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The Strong Property (B) for L_p Spaces

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In memory of Edward Odell.

Abstract. Given a purely non-atomic, finite measure space (Ω, Σ, ν), it is proved that for every closed, infinite-dimensional subspace V of $L_p(\nu)$ $(1 \leq p < \infty)$ there exists a decomposition $L_p(\nu) = X_1 \oplus X_2$, such that both subspaces X_1 and X_2 are isomorphic to $L_p(\nu)$ and both $V \cap X_1$ and $V \cap X_2$ are infinite-dimensional. Some consequences concerning dense, non-closed range operators on L_1 are derived.

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1. Introduction

To generalize a theorem of Dixmier concerning operator ranges into Hilbert spaces [14] to $L_p[0,1]$ $(1 < p < \infty)$, Cross, Ostrovskii, and Shevchik introduced the following notion: *a Banach space* X *is said to have property (B) if for every closed, infinite-dimensional subspace* V of X, *there exists a projection* $P \in \mathcal{L}(X)$, *such that both intersections* $V \cap P(X)$ *and* $V \cap (I_X - P)(X)$ *are infinite-dimensional* [10][Sect. 5]).

On one hand, the aforementioned authors observed in [10] that any space containing an isomorphic copy of the hereditarily indecomposable space X_{GM} of Gowers and Maurey [18] does not have property (B). In particular, since X_{GM} is separable, neither $C[0,1]$ nor $L_\infty[0,1]$ have property (B). On the other hand, they identified two natural classes of Banach spaces for which property (B) holds.

(i) Banach spaces with an unconditional basis;
(ii) subprojective Banach spaces.

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Let us recall that a Banach space X is said to be *subprojective* if every closed, infinite-dimensional subspace Y of X contains a closed, infinite-dimensional subspace Z complemented in X. For more information about subprojective spaces, see [16] and [28].

It is well known that the spaces $L_p[0,1]$ (henceforth, L_p for short) have an unconditional basis if and only if $1 < p < \infty$ (see [1][Theorem 6.1.6] and [29]). Moreover, it is also known that L_p is subprojective if and only if $2 \leq p < \infty$ (see [30] and [1][Corollary 6.4.9]). An example of a subprojective space with basis but without unconditional basis is the quasi-reflexive James space $\mathcal{J}$ [1][Corollary 3.4.7].

It is straightforward from the comments above that L_p has property (B) if $1 < p < \infty$, and does not have it if $p = \infty$. Since L_1 does not belong to neither of types (i) nor (ii), Cross and his coauthors ask if L_1 has property (B) [10][Problem 6.3]).

It is well known that for any decomposition $L_1 = X_1 \oplus X_2$, at least one of the factors X_i is isomorphic to L_1 [15] and the other factor is a $\mathcal{L}_1$-space [8], but it is unknown if it is isomorphic to a space other than L_1, ℓ_1 or a finite-dimensional space. These facts led Odell et al. to introduce the strong property (B) and to ask if L_1 has this property [27][Problem 4], where this property is defined as follows:

Definition 1.1. A Banach space X has the strong property (B) if for every closed, infinite-dimensional subspace V of X, there exists a projection $P \in \mathcal{L}(X)$, such that both spaces $P(X)$ and $(I-P)(X)$ are isomorphic to X and both intersections $V \cap P(X)$ and $V \cap (I-P)(X)$ are infinite-dimensional.

Note that $\mathcal{J}$ does not have the strong property (B), because $\dim \mathcal{J}^{**}/\mathcal{J} = 1$, and therefore, $\mathcal{J}$ cannot be isomorphic to $\mathcal{J} \times \mathcal{J}$.

The central result in this paper (Theorem 3.3) positively answers the question posed by Odell and his coauthors. Even more, it is proved that for any finite measure space (Ω, Σ, ν) with no atom, $L_p(\Omega, \Sigma, \nu)$ has the strong property (B) for all $1 \leq p < \infty$ (Theorem 4.1 and Proposition 4.3). Moreover, as a consequence of Theorem 3.3, it is proved in Corollary 4.2 that the aforementioned generalization of Dixmier's theorem can be extended to L_1.

The results presented in this article are based on three results: the discovery of Aldous that every infinite-dimensional, reflexive subspace E of L_1 contains a subspace isomorphic to ℓ_p for some $1 < p < \infty$ [2], a theorem by Dacunha–Castelle and Krivine that establishes that if that subspace E contains an isomorph of ℓ_p then E contains ℓ_p almost isometrically [11], and a change of density found by Berkes and Rosenthal [5].

The notation to be used throughout this paper is introduced next. Given a topological space (S, τ), an ultrafilter $\mathfrak{U}$ on a set of indices I and $x_0 \in S$, the convergence of $(x_i)_{i \in I} \subset S$ to x_0 following $\mathfrak{U}$ is denoted $x_i \xrightarrow[i \to \mathfrak{U}]{\tau} x_0$ and for short, by $x_i \xrightarrow[\mathfrak{U}]{} x_0$ or $x_0 = \lim_{\mathfrak{U}} x_i$. The notation $x_j \xrightarrow[j]{\tau} x_0$ ($x_j \xrightarrow[j]{} x_0$ if there is no confusion) or $\lim_j x_j = x_0$ is reserved for the usual convergence of a sequence $(x_j)_{j=1}^{\infty}$ to x_0.

Given a pair of Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, the kernel and the range of an operator $T\colon X \longrightarrow Y$ are denoted $N(T)$ and $R(T)$, respectively. Operators are linear and bounded maps; X and Y are isomorphic if there is a bijective operator $T\colon X \to Y$, in which case, it is denoted $X \simeq Y$. Given $1 \leq p < \infty$, $X \oplus_p Y$ denotes the Banach space $X \times Y$ endowed with the norm $\|(x,y)\|_p := (\|x\|_X^p + \|y\|_Y^p)^{1/p}$. A sequence $(x_n)_{n=1}^{\infty}$ contained in X is said to be *equivalent* to a basic sequence $(y_n)_{n=1}^{\infty} \subset Y$ if the linear operator from $\overline{\text{span}}\,\{y_n\}_{n=1}^{\infty}$ into $\overline{\text{span}}\,\{x_n\}_{n=1}^{\infty}$ that maps y_n to x_n is a bijective isomorphism, in which case $(x_n)_{n=1}^{\infty}$ is also basic. For more information about Banach spaces, see [1], [25] and [26].

Given a collection $\mathcal{A}$ of subsets of a set Ω, $\sigma(\mathcal{A})$ denotes the σ-algebra generated by $\mathcal{A}$. Given $A \subset \Omega$, the indicator function associated with A is denoted χ_A.

