J. Math. Anal. Appl. 350 (2009) 29–36

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A local duality principle for the Baire classes of functions ☆

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ARTICLE INFO

Article history:
Received 30 July 2008
Available online 17 September 2008
Submitted by J. Diestel

Keywords:
Local dual subspace
Baire classes of functions
Space of Borel measures
Bidual of the space of continuous functions
Principle of local reflexivity

ABSTRACT

A *local dual* of a Banach space X is a closed subspace of X^* that satisfies the properties that the principle of local reflexivity assigns to X as a subspace of X^{**}. We show that, for every ordinal $1 \leqslant \alpha \leqslant \omega_1$, the spaces $B_\alpha[0,1]$ of bounded Baire functions of class α are local dual spaces of the space $M[0,1]$ of all Borel measures. As a consequence, we derive that each annihilator $B_\alpha[0,1]^\perp$ is the kernel of a norm-one projection.

1. Introduction

The *local dual spaces* of a Banach space X are defined in [8] as those closed subspaces Z of the dual space X^* such that for every pair of finite-dimensional subspaces E of X^* and F of X, and every $\varepsilon > 0$, there exists an operator $L : E \to Z$ which satisfies the following conditions:

(a) $(1-\varepsilon)\|e\| \leqslant \|L(e)\| \leqslant (1+\varepsilon)\|e\|$ for all $e \in E$.
(b) $\langle L(e), x\rangle = \langle e, x\rangle$ for all $e \in E$ and all $x \in F$.
(c) $L(e) = e$ for all $e \in E \cap Z$.

Condition (a) says that X^* is finitely representable in Z; i.e., that every finite-dimensional subspace of X^* is almost isometric to some finite-dimensional subspace of Z. Conditions (b) and (c) imply that Z must be nicely placed inside X^*.

Examples of local dual spaces are provided by the classical principle of local reflexivity [13], which is equivalent to saying that X is a local dual of X^*, and the principle of local reflexivity for ultrapowers [12], which states, for every ultrafilter $\mathfrak{U}$, that $(X^*)_{\mathfrak{U}}$ is a local dual of the ultrapower $X_{\mathfrak{U}}$.

Finding concrete examples of local dual spaces X is very valuable, in particular when no actual representation of X^* is known, because those local dual spaces may tell much about the local structure of X^*. This situation is well featured by the aforementioned principles of local reflexivity and also, in many other cases [8–10]. It is also noticeable that the notion of *local duality* strengthens that of *local complementation* introduced by Kalton [14].

The goal of this article, to be achieved in Theorem 3.4, is to prove that for every ordinal $1 \leqslant \alpha \leqslant \omega_1$, the class $B_\alpha[0,1]$, regarded as a subspace of $C[0,1]^{**}$, is a local dual of $C[0,1]^*$, where $C[0,1]$ stands for the space of all continuous functions on the unit interval, ω_1 is the first uncountable ordinal and $B_\alpha[0,1]$ denotes the set of all the Baire functions of class α

☆ Supported in part by DGI (Spain), Grant MTM2007-67994.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.09.019

on $[0,1]$. Roughly speaking, Theorem 3.4 is telling us that all the Baire classes are locally the same in a very strong way. In fact, the information provided by Theorem 3.4 about the local structure of each class $B_\alpha[0,1]$ is more accurate than the information that can be obtained for the subspaces of $C[0,1]^{**}$ from the only fact that $C[0,1]$ is a local dual of the space of all Borel measures $M[0,1]$.

Let us recall that, despite the Baire classes are similar from a local point of view, they differ in their global structure. Indeed, it is known that every Baire class $B_\alpha[0,1]$ is isometric to a space of continuous functions $C(K_\alpha)$, where K_α is a totally disconnected compact space [2, Theorem 1.4]; however, for $1 \leqslant \alpha < \beta \leqslant \omega_1$, although $B_\alpha[0,1]$ is a subspace of $B_\beta[0,1]$, it is neither isometric to nor complemented in $B_\beta[0,1]$ ([3, Theorem 3.11] and [2, Theorem 1.4]). Moreover, there is no continuous injective operator from $B_\beta[0,1]$ into $B_1[0,1]$ [5], and $B_\alpha[0,1]$ is not isomorphic to $B_{\omega_1}[0,1]$ [3]. Thus there are at least three different isomorphic types among these spaces. Nevertheless, it follows from a result of [17] that the dual space of $B_\alpha[0,1]$ is isometric to ℓ_∞^* for each $1 \leqslant \alpha \leqslant \omega_1$.

In order to reach our goal, we first give, in Section 2, a sort of local version of a representation of $C[0,1]^{**}$ due to Mauldin [15]. Our local representation is achieved in Theorem 2.1 and in its further generalization, Theorem 2.2. We point out that, although Mauldin's representation requires to assume the Continuum Hypothesis (CH), our Theorem 2.2 does not need neither (CH) nor any of its weaker forms. In Section 3, we give the main result of this paper in Theorem 3.4: all the classes $B_\alpha[0,1]$, regarded as subspaces of $C[0,1]^{**}$, are local duals of the space $M[0,1]$ of Borel measures on $[0,1]$. Next, we apply Theorem 3.4 to answer the following question of Godefroy, Kalton and Saphar [7] in a concrete case. Given a Banach space X, let $B_a(X)$ denote the space of its first Baire class elements (that is, the weak* limits in X^{**} of sequences in X). For X a separable Banach space, it is asked in [7, Question 10] if there exists a norm-one projection Q on X^{***} with kernel $B_a(X)^\perp$. Note that there is a positive answer in two cases:

(1) when X contains no copies of ℓ_1, because $B_a(X) = X^{**}$ [16],
(2) when X is weakly sequentially complete, because $B_a(X) = X$.

