Centroids of credal sets: a comparative study⋆

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Abstract. We compare a number of different notions of centroid of a credal set: the Shapley value, that arises in the context of game theory; the average of the extreme points; the incenter with respect to the total variation distance between probability measures; and the limit of a procedure of uniform contraction. We show that these four centers do not coincide in general, give some sufficient conditions for their equality, and analyse their axiomatic properties. Finally, we discuss briefly how to define a notion of centrality measure.

Keywords: Game solutions · credal sets · 2-monotonicity · probability intervals.

1 Introduction

The elicitation of a probability measure within a convex set can be interesting in a wide variety of contexts: we may consider for instance the *core* of a game in coalitional game theory [9], and look for a game solution that provides a way to divide the wealth between the players; we could also consider transformations from imprecise to precise probabilities, as has been done for instance in the context of possibility theory [10]; we might also consider inner approximations of a credal set, so as to look for a more informative model that is compatible with the existing information [16]; or we might also connect the problem with that of transforming second order models with first order ones [24].

In this paper, we examine several possibilities for determining a probability measure that can be considered as the *center* of the credal set. After introducing some preliminary notions in Section 2, in Section 3 we discuss the Shapley value, the average of the extreme points, the incenter of the credal set with respect to the total variation distance and the limit of the contractions of the credal set. These four centroids are compared in Section 4, by showing that they need not coincide in general and establishing sufficient conditions for their equality. In Section 5, we compare the centroids in terms of a number of axiomatic properties; and in Section 6 we discuss how these definitions may lead to a notion of *centrality measure*. Some additional comments are given in Section 7. Due to space limitations, proofs have been omitted.

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2 Preliminary concepts

Consider a finite possibility space $\mathscr{X} = \{x_1, \ldots, x_n\}$, and denote by $\mathbb{P}(\mathscr{X})$ the set of all probability measures on $\mathscr{X}$. A closed and convex subset $\mathscr{M}$ of $\mathbb{P}(\mathscr{X})$ is called a *credal set*. Taking lower envelopes on events, it determines a *coherent* lower and upper probability:

$$\underline{P}(A) = \min_{P \in \mathscr{M}} P(A), \quad \overline{P}(A) = \max_{P \in \mathscr{M}} P(A) \quad \forall A \subseteq \mathscr{X}.$$

A lower probability can be equivalently represented using its Möbius inverse:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \underline{P}(B) \quad \forall A \subseteq \mathscr{X},$$

because $\underline{P}(A) = \sum_{B \subseteq A} m(B)$ for every $A \subseteq \mathscr{X}$.

More generally, we can consider *gambles*, which are functions $f : \mathscr{X} \to \mathbb{R}$; the set of all gambles on $\mathscr{X}$ shall be denoted by $\mathscr{L}$. If for simplicity we use the same symbol P to denote the expectation operator with respect to the probability measure P, then for each gamble $f \in \mathscr{L}$, the credal set $\mathscr{M}$ determines a coherent lower and upper *prevision*:

$$\underline{P}(f) = \min_{P \in \mathscr{M}} P(f), \quad \overline{P}(f) = \max_{P \in \mathscr{M}} P(f) \quad \forall f \in \mathscr{L}.$$

In general, the credal set associated with a coherent lower prevision is not equivalent to that of the coherent lower probability that is its restriction to events. A sufficient condition for their equality is that $\underline{P}$ satisfies the property of 2-monotonicity [5, 22]:

$$\underline{P}(f \vee g) + \underline{P}(f \wedge g) \geq \underline{P}(f) + \underline{P}(g) \quad \forall f, g \in \mathscr{L}.$$

In the case of events, 2-monotonicity means that

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B) \quad \forall A, B \subseteq \mathscr{X}.$$

3 Center points of a credal set

Let us introduce the notions of centroid of a credal set we shall compare in this paper.

3.1 The Shapley value

One of the most popular notions of centroid of a credal set is the *Shapley value*. It was introduced by Shapley [18, 19] in the framework of coalitional game theory, as a 'fair' procedure to distribute some wealth between the players. Later on, it was rediscovered in the context of non-additive probabilities [7] and popularised by Smets as the *pignistic transformation* of a belief function [21].

Definition 1. *Given a credal set $\mathscr{M}$ with associated lower probability $\underline{P}$, its* Shapley value *is defined as the probability measure associated with the following distribution:*

$$\Phi_1^{\mathscr{M}}(\{x\}) = \sum_{x \notin A} \frac{|A|!(n - |A| - 1)!}{n!} \big(\underline{P}(A \cup \{x\}) - \underline{P}(A)\big) \quad \forall x \in \mathscr{X}. \tag{1}$$

It was proven by Smets [20] that, when $\underline{P}$ is a belief function, $\Phi_1^{\mathcal{M}}$ can be equivalently computed as

$$\Phi_1^{\mathcal{M}}(\{x\}) = \sum_{x \in A} \frac{m(A)}{|A|} \quad \forall x \in \mathcal{X}.$$

