ON THE CONNECTEDNESS OF A RANDOM CLOSED SET OF AN EUCLIDEAN SPACE

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Abstract. Our aim is to obtain a suitable characterization of certain topological properties of a random closed set through its *capacity functional*. The main technique mixes two different fields: on the one hand, the abstract simplicial complex associated to a covering, whose topology coincides with the topology of the original set thanks to the *nerve* theorem. On the other hand, the celebrated Choquet-Kendall-Matheron theorem, which states that a random closed set is characterized by its capacity functional.

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1. Introduction

The first concepts and results involving random sets appeared some time ago in both, probabilistic and statistical literature (see for instance [9]) to the first works on the so-called Boolean models. These models appeared early in the area of applied probability, as an attempt to describe random geometrical structures. Its use was however marginal until the publication of Matheron's book in the 70s [10].

In his book, Matheron introduces in a formal way, not only the concept of random closed set, but also establish all the formalism and theoretical machinery required, including the measure formalism used until these days.

From this publication, the theory of random closed sets has experienced a rapid development and a growing interest in different areas as physics, engineering, image processing and economy. For instance, random closed sets are used on the risk analysis of multivariate portfolios [7], the stochastic labelling of images with applications to biology [1] or in the theory of imprecise probabilities [11, 15].

Random closed sets are introduced in [10] (see also [12, 13]) as maps from a complete probability space into the class of closed sets of a topological space, usually denoted by $\mathcal{F}$. The space $\mathcal{F}$ is then endowed with a topology commonly known as the Fell topology, and with its associated σ-algebra. In this context, the probabilities of interest are those related to the probability that a random set intersects a given compact set. To this end it is defined the so-called capacity functional which maps a compact set K with the probability that X intersects (or hits) K.

One of the main results on this regard is the Choquet-Kendall-Matheron theorem, which states that random closed sets are bi-univocally characterized by its capacity functional. Hence, we can interpret that the capacity functional behaves for random closed sets as the probability distributions for a random variable, [12].

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Previous theorem ensures then that the capacity functional should encapsulate all the relevant information of its corresponding random closed set, including its topological information. In fact, in [14] the author was able to characterize the connectedness of a closed random set in terms of its capacity functional, assuming that the associated topological space was $\mathbb{R}$.

Inspired by this result, our aim in this paper is to obtain a similar characterization but in more general random closed sets. In order to achieve this, we will introduce here a technique based on the use of abstract simplicial complexes. As it is well known [8], under some mild hypothesis simplicial complexes share the same topological properties of the corresponding space. Hence, we will be able to make use of the well-known characterizations of connectedness in terms of simplicial complexes, which will be key for our main results.

The contents of this paper are organized as follow: In Section 2 we introduce all the required elements from topological and metric theory that we will require for the rest of the paper, including simplicial complexes. On Section 3 we review the concepts of random closed set, the Fell topology and the capacity functional. We will also define how are we going to define connectedness (and ϵ-connectedness) in this context. Finally, Section 4 includes the main result of the paper, Theorem 4.1, and a discussion of some of its consequences, including a characterization of connectedness in terms of the capacity functional Theorem 4.4.

2. Topological Preliminaries

Along this section, we will introduce the main concepts and results that we will need from the topological theory for the next section. Observe that we will restrict ourselves to work on compact sets K of the euclidean space $\mathbb{R}^n$ even so our results can be easily generalized to more general contexts (for instance, by working on compact Riemannian manifolds).

Let us begin by recalling that there are several ways to define the notion of a random closed set, each one more suitable than the other in different contexts. One of such definitions requires the notion of the *Fell topology* defined over the space of closed (and so, compact) sets inside K, that we will denote by $\mathcal{F}(K)$. One way to introduce this topology is by means of the so-called *Hausdorff topology*, which is defined in the following way: Given two closed sets $F, L \in \mathcal{F}(K)$, we define

$$d_H(F, L) := inf\{\epsilon > 0 : F \subseteq L^\epsilon \text{ and } L \subseteq F^\epsilon\},$$

where

$$F^\epsilon = \{x \in K : d(x, F) < \epsilon\}. \tag{1}$$

(see Figure 1). In particular, the Fell topology is the topology inherited from the metric space $(\mathcal{F}(K), d_H)$.

