An L^p spaces-based formulation yielding a new fully mixed finite element method for the coupled Darcy and heat equations*

Gabriel N. Gatica† Salim Meddahi‡ Ricardo Ruiz-Baier§

Abstract

In this work we present and analyse a new fully-mixed finite element method for the nonlinear problem given by the coupling of the Darcy and heat equations. Besides the velocity, pressure, and temperature variables of the fluid, our approach is based on the introduction of the pseudoheat flux as a further unknown. As a consequence of it, and due to the convective term involving the velocity and the temperature, we arrive at saddle point-type schemes in Banach spaces for both equations. In particular, and as suggested by the solvability of a related Neumann problem to be employed in the analysis, we need to make convenient choices of the Lebesgue and H(div)-type spaces to which the unknowns and test functions belong. The resulting coupled formulation is then written equivalently as a fixed point operator, so that the classical Banach theorem, combined with the corresponding Babuška-Brezzi theory, the Banach-Nečas-Babuška theorem, suitable operators mapping Lebesgue spaces into themselves, regularity assumptions, and the aforementioned Neumann problem, are employed to establish the unique solvability of the continuous formulation. Under standard hypotheses satisfied by generic finite element subspaces, the associated Galerkin scheme is analysed similarly and the Brouwer theorem yields existence of a solution. The respective a priori error analysis is also derived. Then, Raviart-Thomas elements of order $k \geq 0$ for the pseudoheat and the velocity, and discontinuous piecewise polynomials of degree $\leq k$ for the pressure and the temperature are shown to satisfy those hypotheses in the 2D case. Several numerical examples illustrating the performance and convergence of the method are reported, including an application into the equivalent problem of miscible displacement in porous media.

Keywords: Darcy equation, heat equation, Lebesgue spaces, fully-mixed formulation, mixed finite element methods, fixed-point theory, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35K05, 35Q79, 80A20, 76S05

* **Funding**: This work has been partially supported by ANID-Chile through the project Centro de Modelamiento Matemático (AFB170001) of the PIA Program: Concurso Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal; by Centro de Investigación en Ingeniería Matemática (CI²MA), Universidad de Concepción; by Spain's Ministry of Economy through project MTM2017-87162-P; and by the Ministry of Science and Higher Education of the Russian Federation within the framework of state support for the creation and development of World-Class Research Centers "Digital biodesign and personalized healthcare" No. 075-15-2020-926.

† CI²MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: ggatica@ci2ma.udec.cl.

‡ Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, Calvo Sotelo s/n, Oviedo, España, email: salim@uniovi.es.

§ School of Mathematics, Monash University, 9 Rainforest Walk, Melbourne, VIC 3800, Australia; and Institute of Computer Science and Mathematical Modeling, Sechenov University, Moscow, Russian Federation, email: ricardo.ruizbaier@monash.edu.

1 Introduction

In this work we are interested in the distribution of the temperature φ of a fluid in a porous medium occupying a bounded and simply connected Lipschitz-continuous domain Ω in R^n, $n \in \{2,3\}$, which is modelled by the coupling of Darcy's law with a convection diffusion equation depending on the velocity $\mathbf{u}$ of the fluid. More precisely, letting $\Gamma := \partial\Omega$ with unit outward normal vector $\boldsymbol{\nu}$, the corresponding system of equations is given by

$$\begin{gathered} \mu(\varphi)\,\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega\,, \quad \operatorname{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u}\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma\,, \\ -\kappa\,\Delta\varphi + \mathbf{u}\cdot\nabla\varphi = f \quad \text{in} \quad \Omega\,, \quad \varphi = 0 \quad \text{on} \quad \Gamma\,, \end{gathered} \tag{1.1}$$

where μ is the temperature-dependent coefficient (representing the porosity times the dynamic viscosity, divided by the permeability, and from now on simply referred to as scaled viscosity), p is the pressure, $\mathbf{f}$ represents an external vector force, κ is the positive thermal conductivity coefficient, and f stands for an external scalar heat production (per unit volume of the porous medium). Suitable hypotheses on the data $\mathbf{f}$ and f are given throughout the analysis below. In turn, concerning the scaled viscosity $\mu : \mathrm{R} \to \mathrm{R}^+$, we assume that this function is uniformly bounded and Lipschitz-continuous, which means that there exist positive constants μ_1, μ_2, and L_μ, such that

$$\mu_1 \,\le\, \mu(t) \,\le\, \mu_2 \quad \forall\, t \in \mathrm{R} \quad \text{and} \quad |\mu(t) - \mu(\widetilde{t})| \,\le\, L_\mu\,|t - \widetilde{t}| \quad \forall\, t,\, \widetilde{t} \in \mathrm{R}\,. \tag{1.2}$$

We note that the same set of coupled equations serves as model for the miscible displacement in porous media [58].

The coupling of the heat equation (or a general convection-diffusion equation) with diverse models in fluid mechanics, such as Stokes, Navier-Stokes, Darcy, Darcy-Forchheimer, Brinkman-Darcy, and others, has been extensively studied in the literature during the last decade by using a variety of numerical methods, which include finite elements, mixed finite elements, discontinuous Galerkin, augmented formulations, and several other procedures. In particular, for nonlinear transport, Boussinesq, and heat-Darcy (or related), we refer for instance to the sets of works (and the references therein) given by [3, 4, 14, 15, 20, 55, 54], [2, 11, 19, 23, 27, 51, 53], and [5, 9, 10, 28, 29], respectively. Regarding the latter model, let us first mention that the case of constant viscosity, but with the exterior force depending on the temperature, has been analysed in [10] by using a spectral method for the corresponding Galerkin scheme. More recently, the model described by (1.1), which assumes a nonlinear viscosity, was considered in [9], where mixed and primal formulations in the Darcy and heat equations, respectively, were employed within a Hilbertian framework. Then, a countable basis of a separable Sobolev space embedded in $\mathrm{L}^\infty(\Omega)$, and the Galerkin method induced by it, were utilised there to prove existence of solution, whereas under smoother exact solution and sufficiently small data, uniqueness was also established. In addition, two finite element methods, one of them stabilised by a suitable additional term, and which are solved using Picard successive approximations, were proposed in [9], and optimal error estimates were derived, all of which was illustrated by several numerical examples. In turn, the a posteriori error analyses of the methods from [9] were developed in [29] (see also [1]). Furthermore, the analysis and results from [9] were complemented in [28] by introducing a new non-stabilised method, and by providing existence and uniqueness of solution without any restriction on the data, but for sufficiently small meshsizes.

On the other hand, during recent years there has been an increasing development of new mixed finite element methods arising from Banach spaces-based variational formulations to solve diverse nonlinear models in continuum mechanics. Among the main advantages of this methodology, we first highlight the non-necessity of any augmentation procedure, technique commonly used within a Hilbertian framework in many previous works (see, e.g., [3, 4, 18, 23]), which, while yielding some benefits, also increases the complexity of the respective continuous and discrete systems. Another advantageous feature of the Banach framework is given by the fact that the spaces to which the unknowns belong are the natural ones that arise from the application of the Cauchy-Schwarz and Hölder inequalities to the terms, suitably tested, of the equations defining the model. From the large amount of works in this direction, we only refer here to [7, 19, 22, 24, 36]. In particular, a dual-mixed formulation for the Stokes equations, in which, differently from [3], the velocity belongs to L^4, is employed in [7] for the coupled flow-transport problem originally studied in [3]. As a consequence, the Cauchy stress is sought in a suitable H(div)-type Banach space, whereas the concentration unknown of the transport equation lies in H^1. In turn, following some ideas from [44, 38], the velocity and a suitable pseudostress tensor are utilised in [16] to study a Banach spaces-based dual-mixed momentum conservative method for the stationary Navier-Stokes problem. Related approaches have been successfully applied as well to the Boussinesq system in [19, 22, 24], and to fluidised beds in [36].

According to the previous discussion, our goal here is to complement the recent theory on the numerical analysis of nonlinear problems and address a new Banach spaces-based mixed finite element formulation for (1.1). As we are interested in employing mixed formulations in both the Darcy and heat equations, we now introduce as an auxiliary unknown the pseudoheat flux (the negative sum of the conductive heat flux and the convective flux)

$$\boldsymbol{\sigma} := \kappa\nabla\varphi - \varphi\,\mathbf{u} \quad \text{in} \quad \Omega\,,$$

which, using the incompressibility condition given by the second equation of the first row of (1.1), implies

$$\operatorname{div}(\boldsymbol{\sigma}) = \kappa\,\Delta\varphi - \mathbf{u}\cdot\nabla\varphi \quad \text{in} \quad \Omega\,.$$

As a consequence, (1.1) can be rewritten, equivalently, as the first-order nonlinear system

$$\begin{aligned} \mu(\varphi)\,\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega\,, \quad \operatorname{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u}\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma\,,\\ \kappa\nabla\varphi - \varphi\,\mathbf{u} = \boldsymbol{\sigma} \quad \text{in} \quad \Omega\,, \quad \operatorname{div}(\boldsymbol{\sigma}) = -f \quad \text{in} \quad \Omega\,, \quad \varphi = 0 \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{1.3}$$

Note that one of the advantages of using also a mixed scheme in the heat equation is the chance of computing another variable of physical interest, such as the gradient of temperature, by means of the simple post-processing formula $\nabla\varphi = \kappa^{-1}\big(\boldsymbol{\sigma} + \varphi\,\mathbf{u}\big)$, and that the method delivers conservative approximations. Another important motivation behind the use of this approach will be explained later on in Section 4.2.

The rest of the paper is organised as follows. At the end of this section we describe standard notations and functional spaces to be utilised throughout the paper. Then, in Section 2 we lay out further details on the governing equations and state preliminary assumptions, and proceed to derive the continuous formulation and analyse its solvability. More precisely, we first collect some definitions and preliminary results, establish the fully-mixed scheme arising from (1.3), and then introduce an equivalent fixed-point strategy to address its solvability. Next, we employ the Babuška-Brezzi theory in Banach spaces and the Banach-Babuška-Nečas theorem to prove the well-posedness of the uncoupled Darcy and heat problems that define the fixed-point operator, and finally apply the Banach fixed-point theorem to conclude the existence of a unique solution. The associated Galerkin scheme, posed in terms of arbitrary finite element subspaces satisfying suitable hypotheses, is set and investigated in Section 3. Similar analytical tools to those employed in Section 2 are employed here. They include a discrete fixed-point strategy, the well-posedness of the respective uncoupled discrete problems, and the application of the Brouwer theorem to conclude existence of solution. This section ends with the corresponding a priori error analysis. Next, in Section 4 we restrict ourselves to the 2D case and define specific finite element subspaces, basically Raviart-Thomas spaces of order $k \geq 0$ for $\boldsymbol{\sigma}$ and $\mathbf{u}$, and discontinuous piecewise polynomials of degree $\leq k$ for p and φ, which are shown to satisfy the abstract assumptions introduced in Section 3. The latter reduce to the discrete inf-sup conditions for each one of the bilinear forms involved in our continuous and discrete formulations. To this end, we need to collect several preliminary results, namely approximation properties of projection and interpolation operators, L^t-stability of the Ritz projector and of the projector on a discrete kernel, a Neumann regularity result, and further properties of the Raviart-Thomas interpolator. For sake of a more concise presentation, some of the above are gathered in three appendices. Section 4 concludes with the rates of convergence of the Galerkin method. We highlight here that, because of the unusual, though natural, norms of the finite element subspaces involved, the discrete inf-sup conditions that are proved have an intrinsic value by themselves since most likely they will be useful in other models. In this regard, we also remark that along the way we identify the only one of them whose validity is, up to our knowledge, an open issue in 3D. Finally, several numerical examples illustrating the performance of the method and confirming the theoretical rates of convergence, are presented in Section 5.

In what follows, given a Lipschitz-continuous domain $\mathcal{O}$ with boundary Γ, we adopt standard notations for Lebesgue spaces $L^t(\mathcal{O})$ and Sobolev spaces $W^{\ell,t}(\mathcal{O})$ and $W_0^{\ell,t}(\mathcal{O})$, with $\ell \geq 0$ and $t \in [1,+\infty)$, whose corresponding norms and seminorm, either for the scalar or vectorial case, are denoted by $\|\cdot\|_{0,t;\mathcal{O}}$, $\|\cdot\|_{\ell,t;\mathcal{O}}$ and $|\cdot|_{\ell,t;\mathcal{O}}$, respectively. Note that $W^{0,t}(\mathcal{O}) = L^t(\mathcal{O})$, and if $t = 2$ we write $H^\ell(\mathcal{O})$ instead of $W^{\ell,2}(\mathcal{O})$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{\ell,\mathcal{O}}$ and $|\cdot|_{\ell,\mathcal{O}}$, respectively. In addition, letting t' be the conjugate of t, that is such that $1/t + 1/t' = 1$, we denote by $W^{1/t',t}(\Gamma)$ the trace space of $W^{1,t}(\mathcal{O})$, and let $W^{-1/t',t'}(\Gamma)$ be the dual of $W^{1/t',t}(\Gamma)$ endowed with the norms $\|\cdot\|_{-1/t',t';\Gamma}$ and $\|\cdot\|_{1/t',t;\Gamma}$, respectively. Furthermore, given a generic scalar functional space S, we denote by $\mathbf{S}$ its vectorial version, examples of which are $\mathbf{L}^t(\mathcal{O}) := [L^t(\mathcal{O})]^n$ and $\mathbf{W}^{\ell,t}(\mathcal{O}) := [W^{\ell,t}(\mathcal{O})]^n$. Finally, we employ C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretisation parameters, which may take different values at different places.

2 The continuous formulation

In this section we introduce and analyse a suitable weak formulation for (1.3). To this end, we first collect some results that will be employed later on, first to derive the right spaces of the continuous formulation, and then to prove some of the inf-sup conditions required along the analysis.

2.1 Preliminary results

We begin by recalling from [39] a theorem that establishes the $W^{1,r}(\Omega)$-solvability, with r in a suitable range contained in $(1,+\infty)$, of the Poisson equation with Neumann boundary conditions.

Theorem 2.1 *Let Ω be as stated at the beginning of Section 1, and let $g \in L^r(\Omega)$, $\mathbf{g} \in \mathbf{L}^r(\Omega)$, and $g_N \in W^{-1/r,r}(\Gamma)$, with $r \in (1,+\infty)$, such that g and g_N satisfy the compatibility condition*

$$\int_\Omega g \,=\, \langle g_N, 1\rangle_\Gamma\,, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $W^{-1/r,r}(\Gamma)$ and $W^{1/r,s}(\Gamma)$, and $s \in (1,+\infty)$ is the conjugate of r, that is $\frac{1}{r} + \frac{1}{s} = 1$. Then, for each $r \in [4/3, 4]$ when $n = 2$, and for each $r \in [3/2, 3]$ when $n = 3$, there exists $u \in W^{1,r}(\Omega)$, unique up to a constant, such that

$$\Delta u \,=\, g + \operatorname{div}(\mathbf{g}) \quad in \quad \Omega\,, \quad \big(\nabla u - \mathbf{g}\big)\cdot\boldsymbol{\nu} \,=\, g_N \quad on \quad \Gamma\,. \tag{2.2}$$

Moreover, there exists a constant $C > 0$, depending only on n, r, and Ω, such that

$$|u|_{1,r;\Omega} \,\le\, C\left\{\|g\|_{0,r;\Omega} \,+\, \|\mathbf{g}\|_{0,r;\Omega} \,+\, \|g_N\|_{-1/r,r;\Gamma}\right\}. \tag{2.3}$$

Proof. It follows by applying [39, Theorem 1.2] to the particular case of the Laplacian operator, and by restricting the full ranges provided for r, which are $(4/3-\varepsilon, 4+\varepsilon)$ and $(3/2-\varepsilon, 3+\varepsilon)$ for $n = 2$ and $n = 3$, respectively, with a constant $\varepsilon > 0$ that arises from the proof, to the present closed intervals. □

In particular, defining for each r in the ranges specified by Theorem 2.1 the space

$$\widetilde{\mathrm{W}}^{1,r}(\Omega) \,:=\, \left\{v \in \mathrm{W}^{1,r}(\Omega) : \quad \int_\Omega v \,=\, 0\right\}, \tag{2.4}$$

we deduce that there exists a unique $u \in \widetilde{\mathrm{W}}^{1,r}(\Omega)$ solution of (2.2). Moreover, since $\|\cdot\|_{1,r;\Omega}$ and $|\cdot|_{1,r;\Omega}$ are equivalent in $\widetilde{\mathrm{W}}^{1,r}(\Omega)$, which follows from the generalised Poincaré inequality (cf. [46, Theorems 5.11.2 and 5.11.3]), the a priori estimate (2.3) becomes

$$\|u\|_{1,r;\Omega} \,\le\, C_r\left\{\|g\|_{0,r;\Omega} \,+\, \|\mathbf{g}\|_{0,r;\Omega} \,+\, \|g_N\|_{-1/r,r;\Gamma}\right\}, \tag{2.5}$$

with a constant $C_r > 0$ depending only on n, r, and Ω, as well. In addition, the corresponding weak formulation of (2.2) reduces to: Find $u \in \widetilde{\mathrm{W}}^{1,r}(\Omega)$ such that

$$\int_\Omega \nabla u\cdot\nabla v \,=\, \int_\Omega \mathbf{g}\cdot\nabla v \,-\, \int_\Omega g\,v \,+\, \langle g_N, v\rangle_\Gamma \qquad \forall\, v \in \mathrm{W}^{1,s}(\Omega)\,. \tag{2.6}$$

In this regard, we notice that actually there is no need to impose the foregoing testing against constant functions v since, in doing so, and thanks to the compatibility condition (2.1), both sides of (2.6) are nullified. Hence, according to the decomposition $\mathrm{W}^{1,s}(\Omega) \,=\, \widetilde{\mathrm{W}}^{1,s}(\Omega) \,\oplus\, \mathrm{R}$, we conclude that (2.6) is equivalent to stating

$$\int_\Omega \nabla u\cdot\nabla v \,=\, \int_\Omega \mathbf{g}\cdot\nabla v \,-\, \int_\Omega g\,v \,+\, \langle g_N, v\rangle_\Gamma \qquad \forall\, v \in \widetilde{\mathrm{W}}^{1,s}(\Omega)\,.$$

Now, it is important to stress that r lies in the ranges indicated in the statement of Theorem 2.1 if and only if s does as well, and therefore the conclusion of that theorem and the above discussion on the respective weak formulations, remain valid if r and s are swapped.

Furthermore, given an arbitrary $t \in (1, +\infty)$, we define for each $\mathbf{z} \in \mathbf{L}^t(\Omega)$ the function

$$\mathcal{J}_t(\mathbf{z}) \,:=\, \begin{cases} |\mathbf{z}|^{t-2}\,\mathbf{z} & \text{if } \mathbf{z} \neq \mathbf{0}\,, \\ \mathbf{0} & \text{otherwise}\,, \end{cases} \tag{2.7}$$

and establish next the mapping properties of the resulting operators $\mathcal{J}_t$.

Lemma 2.2 *Let r, $s \in (1, +\infty)$ such that $\frac{1}{r} + \frac{1}{s} = 1$. Then, for each $\mathbf{z} \in \mathbf{L}^r(\Omega)$ there hold*

$$\mathbf{z}_s \,:=\, \mathcal{J}_r(\mathbf{z}) \,\in\, \mathbf{L}^s(\Omega)\,, \qquad \mathbf{z} \,=\, \mathcal{J}_s(\mathbf{z}_s)\,, \quad \textit{and} \tag{2.8a}$$

$$\int_\Omega \mathbf{z}\cdot\mathbf{z}_s \,=\, \|\mathbf{z}\|_{0,r;\Omega}^r \,=\, \|\mathbf{z}_s\|_{0,s;\Omega}^s \,=\, \|\mathbf{z}\|_{0,r;\Omega}\,\|\mathbf{z}_s\|_{0,s;\Omega}\,, \tag{2.8b}$$

so that $\mathcal{J}_r : \mathbf{L}^r(\Omega) \to \mathbf{L}^s(\Omega)$ and $\mathcal{J}_s : \mathbf{L}^s(\Omega) \to \mathbf{L}^r(\Omega)$ become bijective and inverse to each other.

Proof. It follows straightforwardly from (2.7) and simple algebraic manipulations. $\square$

Next, we recall two integration by parts formulae that will be employed later on, for which, given $r \in (1,+\infty)$, we first introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_r;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^r(\Omega) \Big\}, \tag{2.9a}$$

$$\mathbf{H}^r(\mathrm{div}_r;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^r(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^r(\Omega) \Big\}, \tag{2.9b}$$

which are endowed with the natural norms defined, respectively, as

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_r;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_r;\Omega)\,, \tag{2.10a}$$

$$\|\boldsymbol{\tau}\|_{r,\mathrm{div}_r;\Omega} := \|\boldsymbol{\tau}\|_{0,r;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^r(\mathrm{div}_r;\Omega)\,. \tag{2.10b}$$

Then, proceeding as in [35, eq. (1.43), Section 1.3.4] (see also [17, Section 4.1], [22, Section 3.1]), one can prove that for each $r \geq \frac{2n}{n+2}$ there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\,(\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_r;\Omega)\times \mathrm{H}^1(\Omega)\,, \tag{2.11}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$. In turn, given r, $s \in (1,+\infty)$ such that $\frac{1}{r}+\frac{1}{s}=1$, there also holds (cf. [32, Corollary B. 57])

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle_\Gamma = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\,(\boldsymbol{\tau}, v) \in \mathbf{H}^r(\mathrm{div}_r;\Omega)\times \mathrm{W}^{1,s}(\Omega)\,, \tag{2.12}$$

where, as indicated in the statement of Theorem 2.1, $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathrm{W}^{-1/r,r}(\Gamma)$ and $\mathrm{W}^{1/r,s}(\Gamma)$.

On the other hand, the following lemma introduces a suitable operator mapping $\mathbf{L}^s(\Omega)$ into itself.

Lemma 2.3 *Let r, $s \in (1,+\infty)$ such that $\frac{1}{r}+\frac{1}{s}=1$, with r (and hence s) satisfying the ranges given by Theorem 2.1. Then there exists a linear and bounded operator $D_s : \mathbf{L}^s(\Omega) \to \mathbf{L}^s(\Omega)$ such that*

$$\mathrm{div}\big(D_s(\mathbf{w})\big) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad D_s(\mathbf{w})\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma \qquad \forall\,\mathbf{w} \in \mathbf{L}^s(\Omega)\,. \tag{2.13}$$

In addition, for each $\mathbf{z} \in \mathbf{L}^r(\Omega)$ such that $\mathrm{div}(\mathbf{z}) = 0$ in Ω and $\mathbf{z}\cdot\boldsymbol{\nu} = 0$ on Γ, there holds

$$\int_\Omega \mathbf{z}\cdot D_s(\mathbf{w}) = \int_\Omega \mathbf{z}\cdot\mathbf{w} \qquad \forall\,\mathbf{w} \in \mathbf{L}^s(\Omega)\,. \tag{2.14}$$

Proof. Given $\mathbf{w} \in \mathbf{L}^s(\Omega)$, we let $u \in \widetilde{\mathrm{W}}^{1,s}(\Omega)$ (cf. (2.4)) be the unique solution of problem (2.2) with $g = 0$, $\mathbf{g} = \mathbf{w}$, and $g_N = 0$, that is:

$$\Delta u = \mathrm{div}(\mathbf{w}) \quad \text{in} \quad \Omega\,, \quad \big(\nabla u - \mathbf{w}\big)\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma\,, \quad \int_\Omega u = 0\,. \tag{2.15}$$

Then, the continuous dependence result of (2.15) (cf. (2.5)) guarantees the existence of a constant $C_s > 0$ such that $\|u\|_{1,s;\Omega} \leq C_s\,\|\mathbf{w}\|_{0,s;\Omega}$, and hence, defining $D_s(\mathbf{w}) := \mathbf{w} - \nabla u \in \mathbf{L}^s(\Omega)$, we have that D_s is clearly linear and satisfies

$$\|D_s(\mathbf{w})\|_{0,s;\Omega} \leq \big(1 + C_s\big)\,\|\mathbf{w}\|_{0,s;\Omega}\,,$$

which shows that D_s is bounded. In addition, it is readily seen from (2.15) that $D_s(\mathbf{w})$ satisfies the required conditions in (2.13). Moreover, given $\mathbf{z}$ as indicated in the statement of the lemma, the integration by parts formula (2.12) applied to $\mathbf{z} \in \mathbf{H}^r(\mathrm{div}_r;\Omega)$ and $u \in \mathrm{W}^{1,s}(\Omega)$, yields

$$\int_\Omega \mathbf{z}\cdot\nabla u = -\int_\Omega u\,\mathrm{div}(\mathbf{z}) + \langle \mathbf{z}\cdot\boldsymbol{\nu}, u\rangle_\Gamma = 0\,,$$

whence (2.14) is obtained, thus completing the proof. $\square$

2.2 The fully-mixed formulation

We begin by testing the first equation of the second row of (1.3) against a vector function $\boldsymbol{\tau}$, which formally yields

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} \,-\, \kappa\int_\Omega \nabla\varphi\cdot\boldsymbol{\tau} \,+\, \int_\Omega \varphi\,\mathbf{u}\cdot\boldsymbol{\tau} \,=\, 0\,. \tag{2.16}$$

Then, using the Cauchy-Schwarz and Hölder inequalities, we find that for all $\ell\,,\jmath \in (1,+\infty)$ such that $\frac{1}{\ell}+\frac{1}{\jmath}=1$, there holds

$$\left|\int_\Omega \varphi\,\mathbf{u}\cdot\boldsymbol{\tau}\right| \,\le\, \|\varphi\|_{0,2\ell;\Omega}\,\|\mathbf{u}\|_{0,2\jmath;\Omega}\,\|\boldsymbol{\tau}\|_{0,\Omega}\,, \tag{2.17}$$

which shows that the third term on the left hand side of (2.16) makes sense for $\varphi\in \mathrm{L}^{2\ell}(\Omega)$, $\mathbf{u}\in\mathbf{L}^{2\jmath}(\Omega)$, and $\boldsymbol{\tau}\in\mathbf{L}^2(\Omega)$. Then, knowing where $\boldsymbol{\tau}$ belongs, the first and second terms on the left hand side of (2.16) are finite if $\boldsymbol{\sigma}\in\mathbf{L}^2(\Omega)$ and $\nabla\varphi\in\mathbf{L}^2(\Omega)$, respectively. In addition, in order to be able to apply (2.11) to $\boldsymbol{\tau}$ and φ, so that we obtain

$$\int_\Omega \nabla\varphi\cdot\boldsymbol{\tau} \,=\, -\int_\Omega \varphi\,\mathrm{div}(\boldsymbol{\tau}) \,+\, \langle\boldsymbol{\tau}\cdot\boldsymbol{\nu},\varphi\rangle \,=\, -\int_\Omega \varphi\,\mathrm{div}(\boldsymbol{\tau})\,, \tag{2.18}$$

with $\boldsymbol{\tau}\cdot\boldsymbol{\nu}\in \mathrm{H}^{-1/2}(\Gamma)$ and $\langle\cdot,\cdot\rangle$ denoting the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, it suffices to assume that $\mathrm{div}(\boldsymbol{\tau})\in \mathrm{L}^{(2\ell)'}(\Omega)$, where $(2\ell)' := \frac{2\ell}{2\ell-1}$ is the conjugate of 2ℓ, and that $\mathrm{H}^1(\Omega)$ is continuously embedded in $\mathrm{L}^{2\ell}(\Omega)$. The later is guaranteed for $2\ell\in[1,+\infty)$ when $n=2$, which is always satisfied, and for $2\ell\in[1,6]$ when $n=3$ (cf. [32, Corollary B.43]). On the other hand, since Theorem 2.1 will be applied later on to $r=2\jmath$ or $r=(2\jmath)'$, which will be required to establish some continuous inf-sup conditions, we need that $2\jmath$ lies in the corresponding ranges specified there, that is $2\jmath\le 4$ when $n=2$, and $2\jmath\le 3$ when $n=3$ (note that the respective lower bounds are already satisfied). Then, it is readily seen that $2\jmath\le 4$ (respectively $2\jmath\le 3$) if and only if $2\ell=\frac{2\jmath}{\jmath-1}\ge 4$ (respectively $2\ell=\frac{2\jmath}{\jmath-1}\ge 6$). Thus, from the restrictions on 2ℓ when $n=3$, we deduce that there must hold $2\ell=6$, which yields $2\jmath=3$, $(2\ell)'=6/5$, and $(2\jmath)'=3/2$, so that defining

$$\rho \,=\, 2\ell\,,\quad \varrho \,=\, (2\ell)'\,,\quad r \,=\, 2\jmath\,,\quad\text{and}\quad s \,=\, (2\jmath)'\,, \tag{2.19}$$

we find that the only possible setting for the 3D case is

$$(\rho,\varrho) \,:=\, (6,6/5) \qquad\text{and}\qquad (r,s) \,:=\, (3,3/2)\,. \tag{2.20}$$

In turn, noting that $2\ell\ge 4$ is the only restriction on 2ℓ when $n=2$, at this point we do not consider any particular choice and continue our analysis with a generic value for ℓ, and hence (cf. (2.19)) for ρ, ϱ, r and s as well. We just observe that, being (ρ,ϱ) and (r,s) pairs of conjugate to each other with ρ, $r>2$, there necessarily holds ϱ, $s\in(1,2)$. In addition, it is readily seen that $\rho>r$ when $\rho>4$. According to the above discussion, from now on we look for $\varphi\in\mathrm{L}^\rho(\Omega)$ and $\mathbf{u}\in\mathbf{L}^r(\Omega)$, whereas the test function $\boldsymbol{\tau}\in\mathbf{L}^2(\Omega)$ is such that $\mathrm{div}(\boldsymbol{\tau})\in\mathrm{L}^\varrho(\Omega)$. Later on in Section 4.4, and in order to complete our discrete analysis, we will impose a sharper range for s.

