Experimental evaluation of the drag coefficient of water rockets by a simple free fall test

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Abstract. The flight trajectory of a water rocket can be reasonably calculated if the magnitude of the drag coefficient is known. The experimental determination of this coefficient with enough precision is usually quite difficult, but in this paper we propose a simple free fall experiment for undergraduate students to reasonably estimate the drag coefficient of water rockets made from plastic soft drink bottles. The experiment is performed using relatively small fall distances (only about 14 m) in addition with a simple digital sound recording device. The fall time is inferred from the recorded signal with a quite good precision, and it is subsequently introduced as an input of a Matlab® program that estimates the magnitude of the drag coefficient. This procedure was tested first with a toy ball, obtaining a result with a deviation from the typical sphere value of only about 3%. For the particular water rocket used in the present investigation, a drag coefficient of 0.345 was estimated.

Keywords: water rocket, drag coefficient, free fall tests, educational experiment

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1. Introduction

The study of the motion of water rockets has been used as an appealing problem to teach the students how some general physics laws can be applied to real life [1-4]. A water rocket can be as simple as a plastic bottle for soft drinks. The bottle is partially filled with water and compressed air is subsequently introduced (using for example a hand pump). If the cap of the bottle is released, the compressed air can expel a jet of water through the nozzle. Due to the momentum conservation, and to the difference in density between water and air, the bottle can be propelled to significant heights.

The flight trajectory of a water rocket can be reasonably calculated provided that the effect of the drag force is taken into account. If the flow field around an immersed body is considered, it is known from the dimensional analysis that two non-dimensional coefficients are obtained [5, 6]: the drag coefficient C_d and the Reynolds number Re. The drag coefficient can be expressed as a function of the Reynolds number as shown in figure 1, and the drag force F_d is then derived from the following general formula:

$$F_d = 0.5 C_d \rho A v^2. \quad (1)$$

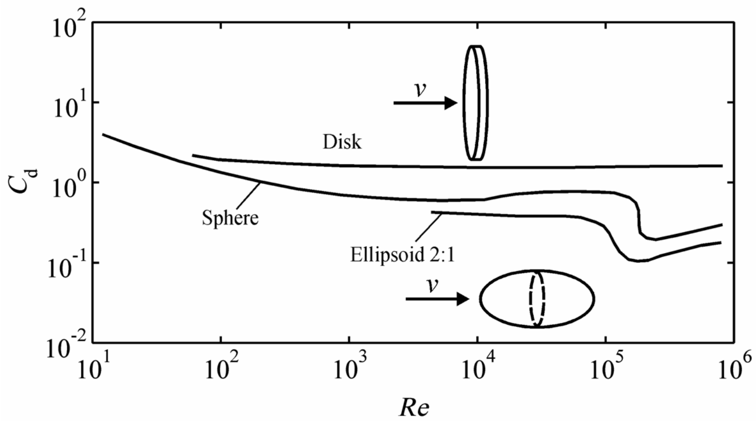

Figure 1. Magnitude of the drag coefficient C_d as a function of the Reynolds number for several 3D bodies of simple shape (adapted from reference [5]).

In the previous equation ρ is the fluid density, A is the cross-sectional area of the body and v is the relative velocity between object and fluid. As seen in figure 1, the drag coefficient C_d remains nearly constant between $10^3<Re<2\cdot10^5$ (the Reynolds number can be obtained as $Re=vD/\nu$, where ν is the kinematic viscosity of the fluid), and hence the amplitude of the drag force within this Reynolds interval is usually obtained with (1) by using a constant value of C_d. However, the drag coefficient is very dependent on the particular shape of the body, and its magnitude for the specific test body usually needs to be experimentally obtained.

The magnitude of the drag coefficient of an object can be inferred with quite good precision from the amplitude of the drag force if a wind tunnel is used [1, 7]. Wind tunnels are usually very expensive, and a real hands-on practice with this type of experimental equipment is difficult to carry out with the students. On the other hand, the drag coefficient can be also obtained by means of free fall tests. In this type of tests the body is dropped and the fall time is measured. The drag force is calculated from the difference between the real fall time and the theoretical inviscid free fall time. If the fall distance is large enough, the body reaches a constant fall velocity (the so-called terminal velocity), thus keeping a constant magnitude of C_d. Several educational experiments reported in the technical literature use this type of approach. The experiments are usually performed with desktop set-ups (small fall distances) and they use a large variety of measurement techniques to synchronize the measurement: a laser beam, an array of mirrors and a photocell [8], photogates [9], sonic motion sensors [10], computer video imaging and high-speed cameras [11], etc. However, these experiments are usually more illustrative than practical because, due to the complexity and cost of the equipment, the tests must be tutorized by the professor, and thus part of the real hands-on experience is lost.