More generally, when $\underline{P}$ is 2-monotone, it follows from [19] that the extreme points of $\mathcal{M}(\underline{P})$ are given by $\{P_\sigma \mid \sigma \in S_n\}$, where S_n denotes the set of permutations of $\{1,\ldots,n\}$, and given $\sigma \in S_n$, P_σ is determined by the equations

$$P_\sigma\big(\{x_{\sigma(1)},\ldots,x_{\sigma(i)}\}\big) = \underline{P}\big(\{x_{\sigma(1)},\ldots,x_{\sigma(i)}\}\big) \quad \forall i = 1,\ldots,n;$$

then the Shapley value can be computed as

$$\Phi_1^{\mathcal{M}}(\{x\}) = \frac{1}{n!} \sum_{\sigma \in S_n} P_\sigma(\{x\}) \quad \forall x \in \mathcal{X}. \tag{2}$$

In fact, the above results can be extended to arbitrary lower probabilities:

Proposition 1. *Let $\underline{P} : \mathcal{P}(\mathcal{X}) \to [0,1]$ be a lower probability with Möbius inverse m, and let $\Phi_1^{\mathcal{M}}$ be given by Eq.* (1)*. Using the notation P_σ in Eq.*(2)*, it holds that:*

$$\Phi_1^{\mathcal{M}}(\{x\}) = \sum_{x \in A} \frac{m(A)}{|A|} = \sum_{\sigma \in S_n} \frac{P_\sigma(\{x\})}{n!} \quad \forall x \in \mathcal{X}.$$

While the Shapley value seems like a reasonable choice as a central point, it has one important drawback: it is only guaranteed to belong to the credal set (i.e., we can only assure that $\Phi_1^{\mathcal{M}} \geq \underline{P}$) when the lower probability $\underline{P}$ of the credal set is 2-monotone.

More generally, when the lower probability $\underline{P}$ of the credal set is not 2-monotone, we can only assure that $\Phi_1^{\mathcal{M}}$ dominates $\underline{P}$ for small cardinalities ($n \leq 4$), as showed by Baroni and Vicig [2, Prop.5]. We refer to [12] for a study of the consistency between the Shapley value and the lower probability.

3.2 Vertex centroid

The second possibility we consider in this paper is the average of the extreme points of the credal set:

Definition 2. *Let $\mathcal{M}$ be a credal set with a finite number of extreme points $\{P_1,\ldots,P_k\}$. The* vertex centroid *[8] is defined as the average of the extreme points:*

$$\Phi_2^{\mathcal{M}} = \frac{\sum_{i=1}^k P_i}{k}.$$

While the definition above is only applicable in the case of polytopes, it is worth remarking that these include in particular the credal sets $\mathcal{M}(\underline{P})$ associated with a lower probability $\underline{P}$ that is coherent [25], and therefore also in the particular cases of 2-monotonicity. Nevertheless, it is not always applicable for arbitrary credal sets, that are associated with coherent lower *previsions*.

The above centroid is sometimes referred to as the 'center of gravity' of the credal set; however, strictly speaking the center of gravity corresponds to the expectation of the set over a uniform probability distribution, and not only over its extreme points. It is not difficult to show that this other notion does not necessarily produce the same centroid as Definition 2. While the center of gravity has the advantage of being applicable over general credal sets, and not only on polytopes, it also has the drawback of being computationally more expensive (see for example [8]).

It follows from Definition 2 that $\Phi_2^{\mathscr{M}}$ always belongs to the credal set $\mathscr{M}$; considering the comments in the previous section, this implies that it need not coincide with Shapley value when the lower envelope $\underline{P}$ of $\mathscr{M}$ is not 2-monotone. As we shall see later on, they need not coincide even under 2-monotonicity: while it has been proven in [19] that the extreme points of $\mathscr{M}(\underline{P})$ are indeed $\{P_\sigma \mid \sigma \in S_n\}$, a key difference is that in the computation of Shapley value in Eq. (2) we are allowing for repetitions of the same extreme point, while in Definition 2 we do not.

3.3 Incenter

Our third approach consists in considering the *incenter* of the credal set, which is the element or elements $P \in \mathscr{M}$ for which we can include the largest ball centered on P in the interior of the credal set. This notion requires the specification of a distance between probability measures from which the balls are defined. We consider here the *total variation distance* [11], which is the one associated with the supremum norm:

$$d_{TV}(P,Q) = \max_{A \subseteq \mathscr{X}} |P(A) - Q(A)|.$$

Our choice of this distance is due to the fact that a ball with the total variation distance is always a polytope, unlike some other distances such as the Euclidean distance. To ease the notation, we will simply denote d_{TV} by d.

Definition 3. *Let $\mathscr{M}$ be a credal set. The* supremum radius *of $\mathscr{M}$ is defined as*

$$\alpha_I = \sup\Big\{\alpha \mid \exists P_0 \in \mathscr{M} \text{ such that } \operatorname{int}\big(B_d^{\alpha}(P_0)\big) \subseteq \operatorname{int}(\mathscr{M})\Big\}. \tag{3}$$

Then any $P \in \mathscr{M}$ such that $\operatorname{int}\big(B_d^{\alpha_I}(P)\big) \subseteq \operatorname{int}(\mathscr{M})$ is called an incenter *of the credal set $\mathscr{M}$. The set of all such P is denoted $\Psi_3^{\mathscr{M}}$.*

The reason why we are requiring the inclusion $\operatorname{int}\big(B_d^{\alpha_I}(P_0)\big) \subseteq \operatorname{int}(\mathscr{M})$ in the above definition is that otherwise we could obtain the counter-intuitive result that an incenter belongs to the boundary of $\mathscr{M}$, which we find incompatible with the underlying idea of centrality (as in 'deepest inside the credal set') we consider in this paper.