Next, replacing the resulting expression from (2.18) into (2.16), and taking into account the definition (2.9a), we arrive at

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} \,+\, \kappa\int_\Omega \varphi\,\mathrm{div}(\boldsymbol{\tau}) \,+\, \int_\Omega \varphi\,\mathbf{u}\cdot\boldsymbol{\tau} \,=\, 0 \qquad \forall\,\boldsymbol{\tau}\in\mathbf{H}(\mathrm{div}_\varrho;\Omega)\,. \tag{2.21}$$

Furthermore, testing now the second equation of the second row of (1.3) against $\psi \in \mathrm{L}^\rho(\Omega)$, which implicitly imposes the unknown $\boldsymbol{\sigma}$ to belong to $\mathbf{H}(\mathrm{div}_\varrho;\Omega)$, assuming that the datum $f \in \mathrm{L}^\varrho(\Omega)$, and multiplying by the constant κ, we obtain

$$\kappa \int_\Omega \psi \,\mathrm{div}(\boldsymbol{\sigma}) \,=\, -\kappa \int_\Omega f\,\psi \qquad \forall\, \psi \in \mathrm{L}^\rho(\Omega)\,. \tag{2.22}$$

Therefore, given $\mathbf{u} \in \mathbf{L}^r(\Omega)$, and setting

$$\mathrm{H} \,:=\, \mathbf{H}(\mathrm{div}_\varrho;\Omega) \quad \text{and} \quad \mathrm{Q} \,:=\, \mathrm{L}^\rho(\Omega)\,, \tag{2.23}$$

the weak formulation of the convection diffusion model reduces to (2.21) and (2.22), that is: Find $(\boldsymbol{\sigma},\varphi) \in \mathrm{H}\times\mathrm{Q}$ such that

$$\begin{aligned} a(\boldsymbol{\sigma},\boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau},\varphi) \,+\, \int_\Omega \varphi\,\mathbf{u}\cdot\boldsymbol{\tau} \,&=\, 0 && \forall\, \boldsymbol{\tau} \in \mathrm{H}\,,\\ b(\boldsymbol{\sigma},\psi) \,&=\, -\kappa \int_\Omega f\,\psi && \forall\, \psi \in \mathrm{Q}\,, \end{aligned} \tag{2.24}$$

where $a : \mathrm{H}\times\mathrm{H} \to \mathrm{R}$ and $b : \mathrm{H}\times\mathrm{Q} \to \mathrm{R}$ are the bilinear forms defined by

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) \,:=\, \int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau} \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau}) \in \mathrm{H}\times\mathrm{H}\,, \tag{2.25a}$$

$$b(\boldsymbol{\tau},\psi) \,:=\, \kappa \int_\Omega \psi\,\mathrm{div}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\psi) \in \mathrm{H}\times\mathrm{Q}\,. \tag{2.25b}$$

It is easily seen that a and b are bounded with respect to the usual norms of $\mathrm{H} := \mathbf{H}(\mathrm{div}_\varrho;\Omega)$ (cf. (2.10a)) and $\mathrm{Q} := \mathrm{L}^\rho(\Omega)$, and that the corresponding boundedness constants are

$$\|a\| \,=\, 1 \qquad \text{and} \qquad \|b\| \,=\, \kappa\,. \tag{2.26}$$

On the other hand, knowing already that $\mathbf{u}$ must belong to $\mathbf{L}^r(\Omega)$, and bearing in mind the incompressibility and boundary conditions, we introduce appropriate trial and test spaces

$$X_2 \,=\, \mathbf{H}^r_0(\mathrm{div}_r;\Omega) := \Big\{ \mathbf{w} \in \mathbf{H}^r(\mathrm{div}_r;\Omega) : \quad \mathbf{w}\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma \Big\}\,, \tag{2.27a}$$

$$X_1 \,=\, \mathbf{H}^s_0(\mathrm{div}_s;\Omega) := \Big\{ \mathbf{v} \in \mathbf{H}^s(\mathrm{div}_s;\Omega) : \quad \mathbf{v}\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma \Big\}\,, \tag{2.27b}$$

which are endowed with the corresponding norms defined by (2.10b). Indeed, given $\varphi \in \mathrm{L}^\rho(\Omega)$, and assuming that the datum $\mathbf{f}$ lies in $\mathbf{L}^r(\Omega)$, we test the first equation of the first row of (1.3) against $\mathbf{v} \in X_1$, so that applying (2.12) to $\mathbf{v} \in \mathbf{H}^s(\mathrm{div}_s;\Omega)$ and $p \in \mathrm{W}^{1,r}(\Omega)$, we obtain

$$\int_\Omega \mu(\varphi)\,\mathbf{u}\cdot\mathbf{v} \,-\, \int_\Omega p\,\mathrm{div}(\mathbf{v}) \,=\, \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\, \mathbf{v} \in X_1\,. \tag{2.28}$$

We notice here that the resulting second term on the left hand side of (2.28) vanishes when p is constant, and hence for sake of uniqueness of solution, the pressure unknown is sought from now on in the space

$$M_1 \,:=\, \mathrm{L}^r_0(\Omega) \,:=\, \Big\{ q \in \mathrm{L}^r(\Omega) : \quad \int_\Omega q \,=\, 0 \Big\}\,.$$

In connection to the above, and thanks to the decomposition $\mathrm{L}^s(\Omega) = \mathrm{L}^s_0(\Omega) \oplus \mathrm{R}$ and the boundary condition satisfied by $\mathbf{u}$, we realise that testing the incompressibility condition (second equation of the first row of (1.3)) against $q \in \mathrm{L}^s(\Omega)$ is equivalent to doing it against $q \in \mathrm{L}^s_0(\Omega)$, so that the associated test space is set as $M_2 := \mathrm{L}^s_0(\Omega)$. Consequently, the weak formulation of Darcy's problem reads: Find $(\mathbf{u}, p) \in X_2 \times M_1$ such that

$$
\begin{aligned}
a_\varphi(\mathbf{u}, \mathbf{v}) \,+\, b_1(\mathbf{v}, p) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in X_1\,, \\
b_2(\mathbf{u}, q) &= 0 \qquad\qquad \forall\, q \in M_2\,,
\end{aligned}
\tag{2.29}
$$

where, given $\psi \in \mathrm{L}^\rho(\Omega)$, $a_\psi : X_2 \times X_1 \to \mathrm{R}$, $b_1 : X_1 \times M_1 \to \mathrm{R}$ and $b_2 : X_2 \times M_2 \to \mathrm{R}$ are the bilinear forms defined as

$$
a_\psi(\mathbf{w}, \mathbf{v}) \,:=\, \int_\Omega \mu(\psi)\, \mathbf{w} \cdot \mathbf{v} \qquad \forall\, (\mathbf{w}, \mathbf{v}) \in X_2 \times X_1\,, \tag{2.30a}
$$

$$
b_i(\mathbf{v}, q) \,:=\, -\int_\Omega q \,\mathrm{div}(\mathbf{v}) \qquad \forall\, (\mathbf{v}, q) \in X_i \times M_i\,, \quad \forall\, i \in \big\{1, 2\big\}\,. \tag{2.30b}
$$

Similarly as for a and b, we observe that, under the assumptions on μ (cf. (1.2)), a_ψ is bounded with boundedness constant $\|a_\psi\| = \mu_2$ for all $\psi \in \mathrm{L}^\rho(\Omega)$, and b_1 and b_2 are bounded as well with $\|b_1\| = \|b_2\| = 1$.

We summarise the previous discussion by stating from (2.24) and (2.29) the weak formulation of the whole coupled problem (1.3): Find $(\boldsymbol{\sigma}, \varphi) \in \mathrm{H} \times \mathrm{Q}$ and $(\mathbf{u}, p) \in X_2 \times M_1$ such that

$$
\begin{aligned}
a(\boldsymbol{\sigma}, \boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau}, \varphi) \,+\, \int_\Omega \varphi\, \mathbf{u} \cdot \boldsymbol{\tau} &= 0 \qquad\qquad \forall\, \boldsymbol{\tau} \in \mathrm{H}\,, \\
b(\boldsymbol{\sigma}, \psi) &= -\kappa \int_\Omega f\, \psi \qquad \forall\, \psi \in \mathrm{Q}\,, \\
a_\varphi(\mathbf{u}, \mathbf{v}) \,+\, b_1(\mathbf{v}, p) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in X_1\,, \\
b_2(\mathbf{u}, q) &= 0 \qquad\qquad \forall\, q \in M_2\,.
\end{aligned}
\tag{2.31}
$$

2.3 The fixed point strategy

In this section we follow similar approaches developed in, e.g., [7, 22, 36, 37], and make use of the variational formulations (2.24) and (2.29) to introduce a fixed-point strategy addressing the solvability of (2.31). Indeed, we first let $\widetilde{T} : \mathrm{L}^\rho(\Omega) \to X_2 \times M_1$ be the operator defined for each $\psi \in \mathrm{L}^\rho(\Omega)$ as $\widetilde{T}(\psi) = (\widetilde{T}_1(\psi), \widetilde{T}_2(\psi)) := (\widetilde{\mathbf{u}}, \widetilde{p})$, where $(\widetilde{\mathbf{u}}, \widetilde{p}) \in X_2 \times M_1$ is the unique solution (to be confirmed below) of (2.29) with ψ instead of φ, that is

$$
\begin{aligned}
a_\psi(\widetilde{\mathbf{u}}, \mathbf{v}) \,+\, b_1(\mathbf{v}, \widetilde{p}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in X_1\,, \\
b_2(\widetilde{\mathbf{u}}, q) &= 0 \qquad\qquad \forall\, q \in M_2\,.
\end{aligned}
\tag{2.32}
$$

In turn, we let $\widehat{T} : \mathbf{L}^r(\Omega) \to \mathrm{H} \times \mathrm{Q}$ be the operator defined for each $\mathbf{w} \in \mathbf{L}^r(\Omega)$ as $\widehat{T}(\mathbf{w}) = (\widehat{T}_1(\mathbf{w}), \widehat{T}_2(\mathbf{w})) := (\widehat{\boldsymbol{\sigma}}, \widehat{\varphi})$, where $(\widehat{\boldsymbol{\sigma}}, \widehat{\varphi}) \in \mathrm{H} \times \mathrm{Q}$ is the unique solution (to be confirmed below as well) of (2.24) with $\mathbf{w}$ instead of $\mathbf{u}$, that is

$$
\begin{aligned}
a(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau}, \widehat{\varphi}) \,+\, \int_\Omega \widehat{\varphi}\, \mathbf{w} \cdot \boldsymbol{\tau} &= 0 \qquad\qquad \forall\, \boldsymbol{\tau} \in \mathrm{H}\,, \\
b(\widehat{\boldsymbol{\sigma}}, \psi) &= -\kappa \int_\Omega f\, \psi \qquad \forall\, \psi \in \mathrm{Q}\,,
\end{aligned}
\tag{2.33}
$$

Thus, defining the composite operator $T : X_2 \to X_2$ as

$$T(\mathbf{w}) \, := \, \widetilde{T}_1\big(\widehat{T}_2(\mathbf{w})\big) \qquad \forall\, \mathbf{w} \in X_2\,, \tag{2.34}$$

we notice that solving (2.31) is equivalent to seeking a fixed point of T, that is $\mathbf{u} \in X_2$ such that

$$T(\mathbf{u}) \; = \; \mathbf{u}\,. \tag{2.35}$$

We end this section by remarking that the above setting certainly requires that both operators $\widetilde{T}$ and $\widehat{T}$ be well defined, that is that the uncoupled problems (2.32) and (2.33) be well-posed, which is precisely the main goal of the following section.

2.4 Well-posedness of the uncoupled problems

2.4.1 Preliminary abstract results

In this section we recall two abstract results that will be applied in what follows. The first one is the classical Babuška-Brezzi theorem, but in Banach spaces.

Theorem 2.4 *Let H_1, H_2, Q_1, and Q_2 be real reflexive Banach spaces, and let $a : H_2 \times H_1 \to \mathrm{R}$ and $b_i : H_i \times Q_i \to \mathrm{R}$, $i \in \{1,2\}$, be bounded bilinear forms with boundedness constants given by $\|a\|$ and $\|b_i\|$, $i \in \{1,2\}$, respectively. In addition, for each $i \in \{1,2\}$, let $\mathcal{K}_i$ be the kernel of the operator induced by b_i, that is*

$$\mathcal{K}_i \, := \, \Big\{ v \in H_i : \quad b_i(v,q) = 0 \quad \forall\, q \in Q_i \Big\}\,.$$

Assume that

i) *there exists $\alpha > 0$ such that*

$$\sup_{\substack{v \in \mathcal{K}_1 \\ v \neq 0}} \frac{a(w,v)}{\|v\|_{H_1}} \, \geq \, \alpha\, \|w\|_{H_2} \qquad \forall\, w \in \mathcal{K}_2\,,$$

ii) *there holds*

$$\sup_{w \in \mathcal{K}_2} a(w,v) \, > \, 0 \qquad \forall\, v \in \mathcal{K}_1\,, \; v \neq 0\,,$$

iii) *for each $i \in \{1,2\}$ there exists $\beta_i > 0$ such that*

$$\sup_{\substack{v \in H_i \\ v \neq 0}} \frac{b_i(v,q)}{\|v\|_{H_i}} \, \geq \, \beta_i\, \|q\|_{Q_i} \qquad \forall\, q \in Q_i\,.$$

Then, for each $(F,G) \in H_1' \times Q_2'$ there exists a unique $(u,p) \in H_2 \times Q_1$ such that

$$\begin{aligned} a(u,v) + b_1(v,p) \; &= \; F(v) \qquad \forall\, v \in H_1\,, \\ b_2(u,q) \; &= \; G(q) \qquad \forall\, q \in Q_2\,, \end{aligned} \tag{2.36}$$

and the following a priori estimates hold:

$$\begin{aligned} \|u\|_{H_2} \; &\leq \; \frac{1}{\alpha}\, \|F\|_{H_1'} + \frac{1}{\beta_2} \left(1 + \frac{\|a\|}{\alpha} \right) \|G\|_{Q_2'}\,, \\ \|p\|_{Q_1} \; &\leq \; \frac{1}{\beta_1} \left(1 + \frac{\|a\|}{\alpha} \right) \|F\|_{H_1'} + \frac{\|a\|}{\beta_1\, \beta_2} \left(1 + \frac{\|a\|}{\alpha} \right) \|G\|_{Q_2'}\,. \end{aligned} \tag{2.37}$$

Moreover, i), ii), *and* iii) *are also necessary conditions for the well-posedness of* (2.36).

Proof. See [8, Theorem 2.1, Corollary 2.1, Section 2.1] for details. In turn, for the particular case given by $H_1 = H_2$, $Q_1 = Q_2$, and $b_1 = b_2$, we also refer to [32, Theorem 2.34]. □

We stress here that, instead of the pair of assumptions given by i) and ii), one could consider the equivalent one arising after exchanging there the roles of $\mathcal{K}_1$ and $\mathcal{K}_2$ (cf. [8, eqs. (2.10) and (2.11)]). Furthermore, it is important to remark that (2.37) is equivalent to an inf-sup condition for the bilinear form arising after adding the left hand sides of (2.36), which means that there exists a constant $C > 0$, depending only on α, β_1, β_2, and $\|a\|$, such that

$$\sup_{\substack{(v,q)\in H_1\times Q_2 \\ (v,q)\neq\mathbf{0}}} \frac{a(u,v) + b_1(v,p) + b_2(u,q)}{\|(v,q)\|_{H_1\times Q_2}} \geq C\,\|(u,p)\|_{H_2\times Q_1} \quad \forall\,(u,p)\in H_2\times Q_1\,. \tag{2.38}$$

The second result is given by the Banach-Nečas-Babuška Theorem (also know as the generalised Lax-Milgram Lemma), which is stated as follows.

Theorem 2.5 *Let H and Q be Banach spaces such that Q is reflexive, and let $A : H\times Q \longrightarrow \mathrm{R}$ be a bounded bilinear form. Assume that*

i) *there exists $\alpha > 0$ such that*

$$\sup_{\substack{v\in Q\\ v\neq 0}} \frac{A(w,v)}{\|v\|_Q} \geq \alpha\,\|w\|_H \qquad \forall\,w\in H\,,$$

ii) *there holds*

$$\sup_{w\in H} A(w,v) > 0 \qquad \forall\,v\in Q,\ v\neq 0\,.$$

Then, for each $F\in Q'$ there exists a unique $u\in H$ such that

$$A(u,v) = F(v) \qquad \forall\,v\in Q\,, \tag{2.39}$$

and the following a priori estimate holds

$$\|u\|_H \leq \frac{1}{\alpha}\,\|F\|_{Q'}\,. \tag{2.40}$$

Moreover, i) *and* ii) *are also necessary conditions for the well-posedness of* (2.39).

Proof. See [32, Theorems 2.6] □

2.4.2 Well-definedness of the operator $\widetilde{T}$

In order to prove that the operator $\widetilde{T}$ is well-defined, we plan to employ some of the preliminary results provided in Section 2.1, and then apply Theorem 2.4. To this end, we first let $\mathcal{K}_i$, $i\in\{1,2\}$, be the kernel of the bilinear form b_i (cf. (2.30b)), that is

$$\mathcal{K}_i := \Big\{\mathbf{v}\in X_i : \quad b_i(\mathbf{v},q) = 0 \qquad \forall\,q\in M_i\Big\}\,,$$

which, according to the definitions of X_1 (cf. (2.27b)), X_2 (cf. (2.27a)), and b_i (cf. (2.30b)), yields

$$\mathcal{K}_1 := \Big\{\mathbf{v}\in \mathbf{H}_0^s(\mathrm{div}_s;\Omega) : \quad \mathrm{div}(\mathbf{v}) = 0 \quad \text{in} \quad \Omega\Big\}\,, \tag{2.41a}$$

$$\mathcal{K}_2 := \Big\{\mathbf{w}\in \mathbf{H}_0^r(\mathrm{div}_r;\Omega) : \quad \mathrm{div}(\mathbf{w}) = 0 \quad \text{in} \quad \Omega\Big\}\,. \tag{2.41b}$$

Then, we have the following continuous inf-sup conditions.

Lemma 2.6 *There exists $\widetilde{\alpha} > 0$ such that for each $\psi \in \mathrm{L}^\rho(\Omega)$ there hold*

$$\sup_{\substack{\mathbf{v}\in\mathcal{K}_1\\ \mathbf{v}\neq\mathbf{0}}} \frac{a_\psi(\mathbf{w},\mathbf{v})}{\|\mathbf{v}\|_{X_1}} \,\geq\, \widetilde{\alpha}\,\|\mathbf{w}\|_{X_2} \qquad \forall\,\mathbf{w}\in\mathcal{K}_2\,, \tag{2.42a}$$

and

$$\sup_{\mathbf{w}\in\mathcal{K}_2} a_\psi(\mathbf{w},\mathbf{v}) \,>\, 0 \qquad \forall\,\mathbf{v}\in\mathcal{K}_1\,,\ \mathbf{v}\neq\mathbf{0}\,. \tag{2.42b}$$

Proof. Given $\psi \in \mathrm{L}^\rho(\Omega)$, we first consider $\mathbf{w} \in \mathcal{K}_2$ (cf. (2.41b)), $\mathbf{w} \neq \mathbf{0}$. Then, recalling that s is the conjugate exponent of r, we let $\mathbf{w}_s := \mathcal{J}_s(\mathbf{w}) \in \mathbf{L}^s(\Omega)$ as defined in (2.7) and Lemma 2.2, which satisfies

$$\int_\Omega \mathbf{w}\cdot\mathbf{w}_s \,=\, \|\mathbf{w}\|_{0,r;\Omega}\,\|\mathbf{w}_s\|_{0,s;\Omega}\,.$$

Thus, applying the lower bound for μ (cf. (1.2)) and Lemma 2.3, we find that

$$\big|a_\psi\big(\mathbf{w},D_s(\mathbf{w}_s)\big)\big| \,\geq\, \mu_1\int_\Omega \mathbf{w}\cdot D_s(\mathbf{w}_s) \,=\, \mu_1\int_\Omega \mathbf{w}\cdot\mathbf{w}_s \,=\, \mu_1\,\|\mathbf{w}\|_{0,r;\Omega}\,\|\mathbf{w}_s\|_{0,s;\Omega}\,,$$

and hence, using that $D_s(\mathbf{w}_s) \in \mathcal{K}_1$ (cf. Lemma 2.3 and (2.41a)), we deduce that

$$\sup_{\substack{\mathbf{v}\in\mathcal{K}_1\\ \mathbf{v}\neq\mathbf{0}}} \frac{a_\psi(\mathbf{w},\mathbf{v})}{\|\mathbf{v}\|_{X_1}} \,\geq\, \frac{\big|a_\psi(\mathbf{w},D_s(\mathbf{w}_s))\big|}{\|D_s(\mathbf{w}_s)\|_{X_1}} \,=\, \frac{\big|a_\psi(\mathbf{w},D_s(\mathbf{w}_s))\big|}{\|D_s(\mathbf{w}_s)\|_{0,s;\Omega}} \,\geq\, \frac{\mu_1}{\|D_s\|}\,\|\mathbf{w}\|_{0,r;\Omega} \,=\, \frac{\mu_1}{\|D_s\|}\,\|\mathbf{w}\|_{X_2}\,,$$

which proves (2.42a) with $\widetilde{\alpha} = \frac{\mu_1}{\|D_s\|}$. In turn, we now take $\mathbf{v} \in \mathcal{K}_1$ (cf. (2.41a)), $\mathbf{v} \neq \mathbf{0}$, and let $\mathbf{v}_r := \mathcal{J}_r(\mathbf{v}) \in \mathbf{L}^r(\Omega)$. In this way, employing again (1.2), Lemmas 2.2 and 2.3, and the fact that $D_r(\mathbf{v}_r) \in \mathcal{K}_2$ (cf. (2.41b)), we obtain

$$\sup_{\mathbf{w}\in\mathcal{K}_2} a_\psi(\mathbf{w},\mathbf{v}) \,\geq\, \mu_1\int_\Omega D_r(\mathbf{v}_r)\cdot\mathbf{v} \,=\, \mu_1\int_\Omega \mathbf{v}_r\cdot\mathbf{v} \,=\, \mu_1\,\|\mathbf{v}\|^s_{0,s;\Omega} \,>\, 0\,,$$

which shows (2.42b) and finishes the proof of the lemma. □

We now establish the continuous inf-sup conditions for the bilinear forms b_i, $i \in \{1,2\}$.

Lemma 2.7 *There exist $\widetilde{\beta}_1$, $\widetilde{\beta}_2 > 0$ such that for each $i \in \{1,2\}$ there holds*

$$\sup_{\substack{\mathbf{v}\in X_i\\ \mathbf{v}\neq\mathbf{0}}} \frac{b_i(\mathbf{v},q)}{\|\mathbf{v}\|_{X_i}} \,\geq\, \widetilde{\beta}_i\,\|q\|_{M_i} \qquad \forall\,q\in M_i\,. \tag{2.43}$$

Proof. It suffices to prove for $i = 1$, since the proof for $i = 2$ follows verbatim by exchanging the roles of r and s. We begin by stressing that (2.7) and Lemma 2.2 are certainly valid for the corresponding scalar version of the operator $\mathcal{J}_t$, $t \in (1,+\infty)$, which we use next. In fact, given $q \in M_1 = \mathrm{L}^r_0(\Omega)$, we first set $q_s := \mathcal{J}_r(q) \in \mathrm{L}^s(\Omega)$ and $q_s^0 := q_s - \frac{1}{|\Omega|}\int_\Omega q_s \in \mathrm{L}^s_0(\Omega)$, and then let $u \in \widetilde{\mathrm{W}}^{1,s}(\Omega)$ be the unique solution of problem (2.2) with $g = q_s^0$, $\mathbf{g} = \mathbf{0}$, and $g_N = 0$, that is:

$$\Delta u \,=\, q_s^0 \quad\text{in}\quad \Omega\,,\quad \nabla u\cdot\boldsymbol{\nu} \,=\, 0 \quad\text{on}\quad \Gamma\,,\quad \int_\Omega u = 0\,. \tag{2.44}$$

Then, the continuous dependence result for (2.44) (cf. (2.5)) implies the existence of a constant $C_s > 0$ such that $\|u\|_{1,s;\Omega} \,\leq\, C_s\,\|q_s^0\|_{0,s;\Omega}$. In turn, there also exists a constant $\widetilde{C}_s > 0$ such that

$\|q_s^0\|_{0,s;\Omega} \,\le\, \widetilde{C}_s\,\|q_s\|_{0,s;\Omega}$. Next, defining $\bar{\mathbf{v}} \,:=\, -\nabla u \in \mathbf{L}^s(\Omega)$, we have that $\mathrm{div}(\bar{\mathbf{v}}) \,=\, -\,q_s^0$ in Ω and $\bar{\mathbf{v}}\cdot\boldsymbol{\nu} = 0$ on Γ, whence $\bar{\mathbf{v}} \in X_1$ (cf. (2.27b)) and

$$\|\bar{\mathbf{v}}\|_{X_1} \,=\, \|\bar{\mathbf{v}}\|_{s,\mathrm{div}_s;\Omega} \,\le\, (1 + C_s)\,\|q_s^0\|_{0,s;\Omega} \,\le\, (1 + C_s)\widetilde{C}_s\,\|q_s\|_{0,s;\Omega}\,.$$

In this way, using that $\int_\Omega q\,q_s^0 = \int_\Omega q\,q_s$, it follows that

$$\sup_{\substack{\mathbf{v}\in X_1\\ \mathbf{v}\neq\mathbf{0}}} \frac{b_1(\mathbf{v},q)}{\|\mathbf{v}\|_{X_1}} \,\ge\, \frac{b_1(\bar{\mathbf{v}},q)}{\|\bar{\mathbf{v}}\|_{X_1}} \,=\, \frac{\displaystyle\int_\Omega q\,q_s}{\|\bar{\mathbf{v}}\|_{X_1}} \,=\, \frac{\|q\|_{0,r;\Omega}\,\|q_s\|_{0,s;\Omega}}{\|\bar{\mathbf{v}}\|_{X_1}} \,\ge\, \big((1 + C_s)\widetilde{C}_s\big)^{-1}\,\|q\|_{0,r;\Omega}\,,$$

which proves (2.43) for $i = 1$ with $\widetilde{\beta}_1 \,=\, \big((1 + C_s)\widetilde{C}_s\big)^{-1}$. As stated at the beginning of this proof, the inf-sup condition for b_2 is proved by taking now $q \in M_2 \,=\, \mathrm{L}_0^s(\Omega)$, setting $q_r := \mathcal{J}_s(q) \in \mathrm{L}^r(\Omega)$ and $q_r^0 := q_r - \frac{1}{|\Omega|}\int_\Omega q_r \in \mathrm{L}_0^r(\Omega)$, and then letting $u \in \widetilde{\mathrm{W}}^{1,r}(\Omega)$ be the unique solution of problem (2.2) with $g = q_r^0$, $\mathbf{g} = \mathbf{0}$, and $g_N = 0$. We omit further details. $\square$

Next, we let $F \in X_1'$ be the functional given by the right hand side of the first equation of (2.32), that is $F(\mathbf{v}) \,:=\, \displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v} \quad \forall\,\mathbf{v} \in X_1$, which satisfies $\|F\|_{X_1'} \,\le\, \|\mathbf{f}\|_{0,r;\Omega}$. Then, we have the following result establishing that the operator $\widetilde{T}$ (cf. (2.32)) is well defined.

Theorem 2.8 *For each $\psi \in \mathrm{L}^\rho(\Omega)$ there exists a unique $(\widetilde{\mathbf{u}},\widetilde{p}) = \widetilde{T}(\psi) \in X_2 \times M_1$ solution to (2.32). Moreover, there hold*

$$\begin{aligned} \|\widetilde{T}_1(\psi)\|_{X_2} \,&=\, \|\widetilde{\mathbf{u}}\|_{X_2} \,\le\, \frac{1}{\widetilde{\alpha}}\,\|\mathbf{f}\|_{0,r;\Omega} \quad \textit{and} \\ \|\widetilde{T}_2(\psi)\|_{M_1} \,&=\, \|\widetilde{p}\|_{M_1} \,\le\, \frac{1}{\widetilde{\beta}_1}\left(1 + \frac{\mu_2}{\widetilde{\alpha}}\right)\|\mathbf{f}\|_{0,r;\Omega}\,. \end{aligned} \tag{2.45}$$

Proof. Thanks to Lemmas 2.6 and 2.7, and bearing in mind that the bilinear forms a_ψ, for each $\psi \in \mathrm{L}^\rho(\Omega)$, b_1, and b_2 are all bounded, as well as that X_1, X_2, M_1, and M_2 are all reflexive Banach spaces, the proof reduces simply to a straightforward application of Theorem 2.4. In particular, the a priori estimates provided by (2.45) follow from (2.37), the upper bound for $\|F\|_{X_1'}$ indicated previously and the fact that the right hand side of the second row of (2.32) is the null functional. $\square$

2.4.3 Well-definedness of the operator $\widehat{T}$

In this section we use a suitable combination of Theorems 2.4 and 2.5 to prove that the operator $\widehat{T}$ is well-defined. More precisely, we first apply Theorem 2.4 to a perturbation of (2.33), and then employ Theorem 2.5 to conclude that the whole problem (2.33) is well-posed. To this end, we begin by letting V be the null space of the operator induced by the bilinear form b, that is

$$\mathrm{V} := \Big\{\boldsymbol{\tau} \in \mathrm{H}: \quad b(\boldsymbol{\tau},\psi) = 0 \quad \forall\,\psi \in \mathrm{Q}\Big\}\,,$$

which, according to the definitions of b (cf. (2.25b)) and the spaces H and Q (cf. (2.23)), yields

$$\mathrm{V} \,:=\, \Big\{\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_\varrho;\Omega): \quad \mathrm{div}(\boldsymbol{\tau}) \,=\, 0\Big\}\,.$$

Then, it is straightforward to see from the definitions of a (cf. (2.25a)) and the norm of $\mathbf{H}(\mathrm{div}_\varrho;\Omega)$ (cf. (2.10a)) that there holds

$$a(\boldsymbol{\tau},\boldsymbol{\tau}) \,=\, \|\boldsymbol{\tau}\|^2_{\mathrm{div}_\varrho;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathrm{V}\,, \tag{2.46}$$

from which one easily deduces that a satisfies the assumptions i) and ii) of Theorem 2.4, the first one with constant $\widehat{\alpha} = 1$.

Furthermore, we prove now that the bilinear form b satisfies the assumption iii) of Theorem 2.4. Indeed, while the corresponding proof is basically already available in the literature (see, e.g. [16, Lemma 3.4], [17, Lemma 2.1], and [36, Lemma 3.5]), we provide it anyway next for sake of completeness of the presentation.

Lemma 2.9 *There exists $\widehat{\beta} > 0$, depending only on Ω, such that*

$$\sup_{\substack{\boldsymbol{\tau}\in \mathrm{H}\\ \boldsymbol{\tau}\neq \mathbf{0}}} \frac{b(\boldsymbol{\tau},\psi)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \,\geq\, \widehat{\beta}\,\|\psi\|_{\mathrm{Q}} \qquad \forall\,\psi\in \mathrm{Q}\,. \tag{2.47}$$

Proof. We begin by using again the scalar version of the operator $\mathcal{J}_t$, $t\in(1,+\infty)$, for which (2.7) and Lemma 2.2 are valid as well. In fact, given $\psi\in \mathrm{Q} := \mathrm{L}^\rho(\Omega)$, we set $\psi_\varrho := \mathcal{J}_\rho(\psi) \in \mathrm{L}^\varrho(\Omega)$, which satisfies

$$\int_\Omega \psi\,\psi_\varrho \,=\, \|\psi\|_{0,\rho;\Omega}\,\|\psi_\varrho\|_{0,\varrho;\Omega}\,. \tag{2.48}$$

Then, we consider the boundary value problem

$$-\Delta w \,=\, \psi_\varrho \quad\text{in}\quad \Omega\,,\quad w \,=\, 0 \quad\text{on}\quad \Gamma\,, \tag{2.49}$$

whose variational formulation, which follows from (2.11) applied to $\nabla w\in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$ and $z\in \mathrm{H}^1_0(\Omega)$, becomes: Find $w\in \mathrm{H}^1_0(\Omega)$ such that

$$\int_\Omega \nabla w\cdot\nabla z \,=\, \int_\Omega \psi_\varrho\, z \qquad \forall\, z\in \mathrm{H}^1_0(\Omega)\,. \tag{2.50}$$

We remark that, thanks to Hölder's inequality and the continuous injection $i_\rho : \mathrm{H}^1(\Omega)\to \mathrm{L}^\rho(\Omega)$, the right hand side of (2.50) defines a functional in $\mathrm{H}^1_0(\Omega)'$. Consequently, a straightforward application of the classical Lax-Milgram Lemma implies the existence of a unique solution $w\in \mathrm{H}^1_0(\Omega)$ to (2.50) (equivalently to (2.49)). Moreover, it follows from (2.50) that

$$|w|_{1,\Omega} \,\leq\, c_P\,\|i_\rho\|\,\|\psi_\varrho\|_{0,\varrho;\Omega}\,, \tag{2.51}$$

where c_P is the positive constant, depending only on Ω, that establishes that $\|v\|_{1,\Omega} \,\leq\, c_P\,|v|_{1,\Omega}$ for all $v\in \mathrm{H}^1_0(\Omega)$, also known as the Poincaré inequality. Then, defining $\widetilde{\boldsymbol{\tau}} := -\nabla w \in \mathbf{L}^2(\Omega)$, we notice that $\mathrm{div}(\widetilde{\boldsymbol{\tau}}) = \psi_\varrho$ in Ω, which says that actually $\widetilde{\boldsymbol{\tau}}\in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$ (cf. (2.9a)), and then, using (2.51), we get

$$\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_\varrho;\Omega} \,=\, \|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} + \|\mathrm{div}(\widetilde{\boldsymbol{\tau}})\|_{0,\varrho;\Omega} \,=\, |w|_{1,\Omega} + \|\psi_\varrho\|_{0,\varrho;\Omega} \,\leq\, \big(1 + c_P\,\|i_\rho\|\big)\|\psi_\varrho\|_{0,\varrho;\Omega}\,. \tag{2.52}$$

In this way, employing now (2.48), we find that

$$\sup_{\substack{\boldsymbol{\tau}\in \mathrm{H}\\ \boldsymbol{\tau}\neq \mathbf{0}}} \frac{b(\boldsymbol{\tau},\psi)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \,\geq\, \frac{b(\widetilde{\boldsymbol{\tau}},\psi)}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_\varrho;\Omega}} \,=\, \frac{\displaystyle\int_\Omega \psi\,\psi_\varrho}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_\varrho;\Omega}} \,=\, \frac{\|\psi\|_{0,\rho;\Omega}\,\|\psi_\varrho\|_{0,\varrho;\Omega}}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_\varrho;\Omega}}\,, \tag{2.53}$$

which, together with (2.52), yields (2.47) with $\widehat{\beta} := \big(1 + c_P\,\|i_\rho\|\big)^{-1}$.