Let (Ω, Σ, ν) be a measure space; a subset $A \in \Sigma$ is said to be an atom if $\nu(A) > 0$ and $\nu(A \cap B)$ is equal to $\nu(A)$ or 0 for all $B \in \Sigma$. A well-known theorem of Sierpinski states that for a measure space (Ω, Σ, ν) with no atom, given $A \in \Sigma$ and given $0 \leq \lambda \leq \nu(A)$, A contains a subset $B \in \Sigma$, such that $\lambda = \nu(B)$. For $A \in \Sigma$, $\Sigma(A) := \{B \cap A\colon B \in \Sigma\}$ is a σ-algebra and $(A, \Sigma(A), \nu|_{\Sigma(A)})$ is a measure space; we will write ν rather than $\nu|_{\Sigma(A)}$ if there is no possible confusion. A *density function* is a measurable function $\varphi\colon \Omega \longrightarrow [0,\infty)$, such that $\int_\Omega \varphi \,\mathrm{d}\nu = 1$; if $\varphi(t) > 0$ for all $t \in \Omega$, φ is said to be a *strictly positive* density function. If $\nu(\Omega) = 1$, (Ω, Σ, ν) is called a *probability space* and every ν-measurable function $f\colon \Omega \longrightarrow \mathbb{R}$ is called a *random variable*. If $\nu(\Omega) < \infty$, then (Ω, Σ, ν) is called a *finite* measure space. Given $1 \leq p < \infty$, the Banach space of all (classes of) real-valued (alt., complex-valued) functions f for which $|f|^p$ is integrable with respect of ν is denoted by $L_p(\Omega, \Sigma, \nu)$ (alt., $L_p(\Omega, \Sigma, \nu, \mathbb{C})$), and in the case $p = 1$, its weak topology is denoted as w. The unit interval $[0,1]$ is denoted by $\mathbb{I}$, the σ-algebra of borelian subsets of $\mathbb{I}$ is denoted by $\mathcal{B}$, and the Lebesgue measure on $\mathcal{B}$ is denoted by μ. Sometimes, if the context is clear, the shorter notation $L_p(\nu)$ will be used instead of $L_p(X, \Sigma, \nu)$. The shorter form L_p will be used to denote $L_p(\mathbb{I}, \mathcal{B}, \mu)$.

Given a random variable f, $\sigma(f)$ denotes the sub-Σ-algebra generated by f, that is, the sub-Σ-algebra generated by all the subsets $\{s \leq f(t) < r\}$ $\{r, s\} \subset \mathbb{R}$. Independent random variables are measurable functions that are independent in the probabilistic sense. Given a random variable f and a sub-σ-algebra $\mathcal{A}$ of Σ, f and $\mathcal{A}$ are said to be independent if $\sigma(f)$ and $\mathcal{A}$ are independent; and if g is another random variable, f and g are independent if so are $\sigma(f)$ and $\sigma(g)$.

Given a probability space (Ω, Σ, ν), the conditional expectation of $f \in L_1(\Omega, \Sigma, \nu)$ relative to a sub-σ-algebra $\mathcal{S}$ of Σ—denoted $\mathbb{E}(f|\mathcal{S})$—is the only function in $L_1(\Omega, \mathcal{S}, \nu|_{\mathcal{S}})$ that satisfies $\int_C f \,\mathrm{d}\nu = \int_C \mathbb{E}(f|\mathcal{S}) \,\mathrm{d}\nu$ for all $C \in \mathcal{S}$; the conditional expectation $\mathbb{E}(\cdot|\mathcal{S})$ is a norm one operator from $L_1(\Omega, \Sigma, \nu)$ onto $L_1(X, \mathcal{S}, \nu|_{\mathcal{S}})$; exactly, $\mathbb{E}(f|\mathcal{S})$ is the Radon–Nikodym derivative of f with respect to $\nu|_{\mathcal{S}}$; f is said to be of zero mean if $\mathbb{E}(f) := \mathbb{E}(f|\{\emptyset, \Omega\}) = 0$, that is, if $\int_\Omega f \,\mathrm{d}\nu = 0$. Section V in [13] and [4] covers all necessary facts about conditional expectations needed here.

2. Ultrapowers of L_1 Spaces

This section begins with a short summary on ultrapowers of L_1 spaces, whose study was initiated by Dacunha–Castelle and Krivine [11]. For proofs and details, see [21] and [17].

Let I be a set of indices. Through this paper, every ultrafilter $\mathfrak{U}$ is a *countably uncomplete* ultrafilter on I, that is, an ultrafilter for which there exists a countable partition $\{I_n\}_{n=1}^{\infty}$ of I disjoint with $\mathfrak{U}$.

Let Ω be any non-empty set and let $\mathcal{B}(I,\Omega)$ denote the set of all families $(t_i)_{i\in I} \subset \Omega$. Let $\sim$ be the equivalence relation in $\mathcal{B}(I,\Omega)$ given by $(t_i)_{i\in I} \sim (s_i)_{i\in I}$ if $\{i \in I \colon t_i = s_i\} \in \mathfrak{U}$. The *theoretic ultrapower* of Ω following $\mathfrak{U}$ is the quotient $\Omega^{\mathfrak{U}} := \mathcal{B}(I,\Omega)/\sim$. The element of $\Omega^{\mathfrak{U}}$ with representative $(t_i)_{i\in I}$ will be denoted $(t_i)^{\mathfrak{U}}$. The theoretic ultraproduct of a family $\{A_i\}_{i\in I}$ where $A_i \subset \Omega$ is the subset $(A_i)^{\mathfrak{U}} := \{(t_i)^{\mathfrak{U}} \colon t_i \in A_i,\ i \in I\}$. In accordance with this definition, $(A_i)^{\mathfrak{U}} \cup (B_i)^{\mathfrak{U}} = (A_i \cup B_i)^{\mathfrak{U}}$ and $(A_i)^{\mathfrak{U}} \cap (B_i)^{\mathfrak{U}} = (A_i \cap B_i)^{\mathfrak{U}}$.

Given a Banach space X, its *ultrapower following* $\mathfrak{U}$ is the quotient Banach space $X_{\mathfrak{U}} := \ell_\infty(I,X)/N$ where N is the subspace of all families $(x_i)_{i\in I}$, such that $\lim_{i\to\mathfrak{U}} x_i = 0$. The element of $X_{\mathfrak{U}}$ with representative $(x_i)_{i\in I}$ is denoted $[x_i]_{\mathfrak{U}}$ or $[x_i]$ for short and its norm can be calculated as $\|[x_i]\| = \lim_{i\to\mathfrak{U}} \|x_i\|$. The space X is canonically embedded into $X_{\mathfrak{U}}$ by means of the isometry $J : X \longrightarrow X_{\mathfrak{U}}$ that maps each x to the constant class $[x]$. For simplicity, J is dropped in the notation, so Jx will be denoted x if the context is clear.

Let (Ω,Σ,μ) be a probability space. The set $\Sigma_{\mathfrak{U}} := \{(A_i)^{\mathfrak{U}} \colon A_i \in \Sigma\}$ is a Boolean algebra. The smallest σ-algebra in $\Omega^{\mathfrak{U}}$ containing $\Sigma_{\mathfrak{U}}$ will be denoted $\sigma(\Sigma_{\mathfrak{U}})$. The measure μ induces a probability measure $\mu_{\mathfrak{U}}$ on $\sigma(\Sigma_{\mathfrak{U}})$ defined on each element $(A_i)^{\mathfrak{U}} \in \Sigma_{\mathfrak{U}}$ as $\mu_{\mathfrak{U}}\big((A_i)^{\mathfrak{U}}\big) := \lim_{i\to\mathfrak{U}} \mu(A_i)$ and extended to each $A \in \sigma(\Sigma_{\mathfrak{U}})$ as

$$\mu_{\mathfrak{U}}(A) := \inf\{\mu_{\mathfrak{U}}(C) \colon A \subset C,\ C \in \Sigma_{\mathfrak{U}}\} = \sup\{\mu_{\mathfrak{U}}(C) \colon C \subset A,\ C \in \Sigma_{\mathfrak{U}}\}.$$