In Corollary 3.6, we prove that this is true for $X = C[0,1]$. We observe that this is a very special case, because $C[0,1]$ is an $\mathcal{L}_{\infty,1+}$-space; however, since every separable Banach space is a subspace of $C[0,1]$ and for a subspace M of X we can identify $B_a(M)$ with $B_a(X) \cap M^{\perp\perp}$ [6, Lemma XIII.7], our result strongly suggests that the answer to the problem should be positive, in general.

Let us recall some facts that will help to introduce some notation. The dual of the space $C[0,1]$ of all the continuous functions on the unit interval can be identified with the space $M[0,1]$ of all the Borel measures on $[0,1]$. Moreover, if $\{A_i\}_{i=1}^n$ is a finite partition of the unit interval $[0,1]$ into borelian sets, the space $M(A_i)$ of all the Borel measures supported by A_i is naturally identified with a subspace of $M[0,1]$. Under these identifications, we get $M[0,1] = M(A_1) \oplus_1 \cdots \oplus_1 M(A_n)$. Therefore, $M[0,1]^* = M(A_1)^* \oplus_\infty \cdots \oplus_\infty M(A_n)^*$. The classes of Baire functions $B_\alpha[0,1]$, $0 \leqslant \alpha \leqslant \omega_1$, are defined by transfinite induction as follows. The class $B_0[0,1]$ is $C[0,1]$, and for each ordinal $1 \leqslant \alpha \leqslant \omega_1$, $B_\alpha[0,1]$ (called the *space of all bounded Baire functions of class* α) is the set of all the bounded functions on $[0,1]$ which are pointwise limits of sequences in $\bigcup_{\beta<\alpha} B_\beta[0,1]$. These spaces were studied as Banach spaces in [2–5]. The class $B_{\omega_1}[0,1]$ coincides with the space of all bounded Borel measurable functions on the unit interval [2]. Moreover, each $B_\alpha[0,1]$, endowed with the supremum norm, is a Banach space that can be isometrically identified with a subspace of $M[0,1]^* \equiv C[0,1]^{**}$, where the duality is given by

$$\langle f, \mu \rangle = \int_0^1 f\, d\mu \quad \text{for } f \in B_{\omega_1}[0,1] \text{ and } \mu \in M[0,1].$$

We also adopt the following notations: given a Banach space X, its closed unit ball is denoted by B_X, its unit sphere by S_X, its first dual by X^*, and its second dual by X^{**}. The action of $f \in X^*$ on $x \in X$ is denoted by $\langle f, x \rangle$. Given $\varepsilon > 0$, we say that $\{x_i\colon\ i \in I\} \subset S_X$ is an *ε-net in S_X* if, for every $x \in S_X$, $\inf_{i\in I} \|x - x_i\| < \varepsilon$. Given a number $\varepsilon > 0$, an operator $T : X \to Y$ is called *ε-isometry* if it satisfies $1 - \varepsilon < \|Tx\| < 1 + \varepsilon$ for all $x \in S_X$.

2. The local Mauldin operator

Mauldin obtained a representation of $C[0,1]^{**}$ in terms of bounded set-functions [15]. In Theorem 2.2, we offer a sort of local version of this result. There are some differences between these two representation results: Mauldin's result has a global character, while ours is merely local. Moreover, we do not use the Continuum Hypothesis or any of its weaker variants, while Mauldin applies CH. Other important differences are that the operator Φ provided by our Theorem 2.2 is linear, and above all, its action over the Borel simple functions (see clause (ii) in Theorem 2.2), which is crucial in order to show that $B_\alpha[0,1]$ satisfies condition (c) in the definition of local dual we stated in the Introduction.

We denote by $\mathcal{B}$ the σ-algebra of all the Borel subsets of $[0,1]$. A finite partition $\mathcal{D}$ of a set $A \in \mathcal{B}$ into borelian, non-void sets is called a *subdivision of* A. We say that a subdivision $\mathcal{D}'$ *refines* $\mathcal{D}$ if each set in $\mathcal{D}'$ is a subset of some set in $\mathcal{D}$. Let $\mathcal{S}$ denote the Banach space of all the bounded set-functions $\Psi : \mathcal{B} \to \mathbb{R}$ endowed with the supremum norm, $\|\Psi\| = \sup_{B\in\mathcal{B}} |\Psi(B)|$.

Theorem 2.1. *Let μ be a norm-one, positive Borel measure concentrated on $A \in \mathcal{B}$. Let E be a finite-dimensional subspace of $M(A)^*$ and let E_0 be its maximal subspace consisting of simple Borel functions. Thus, given a number $\varepsilon > 0$ and a finite set $\{\nu_i\}_{i=1}^n$ of measures absolutely continuous with respect to μ, there exist a linear operator $\Phi : T \in E \to \Phi_T \in \mathcal{S}$ and a subdivision $\mathcal{D}$ of A satisfying the following clauses*:

(i) *for all $T \in E$ and $i \in \{1, \dots, n\}$, and for each refinement $\mathcal{D}'$ of $\mathcal{D}$,*

$$\left| \langle T, \nu_i \rangle - \sum_{B \in \mathcal{D}'} \Phi_T(B) \nu_i(B) \right| < \varepsilon,$$