□

We now let $A : \big(\mathrm{H} \times \mathrm{Q}\big) \times \big(\mathrm{H} \times \mathrm{Q}\big) \to \mathrm{R}$ be the bounded bilinear form arising from (2.33) after adding the left hand sides of the equations, but without including the term depending on the given $\mathbf{w}$, that is

$$A\big((\boldsymbol{\zeta},\phi),(\boldsymbol{\tau},\psi)\big) \,:=\, a(\boldsymbol{\zeta},\boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau},\phi) \,+\, b(\boldsymbol{\zeta},\psi) \tag{2.54}$$

for all $(\boldsymbol{\zeta},\phi)$, $(\boldsymbol{\tau},\psi) \in \mathrm{H}\times\mathrm{Q}$. Note that the boundedness of A follows from those of a and b (cf. (2.26)). Then, denoting by $\mathbf{A} \in \mathcal{L}\big(\mathrm{H}\times\mathrm{Q},(\mathrm{H}\times\mathrm{Q})'\big)$ the operator induced by A, and knowing from (2.46) and Lemma 2.9 that a and b satisfy the hypotheses of Theorem 2.4 with $H_1 = H_2 = \mathrm{H}$, $Q_1 = Q_2 = \mathrm{Q}$, and $b_1 = b_2 = b$, we conclude from a straightforward application of this abstract result that $\mathbf{A}$ is bijective. Moreover, it follows from (2.38) that A satisfies a global inf-sup condition on $\mathrm{H}\times\mathrm{Q}$, which means that there exists a positive constant $\alpha_{\widehat{T}}$, depending only on $\widehat{\alpha}$, $\widehat{\beta}$, and $\|a\|$, such that

$$\sup_{\substack{(\boldsymbol{\tau},\psi)\in\mathrm{H}\times\mathrm{Q}\\(\boldsymbol{\tau},\psi)\neq\mathbf{0}}} \frac{A\big((\boldsymbol{\zeta},\phi),(\boldsymbol{\tau},\psi)\big)}{\|(\boldsymbol{\tau},\psi)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \alpha_{\widehat{T}}\,\|(\boldsymbol{\zeta},\phi)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\boldsymbol{\zeta},\phi) \,\in\, \mathrm{H}\times\mathrm{Q}\,. \tag{2.55}$$

Next, we let $A_{\mathbf{w}} : \big(\mathrm{H} \times \mathrm{Q}\big) \times \big(\mathrm{H} \times \mathrm{Q}\big) \to \mathrm{R}$ be the bounded bilinear form that results after adding the full left hand sides of the equations of (2.33), that is

$$A_{\mathbf{w}}\big((\boldsymbol{\zeta},\phi),(\boldsymbol{\tau},\psi)\big) \,:=\, A\big((\boldsymbol{\zeta},\phi),(\boldsymbol{\tau},\psi)\big) \,+\, \int_\Omega \phi\,\mathbf{w}\cdot\boldsymbol{\tau} \tag{2.56}$$

for all $(\boldsymbol{\zeta},\phi)$, $(\boldsymbol{\tau},\psi) \in \mathrm{H}\times\mathrm{Q}$. We remark that the boundedness of $A_{\mathbf{w}}$ follows from that of A and the estimate (2.17). Furthermore, the formulation (2.33) can be rewritten, equivalently, as: Find $(\widehat{\boldsymbol{\sigma}},\widehat{\varphi}) \in \mathrm{H}\times\mathrm{Q}$ such that

$$A_{\mathbf{w}}\big((\widehat{\boldsymbol{\sigma}},\widehat{\varphi}),(\boldsymbol{\tau},\psi)\big) \,=\, G\big((\boldsymbol{\tau},\psi)\big) \qquad \forall\,(\boldsymbol{\tau},\psi)\in\mathrm{H}\times\mathrm{Q}\,, \tag{2.57}$$

where $G \in (\mathrm{H}\times\mathrm{Q})'$ is defined as

$$G\big((\boldsymbol{\tau},\psi)\big) \,:=\, -\kappa\int_\Omega f\,\psi \qquad \forall\,(\boldsymbol{\tau},\psi)\in\mathrm{H}\times\mathrm{Q}\,. \tag{2.58}$$

Then, it follows from (2.56), (2.55), and (2.17) that

$$\begin{aligned}\sup_{\substack{(\boldsymbol{\tau},\psi)\in\mathrm{H}\times\mathrm{Q}\\(\boldsymbol{\tau},\psi)\neq\mathbf{0}}} \frac{A_{\mathbf{w}}\big((\boldsymbol{\zeta},\phi),(\boldsymbol{\tau},\psi)\big)}{\|(\boldsymbol{\tau},\psi)\|_{\mathrm{H}\times\mathrm{Q}}} &\,\ge\, \alpha_{\widehat{T}}\,\|(\boldsymbol{\zeta},\phi)\|_{\mathrm{H}\times\mathrm{Q}} \,-\, \|\phi\|_{0,\rho;\Omega}\,\|\mathbf{w}\|_{0,r;\Omega}\\ &\ge\, \Big\{\alpha_{\widehat{T}} - \|\mathbf{w}\|_{0,r;\Omega}\Big\}\,\|(\boldsymbol{\zeta},\phi)\|_{\mathrm{H}\times\mathrm{Q}} \quad \forall\,(\boldsymbol{\zeta},\phi)\,\in\,\mathrm{H}\times\mathrm{Q}\,,\end{aligned}$$

and hence, assuming that $\|\mathbf{w}\|_{0,r;\Omega} \,\le\, \dfrac{\alpha_{\widehat{T}}}{2}$, we arrive at

$$\sup_{\substack{(\boldsymbol{\tau},\psi)\in\mathrm{H}\times\mathrm{Q}\\(\boldsymbol{\tau},\psi)\neq\mathbf{0}}} \frac{A_{\mathbf{w}}\big((\boldsymbol{\zeta},\phi),(\boldsymbol{\tau},\psi)\big)}{\|(\boldsymbol{\tau},\psi)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \frac{\alpha_{\widehat{T}}}{2}\,\|(\boldsymbol{\zeta},\phi)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\boldsymbol{\zeta},\phi)\,\in\,\mathrm{H}\times\mathrm{Q}\,. \tag{2.59}$$

Analogously, noting that A is symmetric, and employing again (2.55) and (2.17), we find that

$$\begin{aligned}\sup_{\substack{(\boldsymbol{\zeta},\phi)\in\mathrm{H}\times\mathrm{Q}\\(\boldsymbol{\zeta},\phi)\neq\mathbf{0}}} \frac{A_{\mathbf{w}}\big((\boldsymbol{\zeta},\phi),(\boldsymbol{\tau},\psi)\big)}{\|(\boldsymbol{\zeta},\phi)\|_{\mathrm{H}\times\mathrm{Q}}} &\,\ge\, \alpha_{\widehat{T}}\,\|(\boldsymbol{\tau},\psi)\|_{\mathrm{H}\times\mathrm{Q}} \,-\, \|\boldsymbol{\tau}\|_{0,\Omega}\,\|\mathbf{w}\|_{0,r;\Omega}\\ &\ge\, \Big\{\alpha_{\widehat{T}} - \|\mathbf{w}\|_{0,r;\Omega}\Big\}\,\|(\boldsymbol{\tau},\psi)\|_{\mathrm{H}\times\mathrm{Q}} \quad \forall\,(\boldsymbol{\tau},\psi)\,\in\,\mathrm{H}\times\mathrm{Q}\,,\end{aligned}$$

from which, under the same assumption $\|\mathbf{w}\|_{0,r;\Omega} \,\le\, \dfrac{\alpha_{\widehat{T}}}{2}$, we obtain

$$\sup_{\substack{(\boldsymbol{\zeta},\phi)\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\zeta},\phi)\neq\mathbf{0}}} \frac{A_{\mathbf{w}}\big((\boldsymbol{\zeta},\phi),(\boldsymbol{\tau},\psi)\big)}{\|(\boldsymbol{\zeta},\phi)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \frac{\alpha_{\widehat{T}}}{2}\,\|(\boldsymbol{\tau},\psi)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\boldsymbol{\tau},\psi)\,\in\,\mathrm{H}\times\mathrm{Q}\,. \tag{2.60}$$

Thanks to the foregoing analysis, we are in position to establish next that the operator $\widehat{T}$ (cf. (2.33)) is well-defined.

Theorem 2.10 *For each* $\mathbf{w}\in\mathbf{L}^r(\Omega)$ *such that* $\|\mathbf{w}\|_{0,r;\Omega}\,\le\,\dfrac{\alpha_{\widehat{T}}}{2}$, *there exists a unique* $(\widehat{\boldsymbol{\sigma}},\widehat{\varphi}) = \widehat{T}(\mathbf{w})\in \mathrm{H}\times\mathrm{Q}$ *solution to* (2.33) (*equivalently* (2.57)). *Moreover, there holds*

$$\|\widehat{T}(\mathbf{w})\|_{\mathrm{H}\times\mathrm{Q}} \,=\, \|\widehat{\boldsymbol{\sigma}}\|_{\mathrm{div}_{\varrho};\Omega} + \|\widehat{\varphi}\|_{0,\rho;\Omega} \,\le\, \frac{2}{\alpha_{\widehat{T}}}\,|\kappa|\,\|f\|_{0,\varrho;\Omega}\,. \tag{2.61}$$

Proof. It is clear from (2.59) and (2.60) that $A_{\mathbf{w}}$ satisfies the hypotheses i) and ii) of Theorem 2.5, the first one with $\alpha = \dfrac{\alpha_{\widehat{T}}}{2}$. Hence, bearing in mind that $\mathrm{Q} := \mathrm{L}^\rho(\Omega)$ is a reflexive Banach space, the proof reduces to a straightforward application of the aforementioned result. In particular, the a priori estimate (2.61) follows from (2.40) and the fact that, according to (2.58) and Hölder's inequality, there holds $\|G\| \,\le\, |\kappa|\,\|f\|_{0,\varrho;\Omega}$. □

2.5 Solvability analysis

Knowing that the operators $\widetilde{T}$, $\widehat{T}$, and hence T as well, are well defined, in this section we address the solvability of the fixed point equation (2.35). To this end, in what follows we verify the hypotheses of the respective Banach Theorem. We begin the analysis by establishing a sufficient condition under which T maps a closed ball of X_2 into itself. Indeed, from now on we let

$$S \,:=\, \Big\{\mathbf{w}\in X_2: \quad \|\mathbf{w}\|_{X_2}\,\le\,\frac{\alpha_{\widehat{T}}}{2}\Big\}. \tag{2.62}$$

Then we have the following result.

Lemma 2.11 *Assume that*

$$\|\mathbf{f}\|_{0,r;\Omega}\,\le\,\frac{\widetilde{\alpha}\,\alpha_{\widehat{T}}}{2}\,. \tag{2.63}$$

Then $T(S)\,\subseteq\,S$.

Proof. Given $\mathbf{w}\in S$, we know from Theorem 2.10 that $\widehat{T}(\mathbf{w})$ is well defined. Then, using the a priori estimate for $\widetilde{T}_1$ (cf. (2.45)) we have that

$$\|T(\mathbf{w})\|_{X_2} \,=\, \|\widetilde{T}_1\big(\widehat{T}(\mathbf{w})\big)\|_{X_2} \,\le\, \frac{1}{\widetilde{\alpha}}\,\|\mathbf{f}\|_{0,r;\Omega}\,,$$

which, according to the assumption (2.63), yields $T(\mathbf{w})\in S$ and ends the proof. □

Next, we aim to prove the continuity of T, which will follow from similar properties for the operators $\widehat{T}$ and $\widetilde{T}$. We begin with the corresponding result for $\widehat{T}$.

Lemma 2.12 *There exists a positive constant* $L_{\widehat{T}}$, *depending only on* $\alpha_{\widehat{T}}$, *such that*

$$\|\widehat{T}(\mathbf{w})-\widehat{T}(\mathbf{z})\|_{\mathrm{H}\times\mathrm{Q}} \,\le\, L_{\widehat{T}}\,|\kappa|\,\|f\|_{0,\varrho;\Omega}\,\|\mathbf{w}-\mathbf{z}\|_{0,r;\Omega} \qquad \forall\,\mathbf{w},\,\mathbf{z}\in S\,. \tag{2.64}$$

Proof. Given $\mathbf{w}, \mathbf{z} \in S$, we let $\widehat{T}(\mathbf{w}) := (\widehat{\boldsymbol{\sigma}}, \widehat{\varphi}) \in \mathrm{H} \times \mathrm{Q}$ and $\widehat{T}(\mathbf{z}) := (\bar{\boldsymbol{\sigma}}, \bar{\varphi}) \in \mathrm{H} \times \mathrm{Q}$, which satisfy (2.33) with $\mathbf{w}$ itself and with $\mathbf{w} = \mathbf{z}$, respectively. Then, subtracting the corresponding first and second equations of these systems, we obtain

$$a(\widehat{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \widehat{\varphi} - \bar{\varphi}) = \int_\Omega \big(\bar{\varphi}\,\mathbf{z} - \widehat{\varphi}\,\mathbf{w}\big) \cdot \boldsymbol{\tau} \qquad \forall\, \boldsymbol{\tau} \in \mathrm{H}\,,$$

and

$$b(\widehat{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}, \psi) = 0 \qquad \forall\, \psi \in \mathrm{Q}\,,$$

which, together with the definitions of A (cf. (2.54)) and $A_{\mathbf{w}}$ (cf. (2.56)), yield

$$A\big((\widehat{\boldsymbol{\sigma}}, \widehat{\varphi}) - (\bar{\boldsymbol{\sigma}}, \bar{\varphi}), (\boldsymbol{\tau}, \psi)\big) = \int_\Omega \big(\bar{\varphi}\,\mathbf{z} - \widehat{\varphi}\,\mathbf{w}\big) \cdot \boldsymbol{\tau}$$

and

$$\begin{aligned} A_{\mathbf{w}}\big((\widehat{\boldsymbol{\sigma}}, \widehat{\varphi}) - (\bar{\boldsymbol{\sigma}}, \bar{\varphi}), (\boldsymbol{\tau}, \psi)\big) &= A\big((\widehat{\boldsymbol{\sigma}}, \widehat{\varphi}) - (\bar{\boldsymbol{\sigma}}, \bar{\varphi}), (\boldsymbol{\tau}, \psi)\big) + \int_\Omega \big(\widehat{\varphi} - \bar{\varphi}\big)\,\mathbf{w} \cdot \boldsymbol{\tau} \\ &= \int_\Omega \bar{\varphi}\,\big(\mathbf{z} - \mathbf{w}\big) \cdot \boldsymbol{\tau} \qquad \forall\, (\boldsymbol{\tau}, \psi) \in \mathrm{H} \times \mathrm{Q}\,. \end{aligned}$$

In this way, applying the global inf-sup condition (2.59) to $(\boldsymbol{\zeta}, \phi) := (\widehat{\boldsymbol{\sigma}}, \widehat{\varphi}) - (\bar{\boldsymbol{\sigma}}, \bar{\varphi})$, and then employing the foregoing identity and the Cauchy-Schwarz and Hölder inequalities, the latter with ℓ and j conjugate to each other so that $\rho = 2\ell$ and $r = 2j$, we find that

$$\begin{aligned} \frac{\alpha_{\widehat{T}}}{2}\,\|(\widehat{\boldsymbol{\sigma}}, \widehat{\varphi}) - (\bar{\boldsymbol{\sigma}}, \bar{\varphi})\|_{\mathrm{H}\times\mathrm{Q}} &\le \sup_{\substack{(\boldsymbol{\tau},\psi)\in\mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau},\psi)\neq\mathbf{0}}} \frac{A_{\mathbf{w}}\big((\widehat{\boldsymbol{\sigma}}, \widehat{\varphi}) - (\bar{\boldsymbol{\sigma}}, \bar{\varphi}), (\boldsymbol{\tau}, \psi)\big)}{\|(\boldsymbol{\tau}, \psi)\|_{\mathrm{H}\times\mathrm{Q}}} \\ &= \sup_{\substack{(\boldsymbol{\tau},\psi)\in\mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau},\psi)\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \bar{\varphi}\,\big(\mathbf{z} - \mathbf{w}\big) \cdot \boldsymbol{\tau}}{\|(\boldsymbol{\tau}, \psi)\|_{\mathrm{H}\times\mathrm{Q}}} \le \|\bar{\varphi}\|_{0,\rho;\Omega}\,\|\mathbf{w} - \mathbf{z}\|_{0,r;\Omega}\,, \end{aligned}$$

from which, using the bound for $\|\bar{\varphi}\|_{0,\rho;\Omega} = \|\widehat{T}_2(\mathbf{z})\|_{0,\rho;\Omega}$ provided by (2.61), we arrive at (2.64) with $L_{\widehat{T}} := \dfrac{4}{\alpha_{\widehat{T}}^2}$. □

On the other hand, in order to establish a continuity property for $\widetilde{T}$, we need further regularity assumptions on the solutions of the problems defining the operators $\widehat{T}$ and $\widetilde{T}$. More precisely, from now on we suppose that there exists $\varepsilon \ge \frac{n}{\rho}$ and constants $\widehat{C}_\varepsilon$, $\widetilde{C}_\varepsilon > 0$, such that

($\mathbf{RA}_1$) for each $\mathbf{w} \in S$ there holds $\widehat{T}(\mathbf{w}) := \big(\widehat{T}_1(\mathbf{w}), \widehat{T}_2(\mathbf{w})\big) \in \big(H^\varepsilon(\Omega) \cap \mathrm{H}\big) \times \mathrm{W}^{\varepsilon,\rho}(\Omega)$, and

$$\|\widehat{T}_1(\mathbf{w})\|_{\varepsilon,\Omega} + \|\widehat{T}_2(\mathbf{w})\|_{\varepsilon,\rho;\Omega} \le \widehat{C}_\varepsilon\,|\kappa|\,\|f\|_{0,\varrho;\Omega}\,, \tag{2.65}$$

($\mathbf{RA}_2$) for each $\phi \in \mathrm{W}^{\varepsilon,\rho}(\Omega)$ there holds $\widetilde{T}(\phi) := \big(\widetilde{T}_1(\phi), \widetilde{T}_2(\phi)\big) \in \big(\mathrm{W}^{\varepsilon,r}(\Omega) \cap X_2\big) \times \big(\mathrm{W}^{\varepsilon,r}(\Omega) \cap M_1\big)$, and

$$\|\widetilde{T}_1(\phi)\|_{\varepsilon,r;\Omega} + \|\widetilde{T}_2(\phi)\|_{\varepsilon,r;\Omega} \le \widetilde{C}_\varepsilon\,\|\mathbf{f}\|_{0,r;\Omega}\,. \tag{2.66}$$

The exact reason of the stipulated range for ε will be clarified along the subsequent analysis. Furthermore, we recall that the embedding theorem between fractional Sobolev spaces (cf. [33, Theorem 6, Section 5.6], [41, Theorem 1.4.5.2, part e)]) establishes that whenever $r\varepsilon < n$ there holds $\mathrm{W}^{\varepsilon,r}(\Omega) \subset \mathrm{L}^{\varepsilon^*}(\Omega)$, with continuous injection

$$i_\varepsilon : \mathrm{W}^{\varepsilon,r}(\Omega) \to \mathrm{L}^{\varepsilon^*}(\Omega)\,, \qquad \text{where} \qquad \varepsilon^* = \frac{nr}{n - r\varepsilon}\,. \tag{2.67}$$

Note that $r\varepsilon < n$ is compatible with $\varepsilon \geq \frac{n}{\rho}$ when $\rho > 4$ since in this case there holds $\rho > r$.

We are now in position of proving a continuity property for the first component $\widetilde{T}_1$ of $\widetilde{T}$, which, together with the estimate given by Lemma 2.12, will allow us later on to show that the fixed point operator T is Lipschitz continuous.

Lemma 2.13 *There exists a positive constant $L_{\widetilde{T}}$, depending only on $\widetilde{\alpha}$, L_μ, $\|i_\varepsilon\|$, $\widetilde{C}_\varepsilon$, $|\Omega|$, n, ε, and ρ, such that*

$$\|\widetilde{T}_1(\psi) - \widetilde{T}_1(\phi)\|_{X_2} \;\leq\; L_{\widetilde{T}}\, \|\mathbf{f}\|_{0,r;\Omega}\, \|\psi - \phi\|_{0,\rho;\Omega} \qquad \forall\, \psi,\, \phi \in \mathrm{W}^{\varepsilon,\rho}(\Omega)\,. \tag{2.68}$$

Proof. Given ψ, $\phi \in \mathrm{W}^{\varepsilon,\rho}(\Omega)$, we proceed similarly to the proof of Lemma 2.12 and let $\widetilde{T}(\psi) := (\widetilde{\mathbf{u}}, \widetilde{p}) \in X_2 \times M_1$ and $\widetilde{T}(\phi) := (\bar{\mathbf{u}}, \bar{p}) \in X_2 \times M_1$, which satisfy (2.32) with ψ itself and with $\psi = \phi$, respectively. Then, from the corresponding second equations of these systems we have that both $\widetilde{\mathbf{u}}$ and $\bar{\mathbf{u}}$, and hence $\widetilde{\mathbf{u}} - \bar{\mathbf{u}}$ as well, belong to $\mathcal{K}_2$. In this way, applying the inf-sup condition (2.42a) to the present ψ and to $\mathbf{w} = \widetilde{\mathbf{u}} - \bar{\mathbf{u}}$, we get

$$\widetilde{\alpha}\, \|\widetilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{X_2} \;\leq\; \sup_{\substack{\mathbf{v}\in\mathcal{K}_1 \\ \mathbf{v}\neq\mathbf{0}}} \frac{a_\psi(\widetilde{\mathbf{u}} - \bar{\mathbf{u}}, \mathbf{v})}{\|\mathbf{v}\|_{X_1}}\,, \tag{2.69}$$

where, according to the respective first equations and the definition of a_ψ (cf. (2.30a)), we have

$$\begin{aligned} a_\psi(\widetilde{\mathbf{u}} - \bar{\mathbf{u}}, \mathbf{v}) &= \int_\Omega \mathbf{f}\cdot\mathbf{v} - a_\psi(\bar{\mathbf{u}}, \mathbf{v}) = a_\phi(\bar{\mathbf{u}}, \mathbf{v}) - a_\psi(\bar{\mathbf{u}}, \mathbf{v}) \\ &= \int_\Omega \Big\{\mu(\phi) - \mu(\psi)\Big\}\bar{\mathbf{u}}\cdot\mathbf{v} \qquad \forall\, \mathbf{v} \in \mathcal{K}_1\,. \end{aligned} \tag{2.70}$$

Then, employing the Lipschitz-continuity of μ (cf. (1.2)), and applying Hölder's inequality, first with (r, s), and then with an arbitrary pair of conjugates to each other denoted by (ℓ, j), we obtain from (2.70)

$$\big|a_\psi(\widetilde{\mathbf{u}} - \bar{\mathbf{u}}, \mathbf{v})\big| \;\leq\; L_\mu \int_\Omega |\psi - \phi|\, \|\bar{\mathbf{u}}\|\, \|\mathbf{v}\| \;\leq\; L_\mu\, \|\psi - \phi\|_{0,rj;\Omega}\, \|\bar{\mathbf{u}}\|_{0,r\ell;\Omega}\, \|\mathbf{v}\|_{0,s;\Omega}\,, \tag{2.71}$$

which, replaced back into (2.69), gives

$$\widetilde{\alpha}\, \|\widetilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{X_2} \;\leq\; L_\mu\, \|\psi - \phi\|_{0,rj;\Omega}\, \|\bar{\mathbf{u}}\|_{0,r\ell;\Omega}\,. \tag{2.72}$$

Next, choosing ℓ such that $r\ell = \varepsilon^*$ (cf. (2.67)), we get $\ell = \frac{n}{n - r\varepsilon}$, which yields $rj = \frac{r\ell}{\ell - 1} = \frac{n}{\varepsilon}$, and hence, recalling that $\bar{\mathbf{u}} = \widetilde{T}_1(\phi)$, it follows from the foregoing inequality, the boundedness of i_ε (cf. (2.67)), and the regularity estimate (2.66), that

$$\begin{aligned} \widetilde{\alpha}\, \|\widetilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{X_2} \;&\leq\; L_\mu\, \|\psi - \phi\|_{0,n/\varepsilon;\Omega}\, \|\widetilde{T}_1(\phi)\|_{0,\varepsilon^*;\Omega} \;\leq\; L_\mu\, \|i_\varepsilon\|\, \|\psi - \phi\|_{0,n/\varepsilon;\Omega}\, \|\widetilde{T}_1(\phi)\|_{\varepsilon,r;\Omega} \\ &\leq\; L_\mu\, \|i_\varepsilon\|\, \widetilde{C}_\varepsilon\, \|\mathbf{f}\|_{0,r;\Omega}\, \|\psi - \phi\|_{0,n/\varepsilon;\Omega}\,. \end{aligned} \tag{2.73}$$

Finally, in order to bound $\|\psi - \phi\|_{0,n/\varepsilon;\Omega}$ in terms of $\|\psi - \phi\|_{0,\rho;\Omega}$, it suffices to require that $\frac{n}{\varepsilon} \leq \rho$, that is $\varepsilon \geq \frac{n}{\rho}$, which is precisely our assumption on ε for $(\mathbf{RA}_1)$ and $(\mathbf{RA}_2)$. Thus, a simple algebraic computation shows that $\|\psi - \phi\|_{0,n/\varepsilon;\Omega} \;\leq\; |\Omega|^{\frac{\varepsilon\rho - n}{\rho n}}\, \|\psi - \phi\|_{0,\rho;\Omega}$, which, together with (2.73), leads to the required inequality (2.68) with $L_{\widetilde{T}} := \widetilde{\alpha}^{-1}\, L_\mu\, \|i_\varepsilon\|\, \widetilde{C}_\varepsilon\, |\Omega|^{\frac{\varepsilon\rho - n}{\rho n}}$. □

We stress here that, while it is not necessary for the rest of our analysis, it is also possible to prove the Lipschitz-continuity of $\widetilde{T}_2$. To this end, it suffices to apply the inf-sup condition for b_1 (cf. (2.43)), the first equation of the problem defining $\widetilde{T}$ (cf. (2.32)), and Lemma 2.13. It is also important to remark here that if the viscosity is constant, then the regularity assumptions specified above are not required anymore since the expression that yields (2.73), namely $a_\psi(\widetilde{\mathbf{u}} - \bar{\mathbf{u}}, \mathbf{v})$ (cf. (2.70)), actually vanishes in this case. In other words, the Darcy and heat equations are not coupled, and hence they can be solved sequentially. However, if we keep a constant viscosity and, say for instance, the source term of the Darcy equations is supposed to depend on the temperature, then the model becomes coupled again, but still no extra regularity is needed either in this other case for the respective analysis.

Having proved Lemmas 2.12 and 2.13, we are able to establish now the Lipschitz-continuity of our fixed point operator T in the closed ball S of X_2 (cf. (2.62)).

Lemma 2.14 *There exists a positive constant L_T, depending only on $L_{\widetilde{T}}$ and $L_{\widehat{T}}$, such that*

$$\|T(\mathbf{w}) - T(\mathbf{z})\|_{X_2} \,\le\, L_T\, \|\mathbf{f}\|_{0,r;\Omega}\, |\kappa|\, \|f\|_{0,\varrho;\Omega}\, \|\mathbf{w} - \mathbf{z}\|_{X_2} \qquad \forall\, \mathbf{w},\, \mathbf{z} \in S\,. \tag{2.74}$$

Proof. Given $\mathbf{w}$, $\mathbf{z} \in S$, we first observe, thanks to the regularity of $\widehat{T}$ (cf. $(\mathbf{RA}_1)$), that $\widehat{T}_2(\mathbf{w})$ and $\widehat{T}_2(\mathbf{z})$ belong to $\mathrm{W}^{\varepsilon,\rho}(\Omega)$. Then, according to the definition of T (cf. (2.34)), and employing the Lipschitz-continuity of $\widetilde{T}_1$ (cf. Lemma 2.13) and $\widehat{T}_2$ (cf. Lemma 2.12), we deduce that

$$\begin{aligned}
\|T(\mathbf{w}) - T(\mathbf{z})\|_{X_2} \,&=\, \|\widetilde{T}_1\big(\widehat{T}_2(\mathbf{w})\big) - \widetilde{T}_1\big(\widehat{T}_2(\mathbf{z})\big)\|_{X_2} \\
&\le\, L_{\widetilde{T}}\, \|\mathbf{f}\|_{0,r;\Omega}\, \|\widehat{T}_2(\mathbf{w}) - \widehat{T}_2(\mathbf{z})\|_{0,\rho;\Omega} \\
&\le\, L_{\widetilde{T}}\, \|\mathbf{f}\|_{0,r;\Omega}\, L_{\widehat{T}}\, |\kappa|\, \|f\|_{0,\varrho;\Omega}\, \|\mathbf{w} - \mathbf{z}\|_{0,r;\Omega}\,,
\end{aligned}$$

which yields (2.74) with $L_T := L_{\widetilde{T}}\, L_{\widehat{T}}$. □

Consequently, the main result of this section is stated as follows.