Each $\mathbf{f} = [f_i] \in L_1(\mu_{\mathfrak{U}})$ induces a signed measure $\nu_{\mathbf{f}}$ on $\sigma(\Sigma_{\mathfrak{U}})$ as follows: given $A \in \Sigma_{\mathfrak{U}}$

$$\nu_{\mathbf{f}}(A) := \lim_{i\to\mathfrak{U}} \int_{A_i} f_i \,\mathrm{d}\mu, \quad A = (A_i)^{\mathfrak{U}} \in \Sigma_{\mathfrak{U}}$$

and for each $A \in \sigma(\Sigma_{\mathfrak{U}})$

$$\nu_{\mathbf{f}}(A) := \inf\{\nu_{\mathbf{f}}(C) \colon A \subset C,\ C \in \Sigma_{\mathfrak{U}}\} = \sup\{\nu_{\mathbf{f}}(C) \colon C \subset A,\ C \in \Sigma_{\mathfrak{U}}\}.$$

The space $L_1(\mu_{\mathfrak{U}})$ is canonically embedded into $L_1(\mu)_{\mathfrak{U}}$ by means of the isometry $L \colon L_1(\mu_{\mathfrak{U}}) \longrightarrow L_1(\mu)_{\mathfrak{U}}$ that maps each $\chi_{(A_i)^{\mathfrak{U}}}$ to $[\chi_{A_i}]$. Unless it be necessary, the notation L will be omitted, so, for $\mathbf{f} \in L_1(\mu_{\mathfrak{U}})$, $L\mathbf{f}$ will be also denoted $\mathbf{f}$. In turn, $L_1(\mu)$ embeds into $L_1(\mu_{\mathfrak{U}})$ via the isometry $D \colon L_1(\mu) \longrightarrow L_1(\mu_{\mathfrak{U}})$ that maps χ_A to $\chi_{(A^{\mathfrak{U}})}$ for all $A \in \Sigma$. For simplicity, the notation D will be also omitted, and consequently, the element Df that maps each $(t_i)^{\mathfrak{U}} \in \Omega^{\mathfrak{U}}$ to $\big(f(t_i)\big)^{\mathfrak{U}}$ will be denoted f.

Given $\mathbf{f} = [f_i] \in \big(L_1(\mu)\big)_{\mathfrak{U}}$, consider the Hahn decomposition $\nu_{\mathbf{f}} = w_{\mathbf{f}} + m_{\mathbf{f}}$ where $w_{\mathbf{f}}$ and $m_{\mathbf{f}}$ are the absolutely continuous part and the singular part of $\nu_{\mathbf{f}}$ with respect to $\mu_{\mathfrak{U}}$. Let $g_{\mathbf{f}}$ be the Radon–Nikodym derivative of $w_{\mathbf{f}}$ with respect to $\mu_{\mathfrak{U}}$, so that

$$w_{\mathbf{f}}(A) = \int_A g_{\mathbf{f}} \, \mathrm{d}\mu_{\mathfrak{U}}, \quad A \in \sigma(\Sigma_{\mathfrak{U}}).$$

The operator $Q\colon L_1(\mu)_{\mathfrak{U}} \longrightarrow L_1(\mu)_{\mathfrak{U}}$ that maps each $\mathbf{f}$ to $g_{\mathbf{f}}$ satisfies QL is the identity on $L_1(\mu_{\mathfrak{U}})$. Hence, LQ is a norm one projection from $L_1(\mu)_{\mathfrak{U}}$ on $L_1(\mu_{\mathfrak{U}})$, which determines a decomposition

$$L_1(\mu)_{\mathfrak{U}} = L_1(\mu_{\mathfrak{U}}) \oplus_1 L_1\big(\mu(\mathfrak{U})\big)$$

for certain measure $\mu(\mathfrak{U})$. Given $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$, the two following statements hold [17]:

(a) $\mathbf{f} \in L_1(\mu_{\mathfrak{U}})$ if and only if $\mathbf{f}$ has a relatively weakly compact representative $(f_i)_{i\in I}$;
(b) $\mathbf{f} \in L_1\big(\mu(\mathfrak{U})\big)$ if and only if $\mathbf{f}$ has a representative $(f_i)_{i\in I}$ for which $\lim_{i\to\mathfrak{U}} \mu(\{|f_i| > 0\}) = 0$.

An ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ is said to be a *p-point* if every bounded sequence $(a_n)_{n=1}^{\infty}$ of real numbers contains a convergent subsequence $(a_{k_n})_{n=1}^{\infty}$, such that $\{k_n : n \in \mathbb{N}\} \in \mathfrak{U}$ [7] [Definition 4.6 and Theorem 4.7]; an ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ is said to be *rare* if for every countable partition $\{F_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ into finite sets, there exists $A \in \mathfrak{U}$, such that for every positive integer n, $A \cap F_n$ has at most one element [9]. In the case when $\mathfrak{U}$ is a rare p-point over $\mathbb{N}$, statements (a) and (b) admit the following equivalent forms [17]:

(a') $\mathbf{f} \in L_1(\mu_{\mathfrak{U}})$ if and only if $\mathbf{f}$ has a representative weakly convergent following $\mathfrak{U}$;
(b') $\mathbf{f} \in L_1\big(\mu(\mathfrak{U})\big)$ if and only if $\mathbf{f}$ has a representative $(f_n)_{n\in\mathbb{N}}$, such that $f_n \cdot f_m = 0$ μ-a.e. for all n and all $m \neq n$.

Obviously, if $(f_i)_{i\in I} \subset L_1(\mu)$ is relatively weakly compact, then there exists the weak limit $w\text{-}\lim_{i\to\mathfrak{U}} f_i \in L_1(\mu)$. However, there are ultrafilters $\mathfrak{W}$ on the set of positive integers $\mathbb{N}$ for which there exists $\mathbf{f} \in L_1\big(\mu(\mathfrak{W})\big) \setminus \{0\}$ and each of its representatives is weakly convergent following $\mathfrak{W}$ [17][Proposition 16]; with a similar construction, it is not difficult to find a weakly null family $(f_i)_{i\in\mathbb{N}} \subset L_1(\mu)$ following $\mathfrak{W}$, such that $(f_i)_{i\in J}$ is unbounded for all $J \in \mathfrak{W}$.

Moreover, the existence of rare p-points relies upon set theories accepting the Continuum Hypothesis or some of its weaker forms such as Martin's axiom (see [17][Sect. 5] for a brief discussion).

The first result translates weak convergence in L_1 into ultrapower and probabilistic language. In spite of its simplicity, it is worthy of a proof.

Lemma 2.1. *Let (Ω, Σ, μ) be a probability measure space, let $\mathfrak{U}$ be an ultrafilter on a set I, and let $(f_i)_{i\in I}$ be a relatively weakly compact subset of $L_1(\mu)$, so that $\mathbf{f} := [f_i] \in L_1(\mu_{\mathfrak{U}})$. Then, $f_i \xrightarrow[\mathfrak{U}]{w} f$ if and only if $f = \mathbb{E}(\mathbf{f}|\Sigma)$.*

Proof. By definition of conditional expectation, $f = \mathbb{E}(\mathbf{f}|\Sigma)$ if and only if $\int_A f \,\mathrm{d}\mu = \int_A \mathbf{f} \,\mathrm{d}\mu_{\mathfrak{U}}$ for all $A \in \Sigma$. However, $\mathbf{f} \in L_1(\mu_{\mathfrak{U}})$; hence, $\int_A \mathbf{f} \,\mathrm{d}\mu_{\mathfrak{U}} = \lim_{i\to\mathfrak{U}} \int_A f_i \,\mathrm{d}\mu$. and the result follows. $\square$

According to Berkes and Rosenthal [5], the following result on extraction of sequences can be derived from [11], but they do not offer any proof.

Let us recall that for $1 \leq p \leq 2$, a *p-stable* random variable on a probability space (Ω, Σ, ν) is a random variable f for which there exists a constant $c > 0$, such that $\mathbb{E}(e^{itf(\omega)}) = e^{-c|t|^p}$ for all $t \in \mathbb{R}$.