(ii) *if $T \in E_0$ and $\mathcal{D}'$ is a refinement of $\mathcal{D}$, then $T = \sum_{B \in \mathcal{D}'} \Phi_T(B) \chi_B$,*
(iii) *$|\Phi_T(B)| \leqslant \|T\|$ for all $T \in E$ and all $B \in \mathcal{B}$.*

Proof. Without loss of generality, we assume that $E_0 := \operatorname{span}\{\chi_{A_1}, \dots, \chi_{A_l}\}$ where the sets A_i are non-empty and pairwise disjoint. Rearranging indices if necessary, we also assume that $\mu(A_i) = 0$ for all $i \in \{1, \dots, l'\}$ and $\mu(A_i) > 0$ for all $i \in \{l' + 1, \dots, l\}$. We pick $x_i \in A_i$ for each $i \in \{1, \dots, l'\}$ and consider the set of measures

$$\mathcal{N} = \{\nu_1, \dots, \nu_n, \delta_{x_1}, \dots, \delta_{x_{l'}}\}.$$

Let $\{T_1, \dots, T_l, T_{l+1}, \dots, T_m\}$ be a normalized basis of E with $T_k = \chi_{A_k}$ for $k \in \{1, \dots, l\}$, and let $C := \max\{\sum_{k=1}^m |\lambda_k|\colon \sum_{k=1}^m \lambda_k T_k \in S_E\}$. Notice that $L_1(\mu)$ can be identified with a subspace of $M[0, 1]$, and the restriction $T_k|_{L_1(\mu)}$ determines an element $\mathbf{g}_k \in L_\infty(\mu)$ so that

$$\langle T_k, f \rangle = \int_A \mathbf{g}_k f \, d\mu \quad \text{for all } f \in L_1(\mu).$$

We select a set $\{g_k\}_{k=1}^m$ of bounded Borel functions on $[0, 1]$ as follows: for $k \in \{1, \dots, l\}$, we take $g_k := \chi_{A_k}$, and for $k \in \{l + 1, \dots, m\}$, we choose g_k so that $g_k(t) = \mathbf{g}_k(t)$, μ-a.e. and $g_k(t) = \langle T_k, \delta_{x_i} \rangle$ on all $t \in A_i$ for each $i \in \{1, \dots, l'\}$.

For every $i \in \{1, \dots, n\}$, let f_i be the Radon–Nikodým derivative of the measure ν_i with respect to μ. Let $\mathcal{D}_0$ be a subdivision of A refining $\mathcal{A} := \{A_1, \dots, A_l, A \setminus \bigcup_{i=1}^l A_i\}$ and such that, for each $i \in \{1, \dots, n\}$,

$$\left\| f_i - \sum_{D \in \mathcal{D}_0} \frac{\nu_i(D)}{\mu(D)} \chi_D \right\|_{L_1(\mu)} < \frac{\varepsilon}{C}. \tag{1}$$

We write $\mathcal{C}_0 := \{D \in \mathcal{D}_0\colon\ D \subset A_1 \cup \dots \cup A_{l'}\}$, and for every $k \in \{1, \dots, m\}$ we consider the function $P_{k0} : [0, 1] \to \mathbb{R}$ defined by

$$P_{k0} := \sum_{D \in \mathcal{C}_0} g_k(D) \cdot \chi_D + \sum_{D \in \mathcal{D}_0 \setminus \mathcal{C}_0} \frac{\int_D g_k \, d\mu}{\mu(D)} \cdot \chi_D, \tag{2}$$

under the agreement that $\frac{1}{\mu(D)} \cdot \chi_D = 0$ when $\mu(D) = 0$. Formula (2) yields

$$P_{k0} = \chi_{A_k} \quad \text{for all } k \in \{1, \dots, l\}. \tag{3}$$

Now, for every $k \in \{1, \dots, m\}$ we define the set-function $\Phi_k : \mathcal{B} \to \mathbb{R}$ as

$$\Phi_k(B) = \begin{cases} P_{k0}(D) & \text{if } B \text{ is contained in some } D \in \mathcal{D}_0, \\ 0 & \text{otherwise.} \end{cases}$$

The operator $\Phi : E \to \mathcal{S}$ mapping $T = \sum_{k=1}^m \beta_k T_k$ to $\Phi_T := \sum_{k=1}^m \beta_k \Phi_k$ is linear. Let us prove that Φ satisfies clauses (i)–(iii).

For (i), note that $\nu_i(D) = 0$ for every $D \in \mathcal{C}_0$. So, for every refinement $\mathcal{D}'$ of $\mathcal{D}_0$,

$$\begin{aligned} \sum_{D \in \mathcal{D}'} \Phi_k(D) \nu_i(D) &= \sum_{D \in \mathcal{D}_0 \setminus \mathcal{C}_0} P_{k0}(D) \nu_i(D) = \sum_{D \in \mathcal{D}_0 \setminus \mathcal{C}_0} \frac{\int_D g_k \, d\mu}{\mu(D)} \nu_i(D) \\ &= \sum_{D \in \mathcal{D}_0 \setminus \mathcal{C}_0} \frac{\langle T_k, \chi_D \, d\mu \rangle}{\mu(D)} \nu_i(D) = \left\langle T_k, \sum_{D \in \mathcal{D}_0 \setminus \mathcal{C}_0} \nu_i(D) \frac{\chi_D}{\mu(D)} \, d\mu \right\rangle. \end{aligned}$$