Theorem 2.15 *Assume* $(\mathbf{RA}_1)$, $(\mathbf{RA}_2)$, *and that the data satisfy*

$$\|\mathbf{f}\|_{0,r;\Omega} \,\le\, \frac{\widetilde{\alpha}\, \alpha_{\widehat{T}}}{2} \qquad \text{and} \qquad L_T\, \|\mathbf{f}\|_{0,r;\Omega}\, |\kappa|\, \|f\|_{0,\varrho;\Omega} \,<\, 1\,. \tag{2.75}$$

Then, our coupled problem (2.31) *has a unique solution* $(\boldsymbol{\sigma}, \varphi) \in \mathrm{H} \times \mathrm{Q}$ *and* $(\mathbf{u}, p) \in X_2 \times M_1$ *with* $\mathbf{u} \in S \cap X_2$. *Moreover, there hold*

$$\begin{gathered}
\|(\boldsymbol{\sigma}, \varphi)\|_{\mathrm{H}\times\mathrm{Q}} \,\le\, \frac{2}{\alpha_{\widehat{T}}}\, |\kappa|\, \|f\|_{0,\varrho;\Omega}\,, \qquad \|\mathbf{u}\|_{X_2} \,\le\, \frac{1}{\widetilde{\alpha}}\, \|\mathbf{f}\|_{0,r;\Omega}\,, \\
\text{and} \quad \|p\|_{M_1} \,\le\, \frac{1}{\widetilde{\beta}_1} \Big(1 + \frac{\mu_2}{\widetilde{\alpha}}\Big)\, \|\mathbf{f}\|_{0,r;\Omega}\,.
\end{gathered} \tag{2.76}$$

Proof. We begin by recalling from Lemma 2.11 that the first assumption in (2.75) guarantees that T maps S into itself. Hence, in virtue of the equivalence between (2.31) and (2.35), and bearing in mind the Lipschitz-continuity of T (cf. Lemma 2.14) and the second hypothesis in (2.75), a straightforward application of the Banach fixed point Theorem implies the existence of a unique solution of (2.31) with $\mathbf{u} \in S$. In addition, the fact that $(\mathbf{u}, p) = \widetilde{T}(\varphi)$ and $(\boldsymbol{\sigma}, \varphi) = \widehat{T}(\mathbf{u})$, together with the a priori estimates provided by (2.45) and (2.61), yield (2.76) and conclude the proof. □

3 The Galerkin scheme

In order to approximate the solution of our fully-mixed variational formulation (2.31), we now proceed to introduce and analyse an associated Galerkin scheme. Analogue tools and techniques to those used in Section 2 will be employed here. We begin by considering arbitrary finite element subspaces $\mathrm{H}_h \subseteq \mathrm{H}$, $\mathrm{Q}_h \subseteq \mathrm{Q}$, $X_{2,h} \subseteq X_2$, $M_{2,h} \subseteq M_2$, $X_{1,h} \subseteq X_1$, and $M_{1,h} \subseteq M_1$, whose specific choices satisfying all the required stability conditions will be introduced later on in Section 4. Then, the Galerkin scheme associated with (2.31) reads: Find $(\boldsymbol{\sigma}_h, \varphi_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ and $(\mathbf{u}_h, p_h) \in X_{2,h} \times M_{1,h}$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \varphi_h) + \int_\Omega \varphi_h \, \mathbf{u}_h \cdot \boldsymbol{\tau}_h &= 0 && \forall\, \boldsymbol{\tau}_h \in \mathrm{H}_h \,, \\ b(\boldsymbol{\sigma}_h, \psi_h) &= -\kappa \int_\Omega f \, \psi_h && \forall\, \psi_h \in \mathrm{Q}_h \,, \\ a_{\varphi_h}(\mathbf{u}_h, \mathbf{v}_h) + b_1(\mathbf{v}_h, p_h) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h && \forall\, \mathbf{v}_h \in X_{1,h} \,, \\ b_2(\mathbf{u}_h, q_h) &= 0 && \forall\, q_h \in M_{2,h} \,. \end{aligned} \tag{3.1}$$

3.1 The discrete fixed point strategy

Here we adopt the discrete analogue of the continuous approach applied in Section 2.3 to analyse the solvability of (3.1). Thus, we now let $\widetilde{T}_h : \mathrm{Q}_h \to X_{2,h} \times M_{1,h}$ be the operator defined for each $\psi_h \in \mathrm{Q}_h$ as $\widetilde{T}_h(\psi_h) = (\widetilde{T}_{1,h}(\psi_h), \widetilde{T}_{2,h}(\psi_h)) := (\widetilde{\mathbf{u}}_h, \widetilde{p}_h)$, where $(\widetilde{\mathbf{u}}_h, \widetilde{p}_h) \in X_{2,h} \times M_{1,h}$ is the unique solution (to be confirmed below) of the last two equations of (3.1) with ψ_h instead of φ_h, that is

$$\begin{aligned} a_{\psi_h}(\widetilde{\mathbf{u}}_h, \mathbf{v}_h) + b_1(\mathbf{v}_h, \widetilde{p}_h) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h && \forall\, \mathbf{v}_h \in X_{1,h} \,, \\ b_2(\widetilde{\mathbf{u}}_h, q_h) &= 0 && \forall\, q_h \in M_{2,h} \,. \end{aligned} \tag{3.2}$$

In addition, we also let $\widehat{T}_h : X_{2,h} \to \mathrm{H}_h \times \mathrm{Q}_h$ be the operator defined for each $\mathbf{w}_h \in X_{2,h}$ as $\widehat{T}_h(\mathbf{w}_h) = (\widehat{T}_{1,h}(\mathbf{w}_h), \widehat{T}_{2,h}(\mathbf{w}_h)) := (\widehat{\boldsymbol{\sigma}}_h, \widehat{\varphi}_h)$, where $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\varphi}_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ is the unique solution (to be confirmed below as well) of the first two equations of (3.1) with $\mathbf{w}_h$ instead of $\mathbf{u}_h$, that is

$$\begin{aligned} a(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \widehat{\varphi}_h) + \int_\Omega \widehat{\varphi}_h \, \mathbf{w}_h \cdot \boldsymbol{\tau}_h &= 0 && \forall\, \boldsymbol{\tau}_h \in \mathrm{H}_h \,, \\ b(\widehat{\boldsymbol{\sigma}}_h, \psi_h) &= -\kappa \int_\Omega f \, \psi_h && \forall\, \psi_h \in \mathrm{Q}_h \,, \end{aligned} \tag{3.3}$$

In this way, we now introduce the operator $T_h : X_{2,h} \to X_{2,h}$ as

$$T_h(\mathbf{w}_h) := \widetilde{T}_{1,h}\big(\widehat{T}_{2,h}(\mathbf{w}_h)\big) \qquad \forall\, \mathbf{w}_h \in X_{2,h} \,, \tag{3.4}$$

and realise that solving (3.1) is equivalent to seeking a fixed point of T_h, that is $\mathbf{u}_h \in X_{2,h}$ such that

$$T_h(\mathbf{u}_h) = \mathbf{u}_h \,. \tag{3.5}$$

3.2 Well-posedness of the operators $\widetilde{T}_h$ and $\widehat{T}_h$

In this section we apply the discrete versions of Theorems 2.4 and 2.5 to prove that problems (3.2) and (3.3) are well-posed, equivalently that the discrete operators $\widetilde{T}_h$ and $\widehat{T}_h$ are well-defined. To this end, we need to introduce certain hypotheses concerning the arbitrary spaces H_h, Q_h, $X_{2,h}$, $M_{2,h}$,

$X_{1,h}$, and $M_{1,h}$, and the discrete kernels associated with the bilinear forms b_1, b_2, and b, respectively, that is

$$\mathcal{K}_{1,h} := \Big\{ \mathbf{v}_h \in X_{1,h} : \quad b_1(\mathbf{v}_h, q_h) = 0 \quad \forall\, q_h \in M_{1,h} \Big\}, \tag{3.6a}$$

$$\mathcal{K}_{2,h} := \Big\{ \mathbf{w}_h \in X_{2,h} : \quad b_2(\mathbf{w}_h, q_h) = 0 \quad \forall\, q_h \in M_{2,h} \Big\}, \tag{3.6b}$$

$$\mathrm{V}_h := \Big\{ \boldsymbol{\tau}_h \in \mathrm{H}_h : \quad b(\boldsymbol{\tau}_h, \psi_h) = 0 \quad \forall\, \psi_h \in \mathrm{Q}_h \Big\}. \tag{3.6c}$$

Specific finite element subspaces satisfying the conditions to be described in what follows will be defined later on in Section 4.2. More precisely, from now on we assume the following:

(**H.1**) there exists a constant $\widetilde{\alpha}_{\mathtt{d}} > 0$, independent of h, such that for each $\psi_h \in \mathrm{Q}_h$ there hold

$$\sup_{\substack{\mathbf{v}_h \in \mathcal{K}_{1,h} \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{a_{\psi_h}(\mathbf{w}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{X_1}} \,\geq\, \widetilde{\alpha}_{\mathtt{d}}\, \|\mathbf{w}_h\|_{X_2} \qquad \forall\, \mathbf{w}_h \in \mathcal{K}_{2,h}\,,$$

and

$$\sup_{\mathbf{w}_h \in \mathcal{K}_{2,h}} a_{\psi_h}(\mathbf{w}_h, \mathbf{v}_h) \,>\, 0 \qquad \forall\, \mathbf{v}_h \in \mathcal{K}_{1,h}\,,\ \mathbf{v}_h \neq \mathbf{0}\,,$$

(**H.2**) there exist constants $\widetilde{\beta}_{1,\mathtt{d}}$, $\widetilde{\beta}_{2,\mathtt{d}} > 0$, independent of h, such that for each $i \in \{1, 2\}$ there holds

$$\sup_{\substack{\mathbf{v}_h \in X_{i,h} \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{b_i(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{X_i}} \,\geq\, \widetilde{\beta}_{i,\mathtt{d}}\, \|q_h\|_{M_i} \qquad \forall\, q_h \in M_{i,h}\,,$$

(**H.3**) there holds $\mathrm{div}(\mathrm{H}_h) \subseteq \mathrm{Q}_h$,

(**H.4**) there exists $\widehat{\beta}_{\mathtt{d}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathrm{H}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{b(\boldsymbol{\tau}_h, \psi_h)}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}}} \,\geq\, \widehat{\beta}_{\mathtt{d}}\, \|\psi_h\|_{\mathrm{Q}} \qquad \forall\, \psi_h \in \mathrm{Q}_h\,.$$

Then, as a straightforward consequence of (**H.1**) and (**H.2**), we can establish the following result.

Theorem 3.1 *For each $\psi_h \in \mathrm{Q}_h$ there exists a unique $(\widetilde{\mathbf{u}}_h, \widetilde{p}_h) = \widetilde{T}(\psi_h) \in X_{2,h} \times M_{1,h}$ solution to* (3.2)*. Moreover, there hold*

$$\begin{aligned} \|\widetilde{T}_{1,h}(\psi_h)\|_{X_2} \,&=\, \|\widetilde{\mathbf{u}}_h\|_{X_2} \,\leq\, \frac{1}{\widetilde{\alpha}_{\mathtt{d}}}\, \|\mathbf{f}\|_{0,r;\Omega} \quad \textit{and} \\ \|\widetilde{T}_{2,h}(\psi_h)\|_{M_1} \,&=\, \|\widetilde{p}_h\|_{M_1} \,\leq\, \frac{1}{\widetilde{\beta}_{1,\mathtt{d}}} \left(1 + \frac{\mu_2}{\widetilde{\alpha}_{\mathtt{d}}}\right) \|\mathbf{f}\|_{0,r;\Omega}\,. \end{aligned} \tag{3.7}$$

Proof. Thanks to (**H.1**) and (**H.2**), the proof reduces to a straightforward application of the discrete version of Theorem 2.4 (see, e.g. [8, Corollary 2.2]). In particular, the a priori estimates in (3.7) follow from the discrete analogue of (2.37) (see, e.g. [8, eqs. (2.24), (2.25)]), the upper bound for $\|F\|_{X_1'}$ provided right before the statement of Theorem 2.8, and the fact that the right hand side of the second row of (3.2) is the null functional. □

Next, according to (**H.3**), it readily follows from (3.6c) that

$$\mathrm{V}_h := \Big\{ \boldsymbol{\tau}_h \in \mathrm{H}_h : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \Big\},$$

which yields the discrete analogue of (2.46), that is

$$a(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) = \|\boldsymbol{\tau}_h\|^2_{\mathbf{div}_\varrho;\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathrm{V}_h,$$

and hence the assumptions i) and ii) of the discrete version of Theorem 2.4 (see, e.g. [8, eqs. (2.19), (2.20)]) are satisfied, the first of them with the constant $\widehat{\alpha}_{\mathtt{d}} = 1$. In this way, this fact together with (**H.4**) guarantee the global inf-sup condition for A (cf. (2.54)) when restricted to $\mathrm{H}_h \times \mathrm{Q}_h$, equivalently the discrete analogue of (2.55), which means the existence of a positive constant $\widehat{\alpha}_{\widehat{T},\mathtt{d}}$, depending only on $\widehat{\alpha}_{\mathtt{d}}$, $\widehat{\beta}_{\mathtt{d}}$, and $\|a\|$, such that

$$\sup_{\substack{(\boldsymbol{\tau}_h,\psi_h)\in \mathrm{H}_h\times \mathrm{Q}_h\\ (\boldsymbol{\tau}_h,\psi_h)\neq \mathbf{0}}} \frac{A\big((\boldsymbol{\zeta}_h,\phi_h),(\boldsymbol{\tau}_h,\psi_h)\big)}{\|(\boldsymbol{\tau}_h,\psi_h)\|_{\mathrm{H}\times\mathrm{Q}}} \,\geq\, \alpha_{\widehat{T},\mathtt{d}}\, \|(\boldsymbol{\zeta}_h,\phi_h)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\, (\boldsymbol{\zeta}_h,\phi_h) \,\in\, \mathrm{H}_h\times \mathrm{Q}_h. \tag{3.8}$$

Moreover, proceeding analogously to the analysis developed after (2.55) in Section 2.4.3, we find that for each $\mathbf{w}_h \in X_{2,h}$ such that $\|\mathbf{w}_h\|_{0,r;\Omega} \leq \dfrac{\alpha_{\widehat{T},\mathtt{d}}}{2}$, there holds

$$\sup_{\substack{(\boldsymbol{\tau}_h,\psi_h)\in \mathrm{H}_h\times \mathrm{Q}_h\\ (\boldsymbol{\tau}_h,\psi_h)\neq \mathbf{0}}} \frac{A_{\mathbf{w}_h}\big((\boldsymbol{\zeta}_h,\phi_h),(\boldsymbol{\tau}_h,\psi_h)\big)}{\|(\boldsymbol{\tau}_h,\psi_h)\|_{\mathrm{H}\times\mathrm{Q}}} \,\geq\, \frac{\alpha_{\widehat{T},\mathtt{d}}}{2}\, \|(\boldsymbol{\zeta}_h,\phi_h)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\, (\boldsymbol{\zeta}_h,\phi_h) \,\in\, \mathrm{H}_h\times \mathrm{Q}_h. \tag{3.9}$$

In this way, we conclude that the operator $\widehat{T}_h$ (cf. (3.3)) is well-defined.

Theorem 3.2 *For each $\mathbf{w}_h \in X_{2,h}$ such that $\|\mathbf{w}_h\|_{0,r;\Omega} \leq \dfrac{\alpha_{\widehat{T},\mathtt{d}}}{2}$, there exists a unique $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\varphi}_h) = \widehat{T}_h(\mathbf{w}_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ solution to* (3.3), *equivalently*

$$A_{\mathbf{w}_h}\big((\widehat{\boldsymbol{\sigma}}_h,\widehat{\varphi}_h),(\boldsymbol{\tau}_h,\psi_h)\big) \,=\, G\big((\boldsymbol{\tau}_h,\psi_h)\big) \qquad \forall\, (\boldsymbol{\tau}_h,\psi_h) \in \mathrm{H}_h\times\mathrm{Q}_h.$$

Moreover, there holds

$$\|\widehat{T}_h(\mathbf{w}_h)\|_{\mathrm{H}\times\mathrm{Q}} \,=\, \|\widehat{\boldsymbol{\sigma}}_h\|_{\mathbf{div}_\varrho;\Omega} + \|\widehat{\varphi}_h\|_{0,\rho;\Omega} \,\leq\, \frac{2}{\alpha_{\widehat{T},\mathtt{d}}}\, |\kappa|\, \|f\|_{0,\varrho;\Omega}. \tag{3.10}$$

Proof. Similarly to the proof of Theorem 2.10, it follows from the fact that $A_{\mathbf{w}_h}$ satisfies the hypotheses of the discrete version of Theorem 2.5 (see, e.g. [32, Theorem 2.22]). Indeed, the latter reduces equivalently to fulfil only the corresponding inf-sup condition i), which is precisely (3.9) in this case. We omit further details. □

3.3 Discrete solvability analysis

Having established that the discrete operators $\widetilde{T}_h$, $\widehat{T}_h$, and hence T_h, are all well defined, we now address the solvability of the corresponding fixed point equation (3.5). For this purpose, we first let

$$S_h := \Big\{ \mathbf{w}_h \in X_{2,h} : \quad \|\mathbf{w}_h\|_{X_2} \,\leq\, \frac{\alpha_{\widehat{T},\mathtt{d}}}{2} \Big\}, \tag{3.11}$$

and provide a sufficient condition under which T_h maps S_h into itself. More precisely, we have the following result.

Lemma 3.3 *Assume that*

$$\|\mathbf{f}\|_{0,r;\Omega} \,\leq\, \frac{\widetilde{\alpha}_{\mathtt{d}}\,\alpha_{\widehat{T},\mathtt{d}}}{2}\,. \tag{3.12}$$

Then $T_h(S_h) \,\subseteq\, S_h$*.*

Proof. It proceeds analogously to the proof of Lemma 2.11, noting now from (3.11) and Theorem 3.2 that $\widehat{T}_h(\mathbf{w}_h)$ is well defined for each $\mathbf{w}_h \in S_h$, and using the a priori estimate for $\widetilde{T}_{1,h}$ (cf. (3.7)) and the assumption (3.12). □

Next, we look at the continuity properties of $\widehat{T}_h$ and $\widetilde{T}_h$, and hence at that of T_h. In fact, we first observe that, proceeding analogously to the proof of Lemma 2.12, but now using the discrete inf-sup condition (3.8) and the a priori bound (3.10), we find that

$$\|\widehat{T}_h(\mathbf{w}_h) - \widehat{T}_h(\mathbf{z}_h)\|_{\mathrm{H}\times\mathrm{Q}} \,\leq\, L_{\widehat{T},\mathtt{d}}\,|\kappa|\,\|f\|_{0,\varrho;\Omega}\,\|\mathbf{w}_h - \mathbf{z}_h\|_{0,r;\Omega} \qquad \forall\,\mathbf{w}_h,\,\mathbf{z}_h \in S_h\,, \tag{3.13}$$

where $L_{\widehat{T},\mathtt{d}} = \dfrac{4}{\alpha^2_{\widehat{T},\mathtt{d}}}$. In turn, for the continuity of $\widetilde{T}_{1,h}$ we basically follow the same reasoning of the proof of Lemma 2.13, except that, not being the regularity assumptions ($\mathbf{RA}_1$) and ($\mathbf{RA}_2$) applicable in the present context, we only employ the $\mathrm{L}^{rj}-\mathrm{L}^{r\ell}-\mathrm{L}^{s}$ argument from (2.71), but with different values for j and ℓ, to estimate the discrete version of (2.70). More precisely, we apply the aforementioned tool with j and ℓ conjugate to each other such that $rj = \rho$. As a consequence, given ψ_h, $\phi_h \in \mathrm{Q}_h$, and denoting $\widetilde{T}_h(\psi_h) := (\widetilde{\mathbf{u}}_h, \widetilde{p}_h) \in X_{2,h} \times M_{1,h}$ and $\widetilde{T}_h(\phi_h) := (\bar{\mathbf{u}}_h, \bar{p}_h) \in X_{2,h} \times M_{1,h}$, the discrete analogue of (2.72) becomes

$$\widetilde{\alpha}_{\mathtt{d}}\,\|\widetilde{\mathbf{u}}_h - \bar{\mathbf{u}}_h\|_{X_2} \,\leq\, L_\mu\,\|\psi_h - \phi_h\|_{0,\rho;\Omega}\,\|\bar{\mathbf{u}}_h\|_{0,r\ell;\Omega}\,,$$

which, denoting $L_{\widetilde{T},\mathtt{d}} := \dfrac{L_\mu}{\widetilde{\alpha}_{\mathtt{d}}}$, yields

$$\|\widetilde{T}_{1,h}(\psi_h) - \widetilde{T}_{1,h}(\phi_h)\|_{X_2} \,\leq\, L_{\widetilde{T},\mathtt{d}}\,\|\widetilde{T}_{1,h}(\phi_h)\|_{0,r\ell;\Omega}\,\|\psi_h - \phi_h\|_{0,\rho;\Omega} \qquad \forall\,\psi_h,\,\phi_h \in \mathrm{Q}_h\,. \tag{3.14}$$

In this way, bearing in mind (3.13) and (3.14), it follows from the definition of T_h (cf. (3.4)) and the fact that $\|\mathbf{w}_h - \mathbf{z}_h\|_{0,r;\Omega} \leq \|\mathbf{w}_h - \mathbf{z}_h\|_{X_2}$, that

$$\|T_h(\mathbf{w}_h) - T_h(\mathbf{z}_h)\|_{X_2} \,\leq\, L_{T,\mathtt{d}}\,|\kappa|\,\|f\|_{0,\varrho;\Omega}\,\|T_h(\mathbf{z}_h)\|_{0,r\ell;\Omega}\,\|\mathbf{w}_h - \mathbf{z}_h\|_{X_2} \qquad \forall\,\mathbf{w}_h,\,\mathbf{z}_h \in S_h\,, \tag{3.15}$$

with $L_{T,\mathtt{d}} := L_{\widetilde{T},\mathtt{d}}\,L_{\widehat{T},\mathtt{d}}$. We stress here that (3.15) proves continuity of T_h, but, due to the lack of control of the term $\|T_h(\mathbf{z}_h)\|_{0,r\ell;\Omega}$, it does not necessarily yield neither Lipschitz-continuity and hence nor contractivity of this operator. As a consequence, we are only able to conclude existence but not necessarily uniqueness of a fixed point of T_h.

According to the above, the main result of this section is established as follows.

Theorem 3.4 *Assume that* $\|\mathbf{f}\|_{0,r;\Omega} \,\leq\, \dfrac{\widetilde{\alpha}_{\mathtt{d}}\,\alpha_{\widehat{T},\mathtt{d}}}{2}$*. Then, the Galerkin scheme* (3.1) *has at least one solution* $(\boldsymbol{\sigma}_h, \varphi_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ *and* $(\mathbf{u}_h, p_h) \in X_{2,h} \times M_{1,h}$ *with* $\mathbf{u}_h \in S_h$*. Moreover, there hold*

$$\begin{gathered}\|(\boldsymbol{\sigma}_h, \varphi_h)\|_{\mathrm{H}\times\mathrm{Q}} \,\leq\, \frac{2}{\alpha_{\widehat{T},\mathtt{d}}}\,|\kappa|\,\|f\|_{0,\varrho;\Omega}\,, \qquad \|\mathbf{u}_h\|_{X_2} \,\leq\, \frac{1}{\widetilde{\alpha}_{\mathtt{d}}}\,\|\mathbf{f}\|_{0,r;\Omega}\,, \\ \text{and} \qquad \|p_h\|_{M_1} \,\leq\, \frac{1}{\widetilde{\beta}_{1,\mathtt{d}}}\left(1 + \frac{\mu_2}{\widetilde{\alpha}_{\mathtt{d}}}\right)\|\mathbf{f}\|_{0,r;\Omega}\,.\end{gathered} \tag{3.16}$$

Proof. We first notice from Lemma 3.3 that the assumption on $\|\mathbf{f}\|_{0,r;\Omega}$ guarantees that T_h maps S_h into itself. Then, the aforementioned continuity of T_h, the equivalence between (3.1) and (3.5), and a straightforward application of the Brouwer Theorem (cf. [21, Theorem 9.9-2]) implies the existence of at least one solution of (3.1) with $\mathbf{u}_h \in S_h$. Finally, recalling that $(\mathbf{u}_h, p_h) = \widetilde{T}_h(\varphi_h)$ and $(\boldsymbol{\sigma}_h, \varphi_h) = \widehat{T}_h(\mathbf{u}_h)$, and thanks to the a priori estimates (3.7) and (3.10), we obtain (3.16). □

3.4 A priori error analysis

In this section we derive an a priori error estimate for the Galerkin scheme (3.1) with arbitrary finite element subspaces satisfying the hypotheses introduced in Section 3.2. More precisely, we are interested in establishing a Céa estimate for the error

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{H}} + \|\varphi - \varphi_h\|_{\mathrm{Q}} + \|\mathbf{u} - \mathbf{u}_h\|_{X_2} + \|p - p_h\|_{M_1}\,,$$

where $\big((\boldsymbol{\sigma}, \varphi), (\mathbf{u}, p)\big) \in \big(\mathrm{H} \times \mathrm{Q}\big) \times \big(X_2 \times M_1\big)$ is the unique solution of (2.31) with $\mathbf{u} \in S$ (cf. (2.62)), and $\big((\boldsymbol{\sigma}_h, \varphi_h), (\mathbf{u}_h, p_h)\big) \in \big(\mathrm{H}_h \times \mathrm{Q}_h\big) \times \big(X_{2,h} \times M_{1,h}\big)$ is a solution of (3.1) with $\mathbf{u}_h \in S_h$ (cf. (3.11)). To this end, and in order to employ corresponding Strang estimates, we rewrite (2.31) and (3.1) as the pairs given by a continuous formulation and its associated discrete one, that is

$$\begin{aligned}
a(\boldsymbol{\sigma}, \boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau}, \varphi) &= \mathcal{F}_{\varphi,\mathbf{u}}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathrm{H}\,,\\
b(\boldsymbol{\sigma}, \psi) &= -\kappa \int_\Omega f\,\psi && \forall\, \psi \in \mathrm{Q}\,,\\
a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \,+\, b(\boldsymbol{\tau}_h, \varphi_h) &= \mathcal{F}_{\varphi_h,\mathbf{u}_h}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau}_h \in \mathrm{H}_h\,,\\
b(\boldsymbol{\sigma}_h, \psi_h) &= -\kappa \int_\Omega f\,\psi_h && \forall\, \psi_h \in \mathrm{Q}_h\,,
\end{aligned} \tag{3.17}$$

where

$$\mathcal{F}_{\varphi,\mathbf{u}}(\boldsymbol{\tau}) := -\int_\Omega \varphi\,\mathbf{u}\cdot\boldsymbol{\tau} \quad \forall\,\boldsymbol{\tau} \in \mathrm{H}\,, \quad \text{and} \quad \mathcal{F}_{\varphi_h,\mathbf{u}_h}(\boldsymbol{\tau}_h) := -\int_\Omega \varphi_h\,\mathbf{u}_h\cdot\boldsymbol{\tau}_h \quad \forall\,\boldsymbol{\tau}_h \in \mathrm{H}_h\,,$$

and

$$\begin{aligned}
a_\varphi(\mathbf{u}, \mathbf{v}) \,+\, b_1(\mathbf{v}, p) &= \int_\Omega \mathbf{f}\cdot\mathbf{v} && \forall\, \mathbf{v} \in X_1\,,\\
b_2(\mathbf{u}, q) &= 0 && \forall\, q \in M_2\,,\\
a_{\varphi_h}(\mathbf{u}_h, \mathbf{v}_h) \,+\, b_1(\mathbf{v}_h, p_h) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}_h && \forall\, \mathbf{v}_h \in X_{1,h}\,,\\
b_2(\mathbf{u}_h, q_h) &= 0 && \forall\, q_h \in M_{2,h}\,.
\end{aligned} \tag{3.18}$$

In what follows, given a subspace Z_h of a generic Banach space $(Z, \|\cdot\|_Z)$, we set for each $z \in Z$

$$\mathrm{dist}(z, Z_h) := \inf_{z_h \in Z_h} \|z - z_h\|_Z\,.$$

Then, applying the Strang a priori error estimate from [8, Proposition 2.1, Corollary 2.3, and Theorem 2.3] (see also [22, Lemma 6.1]) to the context given by (3.17), we deduce that there exists a positive constant $\widehat{C}_{\mathtt{S}}$, depending only on $\widehat{\alpha}_{\mathtt{d}} = 1$, $\widehat{\beta}_{\mathtt{d}}$, $\|a\| = 1$, and $\|b\| = |\kappa|$, such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{H}} + \|\varphi - \varphi_h\|_{\mathrm{Q}} \,\le\, \widehat{C}_{\mathtt{S}} \Big\{ \mathrm{dist}(\boldsymbol{\sigma}, \mathrm{H}_h) + \mathrm{dist}(\varphi, \mathrm{Q}_h) + \|\mathcal{F}_{\varphi,\mathbf{u}} - \mathcal{F}_{\varphi_h,\mathbf{u}_h}\|_{\mathrm{H}_h'} \Big\}\,. \tag{3.19}$$

Next, adding and subtracting $\varphi_h \mathbf{u}$, applying the Cauchy-Schwarz and Hölder inequalities, similarly as done in the last part of the proof of Lemma 2.12, and then employing the a priori estimates for $\|\varphi\|_{\mathrm{Q}}$ (cf. (2.76)) and $\|\mathbf{u}_h\|_{X_2}$ (cf. (3.16)), we obtain

$$\begin{aligned}\|\mathcal{F}_{\varphi,\mathbf{u}} - \mathcal{F}_{\varphi_h,\mathbf{u}_h}\|_{\mathrm{H}'_h} \,&=\, \sup_{\substack{\boldsymbol{\tau}_h \in \mathrm{H}_h \\ \boldsymbol{\tau}_h \neq 0}} \frac{\displaystyle\int_\Omega \Big\{ (\varphi_h - \varphi)\, \mathbf{u}_h + \varphi\,(\mathbf{u}_h - \mathbf{u}) \Big\} \cdot \boldsymbol{\tau}_h}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}}} \\ &\le\, \|\mathbf{u}_h\|_{0,r;\Omega} \, \|\varphi - \varphi_h\|_{0,\rho;\Omega} \,+\, \|\varphi\|_{0,\rho;\Omega} \, \|\mathbf{u} - \mathbf{u}_h\|_{0,r;\Omega} \\ &\le\, \frac{1}{\widetilde{\alpha}_{\mathtt{d}}} \, \|\mathbf{f}\|_{0,r;\Omega} \, \|\varphi - \varphi_h\|_{0,\rho;\Omega} \,+\, \frac{2}{\alpha_{\widehat{T}}} \, |\kappa| \, \|f\|_{0,\varrho;\Omega} \, \|\mathbf{u} - \mathbf{u}_h\|_{0,r;\Omega} \,,\end{aligned}$$

which, replaced back into (3.19), yields

$$\begin{aligned}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{H}} + \|\varphi - \varphi_h\|_{\mathrm{Q}} \,&\le\, \widehat{C}_{\mathtt{S}} \Big\{ \mathrm{dist}(\boldsymbol{\sigma}, \mathrm{H}_h) + \mathrm{dist}(\varphi, \mathrm{Q}_h) \Big\} \\ &+\, \frac{\widehat{C}_{\mathtt{S}}}{\widetilde{\alpha}_{\mathtt{d}}} \, \|\mathbf{f}\|_{0,r;\Omega} \, \|\varphi - \varphi_h\|_{0,\rho;\Omega} \,+\, \frac{2\widehat{C}_{\mathtt{S}}}{\alpha_{\widehat{T}}} \, |\kappa| \, \|f\|_{0,\varrho;\Omega} \, \|\mathbf{u} - \mathbf{u}_h\|_{0,r;\Omega} \,.\end{aligned} \tag{3.20}$$