Proposition 2.2. *Let E be a reflexive, infinite-dimensional subspace of L_1. Then, there is $1 < p \leq 2$, there is a normalized sequence $(x_n)_{n=1}^{\infty} \subset E$, and there is a density function u, such that $e^{itx_n} \xrightarrow[n]{w} e^{-|t|^p u^p}$ for all real number t.*

Proof. Let μ denote the Lebesgue measure on $\mathbb{I}$, so that $L_1 = L_1(\mu)$.

The main result in [2] establishes that E contains a subspace isomorphic to ℓ_p for some $1 < p < \infty$ (see [23] for an alternative proof). Thus, Théorèmes 0.1 and 0.2 in [11] provides us with an ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, an element $0 \leq u \in L_1(\mu)$ and a p-stable random variable $\mathbf{v} \in L_1(\mu_{\mathfrak{U}})$, such that $\mathbf{z} := \mathbf{v}u \in E_{\mathfrak{U}} \setminus \{0\}$, $\|\mathbf{z}\| = 1$ and such that $\sigma(\mathbf{v})$ and $\mathcal{B}$ are independent. (The fact that u is non-negative follows from the identity $u = \mathbb{E}(U^p|\mathcal{B})^{1/p}$ where $0 \leq U \in L_1(\mu_{\mathfrak{U}}) \setminus \{0\}$. Indeed, u and $\mathbf{v}$ are, respectively, the functions $\bar{U}$ and $\bar{V}$ that occur at the end of the proof 3⇒4 of Théorème 0.2 in [11], page 348). Hence, $0 \neq u \geq 0$, and dividing by $\int_{\mathbb{I}} u \, \mathrm{d}\mu$ if necessary, u can be assumed to be a density function.

Let $(x_j)_{j\in\mathbb{N}}$ and $(v_j)_{j\in\mathbb{N}}$ be respective representatives of $\mathbf{z}$ and $\mathbf{v}$, such that $\{x_j\}_{j=1}^{\infty}$ is normalized and relatively weakly compact. Thus

$$\|x_j - uv_j\|_{L_1} \xrightarrow[j\to\mathfrak{U}]{} 0. \tag{2.1}$$

Since $\sigma(\mathbf{v})$ and $\mathcal{B}$ are independent sub-σ-algebras of $\sigma(\mathcal{B}_{\mathfrak{U}})$, $\mathbf{v}$ is p-stable and u is $\mathcal{B}$-measurable; it follows:

$$\mathbb{E}(e^{itu\mathbf{v}}|\mathcal{B}) = \mathbb{E}(e^{-|t|^p u^p}|\mathcal{B}) = e^{-|t|^p u^p}, \text{ all } \quad t \in \mathbb{R}.$$

Hence, Lemma 2.1 gives $e^{ituv_j} \xrightarrow[j\to\mathfrak{U}]{w} e^{-|t|^p u^p}$ for all $t \in \mathbb{R}$, and therefore, from (2.1)

$$e^{itx_j} \xrightarrow[j\to\mathfrak{U}]{w} e^{-|t|^p u^p}, \text{ all } \quad t \in \mathbb{R}. \tag{2.2}$$

Let $\{t_n\}_{n=1}^{\infty}$ be a dense, countable subset of $\mathbb{R}$, and for every positive integer n, let $K_n := \{e^{it_n x_j} : j \in \mathbb{N}\} \subset L_1(\mathbb{I}, \mathcal{B}, \mu, \mathbb{C})$.

As each K_n is countable and relatively weakly compact in $L_1(\mathbb{I}, \mathcal{B}, \mu, \mathbb{C})$, the weak topology is metrizable on every $\overline{K_n}^w$ (see the proof of Eberlein–Smulian Theorem in [12]). Moreover, by virtue of (2.2), $e^{-|t_n|^p u^p}$ is a weak limit point of K_n for every natural number n. Hence, for each $n \in \mathbb{N}$, there exists a local basis of decreasing weak neighborhoods $\{\mathcal{V}_i^n\}_{i=1}^{\infty}$ of $e^{-|t_n|^p u^p}$. This means that given a sequence $(y_i)_{i=1}^{\infty} \subset K_n$, if for every $k \in \mathbb{N}$, there exists a natural number i_0, such that $y_i \in \mathcal{V}_k^n$ for all $i \geq i_0$, then

$$y_i \xrightarrow[i]{w} e^{-|t_n|^p u^p}. \tag{2.3}$$

Next, we extract a subsequence $(z_j)_{j=1}^{\infty}$ of $(x_j)_{j=1}^{\infty}$, such that $e^{it_n z_j} \xrightarrow[n]{w} e^{-|t_n|^p u^p}$ for all $n \in \mathbb{N}$ as follows:

Given any pair (k, n) of natural numbers, formula (2.2) yields

$$J_k^n := \{j \in \mathbb{N}\colon e^{it_n x_j} \in \mathcal{V}_k^n\} \in \mathfrak{U}.$$

Fix any $j_1 \in J_1^1$ and assume that the natural numbers $j_1 < j_2 < \ldots < j_{k-1}$ satisfying $j_l \in \bigcap_{m=1}^{l} J_{l+1-m}^m$ for all $1 \le l \le k-1$ have been already chosen. Then, as

$$\emptyset \neq A := \left[\bigcap_{m=1}^{k} J_{k+1-m}^m\right] \cap \{p \in \mathbb{N}\colon p > j_{k-1}\} \in \mathfrak{U},$$

we can pick $j_k \in A$. Continuing this process recursively, as $\{\mathcal{V}_i^n\}_{i=1}^\infty$ is decreasing, we get an increasing sequence $(j_k)_{k=1}^\infty$ of natural numbers, such that for $z_k := x_{j_k}$

$$e^{it_n z_k} \in \mathcal{V}_{k+1-n}^n, \text{ for all } k \ge n, \text{all } n \in \mathbb{N}. \tag{2.4}$$

Then, it follows from (2.3) and (2.4) that:

$$e^{it_n z_j} \xrightarrow[n]{w} e^{-|t_n|^p u^p} \quad \text{for all} \quad t_n. \tag{2.5}$$

Next, we will prove that (2.5) also holds for all $t \in \mathbb{R}$. Fix a real number τ and a balanced, weak neighborhood $\mathcal{W} \subset L_1(\mathbb{I}, \mathcal{B}, \mu, \mathbb{C})$ of 0. Choose $\varepsilon > 0$ and another balanced, weak neighborhood $\mathcal{V} \subset L_1(\mathbb{I}, \mathcal{B}, \mu, \mathbb{C})$ of 0 such that $B(0; \varepsilon) + \mathcal{V} \subset \mathcal{W}$, where $B(0; \varepsilon)$ is the closed ball of $L_1(\mathbb{I}, \mathcal{B}, \mu, \mathbb{C})$ centered at 0 of radius ε. Pick a sequence $(\tau_n)_n$ in $\{t_n\}_{n=1}^\infty$, such that $\tau_n \xrightarrow[n]{} \tau$. The Dominated Convergence Theorem gives a positive integer m_0, such that

$$\|e^{-|\tau|^p u^p} - e^{-|\tau_n|^p u^p}\|_{L_1} < \varepsilon/4 \text{ for all} \quad n \ge m_0. \tag{2.6}$$

