Error estimates for the numerical approximation of optimal control problems with nonsmooth pointwise-integral control constraints*

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Abstract

The numerical approximation of an optimal control problem governed by a semilinear parabolic equation and constrained by a bound on the spatial L^1-norm of the control at every instant of time is studied. Spatial discretizations of the controls by piecewise constant and continuous piecewise linear functions are investigated. Under finite element approximations the sparsity properties of the continuous solutions are preserved in a natural way using piecewise constant approximations of the control, but suitable numerical integration of the objective functional and of the constraint must be used to keep the sparsity pattern when using spatially continuous piecewise linear approximations. We also obtain error estimates and finally present some numerical examples. optimal control; semilinear partial differential equations; discontinuous Galerkin approximations; error estimates.

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1 Introduction

In this paper, we study the numerical approximation of the optimal control problem

$$\text{(P)} \quad \inf_{u\in U_{ad}} J(u) := \frac{1}{2}\int_Q (y_u(x,t) - y_d(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_Q u(x,t)^2 \,\mathrm{d}x\,\mathrm{d}t,$$

where $\kappa > 0$, $Q = \Omega \times (0,T)$, with $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, a convex polygonal/polyhedral domain with boundary Γ and $0 < T < +\infty$ is fixed,

$$U_{ad} = \{u \in L^\infty(Q) : \|u(t)\|_{L^1(\Omega)} \leq \gamma \text{ for a.a. } t \in (0,T)\}$$

with $0 < \gamma < +\infty$. Further y_u is the solution of the semilinear parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + a(x,t,y) = u & \text{in } Q = \Omega \times (0,T), \\ y = 0 \text{ on } \Sigma = \Gamma \times (0,T), & y(0) = y_0 \text{ in } \Omega \end{cases} \tag{1.1}$$

with

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i} y).$$

Precise assumptions on the operator A and the nonlinearity a are given below.

This problem was studied in [8], where the authors proved existence of a solution, and obtained first and second order optimality conditions. As it is emphasized in that paper, there are two special difficulties in the study of (P). The first one is given by the fact that, in order to be able to deal with strong nonlinear terms such as $a(x,t,y) = a_0(x,t)\exp(y)$ with $a_0 \in L^\infty(Q)$, the framework for the control space cannot be $L^2(Q)$, but should be $L^q(Q)$ with q large enough. This implies that the usual techniques to prove existence of a solution fail, rather, a truncation argument on a is used for this purpose. The second difficulty is the nondifferentiability of the constraint. First order optimality conditions are obtained using the convexity of U_{ad}. Second order optimality conditions require a careful setting of the cone of critical directions in order to obtain sufficient conditions with a minimal gap with respect to the necessary ones. With the aid of first order optimality conditions, sparsity properties of the optimal control are derived.

There are numerous references regarding the numerical analysis of problems governed by partial differential equations. Not trying to be exhaustive, and considering only distributed optimal control problems governed by parabolic equations, we can cite [23] (linear equation, no constraints), [13] (semilinear equation, but only dimension 2 and not strong nonlinear terms, no constraints), [1] (discontinuous elements for linear convection-diffusion), [22, 17, 29] (linear, pointwise control-constraints), [20], (space-time spectral discretization), [10, 5] (semilinear, sparsity-promoting term in the functional, no constraints), [11] (semilinear, pointwise control-constraints, no Thikonov regularization), [12] (linear, state constraints), [19] (quasilinear, pointwise state constraints).

The only reference that we have been able to find with a pointwise constraint in time on the norm of the control is [18]. In that reference, the authors impose the differentiable constraint $\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq 1$. However, they do not address the obtention of error estimates for the discrete problems.

Our objectives in this paper are to discretize (P) in such a way that the sparsity properties are preserved, to prove convergence of the discrete solutions to the solutions of the continuous problem, and to obtain error estimates. To discretize the state equation, we use a discontinuous Galerkin scheme, computationally equivalent to the implicit Euler method. For the discretization in space of the state and the adjoint state, continuous piecewise linear finite elements are used, while for the control we study both piecewise constant and continuous piecewise linear approximations. The use of piecewise constant elements leads in a natural way to sparsity properties of the discrete optimal control consistent with those obtained for the continuous problem, but a straightforward discretization of (P) using continuous piecewise linear space-approximations of the control, may result in a loss of the sparsity properties due to the use of a mass matrix. To overcome this difficulty, we discretize the norm in $L^p(\Omega)$, $p = 1, 2$ with the help of the lumped mass matrix and use Carstensen's quasi interpolation operator. A similar technique for problems with sparsity-promoting terms in the functional was used in [6] for a problem governed by a semilinear elliptic equation and in [5] for a problem governed by a semilinear parabolic equation; this technique is also found in the thesis by [26] and in [28].

The plan of this paper is as follows. In Section 2 we recall some results from [8] concerned with the continuous problem. In Section 3 the problem is discretized and the sparsity properties of the discrete solution are established. In Section 4 we prove convergence and obtain error estimates. Finally, in Section 5, numerical examples are presented to illustrate the results obtained in the paper.

2 Assumptions and preliminary results

We make the following assumptions along this paper.

Assumption 1- $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is a convex polygonal/polyhedral domain, and $0 < T < \infty$ is fixed. Γ denotes the boundary of Ω. The coefficients of the operator A satisfy: a_{ij} are Lipschitz functions in $\bar\Omega$ for every $1 \le i, j \le n$, and

$$\Lambda_A|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n \text{ and for a.a. } x \in \Omega \tag{2.1}$$

for some $\Lambda_A > 0$. For the initial state we suppose that $y_0 \in H_0^1(\Omega) \cap C^{0,\alpha}(\bar\Omega)$, where $C^{0,\alpha}(\bar\Omega)$ denotes the space of α-Hölder continuous functions in $\bar\Omega$ with $\alpha \in (0, 1]$.

Assumption 2- We assume that $a : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the last variable satisfying the following properties:

$$\exists C_a \in \mathbb{R} : \frac{\partial a}{\partial y}(x, t, y) \ge C_a \ \forall y \in \mathbb{R}, \tag{2.2}$$

$$a(\cdot, \cdot, 0) \in L^{\hat r}(0, T; L^2(\Omega)), \text{ with } \hat r > \frac{4}{4-n}, \tag{2.3}$$

$$\forall M > 0\ \exists C_{a,M} > 0 : \left|\frac{\partial^j a}{\partial y^j}(x, t, y)\right| \le C_{a,M}\ \forall |y| \le M \text{ and } j = 1, 2, \tag{2.4}$$

$$\begin{aligned}&\forall \rho > 0 \text{ and } \forall M > 0\ \exists \varepsilon > 0 \text{ such that}\\ &\left|\frac{\partial^2 a}{\partial y^2}(x, t, y_1) - \frac{\partial^2 a}{\partial y^2}(x, t, y_2)\right| < \rho \quad \forall |y_1|, |y_2| \le M \text{ with } |y_1 - y_2| < \varepsilon,\end{aligned} \tag{2.5}$$

for almost all $(x, t) \in Q$.

Assumption 3- In the control problem (P), we assume that $\kappa > 0$, $\gamma > 0$, and $y_d \in L^{\hat r}(0, T; L^2(\Omega))$.

As usual we denote $H^{2,1}(Q) = L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$. Then, we have the following result.

Theorem 2.1. *Under Assumptions 1 and 2, for every $u \in L^r(0, T; L^p(\Omega))$ with $\frac{1}{r} + \frac{n}{2p} < 1$ and $r, p \ge 2$ there exists a unique solution $y_u \in C^{0,\beta}(\bar Q) \cap H^{2,1}(Q)$ of (1.1) with $\beta \in (0, \alpha]$. Moreover, the following estimate holds*

$$\begin{cases} \|y_u\|_{C^{0,\beta}(\bar Q)} + \|y_u\|_{H^{2,1}(Q)} \le \eta\big(\|u\|_{L^r(0,T;L^p(\Omega))} + M_{\hat r,0}\big), \\ \|y_u\|_{L^\infty(0,T;L^2(\Omega))} + \|y_u\|_{L^2(0,T;H_0^1(\Omega))} \le C\big(\|u\|_{L^2(Q)} + M_{2,0}\big) \end{cases} \tag{2.6}$$

for a constant C and a monotone nondecreasing function $\eta : [0,\infty) \longrightarrow [0,\infty)$ with $\eta(0) = 0$ independent of u, and

$$M_{\hat r,0} = \|a(\cdot,\cdot,0)\|_{L^{\hat r}(0,T;L^2(\Omega))} + \|y_0\|_{C^{0,\alpha}(\bar\Omega)} + \|y_0\|_{H_0^1(\Omega)},$$
$$M_{2,0} = \|a(\cdot,\cdot,0)\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}.$$

The existence of a unique solution of (1.1) in the space $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ as well as the estimates in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$ were proved in [8]. The $H^{2,1}(Q)$ regularity is a well known consequence of the convexity of Ω and the $H_0^1(\Omega)$ regularity of y_0. The reader is referred to [21, Chap. III-§10] or [15] for the $C^{0,\beta}(\bar Q)$ regularity.

Taking $p = 2$ and $r \in \left(\frac{4}{4-n},\infty\right]$ we have that $\frac{1}{r} + \frac{n}{4} < 1$ and $r > 2$. Then, from Theorem 2.1 we deduce that the mapping $G : L^r(0,T;L^2(\Omega)) \longrightarrow H^{2,1}(Q) \cap L^\infty(Q)$ given by $G(u) = y_u$ is well defined. Further, we have the following differentiability properties.

Theorem 2.2. *The mapping G is of class C^2. For $u, v, v_1, v_2 \in L^r(0,T;L^2(\Omega))$ the derivatives $z_v = G'(u)v$ and $z_{v_1,v_2} = G''(u)(v_1,v_2)$ are the solutions of the equations*

$$\begin{cases} \dfrac{\partial z_v}{\partial t} + Az_v + \dfrac{\partial a}{\partial y}(x,t,y_u)z_v = v \quad \text{in } Q, \\ z_v = 0 \ \text{ on } \Sigma, \quad z_v(0) = 0 \ \text{ in } \Omega, \end{cases} \tag{2.7}$$

$$\begin{cases} \dfrac{\partial z_{v_1,v_2}}{\partial t} + Az_{v_1,v_2} + \dfrac{\partial a}{\partial y}(x,t,y_u)z_{v_1,v_2} + \dfrac{\partial^2 a}{\partial y^2}(x,t,y_u)z_{v_1}z_{v_2} = 0 \quad \text{in } Q, \\ z_{v_1,v_2} = 0 \ \text{ on } \Sigma, \quad z_{v_1,v_2}(0) = 0 \ \text{ in } \Omega \end{cases} \tag{2.8}$$

where $z_{v_i} = G'(u)v_i$, $i = 1,2$.

This theorem was proved in [8] with a change in the range of G, namely $G : L^r(0,T;L^2(\Omega)) \longrightarrow L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \cap L^\infty(Q)$. The proof given there can be adapted using the extra regularity of the data of the state equation and Theorem 2.1.

Theorem 2.2 along with the chain rule leads to the following differentiability properties of the cost functional J.

Corollary 2.3. *If $r > \frac{4}{4-n}$, then $J : L^r(0,T;L^2(\Omega)) \longrightarrow \mathbb{R}$ is of class C^2 and its derivatives are given by the expressions*

$$J'(u)v = \int_Q (\varphi + \kappa u)v \,\mathrm{d}x\,\mathrm{d}t, \tag{2.9}$$

$$J''(u)(v_1,v_2) = \int_Q \Big[\big(1 - \frac{\partial^2 a}{\partial y^2}(x,t,y_u)\varphi\big)z_{v_1}z_{v_2} + \kappa v_1 v_2\Big]\,\mathrm{d}x\,\mathrm{d}t, \tag{2.10}$$

where $z_{v_i} = G'(u)v_i$, $i = 1,2$, and $\varphi \in C^{0,\beta}(\bar Q) \cap H^{2,1}(Q)$ is the solution of the adjoint state equation

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi + \dfrac{\partial a}{\partial y}(x,t,y_u)\varphi = y_u - y_d \quad \text{in } Q, \\ \varphi = 0 \ \text{ on } \Sigma, \quad \varphi(T) = 0 \ \text{ in } \Omega, \end{cases} \tag{2.11}$$

with $A^*\varphi = -\sum_{i,j=1}^n \partial_{x_j}(a_{ji}(x)\partial_{x_i}\varphi)$ the adjoint operator of A.

Concerning the control problem (P), the following theorem and corollaries follow from [8].

Theorem 2.4. *There exists at least one solution of* (P). *Moreover, for every local minimizer* $\bar u$ *in the* $L^r(0,T;L^2(\Omega))$ *sense with* $r > \frac{4}{4-n}$, *there exist* $\bar y, \bar\varphi \in H^{2,1}(Q)\cap C^{0,\beta}(\bar Q)$, *and* $\bar\mu \in L^\infty(Q)$ *such that*

$$\left\{\begin{array}{l} \dfrac{\partial \bar y}{\partial t} + A\bar y + a(x,t,\bar y) = \bar u \quad \text{in } Q,\\ \bar y = 0 \text{ on } \Sigma, \quad \bar y(0) = y_0 \text{ in } \Omega,\end{array}\right. \tag{2.12}$$

$$\left\{\begin{array}{l} -\dfrac{\partial \bar\varphi}{\partial t} + A^*\bar\varphi + \dfrac{\partial a}{\partial y}(x,t,\bar y)\bar\varphi = \bar y - y_d \quad \text{in } Q,\\ \bar\varphi = 0 \text{ on } \Sigma, \quad \bar\varphi(T) = 0 \text{ in } \Omega,\end{array}\right. \tag{2.13}$$

$$\int_Q \bar\mu(u-\bar u)\,\mathrm{d}x\,\mathrm{d}t \le 0 \quad \forall u \in U_{ad}, \tag{2.14}$$

$$\bar\varphi + \kappa\bar u + \bar\mu = 0. \tag{2.15}$$

Let us denote by $\mathrm{Proj}_{B_\gamma} : L^2(\Omega) \longrightarrow B_\gamma \cap L^2(\Omega)$ the $L^2(\Omega)$ projection, where $B_\gamma = \{v \in L^1(\Omega) : \|v\|_{L^1(\Omega)} \le \gamma\}$.