Remark 2.1. Let us remark the fact that, in the set $\mathcal{F}(K)$ we will use two different notions of distance. On the one hand, the Hausdorff distance defined above. On the other hand, on the cases where both sets F, L are disjoint, we will call distance, the gap between sets given by $d_0(F, L) = \min_{x\in F, y\in L} d(x, y)$. Observe that the second one is not truly a distance *per se*, as it is not true that $d_0(F, L) = 0$ implies that $F = L$.

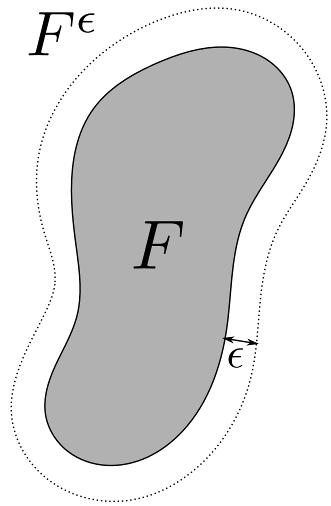


FIGURE 1. Observe that the dotted line encloses the corresponding F^ϵ. Hence, any other closed set L with $d_H(F,L) < \epsilon$ should be included in such a F^ϵ.

We will also make use of some notions and definitions of algebraic topology, in particular the theory of simplicial complexes. Our aim is to extract topological information of our closed sets by studing an appropriate (and more simple) simplicial complex obtained from an cover. Let us begin by fixing the family of covers that we will consider in this paper:

Definition 2.2. A *convex cover* of a compact set K of $\mathbb{R}^n$ is a finite cover $\mathcal{U} = \{U_1, \dots, U_m\}$ of K such that any set of $\mathcal{U}$ is closed and convex[1].

From definition, we have that any convex cover satisfies the following important property: the intersection of an arbitrary number of sets of $\mathcal{U}$ is contractible to a point. In fact, the intersection of an arbitrary number of convex set is again convex and as a consequence, contractible to a point. This fact makes any convex cover a *good cover* (see [3]), and so the *nerve theorem* can be applied.

Let us remind very briefly what the nerve theorem states (we recommend [3, 5, 6, 8] for details). From a finite cover we can build an abstract simplicial complex, the *nerve* $\mathcal{N}(\mathcal{U})$ of $\mathcal{U}$ in the following way: the vertex set is identified with the collection of sets of $\mathcal{U}$, and a set of $k+1$ vertices span a k-simplex if the corresponding U_α's have non-empty intersection. Since the space K covered by $\mathcal{U}$ is compact, the nerve theorem can be recalled to state that K is homotopy equivalent to $\mathcal{N}(\mathcal{U})$. In other words, $\mathcal{N}(\mathcal{U})$ is mathematically equivalent to a simplicial complex of K. From now on, we denote by $\mathcal{N}(\mathcal{U})^i$ the collection of i-simpleces of $\mathcal{N}(\mathcal{U})$.

For a given subset of K, we can also obtain an associated sub-nerve in the following way.

[1] As usual, for convex we mean a set where the segment joining any two points of the set is completely contained in the set. In the context of Riemannian Geometry, this is known as *strongly* convex sets.

Definition 2.3. Let $\mathcal{U}$ be a convex cover of a compact space K of $\mathbb{R}^n$, and let $\mathcal{N}(\mathcal{U})$ be its nerve. For a closed set $A \subseteq K$, define the sub-convex cover $\mathcal{U}|_A$ as $\mathcal{U}|_A := \{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. The sub-convex cover $\mathcal{U}|_A$ yields to a sub-nerve $\mathcal{N}(\mathcal{U})|_A$: its simplices are the simplices of $\mathcal{N}(\mathcal{U})$ with non-empty intersection with A.

It is clear that not always a sub-cover obtained from a subset $A \subset K$ will encapsulate the same homotopical information than A. However, under some natural conditions of the set A, this can be easily achieved:

Proposition 2.4. *Let K be a compact set of $\mathbb{R}^n$ contractible to a point. Let $\mathcal{U}$ be a convex cover of K, with $\mathcal{N}(\mathcal{U})$ its nerve. For a closed convex set $A \subseteq K$, the sub-nerve $\mathcal{N}(\mathcal{U})|_A$ shares with A the same cohomology groups, the trivial ones.*

Proof. Let us prove that $\mathcal{N}(\mathcal{U})|_A$ can be seen as a nerve of A. Observe that the main argument on this proof is the fact that the intersection of convex sets is also convex. Hence, the family $\{U \cap A\}_{U \in \mathbb{N}(\mathcal{U})|_A}$ is a good cover of A and the Nerve theorem applies. As the complex obtained from this cover does not include new simplices neither avoids any one from $\mathcal{N}(\mathcal{U})|_A$, both $\mathcal{N}(\mathcal{U})|_A$ and A shares the same cohomology groups, which are the trivial ones.