Free fall tests can be also carried out using large fall distances and simple equipment as described in [12] and [13]. In this last reference, the authors designed an experiment to measure the drag coefficient of several cork and cast iron spheres of different radii by dropping them into the shafts of two abandoned mines. The magnitude of C_d was then inferred from the shaft depth, the recorded fall time, and the analytic resolution of the equations of motion. The simplicity of this procedure encouraged the authors to propose a full hands-on educational experiment by using smaller fall distances (from a building balcony to the street floor) and a chronometer to measure the fall time, though the precision of the C_d obtained was not very good.

During the last years we have been performing a water rocketry field practice with our undergraduate students of Fluid Mechanics. The students have to design a water rocket from a plastic soft drink bottle of 2 litres in volume and calculate its flight trajectory. They used to estimate the C_d of the rocket by try-and-error, but we thought it would be very illustrative for them to determine the magnitude of the drag coefficient for their specific water rockets. For this purpose, we designed a free fall experiment using as measurement stuff a simple digital recording device (iPod, mp3 player, PDA, mobile phone, etc.) to obtain the fall time. This device records the ambient noise during the fall of the rocket and, specifically, the sound emitted when the rocket is dropped and when it impacts on the ground. The recorded signal can be subsequently loaded onto an audio processing software, and the fall time can be inferred from the signal amplitude with a very good precision.

2. Theoretical background

The free fall of a body is governed by the combined effect of three forces: *i)* the gravity, *ii)* the Archimedes upthrust, and *iii)* the drag force. The first one causes the fall of the body, whereas the Archimedes upthrust and the drag force oppose its motion. Hence, with the application of Newton's second law, the equation of motion can be expressed as:

$$m\,\mathbf{a} = \mathbf{W} - \mathbf{U} - \mathbf{D}, \tag{2}$$

where m and $\mathbf{a}$ are respectively the real mass and the acceleration of the body in motion, $\mathbf{W}$ is the weight, $\mathbf{U}$ is the Archimedes upthrust, and $\mathbf{D}$ is the drag force. Equation (2) can be expressed in the direction of motion as follows:

$$m\frac{\mathrm{d}v_\mathrm{y}}{\mathrm{d}t} = m\,g - V_\mathrm{b}\rho g - 0.5\rho A C_\mathrm{d} v_\mathrm{y}^2 = m^*\,g - 0.5\rho A C_\mathrm{d} v_\mathrm{y}^2. \tag{3}$$

In the above equation v_y is the velocity, t is the time, g is the gravity acceleration, ρ is the air density, V_b is the volume of the body, $m^*=m-V_b\rho$ is the effective mass of the body once the Archimedes upthrust was accounted for, A is the frontal area, and C_d is the drag coefficient. If we define $k_1=m^*/m$, and $k_2=0.5\rho AC_d/m$, and these variables are introduced in (3) we can now write:

$$\frac{d^2y}{dt^2}=k_1g-k_2\left(\frac{dy}{dt}\right)^2. \tag{4}$$

This is a 2nd order Ordinary Differential Equation with the boundary conditions $y=0$, $v=0$ for $t=0$, and $y=h$ for $t=t_F$, where t_F is the fall time and h is the fall distance. This equation was split into a system of two 1st order differential equations by means of a variable change ($y_1=y$, $y_2=dy/dt$). The magnitude of the drag coefficient C_d is assumed as constant for the resolution of the system of equations (further discussion on this assumption will be presented in following sections). The system was solved in Matlab® as a Boundary Value Problem with one unknown parameter (namely the drag coefficient C_d).