Corollary 2.5. *Let* $\bar u$, $\bar\varphi$, *and* $\bar\mu$ *satisfy* (2.12)–(2.15) *and assume that* $\bar u \in U_{ad}$. *Then, the following properties hold*

$$\int_\Omega \bar\mu(t)(v-\bar u(t))\,\mathrm{d}x \le 0 \quad \forall v \in B_\gamma \quad \text{and for a.a. } t \in (0,T), \tag{2.16}$$

$$\bar u(t) = \mathrm{Proj}_{B_\gamma}\big(-\frac{1}{\kappa}\bar\varphi(t)\big) \text{ for a.a. } t \in (0,T), \tag{2.17}$$

$$\left\{\begin{array}{l} \bar u(x,t)\bar\mu(x,t) = |\bar u(x,t)||\bar\mu(x,t)| \text{ for a.a. } (x,t) \in Q,\\ \text{if } \|\bar u(t)\|_{L^1(\Omega)} < \gamma \text{ then } \bar\mu(t) \equiv 0 \text{ in } \Omega \text{ a.e. in } (0,T),\\ \text{if } \|\bar u(t)\|_{L^1(\Omega)} = \gamma \text{ and } \bar\mu(t) \not\equiv 0 \text{ in } \Omega,\\ \qquad\qquad \text{then } \mathrm{supp}(\bar u(t)) \subset \{x \in \Omega : |\bar\mu(x,t)| = \|\bar\mu(t)\|_{L^\infty(\Omega)}\}.\end{array}\right. \tag{2.18}$$

Corollary 2.6. *Let* $\bar u \in U_{ad}\cap L^\infty(Q)$ *satisfy* (2.15) *and* (2.18). *Then, the following identities are fulfilled*

$$\begin{aligned}\bar u(x,t) &= -\frac{1}{\kappa}\,\mathrm{sign}(\bar\varphi(x,t))\big(|\bar\varphi(x,t)| - \|\bar\mu(t)\|_{L^\infty(\Omega)}\big)^+\\ &= -\frac{1}{\kappa}\left\{\Big[\bar\varphi(x,t) + \|\bar\mu(t)\|_{L^\infty(\Omega)}\Big]^- + \Big[\bar\varphi(x,t) - \|\bar\mu(t)\|_{L^\infty(\Omega)}\Big]^+\right\}.\end{aligned} \tag{2.19}$$

Moreover, the regularities $\bar u \in H^1(Q)$ *and* $\bar\mu \in H^1(Q)$ *hold.*

We finish this section by considering the second order optimality conditions. To this end we introduce some notation. We consider the Lipschitz continuous and convex mapping $j : L^1(\Omega) \longrightarrow \mathbb{R}$ defined by $j(v) = \|v\|_{L^1(\Omega)}$. Its directional derivative is given by the expression

$$j'(u;v) = \int_{\Omega_u^+} v(x)\,\mathrm{d}x - \int_{\Omega_u^-} v(x)\,\mathrm{d}x + \int_{\Omega_u^0} |v(x)|\,\mathrm{d}x \quad \forall u,v \in L^1(\Omega), \tag{2.20}$$

where

$$\Omega_u^+ = \{x \in \Omega : u(x) > 0\},\ \Omega_u^- = \{x \in \Omega : u(x) < 0\} \text{ and } \Omega_u^0 = \Omega \setminus (\Omega_u^+ \cup \Omega_u^-).$$

Given an element $\bar u \in U_{ad}$ satisfying the first order optimality conditions (2.12)–(2.15), set

$$I_\gamma = \{t \in (0,T) : j(\bar u(t)) = \gamma\} \quad \text{and} \quad I_\gamma^+ = \{t \in I_\gamma : \bar\mu(t) \not\equiv 0 \text{ in } \Omega\}.$$

Now, we define the cone of critical directions associated with $\bar u$

$$C_{\bar u} = \Big\{ v \in L^2(Q) : J'(\bar u)v = 0 \text{ and } j'(\bar u(t); v(t)) \left\{ \begin{array}{ll} = 0 & \text{if } t \in I_\gamma^+, \\ \le 0 & \text{if } t \in I_\gamma \setminus I_\gamma^+, \end{array} \right. \Big\}.$$

Then, we have the following theorem, whose proof can be found in [8][1].

Theorem 2.7. *Let $\bar u$ be a local solution of* (P) *in the $L^r(0,T;L^2(\Omega))$ sense with $r > \frac{4}{4-n}$. Then, the inequality $J''(\bar u)v^2 \ge 0$ holds for all $v \in C_{\bar u}$. Reciprocally, if $\bar u \in U_{ad}$ satisfies the first order optimality conditions and the second order condition $J''(\bar u)v^2 > 0$ $\forall v \in C_{\bar u} \setminus \{0\}$, then there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$J(\bar u) + \frac{\delta}{2}\|u - \bar u\|^2_{L^2(Q)} \le J(u) \quad \forall u \in U_{ad} \cap B_\varepsilon(\bar u), \tag{2.21}$$

where $B_\varepsilon(\bar u) = \{u \in L^r(0,T;L^2(\Omega)) : \|u - \bar u\|_{L^r(0,T;L^2(\Omega))} \le \varepsilon\}$.

Given $s > 0$ we define the extended cone

$$C^s_{\bar u} = \Big\{ v \in L^2(Q) : |J'(\bar u)v| \le s\|v\|_{L^2(Q)} \text{ and } \left\{ \begin{array}{ll} |j'(\bar u(t); v(t))| \le s\|v\|_{L^2(Q)} & \text{if } t \in I_\gamma^+, \\ j'(\bar u(t); v(t)) \le s\|v\|_{L^2(Q)} & \text{if } t \in I_\gamma \setminus I_\gamma^+, \end{array} \right. \Big\}.$$

Then, we have the following result.

Theorem 2.8. *Let $\bar u \in U_{ad}$ satisfy the first order optimality conditions* (2.12)–(2.15) *and the second order condition $J''(\bar u)v^2 > 0$ $\forall v \in C_{\bar u} \setminus \{0\}$. Then, for every $r \in \big(\frac{4}{4-n}, \infty]$ there exist strictly positive numbers ε, s, λ such that*

$$J''(u)v^2 \ge \lambda \|v\|^2_{L^2(Q)} \quad \forall v \in C^s_{\bar u} \ \text{ and } \ \forall u \in B_\varepsilon(\bar u), \tag{2.22}$$

where $B_\varepsilon(\bar u)$ denotes the $L^r(0,T;L^2(\Omega))$ closed ball.

3 Numerical approximation

In this section we study the numerical discretization of (P) by discontinuous Galerkin finite element methods. To this end we consider a quasi-uniform family of triangulations $\{\mathbb{K}_h\}_{h>0}$ of $\bar\Omega$, cf. [3, Definition (4.4.13)], and a quasi-uniform family of partitions of size τ of $[0,T]$, $0 = t_0 < t_1 < \cdots < t_{N_\tau} = T$. We will denote by N_h and $N_{I,h}$ the number of nodes and interior nodes of the triangulation $\mathbb{K}_h$, $I_j = (t_{j-1}, t_j)$, $\tau_j = t_j - t_{j-1}$, $\tau = \max_{1\le j\le N_\tau} \tau_j$, and $\sigma = (h,\tau)$. Following [24] we make the following assumptions.

Assumption 4- The next properties hold

$$\begin{aligned}
&\exists \theta_1, \theta_2 > 0 \text{ such that } \tau_j \ge \theta_1 \tau^{\theta_2} \ \forall j = 1, \ldots N_\tau, \\
&\exists \varrho > 0 \text{ such that } \tau \le \varrho \tau_j \ \ \forall j = 1, \ldots N_\tau, \\
&\exists \theta_3, \theta_4 > 0 \text{ and } c_{\Omega,T}, C_{\Omega,T} > 0 \text{ such that } c_{\Omega,T} h^{\theta_3} \le \tau \le C_{\Omega,T} h^{\theta_4}, \\
&\tau |C_a| < 1, \text{ where } C_a \text{ satisfies (2.2)},
\end{aligned}$$

where the constants are independent of τ and h. Observe that θ_3 and θ_4 can be arbitrarily large and small, respectively. Hence, it is not a strong restriction.

[1] At the end of the proof of Theorem 5.1 of that paper, the definition of v_k has to be changed to $v_k(x,t) = \frac{v(x,t)}{1 + \frac{1}{k}\|v(t)\|_{L^2(\Omega)}}$ and subsequently $J'(\bar u)v_k = 0$ follows from Lemma 5.1.

3.1 Approximation of the state equation.

Now we define the finite dimensional spaces

$$Y_h = \{y_h \in C(\bar{\Omega}) : \ y_{h|K} \in P_1(K) \ \ \forall K \in \mathbb{K}_h \ \text{ and } \ y_h = 0 \text{ on } \Gamma\},$$

$$\mathcal{Y}_\sigma = \{y_\sigma \in L^2(0,T;Y_h) : y_{\sigma|I_j} \in Y_h \ \ \forall j = 1,\dots,N_\tau\}.$$

The elements of $\mathcal{Y}_\sigma$ can be written as

$$y_\sigma = \sum_{j=1}^{N_\tau} y_{h,j}\chi_j = \sum_{j=1}^{N_\tau}\sum_{i=1}^{N_{I,h}} y_{i,j}e_i\chi_j,$$

where $y_{h,j} \in Y_h$ for $j = 1,\dots,N_\tau$, $y_{i,j} \in \mathbb{R}$ for $i = 1,\dots,N_{I,h}$ and $j = 1,\dots,N_\tau$, $\{e_i\}_{i=1}^{N_{I,h}}$ is the nodal basis associated to the interior nodes $\{x_i\}_{i=1}^{N_{I,h}}$ of the triangulation, and χ_j denotes the characteristic function of the interval $I_j = (t_{j-1}, t_j)$.

For every $u \in L^2(Q)$, we define its associated discrete state as the unique element $y_\sigma(u) \in \mathcal{Y}_\sigma$ such that for $j = 1,\dots,N_\tau$

$$\left\{\begin{array}{l} \displaystyle\int_\Omega (y_{h,j} - y_{h,j-1})z_h \,\mathrm{d}x + \tau_j b(y_{h,j}, z_h) + \int_{I_j}\int_\Omega a(x,t,y_{h,j})z_h \,\mathrm{d}x\,\mathrm{d}t = \int_{I_j}\int_\Omega u z_h \,\mathrm{d}x\,\mathrm{d}t \quad \forall z_h \in Y_h, \\ y_{h,0} = P_h y_0, \end{array}\right. \tag{3.1}$$

where $P_h : L^2(\Omega) \longrightarrow Y_h$ denotes the L^2 projection operator, and $b : H^1(\Omega)\times H^1(\Omega) \longrightarrow \mathbb{R}$ is the bilinear form

$$b(y,z) = \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_{x_i} y \partial_{x_j} z dx \quad \forall y,z \in H^1(\Omega).$$

From a computational point of view, this scheme coincides with the implicit Euler discretization of the system of ordinary differential equations obtained after spatial finite element discretization. The proof of existence and uniqueness of a solution of (3.1) is standard by using Brouwer's fixed point theorem and the assumption $\tau|C_a| < 1$. Moreover, the system (3.1) realizes an approximation of (1.1) in the following sense.

Theorem 3.1. *Let $u \in L^r(0,T;L^2(\Omega))$ hold with $r > \frac{4}{4-n}$. Under the assumptions 1, 2, and 4, there exist $h_0 > 0$, $\tau_0 > 0$, $\delta_0 > 0$, $C > 0$, and a monotone nondecreasing function $\eta_1 : [0,\infty) \longrightarrow [0,\infty)$ independent of u such that for every $\tau < \tau_0$ and $h < h_0$*

$$\|y_u - y_\sigma(u)\|_{L^2(Q)} \le C\big(\|u\|_{L^r(0,T;L^2(\Omega))} + M_{\hat{r},0}\big)(\tau + h^2), \tag{3.2}$$

$$\|y_u - y_\sigma(u)\|_{L^\infty(Q)} \le \eta_1\big(\|u\|_{L^r(0,T;L^2(\Omega))} + M_{\hat{r},0}\big)|\log h|^3 h^{\delta_0}, \tag{3.3}$$

where $M_{\hat{r},0}$ is taken as in Theorem 2.1.

Proof. For the proof of (3.2) the reader is referred to [24, Corollary 6.2]. To prove (3.3) we use [24, Theorem 6.5] to deduce the existence of a constant C_1 independent of u such that

$$\|y_u - y_\sigma(u)\|_{L^\infty(Q)} \le C_1|\log h|\Big(\log\frac{T}{\tau}\Big)^2\|y_u - z_\sigma\|_{L^\infty(Q)} \quad \forall z_\sigma \in \mathcal{Y}_\sigma.$$

Let us select a convenient z_σ. We denote by P_τ the $L^2(0,T)$ projection operator

$$P_\tau w = \sum_{j=1}^{N_\tau}\frac{1}{\tau_j}\int_{I_j} w(t)\,\mathrm{d}t\chi_j \quad \forall w \in L^1(0,T).$$

It is obvious that $\|P_\tau z\|_{L^\infty(Q)} \le \|z\|_{L^\infty(Q)}$ for every $z \in L^\infty(Q)$.

We also set $\Pi_h : C_0(\bar\Omega) \longrightarrow Y_h$ the interpolation operator $\Pi_h z = \sum_{i=1}^{N_{I,h}} z(x_i)e_i$. Then, we take $z_\sigma = P_\tau \Pi_h y_u$. From (2.6) we get

$$\begin{aligned}\|y_u - z_\sigma\|_{L^\infty(Q)} &\le \|y_u - P_\tau y_u\|_{L^\infty(Q)} + \|P_\tau(y_u - \Pi_h y_u)\|_{L^\infty(Q)}\\ &\le \|y_u - P_\tau y_u\|_{L^\infty(Q)} + \|y_u - \Pi_h y_u\|_{L^\infty(Q)}\\ &\le (\tau^\beta + (n+1)h^\beta)\|y_u\|_{C^{0,\beta}(\bar Q)} \le (\tau^\beta + (n+1)h^\beta)\eta\big(\|u\|_{L^r(0,T;L^2(\Omega))} + M_{\hat r,0}\big).\end{aligned}$$

Using the assumption $c_{\Omega,T}h^{\theta_3} \le \tau \le C_{\Omega,\tau}h^{\theta_4}$ and taking $\delta_0 = \min\{1,\theta_4\}\beta$ we deduce (3.3). □

3.2 Approximation of the control problem.

We will consider two different ways to discretize the space of controls:

I - Piecewise constant controls. We introduce the spaces and sets

$$\begin{aligned} U_h &= U_{h,0} = \{u_h \in L^\infty(\Omega) : u_{h|_K} \equiv u_K \in \mathbb{R} \ \ \forall K \in \mathbb{K}_h\},\\ B_{h,\gamma} &= \{u_h = \sum_{K\in\mathbb{K}_h} u_K\chi_K \in U_{h,0} : \sum_{K\in\mathbb{K}_h} |K||u_K| \le \gamma\},\\ \mathbb{U}_\sigma &= \mathbb{U}_{\sigma,0} = \{u_\sigma = \sum_{j=1}^{N_\tau} u_{h,j}\chi_j : u_{h,j} \in U_{h,0} \ \text{ for } j = 1,\dots,N_\tau\},\\ \mathbb{U}_{\sigma,ad} &= \Big\{u_\sigma \in \mathbb{U}_{\sigma,0} : u_{h,j} \in B_{h,\gamma} \ \text{ for } j = 1,\dots,N_\tau\Big\},\end{aligned}$$

where χ_K and χ_j denote the characteristic functions of the sets K and I_j, respectively. It is immediate to check that $\mathbb{U}_{\sigma,ad} = \mathbb{U}_\sigma \cap U_{ad} \subset U_{ad}$.

II - Piecewise linear controls. In this case we take

$$\begin{aligned} U_h &= U_{h,1} = \{u_h \in C(\bar\Omega) : u_{h|_K} \in \mathcal{P}_1(K) \ \ \forall K \in \mathbb{K}_h\},\\ B_{h,\gamma} &= \{u_h = \sum_{i=1}^{N_h} u_i e_i \in U_{h,1} : \sum_{i=1}^{N_h} |u_i| \int_\Omega e_i \,\mathrm{d}x \le \gamma\},\\ \mathbb{U}_\sigma &= \mathbb{U}_{\sigma,1} = \{u_\sigma = \sum_{j=1}^{N_\tau} u_{h,j}\chi_j : u_{h,j} \in U_{h,1} \ \text{ for } j = 1,\dots,N_\tau\},\\ \mathbb{U}_{\sigma,ad} &= \Big\{u_\sigma \in \mathbb{U}_{\sigma,1} : u_{h,j} \in B_{h,\gamma} \ \text{ for } j = 1,\dots,N_\tau\Big\},\end{aligned}$$

where $\mathcal{P}_1(K)$ denotes the space of the polynomials on K of degree ≤ 1. From the inequality

$$\|u_h\|_{L^1(\Omega)} = \int_\Omega \left|\sum_{i=1}^{N_h} u_i e_i\right| \mathrm{d}x \le \sum_{i=1}^{N_h} |u_i| \int_\Omega e_i \,\mathrm{d}x$$

we infer that $\mathbb{U}_{\sigma,ad} \subset U_{ad}$.