□

The result is not satisfied if the convexity assumption of A is avoided, as the following example shows.

Example 2.5. Let us consider the convex cover $\mathcal{U}$ formed by the solid ellipses pictures in the figure 2 (a). Its nerve is pictured next to it, in the figure 2 (b). This abstract simplicial complex has five vertices (0-simplices), represented as crosses; eight edges (1-simplices), represented as the segments; and four faces (2-simplices), which are represented as the blue regions.

Now, we consider the subset A pictured in red in the figure 3 (a). Note that A is contractible to a point, and, consequently, has trivial cohomology groups. Next, we picture the sub-nerve associated to A, in the figure 3 (b). This sub-nerve has four vertices and four edges (and no face). We note that the first cohomology group of the sub-nerve is non-trivial; in fact, this sub-nerve is homotopy equivalent to the circumference $\mathbb{S}^1$.

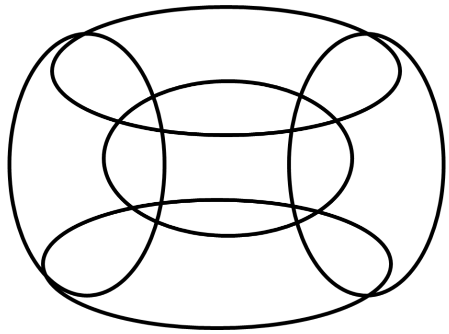
Previous result leads us immediately to the following result, which will be our main technique for this paper:

Proposition 2.6. *Let K be a compact set of $\mathbb{R}^n$ contractible to a point, and let $A \subseteq K$ be a convex closed set. For any convex cover $\mathcal{U}$ of K, the following equation holds:*

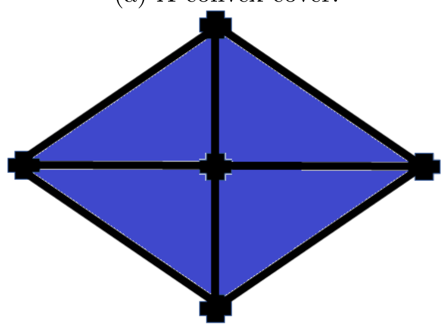
$$\sum_i \sum_{v \in \mathcal{N}(\mathcal{U})^i} (-1)^{i+1}\, \mathbb{I}_A(v) = 1\,, \tag{2}$$

where $\mathbb{I}_A$ denotes the characteristic map *defined by*

$$\mathbb{I}_A(v) := \begin{cases} 1 & \text{if } v \cap A \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$



(a) A convex cover.



(b) Its nerve.

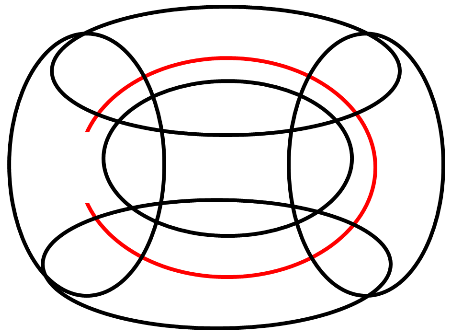
FIGURE 2

Proof. The equation (2) can be expressed in another way: it is the signed sum of the number of i-simplices of the sub-nerve $\mathcal{N}(\mathcal{U})|_A$. This sum is the Euler characteristic of the sub-nerve, [8]. Making use of the proposition 2.4, we have that the sub-nerve is homotopic equivalent to a CW-complex contractible to a point. Since the Euler characteristic is invariant under contractions, and recalling that the Euler characteristic of a point is 1, the result follows. See [5, 8] for further details. □

