A characterization of the property of orthomonotonicity

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In the paper "An inherent difficulty in the aggregation of multidimensional data" [1] recently published in this very journal, the property of orthomonotonicity for functions of the type $A : \mathbb{R}^{d\times n} \to \mathbb{R}^d$ (with $d \geq 2$) is introduced.

Definition 1: A function $A : \mathbb{R}^{d\times n} \to \mathbb{R}^d$ is called orthomonotone if, for any $(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}),(\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(n)}) \in \mathbb{R}^{d\times n}$ and any orthogonal matrix $\mathbf{O} \in \mathbb{R}^{d\times d}$, the fact that $\mathbf{O}\,\mathbf{x}^{(i)} \leq_d \mathbf{O}\,\mathbf{y}^{(i)}$ for any $i \in \{1,\ldots,n\}$, implies that $\mathbf{O}\,A(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}) \leq_d \mathbf{O}\,A(\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(n)})$.

This property is proved to be weaker than $\leq_d$-monotonicity and orthogonal equivariance together (see Proposition 1 in [1]) and to reduce the family of idempotent functions $A : \mathbb{R}^{d\times n} \to \mathbb{R}^d$ to the family of weighted centroids (see Theorem 1 in [1]). However, the paper could have gone much further by providing the following characterization of the property of orthomonotonicity.

Proposition 1: A function $A : \mathbb{R}^{d\times n} \to \mathbb{R}^d$ is orthomonotone if and only if there exist $k_1,\ldots,k_n \geq 0$ such that $A(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}) = A(\mathbf{0},\ldots,\mathbf{0}) + \sum_{i=1}^n k_i\,\mathbf{x}^{(i)}$ for any $(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}) \in \mathbb{R}^{d\times n}$.

Proof. The left-to-right implication follows when we stop the proof of Theorem 1 in [1] one paragraph earlier. Here, we prove the right-to-left implication. Consider any $(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}),(\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(n)}) \in \mathbb{R}^{d\times n}$. If an orthogonal matrix $\mathbf{O} \in \mathbb{R}^{d\times d}$ is such that $\mathbf{O}\mathbf{x}^{(i)} \leq_d \mathbf{O}\mathbf{y}^{(i)}$ for any $i \in \{1,\ldots,n\}$, then it follows that

$$
\begin{aligned}
\mathbf{O}A(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}) &= \mathbf{O}A(\mathbf{0},\ldots,\mathbf{0}) + \sum_{i=1}^n k_i\,\mathbf{O}\mathbf{x}^{(i)} \\
&\leq_d \mathbf{O}A(\mathbf{0},\ldots,\mathbf{0}) + \sum_{i=1}^n k_i\,\mathbf{O}\mathbf{y}^{(i)} \\
&= \mathbf{O}A(\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(n)})\,.
\end{aligned}
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One then concludes that the property of orthomonotonicity is inherent only and exclusively to shifted weighted sums. Note that the term weighted centroid used in [1] is here avoided because the constants $k_1,\ldots,k_n$ are not required to add up to one in case the function is not idempotent.

Interestingly, the above characterization of orthomonotonicity leads to the following equivalences.

Proposition 2: For a function $A : \mathbb{R}^{d\times n} \to \mathbb{R}^d$, the following three statements are equivalent:

(i) A is orthomonotone and $A(\mathbf{0},\ldots,\mathbf{0}) = \mathbf{0}$;
(ii) A is a weighted sum;
(iii) A is orthogonally equivariant and $\leq_d$-monotone.

Proof. (i)⇒(ii). If A is orthomonotone and $A(\mathbf{0},\ldots,\mathbf{0}) = \mathbf{0}$, then we conclude from the above characterization of orthomonotonicity that there exist $k_1,\ldots,k_n \geq 0$ such that $A(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}) = \sum_{i=1}^n k_i\,\mathbf{x}^{(i)}$ for any $(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}) \in \mathbb{R}^{d\times n}$, i.e., A is a weighted sum. (ii)⇒(iii). If A is a weighted sum, then it obviously holds that it is orthogonally equivariant and $\leq_d$-monotone. (iii)⇒(i). If A is orthogonally equivariant and $\leq_d$-monotone, then it follows from Proposition 1 in [1] that A is orthomonotone. The fact that $A(\mathbf{0},\ldots,\mathbf{0}) = \mathbf{0}$ follows straightforwardly from the orthogonal equivariance of A.

Corollary 1: For an idempotent function $A : \mathbb{R}^{d\times n} \to \mathbb{R}^d$, the following three statements are equivalent:

(i) A is orthomonotone;
(ii) A is a weighted centroid;
(iii) A is orthogonally equivariant and $\leq_d$-monotone.

Similarly as in the original paper, we end by concluding that there is an inherent difficulty in the aggregation of multidimensional data and, if weighted sums are to be avoided, one must choose between orthogonal equivariance and $\leq_d$-monotonicity.

References

[1] M. Gagolewski, R. Pérez-Fernández, and B. De Baets, "An inherent difficulty in the aggregation of multidimensional data," *IEEE Transactons on Fuzzy Systems*, vol. 28, pp. 602–606, 2020.