In turn, applying again the Strang a priori error estimate from [8, Proposition 2.1, Corollary 2.3, and Theorem 2.3], but now to the context given by (3.18), and performing some algebraic manipulations in the consistency term determined by $a_\varphi - a_{\varphi_h}$, we find that there exists a positive constant $\widetilde{C}_{\mathtt{S}}$, depending only on $\widetilde{\alpha}_{\mathtt{d}}$, $\widetilde{\beta}_{1,\mathtt{d}}$, $\widetilde{\beta}_{2,\mathtt{d}}$, $\|a_\varphi\| = \|a_{\varphi_h}\| = \mu_2$, and $\|b_1\| = \|b_2\| = 1$, such that

$$\|\mathbf{u} - \mathbf{u}_h\|_{X_2} + \|p - p_h\|_{M_1} \,\le\, \widetilde{C}_{\mathtt{S}} \Big\{ \mathrm{dist}(\mathbf{u}, X_{2,h}) \,+\, \mathrm{dist}(p, M_{1,h}) \,+\, \|(a_\varphi - a_{\varphi_h})(\mathbf{u}, \cdot)\|_{X'_{1,h}} \Big\} . \tag{3.21}$$

Then, proceeding as in the last part of the proof of Lemma 2.13 (cf. (2.73)), and using in particular the regularity estimate (2.66), we get

$$\|(a_\varphi - a_{\varphi_h})(\mathbf{u}, \cdot)\|_{X'_{1,h}} = \sup_{\substack{\mathbf{v}_h \in X_{1,h} \\ \mathbf{v}_h \neq 0}} \frac{\displaystyle\int_\Omega \big(\mu(\varphi) - \mu(\varphi_h) \big) \, \mathbf{u} \cdot \mathbf{v}_h}{\|\mathbf{v}_h\|_{X_1}} \,\le\, \widetilde{L}_{\mathtt{S}} \, \|\mathbf{f}\|_{0,r;\Omega} \, \|\varphi - \varphi_h\|_{0,\rho;\Omega} \,, \tag{3.22}$$

where $\widetilde{L}_{\mathtt{S}} := L_\mu \, \|i_\varepsilon\| \, \widetilde{C}_\varepsilon \, |\Omega|^{\frac{\varepsilon\rho - n}{\rho n}}$. In this way, replacing (3.22) back into (3.21), we arrive at

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_{X_2} + \|p - p_h\|_{M_1} \,&\le\, \widetilde{C}_{\mathtt{S}} \Big\{ \mathrm{dist}(\mathbf{u}, X_{2,h}) \,+\, \mathrm{dist}(p, M_{1,h}) \Big\} \\ &+\quad \widetilde{C}_{\mathtt{S}} \, \widetilde{L}_{\mathtt{S}} \, \|\mathbf{f}\|_{0,r;\Omega} \, \|\varphi - \varphi_h\|_{0,\rho;\Omega} \,.\end{aligned} \tag{3.23}$$

Thus, bounding $\|\mathbf{u} - \mathbf{u}_h\|_{0,r;\Omega}$ in (3.20) by the right hand side of (3.23), the former inequality becomes

$$\begin{aligned}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{H}} + \|\varphi - \varphi_h\|_{\mathrm{Q}} \,&\le\, \widehat{C}_{\mathtt{S}} \Big\{ \mathrm{dist}(\boldsymbol{\sigma}, \mathrm{H}_h) + \mathrm{dist}(\varphi, \mathrm{Q}_h) \Big\} \\ &+\, \widehat{C}_{1,\mathtt{S}} \Big\{ \mathrm{dist}(\mathbf{u}, X_{2,h}) \,+\, \mathrm{dist}(p, M_{1,h}) \Big\} \\ &+\, \Big\{ \widehat{C}_{2,\mathtt{S}} \, \|\mathbf{f}\|_{0,r;\Omega} \,+\, \widehat{C}_{3,\mathtt{S}} \, |\kappa| \, \|f\|_{0,\varrho;\Omega} \, \|\mathbf{f}\|_{0,r;\Omega} \Big\} \, \|\varphi - \varphi_h\|_{0,\rho;\Omega} \,,\end{aligned} \tag{3.24}$$

where

$$\widehat{C}_{1,\mathtt{S}} := \frac{2\widehat{C}_{\mathtt{S}}\widetilde{C}_{\mathtt{S}}}{\alpha_{\widehat{T}}} \, |\kappa| \, \|f\|_{0,\varrho;\Omega} \,, \quad \widehat{C}_{2,\mathtt{S}} := \frac{\widehat{C}_{\mathtt{S}}}{\widetilde{\alpha}_{\mathtt{d}}} \,, \quad \text{and} \quad \widehat{C}_{3,\mathtt{S}} := \frac{2\widehat{C}_{\mathtt{S}}\widetilde{C}_{\mathtt{S}}\widetilde{L}_{\mathtt{S}}}{\alpha_{\widehat{T}}} \,.$$

According to the previous analysis, we are now in a position to establish the announced Céa estimate.

Theorem 3.5 *In addition to the hypotheses of Theorems* 2.15 *and* 3.4, *assume that*

$$\widehat{C}_{2,\mathtt{S}}\,\|\mathbf{f}\|_{0,r;\Omega} \,+\, \widehat{C}_{3,\mathtt{S}}\,|\kappa|\,\|f\|_{0,\varrho;\Omega}\,\|\mathbf{f}\|_{0,r;\Omega} \,\le\, \frac{1}{2} \tag{3.25}$$

Then, there exists a constant $C > 0$, *depending only on* $\widehat{C}_{\mathtt{S}}$, $\widetilde{C}_{\mathtt{S}}$, $\widetilde{L}_{\mathtt{S}}$, $\alpha_{\widehat{T}}$, $|\kappa|$, $\|f\|_{0,\varrho;\Omega}$, *and* $\|\mathbf{f}\|_{0,r;\Omega}$, *such that*

$$\begin{aligned}&\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathrm{H}} + \|\varphi-\varphi_h\|_{\mathrm{Q}} + \|\mathbf{u}-\mathbf{u}_h\|_{X_2} + \|p-p_h\|_{M_1}\\ &\quad\le\; C\,\Big\{\mathrm{dist}(\boldsymbol{\sigma},\mathrm{H}_h) + \mathrm{dist}(\varphi,\mathrm{Q}_h) + \mathrm{dist}(\mathbf{u},X_{2,h}) \,+\, \mathrm{dist}(p,M_{1,h})\Big\}\,.\end{aligned}$$

Proof. It suffices to employ the assumption (3.25) in (3.24), and then combine the resulting estimate with (3.23). □

4 Specific finite element subspaces

In this section we restrict ourselves to the 2D case and define specific finite element subspaces

$$\mathrm{H}_h \subseteq \mathrm{H}\,,\quad \mathrm{Q}_h \subseteq \mathrm{Q}\,,\quad X_{2,h}\subseteq X_2\,,\quad M_{2,h}\subseteq M_2\,,\quad X_{1,h}\subseteq X_1\,,\quad \text{and}\quad M_{1,h}\subseteq M_1\,,$$

satisfying the abstract assumptions (**H.1**), (**H.2**), (**H.3**), and (**H.4**) that were introduced in Section 3.2 for the well-posedness of our Galerkin scheme.

4.1 Preliminaries

We first let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$, which are made of triangles K of diameters h_K, and define the meshsize $h := \max\{h_K : \quad K \in \mathcal{T}_h\}$, which also serves as the index of $\mathcal{T}_h$. Next, given an integer $k \ge 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_k(K)$ be the space of polynomials of degree $\le k$ defined on K with vector version denoted by $\mathbf{P}_k(K)$. In addition, we let $\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \mathrm{P}_k(K)\,\mathbf{x}$ be the local Raviart-Thomas space of order k defined on K, where $\mathbf{x}$ stands for a generic vector in R^2. In turn, we let $\mathrm{P}_k(\mathcal{T}_h)$ and $\mathbf{RT}_k(\mathcal{T}_h)$ be the corresponding global versions of $\mathrm{P}_k(K)$ and $\mathbf{RT}_k(K)$, respectively, that is

$$\mathrm{P}_k(\mathcal{T}_h) := \Big\{ q_h \in \mathrm{L}^2(\Omega) : \quad q_h|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

and

$$\mathbf{RT}_k(\mathcal{T}_h) := \Big\{ \boldsymbol{\tau}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,.$$

Note that there also hold $\mathrm{P}_k(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}(\mathrm{div}_t;\Omega)$ (cf. (2.9a)), and $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}^t(\mathrm{div}_t;\Omega)$ (cf. (2.9b)), for all $t \in [1,+\infty]$, which is implicitly employed below in Section 4.2 to define our specific finite element subspaces. Before doing that, in what follows we provide some useful properties concerning $\mathrm{P}_k(\mathcal{T}_h)$ and $\mathbf{RT}_k(\mathcal{T}_h)$. To this end, we now introduce for each $t \in (1,+\infty)$ the space

$$\mathbf{H}_t \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega) \cup \mathbf{H}(\mathrm{div}_t;\Omega) : \quad \boldsymbol{\tau}|_K \in \mathbf{W}^{1,t}(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

and let $\Pi_h^k : \mathbf{H}_t \longrightarrow \mathbf{RT}_k(\mathcal{T}_h)$ be the global Raviart-Thomas interpolation operator (cf. [12, Section 2.5], [35, Section 3.4]). Then, we recall from [12, Proposition 2.5.2 and eq. (2.5.27)] (see also [35, Lemma 3.7]) the commuting diagram property

$$\mathrm{div}\big(\Pi_h^k(\boldsymbol{\tau})\big) \,=\, \mathcal{P}_h^k\big(\mathrm{div}(\boldsymbol{\tau})\big) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}_t\,, \tag{4.1}$$

where $\mathcal{P}_h^k : \mathrm{L}^1(\Omega) \longrightarrow \mathrm{P}_k(\mathcal{T}_h)$ is the usual orthogonal projector with respect to the $\mathrm{L}^2(\Omega)$-inner product, that is, given $w \in \mathrm{L}^1(\Omega)$, $\mathcal{P}_h^k(w)$ is the unique element in $\mathrm{P}_k(\mathcal{T}_h)$ satisfying

$$\int_\Omega \mathcal{P}_h^k(w)\, q_h \,=\, \int_\Omega w\, q_h \qquad \forall\, q_h \in \mathrm{P}_k(\mathcal{T}_h)\,.$$

Similarly, letting Γ_h be the set of edges $e \subset \Gamma$ that are induced by $\mathcal{T}_h$, and denoting by $\mathrm{P}_k(\Gamma_h)$ the subspace of $\mathrm{L}^2(\Gamma)$ given by the piecewise polynomials of degree $\leq k$ on each $e \in \Gamma_h$, the following property also holds (cf. [12, eq. (2.5.10) in Example 2.5.3], [35, eq. (3.36) in Lemma 3.18])

$$\Pi_h^k(\boldsymbol{\tau}) \cdot \boldsymbol{\nu} \,=\, \mathcal{Q}_h^k(\boldsymbol{\tau} \cdot \boldsymbol{\nu}) \quad \text{on} \quad \Gamma \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}_t\,, \tag{4.2}$$

where $\mathcal{Q}_h^k : \mathrm{L}^1(\Gamma) \longrightarrow \mathrm{P}_k(\Gamma_h)$ is the orthogonal projector with respect to the $\mathrm{L}^2(\Gamma)$-inner product. On the other hand, employing the $\mathrm{W}^{m,t}$ version of the Deny-Lions Lemma (cf. [32, Lemma B.67]) with integer $m \geq 0$ and $t \in (1, +\infty)$, the associated scaling estimates (cf. [32, Lemma 1.101]), and the regularity of $\big\{\mathcal{T}_h\big\}_{h>0}$, we deduce the existence of constants C_1, $C_2 > 0$, independent of h, such that for integers ℓ and m satisfying $0 \leq \ell \leq k+1$ and $0 \leq m \leq \ell$, there hold

$$|w - \mathcal{P}_h^k(w)|_{m,t;\Omega} \,\leq\, C_1\, h^{\ell - m}\, |w|_{\ell,t;\Omega} \qquad \forall\, w \in \mathrm{W}^{\ell,t}(\Omega)\,, \tag{4.3a}$$

and

$$|\mathrm{div}(\boldsymbol{\tau}) - \mathrm{div}\big(\Pi_h^k(\boldsymbol{\tau})\big)|_{m,t;\Omega} \,\leq\, C_1\, h^{\ell - m}\, |\mathrm{div}(\boldsymbol{\tau})|_{\ell,t;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathrm{W}^{1,t}(\Omega) \text{ with } \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{W}^{\ell,t}(\Omega)\,, \tag{4.3b}$$

whereas for integers ℓ and m satisfying $1 \leq \ell \leq k+1$ and $0 \leq m \leq \ell$, there holds

$$|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})|_{m,t;\Omega} \,\leq\, C_2\, h^{\ell - m}\, |\boldsymbol{\tau}|_{\ell,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathrm{W}^{\ell,t}(\Omega)\,. \tag{4.3c}$$

In particular, note that (4.3b) actually follows from (4.1) and a straightforward application of (4.3a) to $w = \mathrm{div}(\boldsymbol{\tau})$. In addition, we remark that (4.3a) is first derived for $1 \leq \ell \leq k+1$, and then using only the scaling estimates one proves the stability of $\mathcal{P}_h^k$, that is the existence of a constant $c > 0$, independent of h, such that

$$\|\mathcal{P}_h^k(w)\|_{0,t;\Omega} \,\leq\, c\, \|w\|_{0,t;\Omega} \qquad \forall\, w \in \mathrm{L}^t(\Omega)\,. \tag{4.4}$$

In turn, employing the triangle inequality and (4.3c) with $\ell = 1$ and $m = 0$, we readily deduce the existence of a constant $C > 0$, independent of h, such that

$$\|\Pi_h^k(\boldsymbol{\tau})|_{0,t;\Omega} \,\leq\, C\, \|\boldsymbol{\tau}\|_{1,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathrm{W}^{1,t}(\Omega)\,. \tag{4.5}$$

Furthermore, taking in particular $(m,t) = (0,2)$ in (4.3c) and $m = 0$ in (4.3b), we readily find that there exists a constant $C_3 > 0$, independent of h, such that for $1 \leq \ell \leq k+1$ there holds

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{\mathrm{div}_t;\Omega} \,\leq\, C_3\, h^\ell \left\{|\boldsymbol{\tau}|_{\ell,\Omega} \,+\, |\mathrm{div}(\boldsymbol{\tau})|_{\ell,t;\Omega}\right\} \tag{4.6}$$

for all $\boldsymbol{\tau} \in \mathbf{H}^\ell(\Omega)$ with $\mathrm{div}(\boldsymbol{\tau}) \in \mathrm{W}^{\ell,t}(\Omega)$. In turn, taking now $m = 0$ in (4.3c) and (4.3b), we deduce the existence of a constant $C_4 > 0$, independent of h, such that for $1 \leq \ell \leq k+1$ there holds

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{t,\mathrm{div}_t;\Omega} \,\leq\, C_4\, h^\ell \left\{|\boldsymbol{\tau}|_{\ell,t;\Omega} \,+\, |\mathrm{div}(\boldsymbol{\tau})|_{\ell,t;\Omega}\right\} \tag{4.7}$$

for all $\boldsymbol{\tau} \in \mathbf{W}^{\ell,t}(\Omega)$ with $\mathrm{div}(\boldsymbol{\tau}) \in \mathrm{W}^{\ell,t}(\Omega)$.

4.2 The finite element subspaces

Our specific finite element subspaces are defined as

$$\mathrm{H}_h := \mathbf{H}(\mathrm{div}_\varrho;\Omega)\cap \mathbf{RT}_k(\mathcal{T}_h) = \Big\{\boldsymbol{\tau}_h\in \mathbf{H}(\mathrm{div}_\varrho;\Omega): \quad \boldsymbol{\tau}_h|_K\in \mathbf{RT}_k(K) \quad \forall\, K\in\mathcal{T}_h\Big\}, \tag{4.8a}$$

$$\mathrm{Q}_h := \mathrm{L}^\rho(\Omega)\cap \mathrm{P}_k(\mathcal{T}_h) = \Big\{\psi_h\in \mathrm{L}^\rho(\Omega): \quad \psi_h|_K\in \mathrm{P}_k(K) \quad \forall\, K\in\mathcal{T}_h\Big\}, \tag{4.8b}$$

$$X_{2,h} := X_2\cap \mathbf{RT}_k(\mathcal{T}_h) = \Big\{\mathbf{w}_h\in \mathbf{H}^r_0(\mathrm{div}_r;\Omega): \quad \mathbf{w}_h|_K\in \mathbf{RT}_k(K) \quad \forall\, K\in\mathcal{T}_h\Big\}, \tag{4.8c}$$

$$M_{2,h} := \mathrm{L}^s_0(\Omega)\cap \mathrm{P}_k(\mathcal{T}_h) = \Big\{q_h\in \mathrm{L}^s_0(\Omega): \quad q_h|_K\in \mathrm{P}_k(K) \quad \forall\, K\in\mathcal{T}_h\Big\}, \tag{4.8d}$$

$$X_{1,h} := X_1\cap \mathbf{RT}_k(\mathcal{T}_h) = \Big\{\mathbf{v}_h\in \mathbf{H}^s_0(\mathrm{div}_s;\Omega): \quad \mathbf{v}_h|_K\in \mathbf{RT}_k(K) \quad \forall\, K\in\mathcal{T}_h\Big\}, \tag{4.8e}$$

$$M_{1,h} := \mathrm{L}^r_0(\Omega)\cap \mathrm{P}_k(\mathcal{T}_h) = \Big\{q_h\in \mathrm{L}^r_0(\Omega): \quad q_h|_K\in \mathrm{P}_k(K) \quad \forall\, K\in\mathcal{T}_h\Big\}. \tag{4.8f}$$

Regarding the above definitions, we first observe that $\mathrm{div}(\mathrm{H}_h) \subseteq \mathrm{Q}_h$, which confirms the verification of the hypothesis (**H.3**). In turn, while the pairs $(\mathrm{H}_h,\mathrm{Q}_h)$, $(X_{2,h},M_{2,h})$ and $(X_{1,h},M_{1,h})$ are topologically different, we stress that they do coincide algebraically. This fact implies that the stiffness matrices associated to the bilinear forms b, b_1, and b_2 are exactly the same, except for the constant factor κ of b, and that those of a and a_φ differ only by the factor $\mu(\varphi)$. The above, being certainly very relevant from the computational point of view, constitutes another advantage of having used a mixed formulation in the heat equation as well.

Furthermore, it is also clear that $\mathrm{div}\big(X_{i,h}\big) \subseteq M_{i,h}$ for all $i\in\{1,2\}$. As a consequence, the corresponding discrete kernels of the bilinear forms b_1 and b_2 (cf. (3.6a), (3.6b)) coincide as well, and it is easily seen that they become the space

$$\mathcal{K}^k_h := \Big\{\mathbf{v}_h\in \mathbf{RT}_k(\mathcal{T}_h): \quad \mathbf{v}_h\cdot\boldsymbol{\nu}=0 \quad \text{on} \quad \Gamma \quad \text{and} \quad \mathrm{div}(\mathbf{v}_h)=0 \quad \text{in} \quad \Omega\Big\}. \tag{4.9}$$

In this way, we now let $\Theta^k_h:\mathbf{L}^1(\Omega)\longrightarrow\mathcal{K}^k_h$ be the $\mathrm{L}^2(\Omega)$-orthogonal projector, that is, given $\mathbf{w}\in\mathbf{L}^1(\Omega)$, $\Theta^k_h(\mathbf{w})$ is the unique element in $\mathcal{K}^k_h$ satisfying

$$\int_\Omega \Theta^k_h(\mathbf{w})\cdot\mathbf{v}_h = \int_\Omega \mathbf{w}\cdot\mathbf{v}_h \qquad \forall\,\mathbf{v}_h\in\mathcal{K}^k_h. \tag{4.10}$$

This operator plays a key role in what follows. Indeed, in order to prove one of the inf-sup conditions required by our discrete analysis, we need to establish a particular stability estimate for Θ^k_h in terms of $\|\cdot\|_{0,t;\Omega}$, with $t\in(1,+\infty)$. This result is provided next in Section 4.3, for which we make use of the related estimates for the Ritz projection that are collected in Appendix A.

4.3 $\mathrm{L}^t(\Omega)$-stability of Θ^k_h

In this section we first characterise the kernel $\mathcal{K}^k_h$ in terms of $\mathrm{P}_{k+1,c}(\mathcal{T}_h)$ (cf. (A.1) in Appendix A), and then establish for each $t\in(1,+\infty)$ the $\mathrm{L}^t(\Omega)$-stability of Θ^k_h when restricted to the space

$$\widetilde{\mathbf{H}}^t_0(\mathrm{div}_t;\Omega) := \Big\{\mathbf{v}\in\mathbf{H}^t_0(\mathrm{div}_t;\Omega): \quad \mathrm{div}(\mathbf{v})=0 \quad \text{in} \quad \Omega\Big\}.$$

More precisely, these results are given by the following two lemmas, whose proofs follow very closely those of [31, Lemma 2.1] and [31, Theorem 3.1], respectively.

Lemma 4.1 *There holds*

$$\mathcal{K}_h^k \;=\; \mathrm{curl}\big(\mathrm{P}_{k+1,c}(\mathcal{T}_h)\big)\,. \tag{4.11}$$

Proof. Let $\mathbf{v}_h \in \mathcal{K}_h^k$, that is $\mathbf{v}_h \in \mathrm{RT}_k(\mathcal{T}_h)$, $\mathrm{div}(\mathbf{v}_h) = 0$ in Ω, and $\mathbf{v}_h \cdot \boldsymbol{\nu} = 0$ on Γ. It follows (see, e.g. [35, proof of Theorem 3.3]) that $\mathbf{v}_h|_K \in \mathbf{P}_k(K)$ for all $K \in \mathcal{T}_h$. In addition, since Ω is simply connected, we deduce from [40, Theorem I.3.1] and the null normal trace of $\mathbf{v}_h$ on Γ that there exists $\phi \in \mathrm{H}_0^1(\Omega)$ such that $\mathbf{v}_h = \mathrm{curl}(\phi)$. Hence, for each $K \in \mathcal{T}_h$ there holds $\mathrm{curl}(\phi)|_K = \mathbf{v}_h|_K \in \mathbf{P}_k(K)$, which implies that $\phi|_K \in \mathrm{P}_{k+1}(K)$. In this way, $\phi \in \mathrm{P}_{k+1,c}(\mathcal{T}_h)$, and therefore $\mathbf{v}_h \in \mathrm{curl}\big(\mathrm{P}_{k+1,c}(\mathcal{T}_h)\big)$. Conversely, let $\mathbf{v}_h \in \mathrm{curl}\big(\mathrm{P}_{k+1,c}(\mathcal{T}_h)\big)$, that is $\mathbf{v}_h = \mathrm{curl}(\phi_h)$ with $\phi_h \in \mathrm{P}_{k+1,c}(\mathcal{T}_h)$. It follows that $\mathbf{v}_h|_K = \mathrm{curl}(\phi_h)|_K \in \mathbf{P}_k(K) \subseteq \mathrm{RT}_k(K)$ for all $K \in \mathcal{T}_h$, and certainly $\mathrm{div}(\mathbf{v}_h) = 0$ in Ω and $\mathbf{v}_h \cdot \boldsymbol{\nu} = 0$ on Γ, which shows that $\mathbf{v}_h \in \mathcal{K}_h^k$. □

Lemma 4.2 *Given $t \in (1, +\infty)$ and an integer $k \geq 0$, there holds*

$$\|\Theta_h^k(\mathbf{w})\|_{0,t;\Omega} \;\leq\; c_t^k\, \|\mathbf{w}\|_{0,t;\Omega} \qquad \forall\, \mathbf{w} \in \widetilde{\mathbf{H}}_0^t(\mathrm{div}_t;\Omega)\,, \tag{4.12}$$

where

$$c_t^k \;:=\; \begin{cases} C_t^k & \text{if } \Omega \text{ is convex}, \\[1ex] \bar{C}_t^k \,\big\{ -\log(h) \big\}^{|1-2/t|} & \text{if } \Omega \text{ is non-convex and } k = 0\,, \\[1ex] \bar{C}_t^k & \text{if } \Omega \text{ is non-convex and } k \geq 1\,. \end{cases} \tag{4.13}$$

Proof. Given $t \in (1, +\infty)$, an integer $k \geq 0$, and $\mathbf{w} \in \widetilde{\mathbf{H}}_0^t(\mathrm{div}_t;\Omega)$, we employ again [40, Theorem I.3.1] and the fact that the normal trace of $\mathbf{w}$ vanishes on Γ, to deduce that there exists $\varphi \in \mathrm{W}_0^{1,t}(\Omega)$ such that $\mathbf{w} = \mathrm{curl}(\varphi)$ in Ω. In turn, according to the identity (4.11) (cf. Lemma 4.1), there exists $\varphi_h \in \mathrm{P}_{k+1,c}(\mathcal{T}_h)$ such that $\Theta_k^k(\mathbf{w}) = \mathrm{curl}(\varphi_h)$, and hence the characterisation (4.10) of $\Theta_h^k(\mathbf{w})$ becomes

$$\int_\Omega \mathrm{curl}(\varphi_h) \cdot \mathrm{curl}(\phi_h) \;=\; \int_\Omega \mathrm{curl}(\varphi) \cdot \mathrm{curl}(\phi_h) \qquad \forall\, \phi_h \in \mathrm{P}_{k+1,c}(\mathcal{T}_h)\,, \tag{4.14}$$

where (4.11) has also been utilised to replace the test functions $\mathbf{v}_h$ of (4.10) by $\mathrm{curl}(\phi_h)$, with $\phi_h \in \mathrm{P}_{k+1,c}(\mathcal{T}_h)$. Next, it is readily seen that, due to the relation between curl and ∇ in the 2D case, (4.14) can be rewritten as

$$\int_\Omega \nabla\varphi_h \cdot \nabla\phi_h \;=\; \int_\Omega \nabla\varphi \cdot \nabla\phi_h \qquad \forall\, \phi_h \in \mathrm{P}_{k+1,c}(\mathcal{T}_h)\,, \tag{4.15}$$

which, invoking (A.2), says that $\varphi_h = \mathcal{R}_h^k(\varphi)$. In this way, bearing in mind again the aforementioned relation, it follows that

$$\|\Theta_h^k(\mathbf{w})\|_{0,t;\Omega} = \|\mathrm{curl}(\varphi_h)\|_{0,t;\Omega} = \|\nabla\varphi_h\|_{0,t;\Omega} = \|\nabla\mathcal{R}_h^k(\varphi)\|_{0,t;\Omega} \tag{4.16}$$

and

$$\|\mathbf{w}\|_{0,t;\Omega} = \|\mathrm{curl}(\varphi)\|_{0,t;\Omega} = \|\nabla\varphi\|_{0,t;\Omega}\,, \tag{4.17}$$

so that these identities, together with (A.5) and (A.9), yield (4.12) - (4.13) and complete the proof.

□

At this point we remark that in 3D, (4.14) and (4.15) do not coincide, and the second equalities of the identities (4.16) and (4.17) do not hold either, which stops us of extending the proof of Lemma 4.2 to an eventual three-dimensional version. Indeed, up to our knowledge, the respective stability estimate (4.12) remains as an open problem in this case, which explains that the discrete inf-sup condition for a_{ψ_h}, to be established below in Lemma 4.3 by making use of (4.12), only holds in 2D. In other words, this latter fact is actually the only reason why the present Section 4 has been restricted to a two-dimensional domain Ω since all the other discrete inf-sup conditions that are required for the discrete analysis, can be proved to be valid in both dimensions.

4.4 The discrete inf-sup conditions for $\widetilde{T}_h$ and $\widehat{T}_h$

In this section we employ the main results provided in Appendices B and C to verify that the specific finite element subspaces that were introduced in Section 4.2 satisfy the assumptions (**H.1**), (**H.2**), and (**H.4**). In other words, in what follows we establish the discrete analogues of Lemmas 2.6, 2.7, and 2.9, for which we suitably adapt their respective proofs to the present context. We begin with the discrete inf-sup condition for a_{ψ_h}, where ψ_h is taken in Q_h.

Lemma 4.3 *For each $\psi_h \in \mathrm{Q}_h$ there hold*

$$\sup_{\substack{\mathbf{v}_h \in \mathcal{K}_h^k \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{a_{\psi_h}(\mathbf{w}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{X_1}} \,\geq\, \widetilde{\alpha}_{\mathtt{d}}\, \|\mathbf{w}_h\|_{X_2} \qquad \forall\, \mathbf{w}_h \in \mathcal{K}_h^k\,, \tag{4.18}$$

with $\widetilde{\alpha}_{\mathtt{d}} := \mu_1/(c_s^k\,\|D_s\|)$ (cf. (1.2), Lemmas 2.3 and 4.2), and

$$\sup_{\mathbf{w}_h \in \mathcal{K}_h^k} a_{\psi_h}(\mathbf{w}_h, \mathbf{v}_h) \,>\, 0 \qquad \forall\, \mathbf{v}_h \in \mathcal{K}_h^k\,, \quad \mathbf{v}_h \neq \mathbf{0}\,. \tag{4.19}$$

Proof. Given $\psi_h \in \mathrm{Q}_h$, we consider $\mathbf{w}_h \in \mathcal{K}_h^k$, $\mathbf{w}_h \neq \mathbf{0}$, define (cf. (2.7)) $\mathbf{w}_{h,s} := \mathcal{J}_r(\mathbf{w}_h) \in \mathbf{L}^s(\Omega)$, and let $\widetilde{\mathbf{v}}_h := \Theta_h^k\big(D_s(\mathbf{w}_{h,s})\big) \in \mathcal{K}_h^k$. Then, thanks to (4.10), (2.14) (cf. Lemma 2.3), and Lemma 2.2, we observe that

$$\int_\Omega \mathbf{w}_h \cdot \widetilde{\mathbf{v}}_h \,=\, \int_\Omega \mathbf{w}_h \cdot D_s(\mathbf{w}_{h,s}) \,=\, \int_\Omega \mathbf{w}_h \cdot \mathbf{w}_{h,s} \,=\, \|\mathbf{w}_h\|_{0,r;\Omega}\, \|\mathbf{w}_{h,s}\|_{0,s;\Omega}\,, \tag{4.20}$$

from which it follows that necessarily $\widetilde{\mathbf{v}}_h \neq \mathbf{0}$. Furthermore, the stability estimate (4.12) (cf. Lemma 4.2) and the boundedness of D_s (cf. Lemma 2.3) yield

$$\|\widetilde{\mathbf{v}}_h\|_{0,s;\Omega} \,\leq\, c_s^k\, \|D_s\|\, \|\mathbf{w}_{h,s}\|_{0,s;\Omega}\,. \tag{4.21}$$

Thus, employing the lower bound for μ (cf. (1.2)), (4.20), and (4.21), we find that

$$\sup_{\substack{\mathbf{v}_h \in \mathcal{K}_h^k \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{a_{\psi_h}(\mathbf{w}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{X_1}} \,\geq\, \frac{\big|a_{\psi_h}(\mathbf{w}_h, \widetilde{\mathbf{v}}_h)\big|}{\|\widetilde{\mathbf{v}}_h\|_{0,s;\Omega}} \,\geq\, \mu_1 \frac{\displaystyle\int_\Omega \mathbf{w}_h \cdot \widetilde{\mathbf{v}}_h}{\|\widetilde{\mathbf{v}}_h\|_{0,s;\Omega}} \,\geq\, \frac{\mu_1}{c_s^k\,\|D_s\|}\, \|\mathbf{w}_h\|_{0,r;\Omega}\,,$$

which yields (4.18) with the indicated constant $\widetilde{\alpha}_{\mathtt{d}}$. Similarly, given $\mathbf{v}_h \in \mathcal{K}_h^k$, $\mathbf{v}_h \neq \mathbf{0}$, we define (cf. (2.7)) $\mathbf{v}_{h,r} := \mathcal{J}_s(\mathbf{v}_h) \in \mathbf{L}^r(\Omega)$, set $\widetilde{\mathbf{w}}_h := \Theta_h^k\big(D_r(\mathbf{v}_{h,r})\big) \in \mathcal{K}_h^k$, and utilise again (1.2), (4.10), (2.14), and Lemma 2.2, to deduce that

$$\begin{aligned}\sup_{\mathbf{w}_h \in \mathcal{K}_h^k} a_{\psi_h}(\mathbf{w}_h, \mathbf{v}_h) \,&\geq\, a_{\psi_h}(\widetilde{\mathbf{w}}_h, \mathbf{v}_h) \,\geq\, \mu_1 \int_\Omega \widetilde{\mathbf{w}}_h \cdot \mathbf{v}_h \\ &=\, \mu_1 \int_\Omega D_r(\mathbf{v}_{h,r}) \cdot \mathbf{v}_h \,=\, \mu_1\, \|\mathbf{v}_h\|_{0,s;\Omega}^s > 0\,,\end{aligned}$$

which proves (4.19) and concludes the proof. $\square$

We stress here that, only when Ω is non-convex and $k=0$ is utilised, the discrete inf-sup constant $\widetilde{\alpha}_{\mathtt{d}}$ depends on the meshsize h, though in a very inoffensive manner. In fact, it is clear from (4.13) (cf. Lemma 4.2) that in that case $\widetilde{\alpha}_{\mathtt{d}} = \mu_1/\big(\bar{C}_s^k \left\{-\log(h)\right\}^{|1-2/s|} \|D_s\|\big)$, where the h-dependent term given by $\left\{-\log(h)\right\}^{|1-2/s|}$ grows very slowly as $h \to 0$, and hence it actually remains reasonably bounded for very small values of h. In particular, taking for instance $s = \frac{8}{5}$ (in Lemma 4.4 below we show that any $s \in (\frac{3}{2}, 2)$ is a feasible choice) and $h \geq 10^{-30}$, then there holds $\left\{-\log(h)\right\}^{|1-2/s|} = \left\{-\log(h)\right\}^{1/4} < 3$.