We observe that $\mathbb{U}_\sigma \subset L^\infty(0,T;U_h)$ in both cases and every element $u_\sigma \in \mathbb{U}_\sigma$ can be written in the form

$$u_\sigma = \sum_{j=1}^{N_\tau} u_{h,j}\chi_j = \begin{cases} \displaystyle\sum_{j=1}^{N_\tau}\sum_{K\in\mathbb{K}_h} u_{K,j}\chi_K\chi_j & \text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,0}, \\ \displaystyle\sum_{j=1}^{N_\tau}\sum_{i=1}^{N_h} u_{i,j}e_i\chi_j & \text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}. \end{cases}$$

Now, we formulate the discrete control problem

$$(\mathrm{P}_\sigma) \qquad \inf_{u_\sigma\in\mathbb{U}_{\sigma,ad}} J_\sigma(u_\sigma) := \frac{1}{2}\int_Q |y_\sigma(u_\sigma) - y_d|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\|u_\sigma\|_\sigma^2,$$

where $y_\sigma(u_\sigma)$ is the solution of (3.1) for $u = u_\sigma$ and

$$\|u_\sigma\|_\sigma^2 = \sum_{j=1}^{N_\tau} \tau_j \|u_{h,j}\|_h^2$$

with $\|\cdot\|_h$ the norm in U_h defined by

$$\|u_h\|_h = \begin{cases} \displaystyle\Big(\sum_{K\in\mathbb{K}_h} |K|u_K^2\Big)^{\frac{1}{2}} & \text{if } U_h = U_{h,0}, \\ \displaystyle\Big(\sum_{i=1}^{N_h}\Big(\int_\Omega e_i(x)\,\mathrm{d}x\Big)u_i^2\Big)^{\frac{1}{2}} & \text{if } U_h = U_{h,1}. \end{cases}$$

We notice that

$$\|u_h\|_{L^2(\Omega)}^2 = \begin{cases} \displaystyle\sum_{K\in\mathbb{K}_h}\int_K |u_K|^2\,\mathrm{d}x = \|u_h\|_h^2 & \text{if } U_h = U_{h,0}, \\ \displaystyle\int_\Omega\Big(\sum_{i=1}^{N_h} u_i e_i(x)\Big)^2\mathrm{d}x \le \int_\Omega\Big(\sum_{i=1}^{N_h} e_i(x)u_i^2\Big)\,\mathrm{d}x = \|u_h\|_h^2 & \text{if } U_h = U_{h,1}, \end{cases} \tag{3.4}$$

where we have used that $0 \le e_i(x) \le 1$ and $\sum_{i=1}^{N_h} e_i(x) = 1$ in Ω.

We also introduce $\|u_\sigma\|_\sigma = \sqrt{(u_\sigma, u_\sigma)_\sigma}$, where the scalar product $(\cdot,\cdot)_\sigma$ in $\mathbb{U}_\sigma$ is defined by

$$(u_\sigma, v_\sigma)_\sigma = \sum_{j=1}^{N_\tau}\tau_j(u_{h,j}, v_{h,j})_h = \begin{cases} \displaystyle (u_\sigma, v_\sigma)_{L^2(Q)} = \sum_{j=1}^{N_\tau}\sum_{K\in\mathbb{K}_h}\tau_j|K|u_{K,j}v_{K,j} & \text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,0}, \\ \displaystyle\sum_{j=1}^{N_\tau}\sum_{i=1}^{N_h}\tau_j\Big(\int_\Omega e_i(x)\,\mathrm{d}x\Big)u_{i,j}v_{i,j} & \text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}. \end{cases}$$

Due to the compactness of $\mathbb{U}_{\sigma,ad}$ in both definitions and the continuity of J_σ, we infer the existence of at least one solution for (P_σ).

Analogously to Corollary 2.3 we have the following differentiability result.

Theorem 3.2. *The functional $J_\sigma : \mathbb{U}_\sigma \longrightarrow \mathbb{R}$ is of class C^2 and its first derivative is given by the expression*

$$J'_\sigma(u_\sigma)v_\sigma = \int_Q \varphi_\sigma v_\sigma\,\mathrm{d}x\,\mathrm{d}t + \kappa(u_\sigma, v_\sigma)_\sigma, \tag{3.5}$$

where $\varphi_\sigma(u_\sigma) \in \mathcal{Y}_\sigma$ is the solution of the adjoint state equation: for $j = N_\tau, \dots, 1$

$$\begin{cases} \displaystyle\int_\Omega (\varphi_{h,j} - \varphi_{h,j+1}) z_h dx + \tau_j b(z_h, \varphi_{h,j}) + \int_{I_j}\int_\Omega \frac{\partial a}{\partial y}(x,t,y_\sigma(u_\sigma))\varphi_{h,j} z_h \,\mathrm{d}x\,\mathrm{d}t \\ \qquad\qquad = \displaystyle\int_{I_j}\int_\Omega (y_\sigma(u_\sigma) - y_d) z_h \,\mathrm{d}x\,\mathrm{d}t \quad \forall z_h \in Y_h, \\ \varphi_{h,N_\tau+1} = 0. \end{cases} \tag{3.6}$$

Now we compare the continuous and discrete adjoint states.

Theorem 3.3. *Let $u \in L^r(0,T;L^2(\Omega))$ hold with $r > \frac{4}{4-n}$, and let us denote by φ_u and $\varphi_\sigma(u)$ the solutions of (2.11) and (3.6) with $y_\sigma(u_\sigma)$ replaced by $y_\sigma(u)$. Under the assumptions 1–4, and taking h_0 and τ_0 as in Theorem 3.1, there exists a monotone nondecreasing function $\eta_2 : [0,\infty) \longrightarrow \mathbb{R}$ independent of u such that for every $\tau < \tau_0$ and $h < h_0$*

$$\|\varphi_u - \varphi_\sigma(u)\|_{L^2(Q)} \le \eta_2\big(\|u\|_{L^r(0,T;L^2(\Omega))} + M_{\hat r,0}\big)(\tau + h^2), \tag{3.7}$$

$$\|\varphi_u - \varphi_\sigma(u)\|_{L^\infty(Q)} \le \eta_2\big(\|u\|_{L^r(0,T;L^2(\Omega))} + M_{\hat r,0}\big)|\log h|^3 h^{\delta_0}. \tag{3.8}$$

Proof. Let $\psi_u \in C^{0,\beta}(\bar Q) \cap H^{2,1}(Q)$ denote the solution of the adjoint state equation

$$\begin{cases} -\dfrac{\partial \psi_u}{\partial t} + A^*\psi_u + \dfrac{\partial a}{\partial y}(x,t,y_\sigma(u))\psi_u = y_\sigma(u) - y_d \ \text{ in } Q, \\ \psi_u = 0 \text{ on } \Sigma, \quad \psi_u(T) = 0 \text{ in } \Omega, \end{cases} \tag{3.9}$$

and set $\varphi_u - \varphi_\sigma(u) = (\varphi_u - \psi_u) + (\psi_u - \varphi_\sigma(u)) = e_u + \xi_u$. Subtracting the equations (2.11) and (3.9) we get

$$\begin{cases} -\dfrac{\partial e_u}{\partial t} + A^* e_u + \dfrac{\partial a}{\partial y}(x,t,y_u)e_u = (y_u - y_\sigma(u)) + \Big[\dfrac{\partial a}{\partial y}(x,t,y_\sigma(u)) - \dfrac{\partial a}{\partial y}(x,t,y_u)\Big]\psi_u \ \text{ in } Q, \\ e_u = 0 \text{ on } \Sigma, \quad e_u(T) = 0 \text{ in } \Omega. \end{cases} \tag{3.10}$$

Setting $M = \|y_u\|_{L^\infty(Q)} + 1$ and taking h_0 and τ_0 small enough, we infer from (3.3) that $\|y_\sigma(u)\|_{L^\infty(Q)} \le M$ for every $\sigma = (h,\tau)$ with $h \le h_0$ and $\tau \le \tau_0$. Then, from (3.9) it follows with (2.4) that

$$\|\psi_u\|_{L^\infty(Q)} \le C_1\big(\|y_\sigma(u)\|_{L^\infty(Q)} + \|y_d\|_{L^{\hat r}(0,T;L^2(\Omega))}\big) \le C_1\big(M + \|y_d\|_{L^{\hat r}(0,T;L^2(\Omega))}\big). \tag{3.11}$$

From (2.4), (3.3), and the mean value theorem we obtain

$$\left|\frac{\partial a}{\partial y}(x,t,y_\sigma(u)(x,t)) - \frac{\partial a}{\partial y}(x,t,y_u(x,t))\right| \le C_{a,M}|y_\sigma(u)(x,t) - y_u(x,t)|. \tag{3.12}$$

From (3.10), (3.11), (3.12), and (3.2), we infer

$$\begin{aligned} \|e_u\|_{L^2(Q)} &\le C_2\big[1 + C_{a,M}C_1\big(M + \|y_d\|_{L^{\hat r}(0,T;L^2(\Omega))}\big)\big]\|y_\sigma(u) - y_u\|_{L^2(Q)} \\ &\le C_M C\big(\|u\|_{L^r(0,T;L^2(\Omega))} + M_{\hat r,0}\big)(\tau + h^2). \end{aligned} \tag{3.13}$$

The constant C_M is a monotone nondecreasing function of M.

Let us estimate ξ_u. Since $\varphi_\sigma(u)$ is the solution of the discretization of the linear equation (3.9), the classical error estimates yield the existence of a constant C_3 such that

$$\|\xi_u\|_{L^2(Q)} \le C_2(\tau + h^2)\big(\|y_\sigma(u)\|_{L^2(Q)} + \|y_d\|_{L^2(Q)}\big). \tag{3.14}$$

Hence, (3.13) and (3.14) along with (2.6) lead to (3.7).

To prove (3.8) we first modify (3.13) as follows

$$\begin{aligned}\|e_u\|_{L^\infty(Q)} &\le C_2\big[1 + C_{a,M}C_1\big(M + \|y_d\|_{L^{\hat r}(0,T;L^2(\Omega))}\big)\big]\|y_\sigma(u) - y_u\|_{L^\infty(Q)}\\ &\le C_M\eta_1\big(\|u\|_{L^r(0,T;L^2(\Omega))} + M_{\hat r,0}\big)|\log h|^3 h^{\delta_0}.\end{aligned} \tag{3.15}$$

Finally, using the linearity of the equations satisfied by ψ_u and arguing as for the estimate (3.3) we infer

$$\|\xi_u\|_{L^\infty(Q)} \le C_3|\log h|^3 h^{\delta_0}.$$

The last inequality and (3.15) imply (3.8). □

3.3 First order optimality conditions.

The goal of this subsection is to prove the first order optimality conditions and their consequences.

Theorem 3.4. *Let $\bar u_\sigma$ be a local minimum of* (P_σ)*. Then there exist $\bar y_\sigma, \bar\varphi_\sigma \in \mathcal{Y}_\sigma$ and $\bar\mu_\sigma \in \mathbb{U}_\sigma$ such that*

$$\begin{cases}\displaystyle\int_\Omega(\bar y_{h,j} - \bar y_{h,j-1})z_h dx + \tau_j b(\bar y_{h,j}, z_h) + \int_{I_j}\int_\Omega a(x,t,\bar y_{h,j})z_h\,dx\,dt\\ \qquad\displaystyle= \int_{I_j}\int_\Omega \bar u_{h,j}z_h\,dx\,dt \quad \forall z_h \in Y_h \text{ and } \forall j = 1,\ldots,N_\tau,\\ \bar y_{h,0} = P_h y_0,\end{cases} \tag{3.16}$$

$$\begin{cases}\displaystyle\int_\Omega(\bar\varphi_{h,j} - \bar\varphi_{h,j+1})z_h dx + \tau_j b(z_h, \bar\varphi_{h,j}) + \int_{I_j}\int_\Omega \frac{\partial a}{\partial y}(x,t,\bar y_\sigma)\bar\varphi_{h,j}z_h\,dx\,dt\\ \qquad\displaystyle= \int_{I_j}\int_\Omega(\bar y_{h,j} - y_d)z_h\,dx\,dt \quad \forall z_h \in Y_h \text{ and } \forall j = N_\tau,\ldots,1,\\ \bar\varphi_{h,N_\tau+1} = 0,\end{cases} \tag{3.17}$$

$$(\bar\mu_\sigma, u_\sigma - \bar u_\sigma)_\sigma \le 0 \quad \forall u_\sigma \in \mathbb{U}_{\sigma,ad}, \tag{3.18}$$

$$\begin{cases}\displaystyle\frac{1}{|K|}\int_K \bar\varphi_{h,j}\,dx + \kappa\bar u_{K,j} + \bar\mu_{K,j} = 0 \;\; \forall K \in \mathbb{K}_h \text{ and } \forall j = 1,\ldots,N_\tau, & \text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,0},\\ \displaystyle\frac{1}{\int_\Omega e_i\,dx}\int_\Omega \bar\varphi_{h,j}e_i\,dx + \kappa\bar u_{i,j} + \bar\mu_{i,j} = 0 \;\; \forall i = 1,\ldots,N_h \text{ and } \forall j = 1,\ldots,N_\tau, & \text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}.\end{cases} \tag{3.19}$$

Proof. Taking $\bar y_\sigma$ and $\bar\varphi_\sigma$ as solutions of (3.16) and (3.17), respectively, and using the convexity of $\mathbb{U}_{\sigma,ad}$ we infer with (3.5)

$$\int_Q \bar\varphi_\sigma(u_\sigma - \bar u_\sigma)\,dx\,dt + \kappa(\bar u_\sigma, u_\sigma - \bar u_\sigma)_\sigma = J'_\sigma(\bar u_\sigma)(u_\sigma - \bar u_\sigma) \ge 0 \;\; \forall u_\sigma \in \mathbb{U}_{\sigma,ad}. \tag{3.20}$$

Now we distinguish the cases $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,0}$ and $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}$.

Case $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,0}$. In this case, (3.20) can be written as follows

$$\sum_{j=1}^{N_\tau}\sum_{K\in\mathbb{K}_h}\tau_j\left(\int_K \bar\varphi_{h,j}\,dx + \kappa|K|\bar u_{K,j}\right)(u_{K,j} - \bar u_{K,j}) \ge 0 \quad \forall u_\sigma \in \mathbb{U}_{\sigma,ad}. \tag{3.21}$$

Then, defining

$$\bar\mu_\sigma = \sum_{j=1}^{N_\tau}\sum_{K\in\mathbb{K}_h}\bar\mu_{K,j}\chi_K\chi_j \quad\text{with}\quad \bar\mu_{K,j} = -\left(\frac{1}{|K|}\int_K\bar\varphi_{h,j}\,\mathrm{d}x + \kappa\bar u_{K,j}\right),$$

we have that the first identity of (3.19) holds. Inequality (3.18) is a consequence of (3.21):

$$\begin{aligned}(\bar\mu_\sigma, u_\sigma - \bar u_\sigma)_\sigma &= \sum_{j=1}^{N_\tau}\sum_{K\in\mathbb{K}_h}\tau_j|K|\bar\mu_{K,j}(u_{K,j}-\bar u_{K,j})\\ &= -\sum_{j=1}^{N_\tau}\sum_{K\in\mathbb{K}_h}\tau_j\left(\int_K\bar\varphi_{h,j}\,\mathrm{d}x + \kappa|K|\bar u_{K,j}\right)(u_{K,j}-\bar u_{K,j}) \le 0 \quad \forall u_\sigma\in\mathbb{U}_{\sigma,ad}.\end{aligned}$$

Case $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}$. From (3.20) and using the definition of $(\cdot,\cdot)_\sigma$ we deduce

$$\sum_{j=1}^{N_\tau}\sum_{i=1}^{N_h}\tau_j\left(\int_\Omega\bar\varphi_{h,j}e_i\,\mathrm{d}x + \kappa\big(\int_\Omega e_i\,\mathrm{d}x\big)\bar u_{i,j}\right)(u_{i,j}-\bar u_{i,j}) \ge 0 \quad \forall u_\sigma\in\mathbb{U}_{\sigma,ad}. \tag{3.22}$$

Now we set

$$\bar\mu_\sigma = \sum_{j=1}^{N_\tau}\sum_{i=1}^{N_h}\bar\mu_{i,j}e_i\chi_j \quad\text{with}\quad \bar\mu_{i,j} = -\left(\frac{1}{\int_\Omega e_i\,\mathrm{d}x}\int_\Omega\bar\varphi_{h,j}e_i\,\mathrm{d}x + \kappa\bar u_{i,j}\right).$$

Then, the second identity of (3.19) is satisfied. We finish the proof by checking (3.18) with the aid of (3.22)

$$\begin{aligned}(\bar\mu_\sigma, u_\sigma - \bar u_\sigma)_\sigma &= \sum_{j=1}^{N_\tau}\sum_{i=1}^{N_h}\tau_j\big(\int_\Omega e_i\,\mathrm{d}x\big)\bar\mu_{i,j}(u_{i,j}-\bar u_{i,j})\\ &= -\sum_{j=1}^{N_\tau}\sum_{i=1}^{N_h}\left(\tau_j\int_\Omega\bar\varphi_{h,j}e_i\,\mathrm{d}x + \kappa\big(\int_\Omega e_i\,\mathrm{d}x\big)\bar u_{i,j}\right)(u_{i,j}-\bar u_{i,j}) \le 0 \quad \forall u_\sigma\in\mathbb{U}_{\sigma,ad}.\end{aligned}$$

□

Let us introduce the following notation:

$$\|u_h\|_{l^\infty} = \begin{cases}\max\limits_{K\in\mathbb{K}_h}|u_K| & \text{if } U_h = U_{h,0},\\ \max\limits_{1\le i\le N_h}|u_i| & \text{if } U_h = U_{h,1},\end{cases}$$

and $j_h : U_h \longrightarrow \mathbb{R}$ is the functional defined by

$$j_h(u_h) = \begin{cases}\sum\limits_{K\in\mathbb{K}_h}|K||u_K| & \text{if } U_h = U_{h,0},\\ \sum\limits_{i=1}^{N_h}|u_i|\displaystyle\int_\Omega e_i\,\mathrm{d}x & \text{if } U_h = U_{h,1}.\end{cases}$$

We have the following corollary.