Two natural questions related to the incenter is whether (i) it exists, and (ii) it is unique. Our next result provides an answer to the first question.

Proposition 2. *Consider a credal set $\mathscr{M}$. Then the value α_I in Eq.* (3) *is a maximum. As a consequence, the set $\Psi_3^{\mathscr{M}}$ is always non-empty, and any of its elements belongs to* $\operatorname{int}(\mathscr{M})$.

Concerning the second question, the set $\Phi_3^{\mathcal{M}}$ may have more than one element, as we show next:

Example 1. Consider the possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$, and the credal set $\mathcal{M}$ determined by $P(\{x_1\}) \in [0.5, 0.8]$, $P(\{x_2\}) \in [0.1, 0.25]$ and $P(\{x_3\}) \in [0.1, 0.35]$. It holds that $\alpha_I = 0.075$ and that the incenter is not unique: $\Psi_3^{\mathcal{M}}$ coincides with the convex combinations of $Q_1 = (0.575, 0.175, 0.25)$ and $Q_2 = (0.65, 0.175, 0.175)$. Hence, there are infinite incenters. Figure 1 shows the graphical representation of $\mathcal{M}$ as well as the balls $B_d^{\alpha_I}(Q_1)$, $B_d^{\alpha_I}(Q_2)$ and $B_d^{\alpha_I}(Q_\beta)$ for $\beta = 0.5$. ♦

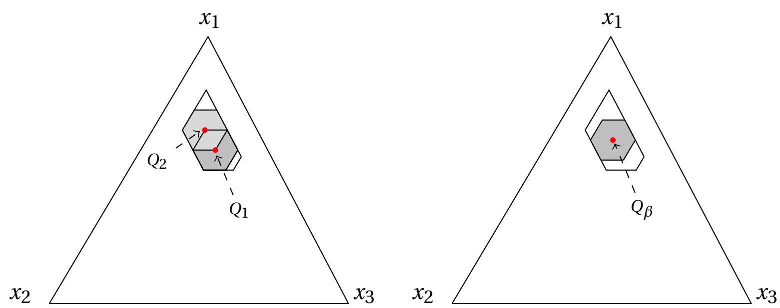

Fig. 1. Credal set $\mathcal{M}$ in Example 1, the incenters Q_1 and Q_2 with the ball they induce (left hand-side figure) and Q_β for $\beta = 0.5$ with the ball it induces (right hand-side figure).

Related to this approach, we may alternatively have considered the *circumcenter* of the credal set, which is the element or elements for which the smallest ball centered on them includes the credal set. That is, we may consider

$$\alpha_C = \inf\{\alpha \mid \exists P_0 \in \mathcal{M} \text{ such that } B_d^{\alpha}(P_0) \supseteq \mathcal{M}\},$$

and then let $\Psi_3'^{\mathcal{M}}$ be the set of those P such that $B_d^{\alpha_C}(P) \supseteq \mathcal{M}$. However, this approach possesses the two drawbacks we have discussed so far: not only it need not lead to a unique solution, but also [1] it may produce values that are outside the credal set.

In this sense, and as suggested by a reviewer, one possibility would be to consider, for each $P \in \mathcal{M}$, the value $\alpha_P = \inf\{\alpha \mid B_d^{\alpha}(P) \supseteq \mathcal{M}\}$, and then call $P \in \mathcal{M}$ a circumcenter of $\mathcal{M}$ if it minimises the value α_P. In this manner we would guarantee that the solution obtained belongs to the credal set, although it may be an element of the boundary. The study of this approach is left as an open problem.

3.4 Contraction centroid

Our fourth approach is motivated by the lack of uniqueness of the incenter. Consider a credal set $\mathcal{M}$ determined by a finite number of constraints. This means that there are two (disjoint) sets of gambles $\mathcal{L}^>$ and $\mathcal{L}^=$ such that the lower and upper previsions $\underline{P},\overline{P}$ associated with $\mathcal{M}$ satisfy $\underline{P}(f)=\overline{P}(f)$ for any $f\in\mathcal{L}^=$ and $\underline{P}(f)<\overline{P}(f)$ for any $f\in\mathcal{L}^>$, and that the credal set can be expressed as:

$$\mathcal{M}=\left\{P\in\mathbb{P}(\mathcal{X})\mid P(f)=\underline{P}(f)\ \ \forall f\in\mathcal{L}^=,\quad P(f)\geq\underline{P}(f)\ \ \forall f\in\mathcal{L}^>\right\}. \tag{4}$$

Note that we can assume without loss of generality that $\{I_A\mid A\subseteq\mathcal{X}\}\subseteq\mathcal{L}^=\cup\mathcal{L}^>$; in that case when $\mathcal{L}^>$ is empty we obtain that $\underline{P}(A)=\overline{P}(A)$ for any $A\subseteq\mathcal{X}$, and thus that $\mathcal{M}$ has only one element.