Since $\|T_k\| = 1$, formula (1) leads to

$$\left| \langle T_k, \nu_i \rangle - \sum_{D \in \mathcal{D}'} \Phi_k(D) \nu_i(D) \right| = \left| \left\langle T_k, \nu_i - \sum_{D \in \mathcal{D}_0 \setminus \mathcal{C}_0} \nu_i(D) \frac{\chi_D}{\mu(D)} \, d\mu \right\rangle \right| \leqslant \frac{\varepsilon}{C}.$$

Therefore, given $T = \sum_{k=1}^{m} \lambda_k T_k \in S_E$, we have

$$\left| \langle T, \nu_i \rangle - \sum_{D \in \mathcal{D}'} \Phi_T(D)\nu_i(D) \right| \leqslant \left| \sum_{k=1}^{m} \lambda_k \left(\langle T_k, \nu_i \rangle - \sum_{D \in \mathcal{D}'} \Phi_k(D)\nu_i(D) \right) \right| \leqslant C \frac{\varepsilon}{C} = \varepsilon,$$

which proves (i).

Now, let $\mathcal{D}'$ be any refinement of $\mathcal{D}_0$ and $k \in \{1, \ldots, l\}$. Applying formula (3), we get

$$\sum_{D \in \mathcal{D}'} \Phi_k(D)\chi_D = \sum_{D \in \mathcal{D}'} P_{k0}(D)\chi_D = \sum_{D \in \mathcal{D}'} \chi_{A_k}\chi_D = \chi_{A_k} = T_k$$

and (ii) is done.

Finally, let us show (iii). Let $T = \sum_{k=1}^{m} \beta_k T_k \in S_E$ and $B \in \mathcal{B}$. There are three possible cases:

First case: *B is not contained in any A_i*. Then $\Phi(B) = 0$.

Second case: *there exists $i \in \{1, \ldots, l'\}$ such that $B \subset A_i$*. Thus we get

$$\Phi_T(B) = \sum_{k=1}^{m} \beta_k \Phi_k(B) = \sum_{k=1}^{m} \beta_k P_{k0}(B) = \sum_{k=1}^{m} \beta_k g_k(A_i) = \sum_{k=1}^{m} \beta_k \langle T_k, \delta_{x_i} \rangle = \langle T, \delta_{x_i} \rangle.$$

Third case: *there exists $i \in \{l'+1, \ldots, l\}$ such that $B \subset A_i$*. Then, we have

$$\Phi_T(B) = \sum_{k=1}^{m} \beta_k P_{k0}(B) = \sum_{k=l'+1}^{m} \beta_k P_{k0}(A_i) = \sum_{k=l'+1}^{m} \beta_k \frac{\int_{A_i} g_k \, d\mu}{\mu(A_i)} = \sum_{k=1}^{m} \beta_k \left\langle T_k, \frac{\chi_{A_i}}{\mu(A_i)} \, d\mu \right\rangle = \left\langle T, \frac{\chi_{A_i}}{\mu(A_i)} \, d\mu \right\rangle.$$

In the three cases it is clear that $|\Phi_T(B)| \leqslant \|T\|$, so the proof is complete. □

Now we apply Theorem 2.1 to obtain our local version of the representation of $C[0,1]^{**}$.

Theorem 2.2. *Let E and F be finite-dimensional subspaces of $C[0,1]^{**}$ and $C[0,1]^{*}$ respectively, and let $\varepsilon > 0$. Let E_0 be the maximal subspace of E consisting of simple Borel functions. Thus there exist a linear operator $\Phi : T \in E \to \Phi_T \in \mathcal{S}$ and a subdivision $\mathcal{D}$ of $[0,1]$ so that the following properties hold*:

(i) *for all $T \in S_E$ and $\nu \in S_F$, and for each refinement $\mathcal{D}'$ of $\mathcal{D}$,*

$$\left| \langle T, \nu \rangle - \sum_{B \in \mathcal{D}'} \Phi_T(B)\nu(D) \right| < \varepsilon,$$

(ii) *if $T \in E_0$, then $T = \sum_{D \in \mathcal{D}'} \Phi_T(D)\chi_D$ for each refinement $\mathcal{D}'$ of $\mathcal{D}$,*

(iii) *$|\Phi_T(B)| \leqslant \|T\|$ for all $T \in E$ and $B \in \mathcal{B}$.*

Proof. Let $\mathcal{M}$ be a maximal set of normalized, positive, mutually disjoint measures on $\mathcal{B}$; let $\{\nu_k\}_{k=1}^{n}$ be a normalized basis of F, and $C := \max\{\sum_{k=1}^{n} |\lambda_k|: \sum_{k=1}^{n} \lambda_k \nu_k \in S_F\}$. Pick two numbers $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $C(2\varepsilon_1 + \varepsilon_2) < \varepsilon$.

Following Artemenko's results in [1] (used by Mauldin in [15]), there exist a countable subset $\{\mu_i\}_{i=1}^{\infty}$ of $\mathcal{M}$ and a family $\{\nu_{ki}\}_{i=1}^{\infty}$ of measures for every $k \in \{1, \ldots, n\}$ such that $\nu_k = \sum_{i=1}^{\infty} \nu_{ki}$ and $\nu_{ki} \ll \mu_i$ for all i. We choose $p \in \mathbb{N}$ so that

$$\left\| \nu_k - \sum_{i=1}^{p} \nu_{ki} \right\| \leqslant \varepsilon_1 \quad \text{for all } k \in \{1, \ldots, n\},$$

and take a subdivision $\{A_i\}_{i=1}^{p}$ of $[0,1]$ such that each μ_i is concentrated on A_i.