Additionally, equation (4) can be solved analytically (as, for instance, in reference [11]), giving some hyperbolic trigonometric function for the velocity. This function can be subsequently integrated to obtain the position as a function of time, and leads to the general solution:

$$y(t)=k_1'\ln\left[\cosh\left(k_2' t\right)\right]. \tag{5}$$

However, we decided to solve equation (4) by programming because, as explained in the introduction, the present work is part of a water rocketry field practice. The students have to program the resolution of the control volume equations (that do not have an analytic solution) to obtain the flight trajectory of the water rocket. Thus, it is more straightforward for them to include the estimation of C_d in the general resolution procedure rather than solving equation (4) analytically.

There are six parameters that need to be known in equation (4) to determine the magnitude of C_d: the real mass m of the falling body, the air density at test conditions ρ, the frontal area A of the body (which can be obtained from the cross-sectional diameter D), the body volume V_b, the fall distance h and the fall time t_F. The magnitude of the effective mass was determined by using a small electronic balance to carry out the tests (m^* is actually the mass obtained in the balance, instead of the real mass m), and the air density ρ was calculated with the help of a barometer and a thermometer and the application of the perfect gas law. The magnitude of the real mass m was obtained from the values of m^*, ρ, and V_b (the volume V_b can be inferred with a ± 1 cm^3 precision by weighting the bottle empty and full of water). The cross-sectional diameter D was measured with a slide gauge, and a laser distance meter was used to obtain the fall distance h. A plumb line can be used instead of a laser meter, but this measurement must be carefully performed because, as discussed later, a small error causes a strong deviation in the magnitude of C_d.

A good precision in the measurement of time t_F is also required. For example, when considering the fall distance used in the present investigation, it was found that a difference of 0.01 s (this precision can be obtained with a conventional chronometer and ultra-rapid reflexes) caused a relative change in the C_d prediction of about 17%. Hence, the measurement of time with a chronometer is not suitable when using small fall distances, and so we had to consider an alternative way to obtain t_F, as explained in the next section.

3. Experimental set-up

The experimental tests were carried out in an interior stairwell of height h=14.360 ±0.002 m (from the base of the bottle to the ground), as depicted in figure 2. The fall time t_F was measured with a digital recording device (a mobile phone was used for the present work), which recorded the ambient noise during the fall of the bottle. The bottle is dropped at time instant t_1 (simultaneously, a sound is emitted), and it impacts on the ground at time instant t_2 (with a crash sound). The sound emitted at t_1 (drop) and also at t_2 (impact) is registered in the receptor as peaks in the recorded noise. Additionally, it must be borne in mind that the sound emitted both at t_1 and at t_2 reaches the receptor with a time delay, and hence the noise peaks are actually registered at time instants t_1' and t_2', as indicated in figure 3. This effect is not negligible: for the fall distance h and a speed of sound c=340 ms^{-1} there is a delay of 0.042 s in time measurement and, as explained in the previous section, this is expected to cause an important change in the estimated C_d.

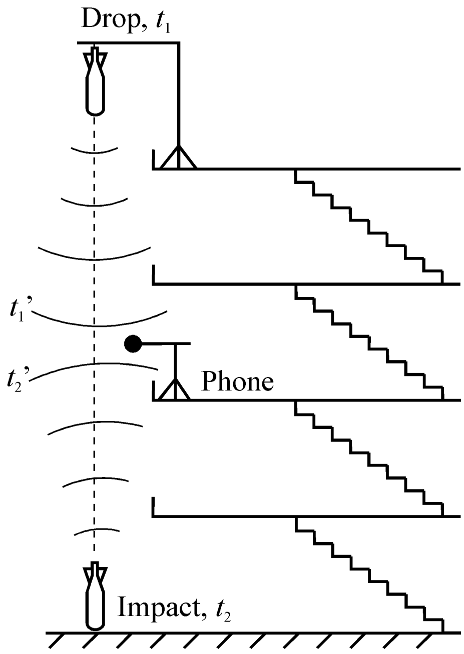

Figure 2. Schematic of the stairwell and experimental set-up.