The idea of this fourth approach is to contract $\mathcal{M}$ in a uniform manner as long as we can, and then proceed iteratively reducing the cardinality of $\mathcal{L}^>$. More specifically, we increase the value of the lower prevision in a constant amount α in all the gambles $f\in\mathcal{L}^>$. Our next proposition gives further insight into this idea:

Proposition 3. *Let $\mathcal{M}$ be a credal set determined by a finite number of gambles $\mathcal{L}^=,\mathcal{L}^>$ by means of Eq.* (4)*. For a given $\alpha>0$, let:*

$$\mathcal{M}_\alpha=\left\{P\in\mathbb{P}(\mathcal{X})\mid P(f)=\underline{P}(f)\ \ \forall f\in\mathcal{L}^=,\quad P(f)\geq\underline{P}(f)+\alpha\ \ \forall f\in\mathcal{L}^>\right\}. \tag{5}$$

Then the $\Lambda=\{\alpha\mid\mathcal{M}_\alpha\neq\emptyset\}$ is non-empty and has a maximum value $\alpha_S=\max\Lambda$. Moreover, there is some $f\in\mathcal{L}^>$ such that $P(f)=\underline{P}(f)+\alpha_S$ for any $P\in\mathcal{M}_{\alpha_S}$.

The above proposition also tells us that when we get to the set $\mathcal{M}_{\alpha_S}$ the size of $\mathcal{L}^>$ decreases; we can therefore iterate the procedure, now starting with $\mathcal{M}_{\alpha_S}$, and after a finite number of steps we end up with a precise credal set: one formed by a single probability measure that we shall call the *contraction centroid*. This means that after a finite number of steps we obtain the values $\alpha_S^1,\dots,\alpha_S^l$ and the chain of nested credal sets:

$$\mathcal{M}\supset\mathcal{M}_{\alpha_S^1}\supset\ldots\supset\mathcal{M}_{\alpha_S^l}=\left\{\Phi_4^{\mathcal{M}}\right\}. \tag{6}$$

Example 2. Consider again the credal set from Example 1. Then $\mathcal{L}^=$ is empty and $\mathcal{L}^>$ contains six gambles, that coincide with the indicator functions of the non-trivial events. Let us see that $\alpha_S^1=\max\Lambda=0.075$. On the one hand, $\mathcal{M}_{\alpha_S^1}$ is non-empty because the probability $P=(0.625,0.175,0.2)$ belongs to $\mathcal{M}_{\alpha_S^1}$. On the other hand, if we increase the lower probability in a quantity α, to keep coherence it should happen that:

$$1\geq\left(\underline{P}(\{x_2\})+\alpha\right)+\left(\underline{P}(\{x_1,x_3\})+\alpha\right)=0.85+2\alpha,$$

so $\alpha\leq 0.075$. Therefore, $\alpha_S^1=0.075$, and this gives rise to the following credal set:

$$\mathcal{M}_{\alpha_S^1}=\left\{P\in\mathbb{P}(\mathcal{X})\mid P(A)\geq\underline{P}(A)+\alpha_S^1\ \ \forall A\neq\emptyset,\mathcal{X}\right\}.$$

If we denote by $\underline{P}_1$ its associated lower probability, it is given by:

$$\begin{aligned}&\underline{P}_1(\{x_1\})=0.575,\quad \underline{P}_1(\{x_2\})=0.175,\quad \underline{P}_1(\{x_3\})=0.175,\\&\underline{P}_1(\{x_1,x_2\})=0.75,\quad \underline{P}_1(\{x_1,x_3\})=0.825,\quad \underline{P}_1(\{x_2,x_3\})=0.35.\end{aligned}$$

In this second step, $\mathcal{L}_1^==\left\{I_{\{x_2\}},I_{\{x_1,x_3\}}\right\}$ and $\mathcal{L}_1^>=\left\{I_{\{x_1\}},I_{\{x_3\}},I_{\{x_1,x_2\}},I_{\{x_2,x_3\}}\right\}$, i.e., there are two events whose probability is now fixed. Iterating the procedure, we obtain $\alpha_S^2=0.0375$ and $\mathcal{M}_{\alpha_S^2}=\left\{\Phi_4^{\mathcal{M}}\right\}$, where $\Phi_4^{\mathcal{M}}=(0.6125,0.175,0.2125)$. Figure 2 shows $\mathcal{M}_{\alpha_S^1}$ (in blue) and $\mathcal{M}_{\alpha_S^2}$ (in red). ♦

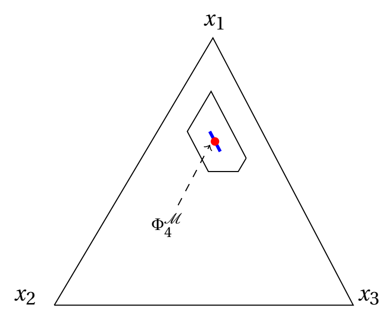

Fig. 2. Graphical representation of the credal sets $\mathcal{M}_{\alpha_S^1}$ in blue and $\mathcal{M}_{\alpha_S^2}$ in red.

It is worth mentioning that the lower envelope of the credal set $\mathcal{M}_\alpha$ in Eq. (5) does not necessarily coincide with $\underline{P}+\alpha$. While by construction it dominates this lower probability, they may not agree on some events because $\underline{P}+\alpha$ need not be coherent. The lower envelope of $\mathcal{M}_\alpha$ corresponds to the *natural extension* [23] of $\underline{P}+\alpha$.