It is also important to highlight at this point that the proof of Lemma 4.3 induces a discrete version of the operator D_s provided by Lemma 2.3. In fact, it suffices to define $D_{s,h} : \mathbf{L}^s(\Omega) \to \mathcal{K}_h^k$ by $D_{s,h}(\mathbf{w}) := \Theta_h^k\big(D_s(\mathbf{w})\big)$ for all $\mathbf{w} \in \mathbf{L}^s(\Omega)$, which satisfies $\int_\Omega \mathbf{w}_h \cdot D_{s,h}(\mathbf{w}) = \int_\Omega \mathbf{w}_h \cdot \mathbf{w}$ for all $\mathbf{w}_h \in X_{2,h}$ such that $\operatorname{div}(\mathbf{w}_h) = 0$ in Ω and $\mathbf{w}_h \cdot \boldsymbol{\nu} = 0$ on Γ.

The discrete inf-sup conditions for the bilinear forms b_i, $i \in \{1,2\}$, are provided next.

Lemma 4.4 *There exist $\widetilde{\beta}_{1,\mathtt{d}}$, $\widetilde{\beta}_{2,\mathtt{d}} > 0$, independent of h, such that for each $i \in \{1,2\}$ there holds*

$$\sup_{\substack{\mathbf{v}_h \in X_{i,h} \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{b_i(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{X_i}} \;\geq\; \widetilde{\beta}_{i,\mathtt{d}} \, \|q_h\|_{M_i} \qquad \forall\, q_h \in M_{i,h}\,. \tag{4.22}$$

Proof. We prove first for $i=1$. In this way, given $q_h \in M_{1,h}$, we set $q_{h,s} := \mathcal{J}_r(q_h) \in \mathrm{L}^s(\Omega)$ and $q_{h,s}^0 := q_{h,s} - \frac{1}{|\Omega|}\int_\Omega q_{h,s} \in \mathrm{L}_0^s(\Omega)$, and let $u \in \widetilde{\mathrm{W}}^{1,s}(\Omega)$ be the unique solution of (2.2) with $g = q_{h,s}^0$, $\mathbf{g} = \mathbf{0}$, and $g_N = 0$, that is

$$\Delta u \,=\, q_{h,s}^0 \quad \text{in} \quad \Omega\,, \quad \nabla u \cdot \boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma\,, \quad \int_\Omega u = 0\,. \tag{4.23}$$

If Ω is convex, then we deduce from [45, Theorem 1.1] that actually $u \in \mathrm{W}^{2,s}(\Omega) \cap \widetilde{\mathrm{W}}^{1,s}(\Omega)$ and that there exists a positive constant $C(s)$ such that

$$\|u\|_{2,s;\Omega} \,\leq\, C(s)\, \|q_{h,s}^0\|_{0,s;\Omega}\,. \tag{4.24}$$

Then, defining $\bar{\mathbf{v}} := -\nabla u \in \mathrm{W}^{1,s}(\Omega)$, we have $\bar{\mathbf{v}} \cdot \boldsymbol{\nu} = 0$ on Γ, $\operatorname{div}(\bar{\mathbf{v}}) = -q_{h,s}^0$ in Ω, and, using (4.24),

$$\|\bar{\mathbf{v}}\|_{1,s;\Omega} \,\leq\, \|u\|_{2,s;\Omega} \,\leq\, C(s)\, \|q_{h,s}^0\|_{0,s;\Omega}\,. \tag{4.25}$$

Thus, letting $\bar{\mathbf{v}}_h := \Pi_h^k(\bar{\mathbf{v}})$ and employing the identities satisfied by the Raviart-Thomas interpolator Π_h^k (cf. (4.2), (4.1)), we observe that $\bar{\mathbf{v}}_h \cdot \boldsymbol{\nu} = \Pi_h^k(\bar{\mathbf{v}}) \cdot \boldsymbol{\nu} = \mathcal{Q}_h^k(\bar{\mathbf{v}} \cdot \boldsymbol{\nu}) = 0$ on Γ, which proves that $\bar{\mathbf{v}}_h \in X_{1,h}$, and

$$\operatorname{div}(\bar{\mathbf{v}}_h) \,=\, \operatorname{div}\big(\Pi_h^k(\bar{\mathbf{v}})\big) \,=\, \mathcal{P}_h^k\big(\operatorname{div}(\bar{\mathbf{v}})\big) \,=\, \mathcal{P}_h^k(-q_{h,s}^0) \quad \text{in} \quad \Omega\,. \tag{4.26}$$

In addition, applying the stability estimates of Π_h^k (cf. (4.5)) and $\mathcal{P}_h^k$ (cf. (4.4)), and thanks to (4.25) and (4.26), we find that

$$\|\bar{\mathbf{v}}_h\|_{0,s;\Omega} \,=\, \|\Pi_h^k(\bar{\mathbf{v}})\|_{0,s;\Omega} \,\leq\, C\, \|\bar{\mathbf{v}}\|_{1,s;\Omega} \,\leq\, \widetilde{C}\, \|q_{h,s}^0\|_{0,s;\Omega}\,, \tag{4.27a}$$

$$\|\operatorname{div}(\bar{\mathbf{v}}_h)\|_{0,s;\Omega} \,=\, \|\mathcal{P}_h^k(q_{h,s}^0)\|_{0,s;\Omega} \,\leq\, c\, \|q_{h,s}^0\|_{0,s;\Omega}\,, \tag{4.27b}$$

from which it follows that (cf. (2.10b))

$$\|\bar{\mathbf{v}}_h\|_{X_1} \,=\, \|\bar{\mathbf{v}}_h\|_{0,s;\Omega} + \|\operatorname{div}(\bar{\mathbf{v}}_h)\|_{0,s;\Omega} \,\leq\, \bar{C}\, \|q_{h,s}^0\|_{0,s;\Omega} \,\leq\, \bar{C}\, \widetilde{C}_s\, \|q_{h,s}\|_{0,s;\Omega}\,, \tag{4.28}$$

where C, c, $\bar{C}$, and $\widetilde{C}_s$ are positive constants independent of h, the latter being specified within the proof of Lemma 2.7. In this way, bearing in mind (4.26) again, the fact that $\int_\Omega q_h\, q^0_{h,s} = \int_\Omega q_h\, q_{h,s}$, the scalar version of (2.8b), and the estimate (4.28), we obtain

$$\begin{aligned} \sup_{\substack{\mathbf{v}_h \in X_{1,h} \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{b_1(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{X_1}} \;\geq\; \frac{-\displaystyle\int_\Omega q_h \operatorname{div}(\bar{\mathbf{v}}_h)}{\|\bar{\mathbf{v}}_h\|_{X_1}} \;=\; \frac{\displaystyle\int_\Omega q_h\, \mathcal{P}^k_h(q^0_{h,s})}{\|\bar{\mathbf{v}}_h\|_{X_1}} \\ =\; \frac{\displaystyle\int_\Omega q_h\, q_{h,s}}{\|\bar{\mathbf{v}}_h\|_{X_1}} \;=\; \frac{\|q_h\|_{0,r;\Omega}\, \|q_{h,s}\|_{0,s;\Omega}}{\|\bar{\mathbf{v}}_h\|_{X_1}} \;\geq\; \frac{1}{\bar{C}\, \widetilde{C}_s}\, \|q_h\|_{0,r;\Omega}\,, \end{aligned} \tag{4.29}$$

which yields (4.22), for $i = 1$ and a convex polygonal domain Ω, with $\widetilde{\beta}_{1,\mathtt{d}} = \frac{1}{\bar{C}\,\widetilde{C}_s}$.

In turn, if Ω is non-convex, and bearing in mind that $s \in (1, 2)$, we observe from Lemma B.1 (cf. (B.2)) that the solution u of (4.23) belongs to $\mathrm{W}^{1+\delta,s}(\Omega) \cap \widetilde{\mathrm{W}}^{1,s}(\Omega)$ for all $\delta \in (0, \delta_0)$, with $\delta_0 = \min\left\{2 - \frac{2}{s}, \frac{\pi}{\omega}\right\}$, and that there exists a positive constant $C(s, \delta)$ such that

$$\|u\|_{1+\delta,s;\Omega} \;\leq\; C(s,\delta)\, \|q^0_{h,s}\|_{0,s;\Omega}\,. \tag{4.30}$$

Thus, defining $\bar{\mathbf{v}} := -\nabla u \in \mathrm{W}^{\delta,s}(\Omega)$, we have $\bar{\mathbf{v}} \cdot \boldsymbol{\nu} = 0$ on Γ, $\operatorname{div}(\bar{\mathbf{v}}) = -q^0_{h,s}$ in Ω, and, using (4.30),

$$\|\bar{\mathbf{v}}\|_{\delta,s;\Omega} \;\leq\; C(s,\delta)\, \|q^0_{h,s}\|_{0,s;\Omega}\,. \tag{4.31}$$

Next, proceeding as in the convex case, we define $\bar{\mathbf{v}}_h := \Pi^k_h(\bar{\mathbf{v}})$ and realise again that $\bar{\mathbf{v}}_h \in X_{1,h}$ and that $\operatorname{div}(\bar{\mathbf{v}}_h) = \mathcal{P}^k_h(-q^0_{h,s})$. Now, in order to apply (C.12b) to $t = s$ and $\boldsymbol{\tau} = \bar{\mathbf{v}}$, which requires, according to Lemma C.1 (cf. (C.3)), that $\delta > \frac{1}{s}$, we need to impose that $\delta_0 > \frac{1}{s}$, or equivalently $2 - \frac{2}{s} > \frac{1}{s}$ and $\frac{\pi}{\omega} > \frac{1}{s}$, that is $s > \frac{3}{2}$ and $\omega < s\pi$. In this way, under these assumptions on s and the maximum interior angle ω of Ω, and thanks to (C.12b) and (4.31), we get

$$\|\bar{\mathbf{v}}_h\|_{0,s;\Omega} = \|\Pi^k_h(\bar{\mathbf{v}})\|_{0,s;\Omega} \;\leq\; c_s\left\{\|\bar{\mathbf{v}}\|_{\delta,s;\Omega} \,+\, h^\delta\, \|\operatorname{div}(\bar{\mathbf{v}})\|_{0,s;\Omega}\right\} \;\leq\; c_s(1 + C(s,\delta))\, \|q^0_{h,s}\|_{0,s;\Omega}\,,$$

where we have simply bounded h^δ by 1. The foregoing inequality together with (4.27b) yield the bound for $\|\bar{\mathbf{v}}_h\|_{X_1}$ in terms of $\|q_{h,s}\|_{0,s;\Omega}$ (cf. (4.28)), and then the rest of the derivation of the discrete inf-sup condition for b_1 follows as (4.29).

On the other hand, for $i = 2$ we consider $q_h \in M_{2,h}$, set $q_{h,r} := \mathcal{J}_s(q_h) \in \mathrm{L}^r(\Omega)$ and $q^0_{h,r} := q_{h,r} - \frac{1}{|\Omega|}\int_\Omega q_{h,r} \in \mathrm{L}^r_0(\Omega)$, and let $u \in \widetilde{\mathrm{W}}^{1,r}(\Omega)$ be the unique solution of

$$\Delta u \;=\; q^0_{h,r} \quad \text{in} \quad \Omega\,, \qquad \nabla u \cdot \boldsymbol{\nu} \;=\; 0 \quad \text{on} \quad \Gamma\,, \qquad \int_\Omega u = 0\,, \tag{4.32}$$

so that in the convex case the proof is almost verbatim to the one for $i = 1$.

In turn, if Ω is non-convex, the fact that $\frac{\pi}{\omega} > \frac{1}{2} > 1 - \frac{2}{r}$ when $s > \frac{3}{2}$ (equivalently, when $r < 3$) allows us to apply Lemma B.1 to $t = r$ without further restrictions. In this way, we conclude that the solution u of (4.32) belongs to $\mathrm{W}^{1+\delta,r}(\Omega) \cap \widetilde{\mathrm{W}}^{1,r}(\Omega)$ for all $\delta \in (0, \delta_0)$, and it satisfies the analogue of (4.30). Note that the hypothesis of Lemma C.1 (cf. (C.3)) is clearly satisfied in this case as well. The rest of the proof proceeds as for the non-convex case of $i = 1$. We omit further details. □

It is important to remark here, as noticed within the previous proof, that in the case of a non-convex Ω, the discrete inf-sup condition for b_1, and hence our whole discrete analysis, is restricted to $s > \frac{3}{2}$

and to those polygonal regions with largest interior angle $\omega < s\pi$. Nevertheless, as illustrated by the numerical results reported later on in Section 5, which even consider $s = \frac{3}{2}$ and domains with $\omega \geq s\pi$, the above constraints seem to be only technical issues of the analysis rather than limitations of the applicability of the method.

We end this section with the discrete analogue of Lemma 2.9. Indeed, while this result is a simple modification of [22, eq. (5.64)], which in turn corresponds essentially to the vector version of [22, Lemma 5.5], in what follows we provide its full proof for sake of completeness of our analysis. Moreover, irrespective of the fact that ρ and its conjugate ϱ are now subject in 2D to the restriction $\rho > 4$, we prove the aforementioned inequality assuming arbitrary $\rho \in (2, +\infty)$ and $\varrho \in (1, 2)$ such that $1/\rho + 1/\varrho = 1$. To this end, we first invoke [22, Lemma 5.4] (with local choices there given by $p = \varrho$, $\ell = 0$, and $n = 2$) to deduce that there exists a constant $C_0 > 0$, independent of h, such that

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{0,\Omega} \,\leq\, C_0\, h^{2(1-1/\varrho)}\, |\boldsymbol{\tau}|_{1,\varrho;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{1,\varrho}(\Omega)\,. \tag{4.33}$$

The announced discrete inf-sup condition for our bilinear form b is proved next.

Lemma 4.5 *There exists $\widehat{\beta}_{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathrm{H}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{b(\boldsymbol{\tau}_h, \psi_h)}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}}} \,\geq\, \widehat{\beta}_{\mathtt{d}}\, \|\psi_h\|_{\mathrm{Q}} \qquad \forall\, \psi_h \in \mathrm{Q}_h\,. \tag{4.34}$$

Proof. Given $\psi_h \in \mathrm{Q}_h$, we let $\mathcal{O}$ be a convex domain contaning $\bar{\Omega}$, and set

$$g := \left\{ \begin{array}{cl} |\psi_h|^{\rho-2}\, \psi_h & \text{in } \Omega\,, \\ 0 & \text{in } \mathcal{O}\backslash\bar{\Omega}\,, \end{array} \right.$$

which is easily seen to belong to $\mathrm{L}^\varrho(\mathcal{O})$, with $\|g\|_{0,\varrho;\mathcal{O}} = \|g\|_{0,\varrho;\Omega} = \|\psi_h\|_{0,\rho;\Omega}^{\rho-1}$. It follows from [34, Corollary 1] that there exists a unique $z \in \mathrm{W}^{2,\varrho}(\mathcal{O}) \cap \mathrm{W}_0^{1,\varrho}(\mathcal{O})$ solution of

$$\Delta z \,=\, g \quad \text{in} \quad \mathcal{O}\,, \quad z = 0 \quad \text{on} \quad \partial\mathcal{O}\,, \tag{4.35}$$

and there exists a constant $C_{\mathtt{reg}} > 0$, depending only on $\mathcal{O}$, such that

$$\|z\|_{2,\varrho;\mathcal{O}} \,\leq\, C_{\mathtt{reg}}\, \|g\|_{0,\varrho;\Omega} = C_{\mathtt{reg}}\, \|\psi_h\|_{0,\rho;\Omega}^{\rho-1}\,. \tag{4.36}$$

Next, we let $\boldsymbol{\zeta} := \nabla z|_\Omega \in \mathrm{W}^{1,\varrho}(\Omega)$ and notice from (4.35) and (4.36) that

$$\mathrm{div}(\boldsymbol{\zeta}) = |\psi_h|^{\rho-2}\, \psi_h \quad \text{in} \quad \Omega \quad \text{and} \quad \|\boldsymbol{\zeta}\|_{1,\varrho;\Omega} \,\leq\, C_{\mathtt{reg}}\, \|\psi_h\|_{0,\rho;\Omega}^{\rho-1}\,. \tag{4.37}$$

Thus, defining $\boldsymbol{\zeta}_h := \Pi_h^k(\boldsymbol{\zeta}) \in \mathrm{H}_h$, applying (4.33) with $\ell = 1$, $n = 2$, and $\varrho \in (1, +\infty)$, and employing the continuous injection i_ϱ of $\mathrm{W}^{1,\varrho}(\Omega)$ into $\mathrm{L}^2(\Omega)$, and the inequality from (4.37), we deduce that

$$\begin{aligned} \|\boldsymbol{\zeta}_h\|_{0,\Omega} \,&\leq\, \|\boldsymbol{\zeta} - \Pi_h^k(\boldsymbol{\zeta})\|_{0,\Omega} \,+\, \|\boldsymbol{\zeta}\|_{0,\Omega} \,\leq\, C_0\, h^{2(1-1/\varrho)}\, |\boldsymbol{\zeta}|_{1,\varrho;\Omega} \,+\, \|i_\varrho\|\, \|\boldsymbol{\zeta}\|_{1,\varrho;\Omega} \\ &\leq\, (C_0 + \|i_\varrho\|)\, \|\boldsymbol{\zeta}\|_{1,\varrho;\Omega} \,\leq\, (C_0 + \|i_\varrho\|) C_{\mathtt{reg}}\, \|\psi_h\|_{0,\rho;\Omega}^{\rho-1}\,, \end{aligned} \tag{4.38}$$

where $h^{2(1-1/\varrho)}$ has been simply bounded by 1. In turn, we have

$$\mathrm{div}(\boldsymbol{\zeta}_h) = \mathcal{P}_h^k\big(\mathrm{div}(\boldsymbol{\zeta})\big) = \mathcal{P}_h^k(|\psi_h|^{\rho-2}\, \psi_h)\,,$$

so that proceeding exactly as for the derivation of (4.27b), we find that

$$\|\mathrm{div}(\boldsymbol{\zeta}_h)\|_{0,\varrho;\Omega} \,\le\, \widehat{C}\,\||\psi_h|^{\rho-2}\,\psi_h\|_{0,\varrho;\Omega} \,=\, \widehat{C}\,\|\psi_h\|_{0,\rho;\Omega}^{\rho-1}\,,$$

which, together with (4.38), give

$$\|\boldsymbol{\zeta}_h\|_{\mathrm{div}_\varrho;\Omega} \,\le\, \Big((C_0 + \|i_\varrho\|)C_{\mathtt{reg}} + \widehat{C}\Big)\,\|\psi_h\|_{0,\rho;\Omega}^{\rho-1}\,. \tag{4.39}$$

Finally, bounding below with $\boldsymbol{\zeta}_h$ and using the orthogonality property of $\mathcal{P}_h^k$, we conclude that

$$\sup_{\substack{\boldsymbol{\tau}_h\in \mathrm{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{b(\boldsymbol{\tau}_h,\psi_h)}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}}} \,\ge\, \frac{b(\boldsymbol{\zeta}_h,\psi_h)}{\|\boldsymbol{\zeta}_h\|_{\mathrm{H}}} \,=\, \frac{\kappa\displaystyle\int_\Omega \psi_h\,\mathcal{P}_h^k(|\psi_h|^{\rho-2}\,\psi_h)}{\|\boldsymbol{\zeta}_h\|_{\mathrm{div}_\varrho;\Omega}} \,=\, \frac{\kappa\|\psi_h\|_{0,\rho;\Omega}^{\rho}}{\|\boldsymbol{\zeta}_h\|_{\mathrm{div}_\varrho;\Omega}}\,,$$

from which, making use of (4.39), we arrive at (4.34) with $\widehat{\beta}_{\mathtt{d}} = \kappa\big((C_0 + \|i_\varrho\|)C_{\mathtt{reg}} + \widehat{C}\big)^{-1}$. □

4.5 The rates of convergence

In this section we provide the rates of convergence of our Galerkin scheme (3.1) with the specific finite element subspaces introduced in Section 4.2. For this purpose, we first collect the approximation properties of H_h, Q_h, $X_{2,h}$, and $M_{1,h}$ (cf. Section 4.2), which follow from (4.3a) (for $m=0$ and $t=\rho,r$), (4.6) (for $t=\varrho$), (4.7) (for $t=r$), and interpolation estimates of Sobolev spaces. More precisely, we have:

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) there exists $C>0$, independent of h, such that for each $\ell\in[1,k+1]$, and for each $\boldsymbol{\tau}\in\mathbf{H}^\ell(\Omega)$ with $\mathrm{div}(\boldsymbol{\tau})\in \mathrm{W}^{\ell,\varrho}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\tau},\mathrm{H}_h) \,:=\, \inf_{\boldsymbol{\tau}_h\in\mathrm{H}_h} \|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_{\mathrm{div}_\varrho;\Omega} \,\le\, C\,h^\ell\,\Big\{\|\boldsymbol{\tau}\|_{\ell,\Omega} \,+\, \|\mathrm{div}(\boldsymbol{\tau})\|_{\ell,\varrho;\Omega}\Big\}\,.$$

($\mathbf{AP}_h^{\varphi}$) there exists $C>0$, independent of h, such that for each $\ell\in[0,k+1]$, and for each $\psi\in \mathrm{W}^{\ell,\rho}(\Omega)$, there holds

$$\mathrm{dist}(\psi,\mathrm{Q}_h) \,:=\, \inf_{\psi_h\in\mathrm{Q}_h} \|\psi-\psi_h\|_{0,\rho;\Omega} \,\le\, C\,h^\ell\,\|\psi\|_{\ell,\rho;\Omega}\,.$$

($\mathbf{AP}_h^{\mathbf{u}}$) there exists $C>0$, independent of h, such that for each $\ell\in[1,k+1]$, and for each $\mathbf{v}\in \mathrm{W}^{\ell,r}(\Omega)$ with $\mathrm{div}(\mathbf{v})\in \mathrm{W}^{\ell,r}(\Omega)$, there holds

$$\mathrm{dist}(\mathbf{v},X_{2,h}) \,:=\, \inf_{\mathbf{v}_h\in X_{2,h}} \|\mathbf{v}-\mathbf{v}_h\|_{r,\mathrm{div}_r;\Omega} \,\le\, C\,h^\ell\,\Big\{\|\mathbf{v}\|_{\ell,r;\Omega} \,+\, \|\mathrm{div}(\mathbf{v})\|_{\ell,r;\Omega}\Big\}\,.$$

($\mathbf{AP}_h^{p}$) there exists $C>0$, independent of h, such that for each $\ell\in[0,k+1]$, and for each $q\in \mathrm{W}^{\ell,r}(\Omega)$, there holds

$$\mathrm{dist}(q,M_{1,h}) \,:=\, \inf_{q_h\in M_{1,h}} \|q-q_h\|_{0,r;\Omega} \,\le\, C\,h^\ell\,\|q\|_{\ell,r;\Omega}\,.$$

Hence, we can state the following main theorem.

Theorem 4.6 *Let* $\big((\boldsymbol{\sigma},\varphi),(\mathbf{u},p)\big) \in \big(\mathrm{H}\times\mathrm{Q}\big)\times\big(X_2\times M_1\big)$ *be the unique solution of* (2.31) *with* $\mathbf{u}\in S$ (*cf.* (2.62)), *and let* $\big((\boldsymbol{\sigma}_h,\varphi_h),(\mathbf{u}_h,p_h)\big)\in\big(\mathrm{H}_h\times\mathrm{Q}_h\big)\times\big(X_{2,h}\times M_{1,h}\big)$ *be a solution of* (3.1) *with* $\mathbf{u}_h\in S_h$ (*cf.* (3.11)), *whose existences are guaranteed by Theorems* 2.15 *and* 3.4, *respectively. Assume that* (3.25) (*cf. Theorem* 3.5) *holds, and that there exists* $\ell\in[1,k+1]$ *such that* $\boldsymbol{\sigma}\in\mathbf{H}^\ell(\Omega)$, $\mathrm{div}(\boldsymbol{\sigma})\in\mathrm{W}^{\ell,\varrho}(\Omega)$, $\varphi\in\mathrm{W}^{\ell,\rho}(\Omega)$, $\mathbf{u}\in\mathrm{W}^{\ell,r}(\Omega)$, $\mathrm{div}(\mathbf{u})\in\mathrm{W}^{\ell,r}(\Omega)$, *and* $p\in\mathrm{W}^{\ell,r}(\Omega)$. *Then, there exists a constant* $C>0$, *independent of* h, *such that*

$$\begin{aligned}&\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathrm{H}}+\|\varphi-\varphi_h\|_{\mathrm{Q}}+\|\mathbf{u}-\mathbf{u}_h\|_{X_2}+\|p-p_h\|_{M_1}\\&\quad\le C\,h^\ell\Big\{\|\boldsymbol{\sigma}\|_{\ell,\Omega}+\|\mathrm{div}(\boldsymbol{\sigma})\|_{\ell,\varrho;\Omega}+\|\varphi\|_{\ell,\rho;\Omega}+\|\mathbf{u}\|_{\ell,r;\Omega}+\|\mathrm{div}(\mathbf{u})\|_{\ell,r;\Omega}+\|p\|_{\ell,r;\Omega}\Big\}.\end{aligned}\tag{4.40}$$

Proof. It follows straightforwardly from Theorem 3.5 and the above approximation properties. □

5 Numerical results

We now address the numerical verification of the convergence properties of the proposed scheme (as stated in Section 4.5), as well as the usability of this new method in problems of applicative interest. In all results reported in this section the linear systems emanating from the Newton-Raphson linearisation have been solved with the unsymmetric multifrontal direct method for sparse matrices (UMFPACK).

The condition of zero-average for pressure needed in (4.8d) and (4.8f) is imposed through a real Lagrange multiplier.