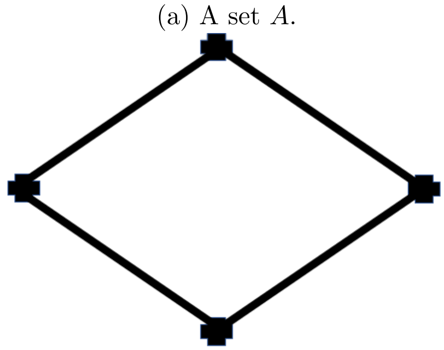
As a corollary of previous result, we can obtain the following characterization:

Proposition 2.7. *Let K be a compact setof $\mathbb{R}^n$ contractible to a point, and let $A \subseteq K$ be a closed set which can be expressed as the disjoint union of m convex sets. The set A is connected (equivalently, $m = 1$) if and only if the equation (2) holds for any convex cover $\mathcal{U}$ of K.*

Proof. We sketch the proof of the single implication that we have to prove. By reductio ad absurdum, assume that A is not connected, and consider d as the induced distance on K from the Euclidean distance. Let $A_1, A_2, \ldots, A_m$ be the connected components of A. Let $\mu := \min_{i,j\in\{1,\ldots,m\}, i\neq j}\left(\min_{x\in A_i, y\in A_j} d(x,y)\right)$. That is, μ is the minimum d_0-distance between the connected components of A. Now, let $\mathcal{U}$ be a convex cover of K whose sets has diameter $\mu/10$ at most. It is clear that the sub-nerve has m components. Each connected component of this sub-nerve is homotopic equivalent to a point, and consequently, for each connected component, the equation (2) holds.



(a) A set A.



(b) Its sub-nerve.

FIGURE 3

However, summing for all the elements of the sub-nerve, finally we find that the right side of the equation (2) equals to m. □

As we can see from previous proof, the result given by the left part of equation (2) will depend strongly on the selected convex cover $\mathcal{U}$: if its sets are *small enough* to distinguish between the different connected components, the result will be m. Otherwise, the convex cover will not detect all the connected components and the result will be smaller. Therefore, it easily follows that:

Corollary 2.8. *Let K be a compact set of $\mathbb{R}^n$ contractible to a point, and let $A \subseteq K$ be a closed set which can be expressed as the disjoint union of m convex sets. For any convex cover $\mathcal{U}$ of K, it holds:*

$$1 \leq \sum_i \sum_{v \in \mathcal{N}(\mathcal{U})^i} (-1)^{i+1}\, \mathbb{I}_A(v) \leq m\,.$$

Let us also observe that Proposition 2.6 yields to other characterization for a different topological property of a closed set. We anticipate that the proof is inspired by the example 2.5.

Proposition 2.9. *Let K be a convex set of $\mathbb{R}^n$, and let $A \subseteq K$ be a connected, closed set. The set A is convex if and only if the equation (2) holds for any convex cover $\mathcal{U}$ of K.*

FIGURE 4. Here we show a rectangle R included in the convex hull of the non-convex set A. The lines of the boundaries of the semispaces H_1, H_2, H_3 and H_4 are depicted, and the semispace H_1 appears shaded.

Proof. Again, we only have to prove the sufficient condition. For this, we assume that A is not convex and we show that there exists a convex cover $\mathcal{U}$ of K for which the equation (2) does not hold. Let us prove it when the underlying space is $\mathbb{R}^2$. After that, we show how to extend this proof to higher dimension.

Denote the convex hull of A by $CH(A)$. From assumptions, there exists a rectangle R satisfying: $R \cap A = \emptyset$ and $C \subset CH(A)$. Consider the semi-spaces H_1, H_2, H_3, H_4 pointing outside of R that each boundary contains each edge of R (see Figure 4 for a graphical representation). We have that $H_1 \cap A \neq \emptyset$, ..., $H_4 \cap A \neq \emptyset$. Otherwise, we would find a contradiction with the aforementioned properties of R. It is clear that $\mathcal{U} := \{R, H_1 \cap K, H_2 \cap K, H_3 \cap K, H_4 \cap K\}$ is a convex cover of K. The sub-nerve $\mathcal{N}(\mathcal{U})|_A$ is equivalent to the simplicial complex represented in 3 (b). The proof in $\mathbb{R}^2$ is complete.