Next, as $\{z_j\}_{j=1}^\infty$ is equiintegrable, there is $K > 0$ such that $\mu(\{\omega\colon |z_j(\omega)| > K\}) < \varepsilon/8$ for all $j \in \mathbb{N}$. Moreover, for every $x \in \mathbb{R}$, $|e^{ix} - 1| = \sqrt{2}\sqrt{1 - \cos x} \le |x|$. Thus

$$\begin{aligned}
&\int_{\mathbb{I}} |e^{i\tau_n z_j(\omega)} - e^{i\tau z_j(\omega)}| \, \mathrm{d}\mu(\omega) = \int_{\mathbb{I}} |e^{i(\tau_n - \tau) z_j(\omega)} - 1| \, \mathrm{d}\mu(\omega) = \\
&\int_{\{|z_j(\omega)| > K\}} |e^{i(\tau_n - \tau) z_j(\omega)} - 1| \, \mathrm{d}\mu(\omega) + \\
&\int_{\{|z_j(\omega)| \le K\}} |e^{i(\tau_n - \tau) z_j(\omega)} - 1| \, \mathrm{d}\mu(\omega) \le \frac{\varepsilon}{4} + |\tau_n - \tau| K.
\end{aligned}$$

Therefore, there is $m \ge m_0$, such that

$$\int_{\mathbb{I}} |e^{i\tau_n z_j(\omega)} - e^{i\tau z_j(\omega)}| \, \mathrm{d}\mu(\omega) \le \varepsilon/2, \text{ for all} \quad n \ge m \text{ and all } j \in \mathbb{N}. \tag{2.7}$$

Next, (2.5) yields j_0, such that

$$e^{i\tau_m z_j} \in e^{-|\tau_m|^p u^p} + \mathcal{V}, \text{for all} \quad j \ge j_0. \tag{2.8}$$

Thus, for every $j \ge j_0$, consecutive applications of (2.7), (2.8) and (2.6) give

$$\begin{aligned}
e^{i\tau z_j} \in e^{i\tau_m z_j} + B(0; \varepsilon/2) \subset e^{-|\tau_m|^p u^p} + \mathcal{V} + B(0; \varepsilon/2) \subset \\
e^{-|\tau|^p u^p} + \mathcal{V} + B(0; \varepsilon) \subset e^{-|\tau|^p u^p} + \mathcal{W},
\end{aligned}$$

and by virtue of (2.3), the proof is done. □

Given a sequence of random variables $\big(x_n(\omega)\big)_n$ on a probability space (Ω, Σ, ν), assume that for each real number t, there exists the limit of the sequence $(e^{itx_n})_n$ in the weak topology of $L_1(\nu)$, and denote this limit by $h(t,\omega)$: the function $h(t,\omega)$ is called the *limit conditional characteristic function* of $(x_n)_n$ (page 498 in[5]).

Remark. The extraction of the subsequence $(z_j)_j$ in the proof of Proposition 2.2 becomes simpler if the ultrafilter $\mathfrak{U}$ were a p-point. Indeed, the metrizability conditions of each (K_n, w) enable to take a subsequence $(x_{k_j})_j$ of $(x_j)_j$, such that $e^{it_n x_{k_j}} \underset{j\to\mathfrak{U}}{\overset{w}{\longrightarrow}} e^{-|t_n|^p u^p}$ and if $\mathfrak{U}$ is a p-point, it can be proved this subsequence satisfies $J_n := \{k_j : j \in \mathbb{N}\} \in \mathfrak{U}$. Thus, relabeling the subsequence $(x_{k_j})_j$ taken from each K_n as $(x_j^n)_j$, we may assume that $(x_j^n)_j \supset (x_j^{n+1})_j$ for all n. Then, a diagonalization argument shows that $(z_j)_j := (x_j^j)_j$ satisfies $e^{it_n z_j} \underset{j}{\overset{w}{\longrightarrow}} e^{-|t_n|^p u^p}$ for all n as required. However, it is necessary to point out that the ultrafilter $\mathfrak{U}$ in the proof of Proposition 2.2 is the one provided in the proof of 3⇒4 of Théorème 0.2 in [11] and that ultrafilter is a product of two ultrafilters, so it is not a p-point. These facts and a comparison of statements (a), (b) to (a') and (b') clearly suggest an explicit proof for Proposition 2.2 was needed.

3. Decompositions of L_1

Let us begin this section recalling that if (Ω, σ, ν) is an atomless finite measure space, such that $L_1(\nu)$ is separable, then $L_1(\nu) \simeq L_1$ (see §41 in[20] or I.B.1 in [31]).

Theorem 3.1. *Let $(f_n)_{n=1}^{\infty} \setminus \{0\}$ be a bounded sequence of independent random variables with zero mean defined on an atomless probability space (Ω, Σ, ν) for which $L_1(\Omega, \Sigma, \nu)$ is separable. Then, there exists a norm one projection $Q\colon L_1(\Omega, \Sigma, \nu) \longrightarrow L_1(\Omega, \Sigma, \nu)$ satisfying the following statements:*

(i) *$R(Q) \simeq L_1$ and $(f_{2n-1})_{n=1}^{\infty} \subset R(Q)$;*
(ii) *$N(Q) \simeq L_1$ and $(f_{2n})_{n=1}^{\infty} \subset N(Q)$.*

Proof. Let $\mathcal{S}_n := \sigma(f_n)$ for all n. Let Σ_e and Σ_o, respectively, denote the sub-σ-algebras of Σ generated by $\cup_{n=1}^{\infty} \mathcal{S}_{2n}$ and by $\cup_{n=1}^{\infty} \mathcal{S}_{2n-1}$.

Denote $L_1(\Sigma_e) := L_1(\Omega, \Sigma_e, \nu|_{\Sigma_e})$ and $L_1(\Sigma_o) := L_1(\Omega, \Sigma_o, \nu|_{\Sigma_o})$.

Since the random variables f_n are independent, neither $(\Omega, \Sigma_o, \mu|_{\Sigma_o})$ nor $(\Omega, \Sigma_e, \mu|_{\Sigma_e})$ has an atom. This fact and the separability of $L_1(\nu)$ yields $L_1 \simeq L_1(\Omega, \Sigma, \nu) \simeq L_1(\Sigma_o) \simeq L_1(\Sigma_e)$. In addition, as Σ_e and Σ_o are independent, then (see [4] or (2.18) and (2.21) in [22])

$$\mathbb{E}(f_{2n}|\Sigma_o) = \mathbb{E}(f_{2n}) \tag{3.1}$$

$$\mathbb{E}(f_{2n-1}|\Sigma_o) = f_{2n-1}, \ n \in \mathbb{N}. \tag{3.2}$$

Let Q be the norm one projection on $L_1(\Omega, \Sigma, \nu)$ given by the conditional expectation $Q(f) := \mathbb{E}(f|\Sigma_o)$.

It is straightforward from (3.2) that

$$L_1 \simeq L_1(\Sigma_o) = R(Q) \subset L_1(\Omega, \Sigma, \nu). \tag{3.3}$$

In particular, $(f_{2n-1})_{n=1}^{\infty} \subset R(Q)$ and (i) is proved.

To prove (ii), note that (3.1) implies that all Σ_e-measurable, integrable functions with zero mean belong to $N(Q)$. In particular, $(f_{2n})_{n=1}^{\infty} \subset N(Q)$, so the proof will be completed as soon as it is demonstrated that $N(Q) \simeq L_1$.