Let $P_i : M[0,1]^* \to M(A_i)^*$ be the projections given by $P_i(T) := T|_{M(A_i)}$. Denote $E_i := P_i(E)$. Passing to a bigger E, if necessary, we can assume that $E = E_1 \oplus_\infty \cdots \oplus_\infty E_p$.

By Theorem 2.1, there exist a subdivision $\mathcal{A}_i$ of each A_i and an operator $\Phi^i : E_i \to \mathcal{S}$ (we will denote $\Phi_T^i := \Phi^i(T)$) satisfying the following properties:

(i′) for all $T \in S_{E_i}$, all ν_{ki} and all the refinements $\mathcal{A}_i'$ of $\mathcal{A}_i$,

$$\left| \langle T, \nu_{ki} \rangle - \sum_{B \in \mathcal{A}_i'} \Phi_T^i(B)\nu_{ki}(D) \right| < \frac{\varepsilon_2}{p},$$

(ii′) if $T \in E_0 \cap E_i$ and $\mathcal{A}_i'$ is a refinement of $\mathcal{A}_i$ then $T = \sum_{D \in \mathcal{A}_i'} \Phi_T^i(D)\chi_D$,

(iii′) $|\Phi_T^i(B)| \leqslant \|T\|$ for all $T \in E_i$ and all $B \in \mathcal{B}$.

Thus $\Phi := \Phi^1 + \cdots + \Phi^p$ is the wanted operator, and $\Phi_T = \Phi^1_{P_1T} + \cdots + \Phi^p_{P_pT}$. Indeed, since $\{A_1, \ldots, A_p\}$ is a subdivision of $[0,1]$ and every $\mathcal{A}_i$ is a subdivision of A_i, then $\mathcal{D} := \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_p$ is a subdivision of the unit interval. Let $\mathcal{D}'$ be a refinement of $\mathcal{D}$, and let $\mathcal{A}'_i$ be the refinement of each $\mathcal{A}_i$ induced by $\mathcal{D}'$ so that $\mathcal{D}' = \mathcal{A}'_1 \cup \cdots \cup \mathcal{A}'_p$.

For part (iii), take $T \in E$ and any $B \in \mathcal{B}$. If for some $i \in \{1, \ldots, p\}$ there is $A \in \mathcal{A}_i$ such that $B \subset A$, then $\Phi_T(B) = \Phi^i_{P_iT}(B)$, so $|\Phi_T(B)| \leqslant \|P_iT\| \leqslant \|T\|$; otherwise, $\Phi_T(B) = 0$ and (iii) is proved.

For part (i), given $T \in S_E$, and bearing in mind that every ν_{ki} is concentrated on A_i, clause (i′) yields

$$\left|\langle T, \nu_{ki}\rangle - \sum_{D\in\mathcal{D}'} \Phi_T(D)\nu_{ki}(D)\right| = \left|\langle P_iT, \nu_{ki}\rangle - \sum_{D\in\mathcal{A}'_i} \Phi^i_{P_iT}(D)\nu_{ki}(D)\right| < \frac{\varepsilon_2}{p}. \tag{4}$$

Thus, since $\|\nu_k - \sum_{i=1}^p \nu_{ki}\| \leqslant \varepsilon_1$ for all $k \in \{1, \ldots, n\}$, we get on the one hand,

$$\left|\left\langle T, \nu_k - \sum_{i=1}^p \nu_{ki}\right\rangle\right| \leqslant \varepsilon_1, \tag{5}$$

and on the other hand, taking into account part (iii) and $\|T\| = 1$,

$$\begin{aligned}\left|\sum_{D\in\mathcal{D}'} \Phi_T(D)\left(\nu_k(D) - \sum_{i=1}^p \nu_{ki}(D)\right)\right| &\leqslant \max_{D\in\mathcal{D}'} |\Phi_T(D)| \cdot \left|\sum_{D\in\mathcal{D}'}\left(\nu_k(D) - \sum_{i=1}^p \nu_{ki}(D)\right)\right| \\ &\leqslant \left\|\nu_k - \sum_{i=1}^p \nu_{ki}\right\| \leqslant \varepsilon_1. \end{aligned} \tag{6}$$

Thus, from formulas (4)–(6),

$$\left|\langle T, \nu_k\rangle - \sum_{D\in\mathcal{D}'} \Phi_T(D)\nu_k(D)\right| \leqslant 2\varepsilon_1 + \left|\sum_{i=1}^p\left(\langle T, \nu_{ki}\rangle - \sum_{D\in\mathcal{D}'} \Phi_T(D)\nu_{ki}(D)\right)\right| \leqslant 2\varepsilon_1 + p\frac{\varepsilon_2}{p} = 2\varepsilon_1 + \varepsilon_2.$$

Therefore, given any measure $\nu = \sum_{k=1}^n \lambda_k\nu_k \in S_E$, we have

$$\left|\langle T, \nu\rangle - \sum_{D\in\mathcal{D}'} \Phi_T(D)\nu(D)\right| \leqslant \sum_{k=1}^n |\lambda_k|\left|\langle T, \nu_k\rangle - \sum_{D\in\mathcal{D}'} \Phi_T(D)\nu_k(D)\right| \leqslant C(2\varepsilon_1 + \varepsilon_2) < \varepsilon,$$

and (i) is proved.

Finally, for part (ii), consider any $T \in E_0$. Thus, by (ii′), $P_iT = \sum_{D\in\mathcal{A}'_i} \Phi^i_T(D)\chi_D$, so

$$T = \sum_{i=1}^p P_iT = \sum_{i=1}^p \sum_{D\in\mathcal{A}'_i} \Phi^i_T(D)\chi_D = \sum_{D\in\mathcal{D}'} \Phi_T(D)\chi_D,$$

which proves (ii). $\square$

The operator Φ obtained in Theorem 2.2 will be called the *local Mauldin operator* associated with (E, F, ε), and for every $T \in E$, we will say that Φ_T is its Mauldin representation (associated with (E, F, ε)).