Corollary 3.5. *Let $\bar u_\sigma$ and $\bar\mu_\sigma$ satisfy* (3.18)*, and assume that $\bar u_\sigma \in \mathbb{U}_{\sigma,ad}$. Then, the following properties hold for every $j = 1,\dots,N_\tau$*

$$(\bar\mu_{h,j}, u_h - \bar u_{h,j})_h \le 0 \quad \forall u_h \in B_{h,\gamma}, \tag{3.23}$$

if $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,0}$ *then*

$$\begin{cases} \bar\mu_{K,j}\bar u_{K,j} = |\bar\mu_{K,j}||\bar u_{K,j}| \quad \forall K \in \mathbb{K}_h, \\ \text{if } j_h(\bar u_{h,j}) < \gamma \ \text{ then } \ \bar\mu_{h,j} = 0, \\ \text{if } j_h(\bar u_{h,j}) = \gamma \ \text{ and } \ \bar\mu_{h,j} \ne 0, \ \text{ then if } \bar u_{K,j} \ne 0 \Rightarrow |\bar\mu_{K,j}| = \|\bar\mu_{h,j}\|_{l^\infty}. \end{cases} \tag{3.24}$$

if $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}$ *then*

$$\begin{cases} \bar\mu_{h,j}\bar u_{h,j} = |\bar\mu_{h,j}||\bar u_{h,j}|, \\ \text{if } j_h(\bar u_{h,j}) < \gamma \ \text{ then } \ \bar\mu_{h,j} = 0, \\ \text{if } j_h(\bar u_{h,j}) = \gamma \ \text{ and } \ \bar\mu_{h,j} \ne 0, \ \text{ then if } \bar u_{i,j} \ne 0 \Rightarrow |\bar\mu_{i,j}| = \|\bar\mu_{h,j}\|_{l^\infty}. \end{cases} \tag{3.25}$$

Proof. Given $1 \le j \le N_\tau$ and $u_h \in B_{h,\gamma}$ we define

$$u_\sigma = \sum_{l=1}^{N_\tau} u_{h,l}\chi_l \ \text{ with } \ u_{h,l} = \begin{cases} \bar u_{h,l} & \text{if } l \ne j, \\ u_h & \text{if } l = j. \end{cases}$$

Then, $u_\sigma \in \mathbb{U}_{\sigma,ad}$ and (3.18) implies

$$\tau_j(\bar\mu_{h,j}, u_h - \bar u_{h,j})_h = (\bar\mu_\sigma, u_\sigma - \bar u_\sigma)_\sigma \le 0,$$

which proves (3.23). The rest of the proof is divided into two cases.

Case $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,0}$. For

$$u_h = \sum_{K' \in \mathbb{K}_h} u_{K'}\chi_{K'} \quad \text{with} \quad u_{K'} = \begin{cases} \bar u_{K',j} & \text{if } K' \ne K, \\ 0 & \text{if } K' = K, \end{cases}$$

(3.23) leads to $|K|\bar\mu_{K,j}\bar u_{K,j} \ge 0$, which implies the first identity of (3.24). To establish the second statement of (3.24), given $K \in \mathbb{K}_h$ arbitrary, we define

$$u_h^\pm = \sum_{K' \in \mathbb{K}_h} u_{K'}\chi_{K'} \quad \text{with} \quad u_{K'} = \begin{cases} \bar u_{K',j} & \text{if } K' \ne K, \\ \bar u_{K,j} \pm \varepsilon & \text{if } K' = K. \end{cases}$$

Then, for ε small enough, due to the fact that $j(\bar u_{h,j}) < \gamma$, we have that $u_h^\pm \in B_{h,\gamma}$. Then, (3.23) leads to $\pm|K|\bar\mu_{K,j}\varepsilon \ge 0$, which implies that $\bar\mu_{K,j} = 0$ for every $K \in \mathbb{K}_h$.

Now, we assume that $j_h(\bar u_{h,j}) = \gamma$ and $\bar\mu_{h,j} \ne 0$. Let $K^0 \in \mathbb{K}_h$ be such that $|\bar\mu_{K^0,j}| = \max_{K' \in \mathbb{K}_h} |\bar\mu_{K',j}|$. If $\bar u_{K,j} \ne 0$ we define

$$u_h = \sum_{K' \in \mathbb{K}_h} u_{K'}\chi_{K'} \quad \text{with} \quad u_{K'} = \begin{cases} \bar u_{K,j} - \dfrac{\varepsilon}{|K|}\operatorname{sign}(\bar u_{K,j}) & \text{if } K' = K, \\ \bar u_{K^0,j} + \dfrac{\varepsilon}{|K^0|}\operatorname{sign}(\bar u_{K^0,j}) & \text{if } K' = K^0, \\ \bar u_{K,j} & \text{otherwise}, \end{cases}$$

where $0 < \varepsilon < |K||\bar u_{K,j}|$. Then, $j_h(u_h) = j_h(\bar u_{h,j}) = \gamma$. Hence, $u_h \in B_{h,\gamma}$ and we get with (3.23) and the first statement of (3.24)

$$\varepsilon|\bar\mu_{K^0,j}| - \varepsilon|\bar\mu_{K,j}| = (\bar\mu_{h,j}, u_h - \bar u_{h,j})_h \le 0.$$

This proves the last statement of (3.24).

Case $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}$. Let $1 \le i \le N_h$ arbitrary and set

$$u_h = \sum_{i'=1}^{N_h} u_{i'} e_{i'} \quad \text{with} \quad u_{i'} = \begin{cases} \bar{u}_{i',j} & \text{if } i' \neq i, \\ 0 & \text{if } i' = i. \end{cases}$$

Then (3.23) implies

$$-\Big(\int_\Omega e_i \,\mathrm{d}x\Big)\bar{\mu}_{i,j}\bar{u}_{i,j} = (\bar{\mu}_{h,j}, u_h - \bar{u}_{h,j})_h \le 0,$$

which proves the first statement of (3.25).

To establish the second statement of (3.25), given $1 \le i \le N_h$ arbitrary, we define

$$u_h^\pm = \sum_{i'=1}^{N_h} u_{i'} e_{i'} \quad \text{with} \quad u_{i'} = \begin{cases} \bar{u}_{i',j} & \text{if } i' \neq i, \\ \bar{u}_{i,j} \pm \varepsilon & \text{if } i' = i. \end{cases}$$

Then, for ε small enough, due to the fact that $j(\bar{u}_{h,j}) < \gamma$, we have that $u_h^\pm \in B_{h,\gamma}$. Then, (3.23) leads to $\pm\big(\int_\Omega e_i \,\mathrm{d}x\big)\bar{\mu}_{i,j}\varepsilon \ge 0$, which implies that $\bar{\mu}_{i,j} = 0$ for every $i = 1, \ldots, N_h$.

Finally, we assume that $j_h(\bar{u}_{h,j}) = \gamma$ and $\bar{\mu}_{h,j} \neq 0$. Let $1 \le i^0 \le N_h$ be such that $|\bar{\mu}_{i^0,j}| = \max_{1\le i'\le N_h} |\bar{\mu}_{i',j}|$. If $\bar{u}_{i,j} \neq 0$ we define

$$u_h = \sum_{i'=1}^{N_h} u_{i'} e_{i'} \quad \text{with} \quad u_{i'} = \begin{cases} \bar{u}_{i,j} - \dfrac{\varepsilon}{\int_\Omega e_i \,\mathrm{d}x} \operatorname{sign}(\bar{u}_{i,j}) & \text{if } i' = i, \\ \bar{u}_{i^0,j} + \dfrac{\varepsilon}{\int_\Omega e_{i^0} \,\mathrm{d}x} \operatorname{sign}(\bar{u}_{i^0,j}) & \text{if } i' = i^0, \\ \bar{u}_{i,j} & \text{otherwise,} \end{cases}$$

where $0 < \varepsilon < \big(\int_\Omega e_i \,\mathrm{d}x\big)|\bar{u}_{i,j}|$. Then, $j_h(u_h) = j_h(\bar{u}_{h,j}) = \gamma$. Hence, $u_h \in B_{h,\gamma}$ and we get with (3.23) and the first statement of (3.25)

$$\varepsilon|\bar{\mu}_{i^0,j}| - \varepsilon|\bar{\mu}_{i,j}| = (\bar{\mu}_{h,j}, u_h - \bar{u}_{h,j})_h \le 0.$$

This proves the last statement of (3.25). □

Corollary 3.6. *Let $\bar{u}_\sigma \in \mathbb{U}_{\sigma,ad}$ satisfy (3.19) and (3.24) or (3.25). Then, the following identities hold for every $j = 1, \ldots, N_\tau$*

$$\text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,0} \text{ then } \bar{u}_{K,j} = -\frac{1}{\kappa}\operatorname{sign}\Big(\int_K \bar{\varphi}_{h,j}\,\mathrm{d}x\Big)\Big(\frac{1}{|K|}\Big|\int_K \bar{\varphi}_{h,j}\,\mathrm{d}x\Big| - \|\bar{\mu}_{h,j}\|_{l^\infty}\Big)^+$$
$$= -\frac{1}{\kappa}\left\{\Big[\frac{1}{|K|}\int_K \bar{\varphi}_{h,j}\,\mathrm{d}x + \|\bar{\mu}_{h,j}\|_{l^\infty}\Big]^- + \Big[\frac{1}{|K|}\int_K \bar{\varphi}_{h,j}\,\mathrm{d}x - \|\bar{\mu}_{h,j}\|_{l^\infty}\Big]^+\right\}, \tag{3.26}$$

$$\text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,1} \text{ then } \bar{u}_{i,j} = -\frac{1}{\kappa}\operatorname{sign}\Big(\int_\Omega \bar{\varphi}_{h,j}e_i\,\mathrm{d}x\Big)\Big(\frac{1}{\int_\Omega e_i\,\mathrm{d}x}\Big|\int_\Omega \bar{\varphi}_{h,j}e_i\,\mathrm{d}x\Big| - \|\bar{\mu}_{h,j}\|_{l^\infty}\Big)^+$$
$$= -\frac{1}{\kappa}\left\{\Big[\frac{1}{\int_\Omega e_i\,\mathrm{d}x}\int_\Omega \bar{\varphi}_{h,j}e_i\,\mathrm{d}x + \|\bar{\mu}_{h,j}\|_{l^\infty}\Big]^- + \Big[\frac{1}{\int_\Omega e_i\,\mathrm{d}x}\int_\Omega \bar{\varphi}_{h,j}e_i\,\mathrm{d}x - \|\bar{\mu}_{h,j}\|_{l^\infty}\Big]^+\right\}. \tag{3.27}$$

Moreover, the following sparsity property is fulfilled for every $j = 1, \ldots, N_\tau$

$$\text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,0} \text{ then } \bar{u}_{K,j} = 0 \Leftrightarrow \frac{1}{|K|}\Big|\int_K \bar{\varphi}_{h,j}\,\mathrm{d}x\Big| \le \|\bar{\mu}_{h,j}\|_{l^\infty}, \quad \forall K \in \mathbb{K}_h, \tag{3.28}$$

$$\text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,1} \text{ then } \bar{u}_{i,j} = 0 \Leftrightarrow \frac{1}{\int_\Omega e_i\,\mathrm{d}x}\Big|\int_\Omega \bar{\varphi}_{h,j}e_i\,\mathrm{d}x\Big| \le \|\bar{\mu}_{h,j}\|_{l^\infty}, \quad \forall i = 1, \ldots, N_h, \tag{3.29}$$

Proof. Let us prove the first identity of (3.26). If $\|\bar\mu_{h,j}\|_{l^\infty} = 0$, then (3.19) implies that

$$\bar u_{K,j} = -\frac{1}{\kappa|K|}\int_K \bar\varphi_{h,j}\,\mathrm{d}x \quad \forall K \in \mathbb{K}_h,$$

which coincides with (3.26). Assume that $\|\bar\mu_{h,j}\|_{l^\infty} \neq 0$. Then, from (3.24) we deduce that $j_h(\bar u_{h,j}) = \gamma$. Then, the third statement of (3.24) implies that $|\bar\mu_{K,j}| = \|\bar\mu_{h,j}\|_{l^\infty}$ if $\bar u_{K,j} \neq 0$. Now, we distinguish three cases.

i) If $\bar u_{K,j} > 0$, (3.19) and the first statement of (3.24) lead to

$$\bar u_{K,j} = -\frac{1}{\kappa}\left\{\frac{1}{|K|}\int_K \bar\varphi_{h,j}\,\mathrm{d}x + \|\bar\mu_{h,j}\|_{l^\infty}\right\},$$

which coincides with (3.26). Indeed, observe that (3.24) and the positivity of $\bar u_{K,j}$ imply $\bar\mu_{K,j} \geq 0$. Hence, we conclude with (3.19) that $\int_K \bar\varphi_{h,j}\,\mathrm{d}x < 0$.

ii) If $\bar u_{K,j} = 0$, using again (3.19) we get

$$\Big|\frac{1}{|K|}\int_K \bar\varphi_{h,j}\,\mathrm{d}x\Big| = |\bar\mu_{K,j}| \leq \|\bar\mu_{h,j}\|_{l^\infty}.$$

Then, the identity (3.26) holds.

iii) If $\bar u_{K,j} < 0$, from the first statement of (3.24) and (3.19) we infer that

$$\bar u_{K,j} = -\frac{1}{\kappa}\left\{\frac{1}{|K|}\int_K \bar\varphi_{h,j}\,\mathrm{d}x - \|\bar\mu_{h,j}\|_{l^\infty}\right\}.$$

Moreover, arguing as in the case i), we deduce that $\int_K \bar\varphi_{h,j}\,\mathrm{d}x > 0$. Hence, (3.26) holds too.