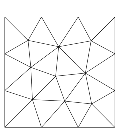
5.1 Test 1: accuracy verification on different domains

The choice of s and the geometry of the underlying domain play key roles in the discrete analysis of the method. More precisely, as pointed out after the proof of Lemma 4.4, the theoretical estimates require in 2D that s be greater than $\frac{3}{2}$ and the largest interior angle ω be less than $s\pi$; and the proofs of stability do not extend readily to 3D domains, as discussed after the proof of Lemma 4.2. In this example we explore these aspects numerically by considering $s=\frac{3}{2}$ and $s=\frac{8}{5}$, that is (cf. (2.19))

$$(\rho,\varrho,r,s)=(6,6/5,3,3/2)\quad\text{and}\quad(\rho,\varrho,r,s)=(8,8/7,8/3,8/5)\,,\tag{5.1}$$

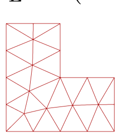
respectively, and using as domains a square Ω_{S}, an L-shaped domain Ω_{L} (having an inner angle of $3\pi/2$), a domain with an inner angle larger than $8\pi/5$, Ω_{V}, and the unit cube Ω_{C}. Following a manufactured solution approach, first we construct a sequence of successively refined unstructured partitions of the given domain (with h tending to zero), and consider the following closed-form synthetic solutions to (1.3) (in all of which we use the specification for temperature-dependent scaled viscosity $\mu(\varphi)=\mu_0+\frac{1}{2}\mu_0\varphi(\mu_1-\varphi)$)

in $\Omega_{\mathrm{S}}:=(-\pi,\pi)^2:\ \varphi=0.5(x_1^2+x_2^2)-0.25\sin(x_1)\cos(x_2),\quad\kappa=0.1,\quad\mu_0=0.5,\quad\mu_1=10,$



$$\mathbf{u}=\frac{1}{10}\begin{pmatrix}\cos(x_1)\sin(x_2)\\-\sin(x_1)\cos(x_2)\end{pmatrix},\quad p=\frac{1}{10}\sin(x_1x_2)e^{-0.1x_1x_2},\quad\boldsymbol{\sigma}=\kappa\nabla\varphi-\varphi\mathbf{u};$$

in $\Omega_{\mathrm{L}}:=(-1,1)^2\setminus(0,1)^2:\ \varphi=1+\sin(x_1)\sin(x_2),\quad\kappa=0.05,\quad\mu_0=0.1,\quad\mu_1=5,$



$$\mathbf{u}=\begin{pmatrix}\cos(x_1)\sin(x_2)\\-\sin(x_1)\cos(x_2)\end{pmatrix},\quad p=x_1^4-x_2^4,\quad\boldsymbol{\sigma}=\kappa\nabla\varphi-\varphi\mathbf{u};$$

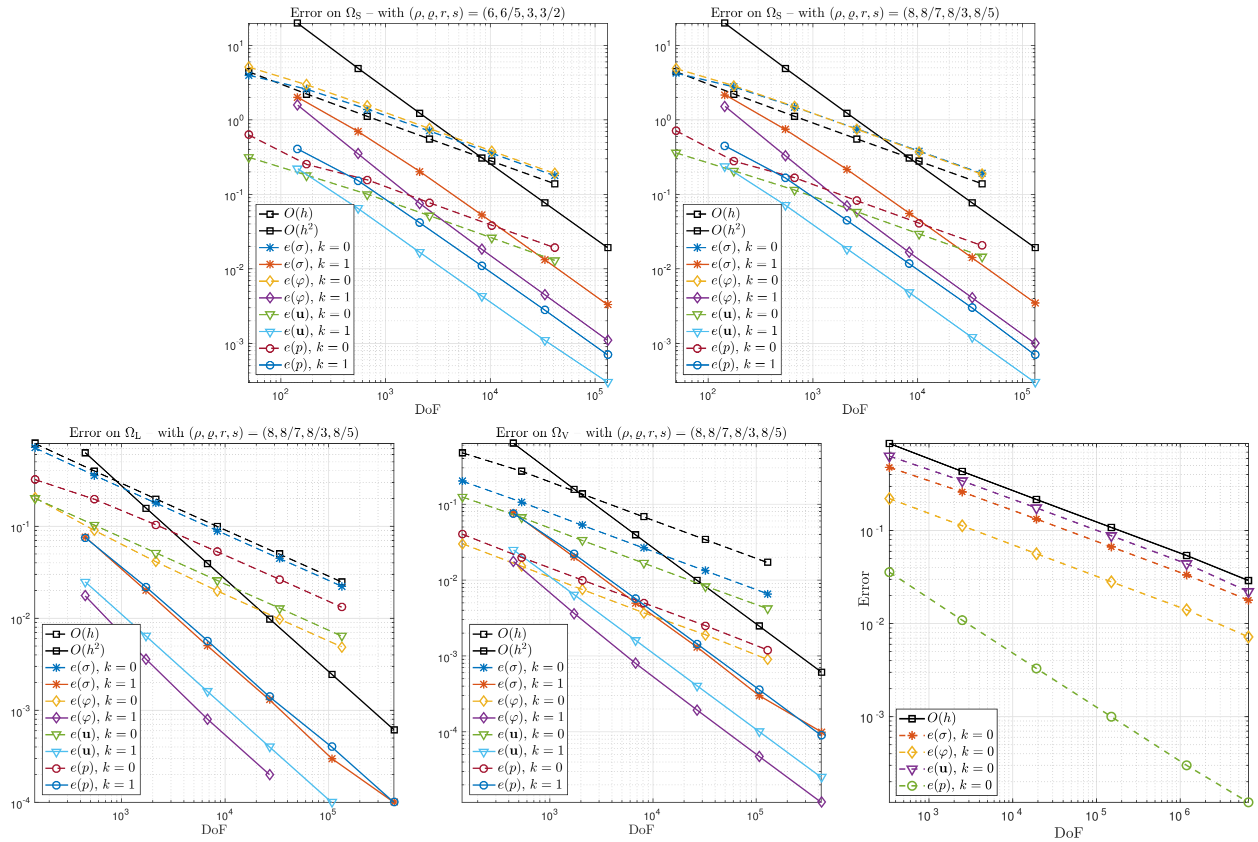
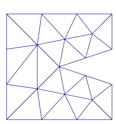


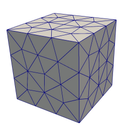
Figure 5.1: Test 1. Error history for pseudoheat flux, temperature, velocity, and pressure, showing convergence of the mixed finite element method on the domains Ω_{S} (top left and top right), Ω_{L} (bottom left), Ω_{V} (bottom centre), and Ω_{C} (bottom right).

$$\text{in } \Omega_{\mathrm{V}} := (0,1)^2 \setminus \Delta((\tfrac{1}{2},\tfrac{1}{2}),(1,\tfrac{1}{3}),(1,\tfrac{2}{3})),\ \varphi = 1 + \frac{3}{4}\cos(\frac{\pi}{4}x_1x_2), \quad \kappa = 0.01, \quad \mu_0 = 0.05, \quad \mu_1 = 3,$$



$$\mathbf{u} = \begin{pmatrix} \sin^2(\pi x_1)\sin^2(\pi x_2)\cos(\pi x_2) \\ -\frac{1}{3}\sin(2\pi x_1)\sin^3(\pi x_2) \end{pmatrix}, \quad p = \sin(x_1x_2)\cos(x_1x_2), \quad \boldsymbol{\sigma} = \kappa\nabla\varphi - \varphi\mathbf{u};$$

$$\text{in } \Omega_{\mathrm{C}} := (0,1)^3:\ \varphi = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) - \frac{1}{4}\sin(x_1)\cos(x_2)\cos(x_3), \quad \kappa = 0.1, \quad \mu_0 = 1, \quad \mu_1 = 10,$$



$$\mathbf{u} = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix}, \ p = \sin(x_1x_2x_3)e^{-0.1x_1x_2x_3}, \ \boldsymbol{\sigma} = \kappa\nabla\varphi - \varphi\mathbf{u}.$$

Source terms (and for the cases that require it, also the non-homogeneous boundary conditions for the normal trace of velocity that are prescribed essentially, and for temperature φ_D which are prescribed weakly by adding the contribution $-\kappa\langle\boldsymbol{\tau}\cdot\boldsymbol{\nu},\varphi_D\rangle_\Gamma$) are imposed using these exact solutions.

The finite element spaces are specified as in (4.8). In addition to RT elements composing (4.8a), (4.8c), (4.8e), we have also tested the convergence with BDM elements, and no substantial differences are observed. We therefore refer only to RT-based families in the plots below.

In Figure 5.1 we collectively show the error history for each case, including computed errors on each refinement level, for two different polynomial degrees $k = 0, 1$, and separating each individual

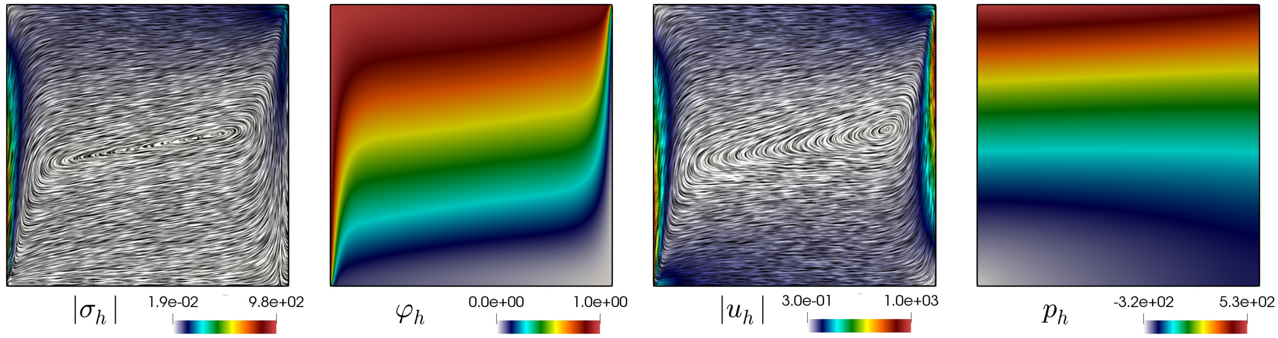


Figure 5.2: Test 2. Porous enclosure heated from the side, with Ra = 1500. Approximate pseudoheat flux and line integral contour, temperature, velocity magnitude with line integral contour, and pressure distribution at $t = 0.5$.

contribution to the total error

$$e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{H}}, \quad e(\varphi) := \|\varphi - \varphi_h\|_{\mathrm{Q}}, \quad e(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{X_2}, \quad e(p) := \|p - p_h\|_{M_1}.$$

And these errors are computed in the norms that use the values from (5.1). For Ω_{S} we use the two sets of values, whereas for $\Omega_{\mathrm{L}}, \Omega_{\mathrm{V}}, \Omega_{\mathrm{C}}$ we use the second set of values (with $s = 8/5$).

The plotted accuracy trends in the top-left panel demonstrate numerically the optimal convergence order anticipated by Theorem 4.6, and a similar conclusion is drawn when testing the accuracy in the domains for which the analysis does not carry over. As usual, a local error decay rate can be obtained, for a generic pair of individual errors $e, \widehat{e}$ generated by the mixed method on meshes associated with meshsizes h and $\widehat{h}$, as $\texttt{rate} = \log\big(e(\cdot)/\widehat{e}(\cdot)\big)[\log(h/\widehat{h})]^{-1}$, and then an average value can be taken for each error history. Alternatively, one can visually compare the convergence against the optimal values in the solid lines of each panel. For instance, for the 3D domain Ω_{C} we can infer a slightly higher convergence for pressure (of about $O(h^{1.45})$). For all these runs, the maximum number of iterations required over the course of the Newton-Raphson loop (which is terminated once the nonlinear residual discrete norm drops below a relative tolerance of 10^{-6}) was 5.

5.2 Test 2: application to buoyancy-driven flow in porous media

In order to study an application into heat and fluid flow in non-isothermal porous media, we extend the model to the classical pseudo-steady case, by adding the rate of change of temperature to the left-hand side of the thermal energy conservation equation: $\partial_t\varphi + \mathbf{u}\cdot\nabla\varphi - \kappa\Delta\varphi = f$ (or $-\partial_t\varphi + \mathrm{div}(\boldsymbol{\sigma}) = -f$ in the context of (1.3)). After non-dimensionalisation, the system regime can be fully described by the Rayleigh number Ra (combining the effects of gravity, permeability, characteristic length, viscosity, and thermal conductivity), and the temperature-dependent viscosity is $\mu(\varphi) = \exp(\varphi)$. The momentum equation has an additional term on the right-hand side, depending on temperature (due to the Boussinesq approximation relating density and temperature), $\mathbf{f} = \varphi\begin{pmatrix}0\\1\end{pmatrix}$. The test configuration and parameter values are taken from [50], where a square porous layer is held between differentially heated sidewalls. The four walls are impermeable, resulting in the condition $\mathbf{u}\cdot\boldsymbol{\nu} = 0$ everywhere on the boundary, and therefore a Lagrange multiplier is used to enforce pressure uniqueness. The temperature boundary conditions adopted for this test are of mixed type, and they differ from those in (1.1): on the left and right sidewalls, normalised temperatures of $\varphi = 1$ and $\varphi = 0$ are imposed, respectively; whereas on the top and bottom walls we set $\boldsymbol{\sigma}\cdot\boldsymbol{\nu} = 0$. The fully-discrete problem

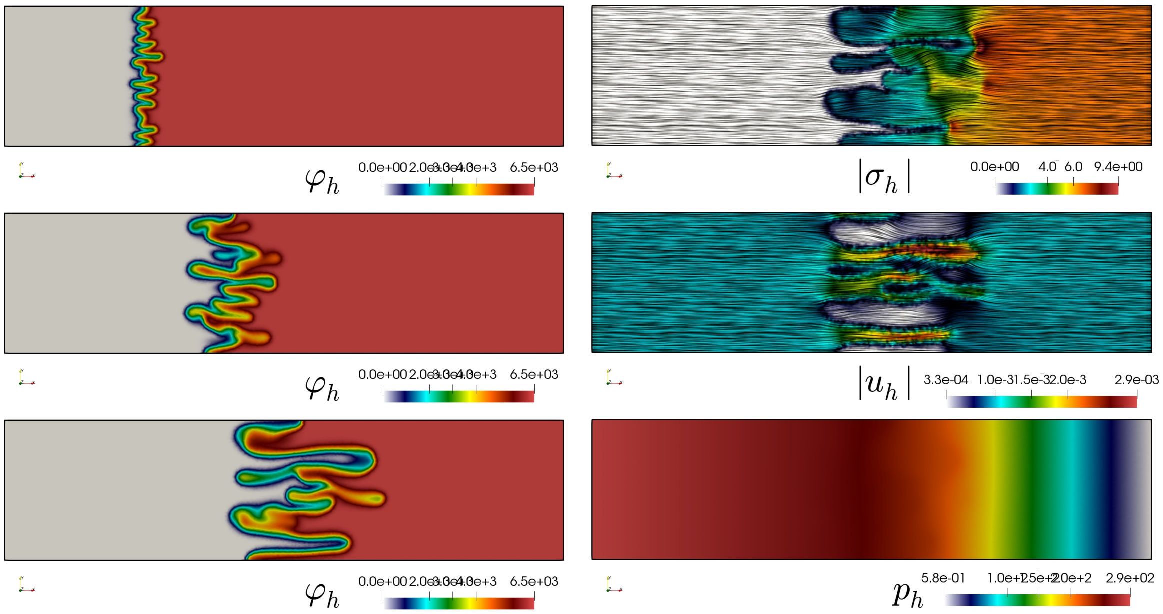


Figure 5.3: Test 3A. Evolution of the concentration in viscous fingering at $t = 5, 10, 15\,\mathrm{s}$ (left panels), and snapshots of total flux, velocity, and pressure at the final time (right column).

resulting from a simple backward Euler time discretisation with constant time step, adopts a form similar to (3.1). The computational domain is the unit square, discretised into a uniform mesh of 40K triangles, and for the lowest-order scheme the method has 320801 DoFs. We use a constant time step $\Delta t = 0.01$ and prescribe a Rayleigh number of 1500, and the approximate solutions on the enclosure heated from the left side, after 50 time steps are shown in Figure 5.2.

5.3 Test 3: applications to Darcian miscible displacement and viscous fingering

To conclude we note that if φ is understood as a species concentration rather than temperature, then equations (1.1) can be used to describe flow displacement in Hele-Shaw cells (see, e.g., [6, 57]) where one injects water into another viscous fluid of different viscosity (and with viscosity ratio of $r = 2$). Starting from the initial distribution of concentration $\varphi(x_1, x_2, 0) = \frac{\varphi_2}{2}\left[1 + \mathrm{erf}\left(\frac{x_1 - 0.01}{4\times 10^{-4}}\right)\right]$, eventually the existing fluid is displaced and so-called viscous fingering instabilities are formed (for this there is no need to prescribe a random perturbation, as the unstructured mesh is sufficient to onset the required instabilities near the initial interface between the two fluids). The computational domain is the channel $\Omega = (0, 0.08) \times (0, 0.02)\,\mathrm{m}^2$. The field φ is now interpreted as concentration of the fluid to be displaced (and measured in $\mathrm{mol/m}^3$). The left side of the domain is the inlet boundary where we impose $\mathbf{u} \cdot \boldsymbol{\nu} = -0.001\mathrm{m/s}$ as inlet velocity and $\varphi = 0$ as inlet concentration (since the second fluid, water, is being injected from that segment). On the horizontal walls of the channel we impose $\mathbf{u} \cdot \boldsymbol{\nu} = \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = 0$ and on the outlet (the right end of the channel) we set $p = 0$ and zero diffusive flux (implying that $[\boldsymbol{\sigma} + \varphi\mathbf{u}] \cdot \boldsymbol{\nu} = 0$). The model parameters are (see, e.g., [6, 58])

$$\kappa = 4 \times 10^{-8}\,\mathrm{m}^2/\mathrm{s}, \quad l_{\mathrm{poro}} = 0.5, \quad l_{\mathrm{mob}} = 2, \quad l_{\mathrm{visc}} = 1\,\mathrm{mPa{\cdot}s}, \quad l_{\mathrm{perm}} = 10^{-6}\,\mathrm{m}^2,$$
$$\varphi_2 = 6500\,\mathrm{mol/m}^3, \quad \mu(\varphi) = \frac{l_{\mathrm{visc}}}{l_{\mathrm{perm}}} \exp(l_{\mathrm{mov}}\varphi/\varphi_2),$$

and they represent diffusivity, porosity, log-mobility ratio, viscosity of the displacing fluid, permeability of the porous medium, reference concentration of the displaced solute, and concentration-dependent

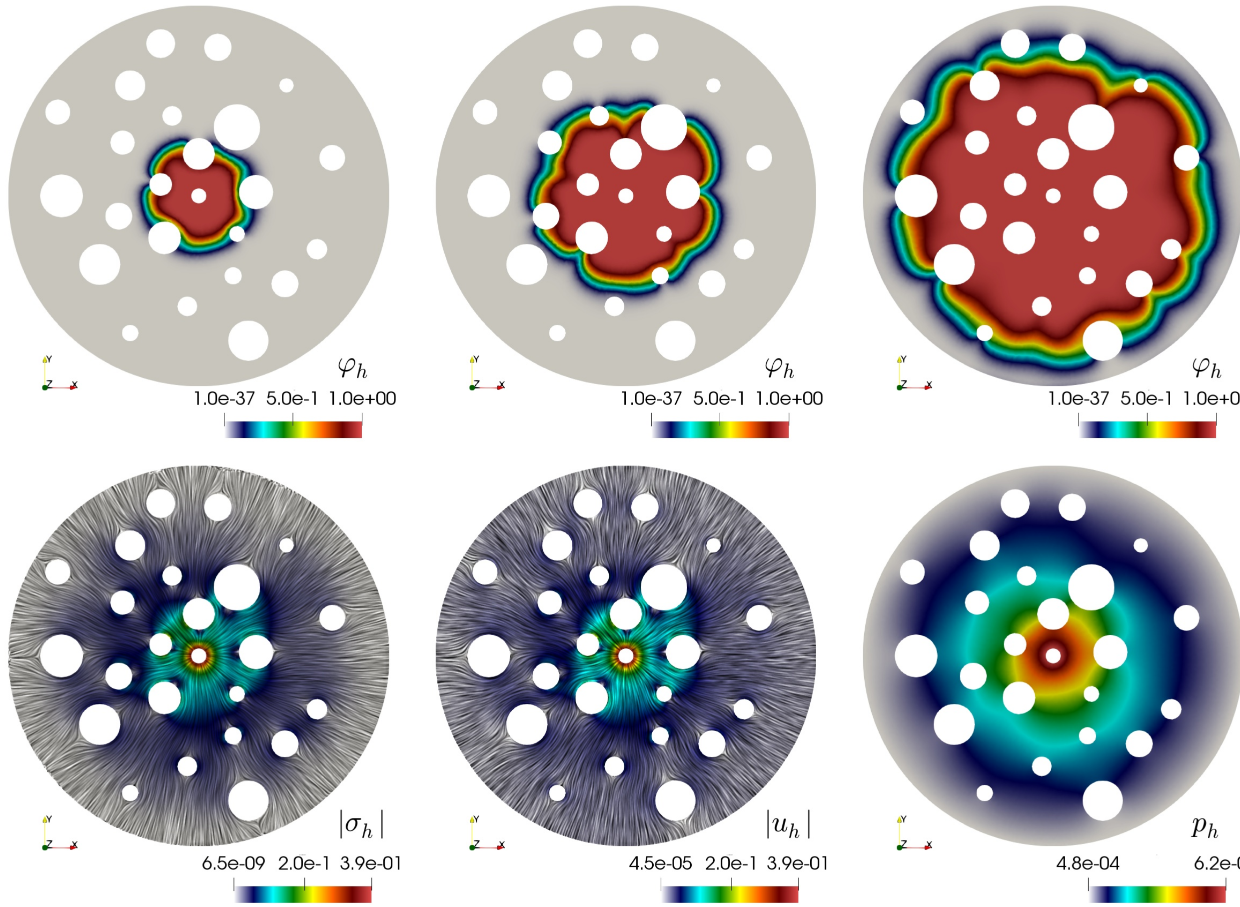


Figure 5.4: Test 3B. Evolution of the concentration in miscible displacement in porous media at adimensional times $t = 10, 30, 90$ (top), and snapshots of total flux, velocity, and pressure at the final time (bottom).

Arrhenius viscosity law (scaled with permeability), respectively. We use an unstructured mesh of 37745 triangles, set a constant timestep of $\Delta t = 0.01\,\mathrm{s}$ and run the simulations until $t = 10\,\mathrm{s}$.

Next we conduct a very similar test but the displaced fluid is water and a fluid with higher viscosity is injected. The domain is an annular region (of radii 0.2 and 5, in adimensional units) with many holes of random location and size. The inlet and outlet are the inner and outer circles, respectively. We prescribe an inlet velocity $\mathbf{u} \cdot \boldsymbol{\nu} = -1$ and inlet concentration $\varphi = 1$, on the outlet we set zero pressure $p = 0$ and zero diffusive flux $[\boldsymbol{\sigma} + \varphi\mathbf{u}] \cdot \boldsymbol{\nu} = 0$, and on the remainder of the boundaries the fluids are allowed to slip, and zero total flux is imposed $\boldsymbol{\sigma} \cdot \boldsymbol{\nu} = 0$. The set of equations is in dimensionless form, depending on the Péclet number Pe$= 750$, from which $\kappa = 1/\mathrm{Pe}$, and the scaled viscosity follows a quarter-power mixing rule $\mu(\varphi) = 1 + (\varphi + 1.18(1-\varphi))^{-4}$. The mesh has 34683 triangular elements, the timestep is $\Delta t = 0.5$ and the computation is evolved until $t = 100$.

The results of both tests are collected in Figures 5.3-5.4, showing snapshots of concentration at different times, as well as examples of total fluxes, velocities and pressures at the final time. And we emphasize that a key benefit offered by the proposed mixed-mixed formulation is the conservativity of the resulting scheme.

Acknowledgements. We are very thankful to Professor Dr. Ricardo Durán for pointing to and clarifying most of the results on the stability of the Ritz projector that are collected in Appendix A.

In addition, we also express our deep gratitude to Professor Dr. Monique Dauge for providing through [25] and [26] most details regarding the regularity result discussed in Appendix B.

A $\mathrm{L}^t(\Omega)$-stability of the Ritz projection

Given an integer $k \geq 0$, we let $\mathrm{P}_{k+1,c}(\mathcal{T}_h)$ be the space of continuous piecewise polynomials of degree $\leq k+1$, that is

$$\mathrm{P}_{k+1,c}(\mathcal{T}_h) := \Big\{ \phi_h \in \mathrm{H}_0^1(\Omega) : \quad \phi_h|_K \in \mathrm{P}_{k+1}(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \tag{A.1}$$

and consider the Ritz projection $\mathcal{R}_h^k : \mathrm{H}_0^1(\Omega) \longrightarrow \mathrm{P}_{k+1,c}(\mathcal{T}_h)$ associated with the Poisson equation under homogeneous Dirichlet boundary conditions. In other words, given $\phi \in \mathrm{H}_0^1(\Omega)$, $\mathcal{R}_h^k(\phi)$ is the unique element in $\mathrm{P}_{k+1,c}(\mathcal{T}_h)$ satisfying

$$\int_\Omega \nabla \mathcal{R}_h^k(\phi) \cdot \nabla \phi_h \,=\, \int_\Omega \nabla \phi \cdot \nabla \phi_h \qquad \forall\, \phi_h \in \mathrm{P}_{k+1,c}(\mathcal{T}_h), \tag{A.2}$$

and hence

$$\|\nabla \mathcal{R}_h^k(\phi)\|_{0,\Omega} \,\leq\, \|\nabla \phi\|_{0,\Omega}. \tag{A.3}$$

Note that the fact that $\mathrm{P}_{k+1,c}(\mathcal{T}_h)$ is contained in $\mathrm{W}^{1,t}(\Omega)$ for all $t \in [1,+\infty]$, guarantees that $\mathcal{R}_h^k$ is actually well-defined in each one of these spaces as well. In this regard, we stress that stability estimates as (A.3), but measured with respect to $\|\cdot\|_{0,t;\Omega}$, $t \neq 2$, though less known, are also available in the literature. One of the first results in this direction goes back to [52, Theorem], where the aforementioned estimate is established for $k = 0$ and $t \in [2,+\infty]$ in the 2D case. More precisely, if Ω is a convex polygonal region of R^2, then for each $t \in [2,+\infty]$ there exists a positive constant C_t^0, independent of h, such that

$$\|\nabla \mathcal{R}_h^0(\phi)\|_{0,t;\Omega} \,\leq\, C_t^0\, \|\nabla \phi\|_{0,t;\Omega} \qquad \forall\, \phi \in \mathrm{W}_0^{1,t}(\Omega). \tag{A.4}$$

Actually, this result is provided in [52, Theorem] in terms of $\|\mathcal{R}_h^0(\phi)\|_{1,t;\Omega}$ and $\|\phi\|_{1,t;\Omega}$, which, due to the equivalence between $\|\cdot\|_{1,t;\Omega}$ and $|\cdot|_{1,t;\Omega}$ in $\mathrm{W}_0^{1,t}(\Omega)$, becomes (A.4). In addition, employing a duality argument (as explained for instance in [13, Section 8.5]), it is not difficult to show that (A.4) is also valid for $t \in (1,2]$. In turn, for the corresponding extension of all the above to any integer $k \geq 1$, we refer to [13, Theorem 8.5.3], whose proof, based on a suitable regularity assumption (cf. [13, eqs. (8.1.2) and (8.1.3)]), follows basically the same technique from [52]. However, whereas the aforementioned hypothesis is satisfied for an arbitrary convex polygonal region in R^2, it requires a maximum interior angle condition in R^3. This difficulty is overcome in [42, eqs. (1.2) and (1.3)] by employing arguments based on Green's functions, which yields the respective stability for $t = +\infty$ (see also [48]). In this way, the interpolation of the latter with (A.3) implies the result for $t \in [2,+\infty]$, and the same duality argument from [13, Section 8.5] allows to extend it to $t \in (1,2]$. Summarising, thanks to the analysis and results from [13], [42], and [52], we know that, given an integer $k \geq 0$ and a convex polygonal (resp. polyhedral) region Ω of R^2 (resp. R^3), for each $t \in (1,+\infty]$ there exists a positive constant C_t^k, independent of h, such that

$$\|\nabla \mathcal{R}_h^k(\phi)\|_{0,t;\Omega} \,\leq\, C_t^k\, \|\nabla \phi\|_{0,t;\Omega} \qquad \forall\, \phi \in \mathrm{W}_0^{1,t}(\Omega). \tag{A.5}$$

For further results on the stability of $\mathcal{R}_h^k$ in convex polygonal regions of R^2, we refer for instance to the recent works [47] and [49], which consider the cases of mixed boundary conditions and graded meshes, respectively.

On the other hand, in the case of arbitrary polygonal domains Ω in R^2, not necessarily convex, one easily proves, starting from [56, eq. (0.7), Theorem 2], that, given an integer $k \geq 0$, there exists a positive constant $\bar{C}^k_\infty$, independent of h, such that

$$\|\nabla \mathcal{R}^k_h(\phi)\|_{0,\infty;\Omega} \,\leq\, \bar{C}^k_\infty \left\{ -\log(h) \right\}^{\mathtt{r}(k)} \|\nabla \phi\|_{0,\infty;\Omega} \qquad \forall\, \phi \in \mathrm{W}^{1,\infty}_0(\Omega)\,, \tag{A.6}$$

where $\mathtt{r}(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}$. Then, interpolating (A.6) with (A.3) we find that for each $t \in [2, +\infty]$ there exists a positive constant $\bar{C}^k_t$, independent of h, such that

$$\|\nabla \mathcal{R}^k_h(\phi)\|_{0,t;\Omega} \,\leq\, \bar{C}^k_t \left\{ -\log(h) \right\}^{\mathtt{r}(k)(1-2/t)} \|\nabla \phi\|_{0,t;\Omega} \qquad \forall\, \phi \in \mathrm{W}^{1,t}_0(\Omega)\,. \tag{A.7}$$

Moreover, applying again the duality argument from [13, Section 8.5], we deduce that for each $t \in (1, 2]$ there exists a positive constant $\bar{C}^k_t$, independent of h, such that

$$\|\nabla \mathcal{R}^k_h(\phi)\|_{0,t;\Omega} \,\leq\, \bar{C}^k_t \left\{ -\log(h) \right\}^{\mathtt{r}(k)(-1+2/t)} \|\nabla \phi\|_{0,t;\Omega} \qquad \forall\, \phi \in \mathrm{W}^{1,t}_0(\Omega)\,, \tag{A.8}$$

so that we summarise (A.7) and (A.8) by simply stating that for each $t \in (1, +\infty]$ there exists a positive constant $\bar{C}^k_t$, independent of h, such that

$$\|\nabla \mathcal{R}^k_h(\phi)\|_{0,t;\Omega} \,\leq\, \bar{C}^k_t \left\{ -\log(h) \right\}^{\mathtt{r}(k)|1-2/t|} \|\nabla \phi\|_{0,t;\Omega} \qquad \forall\, \phi \in \mathrm{W}^{1,t}_0(\Omega)\,. \tag{A.9}$$

B A Neumann regularity result on non-convex domains

We now let Ω be a non-convex polygonal region of R^2, and establish, with $\delta > 0$ and $t \in (1, +\infty)$, a $\mathrm{W}^{1+\delta,t}(\Omega)$-regularity result for the Poisson problem with source term in $\mathrm{L}^t_0(\Omega)$ and homogeneous Neumann boundary conditions. More precisely, defining $\widetilde{\mathrm{H}}^1(\Omega) := \Big\{ v \in \mathrm{H}^1(\Omega) : \int_\Omega v = 0 \Big\}$, and letting $\mathcal{N} : \widetilde{\mathrm{H}}^1(\Omega)' \longrightarrow \widetilde{\mathrm{H}}^1(\Omega)$ be the bounded linear operator that assigns to $f \in \widetilde{\mathrm{H}}^1(\Omega)'$ the unique solution $u_f \in \widetilde{\mathrm{H}}^1(\Omega)$ of the problem

$$\int_\Omega \nabla u_f \cdot \nabla v \,=\, f(v) \qquad \forall\, v \in \widetilde{\mathrm{H}}^1(\Omega)\,,$$

we are interested in providing conditions under which there exists $\delta > 0$ such that $\mathcal{N}$ can also be continuously defined from $\mathrm{L}^t_0(\Omega)$ into $\mathrm{W}^{1+\delta,t}(\Omega)$. Note that this means that for each $q \in \mathrm{L}^t_0(\Omega)$ there exists a unique weak solution $u \in \mathrm{W}^{1+\delta,t}(\Omega) \cap \widetilde{\mathrm{W}}^{1,t}(\Omega)$ of the boundary value problem

$$\Delta u \,=\, q \quad \text{in} \quad \Omega\,, \quad \nabla u \cdot \boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma\,, \quad \int_\Omega u \,=\, 0\,,$$

which satisfies

$$\|u\|_{1+\delta,t;\Omega} \,\leq\, \|\mathcal{N}\|\, \|q\|_{0,t;\Omega}\,.$$

In order to prove this regularity result we basically follow [26] and make use of [25, Corollary (23.5)], which says that $\mathcal{N}$ is continuous from $\mathrm{H}^{s-1}(\Omega)$ to $\mathrm{H}^{s+1}(\Omega)$ for each $s \in [0, \frac{\pi}{\omega})$, where ω stands for the largest interior corner angle of Ω. Indeed, we have the following result.

Lemma B.1 *Assume that $t \in (1, +\infty)$ is such that*

$$\frac{\pi}{\omega} > 1 - \frac{2}{t} \quad \textit{if} \quad t \,\geq\, 2\,, \tag{B.1}$$

and set

$$\delta_0 := \begin{cases} \min\left\{1, \dfrac{\pi}{\omega} + \dfrac{2}{t} - 1\right\} & \textit{if} \quad t \ge 2\,, \\ \min\left\{2 - \dfrac{2}{t}, \dfrac{\pi}{\omega}\right\} & \textit{if} \quad t \in (1,2)\,. \end{cases} \tag{B.2}$$

Then $\mathcal{N} : \mathrm{L}_0^t(\Omega) \longrightarrow \mathrm{W}^{1+\delta,t}(\Omega)$ *is continuous for each* $\delta \in (0, \delta_0)$.