Let us prove now that a similar construction can be made in higher dimension. Let H be a 2-dimensional hyperplane on $\mathbb{R}^n$ ($n \geq 3$) such that $A \cap H$ is not convex on H. Let $\mathcal{U} = \{U_1, \ldots, U_5\}$ the sets that we obtain applying the above procedure on the 2-dimensional Euclidean space H. Split the space as the following orthogonal decomposition: $\mathbb{R}^n = H \times \mathbb{R}^{n-2}$. From this composition, we can obtain $\mathcal{U}' := \{(U_1 \times \mathbb{R}^{n-2}) \cap K, \ldots, (U_1 \times \mathbb{R}^{n-2}) \cap K\}$. It is trivial to show that $\mathcal{U}'$ is a convex cover of K. Furthermore, the sub-nerve $\mathcal{N}(\mathcal{U}')|_A$ is again equivalent to the simplicial complex represented in 3 (b). Now, the whole proof is complete. □

Finally, let us recall that in many practical situations, we will require a more *flexible* notion of connectedness. Maybe a given set A is not connected according to the rigorous definition, but the "gap" between connected components is small enough so it can be disregarded. In this sense, we introduce the following notion:

Definition 2.10. We will say that a set A is ϵ-*connected* if $A^{\epsilon/2}$ (recall (1)) is connected.

Remark 2.11. Observe that, in spite of the concept of connectedness which is purely topological, the notion of ϵ-connectedness depends on the metric. It follows straightforwardly that, if A is *not* ϵ-connected, the distance between at least one of the connected components and the rest of points is bigger than or equal to ϵ. Moreover, if a closed set A with at most a finite number of connected components is not connected, then it will not be ϵ-connected for some ϵ small enough.

We are then in conditions to prove that:

Theorem 2.12. *Let K be a compact set in $\mathbb{R}^n$; and let $A \subset K$ be a closed set which can be expressed as the disjoint union of convex sets. Consider $\mathcal{U}$ be a convex cover so each $U \in \mathcal{U}$ satisfies that* $\operatorname{diam}(U) < \epsilon$. *If equation* (2) *is satisfied, then A is ϵ-connected.*

Proof. Assume that A is not ϵ-connected, and so, that there exists a connected component, say A_0, satisfying that $d(A_0, A \setminus A_0) \geq \epsilon$. Now, as the $\operatorname{diam}(U) < \epsilon$ for all $U \in \mathcal{U}$, any set in $\mathcal{U}$ intersecting A_0 will not intersects $A \setminus A_0$. Hence we can decompose

$\mathcal{N}(U) = \mathcal{N}(U)|_{A_0} \cup \mathcal{N}(U)|_{A \setminus A_0}$. Recalling now that A_0 is connected, Proposition 2.6 and Corollary 2.8, it follows that

$$\sum_i \sum_{v \in \mathcal{N}(\mathcal{U})^i} (-1)^{i+1} \, \mathbb{I}_A(v) \geq 2, \tag{3}$$

a contradiction. □

Hence, we have a simple test to prove when a set A is ϵ-connected: Choose a convex cover which elements have a small diameter, and see if equation (2) is satisfied. A simple way to achieve this is by constructing a grid of K formed by n-dimensional rectangles (i.e., the cartesian product of n intervals).

Remark 2.13. Almost all the results of this section are extensible for compact metric spaces with the appropriate changes. For instance, we need to consider *strong* convexity for the sets: a set U is *strongly convex* if any two points in U can be joined with a unique minimizing curve.

It is straightforward to check that the intersection of two stronlgy convex sets is again a strongly convex set. Even more, a strongly convex set is contractible (see Exercise 4-4 in Chapter 3 of [4]).

There is however only one result (Proposition 2.9) which proof cannot be directly translated to this context. In any case, this result will not be used in the forthcoming sections.

3. On connected and ϵ-connected Random closed sets

In this section we will introduce the basic statistical preliminaries that we will require for our results. Let us begin by giving the definition of Random closed set. As we have mentioned above, there are several analogous ways to define the notion of Random closed set. Here, we will introduce two of them:

Definition 3.1 ([12])**.** A *random closed set* $\widehat{X}$ of $\mathbb{R}^n$ is a map from a complete probability space (Ω, σ, P) into the class of closed sets of $\mathbb{R}^n$ and which satisfies any of the following two equivalent assertions (see [12, Appendix C]):

(i) $\left\{\omega \in \Omega : \widehat{X}(\omega) \cap K\right\} \in \sigma$ for any compact set K.