The measured time can be corrected with the speed of sound and the distance from the drop and impact points to the measurement point. Also, the influence of time delay can be overridden by placing the receptor at the middle of the fall distance h (as we did in our tests). In this way, the magnitude of the drop time delay (t_1'-t_1) is the same as the impact time delay (t_2'-t_2), and the fall time that can be inferred from the recorded signal (t_F=t_2'-t_1') is the same as the real one t_F=t_2-t_1, as shown in figure 3. The experiment can be carried out with only two persons: one at the top of the stairs to drop the bottle and a second one, at the middle of the stairs, who has the task of recording the ambient noise and also to go downstairs and upstairs to pick the bottle (which in turn is a healthy exercise as checked by ourselves).

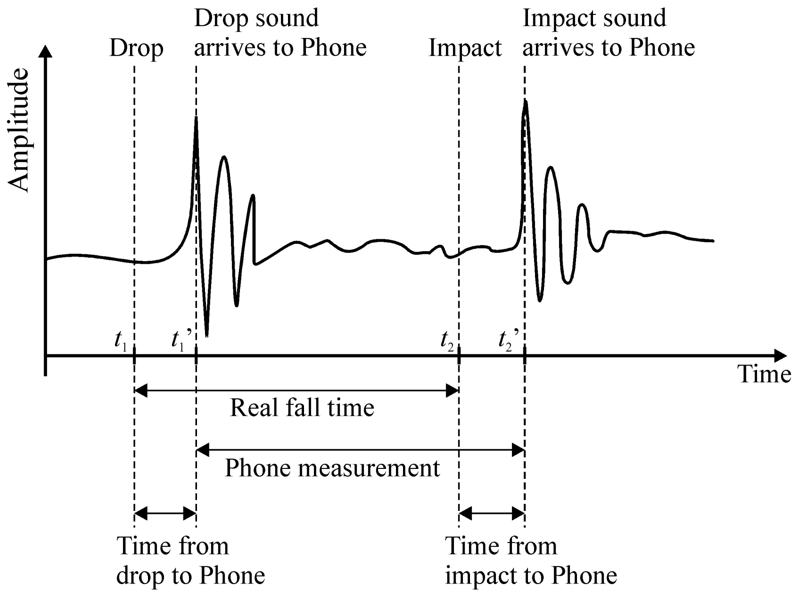

Figure 3. Detail of the time delays in a schematic sound signal.

During the experimental tests we noticed that the way the sound was emitted when dropping the bottle was very important. At first, the bottle was released by hand while shouting go!, but after several tests we discovered that there was an important dispersion among the recorded data. This was presumed to occur due to the time delay in the hand-mouth coordination mechanism (that is, the sound was not emitted exactly at the same time the bottle was dropped), and so we tried to make up a simple automatic system that could help to drop the bottle and to emit the sound simultaneously.

After several attempts we came up with the handcrafted tongs that are depicted schematically in figure 4. When the trigger is activated the tongs arms open and release the bottle and, simultaneously, the arms impact with the bell and a clear sound is emitted. By using this system, the dispersion was substantially smaller than the one observed when releasing the bottle by hand.

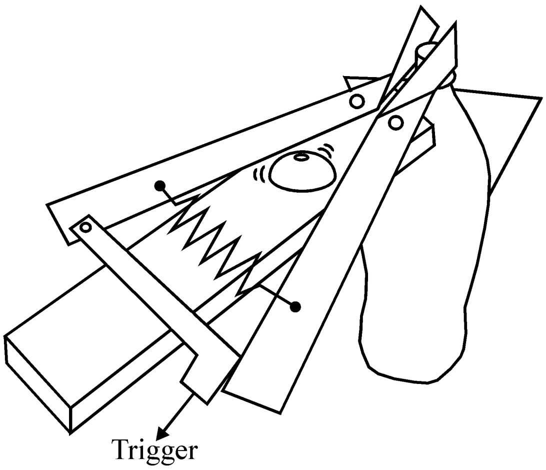

Figure 4. Simplified sketch of the tongs.