3. Main result

In this section, we will show that, for each $1 \leqslant \alpha \leqslant \omega_1$, the space $B_\alpha[0,1]$ of all the bounded Baire functions of class α is a local dual space of the space of Borel measures $M[0,1]$. First, we notice that the equalities of clauses (b) and (c) in the definition of local duality may be relaxed and substituted for inequalities. This assertion is made precise in the following theorem.

Theorem 3.1. *A closed subspace Z of X^* is a local dual of X if and only if for every pair of finite-dimensional subspaces E of X^* and F of X, and every $0 < \varepsilon < 1$, there exists an operator $L : E \to Z$ which satisfies the following conditions*:

(a′) *L is an ε-isometry.*
(b′) *$|\langle L(e), x\rangle - \langle e, x\rangle| < \varepsilon$ for all $e \in S_E$ and all $x \in S_F$.*
(c′) *$\|L(e) - e\| \leqslant \varepsilon\|e\|$ for all $e \in E \cap Z$.*

The proof of Theorem 3.1 is an exercise of linear algebra. A different sort of proof can be found in [8, Theorem 2.5].

In order to prove that an operator $L : E \to X$ is an ε-isometry, it is sufficient to get control over the norms $\|L(x_i)\|$, where $\{x_i\}_{i\in I}$ is a suitable α-net in S_E. This fact is quantitatively settled in the following lemma.

Lemma 3.2. *(See [11, Lemma 2.6].) Let E be a closed subspace of a Banach space X, $\{x_i\}_{i\in I}$ an α-net in S_E with $0<\alpha<1$. Let $\delta>0$ and $L:E\to X$ be a bounded operator such that $1-\delta\leqslant \|L(x_i)\|\leqslant 1+\delta$ for all $i\in I$. Then L is an $(\alpha+\delta)(1-\alpha)^{-1}$-isometry.*

Thus, given $\varepsilon>0$, if α and δ are small enough, then L is an ε-isometry.

The following result collects some results about the spaces $B_\alpha[0,1]$ that we need in order to prove Theorem 3.4.

Theorem 3.3. *(See [2].) Let α be an ordinal with $1\leqslant\alpha\leqslant\omega_1$. There exists a class Ω_α of Borel subsets of $[0,1]$ containing the closed subsets, such that $\{\chi_A\colon A\in\Omega_\alpha\}$ generates a dense subspace of $B_\alpha[0,1]$.*

Here is our main result.

Theorem 3.4. *For every ordinal $1\leqslant\alpha\leqslant\omega_1$, the Baire class $B_\alpha[0,1]$ is a local dual of $M[0,1]$.*

Proof. Let E and F be finite-dimensional subspaces of $M[0,1]^*$ and $M[0,1]$ respectively, and let $0<\varepsilon<1$ be a real number. Our goal is to prove that the clauses (a′), (b′) and (c′) of Theorem 3.1 hold. To do that, let $\{\varepsilon_i\}_{i=1}^6\subset(0,1)$ be real numbers whose exact value will be fixed later. We denote $E_\alpha:=E\cap B_\alpha[0,1]$ and choose a basis of E

$$\{g_1,\dots,g_l,g_{l+1},\dots,g_m\}\subset S_E$$

such that $\{g_1,\dots,g_l\}$ is a basis of E_α. Let $\{\nu_1,\dots,\nu_m\}$ be a subset of $M[0,1]$ so that $\langle g_i,\nu_j\rangle=\delta_{ij}$ for all i and j. Denote $C:=\max_{1\leqslant i\leqslant m}\|\nu_i\|$.

By Theorem 3.3, we can choose a partition $\mathcal{A}=\{A_1,\dots,A_q\}$ of $[0,1]$ and a family of real numbers $\{a_{ir}\}_{i=1}^{l}{}_{r=1}^{q}$ so that $\chi_{A_r}\in B_\alpha[0,1]$ for all r (hence each A_r is a Borel set) and, for each $i\in\{1,\dots,l\}$,

$$\Bigg\|\sum_{r=1}^q a_{ir}\chi_{A_r}-g_i\Bigg\|<\varepsilon_1\quad\text{and}\quad\Bigg\|\sum_{r=1}^q a_{ir}\chi_{A_r}\Bigg\|\leqslant\|g_i\|. \tag{7}$$

We denote $g_i':=\sum_{r=1}^q a_{ir}\chi_{A_r}$ for $i\in\{1,\dots,l\}$, and for $i\in\{l+1,\dots,m\}$, we write $g_i':=g_i$ for the sake of notation.