Consider the norm one projection P defined on $L_1(\Omega, \Sigma, \nu)$ and given by $P(f) := \mathbb{E}(f|\Sigma_e)$. The same arguments given for Q show

$$L_1 \simeq R(P) = L_1(\Sigma_e) \subset L_1(\Omega, \Sigma, \nu) \tag{3.4}$$

$$P(f) = \mathbb{E}(f) \cdot \chi_\Omega \quad \text{for all} f \in L_1(\Sigma_o). \tag{3.5}$$

The combination of (3.3), (3.1), (3.4), and (3.5) readily yields

$$PQ(f) = QP(f) = \mathbb{E}(f), \quad f \in L_1(\Omega, \Sigma, \nu), \tag{3.6}$$

and subsequently, $P\big(R(Q)\big) \subset R(Q)$ and $P\,(N(Q)) \subset N(Q)$. This proves that $P|_{R(Q)}$ and $P|_{N(Q)}$ are projections. Hence

$$L_1 \simeq R(P) = N(Q|_{R(P)}) \oplus R(Q|_{R(P)}) = \big[N(Q) \cap R(P)\big] \oplus R(QP).$$

However, (3.6) yields $\dim R(QP) = 1$, so that $N(Q) \cap R(P)$ is a hyperplane of $L_1(\Omega, \Sigma, \nu)$. Thus, as L_1 is primary [15], then $L_1 \simeq N(Q) \cap R(P)$, and

$$\begin{aligned} N(Q) &= N(P|_{N(Q)}) \oplus R(P|_{N(Q)}) \\ &= \big[N(Q) \cap N(P)\big] \oplus \big[R(P) \cap N(Q)\big] \simeq \big[N(Q) \cap N(P)\big] \oplus L_1. \end{aligned}$$

Therefore, $N(Q)$ is a complemented subspace of $L_1(\Omega, \Sigma, \nu)$ containing a complemented subspace isomorphic to L_1, and by means of Pelczyński decomposition method (Theorem 2.2.3 in[1]), $N(Q) \simeq L_1$. The proof is done. □

Lemma 3.2. *Given a probability space (Ω, Σ, ν), a real number $\varepsilon > 0$ and an equi-integrable sequence $(f_n)_{n=1}^{\infty}$ in $L_1(\Omega, \Sigma, \nu)$, such that $\sum_{n=1}^{\infty} |f_n(t)| < \infty$ for ν-almost every t, there is a subsequence $(f_{k_n})_{n=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$, such that $\|f_{k_n}\| < \varepsilon$ for all n.*

Proof. If the result fails, then there exists $\varepsilon > 0$, such that $\|f_n\|_1 \geq \varepsilon$ for all but for finitely many n. Without loss of generality, we may assume that $\|f_n\|_1 \geq \varepsilon$ for all n. Let

$$A_n := |f_n|^{-1}\big((\varepsilon/2, \infty)\big).$$

Since $\|\chi_{A_n^c} f_n\|_1 \leq \|\chi_{A_n^c} f_n\|_\infty \leq \varepsilon/2$, it follows $\|\chi_{A_n} f_n\|_1 \geq \varepsilon/2$, where $A_n^c := \Omega \backslash A_n$. But $(\chi_{A_n} f_n)_{n=1}^{\infty}$ is equi-itegrable, so there is $\delta > 0$, such that $\nu(A_n) > \delta$ for infinitely many A_n. Pass to a subsequence $(A_{j_n})_{n=1}^{\infty}$, so that $\nu(A_{j_n}) > \delta$ for all n. Thus

$$\nu(\limsup_n A_{j_n}) = \lim_n \nu\left(\bigcup_{k=n}^{\infty} A_{j_n}\right) \geq \delta.$$

Hence, for all $t \in \limsup_n A_{j_n}$, we get $\sum_{n=1}^{\infty} |f_{j_n}(t)| \geq \infty \cdot \varepsilon/2 = \infty$, a contradiction. □

Theorem 3.3. *Let V be an infinite-dimensional, reflexive subspace of L_1. Then, L_1 has two subspaces X and Y satisfying the following properties:*

(i) $L_1 = X \oplus Y$;
(ii) $X \simeq Y \simeq L_1$;
(ii) $X \cap V$ *and* $Y \cap V$ *are infinite-dimensional.*

Proof. By Proposition 2.2, V contains a normalized sequence $(f_n)_{n=1}^{\infty}$ for which there exists a density function ϕ and a real number $1 < p \leq 2$, such that $e^{itf_n} \xrightarrow[n]{w} e^{-|t|^p \phi^p}$ for all $t \in \mathbb{R}$. Let $H(t,\omega) := e^{-|t|^p \phi(\omega)^p}$ and $\Omega := \{\omega \colon \phi(\omega) \neq 0\}$.

Since V is reflexive and $(f_n)_{n=1}^{\infty}$ is norm bounded, it follows that $(f_n)_{n=1}^{\infty}$ is also bounded in probability, and passing to a subsequence if necessary, $(f_n)_{n=1}^{\infty}$ is determining (see the first Definition and Proposition 2.2, both in Sect. 2 of [5]). Thus, as $\mu(\Omega)\phi$ is a strictly positive density on the probability space $(\Omega, \mathcal{B}(\Omega), \mathrm{d}P)$, where $\mathrm{d}P := \mu/\mu(\Omega)$, Lemma 3.3 in [5] yields $(\frac{f_n|_\Omega}{\phi})_n$ is a determining sequence on the probability space $(\Omega, \mathcal{B}(\Omega), \nu)$, where $\nu = \phi\, \mathrm{d}\mu$, whose limit conditional characteristic function is $h(t,\omega) = h(t) := e^{-|t|^p/\mu(\Omega)^p}$.

As $h(t)$ is the characteristic function of a p-stable random variable, Theorem 3.1 in [5] (see its proof and also the comments at the beginning of page 500 in [5]) shows $(f_n)_n$ contains a subsequence $(f_{k_n})_n$ for which there exists a normalized sequence $(g_n)_{n=1}^{\infty}$ of independent, p-stable random variables defined on $(\Omega, \mathcal{B}(\Omega), \nu)$ whose common characteristic function is $e^{-|t|^p/\mu(\Omega)^p}$, and moreover

$$\sum_{n=1}^{\infty} \left| \frac{f_{k_n}}{\phi} - g_n \right| (\omega) < \infty \qquad \text{for } \mu - \text{a.e. } \omega \in \Omega. \tag{3.7}$$

The sequence $(g_n)_n$ is basic; indeed, it is equivalent to the unit vector basis of ℓ_p and the value of its basis constant is 1 (page 182 [26]).

Additionally, as $e^{itf_n|_{\mathbb{I}\setminus\Omega}} \xrightarrow[w]{n} 1$ for all $t \in \mathbb{R}$, where w denotes here the weak topology of $L_1(\Omega^c, \mathcal{B}(\Omega^c), \mathrm{dm})$, $\Omega^c := \mathbb{I} \setminus \Omega$ and $\mathrm{dm} := \mathrm{d}\mu/\mu(\Omega^c)$, then $\mathbb{E}(e^{itf_n|_{\mathbb{I}\setminus\Omega}}) \xrightarrow[n]{} 1$ pointwise in the probability space $(\Omega^c, \mathcal{B}(\Omega^c), \mathrm{dm})$ for all $t \in \mathbb{R}$. This implies that $(f_n \mid_{\mathbb{I}\setminus\Omega})_n$ converges in probability to 0 (Theorem 25.3 in [6]). From here, it is very easy to pick a subsequence $(f_{m_n})_n$ of $(f_{k_n})_n$, such that

$$\sum_{n=1}^{\infty} |f_{m_n}(t)| < \infty \qquad \text{for } \mu - \text{a.e. } t \in \mathbb{I} \setminus \Omega. \tag{3.8}$$

Next, as $(f_n)_{n=1}^{\infty}$ is equi-integrable, an application of Lemma 3.2 on (3.7) and on (3.8) yields a pair of subsequences $(x_n)_{n=1}^{\infty} \subset (f_{m_n})_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty} \subset$