4. Results and discussion

The aforementioned procedure was tested before carrying out the experiments with the water rocket. For these tests we used a small toy ball of mass m^*=68 ±1 g and diameter D=126.0 ±0.1 mm. The air density at test conditions (1012 mbar, 15ºC) was estimated as ρ=1.225 ±0.004 kg m^{-3} and its kinematic viscosity as ν=(1.453 ±0.005)·10^{-5} m^{2}s^{-1}. As previously explained, the fall time t_F was obtained by using the recorded signal of the ambient noise and an audio processing software. One of these recorded signals is presented in figure 5 as an example of time measurement. As seen in this figure, there is at first a low level of ambient noise (we recommend to carry out this experiment within a quite environment) until the drop sound reaches the receptor at time t_1', where a strong rise in the amplitude of the sound signal is observed. From this time on, there is still a relatively high level of ambient noise due to sound reverberation, but this level decreases continuously as the ball falls down. When the ball impacts on the floor at time t_2' a new rise in amplitude is observed. The audio processing software can be used to obtain the fall time t_F from the difference t_2'-t_1' with a precision of 2·10^{-4} s (the precision of a digital sound recording device is usually well above 10 kHz, that is, much higher than that of a conventional chronometer) provided that there is not an important temperature variation along the fall distance that could cause a change in the magnitude of the speed of sound. The time delays of figure 3 do not cancel each other if the speed of sound changes significantly between the drop and impact points. We did some temperature measurements at the top and at the bottom of the stairs, and a temperature variation less than 1ºC was found. This variation implies a maximum time difference between the two time signals of about $4 \cdot 10^{-5}$ s (the speed of sound can be calculated as $c = \sqrt{\gamma R T}$), which is about one order of magnitude below the precision of the recording device.

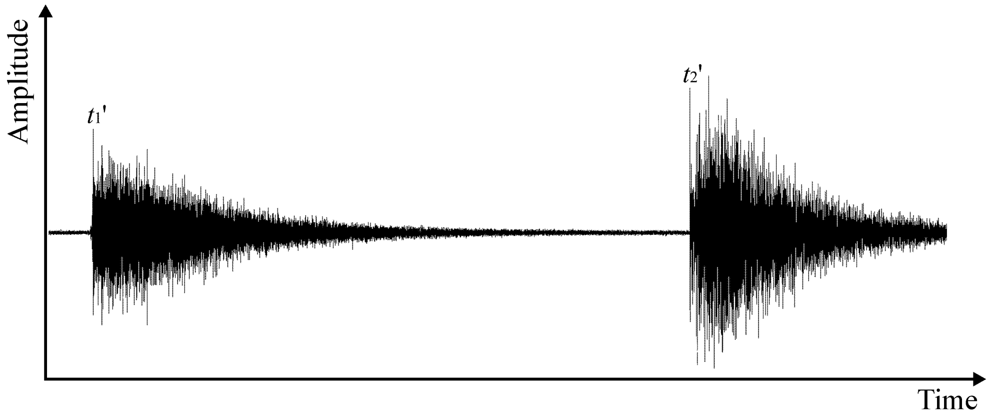

Figure 5. Example of the sound signal recorded during the fall of the toy ball.

We carried out several tests with the toy ball. The measured fall times were averaged and a time of t_F=1.9539 s was obtained. The experimentally determined value of t_F was used as an input for the Matlab® program, which estimated a drag coefficient of C_d=0.484. This estimation is in very good agreement with the typical bibliography results, which usually show a drag coefficient for a sphere of about 0.47 (see for instance reference [5]). Hence, the test procedure was considered adequate, and it was used to obtain the drag coefficient of a water rocket (borrowed from one of our students). This specific bottle has an effective mass of m^*=133 ±1 g and a maximum cross-sectional diameter of D=106.0 ±0.1 mm. We carried out 10 tests with this rocket and an averaged fall time of t_F=1.7842 s was obtained. The averaged fall time was introduced in the program as in the previous case, and the magnitude of the drag coefficient of the rocket obtained from the program resulted C_d=0.345.

The computer program can be also used to obtain the evolution of the position h and of the fall velocity v as a function of time. This is shown in figure 6 for the water rocket and the toy ball and, additionally, the evolution for a free fall without drag is also plotted in the same figure. As observed, the magnitude of h decreases exponentially with time and, for any considered time t, the distance covered by the water rocket is higher than that covered by the toy ball due to the lower magnitude of rocket drag coefficient. The time evolution for the fall without drag shows the influence of C_d: there is a 6% difference in t_F between the experiment and the fall in the absence of drag for the water rocket and, if the toy ball is considered (higher drag coefficient), this difference increases up to 15%.

