills in students that can be used in the broader social environment	0.582	-0.044	0.123	-0.096
25	Students learn because they have completed a teaching/learning process and they are aware of it	0.347	0.171	-0.078	-0.078
26	It is necessary that students look for ways to answer certain questions	0.673	-0.015	0.178	0.008
27	It is important to make students aware of what they do and why when they are learning	0.580	0.062	-0.147	0.138

Table 5 (continued)

Item	Statement	F1	F2	F3	F4
28	Students should maintain a critical attitude to the information that appears in the classroom	0.621	0.020	-0.042	0.131
29	Students should look for alternative sources of information to those used in the classroom	0.589	0.003	0.205	0.072
31	Student curiosity should be provoked in the learning process	0.807	0.135	-0.033	0.083
35	The ideal teacher raises questions for which there are no a priori answers	0.407	0.024	-0.103	0.067
4	The source of information in a classroom should be the textbook	-0.040	0.619	0.332	-0.153
11	Following a textbook is an appropriate approach on the part of the teacher	-0.021	0.796	0.164	0.026
24	Students should be part of the design of teaching planning (e.g., for a lesson/unit)	0.135	0.341	-0.102	0.166
30	The textbook reflects the contents by which the teacher should be guided	-0.058	0.675	0.119	0.158
3	The source of information in a classroom should be the teacher	0.179	0.271	0.524	0.042
18	It is important for teachers to transmit knowledge in the classroom	-0.137	0.062	0.554	0.146
19	The teacher should be the one who has the greatest importance in the classroom	0.304	0.218	0.411	0.287
32	The ideal teacher should validate/justify the ideas that arise in the classroom	0.260	0.162	-0.282	0.175
33	The teacher should always give the correct information	-0.029	0.081	0.637	0.126
1	Mathematical activity in the classroom includes practice exercises	-0.128	0.294	0.248	0.343
6	Once established, learning objectives can be modified by the teacher	0.263	0.028	0.174	0.397
7	The established curricular programming should be followed	-0.139	0.243	0.199	0.460
13	The teacher should set a level in the classroom and teach students according to this level	0.168	0.306	0.258	0.374
14	The teacher should adapt the lesson plan if it is detected that there are different interests or levels amongst the students	0.042	0.131	-0.045	0.342
20	When students work individually, this improves learning	-0.056	0.006	0.071	0.334
34	The teacher should go to class with prior knowledge about the characteristics of the students	0.301	-0.107	-0.138	0.394

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. LJRM, AAG and LMR are supported by Grant TIN2017-87600-P, from Ministerio de Ciencia e Innovación of Spain, and they belong to the Red MTSK.

Data availability Data are available upon request to the corresponding author.

Code availability Not applicable.

Declarations

Conflict of interest The authors declare no competing interests.

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