4 Relationships between the centroids

In our previous section we have introduced four different notions of the center of a credal set. Let us begin by showing that these four notions are indeed different:

Example 3. Consider the credal set in Example 1; there, we gave the set of incenters $\Psi_3^{\mathcal{M}}$, while the contraction centroid $\Phi_4^{\mathcal{M}}$ was given in Example 2. The extreme points of the credal set $\mathcal{M}$ are given by:

	x_1	x_2	x_3
P_1	0.55	0.1	0.35
P_2	0.5	0.15	0.35
P_3	0.5	0.25	0.25

	x_1	x_2	x_3
P_4	0.65	0.25	0.1
P_5	0.8	0.1	0.1

It follows that the average of the extreme points and the Shapley value are given by $\Phi_2^{\mathcal{M}}=(0.6,0.17,0.23)$ and $\Phi_1^{\mathcal{M}}=(0.6\overline{3},0.158\overline{3},0.208\overline{3})$. We conclude, taking into account also Examples 1 and 2, that the four approaches lead to different results. ⧫

While the four approaches do not lead to the same solution in general, in the following subsections we give some sufficient conditions for their equality.

4.1 Probability intervals

A probability interval [4] is an uncertainty model that gives lower and upper bounds to the probability of the singletons: $\mathcal{I}=\{[l_i,u_i] \mid l_i\le u_i \ \ \forall i=1,\dots,n\}$. It determines a credal set given by:

$$\mathcal{M}(\mathcal{I})=\left\{P\in\mathbb{P}(\mathcal{X}) \mid l_i\le P(\{x_i\})\le u_i \ \ \forall i=1,\dots,n\right\}.$$

Taking lower and upper envelopes of $\mathscr{M}(\mathscr{I})$ we obtain a lower and upper prevision, and the probability interval is called *coherent* when $\underline{P}(\{x_i\}) = l_i$ and $\overline{P}(\{x_i\}) = u_i$ for every $i = 1,\dots,n$; in that case, the values of $\underline{P}$ for events can be easily computed using the results in [4]. When the credal set is determined by a coherent probability interval, we can give an explicit formula for the value α_S in the contraction method.

Proposition 4. *Let $\mathscr{M}$ be a credal set determined by a coherent probability interval $\mathscr{I} = \{[l_i, u_i] \mid \forall i = 1,\dots,n\}$. Let $\mathscr{I}^> = \{i \in \{1,\dots,n\} \mid l_i < u_i\}$ and $\mathscr{I}^= = \{i \in \{1,\dots,n\} \mid l_i = u_i\}$. Then:*

1. *The value $\alpha_S = \max\Lambda$ is given by:*
$$\alpha_S = \min\left\{\frac{1}{|\mathscr{I}^>|}\left(1-\sum_{i=1}^n l_i\right), \frac{1}{|\mathscr{I}^>|}\left(\sum_{i=1}^n u_i - 1\right), \frac{1}{2}\min_{i\in\mathscr{I}^>}(u_i - l_i)\right\}.$$
2. *If $\alpha_S = \frac{1}{|\mathscr{I}^>|}\left(1-\sum_{i=1,\dots,n} l_i\right)$ or $\alpha_S = \frac{1}{|\mathscr{I}^>|}\left(\sum_{i=1,\dots,n} u_i - 1\right)$, then $\mathscr{M}_{\alpha_S} = \{\Phi_4^{\mathscr{M}(\mathscr{I})}\}$.*

From Proposition 4 we can also deduce an explicit formula for the value α_S for the *Linear Vacuous* (LV) and the *Pari Mutuel Model* (PMM), which constitute particular instances of *distortion models* [14, 15] and *nearly linear models* [3]. The PMM [13, 17, 23] is determined by the coherent lower probability $\underline{P}_{PMM}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\}$ for any $A \subseteq \mathscr{X}$, where $P_0 \in \mathbb{P}(\mathscr{X})$ is a given probability measure and $\delta > 0$. Similarly, the LV [23] is defined by the coherent lower probability given by $\underline{P}_{LV}(A) = (1-\delta)P_0(A)$, for any $A \subset \mathscr{X}$ and $\underline{P}_{LV}(\mathscr{X}) = 1$, where $P_0 \in \mathbb{P}(\mathscr{X})$ and $\delta \in (0,1)$. Both the PMM and the LV are instances of probability intervals, where:

$$\mathscr{I}_{PMM} = \Big\{\big[\max\{(1+\delta)P_0(\{x_i\}) - \delta, 0\}, \min\{(1+\delta)P_0(\{x_i\}), 1\}\big] \;\big|\; i = 1,\dots,n\Big\},$$
$$\mathscr{I}_{LV} = \Big\{\big[(1-\delta)P_0(\{x_i\}), \min\{(1-\delta)P_0(\{x_i\}) + \delta, 1\}\big] \;\big|\; i = 1,\dots,n\Big\}.$$

Thus we can apply Proposition 4 for computing the value α_S. In fact, when both P_0 and the lower probability only take the values 0 and 1 for the impossible and sure events, respectively, the computation of α_S can be simplified and the procedure of contracting the credal set finishes in only one step.