The second identity of (3.26) is obvious. Following the same arguments as above, (3.27) is proved. Finally, (3.28) and (3.29) are immediate consequences of (3.26) and (3.27), respectively. □

4 Convergence analysis and error estimates

There are two goals in this section. First we prove that the discrete problems (P$_\sigma$) provide an approximation of (P). Second we establish error estimates in terms of $\sigma = (h, \tau)$ for the difference between the discrete and continuous optimal controls.

Theorem 4.1. *For every σ let $\bar u_\sigma$ be a solution of* (P$_\sigma$)*. Then, there exists $\sigma_0 = (h_0, \tau_0)$ such that the family $\{\bar u_\sigma\}_\sigma$ with $h < h_0$ and $\tau < \tau_0$ is bounded in $L^\infty(Q)$. If $\bar u_\sigma \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(Q)$ for a sequence of σ converging to zero, we have that $\bar u$ is a solution of* (P)*, and the following convergence properties hold*

$$\lim_{\sigma\to 0}\|\bar u_\sigma - \bar u\|_{L^r(0;T;L^2(\Omega))} = 0 \quad \forall r \in [1,\infty) \quad \text{and} \quad \lim_{\sigma\to 0} J_\sigma(\bar u_\sigma) = J(\bar u). \tag{4.1}$$

To prove this theorem we need the following stability property for the solution of the system (3.1).

Lemma 4.2. *Let us assume that $4|C_a|\tau < 1$. Then, given $u \in L^2(Q)$ and denoting by $y_\sigma \in \mathcal{Y}_\sigma$ the solution of* (3.1)*, we have the stability estimate*

$$\|y_\sigma\|_{L^\infty(0,T;L^2(\Omega))} + \|y_\sigma\|_{L^2(0,T;H_0^1(\Omega))} \leq C\big(\|u - a(\cdot,\cdot,0)\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}\big) \tag{4.2}$$

for some constant C independent of u and σ.

Proof. For $j = 1, \dots, N_\tau$ we take $z_h = y_{h,j}$ in (3.1), which leads to

$$\int_\Omega (y_{h,j} - y_{h,j-1}) y_{h,j} dx + \tau_j b(y_{h,j}, y_{h,j}) + \int_{I_j}\int_\Omega [a(x,t,y_{h,j}) - a(x,t,0)] y_{h,j}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{I_j}\int_\Omega [u - a(x,t,0)] y_{h,j}\,\mathrm{d}x\,\mathrm{d}t.$$

Using (2.1) and (2.2) along with Young's inequality we deduce from the above identity

$$\frac{1}{2}\|y_{h,j}\|^2_{L^2(\Omega)} + \frac{1}{2}\|y_{h,j} - y_{h,j-1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|y_{h,j-1}\|^2_{L^2(\Omega)} + \tau_j \Lambda_A \|y_{h,j}\|^2_{H^1_0(\Omega)} + C_a \tau_j \|y_{h,j}\|^2_{L^2(\Omega)}$$
$$\le \|u - a(\cdot,\cdot,0)\|_{L^2(\Omega\times I_j)} \sqrt{\tau_j} \|y_{h,j}\|_{L^2(\Omega)} \le C_1 \|u - a(\cdot,\cdot,0)\|^2_{L^2(\Omega\times I_j)} + \tau_j \frac{\Lambda_A}{2} \|y_{h,j}\|^2_{H^1_0(\Omega)}.$$

From here we infer

$$\|y_{h,j}\|^2_{L^2(\Omega)} + \tau_j \Lambda_A \|y_{h,j}\|^2_{H^1_0(\Omega)} \le 2C_1 \|u - a(\cdot,\cdot,0)\|^2_{L^2(\Omega\times I_j)} + 2|C_a|\tau_j \|y_{h,j}\|^2_{L^2(\Omega)} + \|y_{h,j-1}\|^2_{L^2(\Omega)}. \quad (4.3)$$

With the discrete Gronwall's inequality and the fact that $\|y_{h,0}\|_{L^2(\Omega)} \le \|y_0\|_{L^2(\Omega)}$ and $\tau_j \le \tau$ for every $j = 1, \dots, N_\tau$ we get

$$\|y_{h,j}\|^2_{L^2(\Omega)} \le (1 - 2|C_a|\tau)^{-j} \left(\|y_0\|^2_{L^2(\Omega)} + 2C_1 \sum_{k=0}^{j-1} (1 - 2|C_a|\tau)^k \|u - a(\cdot,\cdot,0)\|^2_{L^2(\Omega\times I_{k+1})} \right); \quad (4.4)$$

see, for instance, [16]. From our assumptions $4|C_a|\tau < 1$ and $\tau \le \rho\tau_k$ for every k, and using that

$$\frac{1}{1 - 2|C_a|\tau} = 1 + \frac{2|C_a|\tau}{1 - 2|C_a|\tau} \le \exp\Big(\frac{2|C_a|\tau}{1 - 2|C_a|\tau}\Big)$$

we obtain

$$(1 - 2|C_a|\tau)^{-j} \le \exp\Big(\frac{2|C_a|\tau j}{1 - 2|C_a|\tau}\Big) \le \exp(4\rho|C_a|T).$$

Then, (4.4) yields

$$\|y_{h,j}\|^2_{L^2(\Omega)} \le \exp(4\rho|C_a|T) \left(\|y_0\|^2_{L^2(\Omega)} + 2C_1 \sum_{k=0}^{j-1} \|u - a(\cdot,\cdot,0)\|^2_{L^2(\Omega\times I_{k+1})} \right)$$
$$\le \exp(4\rho|C_a|T) \Big(\|y_0\|^2_{L^2(\Omega)} + 2C_1 \|u - a(\cdot,\cdot,0)\|^2_{L^2(Q)} \Big)$$

and consequently

$$\|y_\sigma\|_{L^\infty(0,T;L^2(\Omega))} = \max_{1\le j\le N_\tau} \|y_{h,j}\|_{L^2(\Omega)}$$
$$\le \exp(2\rho|C_a|T) \max\{1, \sqrt{2C_1}\} \big(\|y_0\|_{L^2(\Omega)} + \|u - a(\cdot,\cdot,0)\|_{L^2(Q)} \big). \quad (4.5)$$

Adding the inequalities (4.3) for $j = 1, \dots, N_\tau$ we deduce

$$\Lambda_A \|y_\sigma\|^2_{L^2(0,T;H^1_0(\Omega))} = \Lambda_A \sum_{j=1}^{N_\tau} \tau_j \|y_{h,j}\|^2_{H^1_0(\Omega)}$$
$$\le 2C_1 \|u - a(\cdot,\cdot,0)\|^2_{L^2(Q)} + \Big(2|C_a|T \|y_\sigma\|^2_{L^\infty(0,T;L^2(\Omega))} + \|y_0\|^2_{L^2(\Omega)} \Big).$$

Finally, (4.2) follows from this inequality and (4.5). □

Proof of Theorem 4.1. We divide the proof into three steps.

Step I. $\{\bar{u}_\sigma\}_\sigma$ *is bounded in* $L^\infty(Q)$. Let us assume that τ satisfies the condition of Lemma 4.2 and $\tau \le \tau_0$, given by Theorem 3.1. Since the null control $u_0 \equiv 0$ is admissible for every problem (P$_\sigma$), we deduce from the optimality of $\bar{u}_\sigma$:

$$\frac{\kappa}{2}\|\bar{u}_\sigma\|_\sigma^2 \le J_\sigma(u_0) = \frac{1}{2}\|y_\sigma^0 - y_d\|^2_{L^2(\Omega)},$$

where y_σ^0 denotes the discrete state associated with u_0. From Lemma 4.2 we infer that $\{y_\sigma^0\}_\sigma$ is bounded in $L^2(Q)$. Hence, with (3.4) we deduce the existence of a constant C_1 independent of $\bar{u}_\sigma$ such that

$$\|\bar{u}_\sigma\|_{L^2(Q)} \le \|\bar{u}_\sigma\|_\sigma \le \frac{1}{\sqrt{\kappa}}\|y_\sigma^0 - y_d\|_{L^2(Q)} \le C_1.$$

We denote by $\bar{y}_\sigma$ and $\bar{\varphi}_\sigma$ the state and adjoint state associated with $\bar{u}_\sigma$. Using again Lemma 4.2 we obtain

$$\|\bar{y}_\sigma\|_{L^\infty(0,T;L^2(\Omega))} \le C\big(\|\bar{u}_\sigma - a(\cdot,\cdot,0)\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}\big) \le C\big(C_1 + \|a(\cdot,\cdot,0)\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}\big) = C_2.$$

Arguing as in the proof of Lemma 4.2 we deduce the stability estimate for the solution of (3.6) corresponding to $\bar{u}_\sigma$

$$\|\bar{\varphi}_\sigma\|_{L^\infty(0,T;L^2(\Omega))} \le C_3\|\bar{y}_\sigma - y_d\|_{L^2(Q)} \le C_3\big(C_2\sqrt{T} + \|y_d\|_{L^2(Q)}\big) = C_4.$$

Next we prove the estimate

$$\|\bar{u}_\sigma\|_{L^\infty(0,T;L^2(\Omega))} \le \frac{C_4}{\kappa}. \tag{4.6}$$

We distinguish two cases according to the definition of $\mathbb{U}_\sigma$.

Case $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,0}$. From (3.26) we get for every $j = 1,\ldots,N_\tau$

$$\begin{aligned}\|\bar{u}_{h,j}\|_{L^2(\Omega)} &= \Big(\sum_{K\in\mathbb{K}_h}|K|\bar{u}_{K,j}^2\Big)^{\frac12} \le \frac{1}{\kappa}\Big(\sum_{K\in\mathbb{K}_h}\frac{1}{|K|}\Big(\int_K\bar{\varphi}_{h,j}\,\mathrm{d}x\Big)^2\Big)^{\frac12}\\ &\le \frac{1}{\kappa}\Big(\sum_{K\in\mathbb{K}_h}\|\bar{\varphi}_{h,j}\|^2_{L^2(K)}\Big)^{\frac12} = \frac{1}{\kappa}\|\bar{\varphi}_{h,j}\|_{L^2(\Omega)}.\end{aligned}$$

This inequality implies (4.6).

Case $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}$. This time we use (3.27) to deduce

$$\begin{aligned}\|\bar{u}_{h,j}\|_{L^2(\Omega)} &= \Big(\int_\Omega\Big(\sum_{i=1}^{N_h}\bar{u}_{i,j}e_i\Big)^2\mathrm{d}x\Big)^{\frac12} \le \Big(\int_\Omega\sum_{i=1}^{N_h}\bar{u}_{i,j}^2e_i\,\mathrm{d}x\Big)^{\frac12}\\ &\le \frac{1}{\kappa}\Big(\sum_{i=1}^{N_h}\frac{1}{\int_\Omega e_i\,\mathrm{d}x}\Big(\int_\Omega\bar{\varphi}_{h,j}e_i\,\mathrm{d}x\Big)^2\Big)^{\frac12} \le \frac{1}{\kappa}\Big(\sum_{i=1}^{N_h}\int_\Omega\bar{\varphi}_{h,j}^2e_i\,\mathrm{d}x\Big)^{\frac12} = \frac{1}{\kappa}\|\bar{\varphi}_{h,j}\|_{L^2(\Omega)}.\end{aligned}$$

Hence, (4.6) is satisfied as well in this case. Then combining (2.6) and (3.3) along with the estimate (4.6) we obtain

$$\|\bar{y}_\sigma\|_{L^\infty(Q)} \le \|y_{\bar{u}_\sigma}\|_{L^\infty(Q)} + \|y_{\bar{u}_\sigma} - \bar{y}_\sigma\|_{L^\infty(Q)} \le C_5$$

for every $\sigma = (h, \tau)$ with $h < h_0$ and $\tau < \tau_0$. From this estimate, (4.6) and (3.8) we deduce the existence of C_6 independent of σ such that $\|\bar\varphi_\sigma\|_{L^\infty(Q)} \le C_6$ for the same range of σ as before. Now using again (3.26) and (3.27) we conclude that

$$\|\bar u_\sigma\|_{L^\infty(Q)} \le \frac{C_6}{\kappa} \tag{4.7}$$

for $h < h_0$ and $\tau < \tau_0$.

Take a sequence such that $\bar u_\sigma \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(Q)$ as $\sigma \to 0$.

Step II: $\bar u \in U_{ad}$. It is immediate to check that $\|u_h\|_{L^1(\Omega)} = j_h(u_h)$ if $u_h \in U_h = U_{h,0}$ and $\|u_h\|_{L^1(\Omega)} \le j_h(u_h)$ if $u_h \in U_h = U_{h,1}$. Therefore, we have

$$\|\bar u_\sigma\|_{L^\infty(0,T;L^1(\Omega))} = \max_{1\le j\le N_h} \|\bar u_{h,j}\|_{L^1(\Omega)} \le \max_{1\le j\le N_h} j_h(\bar u_{h,j}) \le \gamma,$$

and thus $\{\bar u_\sigma\}_\sigma \subset U_{ad}$. Since U_{ad} is convex and closed in $L^2(Q)$, it is weakly closed as well. Then, the weak convergence $\bar u_\sigma \rightharpoonup \bar u$ in $L^2(Q)$ implies that $\bar u \in U_{ad}$.

Step III: $\bar u$ *is a solution of* (P). Let $\tilde u$ be a solution of (P). For every σ we define

$$u_\sigma = \begin{cases} P_\tau P_h \tilde u = \displaystyle\sum_{j=1}^{N_\tau} \sum_{K\in\mathbb{K}_h} \frac{1}{\tau_j |K|} \int_{I_j}\int_K \tilde u(x,t)\,\mathrm{d}x\,\mathrm{d}t \chi_j \chi_K & \text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,0}, \\ P_\tau E_h \tilde u = \displaystyle\sum_{j=1}^{N_\tau} \sum_{i=1}^{N_h} \frac{1}{\tau_j \int_\Omega e_i\,\mathrm{d}x} \int_{I_j}\int_\Omega \tilde u(x,t) e_i(x)\,\mathrm{d}x\,\mathrm{d}t \chi_j e_i & \text{if } \mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}, \end{cases} \tag{4.8}$$

where P_τ is the $L^2(0,T)$ projection operator defined in the proof of Theorem 3.1, $P_h : L^2(\Omega) \longrightarrow U_{h,0}$ is the $L^2(\Omega)$ projection operator, and $E_h : L^1(\Omega) \longrightarrow U_{h,1}$ is the Carstensen quasi-interpolation operator; see [4]. First we prove that $u_\sigma \in \mathbb{U}_{\sigma,ad}$. In case $U_h = U_{h,0}$ we have $u_\sigma = \sum_{j=1}^{N_\tau} u_{h,j}\chi_j$ and for every $j = 1, \ldots, N_\tau$

$$j_h(u_{h,j}) = \sum_{K\in\mathbb{K}_h} |K| \left| \frac{1}{\tau_j|K|} \int_{I_j}\int_K \tilde u(x,t)\,\mathrm{d}x\,\mathrm{d}t \right| \le \sum_{K\in\mathbb{K}_h} \frac{1}{\tau_j} \int_{I_j}\int_K |\tilde u(x,t)|\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{\tau_j}\int_{I_j} \|\tilde u(t)\|_{L^1(\Omega)}\,\mathrm{d}t \le \gamma.$$

This implies that $u_\sigma \in \mathbb{U}_{\sigma,ad}$. In the case $U_h = U_{h,1}$, we have

$$\begin{aligned} j_h(u_{h,j}) &= \sum_{i=1}^{N_h} \left| \frac{1}{\tau_j \int_\Omega e_i\,\mathrm{d}x} \int_{I_j}\int_\Omega \tilde u(x,t) e_i(x)\,\mathrm{d}x\,\mathrm{d}t \right| \int_\Omega e_i\,\mathrm{d}x \le \frac{1}{\tau_j}\int_{I_j} \left(\int_\Omega |\tilde u(x,t)| \sum_{i=1}^{N_h} e_i\,\mathrm{d}x \right) \mathrm{d}t \\ &= \frac{1}{\tau_j}\int_{I_j} \|\tilde u(t)\|_{L^1(\Omega)}\,\mathrm{d}t \le \gamma. \end{aligned}$$

Using that

$$\sum_{j=1}^{N_\tau}\sum_{K\in\mathbb{K}_h} \chi_j\chi_K = \sum_{j=1}^{N_\tau}\sum_{i=1}^{N_h} e_i\chi_j = 1 \quad \text{in} \quad Q,$$

we get $\|u_\sigma\|_{L^\infty(Q)} \le \|\tilde u\|_{L^\infty(Q)}$ for every σ.