$(g_n)_{n=1}^{\infty}$, such that

$$\sum_{n=1}^{\infty} \int_{\Omega} \left| \frac{x_n}{\phi} - y_n \right| \phi \, \mathrm{d}\mu < \infty \tag{3.9}$$

$$\sum_{n=1}^{\infty} \int_{\mathbb{I}\setminus\Omega} |x_n| \, \mathrm{d}\mu < \infty. \tag{3.10}$$

Fix a sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive numbers, such that $\sum_{n=1}^{\infty} \varepsilon_n < 1/4$. Passing to subsequences if necessary, we can assume by virtue of (3.9) and (3.10) that

$$\int_{\Omega} \left| \frac{x_n}{\phi} - y_n \right| \phi \, \mathrm{d}\mu < \varepsilon_n, \qquad n \in \mathbb{N} \tag{3.11}$$

$$\int_{\mathbb{I}\setminus\Omega} |x_n| \, \mathrm{d}\mu < \varepsilon_n, \quad n \in \mathbb{N}. \tag{3.12}$$

The inequality (3.11) and the principle of small perturbations (Proposition 1.a.9 in [25]) prove that $(\frac{x_n}{\phi}|_{\Omega})_n$ is a basic sequence equivalent to $(y_n)_n$ as sequences in $L_1(\Omega, \mathcal{B}(\Omega), \nu)$. However, the operator $G\colon L_1(\Omega, \mathcal{B}(\Omega), \mu) \longrightarrow L_1(\Omega, \mathcal{B}(\Omega), \nu)$ that maps each f to f/ϕ is a bijective isometry, so $(x_n \chi_\Omega)_n$ is a basic sequence in $L_1(\mathbb{I}, \mathcal{B}, \mu)$. Therefore, a second application of the principle of small perturbations in combination with (3.12) yields a positive integer n_0, such that $(x_n)_n = (x_n \chi_\Omega + x_n \chi_{\mathbb{I}\setminus\Omega})_{n=n_0}^{\infty}$ is basic in $L_1(\mathbb{I}, \mathcal{B}, \mu)$. Let $(a_n^*)_{n=n_0}^{\infty}$ be the sequence of coordinate functionals on span $\{x_n\}_{n=n_0}^{\infty}$ associated with $(x_n)_{n=n_0}^{\infty}$ and take a Hahn–Banach extension $x_n^* \in L_1(\mathbb{I})^*$ of each a_n^*, so that $(x_n^*)_{n=n_0}^{\infty}$ is bounded.

As G is a surjective isometry, so is the operator

$$J\colon L_1(\mathbb{I}) \longrightarrow L_1(\Omega, \mathcal{B}(\Omega), \nu) \oplus_1 L_1(\mathbb{I}\setminus\Omega, \mathcal{B}(\mathbb{I}\setminus\Omega), \mu)$$

that maps each f to $\left(\frac{f}{\phi}|_{\Omega}, f|_{\mathbb{I}\setminus\Omega}\right)$. For every $n \in \mathbb{N}$, let

$$z_n := \left(y_n - \frac{x_n}{\phi}|_{\Omega} \, , \, -x_n|_{\mathbb{I}\setminus\Omega} \right) \in L_1(\Omega, \nu) \oplus_1 L_1(\mathbb{I}\setminus\Omega, \mu).$$

It follows from (3.11) and (3.12) that $\|z_n\| < 2\varepsilon_n$ for all n.

Fix $0 < \varepsilon < 1$. Thus, as J is a bijective isometry, $J + K$ is a surjective isomorphism for every operator K from $L_1(\mathbb{I})$ into $L_1(\Omega, \nu) \oplus_1 L_1(\mathbb{I}\setminus\Omega, \mu)$ with $\|K\| < \varepsilon$. For $M = \sup\{\|x_n^*\|\}_{n=1}^{\infty}$, we can take a positive integer $n_1 \geq n_0$ large enough, so that $\sum_{n=n_1}^{\infty} 2\varepsilon_n < \varepsilon/M$, and therefore, the operator $K\colon L_1(\mathbb{I}) \longrightarrow L_1(\Omega, \nu) \oplus_1 L_1(\mathbb{I}\setminus\Omega, \mu)$ that maps each x to $\sum_{n=n_1}^{\infty} \langle x_n^*, x\rangle z_n$ is well defined and $\|K\| < \varepsilon$. Hence, $J + K$ is a bijective isomorphism and $(J + K)(x_n) = (y_n, 0)$ for all $n \geq n_1$.

Next, Theorem 3.1 provides a decomposition $L_1(\Omega, \nu) = Z_1 \oplus Z_2$, such that $Z_1 \simeq Z_2 \simeq L_1(\mathbb{I})$ and each Z_i contains infinitely many elements y_n. Therefore, the subspaces

$$X_1 := (J + K)^{-1}\big(Z_1 \oplus_1 L_1(\mathbb{I}\setminus\Omega, \mu)\big),$$
$$X_2 := (J + K)^{-1}(Z_2)$$

are both isomorphic to $L_1(\mathbb{I})$ and each X_i contains infinitely many elements x_n. Hence, $\dim X_i \cap V = \infty$ for all $1 \le i \le 2$, and the proof is done. □

4. The Strong Property (B) on L_p Spaces

We first prove that each $L_p(\mathbb{I})$, $1 \le p < \infty$, has the strong property (B).

Theorem 4.1. *Let $1 \le p < \infty$ and let V be an infinite-dimensional subspace of L_p. Then, there exists a decomposition $L_p = X_1 \oplus X_2$, such that both subspaces X_1 and X_2 are isomorphic to L_p and both intersections $V \cap X_1$ and $V \cap X_2$ are infinite-dimensional.*

Proof. The proof is organized into the cases (a) $p = 1$; (b) $p = 2$; (c) $p \in (1, 2) \cup (2, \infty)$.
(a) It is well known that an infinite-dimensional subspace of L_1 is either reflexive or contains a subspace isomorphic to ℓ_1 and complemented in L_1 [1] [Proposition 5.6.2]. In the first case, when V is reflexive, the result follows from Theorem 3.3. In the second case, V has a pair of subspaces E and Y, such that $E \simeq \ell_1$ and $L_1 = E \oplus Y$. Since L_1 is primary (Corollary 5.5 in [15]), Y is isomorphic to L_1. Hence, E and Y can be decomposed as $E = E_1 \oplus E_2$ and $Y = Y_1 \oplus Y_2$ where $\ell_1 \simeq E_1 \simeq E_2$ and $L_1 \simeq Y_1 \simeq Y_2$. Then, $X_i := E_i \oplus Y_i$ for $i = 1, 2$ are the required spaces in the statement.

It is straightforward that $L_1 = X_1 \oplus X_2$, $L_1 \simeq L_1 \oplus \ell_1 \simeq X_i$ and E_i is an infinite-dimensional subspace of $E \cap X_i$ for $i = 1, 2$.
(b) The proof for this case is immediate, because L_2 is a Hilbert space, and therefore, each of its infinite-dimensional subspaces is complemented and isomorphic to L_2.
(c) Take a normalized basic sequence $(x_n)_{n=1}^{\infty}$ in V and a sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive real numbers, such that $\sum_{n=1}^{\infty} \varepsilon_n < 1/4$. As $(x_n)_{n=1}^{\infty}$ is weakly null, it contains a subsequence $(x'_n)_{n=1}^{\infty}$ which can be approximated by a block basic sequence of the Haar basis $(h_i)_{i=0}^{\infty}$ in L_p, that is, there exist real numbers λ_i and an increasing sequence $(k_n)_{n=1}^{\infty}$ of non-negative integers with $k_1 = 0$, such that for $y_n := \sum_{i=k_n}^{k_{n+1}-1} \lambda_i h_i$

$$\|x'_n - y_n\| < \varepsilon_n \qquad \text{for all } n. \tag{4.1}$$

Let $(y_n^*)_{n=1}^{\infty} \subset L_p^*$ be a sequence of Hahn-Banach extensions of the coordinate functionals $(a_n^*)_{n=1}^{\infty} \subset \overline{\mathrm{span}}\,\{y_n\}_{n=1}^{\infty}$ associated with the basic sequence $(y_n)_{n=1}^{\infty}$.