Proof. Let us first assume that $t \ge 2$. Then the continuous embeddings $i_0 : \mathrm{L}_0^t(\Omega) \longrightarrow \mathrm{L}^2(\Omega)$ and $i_s : \mathrm{L}^2(\Omega) \longrightarrow \mathrm{H}^{s-1}(\Omega)$, for $s \le 1$, are straightforward. In addition, employing the aforementioned regularity result for $\mathcal{N}$, and noting that for the non-convex domain Ω there holds $\frac{\pi}{\omega} \le 1$, we deduce the continuity of $\mathcal{N} : \mathrm{L}_0^t(\Omega) \longrightarrow \mathrm{H}^{s+1}(\Omega)$ for each $s \in [0, \frac{\pi}{\omega})$, which is depicted by the following sequence

$$\mathrm{L}_0^t(\Omega) \xrightarrow{i_0} \mathrm{L}^2(\Omega) \xrightarrow{i_s} \mathrm{H}^{s-1}(\Omega) \xrightarrow{\mathcal{N}} \mathrm{H}^{s+1}(\Omega)\,.$$

In turn, according to the embedding between fractional Sobolev spaces (cf. [33, Theorem 6, Section 5.6], [41, Theorem 1.4.5.2, part e)]), we know that $i_\delta : \mathrm{H}^{s+1}(\Omega) \longrightarrow \mathrm{W}^{1+\delta,t}(\Omega)$ is continuous if

$$s \,=\, 1 + \delta - \frac{2}{t} \quad \text{and} \quad s \,\ge\, \delta\,.$$

The former holds for some $\delta > 0$ if $s > 1 - \frac{2}{t}$, whereas the later is guaranteed by the former and the fact that $t \ge 2$. Hence, bearing in mind our hypothesis on t, the feasible range for s becomes the interval $(1 - \frac{2}{t}, \frac{\pi}{\omega})$, equivalently $\delta := s - \big(1 - \frac{2}{t}\big) \in (0, \frac{\pi}{\omega} + \frac{2}{t} - 1)$, which, together with the fact that $\delta \le s \le 1$, yields $\delta \in (0, \delta_0)$, and hence the required continuity of $\mathcal{N}$ follows from the diagram

$$\mathrm{L}_0^t(\Omega) \xrightarrow{\mathcal{N}} \mathrm{H}^{s+1}(\Omega) \xrightarrow{i_\delta} \mathrm{W}^{1+\delta,t}(\Omega)\,.$$

Furthermore, given $t \in (1,2)$, we employ again [33, Theorem 6, Section 5.6] (see also [41, Theorem 1.4.5.2, part e)]) to observe that the injection $i_s : \mathrm{L}_0^t(\Omega) \longrightarrow \mathrm{H}^{-s}(\Omega)$ is continuous if $s \ge \frac{2}{t} - 1$, that is $1 - s \le 2 - \frac{2}{t}$. In turn, $\mathcal{N} : \mathrm{H}^{-s}(\Omega) \longrightarrow \mathrm{H}^{2-s}(\Omega)$ is continuous if $1 - s \in [0, \frac{\pi}{\omega})$, whereas $\mathrm{H}^{2-s}(\Omega)$ is continuously embedded in $\mathrm{H}^{1+t}(\Omega)$ if $2 - s \ge 1 + \delta$, that is $1 - s \ge \delta$. Hence, noticing from the present range of t that $\mathrm{H}^{1+t}(\Omega)$ is continuously embedded in $\mathrm{W}^{1+\delta,t}(\Omega)$, we conclude that $i_\delta : \mathrm{H}^{2-s}(\Omega) \longrightarrow \mathrm{W}^{1+\delta,t}(\Omega)$ is continuous as well. In this way, the announced continuity of $\mathcal{N}$ follows from the above constraints on $1 - s$ and δ, and the sequence

$$\mathrm{L}_0^t(\Omega) \xrightarrow{i_s} \mathrm{H}^{-s}(\Omega) \xrightarrow{\mathcal{N}} \mathrm{H}^{2-s}(\Omega) \xrightarrow{i_\delta} \mathrm{W}^{1+\delta,t}(\Omega)\,.$$

□

C Further properties of the Raviart-Thomas interpolators

In this appendix we establish additional stability and approximation properties of the local and global Raviart-Thomas interpolation operators. To this end, we now denote the reference triangle of $\mathcal{T}_h$ by $\widehat{K}$, so that, given $K \in \mathcal{T}_h$, we let $F_K : \widehat{K} \longrightarrow K$ be the bijective affine mapping defined by $F_K(\mathbf{x}) := B_K\,\mathbf{x} + b_K \quad \forall\,\mathbf{x} \in \widehat{K}$, with $B_K \in \mathrm{R}^{2\times 2}$ invertible and $b_K \in \mathrm{R}^2$. Next, given an integer $k \ge 0$ and a side $\widehat{F}$ of $\partial\widehat{K}$, we let d_k and $\big\{\widehat{\varphi}_{\ell,\widehat{F}}\big\}_{\ell=1}^{d_k}$ be the dimension and a basis of $\mathrm{P}_k(\widehat{F})$, respectively. Similarly, when $k \ge 1$, we let r_k be the dimension of $\mathbf{P}_{k-1}(\widehat{K})$ and denote by $\big\{\widehat{\boldsymbol{\psi}}_\ell\big\}_{\ell=1}^{r_k}$ a corresponding basis. Then, for each $\widehat{\boldsymbol{\tau}} \in \mathrm{W}^{1,t}(\widehat{K})$, with $t \in (1, +\infty)$, we formally define the $\widehat{F}$-moments for $k \ge 0$ as

$$m_{\ell,\widehat{F}}(\widehat{\boldsymbol{\tau}}) \,:=\, \int_{\widehat{F}} \widehat{\boldsymbol{\tau}} \cdot \boldsymbol{\nu}\, \widehat{\varphi}_{\ell,\widehat{F}} \qquad \forall\,\ell \in \big\{1, 2, \ldots, d_k\big\}\,, \tag{C.1}$$

whereas the $\widehat{K}$-moments for $k \geq 1$ are given by

$$m_{\ell,\widehat{K}}(\widehat{\boldsymbol{\tau}}) := \int_{\widehat{K}} \widehat{\boldsymbol{\tau}} \cdot \widehat{\boldsymbol{\psi}}_\ell \qquad \forall\, \ell \in \{1, 2, \ldots, r_k\}.$$

In addition, gathering all the $\widehat{F}$ and $\widehat{K}$ moments in the set of linear functionals $\widehat{m}_j$, $j \in \{1, 2, \ldots, N_k\}$, with $N_k := 3\, d_k + r_k$, for each $i \in \{1, 2, \ldots, N_k\}$ we let $\widehat{\boldsymbol{\tau}}_i$ be the unique function in $\mathrm{RT}_k(\widehat{K})$ such that

$$\widehat{m}_j(\widehat{\boldsymbol{\tau}}_i) = \delta_{ij} \quad \forall\, j \in \{1, 2, \ldots, N_k\},$$

and introduce the reference Raviart-Thomas interpolation operator $\Pi^k_{\widehat{K}} : \mathrm{W}^{1,t}(\widehat{K}) \longrightarrow \mathrm{RT}_k(\widehat{K})$ as

$$\Pi^k_{\widehat{K}}(\widehat{\boldsymbol{\tau}}) := \sum_{j=1}^{N_k} \widehat{m}_j(\widehat{\boldsymbol{\tau}})\, \widehat{\boldsymbol{\tau}}_j \qquad \forall\, \widehat{\boldsymbol{\tau}} \in \mathrm{W}^{1,t}(\widehat{K}). \tag{C.2}$$

Proceeding analogously, one defines on each $K \in \mathcal{T}_h$ the local Raviart-Thomas interpolation operator $\Pi^k_K : \mathrm{W}^{1,t}(K) \longrightarrow \mathrm{RT}_k(K)$, which is related to $\Pi^k_{\widehat{K}}$ through the identity

$$\Pi^k_{\widehat{K}}(\widehat{\boldsymbol{\tau}}) = \widehat{\Pi^k_K(\tau)} := |\det(B_K)|\, B_K^{-1}\, \Pi^k_K(\boldsymbol{\tau}) \circ F_K \quad \forall\, \boldsymbol{\tau} \in \mathrm{W}^{1,t}(K),$$

where $\widehat{\ }$ denotes from now on the Piola transformation.

The stability and approximation properties of Π^k_K, measured with respect to $\mathrm{W}^{m,t}(K)$-norms, with integer $m \geq 0$ and $t \in (1, +\infty)$, are well-known for sufficiently smooth functions (see, e.g. Section 4.1 for the corresponding global versions of them). Here we are interested in establishing similar estimates measured in $\mathrm{L}^t(K)$-norms, but for less smooth functions. For this purpose, we need the result provided by the following lemma.

Lemma C.1 *Let $t \in (1, +\infty)$, $t \neq 2$, and $\delta \in [0, 1]$ such that*

$$\begin{cases} \delta > \dfrac{1}{t} & \text{if } t \in (1, 2), \\ \delta \geq 0 & \text{if } t \in (2, +\infty). \end{cases} \tag{C.3}$$

Then, there exists a constant $\widehat{C} > 0$, independent of h, such that

$$\|\Pi^k_{\widehat{K}}(\widehat{\boldsymbol{\tau}})\|_{0,t;\widehat{K}} \leq \widehat{C} \Big\{ \|\widehat{\boldsymbol{\tau}}\|_{\delta,t;\widehat{K}} + \|\mathrm{div}(\widehat{\boldsymbol{\tau}})\|_{0,t;\widehat{K}} \Big\} \qquad \forall\, \widehat{\boldsymbol{\tau}} \in \mathbf{W}^{\delta,t}(\widehat{K}) \cap \mathbf{H}^t(\mathrm{div}_t; \widehat{K}). \tag{C.4}$$

Proof. We first realise that the moments $\widehat{m}_j$, $j \in \{1, 2, \ldots, N_k\}$, are well-defined and constitute bounded linear functionals in $\mathbf{W}^{\delta,t}(\widehat{K}) \cap \mathbf{H}^t(\mathrm{div}_t; \widehat{K})$. In fact, the above is straightforward for the $\widehat{K}$-moments since $m_{\ell,\widehat{K}}$ is clearly linear for each $\ell \in \{1, 2, \ldots, r_k\}$, and, thanks to Hölder's inequality, there holds

$$|m_{\ell,\widehat{K}}(\widehat{\boldsymbol{\tau}})| \leq \|\widehat{\boldsymbol{\tau}}\|_{0,t;\widehat{K}}\, \|\widehat{\psi}_\ell\|_{0,t';\widehat{K}} \qquad \forall\, \widehat{\boldsymbol{\tau}} \in \mathbf{W}^{\delta,t}(\widehat{K}) \cap \mathbf{H}^t(\mathrm{div}_t; \widehat{K}), \tag{C.5}$$

where t' is the conjugate of t. In turn, for the case of the $\widehat{F}$-moments, which are all linear as well, we separate the analysis according to (C.3). If $t \in (1, 2)$ and $\delta > \frac{1}{t}$, then the trace theorem (cf. [41, Theorem 1.5.1.2]) establishes that $\widehat{\boldsymbol{\tau}}|_{\partial\widehat{K}} \in \mathbf{W}^{\delta-\frac{1}{t},t}(\partial\widehat{K})$ for all $\widehat{\boldsymbol{\tau}} \in \mathbf{W}^{\delta,t}(\widehat{K})$. Hence, given $\ell \in \{1, 2, \ldots, d_k\}$, it follows from Hölder's inequality, the continuous embedding of $\mathbf{W}^{\delta-\frac{1}{t},t}(\partial\widehat{K})$ into $\mathbf{L}^t(\partial\widehat{K})$, and the trace inequality for $\mathbf{W}^{\delta,t}(\widehat{K})$, that

$$\begin{aligned} |m_{\ell,\widehat{F}}(\widehat{\boldsymbol{\tau}})| &\leq \|\widehat{\boldsymbol{\tau}}\|_{0,t;\widehat{F}}\, \|\widehat{\varphi}_{\ell,\widehat{F}}\|_{0,t';\widehat{F}} \leq \|\widehat{\boldsymbol{\tau}}\|_{0,t;\partial\widehat{K}}\, \|\widehat{\varphi}_{\ell,\widehat{F}}\|_{0,t';\widehat{F}} \\ &\leq C\, \|\widehat{\boldsymbol{\tau}}\|_{\delta-\frac{1}{t},t;\partial\widehat{K}}\, \|\widehat{\varphi}_{\ell,\widehat{F}}\|_{0,t';\widehat{F}} \leq C\, \|\widehat{\boldsymbol{\tau}}\|_{\delta,t;\widehat{K}}\, \|\widehat{\varphi}_{\ell,\widehat{F}}\|_{0,t';\widehat{F}}. \end{aligned} \tag{C.6}$$

Next, we take $t \in (2,+\infty)$ and $\delta \geq 0$, so that, in particular, $t' \in (1,2)$. Then, given $\ell \in \big\{1,2,\dots,d_k\big\}$, and noticing that certainly $\widehat{\varphi}_{\ell,\widehat{F}} \in \mathrm{W}^{\frac{1}{t},t'}(\widehat{F})$, it follows from [41, Theorem 1.5.2.3, part (a)] that its extension by zero to $\partial\widehat{K}\backslash\widehat{F}$, say $\widehat{\varphi}^0_{\ell,\widehat{F}}$, belongs to $\mathrm{W}^{\frac{1}{t},t'}(\partial\widehat{K})$, and therefore we can redefine $m_{\ell,\widehat{F}}$ (cf. (C.1)) as

$$m_{\ell,\widehat{F}}(\widehat{\boldsymbol{\tau}}) \,:=\, \langle \widehat{\boldsymbol{\tau}}\cdot\boldsymbol{\nu}, \widehat{\varphi}^0_{\ell,\widehat{F}}\rangle_{\partial\widehat{K}} \qquad \forall\,\widehat{\boldsymbol{\tau}} \in \mathbf{H}^t(\mathrm{div}_t;\widehat{K})\,, \tag{C.7}$$

where $\langle\cdot,\cdot\rangle_{\partial\widehat{K}}$ denotes the duality pairing between $\mathrm{W}^{-\frac{1}{t},t}(\partial\widehat{K})$ and $\mathrm{W}^{\frac{1}{t},t'}(\partial\widehat{K})$. Moreover, applying now [41, Theorem 1.5.1.3], we deduce the existence of $\widehat{v}_{\ell,\widehat{F}} \in \mathrm{W}^{1,t'}(\widehat{K})$ such that $\widehat{v}_{\ell,\widehat{F}}|_{\partial\widehat{K}} \,=\, \widehat{\varphi}^0_{\ell,\widehat{F}}$ and

$$\|\widehat{v}_{\ell,\widehat{F}}\|_{1,t';\widehat{K}} \,\leq\, c\,\|\widehat{\varphi}^0_{\ell,\widehat{F}}\|_{\frac{1}{t},t';\partial\widehat{K}}\,. \tag{C.8}$$

In this way, starting from (C.7), and then employing the integration by parts formula (2.12), the Hölder inequality, and the trace estimate (C.8), we find that

$$\begin{aligned} |m_{\ell,\widehat{F}}(\widehat{\boldsymbol{\tau}})| \,&=\, |\langle \widehat{\boldsymbol{\tau}}\cdot\boldsymbol{\nu}, \widehat{\varphi}^0_{\ell,\widehat{F}}\rangle_{\partial\widehat{K}}| \,=\, |\langle \widehat{\boldsymbol{\tau}}\cdot\boldsymbol{\nu}, \widehat{v}_{\ell,\widehat{F}}\rangle_{\partial\widehat{K}}| \,=\, \Big|\int_{\widehat{K}}\Big\{\widehat{\boldsymbol{\tau}}\cdot\nabla\widehat{v}_{\ell,\widehat{F}} \,+\, \widehat{v}_{\ell,\widehat{F}}\,\mathrm{div}(\widehat{\boldsymbol{\tau}})\Big\}\Big| \\ &\leq\, C\,\|\widehat{\boldsymbol{\tau}}\|_{t,\mathrm{div}_t;\widehat{K}}\,\|\widehat{v}_{\ell,\widehat{F}}\|_{1,t';\widehat{K}} \,\leq\, C\,\|\widehat{\boldsymbol{\tau}}\|_{t,\mathrm{div}_t;\widehat{K}}\,\|\widehat{\varphi}^0_{\ell,\widehat{F}}\|_{\frac{1}{t},t';\partial\widehat{K}}\,. \end{aligned} \tag{C.9}$$

Finally, given $\widehat{\boldsymbol{\tau}} \in \mathbf{W}^{\delta,t}(\widehat{K})\cap\mathbf{H}^t(\mathrm{div}_t;\widehat{K})$, we have from (C.2)

$$\|\Pi^k_{\widehat{K}}(\widehat{\boldsymbol{\tau}})\|_{0,t;\widehat{K}} \,\leq\, \sum_{j=1}^{N_k}|\widehat{m}_j(\widehat{\boldsymbol{\tau}})|\,\|\widehat{\boldsymbol{\tau}}_j\|_{0,t;\widehat{K}}\,,$$

which, together with the bounds (C.5), (C.6), and (C.9), and the fact that $\|\widehat{\boldsymbol{\tau}}\|_{0,t;\widehat{K}} \,\leq\, \|\widehat{\boldsymbol{\tau}}\|_{\delta,t;\widehat{K}}$, yield the required estimate (C.4) with $\widehat{C}$ depending on the sets $\big\{\|\widehat{\psi}_\ell\|_{0,t';\widehat{K}}\big\}_{\ell=1}^{r_k}$, $\big\{\|\widehat{\boldsymbol{\tau}}_j\|_{0,t;\widehat{K}}\big\}_{j=1}^{N_k}$, $\big\{\|\widehat{\varphi}_{\ell,\widehat{F}}\|_{0,t';\widehat{F}}\big\}_{\ell=1}^{d_k}$, and $\big\{\|\widehat{\varphi}^0_{\ell,\widehat{F}}\|_{\frac{1}{t},t';\partial\widehat{K}}\big\}_{\ell=1}^{d_k}$, for all the sides $\widehat{F}\subset\partial\widehat{K}$. □

Having proved Lemma C.1, we now establish an approximation property of Π^k_K. More precisely, we have the following result.

Lemma C.2 *Assume that t and δ are as stated in Lemma* C.1*. Then, there exists a constant $C>0$, independent of h, such that for each $K\in\mathcal{T}_h$ there holds*

$$\|\boldsymbol{\tau}-\Pi^k_K(\boldsymbol{\tau})\|_{0,t;K} \,\leq\, C\,h_K^\delta\Big\{|\boldsymbol{\tau}|_{\delta,t;K} \,+\, \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;K}\Big\} \qquad \forall\,\boldsymbol{\tau}\in\mathbf{W}^{\delta,t}(K)\cap\mathbf{H}^t(\mathrm{div}_t;K)\,. \tag{C.10}$$

Proof. It proceeds analogously to the proof of [35, Lemma 3.19], using now the estimate (C.4), and employing the Deny-Lions Lemma for fractional Sobolev spaces (cf. [30, Theorem 6.1]), and the scaling properties of the corresponding semi-norms (cf. [43, Lemmas 2.8 and 2.9]). We omit further details.

□

As a straightforward consequence of the triangle inequality, (C.10), and the fact that both, $\|\cdot\|_{0,t;\mathrm{K}}$ and $|\cdot|_{\delta,t;K}$, are bounded by $\|\cdot\|_{\delta,t;\mathrm{K}}$, we readily deduce the existence of a constant $c>0$, independent of h, such that for each $K\in\mathcal{T}_h$ there holds

$$\|\Pi^k_K(\boldsymbol{\tau})\|_{0,t;K} \,\leq\, c\,\Big\{\|\boldsymbol{\tau}\|_{\delta,t;K} \,+\, h_K^\delta\,\|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;K}\Big\} \qquad \forall\,\boldsymbol{\tau}\in\mathbf{W}^{\delta,t}(K)\cap\mathbf{H}^t(\mathrm{div}_t;K)\,. \tag{C.11}$$

Finally, it is not difficult to see that the global versions of (C.10) and (C.11) become

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{0,t;\Omega} \,\le\, C_t\, h^\delta \Big\{ |\boldsymbol{\tau}|_{\delta,t;\Omega} \,+\, \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \Big\} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{\delta,t}(\Omega) \cap \mathbf{H}^t(\mathrm{div}_t;\Omega)\,, \tag{C.12a}$$

$$\|\Pi_h^k(\boldsymbol{\tau})\|_{0,t;\Omega} \,\le\, c_t \Big\{ \|\boldsymbol{\tau}\|_{\delta,t;\Omega} \,+\, h^\delta\, \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \Big\} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{\delta,t}(\Omega) \cap \mathbf{H}^t(\mathrm{div}_t;\Omega)\,, \tag{C.12b}$$

respectively, with constants C_t, $c_t > 0$, independent of h, but depending on t.

References

[1] K. Allali, *A priori and a posteriori error estimates for Boussinesq equations.* Int. J. Numer. Anal. Model. 2 (2005), no. 2, 179–196.

[2] J.A. Almonacid and G.N. Gatica, *A fully-mixed finite element method for the n-dimensional Boussinesq problem with temperature-dependent parameters.* Comput. Methods Appl. Math. 20 (2020), no. 2, 187–213.

[3] M. Álvarez, G.N. Gatica and R. Ruiz-Baier, *An augmented mixed-primal finite element method for a coupled flow-transport problem.* ESAIM Math. Model. Numer. Anal. 49 (2015), no. 5, 1399–1427.

[4] M. Álvarez, G.N. Gatica and R. Ruiz-Baier, *A mixed-primal finite element approximation of a steady sedimentation-consolidation system.* Math. Models Methods Appl. Sci. 26 (2016), no. 5, 867–900.

[5] M. Amara, M., D. Capatina and L. Lizaik, *Coupling of Darcy-Forchheimer and compressible Navier-Stokes equations with heat transfer.* SIAM J. Sci. Comput. 31 (2008/09), no. 2, 1470–1499.

[6] J. Bear, Dynamics of Fluids in Porous Media, Courier Corporation, 1988.

[7] G.A. Benavides, S. Caucao, G.N. Gatica and A. Hopper, *A Banach spaces-based analysis of a new mixed-primal finite element method for a coupled flow-transport problem.* Comput. Methods Appl. Mech. Engrg. 371 (2020), 113285, 29 pp.

[8] C. Bernardi, C. Canuto and Y. Maday, *Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem.* SIAM J. Numer. Anal. 25 (1988), no. 6, 1237–1271.

[9] C. Bernardi, S. Dib, V. Girault, F. Hecht, F. Murat and T. Sayah, *Finite element methods for Darcy's problem coupled with the heat equation.* Numer. Math. 139 (2018), no. 2, 315–348.

[10] C. Bernardi, S. Maarouf, and D. Yakoubi, *Spectral discretization of Darcy's equations coupled with the heat equation.* IMA J. Numer. Anal. 36 (2016), no. 3, 1193–1216.

[11] C. Bernardi, B. Métivet and B. Pernaud-Thomas, *Couplage des équations de Navier–Stokes et de la chaleur: le modèle et son approximation par éléments finis.* RAIRO Modél. Math. Anal. Numér. 29 (1995), no. 7, 871–921.

[12] D. Boffi, F. Brezzi and M. Fortin, Mixed Finite Element Methods and Applications. Springer Series in Computational Mathematics, 44. Springer, Heidelberg, 2013.

[13] S. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods. Third edition. Texts in Applied Mathematics, 15. Springer, New York, 2008.

[14] R. Bürger, S. Kumar and R. Ruiz-Baier, *Discontinuous finite volume element discretization for coupled flow-transport problems arising in models of sedimentation.* J. Comput. Phys. 299 (2015), 446–471.

[15] R. Bürger, P.E. Méndez and R. Ruiz-Baier, *On H(div)-conforming methods for double-diffusion equations in porous media.* SIAM J. Numer. Anal., 57(2019), no. 3, 1318–1343.

[16] J. Camaño, C. García and R. Oyarzúa, *Analysis of a momentum conservative mixed-FEM for the stationary Navier-Stokes problem.* Numer. Methods Partial Differential Equations, https://doi.org/10.1002/num.22789.

[17] J. Camaño, C. Muñoz and R. Oyarzúa, *Numerical analysis of a dual-mixed problem in non-standard Banach spaces.* Electron. Trans. Numer. Anal. 48 (2018), 114–130.

[18] S. Caucao, G.N. Gatica and R. Oyarzúa, *Analysis of an augmented fully-mixed formulation for the coupling of the Stokes and heat equations.* ESAIM Math. Model. Numer. Anal. 52 (2018), no. 5, 1947–1980.

[19] S. Caucao, R. Oyarzúa and S. Villa-Fuentes, *A new mixed*-FEM *for steady-state natural convection models allowing conservation of momentum and thermal energy.* Calcolo 57 (2020), no. 4, paper no. 36, 39 pp.

[20] S.-H. Chou and Q. Li, *Numerical analysis of a mixed finite element method for a flow-transport problem.* Numer. Methods Partial Differential Equations 12 (1996), no. 2, 221–233.

[21] P. Ciarlet, Linear and Nonlinear Functional Analysis with Applications. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[22] E. Colmenares, G.N. Gatica and S. Moraga, *A Banach spaces-based analysis of a new fully-mixed finite element method for the Boussinesq problem.* ESAIM Math. Model. Numer. Anal. 54 (2020), no. 5, 1525–1568.

[23] E. Colmenares, G.N. Gatica and R. Oyarzúa, *Analysis of an augmented mixed-primal formulation for the stationary Boussinesq problem.* Numer. Methods Partial Differential Equations 32 (2016), no. 2, 445–478.

[24] E. Colmenares and M. Neilan, *Dual-mixed finite element methods for the stationary Boussinesq problem.* Comput. Math. Appl. 72 (2016), no. 7, 1828–1850.

[25] M. Dauge, Elliptic Boundary Value Problems on Corner Domains. Smoothness and Asymptotics of Solutions. Lecture Notes in Mathematics, 1341. Springer-Verlag, Berlin, 1988.

[26] M. Dauge, *Private communication.* (2020).

[27] J. Deteix, A. Jendoubi and D. Yakoubi, *A coupled prediction scheme for solving the Navier–Stokes and convection–diffusion equations.* SIAM J. Numer. Anal. 52 (2014), no. 5, 2415–2439.

[28] D. Dib, S. Dib and T. Sayah, *New numerical studies for Darcy's problem coupled with the heat equation.* Comput. Appl. Math. 39 (2020), no. 1, paper no. 1, 16 pp.

[29] S. Dib, V. Girault, F. Hecht, and T. Sayah, *A posteriori error estimates for Darcy's problem coupled with the heat equation*. ESAIM Math. Model. Numer. Anal. 53 (2019), no. 6, 2121–2159.

[30] T. Dupont and L.R. Scott, *Polynomial approximation of functions in Sobolev spaces*. Math. Comp. 34 (1980), no. 150, 441–463.

[31] R. Durán, *Error analysis in* L^p, $1 \leq p \leq \infty$, *for mixed finite element methods for linear and quasi-linear elliptic problems*. RAIRO Modél. Math. Anal. Numér. 22 (1988), no. 3, 371–387.

[32] A. Ern and J.-L Guermond, Theory and Practice of Finite Elements. Applied Mathematical Sciences, 159. Springer-Verlag, New York, 2004.

[33] L.C. Evans, Partial Differential Equations. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[34] S.J. Fromm, *Potential space estimates for Green potentials in convex domains*. Proc. Amer. Math. Soc. 119 (1993), no. 1, 225–233.

[35] G.N. Gatica, A Simple Introduction to the Mixed Finite Element Method. Theory and Applications. SpringerBriefs in Mathematics. Springer, Cham, 2014.

[36] G.N. Gatica, R. Oyarzúa, R. Ruiz-Baier and Y.D. Sobral, *Banach spaces-based analysis of a fully-mixed finite element method for the steady-state model of fluidized beds*. Comput. Math. Appl. 84 (2021), 244–276.

[37] G.N. Gatica, R. Ruiz-Baier and G. Tierra, *A mixed finite element method for the Darcy equations with pressure-dependent porosity*. Math. Comp. 85 (2016), no. 297, 1–33.

[38] G.N. Gatica, R. Ruiz-Baier and G. Tierra, *A posteriori error analysis of an augmented mixed method for the Navier-Stokes equations with nonlinear viscosity*. Comput. Math. Appl. 72 (2016), no. 9, 2289–2310.

[39] J. Geng, $W^{1,p}$ *estimates for elliptic problems with Neumann boundary conditions in Lipschitz domains*. Adv. Math. 229 (2012), no. 4, 2427–2448.

[40] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms. Springer Series in Computational Mathematics, 5. Springer-Verlag, Berlin, 1986.

[41] P. Grisvard, Elliptic Problems in Nonsmooth Domains. Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[42] J. Guzmán, D. Leykekhman, J. Rossmann and A.H. Schatz, *Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods*. Numer. Math. 112 (2009), no. 2, 221–243.

[43] N. Heuer, *On the equivalence of fractional-order Sobolev semi-norms*. J. Math. Anal. Appl. 417 (2014), no. 2, 505–518.

[44] J.S. Howell and N.J. Walkington, *Dual-mixed finite element methods for the Navier–Stokes equations*. ESAIM Math. Model. Numer. Anal. 47 (2013), no. 3, 789–805.

[45] T. Jakab, I. Mitrea and M. Mitrea, *Sobolev estimates for the Green potential associated with the Robin-Laplacian in Lipschitz domains satisfying a uniform exterior ball condition*. Sobolev Spaces in Mathematics. II, 227–260, Int. Math. Ser. (N. Y.), 9, Springer, New York, 2009.

[46] A. Kufner, O. John and S. Fučík, Function Spaces. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis. Noordhoff International Publishing, Leyden; Academia, Prague, 1977.

[47] D. Leykekhman and B. Li, *Maximum-norm stability of the finite element Ritz projection under mixed boundary conditions*. Calcolo 54 (2017), no. 2, 541–565.

[48] D. Leykekhman and B. Vexler, *Finite element pointwise results on convex polyhedral domains*. SIAM J. Numer. Anal. 54 (2016), no. 2, 561–587.

[49] H. Li, *The* W_p^1 *stability of the Ritz projection on graded meshes*. Math. Comp. 86 (2017), no. 303, 49–74.

[50] D.A. Nield and A. Bejan, Convection in Porous Media, Springer Science + Business Media, New York, 2013.

[51] R. Oyarzúa and P. Zúñiga, *Analysis of a conforming finite element method for the Boussinesq problem with temperature-dependent parameters*. J. Comput. Appl. Math. 323 (2017), 71–94.

[52] R. Rannacher and R. Scott, *Some optimal error estimates for piecewise linear finite element approximations*. Math. Comp. 38 (1982), no. 158, 437–445.

[53] B.M. Rivière and N.J. Walkington, *Convergence of a discontinuous Galerkin method for the miscible displacement equation under low regularity*. SIAM J. Numer. Anal. 49 (2011), no. 3, 1085–1110.

[54] R. Ruiz-Baier and I. Lunati, *Mixed finite element – discontinuous finite volume element discretization of a general class of multicontinuum models*. J. Comput. Phys. 322 (2016), 666–688.

[55] R. Ruiz-Baier and H. Torres, *Numerical solution of a multidimensional sedimentation problem using finite volume-element methods*. Appl. Numer. Math. 95 (2015), 280–291.

[56] A.H. Schatz, *A weak discrete maximum principle and stability of the finite element method in* L_∞ *on plane polygonal domains*. I. Math. Comp. 34 (1980), no. 149, 7–91.

[57] G. Scovazzi, H. Huang, S. Collis, and J. Yin, *A fully-coupled upwind discontinuous Galerkin method for incompressible porous media flows: High-order computations of viscous fingering instabilities in complex geometry*. J. Comput. Phys. 252 (2013), 86–108.

[58] M.F. Wheeler, Numerical simulation in oil recovery, Springer-Verlag New York Inc., New York, NY, 1988.