(ii) For a given open set $\mathcal{O}$ of the Fell topology, $\widehat{X}^{-1}(\mathcal{O}) \in \sigma$.

Let us denote by $\mathcal{K}$ the class of compact sets of $\mathbb{R}^n$. From the definition, we have that the following map is well-defined,

$$\begin{aligned} T : \mathcal{K} &\rightarrow [0,1] \\ K &\mapsto T(K) = P(\widehat{X} \cap K \neq \emptyset) \,. \end{aligned}$$

The map T is the *capacity functional* of $\widehat{X}$, [12, 13]. The Choquet-Kendall-Matheron theorem states that a random closed set is characterized by its capacity functional, (see [12] and references therein for further details). For this reason, it is commonly interpreted that the capacity functional plays a similar role than the probability distribution for a random variable.

In the following definition we will take into account all the properties that we will be interested on Random closed sets:

Definition 3.2. A random closed set $\widehat{X}$ of $\mathbb{R}^n$ is said to be

- *bounded* if there exists a compact set K such that $P(\widehat{X} \subset K) = 1$. We will say that K is a *good support* for $\widehat{X}$.
- *convex* (resp. *convex by components*) if for almost all[2] $\omega \in \Omega$, the set $\widehat{X}(\omega)$ is convex (resp. the disjoint union of a finite number of convex sets).
- *(ϵ-)connected* if for almost all $\omega \in \Omega$, the set $\widehat{X}(\omega)$ is (ϵ-)connected.

As we can see, the notion of Random closed set ensures that we will always be able to measure the open sets of the Fell topology. The following condition will also ensure that if we have a non-trivial open set of the Fell topology (in the sense that the inverse image of the open set is non-empty), it will have some positive measure:

Definition 3.3. We will say that $\widetilde{X}$ is a *continuous random closed set* if given an open set $\mathcal{O}$ of the Fell topology with $\widetilde{X}^{-1}(\mathcal{O}) \neq \emptyset$, $P(\widetilde{X}^{-1}(\mathcal{O})) > 0$.

Observe that continuous random closed set includes the case where the underlying probability space is discrete.

4. Main results

We are now in conditions to present the main results of this paper:

Theorem 4.1. *Let $\widehat{X}$ be a bounded random closed set, with K a good support. Assume that $\widehat{X}$ is convex. The following equation holds for any convex cover $\mathcal{U}$ of K:*

$$\sum_i \sum_{v \in \mathcal{N}(\mathcal{U})^i} (-1)^{i+1}\, T(v) = 1\,. \tag{4}$$

Proof. The random set $\widehat{X}$ yields to a random sub-nerve $\widehat{Y}$ of $\mathcal{N}(\mathcal{U})$; it is a map from the same probability space associating each $\omega \in \Omega$ with the sub-nerve $\widehat{Y}(\omega)$ obtained as described in the preliminaries section. Denote by $\mathbb{I}_\omega$ the characteristic map of $\widehat{Y}(\omega)$ in $\mathcal{N}(\mathcal{U})$. The equation (2) implies then:

$$\int_\Omega \left[\sum_i \sum_{v \in \mathcal{N}(\mathcal{U})^i} (-1)^{i+1}\, \mathbb{I}_\omega(v) \right] dP(\omega) = 1\,.$$

From the linearity of the above expectation, we can make use of the Robbin's theorem as in the proof of the Theorem 1 of [14] to obtain the announced equation. □

Remark 4.2. Formally the right side of equation (4) can be identified with the canonical expansion of $T(\cup U_\alpha)$ in terms of the elements of the form $T(\cap U_\alpha)$, as in Poincaré's inequalities (see [13, p. 12] for instance). Moreover, observe that the random closed set $\widehat{X} := \{X\}$ built from a random variable X must satisfy the equation (4) trivially.

[2]Here, by *for almost all* we mean for all $\omega \in \Omega$ except maybe a set of null probability.