In the case $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,0}$, we have that $P_\tau P_h : L^2(Q) \longrightarrow \mathbb{U}_{\sigma,0}$ is the $L^2(Q)$ projection operator, hence $u_\sigma \to \tilde u$ in $L^2(Q)$ when $\sigma \to 0$. If $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}$, then we have

$$\|\tilde u - u_\sigma\|_{L^2(Q)} \le \|\tilde u - P_\tau\tilde u\|_{L^2(Q)} + \|P_\tau(\tilde u - E_h\tilde u)\|_{L^2(Q)} \le \|\tilde u - P_\tau\tilde u\|_{L^2(Q)} + \|\tilde u - E_h\tilde u\|_{L^2(Q)} \to 0 \text{ as } \sigma \to 0.$$

Indeed Corollary 2.6 implies that $\tilde u \in H^1(Q)$. Hence, from the convergence properties of the Carstensen operator E_h we infer the convergence of the last term in the above expression. The boundedness of $\{u_\sigma\}_\sigma$

in $L^\infty(Q)$ and its strong convergence to $\tilde u$ in $L^2(Q)$ imply the strong convergence in every $L^p(0,T;L^q(\Omega))$ space with $1 \le p,q < \infty$.

Next we prove that $\limsup_{\sigma\to 0} J_\sigma(u_\sigma) \le J(\tilde u)$. Let us denote by $\tilde y$ and y_{u_σ} the continuous states corresponding to $\tilde u$ and u_σ, respectively. We also denote by y_σ the discrete state associated with u_σ. Then, using the established convergence $u_\sigma \to \tilde u$, (2.6), and (3.3) we can easily prove

$$\|\tilde y - y_\sigma\|_{L^\infty(Q)} \le \|\tilde y - y_{u_\sigma}\|_{L^\infty(Q)} + \|y_{u_\sigma} - y_\sigma\|_{L^\infty(Q)} \to 0.$$

The proved convergences of $\{y_\sigma\}_\sigma$ and $\{u_\sigma\}_\sigma$ imply that $J_\sigma(u_\sigma) \to J(\tilde u)$ as $\sigma \to 0$ if $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,0}$; see (3.4). For the case $\mathbb{U}_\sigma = \mathbb{U}_{\sigma,1}$ we have

$$\begin{aligned}\|u_\sigma\|_\sigma^2 &= \sum_{j=1}^{N_\tau}\sum_{i=1}^{N_h}\tau_j\Big(\int_\Omega e_i\,\mathrm{d}x\Big)u_{i,j}^2 = \sum_{j=1}^{N_\tau}\sum_{i=1}^{N_h}\frac{1}{\tau_j\int_\Omega e_i\,\mathrm{d}x}\Big(\int_{I_j}\int_\Omega \tilde u e_i\,\mathrm{d}x\,\mathrm{d}t\Big)^2\\ &\le \sum_{j=1}^{N_\tau}\sum_{i=1}^{N_h}\int_{I_j}\int_\Omega |\tilde u|^2 e_i\,\mathrm{d}x\,\mathrm{d}t = \|\tilde u\|^2_{L^2(Q)},\end{aligned}$$

which leads to the desired inequality $\limsup_{\sigma\to 0} J_\sigma(u_\sigma) \le J(\tilde u)$.

Using the same arguments as above, we deduce that $\bar y_\sigma \to \bar y$ strongly in $L^\infty(Q)$, where $\bar y$ denotes the continuous state associated with $\bar u$. Finally, from the optimality of $\bar u_\sigma$ and the established convergence properties we obtain with (3.4)

$$\begin{aligned}J(\bar u) &\le \liminf_{\sigma\to 0}\Big\{\frac12\|\bar y_\sigma - y_d\|^2_{L^2(Q)} + \frac{\kappa}{2}\|\bar u_\sigma\|^2_{L^2(Q)}\Big\} \le \liminf_{\sigma\to 0} J_\sigma(\bar u_\sigma) \le \limsup_{\sigma\to 0} J_\sigma(\bar u_\sigma)\\ &\le \limsup_{\sigma\to 0} J_\sigma(u_\sigma) \le J(\tilde u) = \inf\,(\mathrm{P}).\end{aligned}$$

These inequalities imply that $\bar u$ is a solution of (P). Moreover, since the identity $J(\bar u) = J(\tilde u)$ holds, we conclude that $J_\sigma(\bar u_\sigma) \to J(\bar u)$. Further we have

$$\begin{aligned}J(\bar u) &\le \liminf_{\sigma\to 0}\Big\{\frac12\|\bar y_\sigma - y_d\|^2_{L^2(Q)} + \frac{\kappa}{2}\|\bar u_\sigma\|^2_{L^2(Q)}\Big\} \le \limsup_{\sigma\to 0}\Big\{\frac12\|\bar y_\sigma - y_d\|^2_{L^2(Q)} + \frac{\kappa}{2}\|\bar u_\sigma\|^2_{L^2(Q)}\Big\}\\ &\le \limsup_{\sigma\to 0} J_\sigma(\bar u_\sigma) \le \limsup_{\sigma\to 0} J_\sigma(u_\sigma) \le J(\tilde u) = J(\bar u).\end{aligned}$$

This property and the strong convergence $\bar y_\sigma \to \bar y$ in $L^2(Q)$ yield that $\|\bar u_\sigma\|_{L^2(Q)} \to \|\bar u\|_{L^2(Q)}$. Together with the weak* convergence $\bar u_\sigma \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(Q)$ this implies the strong convergence $\bar u_\sigma \to \bar u$ in $L^r(0,T;L^2(\Omega))$ for every $r < \infty$. Thus, (4.1) is proved.

The following theorem can be considered as a converse of Theorem 4.1.

Theorem 4.3. *Let $\bar u$ be a strict local minimum of* (P) *in the $L^r(0,T;L^2(\Omega))$ sense with $r \in \big(\frac{4n}{4-n},\infty\big)$. Then, there exist positive numbers τ_0, h_0, ε_0, and a sequence $\{\bar u_\sigma\}_\sigma \subset B_{\varepsilon_0}(\bar u)$ of local minima of* (P_σ) *such that* (4.1) *holds and*

$$J_\sigma(\bar u_\sigma) = \min_{u_\sigma \in \mathbb{U}_{\sigma,ad}\cap B_{\varepsilon_0}(\bar u)} J_\sigma(u_\sigma) \quad \text{for } \tau < \tau_0 \quad \text{and} \quad h < h_0, \tag{4.9}$$

where $B_{\varepsilon_0}(\bar u)$ is the closed ball of $L^r(0,T;L^2(\Omega))$ centered at $\bar u$ with radius ε_0.

Proof. Since $\bar u$ is a strict local minimum of (P) in the $L^r(0,T;L^2(\Omega))$ sense, there exists $\varepsilon_0 > 0$ such that $\bar u$ is the only solution of the problem

$$(\mathrm{Q}) \qquad \inf_{u\in U_{ad}\cap B_{\varepsilon_0}(\bar u)} J(u).$$

Now, we consider the problems

$$(\mathrm{Q}_\sigma) \qquad \inf_{u_\sigma \in \mathbb{U}_{\sigma,ad} \cap B_{\varepsilon_0}(\bar u)} J_\sigma(u_\sigma).$$

If we define u_σ by (4.8) with $\tilde u = \bar u$, then $u_\sigma \in \mathbb{U}_{\sigma,ad}$ and $u_\sigma \to \bar u$ in $L^r(0,T;L^2(\Omega))$. Therefore, there exist $\tau_1 > 0$ and $h_1 > 0$ such that $u_\sigma \in B_{\varepsilon_0}(\bar u)$ for every σ with $\tau < \tau_1$ and $h < h_1$. Hence, $\mathbb{U}_{\sigma,ad} \cap B_{\varepsilon_0}(\bar u)$ is a compact nonempty set for every $\tau < \tau_1$ and $h < h_1$. Then, the continuity of J_σ implies the existence of at least one solution $\bar u_\sigma$ of (Q_σ) for every σ with τ and h satisfying the previous conditions. Since $\{\bar u_\sigma\}_\sigma$ is bounded in $L^r(0,T;L^2(\Omega))$, taking a subsequence if necessary, we can assume that $\bar u_\sigma \rightharpoonup \hat u$ in $L^r(0,T;L^2(\Omega))$ for some $\hat u$. Due to the inclusion $\mathbb{U}_{\sigma,ad} \subset U_{ad}$ we deduce that $\hat u \in U_{ad} \cap B_{\varepsilon_0}(\bar u)$. Moreover, we have

$$J(\hat u) \le \liminf_{\sigma\to 0} J_\sigma(\bar u_\sigma) \le \limsup_{\sigma\to 0} J_\sigma(\bar u_\sigma) \le \limsup_{\sigma\to 0} J_\sigma(u_\sigma) \le J(\bar u).$$

Since $\bar u$ is the unique solution of (Q), this inequality is only possible if $\hat u = \bar u$. Consequently, the whole family $\{\bar u_\sigma\}_\sigma$ converges weakly to $\bar u$ in $L^r(0,T;L^2(\Omega))$ as $\sigma \to 0$ and $J_\sigma(\bar u_\sigma) \to J(\bar u)$. Arguing as in the proof of Theorem 4.1, we deduce the strong convergence $\bar u_\sigma \to \bar u$ in $L^r(0,T;L^2(\Omega))$. This leads to the existence of $\tau_0 \le \tau_1$ and $h_0 \le h_1$ such that $\bar u_\sigma$ belongs to the interior of the ball $B_{\varepsilon_0}(\bar u)$ for every $\sigma = (\tau,h)$ with $\tau < \tau_0$ and $h < h_0$. Hence, every of these $\bar u_\sigma$ is a local minimum of (P_σ) satisfying (4.9). □

The rest of this section is dedicated to the proof of the following theorem.

Theorem 4.4. *Let us assume that $\bar u$ is a local solution of* (P) *in the $L^r(0,T;L^2(\Omega))$ sense with $r \in \big(\frac{4}{4-n},\infty\big)$. We also assume that $J''(\bar u)v^2 > 0$ $\forall v \in C_{\bar u} \setminus \{0\}$. Let $\{\bar u_\sigma\}_\sigma$ be a family of local solutions of problems (P_σ) such that $\bar u_\sigma \to \bar u$ in $L^r(0,T;L^2(\Omega))$; see Theorem 4.3. Then, there exist positive numbers δ_0, τ_0, and C such that the following inequality holds:*

$$\|\bar u_\sigma - \bar u\|_{L^2(Q)} \le C(h+\tau) \quad \text{for every } \sigma = (h,\tau) \text{ with } h < h_0 \text{ and } \tau < \tau_0. \tag{4.10}$$

We prove this theorem arguing by contradiction. If (4.10) does not hold, then there exists a sequence $\{\bar u_{\sigma_k}\}_{k=1}^\infty$ such that $\sigma_k = (h_k,\tau_k) \to 0$ as $k \to \infty$, $h_k > 0$ and $\tau_k > 0$, and

$$\|\bar u_{\sigma_k} - \bar u\|_{L^2(Q)} > k(h_k + \tau_k) \quad \forall k \ge 1. \tag{4.11}$$

We will get a contradiction for this sequence. First we prove the next lemma.

Lemma 4.5. *Let λ be as in* (2.22). *There exists k_0 such that*

$$(J'(\bar u_{\sigma_k}) - J'(\bar u))(\bar u_{\sigma_k} - \bar u) \ge \frac{1}{2}\min\{\lambda,\kappa\}\|\bar u_{\sigma_k} - \bar u\|^2_{L^2(Q)} \quad \forall k \ge k_0. \tag{4.12}$$

Proof. Applying the mean value theorem, we get for some $\hat u_k = \bar u + \theta_k(\bar u_{\sigma_k} - \bar u)$

$$(J'(\bar u_{\sigma_k}) - J'(\bar u))(\bar u_{\sigma_k} - \bar u) = J''(\hat u_k)(\bar u_{\sigma_k} - \bar u)^2. \tag{4.13}$$

Set $v_k = \frac{\bar u_{\sigma_k} - \bar u}{\|\bar u_{\sigma_k} - \bar u\|_{L^2(Q)}}$. Taking a subsequence, if necessary, we can suppose that $v_k \rightharpoonup v$ in $L^2(Q)$. Below we prove that $v \in C_{\bar u}$. Assuming that this is true, then we argue as follows. From (2.10), the fact that

$\|v_k\|_{L^2(Q)} = 1$, and (2.22) we infer

$$\lim_{k\to\infty} J''(\hat u_k)v_k^2 = \lim_{k\to\infty}\left\{\int_Q\Big(1-\frac{\partial^2 a}{\partial y^2}(x,t,y_{\hat u_k})\varphi_{\hat u_k}\Big)z^2_{\hat u_k,v_k}\,\mathrm{d}x\,\mathrm{d}t+\kappa\right\}$$
$$=\int_Q\Big(1-\frac{\partial^2 a}{\partial y^2}(x,t,\bar y)\bar\varphi\Big)z_v^2\,\mathrm{d}x\,\mathrm{d}t+\kappa = J''(\bar u)v^2+\kappa\big(1-\|v\|^2_{L^2(Q)}\big)\ge \kappa+(\lambda-\kappa)\|v\|^2_{L^2(Q)}.$$

Above, we denoted $z_{\hat u_k,v_k} = G'(\hat u_k)v_k$ and $z_v = G'(\bar u)v$, where $G : L^r(0,T;L^2(\Omega)) \longrightarrow H^{2,1}(Q)\cap L^\infty(Q)$ is the mapping associating to each control the associated state. Since $\|v\|_{L^2(Q)} \le 1$, the above inequality proves that

$$\lim_{k\to\infty} J''(\hat u_k)v_k^2 \ge \min\{\lambda,\kappa\}.$$

Therefore, there exists $k_0 > 0$ such that

$$J''(\hat u_k)v_k^2 \ge \frac12\min\{\lambda,\kappa\}\quad \forall k\ge k_0,$$

or equivalently

$$J''(\hat u_k)(\bar u_{\sigma_k}-\bar u)^2 \ge \frac12\min\{\lambda,\kappa\}\|\bar u_{\sigma_k}-\bar u\|^2_{L^2(Q)}\quad \forall k\ge k_0.$$

This inequality along with (4.13) leads to (4.12).