Since the Haar basis of L_p is monotone, so is $(y_n)_{n=1}^{\infty}$ and, therefore, as $\|y_n\| < 1 + 1/4$, we have $\|y_n^*\| < 2 + 1/2$ for all n.

Thus, the operator $K\colon L_p \longrightarrow L_p$ that maps each x to $\sum_{n=1}^{\infty} \langle y_n^*, x\rangle (x'_n - y_n)$ is well defined and $\|K\| \le \sum_{n=1}^{\infty} \varepsilon_n \|y_n^*\| < 1/2 + 1/8 < 1$.

Consider the subspaces of L_p given by

$$Z_1 := \overline{\text{span}}\{h_i \colon k_{2n-1} \le i \le k_{2n} - 1,\ n \in \mathbb{N}\}$$
$$Z_2 := \overline{\text{span}}\{h_i \colon k_{2n} \le i \le k_{2n+1} - 1,\ n \in \mathbb{N}\}$$
$$Y_1 := \overline{\text{span}}\{y_{2n-1} \colon n \in \mathbb{N}\} \subset Z_1$$
$$Y_2 := \overline{\text{span}}\{y_{2n} \colon n \in \mathbb{N}\} \subset Z_2$$
$$V_1 := \overline{\text{span}}\{x'_{2n-1} \colon n \in \mathbb{N}\}$$
$$V_2 := \overline{\text{span}}\{x'_{2n} \colon n \in \mathbb{N}\}.$$

Note that $(I+K)(y_n) = x'_n$. Hence, $(I+K)(Y_1) = V_1$ and $(I+K)(Y_2) = V_2$. As $\|K\| < 1$, it follows that $I + K$ is a bijective isomorphism, and therefore, $Y_1 \simeq V_1$ and $Y_2 \simeq V_2$; analogously, for $W_1 := (I + K)(Z_1)$ and $W_2 := (I + K)(Z_2)$, we have $W_i \simeq Z_i$ for $i = 1, 2$. Moreover, since $(h_i)_{i=0}^{\infty}$ is unconditional, then $L_p = W_1 \oplus W_2$ and one of the following two cases happen (see [19] and also [3][Theorem 1, page 129]:)

- (i) both subspaces W_1 and W_2 are isomorphic to L_p;
- (ii) one of the subspaces W_1, W_2 is isomorphic to L_p and the other is isomorphic to ℓ_p.

If (i) holds, since $V_i \subset W_i$ for $i = 1, 2$, the subspaces $X_1 := W_1$ and $X_2 := W_2$ satisfy the statement of the theorem.

If (ii) holds, assume $W_1 \simeq L_p$ and $W_2 \simeq \ell_p$. Then, a similar argument to one given in case (a) works: since $V_2 \subset W_2 \simeq \ell_p$, W_2 can be decomposed as $W_2 = F_1 \oplus F_2 \oplus G$ where $F_1 \simeq F_2 \simeq \ell_p$ and $F_1 \oplus F_2 \subset V_2$ [1][Proposition 2.2.1]). Decompose W_1 as $W_1 = H_1 \oplus H_2$ with $H_1 \simeq H_2 \simeq L_p$. The subspaces $X_1 := H_1 \oplus F_1$ and $X_2 := H_2 \oplus F_2 \oplus G$ are both isomorphic to L_p, $L_p = X_1 \oplus X_2$ and $F_i \subset V \cap X_i$ are infinite-dimensional for $i = 1, 2$. The proof is complete. □

Cross and his coauthors proved the following result for L_p with $1 < p < \infty$ from the fact that these spaces have property (B). Thus, their proof also works for L_1:

Corollary 4.2. *Given any Banach space Y and an operator $T \in \mathcal{L}(L_1, Y)$, such that $R(T)$ is dense and non-closed in Y, there exists a pair of operators T_1 and T_2 in $\mathcal{L}(L_1, Y)$ satisfying the following conditions:*

- (i) *$R(T_i)$ is dense and non-closed in Y for $i = 1$ and $i = 2$;*
- (ii) *$R(T) = R(T_1) + R(T_2)$;*
- (iii) *$R(T_1) \cap R(T_2) = \{0\}$.*

Proof. Since L_1 has property (B) and is separable, Theorem 5.3 in [10] yields the desired result. □

Theorem 4.1 can be extended to finite measure spaces with no atom.

Proposition 4.3. *For every purely non-atomic, finite measure space (Ω, Σ, ν) and every $1 \le p < \infty$, the space $L_p(\Omega, \Sigma, \nu)$ has the strong property (B).*

Proof. Let V be an infinite-dimensional subspace of $L_p(\Omega, \Sigma, \nu)$, and let E be an infinite-dimensional, separable subspace of V. Let $\{x_j\}_{j\in\mathbb{N}}$ be a countable dense subset of E. Let $A_{j,a,b} := \{\omega \in \Omega\colon a < x_j(\omega) < b\}$ and let $\mathcal{M}$ be the sub-σ-algebra of Σ generated by $\{A_{j,a,b}\colon j \in \mathbb{N},\ a \in \mathbb{Q},\ b \in \mathbb{Q}\}$. Let $M := L_p(\Omega, \mathcal{M}, \nu|_{\mathcal{M}})$, so that $M \simeq L_p(\mathbb{I})$.

As observed in Proposition III.A.2 in [31], E is a separable subspace of M, and M is a complemented subspace of $L_p(\Omega, \Sigma, \nu)$ via the projection $\mathbb{E}(\cdot|\mathcal{M})$. Therefore, $E \subset M \oplus N = L_p(\Omega, \Sigma, \nu)$.

It is immediate from Theorem 9, page 127 in [24] that $L_p(\mathbb{I}) \times L_p(\Omega, \Sigma, \nu) \simeq L_p(\Omega, \Sigma, \nu)$ and $L_p(\Omega, \Sigma, \nu) \times L_p(\Omega, \Sigma, \nu) \simeq L_p(\Omega, \Sigma, \nu)$.

Hence, by virtue of Theorem 4.1, M can be decomposed as $M = A_1 \oplus A_2 \oplus B$ where $A_1 \simeq A_2 \simeq B \simeq L_p(\mathbb{I})$ and the intersections $E \cap A_1$ and $E \cap A_2$ are infinite-dimensional.

Thus

$$E \subset L_p(\Omega, \Sigma, \nu) = A_1 \oplus A_2 \oplus (B \oplus N).$$

However, $B \oplus N$ can be decomposed as $D_1 \oplus D_2$ where $D_1 \simeq D_2 \simeq L_p(\Omega, \Sigma, \nu)$, which leads to

$$E \subset A_1 \oplus A_2 \oplus D_1 \oplus D_2 = (A_1 \oplus D_1) \oplus (A_2 \oplus D_2) = L_p(\Omega, \Sigma, \nu).$$

Clearly, $L_p(\Omega, \Sigma, \nu) \simeq A_i \oplus D_i$ and $V \cap (A_i \oplus D_i)$ is infinite-dimensional for $i = 1, 2$, which proves that $L_p(\Omega, \Sigma, \nu)$ has the strong property (B). $\square$

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