Corollary 1. *Consider a credal set $\mathscr{M}$ associated with either a PMM $\underline{P}_{PMM}$ or a LV $\underline{P}_{LV}$ determined by $P_0 \in \mathbb{P}(\mathscr{X})$ and the distortion parameter δ. Assume that $P_0(A)$, $\underline{P}_{PMM}(A)$ and $\underline{P}_{LV}(A)$ belong to $(0,1)$ for any $A \neq \emptyset, \mathscr{X}$. Then:*

1. *For the PMM, $\alpha_S = \max\Lambda = \frac{\delta}{n}$ and $\mathscr{M}_\alpha = \{\Phi_4^{\mathscr{M}}\}$ where $\Phi_4^{\mathscr{M}}(\{x_i\}) = (1+\delta)P_0(\{x_i\}) - \frac{\delta}{n}$ for any $i = 1,\dots,n$.*
2. *For the LV, $\alpha_S = \max\Lambda = \frac{\delta}{n}$ and $\mathscr{M}_\alpha = \{\Phi_4^{\mathscr{M}}\}$ where $\Phi_4^{\mathscr{M}}(\{x_i\}) = (1-\delta)P_0(\{x_i\}) + \frac{\delta}{n}$ for any $i = 1,\dots,n$.*
3. *In both cases, there is a unique incenter (i.e., $\Psi_3^{\mathscr{M}} = \{\Phi_3^{\mathscr{M}}\}$) and $\Phi_1^{\mathscr{M}} = \Phi_2^{\mathscr{M}} = \Phi_3^{\mathscr{M}} = \Phi_4^{\mathscr{M}}$.*

In this respect, it is worth remarking that (i) the good behaviour of these two distortion models is in line of other desirable properties they possess, as discussed in [6, 14, 15]; and (ii) the centroid of the LV and PMM models does not coincide with P_0, because the distortion is not done uniformly in all directions of the simplex. This was already shown in [12, Sec.4.3] for the particular case of the Shapley value.

4.2 Connections between the incenter and the contraction centroid

Next we prove a connection between contraction centroid and the set of incenters.

Proposition 5. *Let $\mathcal{M}$ be a credal set. For any $\alpha \le \alpha_S$, if $P_0 \in \mathcal{M}_\alpha$, then $B_d^\alpha(P_0) \subseteq \mathcal{M}$. When $\mathcal{M}$ is determined by its restriction to events, then the converse also holds. As a consequence, in that case $\mathcal{M}_{\alpha_S} = \Psi_3^{\mathcal{M}}$.*

From this result we deduce that, when $\mathcal{M}$ is associated with a coherent lower probability, the credal set $\mathcal{M}_{\alpha_S}$ obtained contracting the initial credal set coincides with the set of incenters. However, when the credal set is associated with a lower prevision, this equivalence does not hold in general, as the next example shows.

Example 4. Consider $\mathcal{X} = \{x_1, x_2\}$, the gamble f given by $f(x_1) = 1$ and $f(x_2) = 0.9$ and consider the credal set $\mathcal{M}$ given by:

$$\mathcal{M} = \{P \in \mathbb{P}(\mathcal{X}) \mid 0.91 \le P(f) \le 0.99\}.$$

Equivalently, this credal set is determined by the extreme points $P_1 = (0.1, 0.9)$ and $P_2 = (0.9, 0.1)$. Consider now $P_0 = (0.2, 0.8)$ and $\alpha = 0.015$. Then, the ball $B_d^\alpha(P_0)$ has two extreme points, $Q_1 = (0.185, 0.815)$ and $Q_2 = (0.215, 0.785)$, so $B_d^\alpha(P_0) \subseteq \mathcal{M}$. However, $P_0(f) = 0.92 \notin [0.925, 0.975] = [\underline{P}(f) + \alpha, \overline{P}(f) - \alpha]$. ♦

5 Properties of the centroids

Next we compare the different centroids in terms of the axiomatic properties they satisfy. In this respect, it is worth recalling that Shapley value was characterised in the context of coalitional game theory as the unique probability distribution satisfying the following axioms:

Efficiency: $\sum_{i=1}^n \Phi^{\mathcal{M}}(\{x_i\}) = 1$.
Symmetry: $\underline{P}\big(A \cup \{x_i\}\big) = \underline{P}\big(A \cup \{x_j\}\big)$ for any $A \subseteq \mathcal{X} \setminus \{x_i, x_j\}$ implies that $\Phi^{\mathcal{M}}(\{x_i\}) = \Phi^{\mathcal{M}}(\{x_j\})$.
Linearity: $\Phi^{\mathcal{M}(\lambda_1 \underline{P}_1 + \lambda_2 \underline{P}_2)} = \lambda_1 \Phi^{\mathcal{M}(\underline{P}_1)} + \lambda_2 \Phi^{\mathcal{M}(\underline{P}_2)}$ for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and every $\underline{P}_1, \underline{P}_2$.
Null player: $\underline{P}\big(A \cup \{x_i\}\big) = \underline{P}(A)$ for any $A \subseteq \mathcal{X} \setminus \{x_i\}$ implies $\Phi^{\mathcal{M}}(\{x_i\}) = 0$.

Let us study these properties for the other centroids considered in this paper. Note that in the case of the incenter, they should be required to any $\Phi_3^{\mathcal{M}} \in \Psi_3^{\mathcal{M}}$.