Passing to a bigger subspace F, if necessary, we take an ε_2-net $\{\mu_1,\dots,\mu_p\}$ in S_F such that for every $g\in E$,

$$(1-\varepsilon_3)\|g\|\leqslant\sup_{1\leqslant j\leqslant p}\langle g,\mu_j\rangle. \tag{8}$$

Let Φ be the local Mauldin operator associated with (E,F,ε_4). For each $i\in\{1,\dots,m\}$, let us denote $\Phi_i:=\Phi(g_i'):\mathcal{B}\to\mathbb{R}$ the corresponding Mauldin representation of g_i'. Thus, by Theorem 2.2, there exists a subdivision $\mathcal{D}=\{D_1,\dots,D_s\}$ finer than the subdivision $\mathcal{A}$ such that, for every refinement $\mathcal{D}'$ of $\mathcal{D}$, we have

$$\Bigg|\sum_{D\in\mathcal{D}'}\Phi_i(D)\mu_j(D)-\langle\mu_j,g_i'\rangle\Bigg|<\varepsilon_4\quad\text{for all } i\in\{1,\dots,m\}\text{ and } j\in\{1,\dots,n\}, \tag{9}$$

$$g_i'=\sum_{D\in\mathcal{D}'}\Phi_i(D)\chi_D\quad\text{for all } i\in\{1,\dots,l\}, \tag{10}$$

$$\big|\Phi_i(D)\big|\leqslant\big\|g_i'\big\|\quad\text{for all } i\in\{1,\dots,m\}\text{ and } D\in\mathcal{B}. \tag{11}$$

Now, for each $i\in\{1,\dots,m\}$ we define a function $h_i:[0,1]\to\mathbb{R}$ by

$$h_i:=\sum_{k=1}^s\Phi_i(D_k)\chi_{D_k}.$$

By formula (10), $g_i'=h_i$ for all $i\in\{1,\dots,l\}$.

Next, we consider a new indexation of the elements of the subdivision $\mathcal{D}$:

$$\mathcal{D}=\{A_{11},\dots,A_{1m_1};A_{21},\dots,A_{2m_2};\dots;A_{q1},\dots,A_{qm_q}\},$$

where, for every $i\in\{1,\dots,q\}$, the family $\{A_{i1},\dots,A_{im_i}\}$ is a partition of A_i. Note that we can (and do) assume $m_i>1$ for every i.

For each $i\in\{1,\dots,q\}$ and each $k\in\{2,\dots,m_i\}$, we take a non-empty closed subset B_{ik} of $[0,1]$ contained in A_{ik} such that, for

$$B_{i1}:=A_{i1}\cup(A_{i2}\setminus B_{i2})\cup\dots\cup(A_{im_i}\setminus B_{im_i}),$$

we have $|\mu_j|(B_{i1}\setminus A_{i1})<\varepsilon_5/2q$, for every $j\in\{1,\dots,p\}$; hence

$$\sum_{k=1}^{m_i} |\mu_j|(A_{ik} \triangle B_{ik}) < \frac{\varepsilon_5}{q}, \quad \text{for every } i \in \{1, \dots, q\} \text{ and } j \in \{1, \dots, p\}. \tag{12}$$

Note that for every $i \in \{1, \dots, q\}$, the sets $\{B_{i1}, \dots, B_{im_i}\}$ form a partition of A_i, so

$$\{B_{11}, \dots, B_{1m_1}; B_{21}, \dots, B_{2m_2}; \dots; B_{q1}, \dots, B_{qm_q}\}$$

is a subdivision of $[0, 1]$ with $\chi_{B_{ik}} \in B_\alpha[0, 1]$ for all i and all k. Thus, the equalities

$$U(\chi_{A_{ik}}) := \chi_{B_{ik}}, \quad i \in \{1, \dots, q\}, \ k \in \{1, \dots, m_i\}$$

define an isometry $U : \operatorname{span}\{\chi_D\colon\ D \in \mathcal{D}\} \to B_\alpha[0, 1]$ that satisfies $U(\chi_{A_i}) = \chi_{A_i}$ for all $A_i \in \mathcal{A}$. Hence $U(g_i') = g_i'$, for all $i \in \{1, \dots, l\}$.

Now we define an operator $L : E \to B_\alpha[0, 1]$ by

$$L(g_i) := U(h_i), \quad \text{for all } i \in \{1, \dots, m\},$$

and show that L satisfies clauses (a′), (b′) and (c′) in Theorem 3.1 for the subspaces E and F and the number $\varepsilon > 0$. In order to prove it, we take an ε_6-net $\{e_1, \dots, e_q\}$ in S_E. Note that $e_k = \sum_{i=1}^m c_{ki} g_i$ for all $k \in \{1, \dots, q\}$, where $c_{ki} = \langle \nu_i, e_k\rangle$. Let us denote $e_k' := \sum_{i=1}^m c_{ki} g_i'$ for all $k \in \{1, \dots, q\}$.

By linearity, the Mauldin representation of every e_k' equals $\sum_{i=1}^m c_{ki}\Phi_i$. Therefore, for each $k \in \{1, \dots, q\}$, Theorem 2.2 gives

$$\|L(e_k)\| = \left\| \sum_{i=1}^m c_{ki} U(h_i) \right\| = \left\| \sum_{i=1}^m c_{ki} h_i \right\| = \sup_{D \in \mathcal{D}} \left| \sum_{i=1}^m c_{ki} \Phi_i(D) \right| \leqslant \|e_k'\|.$$

Since $\|e_k' - e_k\| = \|\sum_{i=1}^m \langle \nu_i, e_k\rangle (g_i - g_i')\| \leqslant mC\varepsilon_1$, we get

$$\|L(e_k)\| \leqslant 1 + mC\varepsilon_1. \tag{13}$$

Given $j \in \{1, \dots, p\}$, on the one hand, for $i \in \{1, \dots, l\}$, we have $L(g_i) = g_i'$, so

$$\left|\langle \mu_j, L(g_i)\rangle - \langle \mu_j, g_i\rangle\right| \leqslant \|g_i' - g_i\|_\infty \leqslant \varepsilon_1. \tag{14}$$