For the converse, let us recall that when a set is not (ϵ-)connected, Theorem 2.12 ensures that (2) cannot be satisfied for a small enough cover. In order to apply this result in the context of closed random sets, we require then not only that $\widetilde{X}(\omega)$ is not ϵ-connected for some ω, but that the set of ω where ϵ-connectedness is not satisfied has positive measure. In this sense, we can prove:

Proposition 4.3. *Let $\widehat{X}$ be a bounded and continuous random closed set, with K a good support. Assume that $\widehat{X}$ is convex by components. If equation (4) is satisfied for some good cover $\mathcal{U}$ of K with elements $U \in \mathcal{U}$ satisfying that $\operatorname{diam}(U) < \epsilon/2$, then $\widehat{X}$ is at least ϵ-connected.*

Proof. Let us assume by contradiction that $\widehat{X}$ is not ϵ-connected, and so, that there exists $\omega \in \Omega$ so $\widehat{X}(\omega)$ is not ϵ-connected (as a set). It is possible then to consider an open set $\widehat{X}(\omega) \in \mathcal{O}$ for the Fell Topology so any $A \in \mathcal{O}$ is not $\epsilon/2$-connected (consider for instance a ball with the Hausdorff distance d_H centered on $\widehat{X}(\omega)$ and with radius $\epsilon/4$).

Following the proof of Theorem 2.12 we deduce that for any good cover under the hypotheses and any $A \in \mathcal{O}$, it follows equation (3). Now recalling that $P(\widehat{X}^{-1}(\mathcal{O})) > 0$ ($\widetilde{X}$ is a continuous random closed set) and Corollary 2.8, we have that:

$$\int_{\Omega} \left[\sum_{i} \sum_{v \in \mathcal{N}(\mathcal{U})^i} (-1)^{i+1} \, \mathbb{I}_{\omega}(v) \right] dP(\omega) = \sum_{i} \sum_{v \in \mathcal{N}(\mathcal{U})^i} (-1)^{i+1} \, T(v) > 1 \,,$$

a contradiction. □

As a direct consequence of previous result, we can obtain (see also Remark 2.11):

Theorem 4.4. *Let $\widehat{X}$ be a continuous random closed set, bounded and convex by components, with K a good support of $\widehat{X}$. The random set $\widehat{X}$ is connected if and only if the equation (4) holds for any convex cover of K.*

Example 4.5. Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two random points of $K = [0, A] \times [0, B] \subset \mathbb{R}^2$, not necessarily independent. We want to compute $\mathbb{P}\left(d_\infty(X, Y) \le \delta\right)$ for some positive real δ, where d_∞ is the Chebyshev distance (that is, $d_\infty\left((x_1, x_2), (y_1, y_2)\right) = \max\left\{|x_1 - y_1|, |x_2 - y_2|\right\}$ for points (x_1, x_2) and (y_1, y_2); this distance can be replaced with another one). This problem has several motivations and has multiple applications (see [2] and references therein, for instance). For this end, simulation may be quite difficult -assume complexity in the joint probability law. Let us see how Theorem 4.4 can be used to this end.

In order to use our results on connectedness, let us define the random closed set $\widehat{X}$ by $\widehat{X}(w) = [X_1(w) - \delta, X_1(w) + \delta] \times [X_2(w) - \delta, X_2(w) + \delta] \cup \{Y(w)\}$. Clearly, $\mathbb{P}\left(d_\infty(X, Y) \le \delta\right)$ equals to the probability that $\widehat{X}$ is connected.

In order to compute such a probabilty, let us recall some facts: On the one hand, $\widehat{X}$ has, at most, two connected components. Hence, if we denote with $\#\widehat{X}$ the random variable given by the number of connected components, its expectation is then:

$$\begin{aligned} \mathbb{E}(\#\widehat{X}) = &\ \mathbb{P}(\widehat{X} \text{ is connected}) + 2\,\mathbb{P}(\widehat{X} \text{ is not connected}) \\ = &\ 2 - \mathbb{P}(\widehat{X} \text{ is connected}) \end{aligned}$$

On the other hand, let us consider $\mathcal{U}$ some convex cover, and let $\widehat{Y}$ be the random subnerve of $\mathcal{N}(\mathcal{U})$ associated to $\widehat{X}$ (as defined in Theorem 4.1). Applying then Robbin's theorem to $\#\widehat{Y}$ it follows that:

$$\mathbb{E}(\#\widehat{Y}) = \sum_i \sum_{v\in\mathcal{N}(U)^i} (-1)^{i+1} T(v)$$