In this respect, since in the framework of this paper any center of a credal set shall be a probability measure, the efficiency property is trivially satisfied. With respect to the other properties, it is not difficult to establish the following:

Proposition 6. $\Phi_2^{\mathcal{M}}$, $\Phi_4^{\mathcal{M}}$ *and any* $\Phi_3^{\mathcal{M}} \in \Psi_3^{\mathcal{M}}$ *satisfy the symmetry and null-player properties, but none of them satisfies linearity.*

Next we consider other desirable properties of a centroid.

Definition 4. *Let $\Phi^{\mathcal{M}}$ be a centroid of a credal set $\mathcal{M}$. We say that it satisfies:*

- Consistency *if* $\Phi^{\mathscr{M}} \in \mathscr{M}$.
- Continuity *if for any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $d(\underline{P}_1, \underline{P}_2) := \max_{A \subseteq \mathscr{X}} |\underline{P}_1(A) - \underline{P}_2(A)| < \delta$ *implies* $d\big(\Phi^{\mathscr{M}(\underline{P}_1)}, \Phi^{\mathscr{M}(\underline{P}_2)}\big) < \varepsilon$.
- Ignorance preservation *if* $\mathscr{M} = \mathbb{P}(\mathscr{X})$ *implies that* $\Phi^{\mathscr{M}}$ *is the uniform distribution.*

When dealing with the incenter, the previous properties should be slightly rewritten due to its lack of uniqueness: the incenter satisfies consistency when $\Psi_3^{\mathscr{M}} \subseteq \mathscr{M}$; it satisfies ignorance preservation if $\mathscr{M} = \mathbb{P}(\mathscr{X})$ implies that the only element of $\Psi_3^{\mathscr{M}}$ is the uniform distribution; and it satisfies continuity when for any $\varepsilon > 0$ there exists some $\delta > 0$ such that $d(\underline{P}_1, \underline{P}_2) < \delta$ implies that $d(\underline{Q}_1, \underline{Q}_2) < \varepsilon$, where $\underline{Q}_1$ and $\underline{Q}_2$ are the lower envelopes of $\Psi_3^{\mathscr{M}(\underline{P}_1)}$ and $\Psi_3^{\mathscr{M}(\underline{P}_2)}$.

Proposition 7. *1.* $\Phi_1^{\mathscr{M}}$ *satisfies continuity and ignorance preservation, but it does not satisfy consistency.*
2. $\Phi_2^{\mathscr{M}}$ *satisfies consistency and ignorance preservation, but not continuity.*
3. $\Psi_3^{\mathscr{M}}, \Phi_4^{\mathscr{M}}$ *satisfy consistency, continuity and ignorance preservation.*

The next table summarises the results from this section:

	Shapley value	Average of extremes	Incenter (d_{TV})	Contraction center
Efficiency	YES	YES	YES	YES
Symmetry	YES	YES	YES	YES
Linearity	YES	NO	NO	NO
Null player	YES	YES	YES	YES
Consistency	NO	YES	YES	YES
Continuity	YES	NO	YES	YES
Ignorance preservation	YES	YES	YES	YES

Finally, it is worth remarking on the ability of these centroids to distinguish between lower previsions and lower probabilities: in general, a credal set $\mathscr{M}$ determines, by means of lower envelopes, a lower prevision $\underline{P}$ on a gambles and a lower probability by considering its restriction to events. However, the credal set $\mathscr{M}'$ determined by the latter, given by $\mathscr{M}' := \{P \mid P(A) \geq \underline{P}(A) \quad \forall A \subseteq \mathscr{X}\}$ is in general a superset of the original credal set $\mathscr{M}$. It would be desirable then that the centroids of $\mathscr{M}$ and $\mathscr{M}'$ do not necessarily coincide, since they correspond to different credal sets. In this respect, it is not difficult to show that $\Psi_3^{\mathscr{M}}, \Phi_4^{\mathscr{M}}$ are capable of distinguishing between lower previsions and lower probabilities, and so does $\Phi_2^{\mathscr{M}}$ (with the restriction that it is only applicable on polytopes). On the other hand, the Shapley value is only defined via the lower probability, and so it does not distinguish between lower probabilities and lower previsions.

6 Centrality measures

More generally, instead of determining which element of the credal set can be considered its center, we may define a *centrality measure*, that allows us to quantify how

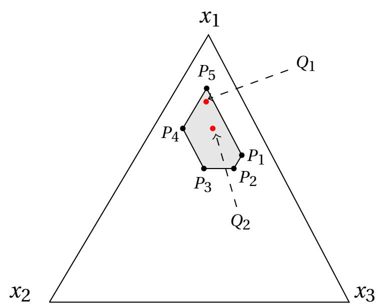

Fig. 3. Graphical representation of the credal set $\mathcal{M}$ in Example 1

deep in the interior of a credal set an element is. Consider for instance the same credal set as in Example 1, depicted in Figure 3. Intuitively, given the probability measures $Q_1 = (0.75, 0.125, 0.125)$ and $Q_2 = (0.65, 0.15, 0.2)$, emphasised in red in Figure 3, Q_2 should have a greater *centrality degree* than Q_1.

This simple example suggests the following definition of centrality measure.