Now, we verify that $v\in C_{\bar u}$. From the optimality of $\bar u$ and the fact that $\bar u_{\sigma_k}\in \mathbb{U}_{\sigma_k,ad}\subset U_{ad}$ we obtain $J'(\bar u)v_k \ge 0$. Then, passing to the limit in this inequality when $k\to\infty$, it follows that $J'(\bar u)v\ge 0$. Let us prove the converse inequality. We consider again the approximations $u_{\sigma_k}\in\mathbb{U}_{\sigma_k,ad}$ defined as in (4.8) with $\tilde u = \bar u$. Then, we have

$$\|u_{\sigma_k}-\bar u\|_{L^2(Q)} \le C_1(h_k+\tau_k)\|\bar u\|_{H^1(Q)}\quad \forall k\ge 1. \tag{4.14}$$

Indeed, if $\mathbb{U}_{\sigma_k} = \mathbb{U}_{\sigma_k,0}$, the above estimate follows from the fact that u_{σ_k} is the $L^2(Q)$ projection of $\bar u$. If $\mathbb{U}_{\sigma_k} = \mathbb{U}_{\sigma_k,1}$, the estimate was proved in [5, Lemma 6.6]. From the local optimality of $\bar u_{\sigma_k}$ we have that $J'_{\sigma_k}(\bar u_{\sigma_k})(u_{\sigma_k}-\bar u_{\sigma_k})\ge 0$. Using this fact we get

$$J'(\bar u)v_k = \frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\left\{J'(\bar u)(\bar u_{\sigma_k}-u_{\sigma_k})+J'(\bar u)(u_{\sigma_k}-\bar u)\right\}$$
$$\le \frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\Big\{[J'(\bar u)-J'(\bar u_{\sigma_k})](\bar u_{\sigma_k}-u_{\sigma_k})+[J'(\bar u_{\sigma_k})-J'_{\sigma_k}(\bar u_{\sigma_k})](\bar u_{\sigma_k}-u_{\sigma_k})+J'(\bar u)(u_{\sigma_k}-\bar u)\Big\}$$
$$= I_{k,1}+I_{k,2}+I_{k,3}.$$

Now, we estimate every $I_{k,i}$ term. For $I_{k,1}$ we use the mean value theorem, the convergence $\bar u_{\sigma_k}\to\bar u$ in $L^r(0,T;L^2(\Omega))$, and (4.14) as follows

$$|I_{k,1}| = \frac{|J''(\bar u+\rho_k(\bar u_{\sigma_k}-\bar u))(\bar u_{\sigma_k}-u_{\sigma_k},\bar u-\bar u_{\sigma_k})|}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}} \le C_2\|\bar u_{\sigma_k}-u_{\sigma_k}\|_{L^2(Q)}$$
$$\le C_2\left\{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}+\|\bar u-u_{\sigma_k}\|_{L^2(Q)}\right\}\to 0 \text{ as } k\to\infty.$$

To estimate $I_{k,2}$ we use (2.9) and (3.5) to get

$$I_{k,2} = \frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\int_Q(\varphi_{\bar u_{\sigma_k}}-\bar\varphi_{\sigma_k})(\bar u_{\sigma_k}-u_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t$$
$$+\frac{\kappa}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\left[\int_Q \bar u_{\sigma_k}(\bar u_{\sigma_k}-u_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t-(\bar u_{\sigma_k},\bar u_{\sigma_k}-u_{\sigma_k})_{\sigma_k}\right]. \tag{4.15}$$

To estimate the first integral in (4.15) we use (3.7) and (4.11) along with the boundedness of $\{\bar u_{\sigma_k}-u_{\sigma_k}\}_{k=0}^\infty$ in $L^2(Q)$ (actually $\|\bar u_{\sigma_k}-u_{\sigma_k}\|_{L^2(Q)}\to 0$) as follows

$$\left|\frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\int_Q(\varphi_{\bar u_{\sigma_k}}-\bar\varphi_{\sigma_k})(\bar u_{\sigma_k}-u_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t\right|\le\frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\|\varphi_{\bar u_{\sigma_k}}-\bar\varphi_{\sigma_k}\|_{L^2(Q)}\|\bar u_{\sigma_k}-u_{\sigma_k}\|_{L^2(Q)}$$
$$\le C_3\frac{h_k^2+\tau_k}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\|\bar u_{\sigma_k}-u_{\sigma_k}\|_{L^2(Q)}\to 0 \text{ as } k\to\infty. \tag{4.16}$$

In the case $\mathbb{U}_{\sigma_k}=\mathbb{U}_{\sigma_k,0}$, the scalar products $(\cdot,\cdot)_{L^2(Q)}$ and $(\cdot,\cdot)_{\sigma_k}$ coincide. Hence, the last two terms of (4.15) cancel and we get from (4.16) that $|I_{k,2}|\to 0$. If $\mathbb{U}_{\sigma_k}=\mathbb{U}_{\sigma_k,1}$, we first observe that

$$(u_\sigma,P_\tau E_h u)_\sigma=\int_Q u_\sigma u\,\mathrm{d}x\,\mathrm{d}t\quad\forall u_\sigma\in\mathbb{U}_\sigma \text{ and }\forall u\in L^1(Q). \tag{4.17}$$

It is immediate to check this identity. Moreover, from (3.4) we have that $\|\bar u_{\sigma_k}\|_{L^2(Q)}\le\|\bar u_{\sigma_k}\|_{\sigma_k}$. This property, (4.17) with $u_\sigma=\bar u_{\sigma_k}$ and $u=\bar u$, (4.14), and (4.11) yield

$$\frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\left\{\int_Q\bar u_{\sigma_k}(\bar u_{\sigma_k}-u_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t-(\bar u_{\sigma_k},\bar u_{\sigma_k}-u_{\sigma_k})_{\sigma_k}\right\}\le\frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\int_Q\bar u_{\sigma_k}(\bar u-u_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t$$
$$\le\frac{\|\bar u-u_{\sigma_k}\|_{L^2(Q)}}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\|\bar u_{\sigma_k}\|_{L^2(Q)}\le C_1\frac{h_k+\tau_k}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\|\bar u_{\sigma_k}\|_{L^2(Q)}\to 0 \text{ as } k\to\infty. \tag{4.18}$$

From (4.15), (4.16), and (4.18) we infer that $\lim_{k\to\infty}I_{k,2}\le 0$. The estimate of the last term $I_{k,3}$ is an immediate consequence of (2.9), (4.11) and (4.14)

$$|I_{k,3}|\le\|\bar\varphi+\kappa\bar u\|_{L^2(Q)}\frac{\|u_{\sigma_k}-\bar u\|_{L^2(Q)}}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\le C_1\|\bar\varphi+\kappa\bar u\|_{L^2(Q)}\frac{h+\tau}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\to 0 \text{ as } k\to\infty.$$

Thus, we have that $J'(\bar u)v=\lim_{k\to\infty}J'(\bar u)v_k\le 0$, and consequently $J'(\bar u)v=0$, which is the first condition to have $v\in C_{\bar u}$.

Now, take $t\in I_\gamma$. This means that $\|\bar u(t)\|_{L^1(\Omega)}=\gamma$. Since $\bar u_{\sigma_k}\in U_{ad}$ we have that $\|\bar u_{\sigma_k}\|_{L^1(\Omega)}\le\gamma$. As a consequence, we get with (2.20) and the convexity of j

$$\begin{aligned}j'(\bar u(t);v_k(t))=&\frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}j'(\bar u(t);\bar u_{\sigma_k}(t)-\bar u(t))\\=&\frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}\lim_{\rho\to 0}\frac{j(\bar u(t)+\rho(\bar u_{\sigma_k}(t)-\bar u(t)))-j(\bar u(t))}{\rho}\\\le&\frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}[j(\bar u_{\sigma_k}(t))-j(\bar u(t))]=\frac{1}{\|\bar u_{\sigma_k}-\bar u\|_{L^2(Q)}}[\|\bar u_{\sigma_k}(t)\|_{L^1(\Omega)}-\gamma]\le 0.\end{aligned}$$

Define now

$$E=\{u\in L^2(Q):\ j'(\bar u(t);u(t))\le 0 \text{ for a.e. } t\in I_\gamma\}.$$

Since the mapping $L^2(Q)\ni u\mapsto j'(\bar u(t);u(t))$ is convex and continuous for a.e. $t\in I_\gamma$, we have that E is closed and convex in $L^2(Q)$, and hence weakly closed. As we have just seen $v_k\in E$ for all k, therefore its weak limit v also belongs to E and we have that $j'(\bar u(t);v(t))\le 0$ for a.e. $t\in I_\gamma$.

It remains to prove that $j'(\bar u(t);v(t))=0$ if $t\in I_\gamma^+$. The inequality $j'(\bar u(t);v(t))\le 0$ for $t\in I_\gamma^+$ implies

$$-\int_{\Omega^+_{\bar u(t)}}v(t)\,\mathrm{d}x+\int_{\Omega^-_{\bar u(t)}}v(t)\,\mathrm{d}x\ge\int_{\Omega^0_{\bar u(t)}}|v(t)|\,\mathrm{d}x. \tag{4.19}$$

Using (2.15), (2.18), and (4.19) we obtain

$$
\begin{aligned}
0 = J'(\bar{u})v &= \int_Q (\bar{\varphi} + \kappa \bar{u}) v \,\mathrm{d}x\,\mathrm{d}t = -\int_Q \bar{\mu} v \,\mathrm{d}x\,\mathrm{d}t = -\int_{I_\gamma^+}\int_\Omega \bar{\mu} v \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{I_\gamma^+}\left[\int_{\Omega^+_{\bar{u}(t)}} \|\bar{\mu}(t)\|_{L^\infty(\Omega)} v \,\mathrm{d}x - \int_{\Omega^-_{\bar{u}(t)}} \|\bar{\mu}(t)\|_{L^\infty(\Omega)} v \,\mathrm{d}x + \int_{\Omega^0_{\bar{u}(t)}} \bar{\mu} v \,\mathrm{d}x\right]\mathrm{d}t \\
&\geq \int_{I_\gamma^+}\int_{\Omega^0_{\bar{u}(t)}} \left[\|\bar{\mu}(t)\|_{L^\infty(\Omega)}|v| - \bar{\mu} v\right]\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

This inequality is possible if and only if $\|\bar{\mu}(t)\|_{L^\infty(\Omega)}|v| = \bar{\mu} v$ in $\Omega^0_{\bar{u}(t)} \times (0,T)$. Now, from this latter identity and the fact that $\bar{\mu}(t) = 0$ if $t \notin I_\gamma^+$ we infer

$$
\begin{aligned}
0 &\geq \int_{I_\gamma^+} \|\bar{\mu}(t)\|_{L^\infty(\Omega)} j'(\bar{u}(t); v(t))\,\mathrm{d}t = \int_{I_\gamma^+} \|\bar{\mu}(t)\|_{L^\infty(\Omega)}\left[\int_{\Omega^+_{\bar{u}(t)}} v(t)\,\mathrm{d}x - \int_{\Omega^-_{\bar{u}(t)}} v(t)\,\mathrm{d}x + \int_{\Omega^0_{\bar{u}(t)}} |v(t)|\,\mathrm{d}x\right]\mathrm{d}t \\
&= \int_{I_\gamma^+}\int_\Omega \bar{\mu} v \,\mathrm{d}x\,\mathrm{d}t = \int_Q \bar{\mu} v \,\mathrm{d}x\,\mathrm{d}t = -J'(\bar{u})v = 0,
\end{aligned}
$$

which implies that $j'(\bar{u}(t); v(t)) = 0$ for almost every $t \in I_\gamma^+$. This concludes the proof of $v \in C_{\bar{u}}$. □

Proof of Theorem 4.4. Let us take k_0 big enough so that (3.7) and (4.12) hold. The goal is to prove that (4.11) is not possible. To this end, we take $u_{\sigma_k} \in \mathbb{U}_{\sigma_k,ad}$ as in the proof of Lemma 4.5. Using the optimality of $\bar{u}_{\sigma_k}$ we get

$$
\begin{aligned}
&0 \leq J'(\bar{u}_{\sigma_k})(u_{\sigma_k} - \bar{u}_{\sigma_k}) \\
&\;= J'(\bar{u}_{\sigma_k})(\bar{u} - \bar{u}_{\sigma_k}) + J'(\bar{u})(u_{\sigma_k} - \bar{u}) + [J'(\bar{u}_{\sigma_k}) - J'(\bar{u})](u_{\sigma_k} - \bar{u}) + [J'_{\sigma_k}(\bar{u}_{\sigma_k}) - J'(\bar{u}_{\sigma_k})](u_{\sigma_k} - \bar{u}_{\sigma_k}).
\end{aligned}
$$

We also have that $J'(\bar{u})(\bar{u}_{\sigma_k} - \bar{u}) \geq 0$. Adding these two inequalities and using (4.12) we infer

$$
\begin{aligned}
&\frac{1}{2}\min\{\lambda,\kappa\}\|\bar{u}_{\sigma_k} - \bar{u}\|^2_{L^2(Q)} \leq [J'(\bar{u}_{\sigma_k}) - J'(\bar{u})](\bar{u}_{\sigma_k} - \bar{u}) \leq J'(\bar{u})(u_{\sigma_k} - \bar{u}) \\
&+ [J'(\bar{u}_{\sigma_k}) - J'(\bar{u})](u_{\sigma_k} - \bar{u}) + [J'_{\sigma_k}(\bar{u}_{\sigma_k}) - J'(\bar{u}_{\sigma_k})](u_{\sigma_k} - \bar{u}_{\sigma_k}) = I_{k,1} + I_{k,2} + I_{k,3}. \qquad (4.20)
\end{aligned}
$$

To estimate $I_{k,1}$ we use the property

$$
\|u_{\sigma_k} - \bar{u}\|_{H^1(Q)^*} \leq C_1(h_k^2 + \tau_k^2)\|\bar{u}\|_{H^1(Q)}; \qquad (4.21)
$$

see [5]. With this inequality we get

$$
|I_{k,1}| \leq \|\bar{\varphi} + \kappa\bar{u}\|_{H^1(Q)}\|u_{\sigma_k} - \bar{u}\|_{H^1(Q)^*} \leq C_1(h_k^2 + \tau_k^2)\|\bar{\varphi} + \kappa\bar{u}\|_{H^1(Q)}. \qquad (4.22)
$$

For the estimate of $I_{k,2}$ we use the mean value theorem, the convergence $\bar{u}_{\sigma_k} \to \bar{u}$ in $L^r(0,T;L^2(\Omega))$, and (4.14) to get

$$
\begin{aligned}
|I_{k,2}| &= |J''(\bar{u} + \rho_k(\bar{u}_{\sigma_k} - \bar{u}))(u_{\sigma_k} - \bar{u}, \bar{u}_{\sigma_k} - \bar{u})| \leq C_2\|u_{\sigma_k} - \bar{u}\|_{L^2(Q)}\|\bar{u}_{\sigma_k} - \bar{u}\|_{L^2(Q)} \\
&\leq C_3(h + \tau)\|\bar{u}_{\sigma_k} - \bar{u}\|_{L^2(Q)}. \qquad (4.23)
\end{aligned}
$$

To deal with $I_{k,3}$ we apply (2.9) and (3.5) to deduce

$$
I_{k,3} = \int_Q (\bar{\varphi}_{\sigma_k} - \varphi_{\bar{u}_{\sigma_k}})(u_{\sigma_k} - \bar{u}_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t + \kappa\left[(\bar{u}_{\sigma_k}, u_{\sigma_k} - \bar{u}_{\sigma_k})_{\sigma_k} - \int_Q \bar{u}_{\sigma_k}(u_{\sigma_k} - \bar{u}_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t\right]. \qquad (4.24)
$$

With (3.7) and (4.14) we obtain

$$
\begin{aligned}
&\left|\int_Q (\bar\varphi_{\sigma_k} - \varphi_{\bar u_{\sigma_k}})(u_{\sigma_k} - \bar u_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t\right| \le \|\bar\varphi_{\sigma_k} - \varphi_{\bar u_{\sigma_k}}\|_{L^2(Q)}\|u_{\sigma_k} - \bar u_{\sigma_k}\|_{L^2(Q)}\\
&\le C_4(h_k^2 + \tau_k)\Big(\|u_{\sigma_k} - \bar u\|_{L^2(Q)} + \|\bar u - \bar u_{\sigma_k}\|_{L^2(Q)}\Big)\\
&\le C_5(h_k^2 + \tau_k)(h_k + \tau_k) + C_4(h_k^2 + \tau_k)\|\bar u - \bar u_{\sigma_k}\|_{L^2(Q)}.
\end{aligned} \tag{4.25}
$$

The last two terms of (4.24) cancel if $\mathbb{U}_{\sigma_k} = \mathbb{U}_{\sigma_k,0}$. In the case $\mathbb{U}_{\sigma_k} = \mathbb{U}_{\sigma_k,1}$, (3.4), (4.17), (4.14), and (4.21) yield

$$
\begin{aligned}
&(\bar u_{\sigma_k}, u_{\sigma_k} - \bar u_{\sigma_k})_{\sigma_k} - \int_Q \bar u_{\sigma_k}(u_{\sigma_k} - \bar u_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t \le \int_Q \bar u_{\sigma_k}(\bar u - u_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_Q (\bar u_{\sigma_k} - \bar u)(\bar u - u_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t + \int_Q \bar u(\bar u - u_{\sigma_k})\,\mathrm{d}x\,\mathrm{d}t\\
&\le \|\bar u_{\sigma_k} - \bar u\|_{L^2(Q)}\|\bar u - u_{\sigma_k}\|_{L^2(Q)} + \|\bar u\|_{H^1(Q)}\|\bar u - u_{\sigma_k}\|_{H^1(Q)^*}\\
&\le C_6(h_k + \tau_k)\|\bar u_{\sigma_k} - \bar u\|_{L^2(Q)} + C_7(h_k^2 + \tau_k^2)\|\bar u\|_{H^1(Q)}.
\end{aligned} \tag{4.26}
$$

The estimates (4.24)-(4.16) lead to

$$
|I_{k,3}| \le C_8(h_k^2 + \tau_k^2) + C_9(h_k + \tau_k)\|\bar u_{\sigma_k} - \bar u\|_{L^2(Q)}. \tag{4.27}
$$

Finally, (4.20), (4.22), (4.23), and (4.27) imply

$$
\|\bar u_{\sigma_k} - \bar u\|_{L^2(Q)} \le C_{10}(h_k + \tau_k) \quad \forall k \ge k_0,
$$

which contradicts (4.11).