Finally, let us recall that, depending on the convex cover $\mathcal{U}$, given some $\omega \in \Omega$, $\widehat{Y}(\omega)$ could be connected even so $\widehat{X}(\omega)$ is not (i.e., the convex cover does not detect the gap between $[X_1(w)-\delta, X_1(w)+\delta]\times[X_2(w)-\delta, X_2(w)+\delta]$ and $\{Y(w)\}$). Therefore, $\mathbb{E}(\#\widetilde{Y})$ should be lower than $\mathbb{E}(\#\widehat{X})$. In conclusion,

$$2 - \mathbb{P}(\widehat{X} \text{ is connected}) \leq \sum_i \sum_{v\in\mathcal{N}(U)^i} (-1)^{i+1} T(v).$$

In order to simplify the computation of $T(v)$ we can provide a simple convex cover: for fixed m_x and m_y, define $\mathcal{U} = \{U_{i,j}\}_{i,j}$ where $i \in \{0,\dots,m_x-1\}$, $j \in \{0,\dots,m_y-1\}$ and

$$U_{ij} = \left[\left(\frac{A}{2m_x+1}\right)(2i), \left(\frac{A}{2m_x+1}\right)(2i+3)\right]\times\left[\left(\frac{B}{2m_y+1}\right)(2j), \left(\frac{B}{2m_y+1}\right)(2j+3)\right].$$

Note that the intersection of two (or more) sets of $\mathcal{U}$ is either empty or a rectangle. Then, the simplices of $\mathcal{N}(\mathcal{U})$ are constituted by rectangles and we only need to evaluate T on them.

Remark 4.6. There is another approach for previous example by defining a random closed set $\widehat{Y}$ as $\widehat{Y}(\omega) = \{X(\omega)\} \cup \{Y(\omega)\}$. In this case, $\mathbb{P}(d_\infty(X,Y) \leq \delta)$ equals the probabilty that $\widehat{Y}$ is $\delta/2$-connected, and then an analogous argument can be applied by using Proposition 4.3 instead of Theorem 4.4.

Finally, we would like to remark that Theorem 4.1 is obtained as a consequence of both, Proposition 2.6 and Robbin's Theorem. Following the same arguments but using Proposition 2.9 instead of Proposition 2.6, we can obtain:

Theorem 4.7. *Let $\widehat{X}$ be a bounded and continuous random closed set, with K a good support. Assume that $\widehat{X}$ is connected. Then $\widehat{X}$ is convex if and only if equation (4) is satisfied.*

Proof. Again, the right implication is a direct consequence of Proposition 2.9 and Robbin's theorem. For the left one, assume that $\widehat{X}$ is not convex, and so, that $\widehat{X}(\omega)$ is not convex for some $\omega \in \Omega$. It is not difficult then to prove that there exists an open set $\mathcal{O}$ for the Fell topology with $\widehat{X}(\omega) \in \mathcal{O}$ and so that any $A \in \mathcal{O}$ is not convex. By Proposition 2.9 it follows then that equation 2 does not hold for any element in $\mathcal{O}$, and so, as from hypothesis $P(\widehat{X}^{-1}(\mathcal{O})) > 0$, equation (4) does not hold either. □

Example 4.8. Let X, Y and Z be three (non-independent) univariate random variables, all of them defined on $(0, A)$, with $A > 0$. Build a random set $\widehat{X}$ as follows: $\widehat{X}(w)$ is the polygon of $\mathbb{R}^2$ whose vertices are $(0,0)$, $(0, X(w))$, $(1, Y(w))$, $(2, Z(w))$ and $(2,0)$. We want to determine whether $\widehat{X}$ is convex. Theorem 4.7 can be applied to this end. Note that it can be used a convex cover as in Example 4.5. Again, we only need evaluations of the capacity functional on rectangles.

5. Conclusions

In this work, several topological and geometrical properties of a random closed set have been characterized. Concretely, connectedness and convexity. A priori, this problem has a hard nature; in the most naive approach, checking whether each $\widehat{X}(w)$ is connected (or convex) needs high computational costs and eventually it cannot be solved in finite time. Our results indicate that some finite computations can be enough to determine such properties of the random closed set. In fact, Theorems 4.4 and 4.5 are quite easy to implement in a computer program. On the other hand, we should observer that a square grid (a lattice) can be a nice convex cover to be considered in this setting.

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