On the other hand, for $i \in \{l+1, \dots, m\}$, we have $L(g_i) = U(h_i)$ and $g_i = g_i'$, so

$$\left|\langle \mu_j, L(g_i)\rangle - \langle \mu_j, g_i\rangle\right| \leqslant \left|\langle \mu_j, U(h_i) - h_i\rangle\right| + \left|\langle \mu_j, h_i - g_i'\rangle\right|. \tag{15}$$

But $h_i = \sum_{k=1}^s \Phi_i(D_k)\chi_{D_k} = \sum_{t=1}^q \sum_{r=1}^{m_q} \Phi_i(A_{tr})\chi_{A_{tr}}$, and by (7), $|\Phi_i(B_{tr})| \leqslant \|g_i'\| \leqslant \|g_i\| = 1$, hence by formula (12), we get

$$\left|\langle \mu_j, U(h_i) - h_i\rangle\right| = \left|\left\langle \mu_j, \sum_{t=1}^q \sum_{r=1}^{m_q} \Phi_i(A_{tr})(\chi_{B_{tr}} - \chi_{A_{tr}})\right\rangle\right| \leqslant \sum_{t=1}^q \sum_{r=1}^{m_q} |\mu_j|(B_{tr} \triangle A_{tr}) \leqslant q\frac{\varepsilon_5}{q} = \varepsilon_5. \tag{16}$$

Moreover, by (9),

$$\left|\langle \mu_j, h_i - g_i'\rangle\right| \leqslant \varepsilon_4. \tag{17}$$

So formulas (15)–(17) yield that

$$\left|\langle \mu_j, L(g_i)\rangle - \langle \mu_j, g_i\rangle\right| \leqslant \varepsilon_5 + \varepsilon_4. \tag{18}$$

Therefore, by (14) and (18), given $g \in S_E$, since $g = \sum_{i=1}^m \langle \nu_i, g\rangle g_i$ and $\sum_{i=1}^m |\langle \nu_i, g\rangle| \leqslant mC$, we get

$$\left|\langle \mu_j, L(g)\rangle - \langle \mu_j, g\rangle\right| \leqslant mC(\varepsilon_1 + \varepsilon_4 + \varepsilon_5). \tag{19}$$

In particular, choosing for every e_i a measure $\mu_j \in \{\mu_i\}_{i=1}^p$ so that $\langle e_i, \mu_j\rangle > 1 - \varepsilon_3$, and taking into account that $C \geqslant 1$, we obtain

$$\|L(e_i)\| \geqslant \langle L(e_i), \mu_j\rangle \geqslant \langle e_i, \mu_j\rangle - mC(\varepsilon_1 + \varepsilon_4 + \varepsilon_5) \geqslant 1 - mC(\varepsilon_1 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5). \tag{20}$$

Therefore, since $\{e_1, \dots, e_q\}$ is an ε_6-net in S_E, Lemma 3.2 and inequalities (13) and (20) yield

$$L \text{ is an } mC\frac{\varepsilon_1 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6}{1 - \varepsilon_6}\text{-isometry}. \tag{21}$$

Moreover, as $\{\mu_1, \dots, \mu_p\}$ is an ε_2-net in S_F, formula (19) shows that for every $\mu \in S_F$ and every $g \in S_E$,

$$\big|\big\langle\mu, L(g)\big\rangle - \langle\mu, g\rangle\big| \leqslant \big(1+\|L\|\big)\varepsilon_2 + mC(\varepsilon_1+\varepsilon_4+\varepsilon_5). \tag{22}$$

Finally, for every $g \in S_{E_\alpha}$, we have

$$\big\|L(g)-g\big\| = \left\|\sum_{i=1}^{l}\langle\nu_i, g\rangle\big(g_i'-g_i\big)\right\| \leqslant lC\varepsilon_1 \leqslant mC\varepsilon_1. \tag{23}$$

Therefore, it follows immediately from formulas (21)–(23) that if the numbers $\{\varepsilon_i\}_{i=1}^6$ are chosen to be small enough, then L satisfies clauses (a′), (b′) and (c′) of Theorem 3.1, and the proof is complete. □

Remark 3.5. Since the characteristic functions in the space $C[0,1] \equiv B_0[0,1]$ do not generate a dense subspace, the proof of Theorem 3.4 is not valid in the case $\alpha = 0$. However, it directly follows from the principle of local reflexivity that the natural copy of $C[0,1]$ in $M[0,1]^* \equiv C[0,1]^{**}$ is a local dual of $M[0,1]$.

Given a Banach space X, let $B_a(X)$ denote the space of its first Baire class elements, that is, the weak* limits in X^{**} of sequences in X. Godefroy, Kalton and Saphar ask in [7, Question 10] which separable spaces X admit a norm-one projection Q on X^{***} with kernel $B_a(X)^\perp$. In the following, we prove that the answer is positive in the case when $X = C[0,1]$ as a consequence of Theorem 3.4.

Corollary 3.6. *There exists a norm-one projection $Q : M[0,1]^{**} \to M[0,1]^{**}$ whose kernel is $B_a(C[0,1])^\perp$ and its range contains $M[0,1]$.*

Proof. Indeed, a subspace Z of a dual space X^* is a local dual of X if and only if there exists a norm-one projection $P : X^{**} \to X^{**}$ whose kernel equals $Z^\perp$ and its range contains X [8, Theorem 2.5]. Thus the result follows immediately from the fact that $B_1[0,1]$ is a local dual of $M[0,1]$, proved in Theorem 3.4, and from the fact that $B_1[0,1]$ is isometrically identifiable with the space $B_a(C[0,1])$. □

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