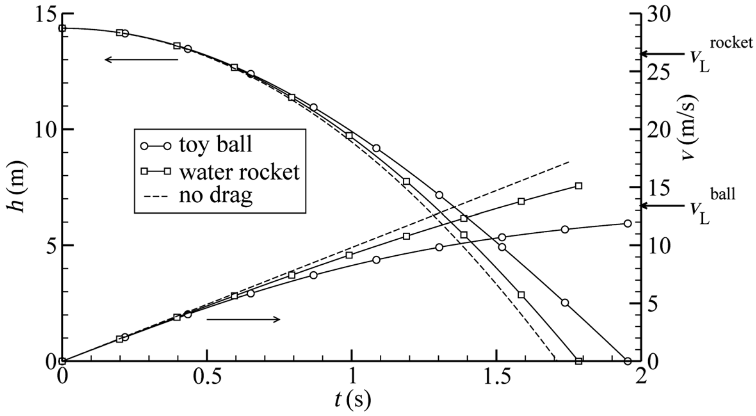

Figure 6. Time evolution of position and velocity for the toy ball and the water rocket. The evolution for a free fall without drag is also plotted in the figure.

Free fall tests are usually carried out using large fall distances to reach terminal velocity as in refs [12, 13]. If the fall time t_F is obtained under terminal conditions, it is assured that the drag coefficient remains constant because the fall velocity is constant. However, for the present investigation a fall distance h>36 m is required for the water rocket to reach a constant fall velocity, and consequently the water rocket did not reach its terminal velocity v_L during the tests (neither did the toy ball, by the way), as observed in figure 6. Nonetheless, as seen in figure 7 (which presents the time evolution of the Reynolds number), a Re=10^3 is reached at about 0.02 s (1.1% of total fall time) and, on the other hand, the Reynolds number keeps below $2\cdot10^5$ during the fall time t_F. Hence, it can be concluded that, although the water rocket and the toy ball did not reach terminal velocity, the magnitude of C_d remained nearly constant (see figure 1) because laminar regime was guaranteed in the tests (the transition to turbulent regime takes place at about Re=$2\text{-}4\cdot10^5$).

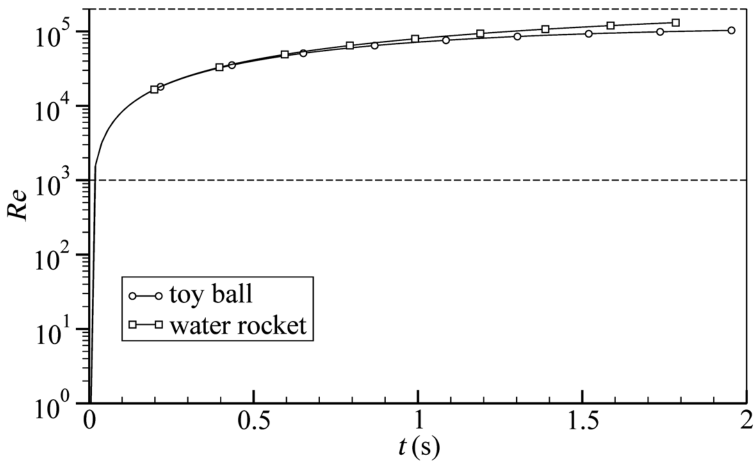

Figure 7. Time evolution of the Reynolds number for the toy ball and the water rocket during the free fall tests.

We investigated whether this coefficient could fit the C_d of a three-dimensional body of simple shape. It was found that the estimated C_d for the water rocket was very close to the drag coefficient of an ellipsoid with an L/D ratio of 1.5, as indicated in figure 8. The typical value of the drag coefficient for an ellipsoid with this ratio in the laminar regime (see for instance [5]) is C_d=0.37.

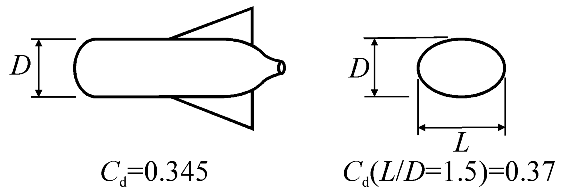

Figure 8. Comparison of the drag coefficient of the rocket with that of an ellipsoid.