Definition 5. *Given a credal set $\mathcal{M}$, a* centrality measure *is a function $\varphi : \mathbb{P}(\mathcal{X}) \to [0,1]$ satisfying the following properties:*

CM1 $\varphi(P) = 0$ *for every $P \notin \mathcal{M}$.*
CM2 *If $P \in ext(\mathcal{M})$, then $\varphi(P) = 0$.*
CM3 *There exists a unique $P_0 \in \mathcal{M}$ satisfying $\varphi(P_0) = 1$. Such P_0 is called* central point *in $\mathcal{M}$ with respect to φ.*
CM4 *Consider $P \in ext(\mathcal{M})$, P_0 the probability given in the previous item and $\lambda, \beta \in [0,1]$ such that $\lambda \geq \beta$. Given $P_1 = \lambda P + (1-\lambda)P_0$ and $P_2 = \beta P + (1-\beta)P_0$, it holds that $\varphi(P_1) \leq \varphi(P_2)$.*

The idea underlying these properties is the following: **CM1** tells us that an element outside the credal set should have degree of centrality zero; from **CM2**, the same should hold for the extreme points of the credal set; **CM3** means that there is a unique probability P_0 with degree of centrality 1; finally, property **CM4** represents the idea that the closer a probability is to P_0, the greater its degree of centrality.

A centrality measure φ allows to define a chain of credal sets $\{\mathcal{M}_\alpha\}_{\alpha \in [0,1]}$, where $\mathcal{M}_\alpha$ is formed by the probabilities with centrality degree of at least α:

$$\mathcal{M}_\alpha = \{P \in \mathcal{M} \mid \varphi(P) \geq \alpha\} \quad \forall \alpha \in [0,1].$$

We next discuss two possible strategies for defining a centrality measure. The former consists in considering a centroid of the credal set, and to measure the distance with respect to it. It requires then to specify both the centroid and the distance. Out of the options considered in the previous section, we would reject $\Phi_1^{\mathcal{M}}$ due to the lack of consistency and $\Psi_3^{\mathcal{M}}$ because of non-uniqueness. With respect to the distance, we will be considering here the total variation, although it would also be possible to consider other options such as the L_1 or the Euclidean distances.

In this sense, if we let $\Phi^{\mathscr{M}}$ be this centroid and take $\beta = \min\Big\{d\big(\Phi^{\mathscr{M}}, P_i\big) : P_i \in ext(\mathscr{M})\Big\}$, then we can define

$$\varphi_1(P) = 1 - \min\left\{\frac{d\big(P, \Phi^{\mathscr{M}}\big)}{\beta}, 1\right\}. \tag{7}$$

A second approach would consist in considering directly a chain $\{\mathscr{M}_\alpha\}_{\alpha\in[0,1]}$ of convex credal sets such that $\mathscr{M}_0 := \mathscr{M}$, $\mathscr{M}_1$ is a singleton determining the central point $\Phi^{\mathscr{M}}$ and where $\mathscr{M}_\alpha$ is included in the interior of $\mathscr{M}$ for any $\alpha > 0$, and letting

$$\varphi_2(P) = \sup\Big\{\alpha \in [0,1] \mid P \in \mathscr{M}_\alpha\Big\}. \tag{8}$$

The chain $\{\mathscr{M}_\alpha\}_{\alpha\in[0,1]}$ of credal sets could be defined, for example, as:

$$\mathscr{M}_\alpha = CH\big(\big\{(1-\alpha)P + \alpha\Phi^{\mathscr{M}} \mid P \in ext(\mathscr{M})\big\}\big) \quad \forall \alpha \in [0,1].$$

Let us show that both approaches lead to a centrality measure.

Proposition 8. *Let $\mathscr{M}$ be a credal set, and let φ_1, φ_2 be given by Eqs.* (7) *and* (8)*. Then φ_1 and φ_2 satisfy conditions (CM1)–(CM4).*

It is also possible to define a centrality measure by considering the chain of credal sets from Eq. (6). For this, note that for each $P \in \mathscr{M}$ there is $j \in \{1,\ldots,l\}$ such that $P \in \mathscr{M}_{\alpha_S^{j-1}} \setminus \mathscr{M}_{\alpha_S^j}$. Also, there is $\alpha \in \Lambda_{j-1}$ such that $P \in \big(\mathscr{M}_{j-1}\big)_\alpha$, but $P \notin \big(\mathscr{M}_{j-1}\big)_{\alpha+\varepsilon}$ for any $\varepsilon > 0$. Then we let:

$$\varphi_3(P) = \frac{\alpha_S^1 + \ldots + \alpha_S^j + \alpha}{\alpha_S^1 + \ldots + \alpha_S^l}. \tag{9}$$

Proposition 9. *The function φ_3 defined in Eq.* (9) *satisfies conditions (CM1)–(CM4).*

7 Conclusions

In this paper, we have analysed four different definitions of centroid for a credal set. These could serve as a representative of the credal set or as a game solution when the credal set is interpreted as the core of a cooperative game. We have analysed their differences, the connection between them as well as their axiomatic properties. Also, we have seen that the problem could be tackled by defining a centrality measure whose unique modal point is interpreted as the centroid.

While the above results give some overview of the properties of the centroids of a credal set, there is still much work to be done in order to have a full picture of this problem. On the one hand, we could consider other possibilities in the context of game solutions, such as the Banzhaf value, or other alternatives to the total variation distance, such as the Euclidean distance or the Kullback-Leibler divergence; secondly, it would be interesting to obtain further conditions for the equality between some of these centroids; and finally, a deeper study of centrality measures and their axiomatic properties would be of interest. We intend to tackle these problems in the near future.

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