5 Numerical Examples

Let Ω be $(0,1)^n$, $n = 1$ or $n = 2$, $A = -\Delta$, $a \equiv 0$, $y_0 \equiv 0$, $T = 1$, $\kappa = 10^{-4}$, and

$$
y_d(x,t) = \exp(-20[(x-0.2)^2 + (t-0.2)^2]) + \exp(-20[(x-0.7)^2 + (t-0.9)^2]) \text{ if } n = 1,
$$

or

$$
\begin{aligned}
y_d(x,t) \;=\; &\exp(-20[(x_1-0.2)^2 + (x_2-0.2)^2 + (t-0.2)^2])\\
&+ \exp(-20[(x_1-0.7)^2 + (x_2-0.7)^2 + (t-0.9)^2]) \text{ if } n = 2.
\end{aligned}
$$

Notice that all the results obtained in the paper are also valid for dimension $n = 1$. For dimension 1, these data correspond to the problem presented in [7, Remark 2.11] and also studied in [10, 5]. The problem in dimension 2 was introduced in [5].

To discretize the problems, we use two families of uniform partitions in space and time, with $h_i = 2^{-i}\sqrt{2^n - 1}$ and $\tau_j = 2^{-j}$, and denote $\sigma_{i,j} = (h_i, \tau_j)$. The discrete problems are solved using a projected gradient algorithm with the Barzilai-Borwein strategy as line search; see [2, eq. (5)]. Projection strategies onto the L^1-ball can be found in [14].

5.1 Sparsity patterns

In Figure 1 we show the solutions obtained for the one-dimensional problem as the bound parameter γ varies in $\{0.5, 1, 2, 3\}$. To discretize the problem we use the control space $\mathbb{U}_{\sigma,1}$ at the discretization level $i = j = 10$. We also plot at the left hand side of the graph the norm in $L^1(\Omega)$ of $\bar{u}_\sigma(\cdot, t)$ for all $t \in [0, 1]$, (this norm is computed with the approximation $j_h(u_{h,j})$). We use a dark green line for the norm and a magenta line for the bound. Notice that when the control constraint is attained, the solution exhibits a sparsity pattern that varies with time. We have coloured in grey the zero-level set of $\bar{u}_\sigma$ to emphasize this behaviour. For $\gamma = 0.5$ and $\gamma = 1$, we have that the control constraint is active for all $t \in [0, 1]$ (green line and magenta line coincide). For $\gamma = 2$, we have that $\|\bar{u}_\sigma(\cdot, t)\|_{L^1(\Omega)} < \gamma$ if $t \in J_1 = (0.4814, 0.5723)$ and if $t > 0.9980$; for $\gamma = 3$, $\|\bar{u}_\sigma(\cdot, t)\|_{L^1(\Omega)} < \gamma$ if $t \in J_2 = (0.4502, 0.6182)$ and if $t > 0.9971$. Black lines are drawn to separate these regions. As soon as the norm constraint is not active we do not observe sparsity, in the sense that there are no subintervals in space where the control is identically zero. This behavior is consistent with the optimality condition expressed in (2.18).

The sparsity behavior obtained by means of the constraint imposed by $u \in U_{ad}$ should also be compared to sparsity phenomena implied by nonsmooth cost-functionals, as considered in [7], for example. The functional in that paper, which is closest to the situation of the present one is given by $u \to \|u\|_{L^2(0,T;L^1(\Omega))}$, ie. it considers the L^2-norm in time, compared to the L^∞-norm used here. In both cases the L^1 norm in space is used. In [7, Figure 1] a numerical result with the same desired state as in Figure 1 of the present paper is presented. It lies in the nature of these two different sparsity enhancing approaches, that the solution in [7, Figure 1] also exhibits intervals of sparsity in the regions corresponding to J_1, J_2.

In Figure 2 we show, at nine different instants of time, the solution obtained for the two-dimensional problem, using the control space $\mathbb{U}_{\sigma,0}$ at the discretization level $i = j = 7$. The control constraint parameter is set to $\gamma = 2$. The norm of the optimal control in $L^1(\Omega)$ is also reported at the indicated time instances. Again, the solution exhibits a sparsity pattern that varies with time, and there is not sparsity if the control constraint is inactive. The subdomains where $\bar{u}_\sigma(x, t)$ vanishes are coloured in grey.

5.2 Convergence rates

We show convergence rates for the problem in dimension 1. In this case we take the bound $\gamma = 4$. Since we do not have the analytic solution, we denote $I = 13$ and take as reference solution the one obtained for $\sigma_{I,I}$.

Three tests are carried out for each of the three discretizations of the control proposed in Section 3.2. In the first test, we take $h_i = \tau_i$, $i = 8, 9, 10$; in the second one, we take a fixed fine discretization in time given by τ_I, $I = 13$, and solve for h_i, $i = 8, 9, 10$; finally, we fix the discretization parameter in space to h_I, $I = 13$, and solve for τ_i, $i = 8, 9, 10$. We measure the experimental order of convergence (EOC) between two consecutive simultaneous refinement levels by setting

$$EOC \quad = \quad \log_2 \|\bar{u}_{\sigma_{I,I}} - \bar{u}_{\sigma_{i-1,i-1}}\|_{L^2(Q)} - \log_2 \|\bar{u}_{\sigma_{I,I}} - \bar{u}_{\sigma_{i,i}}\|_{L^2(Q)},$$

and analogously for the refinement in space and in time, respectively.

Results are shown in Table 1 for simultaneous refinement, Table 2 for refinement in space, and Table 3 for refinement in time. The observed orders of convergence for the control are as predicted in Theorem 4.4, with the exception of the spatial convergence described in Table 2, when we use continuous piecewise approximations of the control. This can be expected from the improved spatial regularity exhibited by the solution; see [27] or [9] for similar situations. However, the method of proof used in the previous references cannot be applied here. For the convenience of the reader, we have also included the experimental orders of convergence for the error in the state variable. A superconvergence phenomenon as the one described in [25] can also be observed in Table 2.

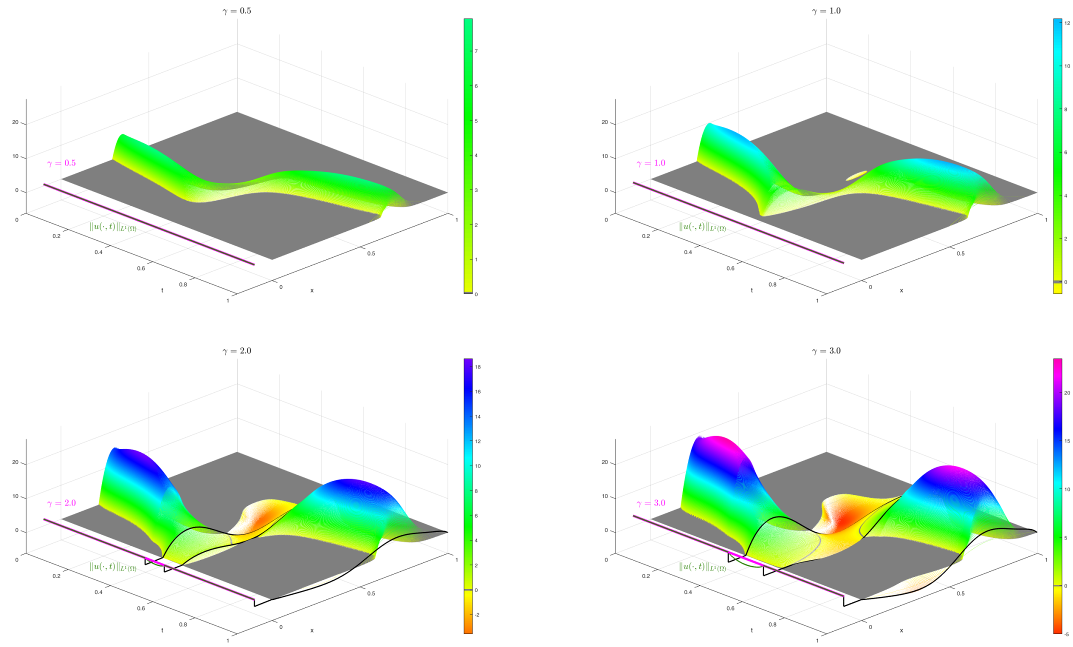

Figure 1: 1D problem. Continuous piecewise linear approximation in space, piecewise constant approximation in time, of the optimal control for different values of γ. The norm in $L^1(\Omega)$ at every instant of time is also shown with a dark green line located on the plane $x = -0.1$ together with the magenta line $z = \gamma$. In grey, the zero-level set of $\bar{u}_\sigma$.

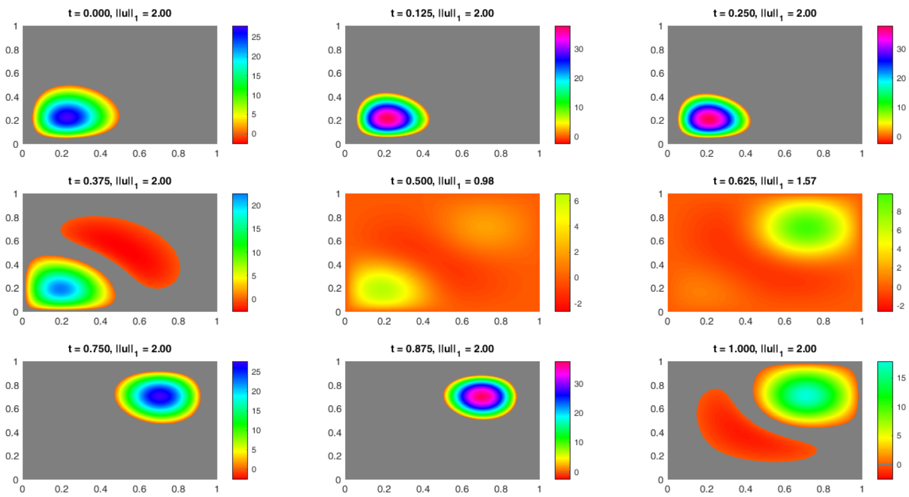

Figure 2: 2D problem. Piecewise constant approximation of the optimal control. In grey, the level sets $\bar{u}_\sigma(\cdot, t_j) = 0$.

	$\mathbb{U}_{\sigma,0}$		$\mathbb{U}_{\sigma,1}$	
$h_i = \tau_i$	$\|\bar{u}_{\sigma_I,I} - \bar{u}_{\sigma_i,i}\|_{L^2(Q)}$	*EOC*	$\|\bar{u}_{\sigma_I,I} - \bar{u}_{\sigma_i,i}\|_{L^2(Q)}$	*EOC*
2^{-8}	$2.01\mathrm{E}-1$	–	$1.76\mathrm{E}-1$	–
2^{-9}	$1.02\mathrm{E}-1$	0.98	$8.93\mathrm{E}-2$	0.98
2^{-10}	$5.11\mathrm{E}-2$	0.99	$4.49\mathrm{E}-2$	0.99

$h_i = \tau_i$	$\|\bar{y}_{\sigma_I,I} - \bar{y}_{\sigma_i,i}\|_{L^2(Q)}$	*EOC*	$\|\bar{y}_{\sigma_I,I} - \bar{y}_{\sigma_i,i}\|_{L^2(Q)}$	*EOC*
2^{-8}	$2.75\mathrm{E}-3$	–	$2.75\mathrm{E}-3$	–
2^{-9}	$1.36\mathrm{E}-3$	1.02	$1.36\mathrm{E}-3$	1.02
2^{-10}	$6.49\mathrm{E}-3$	1.06	$6.49\mathrm{E}-4$	1.06

Table 1: Experimental order of convergence. Simultaneous refinement in space and time.

	$\mathbb{U}_{\sigma,0}$		$\mathbb{U}_{\sigma,1}$	
h_i	$\|\bar{u}_{\sigma_I,I} - \bar{u}_{\sigma_i,I}\|_{L^2(Q)}$	*EOC*	$\|\bar{u}_{\sigma_I,I} - \bar{u}_{\sigma_i,I}\|_{L^2(Q)}$	*EOC*
2^{-8}	$9.86\mathrm{E}-2$	–	$1.10\mathrm{E}-02$	–
2^{-9}	$4.93\mathrm{E}-2$	1.00	$3.87\mathrm{E}-03$	1.51
2^{-10}	$2.45\mathrm{E}-2$	1.01	$1.34\mathrm{E}-03$	1.53

h_i	$\|\bar{y}_{\sigma_I,I} - \bar{y}_{\sigma_i,I}\|_{L^2(Q)}$	*EOC*	$\|\bar{y}_{\sigma_I,I} - \bar{y}_{\sigma_i,I}\|_{L^2(Q)}$	*EOC*
2^{-8}	$1.35\mathrm{E}-5$	–	$1.80\mathrm{E}-05$	–
2^{-9}	$2.79\mathrm{E}-6$	2.28	$4.85\mathrm{E}-06$	1.90
2^{-10}	$5.85\mathrm{E}-7$	2.25	$1.21\mathrm{E}-06$	2.00

Table 2: Experimental order of convergence. Refinement in space.

	$\mathbb{U}_{\sigma,0}$		$\mathbb{U}_{\sigma,1}$	
τ_i	$\|\bar{u}_{\sigma_I,I} - \bar{u}_{\sigma_I,i}\|_{L^2(Q)}$	*EOC*	$\|\bar{u}_{\sigma_I,I} - \bar{u}_{\sigma_I,i}\|_{L^2(Q)}$	*EOC*
2^{-8}	$1.76\mathrm{E}-1$	–	$1.76\mathrm{E}-1$	–
2^{-9}	$8.93\mathrm{E}-2$	0.98	$8.93\mathrm{E}-2$	0.98
2^{-10}	$4.49\mathrm{E}-2$	0.99	$4.49\mathrm{E}-2$	0.99

τ_i	$\|\bar{y}_{\sigma_I,I} - \bar{y}_{\sigma_I,i}\|_{L^2(Q)}$	*EOC*	$\|\bar{y}_{\sigma_I,I} - \bar{y}_{\sigma_I,i}\|\|_{L^2(Q)}$	*EOC*
2^{-8}	$2.75\mathrm{E}-3$	–	$2.75\mathrm{E}-3$	–
2^{-9}	$1.35\mathrm{E}-3$	1.02	$1.35\mathrm{E}-3$	1.02
2^{-10}	$6.49\mathrm{E}-2$	1.06	$6.49\mathrm{E}-4$	1.06

Table 3: Experimental order of convergence. Refinement in time.

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