5. Sensitivity analysis of the predictions

Once the experimental tests with the water rocket were completed, a sensitivity analysis of the equations with respect to the different parameters was carried out. The purpose of this analysis is to estimate the precision of the predicted C_d of the rocket; the results of the study are shown in figure 9 for each measured variable. In this figure, the vertical axis presents the relative change of C_d (only positive changes have been plotted) as a function of the relative change of the generic variable x, where x can be any of the parameters (V_b, m^*, ρ, D, h, and t_F).

As seen in figure 9, the relation between the change in C_d and the change in any of the measured variables follows a linear trend for small values of the change in the parameters. It is clear that the precision of V_b has low influence in the prediction of C_d to estimate the drag coefficient with at least a 10% precision. The magnitude of m^* and ρ should be obtained with a precision higher than 7.8% and 7.2% respectively. The cross-sectional diameter of the body D should be measured at least with a precision of 4.6%. Finally, it is also observed in figure 9 that the fall distance h and the fall time t_F are the variables that most influence the prediction of C_d: they should be measured with a precision of 0.6% and 0.3% respectively.

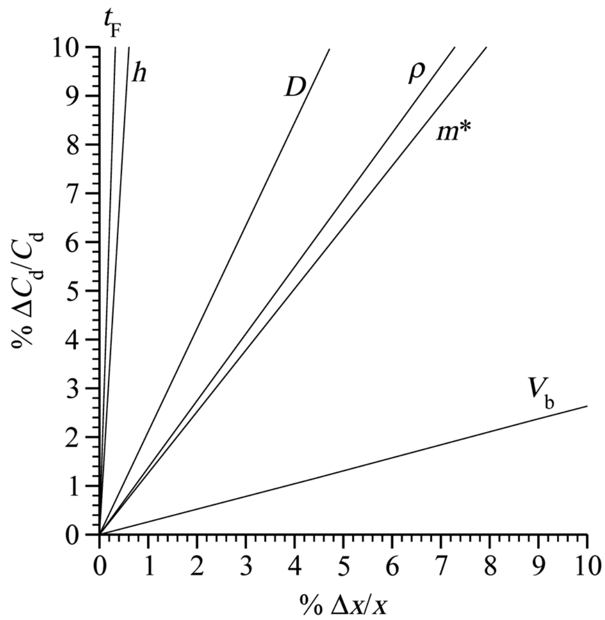

Figure 9. Results of the sensitivity analysis of the predictions.

The relative precision of the measurement of each variable for our water rocket tests is summarized in table 1. Additionally, in its last column, this table presents the precision of the predicted C_d in accordance with the sensitivity analysis previously presented.

Table 1. Summarized measurement and estimated precision.

Variable	Measurement precision (%)	C_d precision, e_i (%)
V_b	0.05	0.01
m^*	0.75	0.95
ρ	0.33	0.45
D	0.09	0.20
h	0.01	0.23
t_F	0.01	0.34

If we assume that the measured variables are independent among them, then the global precision can be calculated as follows:

$$e = \sqrt{\sum_{\mathrm{i}} e_{\mathrm{i}}^2} = 1.14\%, \qquad (6)$$

and hence the drag coefficient of the water rocket can be estimated as C_d=0.345 ±0.004 for values of the Reynolds number above 10^3 and below $2 \cdot 10^5$.

6. Conclusions

The drag coefficient of water rockets made from plastic soft drink bottles can be experimentally determined with a very good precision by means of simple free fall tests. In the educational experiment reported in the present paper we used a fall distance of only about 14 m, and the fall time was measured with a simple sound recording device (namely a mobile phone). The fall time was inferred from the changes in amplitude recorded in the sound signal when the bottle was dropped and when it impacted on the ground. The measured fall time was used as an input of a Matlab® program that solved the differential equations of motion thus calculating the magnitude of C_d. This procedure was validated first by using a spherical body (a small toy ball) in the free fall tests. It was found that the predicted C_d of the ball only deviated about 3% from the typical values found in the bibliography. A sensitivity analysis was also carried out to estimate the precision of the predicted C_d. The experimental tests performed produced an estimation of the drag coefficient of the water rocket of C_d=0.345 ±0.004 within the Reynolds interval $10^3 < Re < 2 \cdot 10^5$. It was found that the magnitude of this coefficient fits reasonably well with that of an ellipsoid with a length/diameter ratio of 1.5.

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